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Proofs and Certificates for Max-SAT

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Proofs for SAT

SAT Problem

Let ϕ be a CNF formula. Solving the Satisfiability (SAT) problem consists in determining whether there exists an assignments which satisfies ϕ .

Resolution (Robinson, 1965)

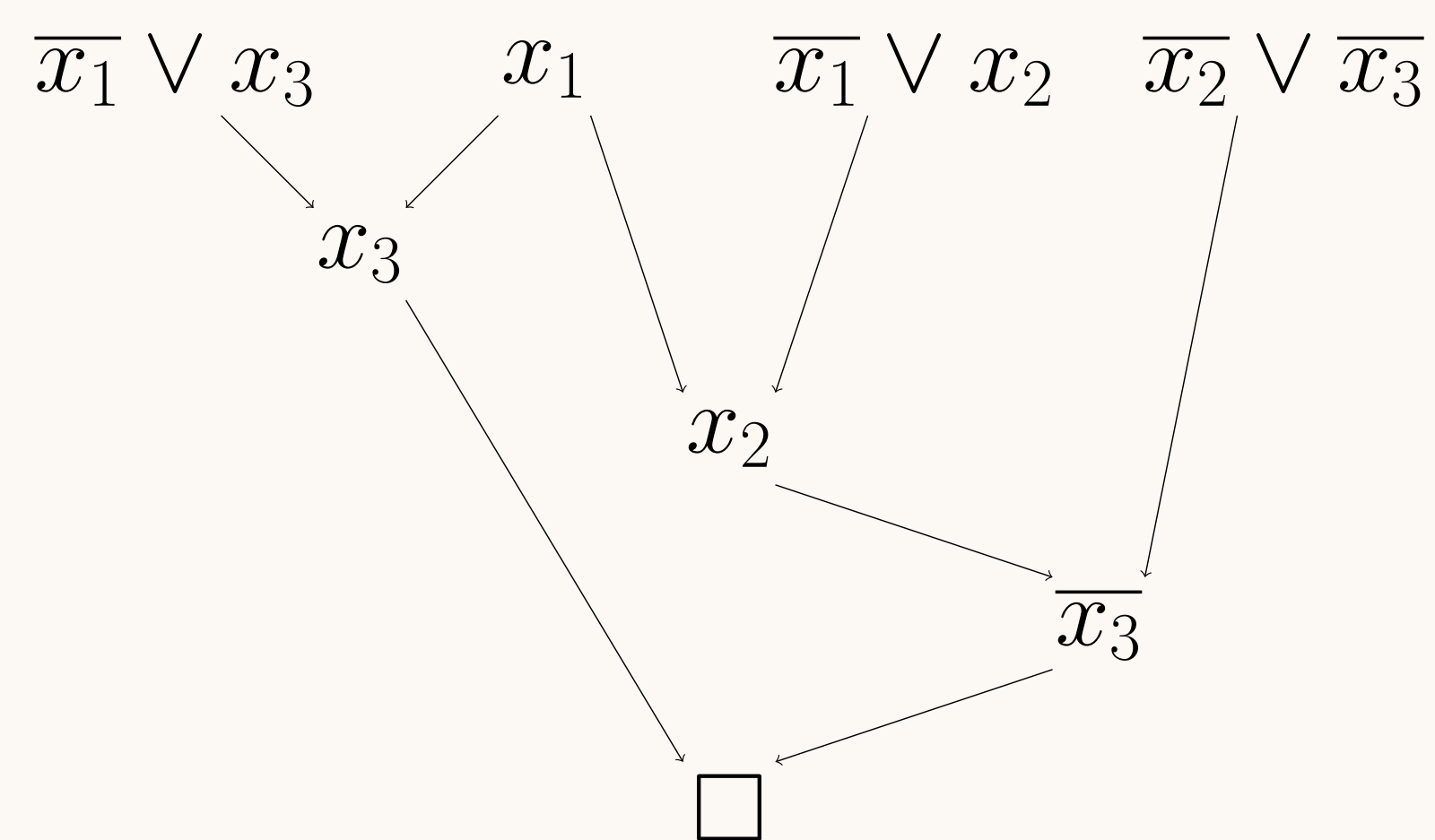
Let $x \vee A$ and $\bar{x} \vee B$ two clauses. The resolution rule deduces a third additional clause as follows:

$$\frac{c_1 = x \vee A \quad c_2 = \bar{x} \vee B}{c_3 = A \vee B}$$

Resolution Refutation

Let ϕ be an unsatisfiable formula. A resolution refutation is a sequence of resolution steps deducing the empty clause \square .

Let $\phi = (\bar{x}_1 \vee x_3) \wedge (x_1) \wedge (\bar{x}_1 \vee x_2) \wedge (\bar{x}_2) \wedge (\bar{x}_2 \vee \bar{x}_3)$. This formula is unsatisfiable and a resolution refutation is given below.



Proofs for Max-SAT

Max-SAT Problem

Let ϕ be a CNF formula. Solving the Maximum Satisfiability (Max-SAT) problem consists in determining the minimum number of clauses in ϕ that each assignment must falsify.

Max-SAT Resolution (Bonet et al., 2007)

Given two clauses $c_1 = x \vee A$ and $c_2 = \bar{x} \vee B$ where $A = a_1 \vee \dots \vee a_s$ and $B = b_1 \vee \dots \vee b_t$, the Max-SAT resolution rule replaces these two clauses by a set of new clauses as follows:

$$\frac{c_1 = x \vee A \quad c_2 = \bar{x} \vee B}{\begin{array}{l} c_r = A \vee B \\ cc_1 = x \vee A \vee \bar{b}_1 \\ \vdots \\ cc_t = x \vee A \vee b_1 \vee \dots \vee b_{t-1} \vee \bar{b}_t \\ cc_{t+1} = \bar{x} \vee B \vee \bar{a}_1 \\ \vdots \\ cc_{t+s} = \bar{x} \vee B \vee a_1 \vee \dots \vee a_{s-1} \vee \bar{a}_s \end{array}}$$

Split Rule

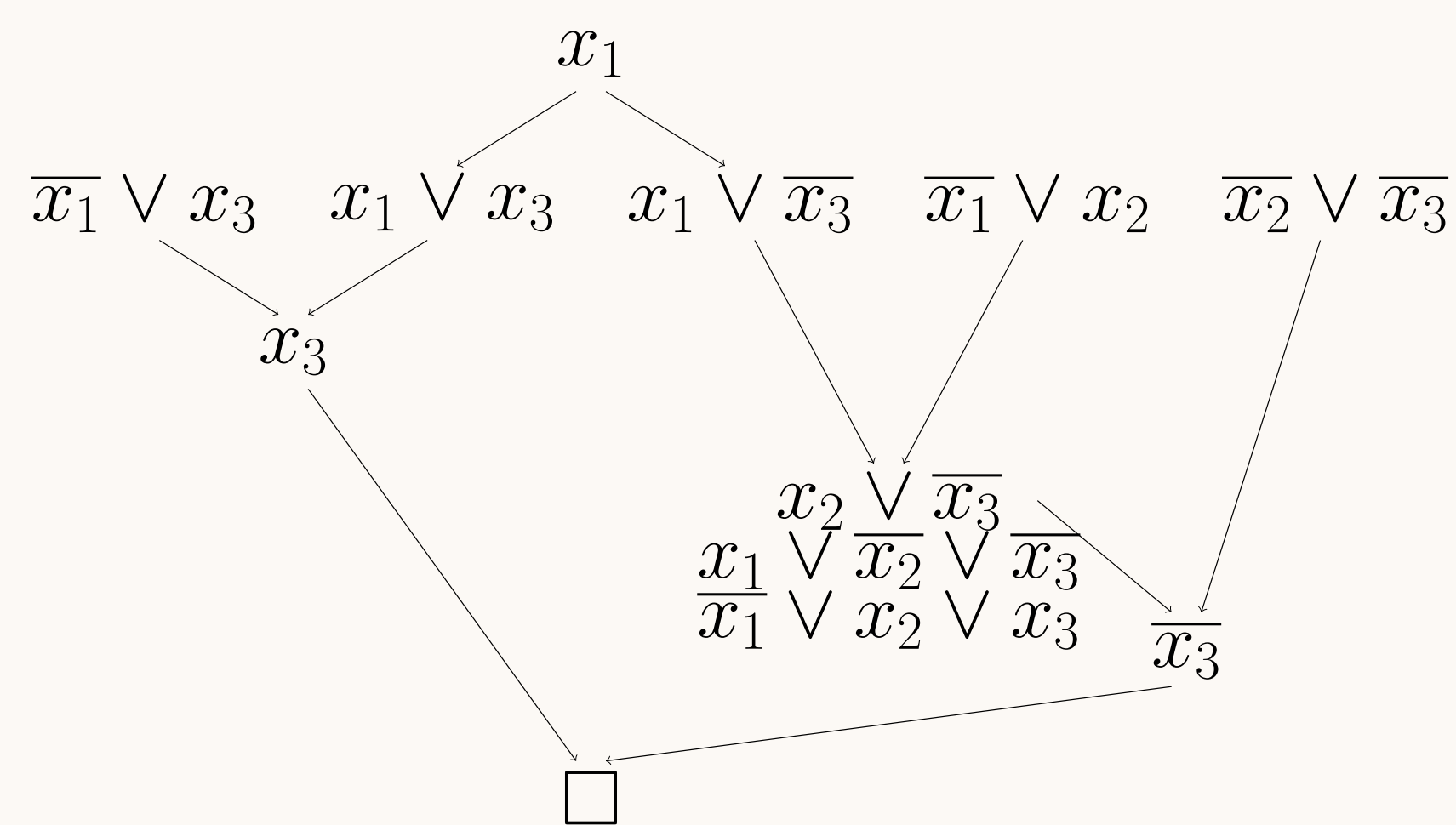
Given a clause $c_1 = A$, the split rule replaces the clause c_1 by two new clauses as follows:

$$\frac{c_1 = A}{c_2 = x \vee A \quad c_3 = \bar{x} \vee A}$$

Max-Refutation

Let ϕ an unsatisfiable formula. A Max-SAT refutation (or simply max-refutation) is a sequence of Max-SAT inference steps deducing the empty clause \square .

Let $\phi = (\bar{x}_1 \vee x_3) \wedge (x_1) \wedge (\bar{x}_1 \vee x_2) \wedge (\bar{x}_2) \wedge (\bar{x}_2 \vee \bar{x}_3)$. This formula is unsatisfiable and a max-refutation is given below.



Open Problems

- Can we adapt a resolution refutation into a max-refutation without considerably increasing its size?
- Can we effectively construct certificates, i.e. proofs of the optimum, for the Max-SAT Problem?

From Resolution Refutations to Max-Refutations

→ We introduce linear adaptations for several classes of resolution refutations and a general adaptation which is exponential in the worst case as shown below.

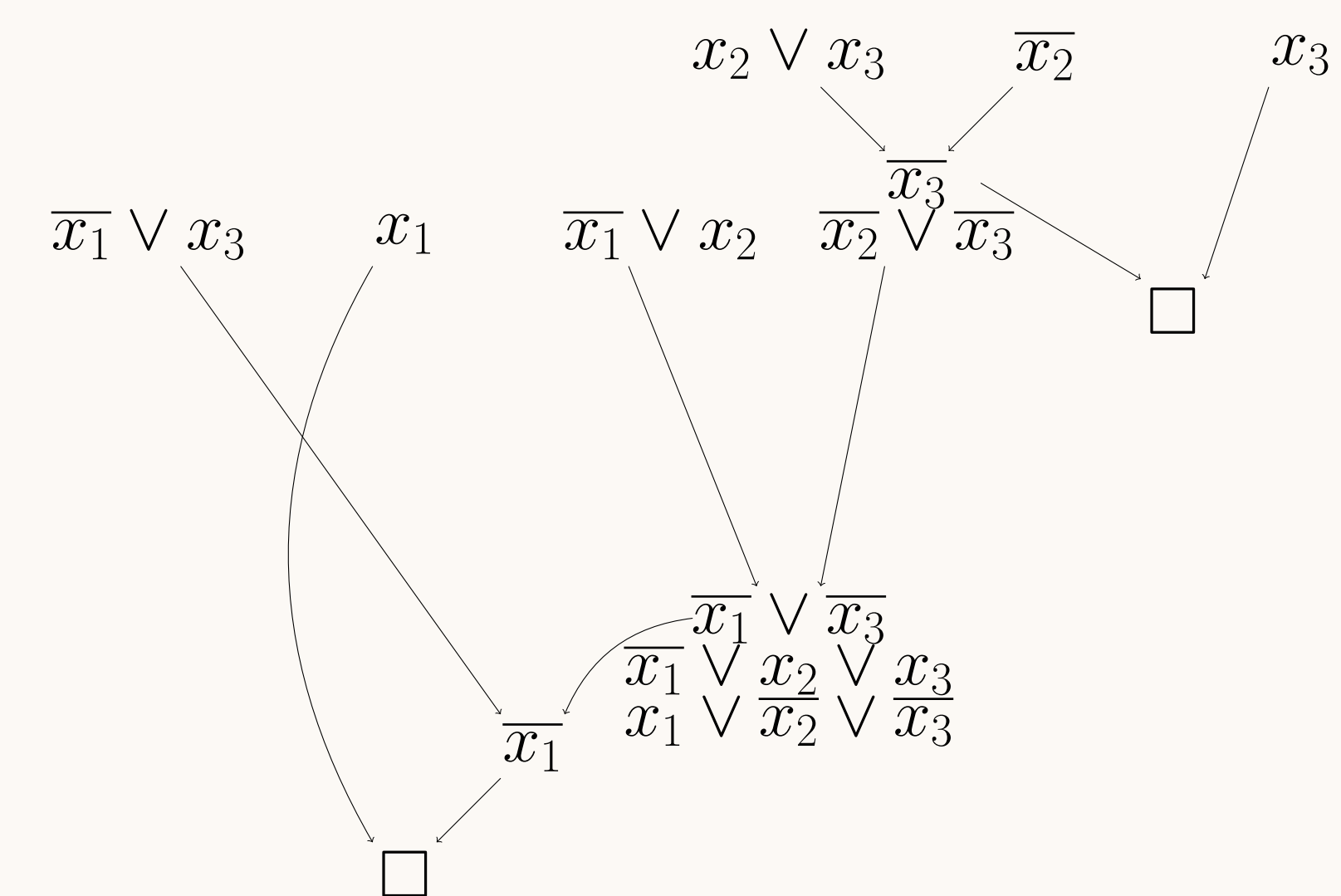
| Resolution Refutation Class | Resulting Max-Refutation Size |
|-----------------------------|-------------------------------|
| Read-Once | Linear (Bonet et al., 2007) |
| Semi-Read-Once | Linear |
| Tree-Like Regular | Linear |
| Tree-Like | Linear |
| Semi-Tree-Like | Linear |
| Unrestricted | Exponential |

Let $\phi = (\bar{x}_1 \vee x_3) \wedge (x_1) \wedge (\bar{x}_1 \vee x_2) \wedge (\bar{x}_2) \wedge (\bar{x}_2 \vee \bar{x}_3)$. The given resolution refutation on the left is tree-like and can be adapted into the max-refutation given after it on the left.

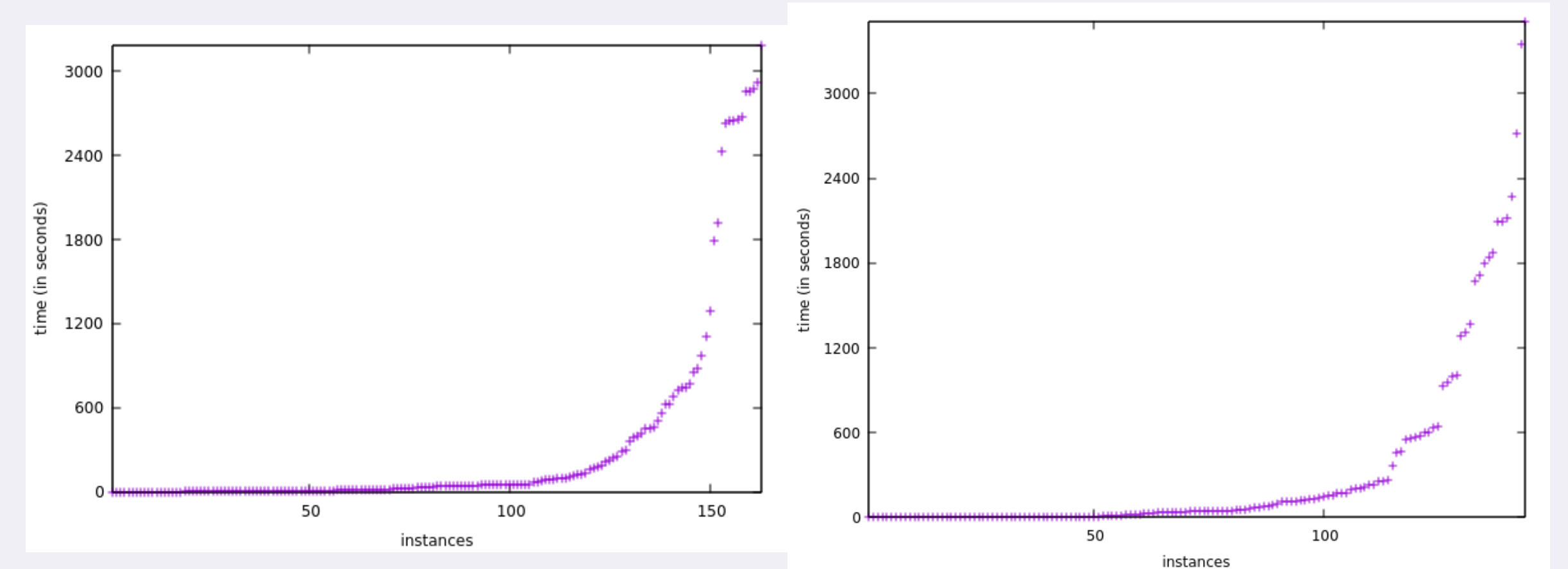
Building Certificates for Max-SAT

→ We introduce a new tool called MS-Builder which iteratively calls a SAT oracle to extract resolution refutations which are then adapted into max-refutations using our results above.

Let $\phi = (\bar{x}_1 \vee x_3) \wedge (x_1) \wedge (\bar{x}_1 \vee x_2) \wedge (\bar{x}_2) \wedge (\bar{x}_3) \wedge (x_2 \vee x_3)$. A Max-SAT certificate of this formula (whose optimum is 2) is composed of the transformation below and the assignment $\alpha = \{\bar{x}_1, \bar{x}_2, \bar{x}_3\}$ which falsifies exactly two clauses. The transformation includes two max-refutations adapted from resolution refutations extracted by the SAT oracle.



→ We conduct an empirical study on the 2020 Max-SAT Evaluation benchmark. MS-Builder has succeeded to construct full proofs for 163 unweighted instances over 576 and for 144 weighted instances over 600 in reasonable running time as shown in the figures below. The tool has also succeed in building at least half of the proofs (with respect to the optimum value) of 302 unweighted instances over 463 and weighted 326 instances over 489 for which the optimum cost is known. The table below shows the distribution of encountered resolution types.



| Type of resolution refutation | Unweighted instances | | Weighted instances | |
|-------------------------------|----------------------|------------|--------------------|------------|
| | Number | Percentage | Number | Percentage |
| read-once | 169,239 | 83.7 % | 135,594 | 35.60 % |
| semi-read-once | 24,556 | 12.1 % | 87,748 | 23.04 % |
| tree-like regular | 2,879 | 1.4 % | 23,612 | 6.20 % |
| tree-like | 1,795 | 0.9 % | 87,529 | 22.99 % |
| unrestricted | 3,799 | 1.9 % | 46,337 | 12.17 % |

References

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