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Functional Interval Estimation for Continuous-Time Linear Systems^{*}

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Abstract: This paper presents a functional interval estimation method for continuous-time systems subject to disturbances and noise. First, a robust functional observer is synthesised to attenuate the effects of uncertainties. Then, based on interval analysis, an interval-based method is proposed to achieve tight functional interval estimation. The proposed method can provide accurate functional interval estimation results for continuous-time systems without using the stringent cooperativity condition and the traditionally required condition on the time derivative of the measurement noise. Simulation results are given to illustrate the effectiveness and superiority of the proposed method.

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Keywords: Functional interval estimation, continuous-time linear systems, robust functional observer, interval analysis

1. INTRODUCTION

A functional observer is an estimator of a functional of the system variables (Fernando et al. (2010)). Different from the traditional state observer, it can estimate the desired functional directly rather than estimating all the individual states and then combining them to get the sought estimation of the linear functional. Moreover, the functional observer is a special form of reduced order observer (Fernando and Trinh (2014)), which has relaxed design conditions and low computational complexity (Darouach (2000)). Thanks to these advantages, many excellent works related to functional observers have been reported in the literature during the past decades (Xiong and Saif (2003)).

However, in practice, it is impossible to obtain an exact point estimation due to the presence of uncertainties. Therefore, in recent years, the interval estimation methods have received considerable attention and have been applied in various fields (Efimov et al. (2013b); Wang et al. (2022)). In the literature, there are mainly two kinds of interval estimation methods, namely the set-based methods and interval observers. Both of them are based on the assumption that the uncertainties are unknown but bounded. The set-based method aims to construct compact sets to enclose all the possible values of the actual state vector inside via ellipsoids (Zhang et al. (2021)), zonotopes (Wang et al. (2018); Li et al. (2020)) and so on. Due to

its recursive set operations, the set-based method is only suitable for discrete-time systems (Tang et al. (2020)). The basic idea of interval observers is to construct two coupled sub-observers to estimate the upper and lower bounds of system state (Li et al. (2019)). As pointed out in Efimov et al. (2013a), interval observers have less computational complexity than the set-based method and can be used for both discrete-time and continuous-time systems. The most commonly adopted method to design interval observers is to make the error dynamics cooperative based on the monotone system theory (Wang et al. (2015)). Unfortunately, it is not a trivial work to construct cooperative error dynamics. To handle this problem, the coordinate transformation approach was used to relax the design conditions (Raïssi et al. (2011)). However, this method may cause additional conservatism and it still suffers from the cooperativity constraint (Tang et al. (2019)).

The functional interval estimation, which combines the advantages of functional observer and interval estimation, aims to achieve the interval estimation of the linear functional of the system state Huang et al. (2021). Only few works have been investigated on functional interval estimation methods for continuous-time systems (Gu et al. (2018); Meyer (2019); Meyer et al. (2020)). In Gu et al. (2018), by solving Sylvester equations, a functional interval observer design method was presented for continuous-time systems with disturbances. However, this method ignores the existence of measurement noise. In Meyer (2019), a design method of functional interval observers for continuous-time systems affected by both disturbances and noise was proposed based on the classical pole place-

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ment technique. Thereafter, in Meyer et al. (2020), a novel interval state estimation method is introduced and then extended to deal with the case of functional interval estimation. This new approach has the merit of relaxing the restrictive cooperative requirements. However, both of them assumed that the bounds of the derivative or high-order derivatives of the measurement noise are known, which is difficult to be satisfied in real world situations.

Based on the above discussions, it is meaningful to develop a functional interval estimation method for continuous-time systems subject to disturbances and noise. Herein the bounds of the derivative of noise are assumed unknown. Recently, under the classical detectability condition, a set-integration method based on interval arithmetic was presented in Meslem et al. (2020) to realize interval estimation for uncertain continuous-time systems. Inspired by Meslem et al. (2020), we present an interval-based functional interval estimation method by combining robust functional observer design with interval analysis.

The contributions of this paper lie in the following aspects: First, motivated by Meslem et al. (2020), a novel functional interval estimation method is presented by allied H_∞ functional observer design with interval-based functional interval estimation. The proposed method can enhance the estimation performance and will be applied for continuous systems without using the stringent cooperativity condition. Second, by using an auxiliary estimation error, the measurement noise is considered in the proposed functional interval estimation method. To relax the traditionally required conditions on the time derivative of the considered measurement noise, we consider here that its bounds are unknown but finite.

The remainder of this paper is organized as follows: Section 2 presents some preliminaries and briefly introduces the problem to be studied. The main results of H_∞ functional observer design is presented in Section 3. In section 4, we present the interval-based functional interval estimation method based on interval analysis. A numerical example is utilized in Section 5 to demonstrate the effectiveness and superiority of the proposed method. Section 6 gives the final conclusion.

2. PRELIMINARIES AND PROBLEM STATEMENT

Notation: \mathbb{R}^n and $\mathbb{R}^{m \times n}$ represent the n and $m \times n$ dimensional Euclidean spaces, respectively. I denotes the identity matrices of appropriate dimensions. For a vector $x \in \mathbb{R}^n$, x_i denotes its i -th element. The operators $>$, $<$, \leq and \geq should be understood elementwise. $\mathbb{R}_+^n = \{x \in \mathbb{R}^n : x > 0\}$. For a matrix $A \in \mathbb{R}^{m \times n}$, $\text{rank}(A)$, $\lambda_i(A)$, A^T and A^\dagger stands for its rank, i -th eigenvalue, transpose and pseudo-inverse, respectively. Meanwhile, define $\mathbf{He}(A) = A + A^T$, $A^+ = \max(A, 0)$ and $A^- = A^+ - A$ so that $|A| = A^+ + A^-$. For a square matrix $P \in \mathbb{R}^{n \times n}$, $P \succ 0$, $P \prec 0$ and $P \preceq 0$ are used to indicate that P is positive definite, negative definite and negative semidefinite, respectively. In a symmetric matrix, \star denotes the term that can be determined by symmetry.

The following definitions and lemmas are essential in this paper.

Definition 1. For a signal $x(t)$, $\|x(t)\|$ denotes its \mathcal{L}_2 - norm, i.e.

$$\|x(t)\| = \sqrt{\int_0^\infty x^T(t)x(t)dt}. \quad (1)$$

Definition 2. For a matrix $A \in \mathbb{R}^{n \times n}$, $\|A\|_2$ denotes its 2 - norm, i.e.

$$\|A\|_2 = \sqrt{\max(\lambda_i(A^T A))}, \quad i = 1, \dots, n. \quad (2)$$

Lemma 1. (Efimov et al. (2013a)) Given a real vector $x \in \mathbb{R}^n$ satisfying $\underline{x} \leq x \leq \bar{x}$ and a real matrix A of appropriate dimensions, the following inequality holds

$$A^+ \underline{x} - A^- \bar{x} \leq Ax \leq A^+ \bar{x} - A^- \underline{x}. \quad (3)$$

Lemma 2. (Ben-Israel and Greville (2003)) Given matrices $Y \in \mathbb{R}^{b \times c}$ and $S \in \mathbb{R}^{a \times c}$, if $\text{rank}(Y) = \text{rank}(S)$, then the solution of $XY = S$ is

$$X = SY^\dagger + Z(I - YY^\dagger), \quad (4)$$

where $Z \in \mathbb{R}^{a \times b}$ is an arbitrary matrix.

Lemma 3. (Zhou and Khargonekar (1988)) For any matrices X, Y of appropriate dimensions and a positive scalar ρ , we have

$$XY^T + YX^T \preceq \frac{1}{\rho}XX^T + \rho YY^T. \quad (5)$$

Consider the following continuous linear time-invariant system.

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) + D_w w(t) \\ y(t) = Cx(t) + D_v v(t) \\ z(t) = Lx(t) \end{cases}, \quad (6)$$

where $x(t) \in \mathbb{R}^{n_x}$, $u(t) \in \mathbb{R}^{n_u}$, $y(t) \in \mathbb{R}^{n_y}$, $w(t) \in \mathbb{R}^{n_w}$ and $v(t) \in \mathbb{R}^{n_v}$ represent the state vector, the input vector, the output vector, the time-varying disturbances, and the measurement noise, respectively. $z(t) \in \mathbb{R}^{n_z}$ is the vector to be estimated and it denotes the linear functional of $x(t)$. A, B, C, D_w, D_v , and L are known matrices of appropriate dimensions. Herein, A, C and L are assumed to satisfy the condition for the existence of functional observers in Darouach (2000).

In this paper, $x(0)$, $w(t)$, and $v(t)$ are assumed to be unknown but bounded as follows

$$|x(0) - c_0| \leq \bar{x}_0, \quad |w(t)| \leq \bar{w}, \quad |v(t)| \leq \bar{v}, \quad (7)$$

where $c_0 \in \mathbb{R}^{n_x}$, $\bar{x}_0 \in \mathbb{R}_+^{n_x}$, $\bar{w} \in \mathbb{R}_+^{n_w}$, and $\bar{v} \in \mathbb{R}_+^{n_v}$ are known vectors. Additionally, $\dot{v}(t)$ satisfies

$$|\dot{v}(t)| \leq \nu, \quad (8)$$

where $\nu \in \mathbb{R}_+^{n_v}$ is an unknown but finite vector.

3. ROBUST FUNCTIONAL OBSERVER DESIGN

In this section, we apply an H_∞ synthesis method to assess the estimation performance in terms of uncertainty attenuation. A sufficient H_∞ functional observer design condition is given to provide robust point estimation of the linear functional.

Motivated by Darouach (2000), the following functional observer is usually used for system (6):

$$\begin{cases} \dot{\eta}(t) = N\eta(t) + Hu(t) + Jy(t) \\ \hat{z}(t) = \eta(t) + Ey(t) \end{cases}, \quad (9)$$

where $\eta(t)$ is the intermediate variable, $\hat{z}(t) \in \mathbb{R}^{n_z}$ is the point estimation of $z(t)$. $N \in \mathbb{R}^{n_z \times n_z}$, $H \in \mathbb{R}^{n_z \times n_u}$,

$J \in \mathbb{R}^{n_z \times n_y}$ and $E \in \mathbb{R}^{n_z \times n_y}$ are matrices to be designed and satisfy the following conditions

$$\begin{aligned} H &= TB, \\ TA - NT - JC &= 0, \end{aligned} \quad (10)$$

where

$$T = L - EC. \quad (11)$$

Define the estimation error as $e(t) = z(t) - \hat{z}(t)$, then from (6) and its associated functional observer (9), we obtain the error dynamics as follows:

$$\dot{e}(t) = Ne(t) + TD_w w(t) - KD_v v(t) - ED_v \dot{v}(t), \quad (12)$$

where

$$K = J - NE. \quad (13)$$

For the sake of simplicity, we rewrite (12) as

$$\dot{e}(t) = Ne(t) + Dd(t), \quad (14)$$

where $D = [TD_w - KD_v - ED_v]$, and the uncertainties are gathered into $d(t) = [w^T(t) v^T(t) \dot{v}^T(t)]^T$. Herein, $d(t)$ is unknown but bounded as follows:

$$|d(t)| \leq \bar{d}, \quad (15)$$

where $\bar{d} = [\bar{w}^T \bar{v}^T \nu^T]^T$.

After introducing the dynamics of the estimation error, we applied the H_∞ technique to attenuate the impact of uncertainties on the accuracy of the estimates. To this end, the following theorem is proposed to give a sufficient design condition for H_∞ functional observer (9).

Theorem 1. Given a constant $\delta > 0$, if there exists a scalar $\gamma > 0$, a symmetric matrix $P \in \mathbb{R}^{n_z \times n_z}$ and a matrix $W \in \mathbb{R}^{n_z \times 2n_y}$ such that

$$P \succ 0, \quad (16)$$

$$\begin{bmatrix} \Pi_1 & \Pi_2 & \Pi_3 & \Pi_4 \\ \star & -\gamma^2 I & 0 & 0 \\ \star & \star & -\gamma^2 I & 0 \\ \star & \star & \star & -\gamma^2 I \end{bmatrix} \prec 0, \quad (17)$$

$$\begin{bmatrix} -\delta^2 I & W^T \\ \star & -2P + I \end{bmatrix} \prec 0, \quad (18)$$

in which

$$\begin{aligned} \Pi_1 &= \mathbf{He}(PF - WG) + I, \\ \Pi_2 &= PLD_w - P\Xi\alpha_1 CD_w - W\Psi\alpha_1 CD_w, \\ \Pi_3 &= -P\Xi\alpha_2 D_v - W\Psi\alpha_2 D_v, \\ \Pi_4 &= -P\Xi\alpha_1 D_v - W\Psi\alpha_1 D_v, \\ \alpha_1 &= \begin{bmatrix} I \\ 0 \end{bmatrix}, \quad \alpha_2 = \begin{bmatrix} 0 \\ I \end{bmatrix}, \quad \Gamma = \begin{bmatrix} CA \\ C \end{bmatrix}, \\ \Theta &= I - L^\dagger L, \quad \Sigma = \Gamma\Theta, \\ \Xi &= LA\Theta\Sigma^\dagger, \quad \Psi = I - \Sigma\Sigma^\dagger, \\ F &= LAL^\dagger - \Xi\Gamma L^\dagger, \quad G = \Psi\Gamma L^\dagger, \end{aligned} \quad (19)$$

then (9) is a robust functional observer for (6) and satisfies the H_∞ performance, i.e.

$$\|e(t)\| \leq \gamma \|d(t)\|. \quad (20)$$

Moreover, if (16)-(18) are solvable, then the matrix Z is determined by

$$Z = P^{-1}W, \quad (21)$$

which satisfies

$$\|Z\|_2 < \delta. \quad (22)$$

Furthermore, the design matrices of functional observer (9) can be calculated as follows:

$$\begin{aligned} E &= \Xi\alpha_1 + Z\Psi\alpha_1, \quad K = \Xi\alpha_2 + Z\Psi\alpha_2, \\ N &= F - ZG, \quad J = K + NE. \end{aligned} \quad (23)$$

Proof. Based on (10) and (11), the relationship between the design matrices of (9) is derived firstly. Since L is pseudo-invertible, by multiplying (10) with $[L^\dagger \Theta]$, we have

$$(TA - NT - JC) [L^\dagger \Theta] = 0. \quad (24)$$

Substituting (11) into (24) yields that

$$N = TAL^\dagger - KCL^\dagger \quad (25)$$

and

$$TA\Theta = KC\Theta. \quad (26)$$

Note that (26) is equivalent to

$$[E \ K]\Sigma = LA\Theta. \quad (27)$$

From Lemma 2, the general solution of (27) is determined as follows:

$$\begin{aligned} E &= LA\Theta\Sigma^\dagger\alpha_1 + Z(I - \Sigma\Sigma^\dagger)\alpha_1, \\ K &= LA\Theta\Sigma^\dagger\alpha_2 + Z(I - \Sigma\Sigma^\dagger)\alpha_2. \end{aligned} \quad (28)$$

Meanwhile, (25) can be rewritten as

$$\begin{aligned} N &= LAL^\dagger - [E \ K] \begin{bmatrix} CAL^\dagger \\ CL^\dagger \end{bmatrix} \\ &= LAL^\dagger - LA\Theta\Sigma^\dagger \begin{bmatrix} CAL^\dagger \\ CL^\dagger \end{bmatrix} - Z(I - \Sigma\Sigma^\dagger) \begin{bmatrix} CAL^\dagger \\ CL^\dagger \end{bmatrix}. \end{aligned} \quad (29)$$

Based on (10)-(11), (13), (28), and (29), we obtain (23).

The functional observer can be determined as long as Z has been designed. To determine Z such that (9) is a functional observer for (6) with H_∞ performance, we rewrite (14) as follows:

$$\begin{cases} \dot{e}(t) = Ne(t) + Dd(t) \\ e(t) = Ie(t) \end{cases}. \quad (30)$$

Let $W = PZ$. From (17), we have

$$\begin{bmatrix} \mathbf{He}(PN) + I & PD \\ \star & -\gamma^2 I \end{bmatrix} \prec 0. \quad (31)$$

Based on the well-known bounded-real lemma (Boyd et al. (1994)), (31) implies (20).

Additionally, from Lemma 3, by setting $X = P$, $Y = I$ and $\rho = 1$, we deduce that

$$-PP^T \preceq -2P + I. \quad (32)$$

Therefore, (18) is a sufficient condition for

$$\begin{bmatrix} -\delta^2 I & W^T \\ \star & -PP^T \end{bmatrix} \prec 0. \quad (33)$$

Based on (21) and the Schur complement lemma (Xu and Lam (2006)), (33) is equivalent to

$$Z^T Z - \delta^2 I < 0. \quad (34)$$

which implies that all eigenvalues of $Z^T Z$ are less than δ^2 . According to Definition 2, (22) is obtained. \square

Remark 1. The H_∞ performance index γ should be minimized to minimize the energy gain from $d(t)$ to $e(t)$, which can enhance the estimation accuracy. A minimal γ can be obtained by solving the following optimization problem.

$$\min \gamma^2, \quad s.t. (16) - (18). \quad (35)$$

Remark 2. Note that (18) is a sufficient condition for (22). Different δ usually leads to different observer parameters, which gives more degrees of design freedom. It is worth

pointing that sometimes solving (16) and (17) leads to Z with no limit in some of its entries so that Z will be unsolvable. Thus, the main idea behind (22) is to avoid this numerical issue by limiting $\|Z\|_2$.

The designed functional observer (9) can provide the point estimation of the linear functional $z(t)$. It is obvious that by combining $\hat{z}(t)$ with the bounds of $e(t)$, we can achieve interval estimation of $z(t)$. Therefore, further analysis is needed to find out the bounds of the estimation error.

4. FUNCTIONAL INTERVAL ESTIMATION

In the previous section, we have described the dynamics of the estimation error by (12). However, since the bounds of $\dot{v}(t)$ are unknown, this dynamical model cannot be used directly to compute the bounds of $e(t)$. In this section, another form of estimation error dynamics is derived to handle this problem. Based on this error dynamics, an interval-based method is presented to estimate the bounds of the linear functional.

From (12), it is clear that the estimation error dynamics depends on $\dot{v}(t)$. To handle this problem, we define an auxiliary error as

$$\epsilon(t) = e(t) + ED_v v(t). \tag{36}$$

Then, the following auxiliary error dynamic is established

$$\begin{cases} \dot{\epsilon}(t) = N\epsilon(t) + TD_w w(t) - JD_v v(t) \\ \epsilon(t) = \epsilon(t) - ED_v v(t) \end{cases} \tag{37}$$

Note that if we find out the bounds $\underline{e}(t)$ and $\bar{e}(t)$ of $e(t)$ such that

$$\underline{e}(t) \leq e(t) \leq \bar{e}(t), \tag{38}$$

then by using the fact that $e(t) = z(t) - \hat{z}(t)$, we can obtain the interval estimation of $z(t)$ such that

$$\underline{z}(t) \leq z(t) \leq \bar{z}(t), \tag{39}$$

where

$$\begin{cases} \bar{z}(t) = \hat{z}(t) + \bar{e}(t) \\ \underline{z}(t) = \hat{z}(t) + \underline{e}(t) \end{cases} \tag{40}$$

To compute the bounds of $e(t)$, the initial estimate is chosen as $\hat{x}(0) = c_0$, which follows that

$$e(0) = z(0) - \hat{z}(0) = Lx(0) - Lc_0 \tag{41}$$

and

$$\epsilon(0) = e(0) + ED_v v(0). \tag{42}$$

Based on (37) and (42), an interval-based method motivated by Meslem et al. (2020) is presented in the following theorem.

Theorem 2. A tight interval estimation of the linear functional $z(t)$ such that $\underline{z}(t) \leq z(t) \leq \bar{z}(t)$ is provided by

$$\begin{cases} \bar{z}(t) = \hat{z}(t) + \bar{e}(t) \\ \underline{z}(t) = \hat{z}(t) + \underline{e}(t) \end{cases}, \tag{43}$$

where

$$\begin{cases} \bar{e}(t) = |\Phi(t)L|\bar{x}_0 + |\Phi(t)ED_v|\bar{v} + |ED_v|\bar{v} + \bar{\zeta}(t) \\ \underline{e}(t) = -|\Phi(t)L|\bar{x}_0 - |\Phi(t)ED_v|\bar{v} - |ED_v|\bar{v} + \underline{\zeta}(t) \end{cases} \tag{44}$$

in which $\Phi(t) = e^{Nt}$. $\bar{\zeta}(t)$ and $\underline{\zeta}(t)$ are the solutions to

$$\begin{cases} \dot{\bar{\zeta}}(t) = |\Phi(t)TD_w|\bar{w} + |\Phi(t)JD_v|\bar{v} \\ \dot{\underline{\zeta}}(t) = -|\Phi(t)TD_w|\bar{w} - |\Phi(t)JD_v|\bar{v} \end{cases} \tag{45}$$

and $\bar{\zeta}(0) = \underline{\zeta}(0) = 0$.

Proof. From (36), the explicit expression of $\epsilon(t)$ has the following form

$$\epsilon(t) = \Phi(t)\epsilon(0) + \zeta_w(t) + \zeta_v(t), \tag{46}$$

where

$$\zeta_w(t) = \int_0^t \Phi(t-\tau)TD_w w(\tau)d\tau, \tag{47}$$

$$\zeta_v(t) = -\int_0^t \Phi(t-\tau)JD_v v(\tau)d\tau. \tag{48}$$

From (36), (42) and (46), the explicit expression of the estimation error is formulated as

$$\begin{aligned} e(t) = & \Phi(t)L(x(0) - c_0) + \Phi(t)ED_v v(0) - ED_v v(t) \\ & + \zeta_w(t) + \zeta_v(t). \end{aligned} \tag{49}$$

Using Lemma 1, we have

$$-|\Phi(t)L|\bar{x}_0 \leq \Phi(t)L(x(0) - c_0) \leq |\Phi(t)L|\bar{x}_0 \tag{50}$$

$$-|\Phi(t)ED_v|\bar{v} \leq \Phi(t)ED_v v(0) \leq |\Phi(t)ED_v|\bar{v} \tag{51}$$

$$-|ED_v|\bar{v} \leq -ED_v v(t) \leq |ED_v|\bar{v} \tag{52}$$

Meanwhile, substituting $\beta = t - \tau$ into $\zeta_w(t)$ and using the unknown but bounded condition (7), we have

$$\zeta_w(t) \in \int_0^t \Phi(\beta)TD_w[-\bar{w}, \bar{w}]d\beta, \tag{53}$$

which follows that

$$\dot{\zeta}_w(t) \in \Phi(t)TD_w[-\bar{w}, \bar{w}] \tag{54}$$

and

$$\bar{\zeta}_w(0) = \underline{\zeta}_w(0) = \zeta_w(0) = 0, \tag{55}$$

where $[-\bar{w}, \bar{w}]$ denotes the interval vector whose endpoints are $-\bar{w}$ and \bar{w} . Based on Lemma 1, it is not difficult to establish the following inequality

$$-|\Phi(t)TD_w|\bar{w} \leq \dot{\zeta}_w(t) \leq |\Phi(t)TD_w|\bar{w}. \tag{56}$$

Similarly, from (48), we have

$$-|\Phi(t)JD_v|\bar{v} \leq \dot{\zeta}_v(t) \leq |\Phi(t)JD_v|\bar{v} \tag{57}$$

and $\bar{\zeta}_v(0) = \underline{\zeta}_v(0) = 0$.

From (49) and (50)-(57), we have

$$\underline{e}(t) \leq e(t) \leq \bar{e}(t), \tag{58}$$

where $\bar{e}(t)$ and $\underline{e}(t)$ are obtained by (44).

Based on (40) and (44), we obtain the bounds of the linear functional $z(t)$ in (43). \square

Remark 3. It should be noted that the proposed method is not a straightforward extension of the method in Meslem et al. (2020). We have revised and improved it based on the demand for functional interval estimation. To illustrate this point, we give the following functional interval estimation results provided by the direct application of the former method

$$\begin{cases} \bar{z}(t) = \hat{z}(t) + \bar{e}_\psi(t) \\ \underline{z}(t) = \hat{z}(t) + \underline{e}_\psi(t) \end{cases} \tag{59}$$

where

$$\begin{cases} \bar{e}_\psi(t) = |\Phi(t)||L|\bar{x}_0 + |\Phi(t)||ED_v|\bar{v} + |ED_v|\bar{v} + \bar{\psi}(t) \\ \underline{e}_\psi(t) = -|\Phi(t)||L|\bar{x}_0 - |\Phi(t)||ED_v|\bar{v} - |ED_v|\bar{v} + \underline{\psi}(t) \end{cases} \tag{60}$$

in which $\bar{\psi}(t)$ and $\underline{\psi}(t)$ are the solutions to

$$\begin{cases} \dot{\bar{\psi}}(t) = |\Phi(t)||TD_w|\bar{w} + |\Phi(t)||JD_v|\bar{v} \\ \dot{\underline{\psi}}(t) = -|\Phi(t)||TD_w|\bar{w} - |\Phi(t)||JD_v|\bar{v} \end{cases} \tag{61}$$

and $\bar{\psi}(0) = \psi(0) = 0$. To measure the tightness of the functional interval estimation results, we define the estimation gap as $z_g(t) = \bar{z}(t) - z(t)$. We denote by $z_g^F(t)$ the estimated vector $z_g(t)$ provided by the former method. Using (59)-(61), we have

$$z_g^F(t) = 2|\Phi(t)||L|\bar{x}_0 + 2|\Phi(t)||ED_v|\bar{v} + 2|ED_v|\bar{v} + 2\bar{\psi}(t). \tag{62}$$

Meanwhile, we denote by $z_g^T(t)$ the estimated vector $z_g(t)$ yielded by the proposed method. Then, we obtain

$$z_g^T(t) = 2|\Phi(t)L|\bar{x}_0 + 2|\Phi(t)ED_v|\bar{v} + 2|ED_v|\bar{v} + 2\bar{\zeta}(t) \tag{63}$$

Since $|\Phi(t)L| \leq |\Phi(t)||L|$, $|\Phi(t)ED_v| \leq |\Phi(t)||ED_v|$, $|TD_w| \leq |T||D_w|$, $|JD_v| \leq |J||D_v|$, and \bar{x}_0 , \bar{w} and \bar{v} are positive vectors, it is not difficult to verify that

$$z_g^T(t) \leq z_g^M(t), \tag{64}$$

which indicates that the functional interval estimation provided by the proposed method is tighter than that provided by the direct application of the former method. In fact, in these interval-based estimation methods, the introduced conservatism is mainly depending on the number of the times Lemma 1 is applied. Thus, it is straightforward to check that the proposed method uses less Lemma 1 than the direct extension of the former method.

5. SIMULATION

In this section, a numerical example from Meyer (2019) is utilized to illustrate the effectiveness and superiority of the proposed method. Consider the continuous linear time-invariant system (6) with the following system matrices

$$A = \begin{bmatrix} -2 & 1 & 0 \\ 1 & -3 & 1 \\ 0 & 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

$$L = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}, \quad D_w = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad D_v = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

Meanwhile, the bounded initial state and uncertainties satisfy

$$|x(0)| \leq \begin{bmatrix} 20 \\ 20 \\ 20 \end{bmatrix}, \quad |w(t)| \leq \begin{bmatrix} 0.5 \\ 0.5 \\ 0.5 \end{bmatrix}, \quad |v(t)| \leq \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix}.$$

It should be noted that in this simulation, the bounds of the derivative of the measurement noise are unknown.

By choosing $\delta = 5$ and solving the optimization problem (35), we obtain

$$Z = \begin{bmatrix} 0.0982 & 0.1416 & 1.0981 & 1.3699 \\ -0.0326 & 0.0156 & 0.2308 & 0.3909 \end{bmatrix}$$

with the minimal $\gamma = 2.5572$. Based on Theorem 1, the design matrices of (9) can be determined as follows:

$$E = \begin{bmatrix} -0.0129 & 0.2898 \\ -0.0326 & 0.0156 \end{bmatrix}, \quad K = \begin{bmatrix} 1.1352 & 1.3328 \\ 0.2308 & 0.3909 \end{bmatrix}$$

$$N = \begin{bmatrix} -2.4507 & 3.6116 \\ -0.3116 & -0.3923 \end{bmatrix}, \quad J = \begin{bmatrix} 1.0489 & 0.6789 \\ 0.2476 & 0.2945 \end{bmatrix},$$

$$T = \begin{bmatrix} 1.0129 & 0.7102 & 2 \\ 0.0326 & -0.0156 & 1 \end{bmatrix}, \quad H = \begin{bmatrix} -0.9871 & 0.7102 \\ -0.9674 & -0.0156 \end{bmatrix}.$$

In this simulation, the input vector is formulated as follows

$$u(t) = [4 \cos(1.5t) \ 5 \sin(t)]^T.$$

The initial estimate $\hat{x}(0)$ is chosen as $\hat{x}(0) = [0 \ 0 \ 0]^T$.

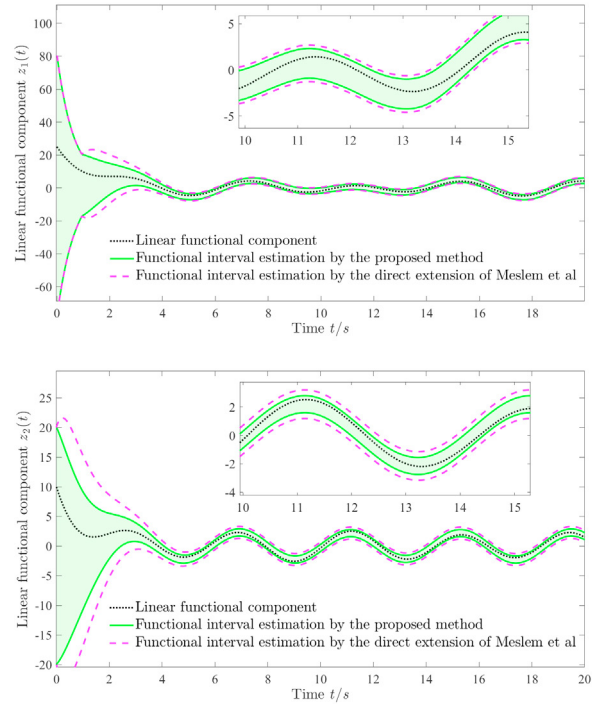


Fig. 1. Functional interval estimation results provided by the proposed method and the direct extension of the method in Meslem et al. (2020).

Since the method in Meyer (2019) requires the knowledge on the bounds of $\dot{v}(t)$, which is unknown in this simulation, we cannot compute the interval estimation results by using this method. Therefore, in Fig. 1, we only plot the functional interval estimation results by the proposed method and the direct extension of the method in Meslem et al. (2020). As it is shown in Fig. 1, the applied two methods can enclose the real linear functional effectively. Moreover, the results provided by the proposed method are tighter than those obtained by the direct extension of the former method. These simulation results illustrate that our proposed method can handle the situation when the bounds of $\dot{v}(t)$ is unknown and have a better functional interval estimation performance.

Now, in order to acquire the functional interval estimation results provided by the method in Meyer (2019), a condition $|\dot{v}(t)| \leq [0.7 \ 0.4]^T$ is added. The functional interval estimation results provided by the proposed method and the method in Meyer (2019) are plotted in Fig. 2. From Fig. 2, both of them yield accurate interval estimation results. In addition, for the components of the linear functional, the method in Meyer (2019) can produce tighter bounds at the beginning, whereas the results provided by our methods becomes tighter ones as time goes by. This is mainly because we adopt an auxiliary estimation error to relax the condition on the time derivative of the measurement noise, which will lead to an overestimation in the initial stage. Fortunately, the conservatism caused by this overestimation will decrease quickly over time. At the steady stage, less conservative results are acquired according to our proposed methods. These simulation results demonstrate the superiority of our proposed method.

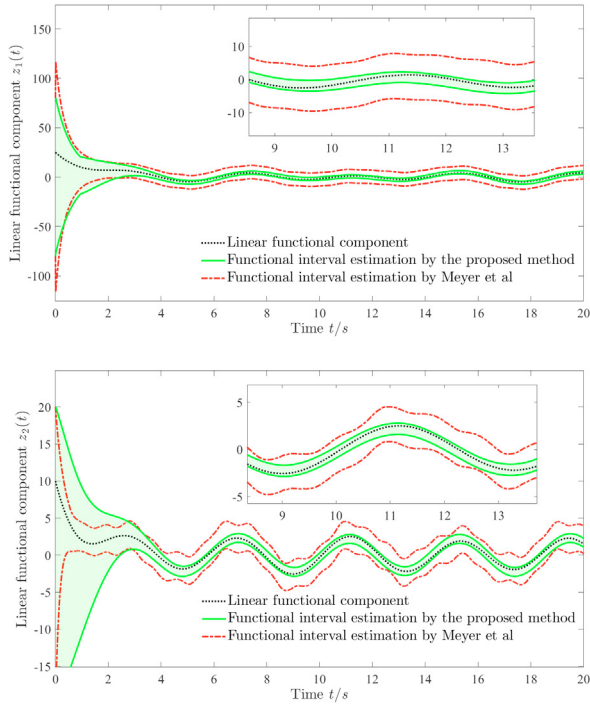


Fig. 2. Functional interval estimation results provided by the proposed method and the method in Meyer (2019).

6. CONCLUSION

The functional interval estimation for continuous-time systems subject to disturbances and measurement noise has been considered in this paper by combining functional observer design with interval analysis. Based on the H_∞ technique, a new sufficient condition to design a robust functional observer is derived to enhance the estimation performance. Then, an interval-based method is presented to compute comprehensive functional interval estimation. Simulation results have illustrated the validity and superiority of the proposed method.

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