Higher Bernstein Polynomials and Multiple Poles of $\frac{1}{\Gamma(\lambda)}\int_X |f|^{2\lambda}\bar{f}^{-h}\rho\omega\wedge\bar{\omega}'$ version 2

Daniel Barlet*

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Les hommes dérivent pour comprendre
. *alors que Dieu se contente d'intégrer.*

Abstract. The goal of this paper is to give a converse to the main result of my previous paper [12], so to prove the existence of a pole with an hypothesis on the Bernstein polynomial of the (a,b)-module generated by the germ $\omega \in \Omega_0^{n+1}$. A difficulty to prove such a result comes from the use of the formal completion in f of the Brieskorn module of the holomorphic germ $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ which does not give access to the cohomology of the Milnor's fiber of f, which by definition, is outside $\{f = 0\}$. This leads to introduce convergent (a,b)-modules which allow this passage. In order to take in account Jordan blocs of the monodromy in our result we introduce the semi-simple filtration of a (convergent) geometric (a,b)-module and define the higher order Bernstein polynomials in this context which corresponds to a decomposition of the "standard" Bernstein polynomial in the case of frescos. Our main result is to show that the existence of a root in $-\alpha - \mathbb{N}$ for the p-th Bernstein polynomial of the fresco generated by a holomorphic form $\omega \in \Omega_0^{n+1}$ in the (convergent) Brieskorn (a,b)-module H_0^{n+1} associated to f, under the hypothesis that f has an isolated singularity at the origin relative to the eigenvalue $\exp(2i\pi\alpha)$ of the monodromy, produces poles of order at least p for the meromorphic extension of the (conjugate) analytic functional, for some $h \in \mathbb{Z}$:

$$\omega' \in \Omega_0^{n+1} \mapsto \frac{1}{\Gamma(\lambda)}\int_{\mathbb{C}^{n+1}} |f|^{2\lambda}\bar{f}^{-h}\rho\omega\wedge\bar{\omega}'$$

at points $-\alpha - N$ for N and h well chosen integers. This result is new, even for $p = 1$. As a corollary, this implies that in this situation the existence of a root in

*Barlet Daniel, Institut Elie Cartan UMR 7502
Université de Lorraine, CNRS, INRIA et Institut Universitaire de France,
BP 239 - F - 54506 Vandoeuvre-lès-Nancy Cedex.France.
e-mail : daniel.barlet@univ-lorraine.fr

$-\alpha - \mathbb{N}$ for the p-th Bernstein polynomial of the fresco generated by a holomorphic form $\omega \in \Omega_0^{n+1}$ implies the existence of at least p roots (counting multiplicities) for the usual reduced Bernstein polynomial of the germ $(f, 0)$.
In the case of an isolated singularity we obtain that for each $\alpha \in]0, 1] \cap \mathbb{Q}$ the biggest root $-\alpha - m$ of the reduced Bernstein polynomial of f in $-\alpha - \mathbb{N}$ produces a pole at $-\alpha - m$ for some $h \in \mathbb{Z}$ for the meromorphic extension of the distribution

$$\square \longrightarrow \frac{1}{\Gamma(\lambda)} |f|^{2\lambda} \bar{f}^{-h} \square.$$

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Contents

1 Introduction

The roots of the reduced Bernstein polynomial $b_{f,0}$ of the germ of holomorphic function at the origin in $\mathbb{C}^{n+1}$ control the poles of the meromorphic extension of the distribution

$$\square \longrightarrow \frac{1}{\Gamma(\lambda)} \int_{\mathbb{C}^{n+1}} |f|^{2\lambda} \bar{f}^{-h} \square$$

which is defined in a neighborhood of $0 \in \mathbb{C}^{n+1}$ (see for instance [2] or [15]).
The first goal of this article is to show that, assuming that 0 is an isolated singularity for the eigenvalue $\exp(2i\pi\alpha)$ of the monodromy (this corresponds to our hypothesis $H(\alpha, 1)$), the roots of the Bernstein polynomial of the (a,b)-module generated by the germ ω of holomorphic $(n+1)$-form at the origin in the Brieskorn (a,b)-module H_0^{n+1} of f at 0, control the poles of the (conjugate) analytic functional defined on Ω_0^{n+1} by polar parts of poles in $-\alpha - \mathbb{N}$ of the meromorphic functions

$$\omega' \mapsto \frac{1}{\Gamma(\lambda)} \int_X |f|^{2\lambda} \bar{f}^{-h} \rho\omega \wedge \bar{\omega}'$$

where $\rho \in \mathscr{C}_c^\infty(\mathbb{C}^{n+1})$ is identically 1 near 0 and with a sufficiently small support (note that the polar parts of these meromorphic extensions at points in $-\alpha - \mathbb{N}$ are

independent of the choices of ρ thanks to our hypothesis $H(\alpha, 1)$).

Our second goal is to give a sufficient condition, still on the (a,b)-module generated by the germ ω, to obtain higher order poles for such integrals.
The difficulty commes now from the fact that it is not clear when, for instance, two roots, $-\alpha - m$ and $-\alpha - m'$ with $m, m' \in \mathbb{N}$, of the Bernstein polynomial give a simple pole or a double pole for such a meromorphic extension at points $-\alpha - N$, for some choice of ω' and for some integers N and h well chosen.
So we try to understand when such a pair of roots are "linked", so produces a double pole for some choice of ω', h and N, or are "independent", so produces at most a simple pole for any choices of ω', h and N.
As it is known that the nilpotent part of the monodromy is related to this phenomenon (see [3] and [4]) we consider the action of the monodromy on geometric[1] (a,b)-modules and show that the natural semi-simple filtration of a geometric (a,b)-module $\mathcal{E}$ is related to the filtration induced by the nilpotent part of the action of the monodromy on its saturation $\mathcal{E}^\sharp$ by $b^{-1}a$.
This allows to show that, in the case of a fresco $\mathcal{F}$ (see Section 5 for the definition), the Bernstein polynomial is a product of the Bernstein polynomials (with suitable shifts for the roots) of the graduate pieces of its semi-simple filtration (which are semi-simple frescos) and these Bernstein polynomials define the "higher order Bernstein polynomials" of the fresco $\mathcal{F}$.
Then we show that, under our hypothesis $H(\alpha, 1)$, the existence of a root in $-\alpha - \mathbb{N}$ for the j-th Bernstein polynomial of the fresco generated by ω, produces a pole of order at least j at some point $-\alpha - N$ for the meromorphic extension of the integral

$$\frac{1}{\Gamma(\lambda)} \int_{\mathbb{C}^{n+1}} |f|^{2\lambda} \bar{f}^{-h} \rho\omega \wedge \bar{\omega}' \qquad (\star)$$

for some $\omega' \in \Omega_0^{n+1}$ and some integers N and h well chosen, where ρ is as above.
An interesting consequence of this result is the fact that under our hypothesis that the origin is an isolated singularity of f for the eigenvalue $\exp(2i\pi\alpha)$ of the monodromy, the existence of a root $-\alpha - m$ for $B_j(f, \omega)$, the j-th Bernstein polynomial of the fresco $\mathcal{F}_{f,\omega}$ associated to the pair (f, ω), implies the existence of at least j roots in $-\alpha - \mathbb{N}$ (counting multiplicities) for the (usual) reduced Bernstein polynomial $b_{f,0}$ of the germ of f at the origin. This means that at least j such roots of $b_{f,0}$ and "linked" in the sense that they contribute to increase the order of the poles of the integral $(\star)$ at points in $-\alpha - N$ for N and h large enough for some choice of $\omega' \in \Omega_0^{n+1}$.
We obtain also that, in the isolated singularity case (for each $\exp(2i\pi\alpha)$), the biggest root $-\alpha - m$ in $-\alpha - \mathbb{N}$ of $\tilde{b}_{f,0}$, the reduced Bernstein polynomial of f, always produces a pole at the point $-\alpha - m$ for the meromorphic extension of the distribution $\frac{1}{\Gamma(\lambda)}|f|^{2\lambda}\bar{f}^{-h}$ for some $h \in \mathbb{Z}$.

[1]The (a,b)-modules deduced from the Gauss-Manin connection which appear here are always geometric. See section 5

Description of the content.

- In section 2 we establish several basic estimates to work with the algebra $B := \mathbb{C}\{\{b\}]$ which is a closed sub-algebra of the algebra of continuous $\mathbb{C}$-linear endomorphisms of the algebra $A := \mathbb{C}\{s\}$ of holomorphic germs at the origin in $\mathbb{C}$.
  Then we introduce the algebra $\tilde{\mathcal{A}}_{conv.}$ which contains A and B with the commutation relation $ab - ba = b^2$. We show that $\tilde{\mathcal{A}}_{conv.}$ is local and we prove the Division Theorem which allows to show the equivalence of considering a geometric (convergent) (a,b)-module as a left module on the sub-algebra $B[a] \subset \tilde{\mathcal{A}}_{conv.}$ or as a left $\tilde{\mathcal{A}}_{conv.}$-module (see Theorem 7.1.1).

- Section 3 is devoted to extend the standard properties of regular (formal) (a,b)-modules (see [6]) to the convergent case.

- The purpose of Section 4 is to define the semi-simple filtration of a regular convergent (a,b)-module and to establish the basic properties of this filtration.

- In Section 5 we show that convergent geometric (a,b)-modules are simply convergent sub-(a,b)-modules of classical asymptotic expansions modules. The Embedding Theorem shows that the geometric (a,b)-modules are exactly regular (a,b)-modules such that the roots of their Bernstein polynomial are negative and rational (compare with with the positivity Theorem of B. Malgrange [19] and the Rationality Theorem of M. Kashiwara [18]).

- Section 6 explains the relation between the semi-simple filtration and the nilpotent part of the logarithm of the monodromy, logarithm which is naturally defined on a simple pole geometric (a,b)-module.

- After some general results on convergent frescos (these are geometric (a,b)-modules with one generator as left $B[a]$-module) extending to the convergent case some results in [9], we are able in Section 7 to define the higher order Bernstein polynomials for a fresco and we obtain a precise link with the standard Bernstein polynomial.

- The first part of Section 8, after explaining how to use the tools introduced above, gives an improvement of the results in [12] which links the existence, for some choices of $\omega' \in \Omega_0^{n+1}$ and some integer h, of an order p pole at a point in $-\alpha - \mathbb{N}$ of the meromorphic function $F_h^{\omega,\omega'}(\lambda)$ to the existence of a root for the p-th Bernstein polynomial of the fresco $\mathcal{F}_{f,\omega}$ associated to the pair (f, ω) when f has an isolated singularity for the eigenvalue $\exp(-2i\pi\alpha)$ of the monodromy.
  The rest of Section 8 is devoted to the proof of our main result which gives the link (in the other direction) between existence of roots for the higher order Bernstein polynomials of the fresco associated to a given form $\omega \in \Omega_0^{n+1}$ and the higher order poles of the hermitian periods associated to ω under our

hypothesis $H(\alpha, 1)$. We also give, in the case of an isolated singularity for f, some precise links to the reduced Bernstein polynomial of f at the origin.

NOTATIONS. In the sequel we shall use the following complex unitary algebras:

1. The algebra $A := \mathbb{C}\{a\} \simeq \mathbb{C}\{s\}$ of germs of holomorphic functions at the origin of $\mathbb{C}$. The letter a represent the multiplication by s.
2. The algebra $B := \mathbb{C}\{\{b\}\}$ which is the closed sub-algebra of continuous linear operators on A generated by b, the primitive without constant in $\mathbb{C}\{s\}$.
3. The algebra $\mathcal{A}$ which is the algebra of polynomials in a and b inside the algebra of continuous linear operators on A. So they satisfy $ab - ba = b^2$.
4. The algebra $B[a]$ of polynomials in a with coefficients in B, with the commutation relations $aS(b) - S(b)a = b^2 S'(b)$ for each $S \in B$.
5. The algebra $\tilde{\mathcal{A}}_{conv.}$ is defined in Section 2.4.
6. The algebra $\widehat{A} := \mathbb{C}[[a]]$
7. The algebra $\widehat{B} := \mathbb{C}[[b]]$
8. The algebra $\tilde{\mathcal{A}} := \widehat{B}[a]$ of polynomials in a with coefficients in $\widehat{B}$, with the commutation relations $aS(b) - S(b)a = b^2 S'(b)$ for each $S \in \widehat{B}$.
9. The algebra $\widehat{\mathcal{A}}$ which the algebra of formal power series in a and b with the commutation relation $ab - ba = b^2$.

2 Convergent (a,b)-modules

2.1 The algebra $\mathbb{C}_r\{\{b\}\}$

For each real number $r \in]0, 1[$ let $\mathbb{C}_r\{\{b\}\}$ the sub-vector space of $\mathbb{C}[[b]]$ defined by

$$\mathbb{C}_r\{\{b\}\} := \{S = \sum_{j=0}^{\infty} s_j b^j \ / \ \exists R \in]1, 1/r[\quad \exists C_R \ s.\ t.\ \forall j \in \mathbb{N} \quad |s_j| \leq C_R R^j j!\}. \quad (1)$$

REMARKS.

1. If the condition above holds for some $R_0 \in]1, 1/r[$ then it holds for any R in $[R_0, 1/r[$ and with the constant $C_R = C_{R_0}$.

2. If for some positive integer q and for some $S = \sum_{j=0}^{\infty} s_j b^j$ we have:

$$\exists R \in]1, 1/r[\quad \exists C_R \text{ such that } \quad \forall j \in \mathbb{N} \quad |s_j| \leq C_R R^j (j+q)!$$

then S is in $\mathbb{C}_r\{\{b\}\}$ because

$$\frac{(j+q)!}{j!} \leq (1+q)^q j^q$$

and for $\rho \in]R, 1/r[$ the sequence $j \mapsto (R/\rho)^j j^q$ is bounded for a given q.

3. The vector space $\mathbb{C}_r\{\{b\}\}$ is stable by derivation because if $S = \sum_{j=0}^{\infty} s_j b^j$ is in $\mathbb{C}_r\{\{b\}\}$ then $S' := \sum_{j=0}^{\infty} (j+1) s_{j+1} b^j$ satisfies

$$\exists R \in]1, 1/r[\quad s.\ t. \quad |(j+1)s_{j+1}| \leq (j+1) C_R R^{j+1} (j+1)! \leq R C_R R^j (j+2)!$$

and we may apply the previous remark with $q = 2$ to S'.

We have a vector spaces isomorphism

$$\mathbb{C}_r\{\{b\}\} \to \mathbb{C}_r\{z\} := \{\sum_{j=0}^{\infty} c_j z^j \ / \ \exists R \in]1, 1/r[\quad \exists C_R \ s.\ t.\ \forall j \in \mathbb{N} \quad |c_j| \leq C_R R^j\}$$

which is given by $s_j \mapsto c_j = s_j/j!$. As $\mathbb{C}_r\{z\}$ is the algebra of germs of holomorphic functions around the closed disc $\bar{D}_r$ in $\mathbb{C}$, it is a **dual Fréchet** (in short DF) topological vector space and we shall define the topology on $\mathbb{C}_r\{\{b\}\}$ via this isomorphism.

Lemma 2.1.1 *For each $r \in]0, 1[$, $\mathbb{C}_r\{\{b\}\}$ is a dual Fréchet sub-algebra of $\mathbb{C}[[b]]$ which is stable by derivation.*

PROOF. Let $S := \sum_{j=0}^{\infty} s_j b^j$ and $T := \sum_{j=0}^{\infty} t_j b^j$ be in $\mathbb{C}_r\{\{b\}\}$. The product ST in $\mathbb{C}[[b]]$ is given by

$$ST = \sum_{j=0}^{\infty} u_j b^j \quad \text{where} \quad u_j := \sum_{p=0}^{j} s_{j-p} t_p.$$

For some $R \in]1, 1/r[$ there exists $C_R > 0$ and $D_R > 0$ such that $|s_j| \leq C_R R^j j!$ and $|t_j| \leq D_R R^j j!$ by definition. This implies

$$|u_j| \leq C_R D_R R^j j! \Big(\sum_{p=0}^{j} \frac{(j-p)!p!}{j!} \Big).$$

The following easy lemma will be used several times in the sequel.

Lemma 2.1.2 *For any $j \geq 0$ we have $\sum_{p=0}^{j} (j-p)!p! \leq 3(j!)$.*

PROOF. For $j \geq 2$ and $p \in [1, j-1]$ we have the estimates $(j-p)!p! \leq (j-1)!$ which implies for $j \geq 2$:

$$\sum_{p=0}^{j}(j-p)!p! \leq 2(j!) + \sum_{p=1}^{j-1}(j-p)!p! \leq 2(j!) + (j-1)((j-1)!) \leq 3(j!).$$

As the cases $j = 0, 1$ are obvious, so the proof is complete. ■

END OF PROOF OF LEMMA 2.1.1. So we obtain that $|u_j| \leq 3C_R D_R R^j j!$ showing that ST is in $\mathbb{C}_r\{\{b\}\}$.
The countable family given by

$$||S||_R := \sup_j |s_j|/(R^j j!)$$

for $R \in \mathbb{Q} \cap]1, 1/r[$ defines the dual Fréchet topology on $\mathbb{C}_r\{\{b\}\}$ and the previous computation shows that $||ST||_R \leq 3||S||_R||T||_R$ for each $R]1, 1/r[$.
The stability by derivation is explained in Remark 3 above. ■

For each real number $r \in [0, 1[$ define the operator b on the algebra $\mathbb{C}_r\{z\}$ of germs of holomorphic functions around the closed disc $\bar{D}_r$ with center 0 and radius r in $\mathbb{C}$, as the linear continuous operator given by the primitive vanishing at the origin, that is to say:

$$b(f)(z) := \int_0^z f(t)dt = z\int_0^1 f(tz)dt;$$

Proposition 2.1.3 *This action of $\mathbb{C}[b]$ on $\mathbb{C}_r\{z\}$ extends to a continuous action of $\mathbb{C}_r\{\{b\}\}$ on $\mathbb{C}_r\{z\}$. It induces an isomorphism of DF-algebras between $\mathbb{C}_r\{\{b\}\}$ and the commutant of b in the algebra of continuous endomorphism of $\mathbb{C}_r\{z\}$.*

Note that this commutant is a closed sub-algebra of the algebra of continuous endomorphism of $\mathbb{C}_r\{z\}$.

PROOF. Let $S := \sum_{j=0}^{\infty} s_j b^j$ and $f(z) := \sum_{\nu=0}^{\infty} \gamma_\nu z^\nu$ where we may assume that there exists $R \in]1, 1/r[$ such that the following estimates hold:

$$|s_j| \leq C_R R^j j! \quad \text{and} \quad |\gamma_\nu| \leq \Gamma_R R^\nu.$$

Since $b^j[z^\nu] = \frac{\nu! z^{\nu+j}}{(\nu+j)!}$ we obtain if we write $S[f](z) = \sum_{q=0}^{\infty} \delta_q z^q$

$$|\delta_q| = |\sum_{j=0}^{q} s_j \gamma_{q-j} \frac{(q-j)!j!}{q!}| \leq \frac{C_R \Gamma_R R^q}{q!} \sum_{j=0}^{q} j!(q-j)!.$$

We obtain, thanks to estimates given in Lemma 2.1.2

$$|\delta_q| \leq 3C_R \Gamma_R R^q$$

This shows that $S[f]$ is in $\mathbb{C}_r\{z\}$ and that S acts continuously on this DF space. Then it is clear that the image of $\mathbb{C}_r\{\{b\}\}$ is a commutative sub-algebra of the commutant of b in the algebra of continuous endomorphisms of $\mathbb{C}_r\{z\}$.
Let us show the converse. If Φ is a continuous endomorphisms of $\mathbb{C}_r\{z\}$ commuting with b, put $\Phi(1) = \sum_{j=0}^{\infty} \gamma_j z^j$ and define $S := \sum_{j=0}^{\infty} j!\gamma_j b^j$. Then S is in $\mathbb{C}_r\{\{b\}\}$. We shall compare $\Phi[z^p]$ and $S[z^p]$ for $p \in \mathbb{N}$:

$$\Phi[z^p] = \Phi[p!b^p(1)] = p!b^p[\Phi(1)] = \sum_{j=0}^{\infty} p!\gamma_j z^{p+j} \frac{j!}{(p+j)!}$$

and

$$S[z^p] = \sum_{j=0}^{\infty} j!\gamma_j b^j(z^p) = \sum_{j=0}^{\infty} j!\gamma_j \frac{p!z^{p+j}}{(p+j)!}.$$

As the linear combinations of the z^p, $p \in \mathbb{N}$, are dense in $\mathbb{C}_r\{z\}$ we conclude that Φ coincides with the image of S in the algebra of continuous endomorphisms of $\mathbb{C}_r\{z\}$. So the image of $\mathbb{C}_r\{\{b\}\}$ is the commutant sub-algebra of b. ■

Note that the action of $\mathbb{C}_r\{\{b\}\}$ extends to the Banach algebra of continuous functions on $\bar{D}_r$ using the formula

$$b(f)(z) := z \int_0^1 f(tz)dt :$$

iI is easy to see that for a continuous function defined on $\bar{D}_r$ we have the estimate

$$||b^j(f)||_r \leq ||f||r^j/j!.$$

It is not difficult to extend the result above to this action using the density of polynomial in z and $\bar{z}$ in the Banach algebra of continuous functions on $\bar{D}_r$.

For each real number $s \in]0, r[$ we have a continuous inclusion $\mathbb{C}_s\{\{b\}\} \subset \mathbb{C}_r\{\{b\}\}$ of DF-algebras and we define B as the algebra $\mathbb{C}_0\{\{b\}\}$ which is the union for all positive r of the algebras $\mathbb{C}_r\{\{b\}\}$.
Note that the algebra B is defined by

$$B := \{S = \sum_{j=0}^{\infty} s_j b^j \ / \ \exists R > 1 \text{ and } \exists C_R > 0 \text{ such that } \quad \forall j \in \mathbb{N} \quad |s_j| \leq C_R R^j j!\}$$

and that B acts on $A := \mathbb{C}_0\{z\}$ the algebra of holomorphic germs at the origin which is the union for all $r \in]0, 1[$ of the algebra $\mathbb{C}_r\{z\}$ and is defined by

$$A := \{f = \sum_{j=0}^{\infty} \gamma_j z^j \ / \ \exists R > 1 \text{ and } \exists C_R > 0 \text{ such that } \quad \forall j \in \mathbb{N} \quad |\gamma_j| \leq C_R R^j\}.$$

Proposition 2.1.4 *The algebra B is local.*

This result is a consequence of the following lemma. ■.

Lemma 2.1.5 *Let S be in $\mathbb{C}_r\{\{b\}\}$ such that $S(0) \neq 0$. Then there exists $t \in]0, r[$ such that S is invertible in $\mathbb{C}_t\{\{b\}\}$. So the algebra $B := \mathbb{C}_0\{\{b\}\}$ has a unique maximal ideal which is generated by b.*

PROOF. The inverse of S in $\mathbb{C}[[b]]$ is given by $T := \sum_{j=0}^{\infty} t_j b^j$ with $t_0 := 1/s_0$ and $t_q := -(1/s_0)(\sum_{j=1}^{q} s_j t_{q-j})$ for $q \geq 1$. For some $R \in]1, 1/r[$ there exists $C_R > 0$ such that $|s_j| \leq C_R R^j j! \quad \forall j \geq 0$. Now assume that for some

$$\rho > \sup\{R, \frac{3RC_R}{|s_0|}\},$$

some D_ρ and some $q \in \mathbb{N}^*$ we have for each integer $h \in [0, q-1]$ the estimates

$$|t_h| \leq D_\rho \rho^h h!.$$

Then we obtain

$$|t_q| \leq \frac{1}{|s_0|} C_R D_\rho \sum_{j=1}^{q} R^j \rho^{q-j} j!(q-j)!$$

$$|t_q| \leq \frac{1}{|s_0|} C_R D_\rho \rho^q q! \sum_{j=1}^{q} (\frac{R}{\rho})^j \frac{j!(q-j)!}{q!}$$

$$|t_q| \leq \frac{1}{|s_0|} C_R D_\rho \rho^q q! \frac{3R}{\rho} \leq D_\rho \rho^q q!$$

So the estimates $|t_h| \leq D_\rho \rho^h h!$ will be valid $\forall h \geq 0$ as soon as it is true for $h = 0$. We conclude the proof by defining $D_\rho := 1/|s_0|$ and by choosing $t < 1/\rho$. ■

2.2 The r-convergent (a,b)-modules

Fix a real number $r \in [0, 1[$.

Definition 2.2.1 *A free finite rank $\mathbb{C}_r\{\{b\}\}$-module $\mathcal{E}_r$ endowed with a continuous $\mathbb{C}$-linear endomorphism a which satisfies*

$$ab - ba = b^2$$

will be called a **r-convergent (a,b)-module**. *In the case $r = 0$ we simply call $\mathcal{E} := \mathcal{E}_0$ a* **convergent (a,b)-module**.
We say that $\mathcal{E}_r$ has a **simple pole** *when it satisfies $a\mathcal{E}_r \subset b\mathcal{E}_r$.*

REMARK. The dual Fréchet topology on a free finite rank $\mathbb{C}_r\{\{b\}\}$-module $\mathcal{E}_r$ given by the choice of a $\mathbb{C}_r\{\{b\}\}$-basis e of $\mathcal{E}_r$ is independent of this choice because if $\varepsilon := M(b)e$ is an other basis the linear bijective map corresponding to the change of basis is continuous, thanks to the continuity of the product in the $\mathbb{C}$-algebra $\mathbb{C}_r\{\{b\}\}$. Then it is an isomorphism of dual Fréchet spaces.

Note that the continuity of a for the natural dual Fréchet topology deduced from any $\mathbb{C}_r\{\{b\}\}$-basis of $\mathcal{E}_r$ implies that for any $S(b) := \sum_{j=0}^{+\infty} s_j b^j$ in $\mathbb{C}_r\{\{b\}\}$ and for any $x \in \mathcal{E}_r$ we have

$$a(S(b)x) = \lim_{N\to\infty} \sum_{j=0}^{N} s_j a(b^j x) = \lim_{N\to\infty} \sum_{j=0}^{N} s_j (b^j a + j b^{j+1})x = S(b)ax + b^2 S'(b)x.$$

Of course, to each r-convergent (a,b)-module we can associate a s-convergent (a,b)-module for any $s \in [0, r]$ and so a convergent (a,b)-module in the case $s = 0$, via the correspondence:

$$\mathcal{E}_r \mapsto \mathcal{E}_r \otimes_{\mathbb{C}_r\{\{b\}\}} \mathbb{C}_s\{\{b\}\}$$

where the action of a is defined by

$$a(x \otimes S) = ax \otimes S + x \otimes b^2 S'$$

for $x \in \mathcal{E}_r$.

Lemma 2.2.2 *For any convergent (a,b)-module with a simple pole $\mathcal{E}$ there exists $r > 0$ and a r-convergent (a,b)-module $\mathcal{E}_r \subset \mathcal{E}$ such that $\mathcal{E} = \mathcal{E}_r \otimes_{\mathbb{C}_r\{\{b\}\}} B$ as a B-module and such that the equality*

$$a(x \otimes S(b)) = ax \otimes S(b) + x \otimes b^2 S'(b)$$

holds for each $S \in B$ and each $x \in \mathcal{E}_r$.

PROOF. Let $e := (e_1, \ldots, e_k)$ be a B-basis of $\mathcal{E}$ and write $ae = M(b)e$ where M is in $B \otimes_{\mathbb{C}} End_{\mathbb{C}}(\mathbb{C}^k)$ satisfies $M(0) = 0$. Choose now $r > 0$ such that M is in fact in $\mathbb{C}_r\{\{b\}\} \otimes_{\mathbb{C}} End_{\mathbb{C}}(\mathbb{C}^k)$ and define $\mathcal{E}_r := \oplus_{j=1}^{k} \mathbb{C}_r\{\{b\}\}e_j \subset \mathcal{E}$. Now the $\mathbb{C}$-linear endomorphism of $\mathcal{E}_r$ induced by a is clearly continuous for the dual Fréchet topology of $\mathcal{E}_r$, thanks to the continuity of the product in $\mathbb{C}_r\{\{b\}\}$, and satisfies $ab - ba = b^2$. It is easy to verify that the r-convergent (a,b)-module $\mathcal{E}_r$ satisfies the lemma. ■

2.3 Construction of $\mathcal{E}(\Theta)$

We shall construct now an important family of examples of simple poles convergent (a,b)-modules.
We begin by a very simple but useful lemma:

Lemma 2.3.1 *For $x \in [1, +\infty[$ and any integer $k \geq 0$ we have*

$$\frac{x(x+1)\dots(x+k)}{(k+1)!} \leq x^{k+1}.$$

PROOF. Obvious because for $x \geq 1$ we have $x + j \leq x(j+1)$ for each $j \geq 0$. ■

Note that for $x \in [0, 1]$ we have $\frac{x(x+1)\dots(x+k)}{(k+1)!} \leq 1$.

As an easy consequence, we obtain that for any endomorphism $\Theta \in L(\mathbb{C}^p, \mathbb{C}^p)$ we have the estimates

$$||\Theta \circ (\Theta + 1) \circ \cdots \circ (\Theta + k)|| \leq (k+1)!\, \theta^{k+1}$$

where $\theta := sup\{1, ||\Theta||\}$. where the norm $||\quad||$ satisfies $||x \circ y|| \leq ||x||.||y||$ for all $x, y \in L(\mathbb{C}^p, \mathbb{C}^p)$.

Then if we have a basis e of a rank p convergent (a,b)-module which satisfies

$$ae = \Theta be$$

where Θ is in $L(\mathbb{C}^p, \mathbb{C}^p)$ we shall have, by an easy induction on $k \geq 1$:

$$a^k e = \Theta \circ (\Theta + 1) \circ \cdots \circ (\Theta + k) b^k e$$

because $ab^k = b^k a + kb^{k+1}$.

Then assume now that we have $T \in \mathbb{C}\{a\} \otimes_{\mathbb{C}} L(\mathbb{C}^p, \mathbb{C}^p)$ and $x = Te$, write $T := \sum_{k=0}^{\infty} T_k a^k$ with a positive radius of convergence. We obtain that

$$x = \sum_{k=0}^{\infty} T_k \Theta \circ (\Theta + 1) \circ \cdots \circ (\Theta + k) b^k e$$

and the estimates above implies that x lies inside the B-module generated by e, because the series $\frac{1}{k!} T_k \Theta \circ (\Theta+1) \circ \cdots \circ (\Theta+k) b^k$ has a positive radius of convergence.

Proposition 2.3.2 *Let Θ be an invertible $(k,k)-$matrix with complex entries such that the spectrum of Θ is disjoint from $-\mathbb{N}$. Then we define the convergent (a,b)-module $\mathcal{E}(\Theta)$ as follows:*

1. *$\mathcal{E}(\Theta)$ is the free, rank k, B-module with basis $e := (e_1, \dots, e_k)$, the standard basis of $\mathbb{C}^k$.*
2. *The $\mathbb{C}$-linear* **continuous** *endomorphism $a : \mathcal{E}(\Theta) \to \mathcal{E}(\Theta)$ is defined by*

$$ae = \Theta be \quad \text{and} \quad ab - ba = b^2.$$

Then $\mathcal{E}$ is a simple pole convergent (a,b)-module.
Moreover, the action of $\mathbb{C}[a]$ on $\mathcal{E}(\Theta)$ extends to a continuous action of $A = \mathbb{C}\{a\}$ on $\mathcal{E}(\Theta)$ and under this action $\mathcal{E}(\Theta)$ is a free, rank k, A-module with basis e.

PROOF. In fact we shall begin by proving that for any $r \in]0,1[$ the $\mathbb{C}_r\{\{b\}\}-$submodule $\mathcal{E}_r(\Theta) := \oplus_{j=1}^k \mathbb{C}_r\{\{b\}\}e_j \subset \mathcal{E}(\Theta)$ which is clearly stable by a, is a r-convergent (a,b)-module with a simple pole. The continuity of a for the dual Fréchet topology of $\mathcal{E}_r$ follows from the continuity of the derivation in $\mathbb{C}_r\{\{b\}\}$ and the formula $a(S(b)e) = S(b)\Theta e + b^2 S'(b)e$ for any matrix $S \in \mathbb{C}_r\{\{b\}\} \otimes_{\mathbb{C}} End_{\mathbb{C}}(\mathbb{C}^k)$.
The last assertion is an easy consequence of the following equality, valid for all j and p in $\mathbb{N}$:

$$a^j b^p e = (\Theta + pId) \circ \cdots \circ (\Theta + (p+j-1)Id) b^{(j+p)} e. \tag{2}$$

As this equality is clear for $j = 0, 1$ and any $p \in \mathbb{N}$, thanks to the commutation relation $ab^p = b^p a + pb^{p+1}$, assume that it is true for j and compute $a^{j+1}b^p e$. The matrix Θ has complex entries so commutes with the action of a, so the commutation relation above is enough to conclude.
Now if $U := \sum_{j=0}^{+\infty} u_j a^j$ is in $\mathbb{C}_r\{a\}$, for some $R \in]1, 1/r[$ we may find a constant $C_R > 0$ such that $|u_j| \leq C_R R^j \quad \forall j \geq 0$.
Also if $S := \sum_{j=0}^{+\infty} s_j b^j$ is in $\mathbb{C}_r\{\{b\}\}^k$ we may find, choosing a bigger $R \in]1, 1/r[$ if necessary, a constant D_R such that

$$||s_j|| \leq D_R R^j j!.$$

Then we have

$$U(S(b)e) = \sum_{p=0}^{+\infty} X_p b^p e$$

where X_p in $\mathbb{C}^k$ is given by

$$X_p = \sum_{j=0}^{p} s_j u_{p-j}.$$

So we obtain the estimates, for some $R \in]1, 1/r[$, where $\theta := sup\{1, ||\Theta||\}$, assuming that $||\ ||$ is a multiplicative norm on the complex (k,k)-matrices and that q is a positive integer bigger than θ

$$||X_p|| \leq C_R D_R R^p \Big(\sum_{j=0}^{p} \frac{(p+q-1))!}{(q+j-1)!} \Big) \leq (1/(q-1)!) C_R D_R R^p (p+q)!$$

since we have

$$\sum_{j=0}^{p} (q+p-1)!/(q+j-1)! \leq (p+q)!/(q-1)!$$

for each integer $p \geq 0$. So $U(S(b)e)$ is in $\mathcal{E}_r(\Theta)$. This implies that $\mathcal{E}_r(\Theta)$ is a $\mathbb{C}_r\{a\}$-module.
To show that $\mathcal{E}(\Theta)$ is free A-module with basis e, since it has no a-torsion thanks to our hypothesis that $Spec(\Theta) \cap -\mathbb{N} = \emptyset$, it is enough to show that, for any $S \in B$,

$S(b)e$ is in the A-module generated by $e_1, \ldots, e_k$. Using the formula above with $p = 0$ we obtain, with the notation $(\Theta + qId)^{-1} := H_q$

$$b^j e = H_{j-1} \circ \cdots \circ H_0 a^j e$$

for each $j \geq 0$ and then $S(b)e = \sum_{p=0}^{+\infty} Y_p a^p e$ with $Y_p := s_p H_{p-1} \circ \cdots \circ H_0$.
But since the sequence (Θ/q) converges to 0 when $q \to +\infty$, there exists a constant $\Gamma > 1$ such that $(\Theta+qId)^{-1} = (1/q)(1+\Theta/q)^{-1}$ has its norm bounded by $\Gamma/(q+1)$ for each $q \geq 0$. This implies $||H_{p-1} \circ \cdots \circ H_0|| \leq \Gamma^p/p!$ and we obtain the estimates, since for some $R \in]1, 1/r[$ there exists a constant C_R such that $||s_p|| \leq C_R R^p p! \quad \forall p \geq 0$,

$$|Y_p| \leq C_R R^p \Gamma^p$$

which implies that $\sum_{p=0}^{+\infty} Y_p a^p$ is in $\mathbb{C}_s\{a\} \subset A$ with $s = r/\Gamma$. ■

EXAMPLE. For $\alpha \in]0, 1] \cap \mathbb{Q}$, the convergent (a,b)-module $\Xi_\alpha^{(N)}$ of asymptotic expansions (see section 5 below) is equal to $\mathcal{E}(\Theta_\alpha)$ where the matrix Θ_α is the following $(N+1, N+1)$ matrix:

$$J_{\alpha,N+1} := \begin{pmatrix} \alpha & 0 & 0 & 0 & \ldots & 0 \\ 1 & \alpha & 0 & 0 & \ldots & 0 \\ 0 & 1 & \alpha & 0 & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & 0 \\ 0 & \ldots & \ldots & 1 & \alpha & 0 \\ 0 & \ldots & 0 & 0 & 1 & \alpha \end{pmatrix}.$$

2.4 The algebra $\tilde{\mathcal{A}}_{conv.}$

Recall that the algebra $\mathcal{A}$ is the unitary $\mathbb{C}$-algebra generated by the two variables a and b with the commutation relation $ab - ba = b^2$.
For each $S \in \mathbb{C}[b]$ we have in $\mathcal{A}$ the relation $aS(b) = S(b)a + b^2 S'(b)$ where S' is the derivative of the polynomial S.

EXERCISE.

1. Verify that the vector space of homogenous polynomials in $a, b)$ of degree d in $\mathcal{A}$ admits either $a^{d-j}b^j, j \in [0, d]$ or $b^j a^{d-j}, j \in [0, d]$ as a basis.

2. Show that each homogeneous polynomial of degree d and monic in a in $\mathcal{A}$ may be written
$$P = (a - \lambda_1 b) \ldots (a - \lambda_d b)$$
where $\lambda_1, \ldots, \lambda_d$ are complex numbers (but such an expression in not unique, in general).
(*Hint*: Show that $b^{-d}P$ is a degree d monic polynomial in $b^{-1}a$). □

Let define the numbers $\Gamma^j_{p,q}$ for $p, q \in \mathbb{N}$ and for $j \in [0, p]$ by the equation in $\mathcal{A}$

$$a^p b^q = \sum_{j=0}^{p} \Gamma^j_{p,q} b^{q+j} a^{p-j} \tag{3}$$

It will be convenient to define $\Gamma^j_{p,q} = 0$ for $j > p$.

Lemma 2.4.1 *We have the following recursion relations:*

$$\begin{aligned}
&\Gamma^j_{p+1,q} = \Gamma^j_{p,q} + (q + j - 1)\Gamma^{j-1}_{p,q} \qquad \forall p, q \geq 0 \text{ and } \forall j \in [1, p] \\
&\Gamma^j_{p,q} = \Gamma^j_{p-1,q} + q\Gamma^{j-1}_{p-1,q+1} \qquad \forall p \geq 1, q \geq 0 \text{ and } \forall j \in [1, p+1]. \\
&\text{So} \quad \Gamma^j_{p+1,q} = \Gamma^j_{p,q} + q\Gamma^{j-1}_{p,q+1} \qquad \forall p, q \geq 0 \text{ and } \forall j \in [1, p+1].
\end{aligned}$$

PROOF. To prove the first relation multiply on the left the equality (3) by a. This gives that $\Gamma^j_{p+1,q}$ is the coefficient of $b^{q+j}a^{p+1-j}$ in the sum

$$\begin{aligned}
\sum_{h=0}^{p} \Gamma^h_{p,q} a b^{q+h} a^{p-h} &= \sum_{h=0}^{p} \Gamma^h_{p,q} (b^{q+h} a + (q+h) b^{q+h+1}) a^{p-h} \\
&= \sum_{h=0}^{p} \Gamma^h_{p,q} b^{q+h} a^{p-h+1} + \sum_{h=0}^{p} \Gamma^h_{p,q} (q+h) b^{q+h+1}) a^{p-h}
\end{aligned}$$

and we obtain the first relation.
For the second relation, write if $p \geq 1$ and $q \geq 1$:

$$a^p b^q = a^{p-1}(ab)b^{q-1} = a^{p-1}(ba + b^2)b^{q-1} = a^{p-1}b(b^{q-1}a + (q-1)b^q) + a^{p-1}b^{q+1} = a^{p-1}b^q a + q a^{p-1} b^{q+1}.$$

Then looking for the coefficient of $b^{q+j}a^{p-j}$ in the sum

$$\sum_{h=0}^{p-1} \Gamma^h_{p-1,q} b^{q+h} a^{p-h} + q \sum_{h=0}^{p-1} \Gamma^h_{p-1,q+1} b^{q+h+1} a^{p-h-1}$$

we obtain the second relation.
Now writing the second relation for $p + 1$ gives the third relation. ■

Comparing the third relation with the first one gives:

$$q\Gamma^{j-1}_{p,q+1} = (q + j - 1)\Gamma^{j-1}_{p,q} \qquad \forall p \geq 1 \quad \forall q \geq 0 \quad \forall j \in [1, p+1]$$

which leads to

$$\Gamma^j_{p,q+1} = \frac{(q+j)!}{j!q!} \Gamma^j_{p,1} \qquad \forall p \geq 1 \quad \forall q \geq 0 \quad \forall j \in [0, p].$$

As have

$$\Gamma^j_{p,0} = 1 \text{ for } j = 0 \quad \text{and } \Gamma^j_{p,0} = 0 \text{ for } j \in [1, p]$$

the first relation in the lemma above gives

$$\Gamma^j_{p,1} = \frac{p!}{(p-j)!}$$

So we have proved the following formula:

$$\Gamma^j_{p,q} = \frac{(q+j-1)!}{j!(q-1)!}\frac{p!}{(p-j)!} \qquad \forall q \geq 1 \quad \forall p \geq 0 \quad \text{and } \forall j \in [0,p] \tag{4}$$

REMARK. There is a $\mathbb{C}$-linear anti-automorphism F of $\mathcal{A}$ defined by the following conditions

1. $F(xy) = F(y)F(x) \quad \forall x, y \in \mathcal{A}$.
2. $F(a) = a, F(b) = -b \quad \text{and } F(1) = 1$

Then apply F to the relation (3) gives

$$b^q a^p = \sum_{j=0}^{p} (-1)^j \Gamma^j_{p,q} a^{p-j} b^{q+j} \tag{5}$$

which inverts the relation (3) when we consider it as the base change in the vector space of homogeneous polynomials of degree m in (a,b) inside $\mathcal{A}$ for the basis $(a^p b^q)_{p+q=m}$ to the basis $(b^q a^p)_{p+q=m}$.

Corollary 2.4.2 *For each $x \in \mathbb{C}$ we have the equality*

$$(a+xb)^p = a^p + \sum_{j=1}^{p} \gamma_j(x) C^j_p b^j a^{p-j} \tag{6}$$

in the algebra $\mathcal{A}$, where $\gamma_j(x) := (x+j-1)(x+j-2)\dots x$ for $j \in [1,p]$.

PROOF. Since both sides are degree p polynomials in x with coefficients homogeneous of degree p in (a,b) in the $\mathbb{C}$-algebra $\mathcal{A}$, it is enough to prove this formula for each $x = q \in \mathbb{N}$. Let us show first that we have the identity $b^q(a+qb)^p = a^p b^q$ in $\mathcal{A}$: This is an easy consequence of the fact that $b(a+qb) = ab - b^2 + qb^2 = (a+(q-1)b)b$. Then the formulas (4) and (5) imply (6) for $x = q \in \mathbb{N}$, because b is not a zero divisor in $\mathcal{A}$. ■

Note that for $x = q \in \mathbb{N}^*$ we have $\gamma_j(q) = (q+j-1)!/(q-1)!$.

AN EXAMPLE. As $\gamma_j(-1) = 0$ for $j \geq 2$ we obtain in $\mathcal{A}$ the relations:

$$(a-b)^p = a^p - pba^{p-1} \quad \forall p \geq 2$$

This equality is easy to prove directly by induction on $p \in \mathbb{N}$, for instance using the identities $(a-b)^p b = ba^p$ and $a^p b = ba^p + pba^{p-1}b$.

EXERCISE. Show that for each integers $p, q \geq 0$ we have

$$(a + xb)^p b^q = b^q (a + (x + q)b)^p.$$

Then for $x = -q$ this gives $(a - qb)^p b^q = b^q a^p$. □

Theorem 2.4.3 *Let $\tilde{\mathcal{A}}_{conv.}$ be the $\mathbb{C}-$vector space*

$$\tilde{\mathcal{A}}_{conv.} := \{\sum_{p,q} \gamma_{p,q} a^p b^q \quad \exists R > 1,\ \exists C_R \quad s.\ t. \quad |\gamma_{p,q}| \leq C_R R^{p+q} q!\}.$$

Assume that the variables a and b satisfies the commutation relation $ab - ba = b^2$. Then $\tilde{\mathcal{A}}_{conv.}$ is a sub-algebra of $\widehat{\mathcal{A}}$ of formal power series in the variables (a, b) with the relation $ab - ba = b^2$. Moreover $\tilde{\mathcal{A}}_{conv.}$ is also described as the vector space:

$$\tilde{\mathcal{A}}_{conv.} := \{\sum_{p,q} \delta_{p,q} b^q a^p \quad \exists R > 1,\ \exists D_R \quad s.\ t. \quad |\delta_{p,q}| \leq D_R R^{p+q} q!\}.$$

PROOF. Let $X := \sum_{p,q} \gamma_{p,q} a^p b^q$ and $Y := \sum_{p',q'} \delta_{p',q'} a^{p'} b^{q'}$ be in $\tilde{\mathcal{A}}_{conv.}$. Then the product XY in $\widehat{\mathcal{A}}$ is given by

$$XY = \sum_{m,n} \varepsilon_{m,n} a^m b^n$$

where

$$\varepsilon_{m,n} = \sum_{p+p'-j=m, q+q'+j=n} (-1)^j \Gamma^j_{p',q} \gamma_{p,q} \delta_{p',q'}$$

using Formula (5) to compute $b^q a^{p'}$.
There exists positive constants R and C_R large enough such that the following estimates hold true

$$|\gamma_{p,q}| \leq C_R R^{p+q} q! \quad \text{and} \quad |\delta_{p,q}| \leq C'_R R^{p+q} q!$$

and then we obtain

$$|\varepsilon_{m,n}| \leq C_R C'_R R^{m+n} n! \sum_{p+p'-j=m, q+q'+j=n} \frac{(q + j - 1)! p'! q! q'!}{n!(q-1)! j! (p'-j)!}$$

In order to estimates the sum above, first note that q, q', j are at most equal to n so that the triple (q, q', j) takes at most $(n+1)^2$ values. Then p and p' are at most equal to $m + n$ and so the sum has at most $(n+1)^2(m+n+1)$ terms. Each term is now bounded by

$$q \frac{C^j_{p'}}{n C^{q'}_{n-1}} \leq C^j_{p'} \leq 2^{m+n}$$

using the fact that $q+j-1=n-q'-1$. So there exists a constant D_{3R} such that

$$C_R C'_R (2R)^{m+n}(n+1)^2(m+n+1) \leq D_{3R}(3R)^{m+n} \quad \forall m,n \in \mathbb{N}$$

which implies

$$|\varepsilon_{m,n}| \leq D_{3R}(3R)^{m+n} n!$$

showing that XY is in $\tilde{\mathcal{A}}_{conv.}$.
In order to write $X = \sum_{m,n} c_{m,n} b^n a^m$ in $\widehat{\mathcal{A}}$ we use the formula (4) and we obtain

$$c_{m,n} = \sum_{p-j=m, q+j=n} (-1)^q \Gamma^j_{p,q} \gamma_{p,q}.$$

The sum has at most $(n+1)^2(m+n+1)$ terms and we have

$$|\Gamma^j_{p,q}\gamma_{p,q}| \leq C_R.R^{p+q} q! \frac{(q+j-1)!p!}{(q-1)!j!(p-j)!} \leq C_R R^{m+n} n! C^j_p \leq C_R (2R)^{m+n} n!$$

and we conclude as above that there exists a constant D_R such that the following estimates hold true $\quad |c_{m,n}| \leq D_R(3R)^{m+n} n! \quad \forall (m,n) \in \mathbb{N}^2$. ■

REMARK. For $1 < S < R$ and any positive integers m and n there is a positive constant $C_{m,n}$ such that

$$(\frac{S}{R})^{p+q} \frac{(q+m)!}{q!}(1+p)^n \leq C_{m,n} \quad \forall (p,q) \in \mathbb{N}^2.$$

So if $X := \sum_{p,q} x_{p,q} a^p b^q$ satisfies, for some given integers m and n, the estimates

$$|x_{p,q}| \leq C_R R^{p+q}(q+m)!(1+p)^n \quad \forall (p,q) \in \mathbb{N}^2$$

then X is in $\tilde{\mathcal{A}}_{conv.}$.
In particular, if some $X \in \widehat{\mathcal{A}}$ is such that Xb^m is in $\tilde{\mathcal{A}}_{conv.}$ then X is also in $\tilde{\mathcal{A}}_{conv.}$. Moreover we may find $Y \in \tilde{\mathcal{A}}_{conv.}$ such that $Xb^m = b^m Y$. And the same is true if $b^m X$ is in $\tilde{\mathcal{A}}_{conv.}$, then X is also in $\tilde{\mathcal{A}}_{conv.}$ and we may find $Z \in \tilde{\mathcal{A}}_{conv.}$ such that $b^m X = Z b^m$.
The proof of the previous assertions is easy as it is enough to treat the case $m=1$, using the second part of the previous theorem.
Also, if some $X \in \widehat{\mathcal{A}}$ is such that Xa or aX is in $\tilde{\mathcal{A}}_{conv.}$ then X is also in $\tilde{\mathcal{A}}_{conv.}$. But contrary to $b\tilde{\mathcal{A}}_{conv.} = \tilde{\mathcal{A}}_{conv.}b$ the left and right ideals $a\tilde{\mathcal{A}}_{conv.}$ and $\tilde{\mathcal{A}}_{conv.}a$ are different. □

Lemma 2.4.4 *For any $X := \sum_{p,q} x_{p,q} a^p b^q \in \tilde{\mathcal{A}}_{conv.}$ define for $f(z) := \sum_{m=0}^{\infty} t_m z^m$ in $\mathbb{C}\{z\}$*

$$X(f) := \sum_{m=0}^{\infty} u_m z^m \quad \text{where} \quad u_m := \sum_{p+q+r=m} \frac{r!}{(q+r)!} x_{p,q} t_r.$$

Then $X(f)$ is in $\mathbb{C}\{z\}$.
The corresponding map of $\mathbb{C}$-algebras which sends $\tilde{\mathcal{A}}_{conv.}$ to the algebra of continuous endomorphisms of $\mathbb{C}\{z\}$, is continuous and injective.

PROOF. For $R > 1$ large enough there exists $C_R > 0$ and $D_R > 0$ such that

$$|t_r| \leq C_R R^r \quad \text{and} \quad |x_{p,q}| \leq D_R R^{p+q} q!.$$

Then we obtain

$$|u_m| \leq C_R D_R R^m \sum_{p+q+r=m} \frac{r!q!}{(q+r)!}.$$

The sum above has at most $(m+1)^2$ terms and each one is bounded by 1 so for any $\rho > R$ there exists a constant C_ρ such that $|u_m| \leq C_\rho \rho^m$, and this allows to conclude that $X(f)$ is in $\mathbb{C}\{z\}$.
The continuity of this map follows from the estimates above.
Consider now X such that $X(z^r) = 0$ for each $r \in \mathbb{N}$. The coefficient of z^{m+r} in $X(z^r)$ is given by $P_m(z^r)$ where $P_m := \sum_{p+q=m} \gamma_{p,q} a^p b^q$ is the homogeneous part of degree m in (a,b) in X. We know that a non zero homogeneous degree m in (a,b) element in $\mathcal{A}$ may be written $b^j(a-\lambda_1 b)\dots(a-\lambda_{m-j}b)$ where $\lambda_1, \dots, \lambda_{m-j}$ are complex numbers (see exercise 2 at the beginning of this section). Also b is injective on $\mathbb{C}\{z\}$, and we have

$$(a - \lambda b)(z^r) = (1 - \lambda/(r+1))z^{r+1}$$

which vanishes only when $r+1 = \lambda$. So, for any given m there exists r large enough such that $P_m(z^r) \neq 0$. Then, for $X \neq 0$ there exists $m \in \mathbb{N}$ such that $P_m \neq 0$ and for r large enough the coefficient of z^{r+m} in $X(z^r)$ is not zero, and the conclusion follows. ■

Lemma 2.4.5 *For each non negative integer N the vector space defined as the subset of series $X := \sum_{(p,q)\geq 0} x_{p,q} a^p b^q$ in $\widehat{\mathcal{A}}$ satisfying*

$$\exists R > 1,\ \exists C_R \quad s.\ t. \quad |x_{p,q}| \leq C_R R^{p+q} \frac{(p+q)!}{p!}(1+p)^N \quad \forall p, q \geq 0$$

is equal to the algebra $\tilde{\mathcal{A}}_{conv.}$.

PROOF. The inclusion in $\tilde{\mathcal{A}}_{conv.}$ is clear because $p!q! \leq (p+q)!(1+p)^N$ for any integers $p, q, N \geq 0$. Conversely, if X is in $\mathcal{S}$ we have

$$|x_{p,q}| \leq C_R R^{p+q} C^q_{p+q} q! (1+p)^N \leq \Gamma_N C_R (4R)^{p+q} q!$$

because $C^q_{p+q} \leq 2^{p+q}$ and for each N there exists Γ_N large enough such that $(1+p)^N \leq \Gamma_N 2^p$ for any $p \geq 0$. ■

Proposition 2.4.6 *Let X be in $\tilde{\mathcal{A}}_{conv.}$ and assume that $x_{0,0} = 1$. Then X is invertible in the algebra $\tilde{\mathcal{A}}_{conv.}$.*

PROOF. Using the formula for the product in $\tilde{\mathcal{A}}_{conv.}$ obtained in the proof of Theorem 2.4.3 we see that, if $Y := \sum_{p',q' \geq 0} y_{p',q'} a^{p'} b^{q'}$ is a formal inverse of X, then Y is given by $y_{0,0} = 1$ and for $(m,n) \neq (0,0)$ by

$$y_{m,n} = \sum_{(j,p,q) \in A(m,n)} (-1)^j \Gamma^j_{p',q} x_{p,q} y_{p',q'}$$

where we define

$$A(m,n) = \{(p,q,j) \in \mathbb{N}^3 \ / \ \exists (p',q') \neq (m,n) \ such \ that \ p+p'-j = m, q+q'+j = n\}.$$

We shall note by $B_{m,n}$ the set of (p',q') such that there exists $(p,q,j) \in A_{m,n}$ such that $p' = m+j-p, q' = n-q-j$. Choose $R > 1$ and C_R such that

$$|x_{p,q}| \leq C_R R^{p+q} \frac{(p+q)!}{p!} \quad \forall p,q \geq 0$$

and assume that $S > R$ is large enough to satisfy satisfies

$$C_R R/(S-R) \leq 1.$$

We shall prove by induction on $p'+q' \geq 0$ the estimates

$$|y_{p',q'}| \leq D_S S^{p'+q'} \frac{(p'+q')!}{p'!}.$$

So assume that this estimates has been obtained for $p'+q' \leq d-1$ with $d \geq 1$ (for $d = 0$ we have only to ask that $D_S \geq 1$) and fix any (m,n) such that $m+n = d$. We shall describe $A_{m,n}$ by fixing j,q and $r := p+q$ so we have $p = r-q$, $p' = m+j-(r-q)$ and $q' = n-j-q$, and

$$\sum_{p,q,j \in A_{m,n}} = \sum_{r=1}^{m+n} \sum_{q=0}^{n} \sum_{j=0}^{n-q}.$$

Then remark that, as $(p',q') \in B_{m,n}$ implies that $p+q+p'+q' = m+n = d$ with $(p,q) \neq (0,0)$ we have $p'+q' \leq d-1$ for $(p',q') \in B_{m,n}$; so for $(p',q') \in B_{m,n}$, $\beta_{p',q'}$ satisfies our inductive estimates.
Then we obtain the estimates:

$$|y_{m,n}| \leq D_S S^{m+n} \frac{(m+n)!}{m!} \sum_{p,q,j \in A_{m,n}} C_R (R/S)^{p+q} \frac{(p+q)!(p'+q')!}{p!p'!} \frac{m!}{(m+n)!} \Gamma^j_{p',q}$$

$$\leq D_S S^{m+n} \frac{(m+n)!}{m!} \big[\sum_{r=1}^{m+n} \frac{C_R (R/S)^r}{C^r_{m+n}} \Phi(r) \big]$$

where, fixing $r \in [1, m+n]$, we have to estimate, using that $\Gamma^j_{p',q} = C^{q-1}_{q+j-1} \frac{p'!}{(p'-j)!}$, the quantity

$$\Phi(r) := \sum_{q=0}^{n} \sum_{j=0}^{n-q} \frac{m!}{p!p'!} \Gamma^j_{p',q} = \sum_{q=0}^{n} \sum_{j=0}^{n-q} \frac{m!}{p!p'!} C^{q-1}_{q+j-1} \frac{p'!}{(p'-j)!}.$$

Using the equality $m = p + p' - j$ we obtain

$$\Phi(r) = \sum_{q=0}^{n} \sum_{j=0}^{n-q} \frac{m!}{p!(p'-j)!} C_{q+j-1}^{q-1} = \sum_{q=0}^{n} \sum_{j=0}^{n-q} C_m^{r-q} C_{q+j-1}^{q-1}. \qquad (@)$$

Now we shall use the following elementary lemma.

Lemma 2.4.7 *For any positive integers x, y we have*

$$\sum_{j=0}^{y} C_{x+j}^{j} = C_{x+y+1}^{x+1}.$$

PROOF. As $C_{x+j}^{j} = C_{x+j}^{x}$ we are computing the coefficient of a^x in the polynomial $\sum_{j=0}^{y}(1+a)^{x+j}$ which is equal to

$$(1+a)^x \frac{(1+a)^{y+1} - 1}{a}.$$

So our sum is equal to the coefficient of a^{x+1} in the polynomial

$$(1+a)^{x+y+1} - (1+a)^x$$

and it is equal to C_{x+y+1}^{x+1}. ■

END OF PROOF OF PROPOSITION 2.4.6. So, thanks to the equality above, for $x := q - 1$ and $y := n - q$, which gives $\sum_{j=0}^{n-q} C_{q+j-1}^{q-1} = C_n^q$, the sum in (@) admit the estimates

$$\Phi(r) \leq \sum_{q=0}^{n} C_m^{r-q} C_n^q \leq C_{m+n}^r$$

because $\sum_{p=0}^{r} C_m^p C_n^{r-p} = C_{m+n}^r$. Then the estimates (@) gives

$$|y_{m,n}| \leq C_R D_S S^{m+n}(m+n)! \frac{1}{m!} \sum_{r=1}^{m+n} \frac{C_R (R/S)^r}{C_{m+n}^r} \Phi(r) \leq D_S S^{m+n}(m+n)! \frac{1}{m!}$$

as we choose S in order that

$$C_R \sum_{r=1}^{m+n} (R/S)^r \leq C_R \frac{R}{S-R} \leq 1$$

we obtain that Y is in $\tilde{\mathcal{A}}_{conv.}$. ■

Of course this proposition implies that any $X \in \tilde{\mathcal{A}}_{conv.}$ such that $x_{0,0} \neq 0$ is invertible in $\tilde{\mathcal{A}}_{conv.}$ because writing such an element as $X = S(b)(1 + aZ)$ where S is in B is

invertible (because $S(0) = x_{0,0} \neq 0$) and we may apply the previous proposition to $T := S^{-1}X = 1 + aZ$ which is in $\tilde{\mathcal{A}}_{conv.}$ and satisfies $t_{0,0} = 1$. Then $X^{-1} = T^{-1}S^{-1}$ is in $\tilde{\mathcal{A}}_{conv.}$.
As a consequence, we see that $a\tilde{\mathcal{A}}_{conv.} + b\tilde{\mathcal{A}}_{conv.} = \tilde{\mathcal{A}}_{conv.}b + \tilde{\mathcal{A}}_{conv.}a$ is the unique two sided maximal ideal in $\tilde{\mathcal{A}}_{conv.}$ which is closed and with quotient

$$\tilde{\mathcal{A}}_{conv.}/a\tilde{\mathcal{A}}_{conv.} + b\tilde{\mathcal{A}}_{conv.} \simeq \mathbb{C}.$$

Corollary 2.4.8 *Let Θ be a (k,k)-matrix such that its spectrum is disjoint from $-\mathbb{N}$ and consider the simple pole convergent (a,b)-module $\mathcal{E}(\Theta)$ introduced in section 2.3. Then the continuous actions of the algebras A and B on $\mathcal{E}(\Theta)$ are the restriction of a continuous action of $\tilde{\mathcal{A}}_{conv.}$ on $\mathcal{E}(\Theta)$.*

PROOF. Since the rigth multiplication by an element of B is a continuous endomorphism of left $\tilde{\mathcal{A}}_{conv.}$-module on $\tilde{\mathcal{A}}_{conv.}$, it is enough to define the (left) action of $\tilde{\mathcal{A}}_{conv.}$ on the B-basis $e_j, j \in [1,k]$ of $\mathcal{E}(\Theta)$:
Indeed, if $X \in \tilde{\mathcal{A}}_{conv.}$ and $Z \in B^k$ the action of X on Ze is, by definition, the action of $XZ \in \tilde{\mathcal{A}}^k_{conv.}$ on e where XZ is defined by the left action of $\tilde{\mathcal{A}}_{conv.}$ on $\tilde{\mathcal{A}}^k_{conv.}$ by left multiplication (using the fact that $B \subset \tilde{\mathcal{A}}_{conv.}$) and then we use the action of $\tilde{\mathcal{A}}^k_{conv.}$ on each e defined below.

So, using the second assertion of Theorem 2.4.3, let $X = \sum_{p,q} x_{p,q}b^q a^p \in \tilde{\mathcal{A}}^k_{conv.}$; there exists constants $R > 1$ and $C_R > 0$ with $|x_{p,q}| \leq C_R R^{p+q} q! \quad \forall (p,q) \in \mathbb{N}^2$.
Since we have $a^p e = (\Theta + (p-1)Id) \circ \cdots \circ \Theta b^p e$ we obtain

$$Xe = \sum_{p,q} x_{p,q} b^q a^p e = \sum_{j=0}^{\infty} z_j b^j e$$

where $z_j = \sum_{p+q=j} x_{p,q}(\Theta + (p-1)Id) \circ \cdots \circ \Theta$. We have the estimate, for τ an integer larger than $||\Theta||$

$$|z_j| \leq C_R R^j j! \sum_{p=0}^{j} \frac{(j-p)!(\tau+p-1)!}{j!(\tau-1)!}.$$

For $S > R$ and $\tau \in \mathbb{N}$ given there exists a constant $D > 0$ such that

$$R^j(\tau+p-1)! \leq DS^j(1+p)^\tau(\tau-1)!$$

for each integers j and p so we obtain, using Lemma 2.1.2

$$|z_j| \leq 3C_R D S^j j!(1+j)^\tau \quad \forall j \in \mathbb{N}$$

and then $Z := \sum_{j=0}^{\infty} z_j b^j$ is in B^k and $Xe := Ze$ is well defined in the B-module with basis e.

REMARK. The special case where, for $\alpha \in]0,1] \cap \mathbb{Q}$ and $k \in \mathbb{N}^*$, we consider the (k,k)-matrix Θ_α which gives $\mathcal{E}(\Theta_\alpha) = \Xi_\alpha^{(k-1)}$ will be important since it shows, using the Embedding Theorem 5.1.3, that geometric (a,b)-modules are canonically left $\tilde{\mathcal{A}}_{conv.}$-modules.

2.5 The Division Theorem

First we shall work inside the algebra $\mathcal{A}$ of polynomials in the variables a and b with the commutation relation $ab - ba = b^2$.
Note that this algebra is integral and that any homogeneous element in (a,b) of degree $m \geq 1$ monic in a may be factorized as

$$P_m := (a - \lambda_1 b)(a - \lambda_2 b) \dots (a - \lambda_m b)$$

and that the polynomial $\pi(x) := (x - (\lambda_1 - m) \dots (x - \lambda_m)$ satisfies the relation

$$(-b)^m \pi(-b^{-1}a) = P_m$$

where the computation is made in the algebra $\mathcal{A}[b^{-1}]$.

Lemma 2.5.1 *For each complex number λ and each integer $m \in \mathbb{N}^*$ we have the equality*

$$a^m = Q_{m-1}(\lambda)(a - \lambda b) + R_m(\lambda)$$

where

$$Q_{m-1}(\lambda) = a^{m-1} + \lambda a^{m-2} b \cdots + \lambda(\lambda+1) \dots (\lambda + m - 2) b^{m-1}$$

and

$$R_m(\lambda) = \lambda(\lambda+1) \dots (\lambda + m - 1) b^m.$$

PROOF. For $m = 1$ the relation $a = (a - \lambda b) + \lambda b$ is clear and give $Q_0(\lambda) \equiv 1$ and $R_1(\lambda) = \lambda b$. So assume that the lemma is proved for $m \geq 1$ and multiply on the left by a. We obtain:

$$a^{m+1} = aQ_{m-1}(\lambda) + aR_m(\lambda) = aQ_{m-1}(\lambda) + R_m(\lambda)(a + mb).$$

Writing $R_m(\lambda)(a + mb) = R_m(\lambda)\big((a - \lambda b) + (m + \lambda)b\big)$ we obtain

$$Q_m(\lambda) = aQ_{m-1}(\lambda) + R_m(\lambda) \quad \text{and} \quad R_{m+1}(\lambda) = (\lambda + m) b R_m(\lambda)$$

completing the proof. ■

Proposition 2.5.2 *Let $X := \sum_{(p,q)\in\mathbb{N}^2} x_{p,q} b^q a^p$ be an element in the algebra $\tilde{\mathcal{A}}_{conv.}$. Then for any* **real** *number λ there exists a unique Q in $\tilde{\mathcal{A}}_{conv.}$ and an unique R in B such that the following equality holds in $\tilde{\mathcal{A}}_{conv.}$:*

$$X = Q(a - \lambda b) + R$$

PROOF. To prove the uniqueness we have to show that $Q(a-\lambda b)+R=0$ with $Q\in\tilde{\mathcal{A}}_{conv.}$ and $R\in B$ implies $Q=0$ and $R=0$.
Assume that Q is not 0 and consider the minimal integer $m\geq 1$ such that there is a non zero homogeneous term of degree $m-1$ in (a,b) inside Q, denote it q_{m-1}. Then the assumption implies we have $q_{m-1}(a-\lambda b)+r_m b^m=0$ for some complex number r_m which is the coefficient of b^m in R. For each integer d the vector space of homogeneous elements of degree d in (a,b) admits the basis

$$(a-\lambda b)^d, b(a-\lambda b)^{d-1},\dots,b^{d-1}(a-\lambda b),b^d$$

and the relation above gives a non trivial linear relation between these linearly independent elements. This contradicts the assumption that $q_{m-1}\neq 0$. Then $Q=0$ and so $R=0$.
To prove the existence, write $X=\sum_{p=0}^{\infty}X_p(b)a^p=\sum_{p,q}x_{p,q}b^q a^p$ where X_p is in B for each $p\geq 0$. Then at the formal level[2] we have, thanks to Lemma 2.5.1, the equality

$$X=\big(\sum_{p=0}^{\infty}X_p(b)Q_p\big)(a-\lambda b)+R$$

To complete the proof, we have to show that the formal solution obtained above

$$\big(Q:=\big(\sum_{p=0}^{\infty}X_p(b)Q_p\big),R\big)$$

defines a Q in $\tilde{\mathcal{A}}_{conv.}$ and a R in B. The real number λ will be fixed in the sequel and we put $R=\sum_{p=0}^{\infty}r_p b^p$.
Fix positive constants $S>1$ and C such that

$$|x_{p,q}|\leq CS^{(p+q)}q!$$

The coefficient r_j of b^j in R comes only from the products of $x_{p,j-p-1}b^{j-p-1}a^p$ by $-\lambda b$ for $p\in[0,j-1]$. This gives

$$r_j=-\lambda\sum_{p=0}^{j-1}x_{p,j-p-1}\Gamma^p_{p,1}=-\lambda\sum_{p=0}^{j-1}x_{p,j-p-1}p!$$

so using the estimate $\sum_{p=0}^{j-1}(j-p-1)!p!\leq 3(j!)$ and defining $C_1:=3|\lambda|CS^{-1}$ give

$$|r_j|\leq C_1S^j j!$$

showing that R is in B.
So $Q(a-\lambda b)$ is in $\tilde{\mathcal{A}}_{conv.}$ and we shall prove now that this implies that Q is also in $\tilde{\mathcal{A}}_{conv.}$.

[2]So in the algebra $\widehat{\mathcal{A}}[\lambda]$ the formal completion in (a,b) of $\mathcal{A}[\lambda]$.

The real number λ is fixed and and define the complex numbers $\gamma_{p,q}$ and $\delta_{p,q}$ by

$$Q := \sum_{p,q} \gamma_{p,q} a^p b^q \quad \text{and} \quad Q(a - \lambda b) := \sum_{p,q} \delta_{p,q} a^p b^q.$$

The relation between the $\delta_{p,q}$ and $\gamma_{p,q}$ are given by

$$\delta_{p+1,q+1} = \gamma_{p,q+1} - (q + \lambda)\gamma_{p+1,q} \quad \forall (p, q) \in \mathbb{N}^2 \tag{R}$$

and with $\delta_{0,0} = 0, \delta_{0,q+1} = -(q + \lambda)\gamma_{0,q}$ and $\delta_{p+1,0} = \gamma_{p,0}$.
The following two lemmas allow to reduced to find the estimates we are looking for in the case where λ is in $]0, 1[$ (recall that the case $\lambda = 0$ is trivial; see the remark following Theorem 2.4.3).

Lemma 2.5.3 *Fix $X \in \tilde{\mathcal{A}}_{conv.}$ and the complex number λ. Assume that there exists a positive integer m for which the quotient in the right division of Xb^m by $a - (\lambda - m)b)$ lies in $\tilde{\mathcal{A}}_{conv.}$. Then the quotient in the right division of X by $(a - \lambda b)$ is also in $\tilde{\mathcal{A}}_{conv.}$.*

Proof. The identity $b^m(a-(\lambda-m)b) = (a-\lambda b)b^m$ allows to obtain at the formal level, using the uniqueness of the "formal" division, that if

$$X = Q(a - \lambda b) + R \quad \text{then} \quad Xb^m = Qb^m(a - (\lambda - m)b) + Rb^m.$$

As we assume that Qb^m is in $\tilde{\mathcal{A}}_{conv.}$ we conclude that Q is in $\tilde{\mathcal{A}}_{conv.}$ thanks to the remark following Theorem 2.4.3.

Lemma 2.5.4 *Fix the complex number λ. Assume that there exists a positive integer m such that for any $X \in \tilde{\mathcal{A}}_{conv.}$ the quotient in the right division of X by $a - (\lambda + m)b$ lies in $\tilde{\mathcal{A}}_{conv.}$. Then, for any $X \in \tilde{\mathcal{A}}_{conv.}$ the quotient of X in the right division by $(a - \lambda b)$ is also in $\tilde{\mathcal{A}}_{conv.}$.*

Proof. First take any $X \in \tilde{\mathcal{A}}_{conv.}$ and write $b^m X = Y b^m$ with Y in $\tilde{\mathcal{A}}_{conv.}$ (see the remark following Theorem 2.4.3). Then write $Y = Q(a - (\lambda + m)b) + R$ with $Q \in \tilde{\mathcal{A}}_{conv.}$ and $R \in B$ thanks to our hypothesis. Multiply this equality on the right by b^m gives

$$Y b^m = b^m X = Q(a - (\lambda + m)b)b^m + Rb^m.$$

But we have $(a - (\lambda + m)b)b^m = b^m(a - \lambda b)$ and,using again the remark following Theorem 2.4.3, we may write $Qb^m = b^m Q_1$ where Q_1 is $\tilde{\mathcal{A}}_{conv.}$. So we obtain

$$b^m X = b^m Q_1(a - \lambda b) + b^m R$$

which allows to conclude as left product by b is injective in $\tilde{\mathcal{A}}_{conv.}$. ■

End of proof of Proposition 2.5.2. From now on, the number λ will be real, and thanks to the previous lemmas we may assume that λ is in $]0,1[$.

We fix some positive constants C and $R > 1$ such that $|\delta_{p,q}| \leq CR^{(p+q)}q! \quad \forall (p,q) \in \mathbb{N}^2$.
We fix also some $S := kR$ with $k > 1$.

For $q = 0$ and any $p \geq 0$ we have

$$|\gamma_{p,0}| = |\delta_{p+1,0}| \leq CR^{(p+1)} \leq \Gamma S^p$$

because we shall choose the constant Γ larger than CR.

Let $q \geq 1$ and assume that we have already proved that for any $p \in \mathbb{N}$

$$|\gamma_{p,q-1}| \leq \Gamma S^{(p+q-1)}(q-1)! \quad \forall p \geq 0$$

where we shall choose the constant $\Gamma \geq CR$ later on (but independent on $q \in \mathbb{N}$). Then, as $\gamma_{p,q} = \delta_{p+1,q} + (q + \lambda - 1)\gamma_{p+1,q-1}$ we obtain

$$|\gamma_{p,q}| \leq CR^{(p+q+1)}q! + (q + \lambda - 1)\Gamma S^{(p+q)}(q-1)!$$

and then

$$|\gamma_{p,q}| \leq \Gamma S^{(p+q)}q!\Big[\frac{CR}{\Gamma}k^{-(p+q)} + \frac{q+\lambda-1}{q}\Big].$$

Now we shall choose the constant Γ in order that it satisfies

$$\frac{CR}{\Gamma}k^{-(p+q)} + \frac{q+\lambda-1}{q} \leq 1$$

for any $p \geq 0$ and any $q \geq 1$:
First we shall choose a constant $\alpha := CR/\Gamma$ (by choosing Γ big enough) in order that

$$\frac{CR}{\Gamma}k^{-(p+q)} + \frac{\lambda-1}{q} \leq 0$$

for any $p \geq 0$ and $q \geq 1$. Because qk^{-q} converges to 0 when q goes to $+\infty$ and $\lambda - 1 < 0$, there exists a positive integer q_0 such that $qk^{-(p+q)} + \lambda - 1 < 0$ for all $q \geq q_0$ and all $p \geq 0$. This is possible since $k > 1$ and $\lambda < 1$.
Then, for any given q, as $\alpha q k^{-(p+q)} + \lambda - 1$ goes to $\lambda - 1 < 0$ when α goes to 0^+, there exists $\alpha_0 \in]0,1[$ such that for any $\alpha \in]0,\alpha_0[$ and any $q \leq q_0$ we have $\alpha q k^{-(p+q)} + \lambda - 1 < 0$. Now taking Γ big enough to have $CR/\Gamma < \alpha_0 < 1$ we obtain that

$$\frac{CR}{\Gamma}k^{-(p+q)} + 1 + (\lambda - 1)/q \leq 1$$

for any $p \geq 0$ and any $q \geq 1$. Then

$$|\gamma_{p,q}| \leq \Gamma S^{(p+q)}q!$$

concluding the proof of the proposition 2.5.2. ■

Remarks.

1. For the estimates in the previous proposition, we use in a crucial way that λ is real and in $]0,1[$, the case $\lambda = 0$ is simple and the reduction of λ in $[0,1[$ has been obtained by the lemmas 2.5.3 and 2.5.4.
   The reader will see that, in the sequel, the number λ will be mainly a positive rational number in relation with the theorem of M. Kashiwara [18] on rationality and negativity of the roots of the Bernstein polynomials.

2. For $\lambda \in \mathbb{C} \setminus \mathbb{R}$ it is not clear that the result is true. For the algebras $\mathcal{A}, B[a]$ and also $\tilde{\mathcal{A}} = \widehat{B}[a]$ and $\widehat{\mathcal{A}}$ there is for each $\lambda \in \mathbb{C}$ an automorphism τ_λ which send 1 to 1, b to b and a to $a - \lambda b$. But this is not the case for $\tilde{\mathcal{A}}_{conv.}$:
   Take $x \in]0,1]$ and $X := \sum_{p,q\geq 0} q! a^p b^q$ which is $\tilde{\mathcal{A}}_{conv.}$. Then, by replacing a by $a + xb$ inside X we obtain the element $\sum_{p',q'\geq 0} \xi_{p',q'} a^{p'} b^{q'}$ in $\widehat{\mathcal{A}}$ where, using Corollary 2.4.2, we find

$$\xi_{p',q'} = q'! + \sum_{j=1}^{p'} (q'+j)! \gamma_j(x) C^j_{q'+j} \geq x \sum_{j=1}^{p'} \frac{(j-1)!((q'+j)!)^2}{q'!j!} \geq x \frac{((q'+p')!)^2}{q'!p'}$$

   at least for $p' \geq 1$. Then for $p' = q'$ we find that $\xi_{q',q'} \geq x((2q')!)^2/q'!q'$ and for any $R > 1$ there does not exist C_R such that $((2q')!)^2 \leq q'((q')!)^2 C_R R^{2q'}$.
   So for each $x \in]0,1]$ the image of X in $\widehat{\mathcal{A}}$ by the automorphism τ_{-x} is not in $\tilde{\mathcal{A}}_{conv.}$. So, even for $x \in]0,1[$ we cannot reduce the proof of Proposition 2.5.2 to the trivial cas $\lambda = 0$ as in the formal case.

3. Assume in the situation of Proposition 2.5.2 that X is in $B[a]$. So $Q(a - \lambda b)$ is a polynomial in a. Then Q is also a polynomial in a and its degree in a is the degree of X minus 1.
   It is easy to give a direct proof in this case of the fact that Q is in $B[a]$ (so without using Proposition 2.5.2 and then for any **complex** λ), using the automorphism τ_λ which reduces to the trivial case $\lambda = 0$.

Theorem 2.5.5 *Let $\lambda_1, \ldots, \lambda_k$ be real numbers and $S_1, \ldots, S_k$ be invertible elements in B. Then define $P \in B[a]$ by*

$$P := (a - \lambda_1 b) S_1 (a - \lambda_2 b) S_2 \ldots (a - \lambda_k b) S_k.$$

Then for any X in $\tilde{\mathcal{A}}_{conv.}$ there exists unique $Q \in \tilde{\mathcal{A}}_{conv.}$ and $R \in B[a]$ with $deg_a(R) \leq k-1$ such that

$$X = QP + R.$$

Proof. The uniqueness is clear from the uniqueness statement in Proposition 2.5.2 by an easy induction on k.
We shall prove the existence also by induction on $k \geq 1$. For $k = 1$ it is enough to

apply Proposition 2.5.2 to XS_1^{-1}. So assume the proposition proved for $k-1$ and then write

$$X = Q_0(a-\lambda_2 b)S_2 \dots (a-\lambda_k b)S_k + R_0$$

where R_0 has degree in a at most $k-2$. Now apply Proposition 2.5.2 for $(a-\lambda_1 b)$ to $Q_0 S_1^{-1}$. We obtain

$$Q_0 S_1^{-1} = Q_1(a-\lambda_1 b) + R_1$$

where R_1 has degree 0 in a. Then replacing Q_0 by $Q_1(a-\lambda_1 b)S_1 + R_1 S_1$ in the previous division we obtain

$$X = Q_1(A-\lambda_1 b)S_1(a-\lambda_2 b)S_2 \dots (a-\lambda_k b)S_k + R_0 + R_1 S_1(a-\lambda_2 b)S_2 \dots (a-\lambda_k b)S_k$$

this concludes the proof because

$$R := R_0 + R_1 S_1(a-\lambda_2 b)S_2 \dots (a-\lambda_k b)S_k$$

has degree in a at most equal to $k-1$. ■

REMARK. As a direct consequence, we obtain that the quotient of $\tilde{\mathcal{A}}_{conv.}$ by the (closed) left ideal $\tilde{\mathcal{A}}_{conv.}P$ for such a P, coincides with the quotient $B[a]/B[a]P$ and is a free B-module with basis $1, a, \dots, a^{k-1}$. We shall see that this is the general form of a **convergent frescos** with rank k when the numbers $\lambda_j + j - k$ are rational and positive (see Theorem 7.1.1).
So it will be equivalent, thanks to the previous results to consider geometric (a,b)-modules as left $\tilde{\mathcal{A}}_{conv.}$-modules or as left $B[a]$-modules and any $B[a]$-linear maps between two geometric (a,b)-modules is $\tilde{\mathcal{A}}_{conv.}$-linear.

3 Regular convergent (a,b)-modules

3.1 Basic properties

NOTATION. Recall that we note B and $\widehat{B}$ respectively the algebras $B := \mathbb{C}\{\{b\}\}$ and $\widehat{B} := \mathbb{C}[[b]]$ in the sequel.
For S in B (or in $\widehat{B}$) we shall note S' the usual derivative of S; we mean that $S'(b) = \sum_{j=1}^{\infty} j s_j b^{j-1}$ if $S := \sum_{j=0}^{\infty} s_j b^j$. The algebras B and $\widehat{B}$ are stable by this derivation.
Recall that we note $B[a]$ the unitary B-algebra generated by 1 and a over B with the commutation relation $ab - ba = b^2$. It is the free left B-module with basis $1, a, \dots, a^n, \dots$. Its product is defined by the commutation relation $aS - Sa = b^2 S'$ for $S \in B$. Recall also that $\tilde{\mathcal{A}} = \widehat{B}[a]$ is the unitary $\widehat{B}$-algebra generated by 1 and a with the same commutation relation for $S \in \widehat{B}$.

Definition 3.1.1 *We define a* **convergent (a,b)-module** *as a free, finite rank B-module $\mathcal{E}$ endowed with a continuous*[3] *$\mathbb{C}$-action of $a : \mathcal{E} \to \mathcal{E}$ such that $ab - ba = b^2$. Then we define its* **formal completion** *(in b) of $\mathcal{E}$ as $E := \mathcal{E} \otimes_B \widehat{B}$ on which the action of a on it is defined by*

$$a(x \otimes S) = ax \otimes S + x \otimes b^2 S'$$

for $x \in \mathcal{E}$ and $S \in \widehat{B}$. It is a (formal) (a,b)-module (see [6]).

Remarks.

1. The continuity of a implies that for any $S \in B$ we have $aS - Sa = b^2 S'$ as an equality between $\mathbb{C}$-linear continuous endomorphisms of $\mathcal{E}$.
   Then a convergent (a,b)-module is a left $B[a]$-module.

2. A sub-module of a convergent (a,b)-module $\mathcal{E}$ is, by definition, a sub-B-module of $\mathcal{E}$ which is stable by a. So a sub-module is simply a left sub-$B[a]$-module.
   As any sub-B-module of a free finite rank B-module is again free and finite rank B-module, a sub-module of a convergent (a,b)-module $\mathcal{E}$ is itself a convergent (a,b)-module.

3. Remark that if $\mathcal{F}$ is a sub-module of the convergent (a,b)-module $\mathcal{E}$ the quotient $\mathcal{E}/\mathcal{F}$ is not, in general, a free B-module, because it may have b-torsion. This is the reason to introduce the notion of **normal** sub-module.

Definition 3.1.2 *Let $\mathcal{F} \subset \mathcal{E}$ be a sub-module of the convergent (a,b)-module $\mathcal{E}$. We say that $\mathcal{F}$ is* **normal** *when it satisfies $\mathcal{F} \cap b\mathcal{E} = b\mathcal{F}$.*

This condition is necessary and sufficient in order that the quotient $\mathcal{E}/\mathcal{F}$ has no b-torsion and so that $\mathcal{E}/\mathcal{F}$ is a convergent (a,b)-module.

Lemma 3.1.3 *Let $\mathcal{F} \subset \mathcal{E}$ be a sub-module of the convergent (a,b)-module $\mathcal{E}$. Then define the* **normalization** *$\tilde{\mathcal{F}}$ of $\mathcal{F}$ by the equality:*

$$\tilde{\mathcal{F}} := \{x \in \mathcal{E} / \ / \exists n \in \mathbb{N} \quad \text{such that} \quad b^n x \in \mathcal{F}\}.$$

Then $\tilde{\mathcal{F}}$ is a normal sub-module of $\mathcal{E}$ and it is the smallest normal sub-module in $\mathcal{E}$ which contains $\mathcal{F}$. The quotient $\tilde{\mathcal{F}}/\mathcal{F}$ is a finite dimensional complex vector space.

Proof. It clear that $\tilde{\mathcal{F}}$ is a B-sub-module (so it is free finite rank over B) and its stability by the action of a is consequence the formula $b^N a = ab^N - Nb^{N+1}$ in $B[a]$. It is clearly normal. Let $\mathcal{G}$ be a normal sub-module of $\mathcal{E}$ containing $\mathcal{F}$. Now if x is in $\tilde{\mathcal{F}}$ there exists an integer n such that $b^n x \in \mathcal{F} \subset \mathcal{G}$. Since $\mathcal{G}$ is normal, we have $x \in \mathcal{G}$ and so $\tilde{\mathcal{F}} \subset \mathcal{G}$. Then $\tilde{\mathcal{F}}$ is the smallest normal sub-module in $\mathcal{E}$ containing $\mathcal{F}$.

[3] for the natural topology of B.

Also $\tilde{\mathcal{F}}$ is a free finite rank B-module. Then let $e_1, \ldots, e_k$ be a B-basis of $\tilde{\mathcal{F}}$. For each $j \in [1,k]$ there exists $N_j \in \mathbb{N}$ such that $b^{N_j} e_j$ is $\mathcal{F}$. Then for $N := \sup\{N_j, j \in [1,k]\}$ we have $b^N \tilde{\mathcal{F}} \subset \mathcal{F}$. So $\tilde{\mathcal{F}}/\mathcal{F}$ is a quotient of the finite dimensional complex vector space $\tilde{\mathcal{F}}/b^N\tilde{\mathcal{F}}$, concluding the proof. ■

Definition 3.1.4 *A convergent (a,b)-module $\mathcal{E}$ has a* **simple pole** *when it satisfies $a\mathcal{E} \subset b\mathcal{E}$.*
A convergent (a,b)-module $\mathcal{E}$ is **regular** *when it is a sub-module of some simple pole convergent (a,b)-module.*

Lemma 3.1.5 *Let $\mathcal{E}$ be a regular convergent (a,b)-module. Then there exists a natural injective $B[a]$-linear map $j : \mathcal{E} \to \mathcal{E}^\sharp$ where $\mathcal{E}^\sharp$ is a simple pole convergent (a,b)-module such that any injective $B[a]$-linear map $h : \mathcal{E} \to \mathcal{E}_1$ into a simple pole convergent (a,b)-module $\mathcal{E}_1$ factorizes (uniquely) by a $B[a]$-linear map $H : \mathcal{E}^\sharp \to \mathcal{E}_1$. Moreover, the quotient $\mathcal{E}^\sharp/j(\mathcal{E})$ is a finite dimensional complex space.*

Proof. Let $K := B[b^{-1}]$ and consider on the B-module $\mathcal{E} \otimes_B K$ the $\mathbb{C}$-linear action of a defined by

$$a(x \otimes b^{-p}) := ax \otimes b^{-p} - px \otimes b^{-p+1}$$

for each $p \in \mathbb{N}$. Then it is a $B[a]$-module and $j_0 : \mathcal{E} \to \mathcal{E} \otimes_B K$ given by $j_0(x) := x \otimes 1$ is $B[a]$-linear and injective. Define

$$\mathcal{E}^\sharp := \sum_{p=0}^{\infty} (b^{-1}a)^p j_0(\mathcal{E}) \subset \mathcal{E} \otimes_B K$$

and define j as the induced map. First we want to prove that $\mathcal{E}^\sharp$ is a finitely generated B-module (obviously stable by $b^{-1}a$ so by a).
For this purpose consider an injective $B[a]$-linear map $h : \mathcal{E} \to \mathcal{E}_1$ into a simple pole convergent (a,b)-module $\mathcal{E}_1$. The regularity of $\mathcal{E}$ insures that there exists at least one such map. Now the simple pole assumption on $\mathcal{E}_1$ implies that $b^{-1}a$ acts on $\mathcal{E}_1$. So for each $q \in \mathbb{N}$ we may extend the $B[a]$-linear map h to an injective $B[a]$-linear map $h_q : \sum_{p=0}^{q} (b^{-1}a)^p j_0(\mathcal{E}) \to \mathcal{E}_1$ commuting with $b^{-1}a$. This defines an increasing sequence of B-sub-modules in $\mathcal{E}_1$. So it is stationary for $q \geq q_0$ for some integer q_0. Since h_q is still injective for any q, this implies that

$$\mathcal{E}^\sharp := \sum_{p=0}^{q_0} (b^{-1}a)^p j_0(\mathcal{E})$$

and $\mathcal{E}^\sharp$ is a simple pole convergent (a,b)-module containing $\mathcal{E}$ via the map j induced by j_0. The previous argument shows the universal property of the inclusion map constructed above, $j : \mathcal{E} \to \mathcal{E}^\sharp$, relative to any injective $B[a]$-linear map of $\mathcal{E}$ into a simple pole convergent (a,b)-module.

The only point to conclude the proof if to show the finite dimension of the complex vector space $\mathcal{E}^\sharp/j(\mathcal{E})$. But we already know that $b^{q_0}\mathcal{E}^\sharp \subset j(\mathcal{E})$ and this is enough to conclude, as $\mathcal{E}^\sharp$ is a finite B-module and $\mathcal{E}^\sharp/j(\mathcal{E})$ is a quotient of $\mathcal{E}^\sharp/b^{q_0}\mathcal{E}^\sharp$. ■

REMARK. Consider a short exact sequence of regular convergent (a,b)-modules $0 \to \mathcal{F} \to \mathcal{E} \to \mathcal{G} \to 0$. It gives a surjective map $\mathcal{E}^\sharp \to \mathcal{G}^\sharp$ thanks to the minimality[4] of $\mathcal{G}^\sharp$. But the kernel of this map is, in general, bigger than $\mathcal{F}^\sharp$ although is has a simple pole, because a normal sub-module of a simple pole module has again a simple pole.

WARNING. From now on we omit "convergent" when we consider an (a,b)-module, and we use "script characters" (like $\mathcal{E}$) for these. If we want to consider a "formal" (a,b)-module we shall use "roman characters" (like E) and say "formal (a,b)-module" if we want more precision.

3.2 The Bernstein polynomial of a regular (a,b)-module.

We introduce now a fundamental (numerical) invariant of a regular (a,b)-module.

Definition 3.2.1 *Let $\mathcal{E}$ be a regular (a,b)-module. The Bernstein polynomial of $\mathcal{E}$ is the minimal polynomial of the action of $-b^{-1}a$ on the finite dimensional complex vector space $\mathcal{E}^\sharp/b\mathcal{E}^\sharp$.*

REMARKS.

1. If E is the formal b-completion of the regular (a,b)-module $\mathcal{E}$, so $E := \mathcal{E} \otimes_B \widehat{B}$; then $E^\sharp$ is the formal completion of $\mathcal{E}^\sharp$ and there is a natural isomorphism $\mathcal{E}^\sharp/b\mathcal{E}^\sharp \simeq E^\sharp/bE^\sharp$ which commutes with the respective actions of $b^{-1}a$.
   So the Bernstein polynomial of $\mathcal{E}, \mathcal{E}^\sharp, E^\sharp$ and of E are the same.

2. Let m be a non negative integer. When $\mathcal{E}$ is a simple pole (a,b)-module, the sub-module $b^m\mathcal{E}$ which has finite complex co-dimension in $\mathcal{E}$ is again a simple pole (a,b)-module since $ab^m\mathcal{E} = b^m(a+mb)\mathcal{E} \subset b(b^m\mathcal{E})$. The Bernstein polynomial of $b^m\mathcal{E}$ is then given by
$$B_{b^m\mathcal{E}}(x) = B_{\mathcal{E}}(x-m)$$
since $(b^{-1}a)^q b^m = b^m(b^{-1}a+m)^q$.

3. Let $\pi : \mathcal{E} \to \mathcal{G}$ a surjective $B[a]$-linear map. The map $\mathcal{E}^\sharp \to \mathcal{G}^\sharp$ is surjective (see the remark at the end of section 3.1). Since this map commutes with the respective actions of $b^{-1}a$, the Bernstein polynomial of $\mathcal{G}$ divides the Bernstein polynomial of $\mathcal{E}$.

[4]The image of $\mathcal{E}^\sharp$ in $\mathcal{G}[b^{-1}]$ has a simple pole and contains $\mathcal{G}$.

4. Despite the previous remark, there are, in general, two difficulties to compute the Bernstein polynomial of a regular (a,b)-module $\mathcal{E}$:
   The first one comes from the non left exactness of the functor " $\sharp$ " (see the remark following Lemma 3.1.5).
   The second difficulty comes from the fact that the minimal polynomial of a pair (V,T) of a vector space V with an endomorphism of V also does not behave nicely under injective maps compatible with T.
   This is the reason to introduce below the notion of fresco, which is stable by short exact sequences and for which we shall dispose of a nice behavior of their Bernstein polynomials in short exact sequences and for which we may avoid to compute their saturation by $b^{-1}a$, using only any Jordan-Hölder sequence (see Section 7.1) to obtain the Bernstein polynomial.

Let $\mathscr{A}$ a subset of $\mathbb{C}/\mathbb{Z}$.

Definition 3.2.2 *We say that a regular (a,b)-module $\mathcal{E}$ is* **$\mathscr{A}$-primitive** *when all roots of its Bernstein polynomial $B_{\mathcal{E}}$ are in $-\mathscr{A}+\mathbb{Z}$.*
In the case where $\mathscr{A}=\{\alpha\}$ where α is an element of $\mathbb{C}/\mathbb{Z}$ we say that $\mathscr{E}$ is $[\alpha]$-primitive.

We shall now prove the following important key for the Decomposition Theorem and also for for existence of Jordan-Hölder sequences (see Section 3.4).

Proposition 3.2.3 *Let $0\to\mathcal{F}\to\mathcal{E}\to\mathcal{G}\to 0$ be an exact sequence of simple pole (a,b)-modules. Assume that for each root $-\lambda$ of $B_{\mathcal{F}}$ and each root $-\mu$ of $B_{\mathcal{G}}$ we have $\mu-\lambda\notin\mathbb{N}^*$. Then this short exact sequence splits and there exists a normal sub-module $\mathcal{G}_0$ in $\mathcal{E}$ such that $\mathcal{E}=\mathcal{F}\oplus\mathcal{G}_0$.*

The proof of this proposition uses the following lemma

Lemma 3.2.4 *Let F and G two matrices with complex entries of size (l,l) and (k,k) respectively having no common eigenvalue. Then consider the endomorphisms f and g on the vector space of matrices Z with complex entries and size (k,l) given by left and write multiplication by F and G respectively. Then $f-g$ is bijective.*

PROOF. Since the endomorphisms f and g commute we may find a basis $Z_1,\dots,Z_{kl}$ of the vector space of the (k,l) matrices which makes the matrices of f and g lower triangular. So for each $i\in[1,kl]$ we have $FZ_i=\lambda_iZ_i$ modulo V_{i+1} and $Z_iG=\mu_iZ_i$ modulo V_{i+1} where V_{i+1} is the subspace generated by $Z_{i+1},\dots,Z_{kl}$.
Let $Z:=\sum_{i=1}^{kl}\alpha_iZ_i$ such that $FZ=ZG$ and assume that $Z\neq 0$. Let i_0 be the smallest integer in $[1,kl]$ such that $\alpha_{i_0}\neq 0$. Then we have

$$FZ=\alpha_{i_0}\lambda_{i_0}Z_{i_0}+V_{i_0+1}\quad\text{and}\quad ZG=\alpha_{i_0}\mu_{i_0}Z_{i_0}+V_{i_0+1}$$

which implies that $\lambda_{i_0}=\mu_{i_0}$. But the eigenvalues of f (resp. of g) are eigenvalues of F (resp. of G) because $FZ=\lambda Z$ implies that each column of Z is an eigenvector of

F (or 0) and if Z is not zero, at least one column of Z is not zero (resp. if $ZG = \mu Z$ and Z is not zero, at least one line of Z is a non zero eigenvector for the transpose of G and μ is an eigenvalue for G). So we obtain a contradiction assuming that a non zero Z satisfies $FZ = ZG$. ■

Proof of Proposition 3.2.3. Let $e := (e_1, \dots, e_k)$ be a B-basis of $\mathcal{F}$ and let $(e_1, \dots, e_k, \varepsilon_1, \dots, \varepsilon_l)$ be a B-basis of $\mathcal{E}$ such we have (with matrix notations):

$$a \begin{pmatrix} e \\ \varepsilon \end{pmatrix} = b \begin{pmatrix} F & 0 \\ bX & G \end{pmatrix} \begin{pmatrix} e \\ \varepsilon \end{pmatrix}$$

where F, G and X are respectively $(k,k), (l,l)$ and (l,k) matrices with entries in B. This possible because we have a direct sum decomposition of $\mathcal{E}/b\mathcal{E}$ compatible with the spectral decomposition of the action of $b^{-1}a$ on this finite dimensional vector space. Then write:

$$F := \sum_{j=0}^{\infty} F_j b^j \quad G := \sum_{j=0}^{\infty} G_j b^j \quad X := \sum_{j=0}^{\infty} X_j b^j$$

where F_j, G_j, X_j are matrices with complex coefficients. Choose on the vector spaces of matrices with complex entries and size $(k,k), (l,l), (l,k)$ norms such that when the product is defined we have $|XY| \leq |X| \times |Y|$. Choose also a norm on the vector space of endomorphism of (l,k)-matrices such that $|H(Z)| \leq ||H|| \times |Z|$.
Note that the endomorphism H defined by $H(Z) = ZF_0 - G_0 Z$ has no eigenvalue in $-\mathbb{N}^*$ thanks to our hypothesis and the lemma above applied to $H + jId, j \in \mathbb{N}^*$.
Now we look for a (l,k) matrix Z with entries in B such that we have

$$a(\varepsilon + Ze) = bG(\varepsilon + Ze).$$

Put $Z := \sum_{j=1}^{\infty} Z_j b^j$. Then to find Z with entries in $\widehat{B}$ is equivalent to solve the equation

$$a(\varepsilon + Ze) = b^2 Xe + bG\varepsilon + bZFe + b^2 Z'e = bG\varepsilon + bGZe$$

which is equivalent to

$$ZF - GZ + bZ' = -bX$$

and then is equivalent to the system of equations

$$Z_j F_0 - G_0 Z_j + jZ_j = -X_{j-1} + \sum_{p=1}^{j} \left(G_p Z_{j-p} - Z_{j-p} F_p \right) \quad \forall j \in \mathbb{N}^*. \tag{S}$$

Since the endomorphism $H + jId$ is bijective for each $j \geq 1$, an induction shows that there exists a unique solution Z with entries in $\widehat{B}$.
We want now to show that Z has its entries in B. So fix $R > 1$ and choose a positive constant C_R such that we have the estimates:

$$|F_j| \leq C_R R^j j! \quad |G_j| \leq C_R R^j j! \quad |X_j| \leq C_R R^j j! \tag{7}$$

Let $\rho := ||H||$. Then for $j > \rho$ we have

$$(H + jId)^{-1} = j^{-1}(Id + H/j)^{-1} = j^{-1}\sum_{h=0}^{\infty}(-H/j)^h \tag{8}$$

and this gives the estimates

$$||(H + j.Id)^{-1}|| \leq \frac{1}{j-\rho} \quad \forall j > \rho. \tag{9}$$

Then choose a positive constant $D_R \geq 1$ large enough such that the estimate $|Z_j| \leq D_R R^j j!$ is valid for any $j \leq \rho + 5C_R$.
Now assume that for some $j_0 \geq \rho + 5C_R$ we have proved that $|Z_j| \leq D_R R^j j!$ for any $j \leq j_0 - 1$. We shall prove that this estimate is also valid for j_0.
We have

$$H(Z_{j_0}) + j_0 Z_{j_0} = -X_{j_0-1} + \sum_{p=1}^{j_0} G_p Z_{j_0-p} - Z_{j_0-p}F_p$$

and so, using the estimate (9) for the norm of $(H + jId)^{-1}$:

$$|Z_{j_0}| \leq \frac{1}{j_0-\rho}\big(C_R R^{j_0-1}(j_0-1)! + 2C_R D_R R^{j_0}(j_0)!\sum_{p=1}^{j_0}\frac{(j_0-p)!p!}{j_0!}\big)$$

We obtain, using $j_0 \geq \rho + 5C_R$, $D_R \geq 1/R$ and Lemma 2.1.2:

$$|Z_{j_0}| \leq \frac{1}{j_0-\rho}C_R R^{j_0} j_0!\big(\frac{1}{Rj_0} + 4D_R\big) \leq D_R R^{j_0} j_0!$$

since

$$\frac{1}{j_0-\rho}\big(\frac{1}{Rj_0} + 4D_R\big) \leq 5D_R/5C_R.$$

This completes the proof that Z has its entries in B.
The conclusion follows immediately defining $\mathcal{G}_0$ as the sub-module generated by $\varepsilon + Ze$. ■

REMARK. The uniqueness of the matrix Z in the proposition above implies that the splitting of the exact sequence is unique in this situation. This means that the complement $\mathcal{G}_0$ constructed in the proof is unique and that the decomposition $\mathcal{E} \simeq \mathcal{F} \oplus \mathcal{G}_0$ is "natural".

Corollary 3.2.5 *Let $\mathcal{E}$ be a simple pole convergent (a,b)–module with basis the column $e := {}^t(e_1, \ldots, e_k)$ with $ae = bF(b)e$ where F is a (k,k)-matrix with entries in the algebra $B = \mathbb{C}\{\{b\}\}$. Write $F(b) := \sum_{j=0}^{+\infty} F_j b^j$ and assume that, for a given complex number λ, the spectrum of F_0 does not meet $\{\lambda - \mathbb{N}\}$. Then for any $y \in \mathcal{E}$ there exists a unique $x \in \mathcal{E}$ such that*

$$(a - \lambda b)x = by \quad \text{in } \mathcal{E} \tag{10}$$

PROOF. Let e be a B-basis of $\mathcal{E}$ which satisfies $ae = bFe$ with $F := \sum_{j=0}^{\infty} F_j b^j$ is a (k,k)-matrix with entries in B. Then for a given $= Ye, Y \in B^k$, we to look for $Z \in B^k$ such that $x := Ze$, satisfies $(a - \lambda b)x = by$. This leads to the equation

$$ZbFe + b^2 Z'e - \lambda bZe = bYe$$

which is equivalent to the system of equations, writng $Z = \sum_{j=0}^{\infty} Z_j b_j$:

$$Z_j F_0 - \lambda Z_j + jZ_j = Y_j - \sum_{p=1}^{j} Z_{j-p} F_p \quad \forall j \geq 0$$

For $j \geq 1$ this is the same system as the system (S) in the previous proof by letting $G = G_0 = \lambda Id$. As the equation for $j = 0$ has a unique solution, since our hypothesis implies that $F_0 - \lambda$ is bijective on $\mathcal{E}/b\mathcal{E}$, we find a unique solution Z with entries in B for each $y \in \mathcal{E}$ thanks to the estimates given in the proof of Proposition 3.2.3. ■

REMARK. The case $\lambda = 0$ of the previous corollary shows that a simple pole (a,b)-module $\mathcal{E}$ such that its Bernstein polynomial has no root in $-\mathbb{N}$ satisfies $a\mathcal{E} = b\mathcal{E}$, so $b^{-1}a$ is bijective.

Corollary 3.2.6 *Let $\mathcal{E}$ be a simple pole convergent (a,b)-module and let λ be an eigenvalue of $b^{-1}a$ acting on $\mathcal{E}/b\mathcal{E}$ and having the following properties:*

1. *$\lambda - p$ is not an eigenvalue of $b^{-1}a$ acting on $\mathcal{E}/b\mathcal{E}$ for each $p \in \mathbb{N}^*$.*
2. *There exists a rank k Jordan bloc for the eigenvalue λ acting on $\mathcal{E}/b\mathcal{E}$*

Then there exists elements $\varepsilon_1, \ldots, \varepsilon_k$ in $\mathcal{E}$ which are independent over B and satisfies the relations

$$a\varepsilon_j = \lambda b\varepsilon_j + b\varepsilon_{j+1} \quad \text{for each } j \in [1,k] \quad \text{with the convention} \quad \varepsilon_{k+1} = 0. \qquad (11)$$

PROOF. Thanks to Corollary 3.2.5 we can solve, for any $y \in \mathcal{E}$, the equation $(a - (\lambda - 1)b)x = by$.
Then consider elements $e_1, \ldots, e_k$ in $\mathcal{E}$ such that they induce a k-Jordan bloc for the eigenvalue λ of $b^{-1}a$ acting on $\mathcal{E}/b\mathcal{E}$. Then there exist $y_1, \ldots, y_k$ in $\mathcal{E}$ such the following relations holds

$$\begin{aligned} ae_1 &= \lambda be_1 + be_2 + b^2 y_1 \\ ae_2 &= \lambda be_2 + be_3 + b^2 y_2 \\ &\cdots\cdots \\ ae_k &= \lambda be_k + b^2 y_k \end{aligned}$$

We look now for $x_1, \ldots, x_k$ in $\mathcal{E}$ such that $\varepsilon_j := e_j - bx_j$ satisfy the equations (11). We shall argue by a descending induction on j. Assume that we have already found $x_{j+1}, \ldots, x_{k+1} := 0$ for some $j \in [1, k]$. Then to find x_j we have to solve the equation

$$(a - \lambda b)(e_j - bx_j) = b(e_{j+1} - bx_{j+1})$$

which is equivalent, thanks to the relation $(a - \lambda b)b = b(a - (\lambda - 1)b$ and the injectivity of b, to

$$(a - (\lambda - 1)b)x_j = b(y_j - x_{j+1}).$$

This is enough to conclude the proof. ■

The following corollary is immediate.

Corollary 3.2.7 *A rank 1 regular convergent (a,b)-module is isomorphic to a quotient $B[a]/B[a].(a - \lambda b)$. So it has a B-basis e_λ, and $\mathcal{E}_\lambda := Be_\lambda$ where $ae_\lambda = \lambda be_\lambda$. It has a simple pole, the action of $b^{-1}a$ on the 1-dimensional vector space $\mathcal{E}/b\mathcal{E}$ is given by λ and its Bernstein polynomial is $x + \lambda$.* ■

Note that the previous proposition gives also that if $\mathcal{E}$ is a simple pole rank k (a,b)-module such that $\mathcal{E}/b\mathcal{E}$ has an unique Jordan block of rank k for the eigenvalue λ then $\mathcal{E}$ is isomorphic to the (a,b)-module with B-basis $\varepsilon_1, \ldots, \varepsilon_k$ where the action of a is defined by the relations (11) above.

The following application of Corollary 3.2.6 is rather useful.

Lemma 3.2.8 *Let $\mathcal{F}$ be a submodule of a regular (a,b)-module $\mathcal{E}$. If $-\beta$ is a root of the Bernstein polynomial of $\mathcal{F}$ there exists an integer $m \in \mathbb{N}$ such that $-\beta + m$ is a root of the Bernstein polynomial of $\mathcal{E}$.*

Proof. It is enough to prove the result for the inclusion of $\mathcal{F}^\sharp$ in $\mathcal{E}^\sharp$. Moreover, we may assume that $-\beta$ is the biggest root of the Bernstein polynomial of $\mathcal{F}^\sharp$ in $-\beta+\mathbb{Z}$. So β is an eigenvalue of the action of $b^{-1}a$ on $\mathcal{F}^\sharp/b\mathcal{F}^\sharp$ which is minimal in $\beta+\mathbb{Z}$. Then Corollary 3.2.6 gives the existence of an x in $\mathcal{F}^\sharp \setminus b\mathcal{F}^\sharp$ which satisfies $(a - \beta b)x = 0$. Let $p \in \mathbb{N}$ be the maximal integer such that x is in $b^p\mathcal{E}^\sharp$ and write $x = b^p y$ with $y \in \mathcal{E}^\sharp$. Then y satisfies $(a - (\beta - p)b)y = 0$ and $y \notin b\mathcal{E}^\sharp$ by definition of p. So we see that $-\beta+p$ is a root of the Bernstein polynomial of $\mathcal{E}$ concluding the proof.■

3.3 The Decomposition Theorem

Definition 3.3.1 *For any subset $\mathscr{A} \in \mathbb{Q}/\mathbb{Z}$ we say that a regular (a,b)-module $\mathcal{E}$ is $\mathscr{A}$-**primitive** when the image of all the roots of its Bernstein polynomial in $\mathbb{Q}/\mathbb{Z}$ are in $-\mathscr{A}$.*

The next lemma is a useful tool to discuss $\mathscr{A}$-primitive (a,b)-modules.

Lemma 3.3.2 *Let $\mathcal{E}$ be a regular (a,b)-module and assume that $-\lambda$ is a root of its Bernstein polynomial. Then there exists $m \in \mathbb{N}$ and a non zero x in $\mathcal{E}$ such that $(a-(\lambda+m)b)x = 0$.*

PROOF. Let $-\lambda_1 := -\lambda + m_1$ be biggest root of $B_{\mathcal{E}}$ which is in $-\lambda + \mathbb{N}$ and let $y \in \mathcal{E}^\sharp \setminus b\mathcal{E}^\sharp$ which satisfies $(a-\lambda_1 b)y \in b^2\mathcal{E}^\sharp$. Then put $(a-\lambda_1 b)y = b^2 z$. Since there is no root of $B_{\mathcal{E}}$ in $-\lambda_1 + 1 + \mathbb{N}$, we know that $b^{-1}a - (\lambda_1 - 1)$ is bijective on $\mathcal{E}^\sharp$. Then there exists $x \in \mathcal{E}^\sharp$ satisfying $(b^{-1}a - (\lambda_1 - 1))x = z$ and so we have $b(a-(\lambda_1-1)b)x = (a-\lambda_1 b)bx = b^2 z = (a-\lambda_1 b)y$ and $(a-\lambda_1 b)(y-bx) = 0$. Moreover, as y is not in $b\mathcal{E}^\sharp$, $y - bx$ is also not in $b\mathcal{E}^\sharp$ and then $y - bx \neq 0$.
Let $n \in \mathbb{N}$ such that $b^n\mathcal{E}^\sharp \subset \mathcal{E}$ and $n \geq m_1$. Then $t := b^n(y-bx)$ is in $\mathcal{E} \setminus \{0\}$ and satisfies $(a-(\lambda - m_1 + n)b)t = 0$ with $n - m_1 \in \mathbb{N}$. ■

We give an interesting, but obvious consequence of Corollary 3.2.5.

Corollary 3.3.3 *Let $\mathcal{E}$ be a $\mathscr{A}$-primitive simple pole (a,b)-module. If μ is a complex number such that $[\mu]$ is not in $\mathscr{A}$ then $b^{-1}a - \mu$ is bijective on $\mathcal{E}$.*
So, for any regular (a,b)-module $\mathcal{F}$ such that $-\mu + \mathbb{Z}$ does not contain any root of its Bernstein polynomial, the only solution of $(a-\mu b)x = 0$ is $x = 0$. ■

The next proposition is a first step to the Decomposition Theorem 3.3.7.

Proposition 3.3.4 *Let $\mathscr{A}$ be a subset in $\mathbb{C}/\mathbb{Z}$ and let $\mathcal{E}$ be a regular (a,b)-module. Then there exists a maximal sub-module $\mathcal{E}_{[\mathscr{A}]}$ in $\mathcal{E}$ which is $\mathscr{A}$-primitive. Moreover, this sub-module is normal and the quotient $\mathcal{E}/\mathcal{E}_{[\mathscr{A}]}$ is a $\mathscr{A}^c$-primitive, where $\mathscr{A}^c$ is the complement of $\mathscr{A}$ in $\mathbb{C}/\mathbb{Z}$.*

The proof of this result needs some lemmas.

Lemma 3.3.5 *Fix any subset $\mathscr{A}$ in $\mathbb{C}/\mathbb{Z}$. Let $0 \to \mathcal{F} \to \mathcal{E} \to \mathcal{G} \to 0$ be a short exact sequence of regular (a,b)-modules. Then $\mathcal{E}$ is $\mathscr{A}$-primitive if and only if both $\mathcal{F}$ and $\mathcal{G}$ are $\mathscr{A}$-primitive.*

PROOF. First assume that $\mathcal{E}$ is $\mathscr{A}$-primitive. Let $-\mu$ a root of $B_{\mathcal{F}}$. Then thanks to Lemma 3.3.2 there exists an integer m and a non zero element $x \in \mathcal{F}$ satisfying $(a-(\mu+m)b)x = 0$. Let k the maximal integer such that x is in $b^k\mathcal{E}^\sharp$ and put $x = b^k y$ where $y \in \mathcal{E}^\sharp$. Then we have $(a-(\mu+m-k)b)y = 0$ with $y \notin b\mathcal{E}^\sharp$ so $[\mu]$ is in $\mathscr{A}$. Then $\mathcal{F}$ is $\mathscr{A}$-primitive.
The fact that $\mathcal{G}$ is also $\mathscr{A}$-primitive is a consequence of Remark 3 following Definition 3.2.1.
Let assume now that $\mathcal{F}$ and $\mathcal{G}$ are $\mathscr{A}$-primitive and let $-\mu$ be a root of $B_{\mathcal{E}}$. Assume that $[\mu] \notin \mathscr{A}$. There exists a non zero element $x \in \mathcal{E}$ which satisfies $(a-(\mu-m)b)x = 0$, where m is an integer, thanks to Corollary 3.3.3. Then the same result implies that the image of x in $\mathcal{G}$ is zero. Then x is in $\mathcal{F}$ which contradicts

the non vanishing of x. ■

Lemma 3.3.6 *Let $\mathcal{E} \subset \mathcal{E}'$ be an inclusion of two regular (a,b)-modules such that $\mathcal{E}'/\mathcal{E}$ is a finite dimensional complex vector space. Then for any subset $\mathscr{A}$ in $\mathbb{C}/\mathbb{Z}$, $\mathcal{E}'$ is $\mathscr{A}$-primitive if and only if $\mathcal{E}$ is $\mathscr{A}$-primitive.*

Proof. Thanks to Lemma 3.3.2 and to Corollary 3.3.3 it is enough to prove the equivalence of the existence in $\mathcal{E}$ or in $\mathcal{E}'$ for any $[\mu] \in \mathbb{C}/\mathbb{Z}$ of a non zero solution of the equation $(a-(\mu+m)b)x = 0$ for each $m \in \mathbb{Z}$. But this is an obvious consequence of the existence of an integer N such that $b^N\mathcal{E}' \subset \mathcal{E}$. ■

Note that for any regular (a,b)-module the previous corollary apply to $\mathcal{E}' := \mathcal{E}^\sharp$.

Proof of Proposition 3.3.4. Then let us prove that the sum of two $\mathscr{A}$-primitive sub-modules is again $\mathscr{A}$-primitive. This is obvious for a direct sum, and in general, the sum is a quotient of the direct sum, so we conclude using Lemma 3.3.2. Let $\mathcal{F}$ be a maximal $\mathscr{A}$-primitive sub-module of $\mathcal{E}$. This exists thanks to the Noether's property of B. Since $\mathcal{F}$ has finite co-dimension in its normalization, Lemma 3.3.6 implies that $\mathcal{F}$ is normal.
Assume that the Bernstein polynomial of $\mathcal{E}/\mathcal{F}$ has a root $-\mu$ in $-\mathscr{A}$, then Lemma 3.3.2 gives us a non zero $z \in \mathcal{E}/\mathcal{F}$ such that $(a-(\mu+m)bz = 0$ for an integer $m \in \mathbb{Z}$. Let $\mathcal{G} := Bz \subset \mathcal{E}/\mathcal{F}$. It is a rank 1 $\mathscr{A}$-primitive sub-module and its pull-back in $\mathcal{E}$ is a $\mathscr{A}$-primitive sub-module, thanks to Lemma 3.3.5, which is strictly bigger than $\mathcal{F}$. Contradiction. So $\mathcal{E}/\mathcal{F}$ is $\mathscr{A}^c$-primitive. ■

Theorem 3.3.7 *Let $\mathcal{E}$ be a simple pole (convergent) (a,b)-module and note $-\mathscr{A}$ the image in $\mathbb{C}/\mathbb{Z}$ of the roots of the Bernstein polynomial of $\mathcal{E}$. Then we have a natural isomorphism of (a,b)-modules(so of left $B[a]$-modules):*

$$\mathcal{E} \simeq \oplus_{\alpha\in\mathscr{A}}\ \mathcal{E}_{[\alpha]}.$$

The proof is an obvious consequence of the following corollary of Proposition 3.2.3, using an induction on the cardinal of the set $\mathscr{A}$. ■

For the "naturality" of this decomposition, see the remark following Proposition 3.2.3.

Corollary 3.3.8 *Let $0 \to \mathcal{F} \to \mathcal{E} \to \mathcal{G} \to 0$ be a short exact sequence of regular (a,b)-modules. Assume that $\mathcal{E}$ has a simple pole and that $\mathcal{F}$ and $\mathcal{G}$ are respectively $\mathscr{A}$ and $\mathscr{B}$-primitive with $\mathscr{A} \cap \mathscr{B} = \emptyset$ in $\mathbb{C}/\mathbb{Z}$. Then the exact sequence splits and $\mathcal{E} \simeq \mathcal{F} \oplus \mathcal{G}$.* ■

Note that for a regular (a,b)-module $\mathcal{E}$ which is not a simple pole (a,b)-module, such a short exact sequence does not split in general. For instance, consider the rank 2 (a,b)-module $\mathcal{H}$ with B-basis e_1, e_2 on which the action of a is defined by

$$ae_1 = \lambda be_1 + e_2 \quad ae_2 = \mu be_2 \quad \text{with} \quad \lambda - \mu \notin \mathbb{Z}.$$

It is easy to see that $\mathcal{H}_{[\lambda]} \simeq \mathcal{E}_{\lambda+1}$ is the normal sub-module generated by the element $e_2 + (\lambda - \mu + 1)be_1$ in $\mathcal{H}$ and that $\mathcal{H}_{[\mu]}$ is generated by e_2. But e_1 is not in $\mathcal{H}_{[\lambda]} \oplus \mathcal{H}_{[\mu]}$. The decomposition Theorem implies the following obvious decomposition result for any simple pole (a,b)-module.

Corollary 3.3.9 *Let $\mathcal{E}$ a simple pole (a,b)-module and let $-\mathscr{A}$ be the image in $\mathbb{Q}/\mathbb{Z}$ of the set of roots of its Bernstein polynomial of $\mathcal{E}$. Then we have*

$$B_{\mathcal{E}} = \prod_{\alpha \in \mathscr{A}} B_{\mathcal{E}_{[\alpha]}}. \qquad \blacksquare$$

3.4 Existence of Jordan-Hölder sequences

Since rank 1 regular (a,b)-module are very easy to understand, we introduce now the notion of Jordan-Hölder sequence which allows to describe a regular (a,b)-module from their rank 1 sub-quotients.

Definition 3.4.1 *Let $\mathcal{E}$ be a regular (convergent) (a,b)-module with rank k. We say that a filtration $0 := F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_k := \mathcal{E}$ is a* **Jordan-Hölder sequence** *for $\mathcal{E}$ if, for each $j \in [0,k]$, F_j is a normal sub-module with rank j.*
Then for each $j \in [1,k]$ the quotient F_j/F_{j-1} is a regular rank 1 convergent (a,b)-module so there exists a unique complex number λ_j such that F_j/F_{j-1} is isomorphic to $\mathcal{E}_{\lambda_j} := B[a]/B[a](a - \lambda_j b)$.
In this situation we say that $(\lambda_1, \ldots, \lambda_k)$ is the **characteristic sequence** *of the given J-H. sequence $(F_j), j \in [1,k]$ of $\mathcal{E}$.*

Proposition 3.4.2 *Any regular convergent (a,b)-module $\mathcal{E}$ admits a J-H. sequence. If $(\lambda_1, \ldots, \lambda_k)$ is the characteristic sequence of $\mathcal{E}$ then each root of the Bernstein polynomial of $\mathcal{E}$ is the opposite modulo $\mathbb{Z}$ to some λ_j and conversely, for each λ_j there is a root of $B_{\mathcal{E}}$ in $-\lambda_j + \mathbb{Z}$. Moreover the complex number $\sum_{j=1}^k \lambda_j$ is independent of the chosen J-H. sequence. If $\mathcal{E}'$ is a sub-module of $\mathcal{E}$ such that $\mathcal{E}/\mathcal{E}'$ has finite complex dimension p and if $(\lambda'_1, \ldots, \lambda'_k)$ is the characteristic sequence of the J-H. sequence $F_j \cap \mathcal{E}'$ of $\mathcal{E}'$ we have $\sum_{j=1}^k \lambda'_j = \sum_{j=1}^k \lambda_j + p$.*

Proof. We first consider the case of a simple pole (a,b)-module. We shall prove the existence of a J-H. sequence by induction on the rank. So assume that $\mathcal{E}$ is a simple pole of rank $k \geq 1$ and that the existence of a J-H. sequence is proved for any simple pole rank $k-1$ (a,b)-module. Let $-\lambda_1$ be a root of the Bernstein polynomial of $\mathcal{E}$ which is maximal in its class modulo $\mathbb{Z}$ among all the roots of $B_{\mathcal{E}}$. Then

Corollary 3.2.6 shows the existence of an $x \in \mathcal{E} \setminus b\mathcal{E}$ such that $(a - \lambda_1 b)x = 0$. The sub-module $B[a]x$ in $\mathcal{E}$ is normal with rank 1 and so isomorphic to $B[a]/B[a](a-\lambda_1 b)$ (see Corollary 3.2.7). Then the existence of a J-H. sequence of $\mathcal{E}/B[a]x$ implies the existence of a J-H. sequence for $\mathcal{E}$.
Now, as F_j has a quotient isomorphic to $\mathcal{E}_{\lambda_j}$ there exists, thanks to Remark 3 following Definition 3.2.1 and Corollary 3.2.6, a non zero $x \in F_j$ and an integer $m \in \mathbb{Z}$ such that $(a - (\lambda_j + m)b)x = 0$. If $x = b^k y$ with $y \in \mathcal{E}^\sharp \setminus b\mathcal{E}^\sharp$ we have $(a - (\lambda_j + m - k)b)y = 0$ and $\lambda_j + m - k$ is the opposite of a root of $B_\mathcal{E}$.
Consider now the situation of a sub-module $\mathcal{E}$ in a simple pole (a,b)-module $\mathcal{E}'$ such that $\mathcal{E}'/\mathcal{E}$ is a finite dimensional complex vector space[5]. Let k be the common rank of $\mathcal{E}$ and $\mathcal{E}'$ and let $(F'_j), j \in [1,k]$, be a J-H. sequence for $\mathcal{E}'$. Then $F_j := \mathcal{E} \cap F'_j$, for $j \in [1,k]$, is a J-H. sequence for $\mathcal{E}$ because for each j, F_j has finite co-dimension in F'_j so has rank equal to j. Moreover F_j is normal in $\mathcal{E}$ since F'_j is normal in $\mathcal{E}'$.
Conversely, if $F_j, j \in [1,k]$, is a J-H. sequence for $\mathcal{E}$ define F'_j as the normalization of F_j in $\mathcal{E}'$. It has also rank j and $F'_j, j \in [1,k]$, is a J-H. sequence for $\mathcal{E}'$. So we see that, not only there exists a J-H. sequence for such an $\mathcal{E}$ but any J-H. sequence of $\mathcal{E}$ is the trace of a J-H. sequence of $\mathcal{E}'$ in the this situation. Now, for any given regular $\mathcal{E}$ we may choose $\mathcal{E}' := \mathcal{E}^\sharp$ and obtain a bijective correspondance between J-H. sequences of $\mathcal{E}$ and $\mathcal{E}^\sharp$ in this way.
Let $(\lambda_1, \ldots, \lambda_k)$ be the characteristic sequence of the J-H. sequence $G_1, \ldots, G_k$ of $\mathcal{E}^\sharp$ and define $F_j := G_j \cap \mathcal{E}$ for each $j \in [1,k]$. Then F_j/F_{j-1} is a sub-module of G_j/G_{j-1} which has finite codimension. So if $(\lambda_1, \ldots, \lambda_k)$ is the characteristic sequence of the J-H. sequence $F_j, j \in [1,k]$, of $\mathcal{E}$, there exists non negative integer $p_j, j \in [1,k]$, such that $\lambda_j = \lambda'_j + p_j$. The exact sequence of finite dimensional vector spaces

$$0 \to G_{j-1}/F_{j-1} \to G_j/F_j \to G_j/(G_{j-1} + F_j) \to 0$$

gives for each $j \in [1,k]$ the equality $p_j = p_{j-1} + dim_\mathbb{C}(G_j/(G_{j-1}+F_j))$ which implies $p_j = \sum_{h=1}^{j-1} p_h$ and then

$$\sum_{j=1}^{k} \lambda_j = \sum_{j=1}^{k} \lambda'_j + dim_\mathbb{C}(\mathcal{E}^\sharp/\mathcal{E}).$$

To complete the proof, we have to show that, if, for a simple pole (a,b)-module $\mathcal{E}$, $(\lambda_1, \ldots, \lambda_k)$ is the characteristic sequence of a J-H. sequence $(F_j), j \in [1,k]$, of $\mathcal{E}$, any root of $B_\mathcal{E}$ is equal, modulo $\mathbb{Z}$, to some $-\lambda_j$.
Let $\mathcal{E}$ be a simple pole (a,b)-module which is of minimal rank $k \geq 1$ such that it has a J-H. sequence $F_1, \ldots, F_k$ with characteristic sequence $(\lambda_1, \ldots, \lambda_k)$ and a root $-\alpha$ of its Bernstein polynomial which is not in the union of the $-\lambda_j + \mathbb{Z}$. Then $\mathcal{F} := \mathcal{E}/F_1$ has rank $k-1$ and in the exact sequence

$$0 \to F_1/bF_1 \to \mathcal{E}/b\mathcal{E} \to \mathcal{F}/b\mathcal{F} \to 0$$

[5]Remark that $\mathcal{E}$ is regular.

which is compatible with the respective actions of $b^{-1}a$ we see that $-\alpha$ is a root of $B_{\mathcal{F}}$ because $\alpha \neq \lambda_1$. But since $\mathcal{F}$ admits the J-H. sequence $F_2/F_1, \ldots, F_k/F_1$ whose characteristic sequence is $(\lambda_2, \ldots, \lambda_k)$ we contradict the minimality of $\mathcal{E}$ and the proof is complete. ■

4 The semi-simple filtration

In this section every (convergent) (a,b)-module is assumed to be regular.

4.1 Semi-simple regular (a,b)-modules

It is easy to see that a regular (a,b)-module $\mathcal{E}$ is **simple** (so by definition has no non trivial *normal* sub-module) is either $\mathcal{E} = \{0\}$ or has rank 1.

Definition 4.1.1 *Let $\mathcal{E}$ be a regular (a,b)-module. We say that $\mathcal{E}$ is* **semi-simple** *if it is a sub-module of a finite direct sum of rank 1 regular (a,b)-modules.*

Note that if $\mathcal{E}$ is a sub-module of a regular (a,b)-module it is necessary regular. As a finite direct sum of regular (a,b)-modules is regular, our assumption that $\mathcal{E}$ is regular is superfluous.
It is clear from this definition that a sub-module of a semi-simple (a,b)-module is semi-simple and that a (finite) direct sum of semi-simple (a,b)-modules is again semi-simple.

Remarks.

1. We have already seen that a rank 1 regular (a,b)-module is isomorphic to $\mathcal{E}_\lambda := B[a]/B[a](a - \lambda b)$, and then a convergent (a,b)-module is simple if and only if its formal completion is simple. Then a semi-simple convergent (a,b)-module has a formal completion which is also semi-simple, that is to say which is embeddable in a finite direct sum of formal simple (a,b)-modules.

2. A rather easy consequence of the classification of formal rank 2 (a,b)-modules given in [6] is that the rank 2 (a,b)-modules defined in the $\widehat{B} = \mathbb{C}[[b]]-$basis x, y by the relations :

$$(a - (\alpha + p - 1)b)x = y + b^p y \quad \text{and} \quad (a - \alpha b)y = 0$$

for any $\alpha \in \mathbb{C}$ and any $p \in \mathbb{N}^*$ are not semi-simple. We leave the verification of this point to the reader.
So the analog rank 2 B-module with a defined by the same relations is also not semi-simple.

Let us begin by a characterization of the semi-simple (a,b)-modules which have a simple pole. First we shall prove that a quotient of a semi-simple (a,b)-module is semi-simple. This will be deduced from the following lemma and its corollary.

Lemma 4.1.2 *Let $\mathcal{E}$ be an (a,b)-module which is direct sum of regular rank 1 (a,b)-modules, and let $\mathcal{F} \subset \mathcal{E}$ be a rank 1 normal sub-module. Then $\mathcal{F}$ is a direct factor of $\mathcal{E}$ and we have $\mathcal{E} = \mathcal{F} \oplus \mathcal{H}$ where $\mathcal{H}$ is again a finite direct sum of regular rank 1 (a,b)-modules.*

Corollary 4.1.3 *If $\mathcal{E}$ is a semi-simple regular (a,b)-module and $\mathcal{F}$ a normal sub-module of $\mathcal{E}$, the quotient $\mathcal{E}/\mathcal{F}$ is a (regular) semi-simple (a,b)-module.*

Proof of the lemma. Let $\mathcal{E} = \oplus_{j=1}^{k} \mathcal{E}_{\alpha_j}$ and assume that $\mathcal{F} \simeq \mathcal{E}_\beta$. Let e_j be a standard generator of $\mathcal{E}_{\alpha_j}$ and e be a standard generator of $\mathcal{E}_\beta$. Write

$$e = \sum_{j=1}^{k} S_j(b) e_j$$

and compute $(a - \beta b)e = 0$ using the fact that $e_j, j \in [1,k]$, is a B-basis of $\mathcal{E}$ and the relations $(a - \alpha_j b)e_j = 0$ for each j. We obtain, for each $j \in [1,k]$, the relation

$$bS_j' - (\beta - \alpha_j)S_j = 0.$$

So, if $\beta - \alpha_j$ is not in $\mathbb{N}$, we have $S_j = 0$. When $\beta = \alpha_j + p_j$ with $p_j \in \mathbb{N}$ we obtain $S_j(b) = \rho_j b^{p_j}$ for some $\rho_j \in \mathbb{C}$. As we assume that e is not in $b\mathcal{E}$, there exists at least one $j_0 \in [1,k]$ such that $p_{j_0} = 0$ and $\rho_{j_0} \neq 0$. In the case where there exists only one j_0 with $p_{j_0} = 0$ and ρ_{j_0} it is clear that we have

$$\mathcal{E} = \mathcal{F} \oplus \big(\oplus_{j \neq j_0} \mathcal{E}_{\alpha_j} \big).$$

If there are many such j_0 then we are reduced to the case where $\mathcal{E}$ is the direct sum of several copies of $\mathcal{E}_\beta$ and where e is a (complex) linear combination of the standard generators. This case is also obvious, concluding the proof. ■

Proof of the corollary. We argue by induction on the rank of $\mathcal{F}$. In the rank 1 case, we have $\mathcal{F} \subset \mathcal{E} := \oplus_{j=1}^{k} \mathcal{E}_{\alpha_j}$. Let $\tilde{\mathcal{F}}$ the normalization of $\mathcal{F}$ in $\oplus_{j=1}^{k} \mathcal{E}_{\alpha_j}$. Then the lemma shows that there exists a $j_0 \in [1,k]$ such that

$$\mathcal{E} = \tilde{\mathcal{F}} \oplus \big(\oplus_{j \neq j_0} \mathcal{E}_{\alpha_j} \big).$$

Then, as $\tilde{\mathcal{F}} \cap \mathcal{E} = \mathcal{F}$, the quotient map $\mathcal{E} \to \mathcal{E}/\tilde{\mathcal{F}} \simeq \oplus_{j \neq j_0} \mathcal{E}_{\alpha_j}$ induces an injection of $\mathcal{E}/\mathcal{F}$ in a direct sum of regular rank 1 (a,b)-modules. So $\mathcal{E}/\mathcal{F}$ is semi-simple. Assume now that the result is proved for $\mathcal{F}$ with rank $\leq d-1$ and assume that $\mathcal{F}$ has rank d. Then using a rank 1 normal sub-module $\mathcal{G}$ in $\mathcal{F}$, we obtain that $\mathcal{F}/\mathcal{G}$ is a normal rank $d-1$ sub-module of $\mathcal{E}/\mathcal{G}$. Using the rank 1 case we know that $\mathcal{E}/\mathcal{G}$ is semi-simple, and the induction hypothesis gives that

$$\mathcal{E}/\mathcal{F} = (\mathcal{E}/\mathcal{G}) \Big/ (\mathcal{F}/\mathcal{G})$$

is semi-simple. ■

Proposition 4.1.4 *Let $\mathcal{E}$ be a simple pole semi-simple (a,b)-module. Then $\mathcal{E}$ is a direct sum of rank 1 simple pole (a,b)-modules.*

PROOF. We shall prove the proposition by induction on the rank of $\mathcal{E}$. The case of rank 1 being clear, assume the result proved in rank $k-1$ with $k \geq 2$ and let $\mathcal{E}$ be a simple pole semi-simple (a,b)-module with rank k. Now, thanks to Corollary 3.2.6 there exists a normal rank 1 sub-module F_1 in $\mathcal{E}$. Let $j : \mathcal{E} \to \mathcal{G}$ be an embedding of $\mathcal{E}$ in a finite direct sum $\mathcal{G}$ of rank 1 regular (a,b)-modules. Let $\tilde{F}_1$ be the normalization of $j(F_1)$ in $\mathcal{G}$. Then thanks to Lemma 4.1.2 we have a direct sum decomposition $\mathcal{G} = \tilde{F}_1 \oplus \mathcal{H}$ where $\mathcal{H}$ is again a finite direct sum of rank 1 regular (a,b)-modules. Then, since $j^{-1}(\tilde{F}_1) = F_1$ we have $\mathcal{E} = F_1 \oplus j^{-1}(\mathcal{H})$ where $j^{-1}(\mathcal{H})$ is a normal sub-module of $\mathcal{E}$ which is semi-simple of rank $k-1$ and has a simple pole (by normality). The induction hypothesis allows to conclude. ■

REMARK. If a regular (a,b)-module is semi-simple, its Bernstein polynomial has only simple roots.

Definition 4.1.5 *Let $\mathcal{E}$ and $\mathcal{F}$ two (convergent) (a,b)-modules. Then, endowed with the action of a given by $a(x \otimes y) = ax \otimes y + x \otimes ay$, the B-module $\mathcal{E} \otimes_B \mathcal{F}$ becomes an (a,b)-module which will be noted $\mathcal{E} \otimes_{a,b} \mathcal{F}$ and called the* **tensor product** *of the (a,b)-modules $\mathcal{E}$ and $\mathcal{F}$.*

REMARKS.

1. As $\mathcal{E}$ and $\mathcal{F}$ are two free finite type B-modules, so is $\mathcal{E} \otimes_B \mathcal{F}$. Then for any $S(b) \in B$ we have in $\mathcal{E} \otimes_{a,b} \mathcal{F}$

$$\begin{aligned} aS(b)(x \otimes y) &= aS(b)x \otimes y + S(b)x \otimes ay \\ &= S(b)ax \otimes y + b^2 S'(b)x \otimes y + S(b)x \otimes ay \\ &= S(b)a(x \otimes y) + b^2 S'(b)(x \otimes y) \end{aligned}$$

   and then $\mathcal{E} \otimes_{a,b} \mathcal{F}$ is an (a,b)-module.

2. For each $\lambda \in \mathbb{C}$ the (a,b)-module $\mathcal{E} \otimes_{a,b} \mathcal{E}_\lambda$ is equal to the B-module $\mathcal{E}$ endowed with the action of a given by $x \mapsto (a + \lambda b)x$. Note that this corresponds to the action of $B[a]$ on $\mathcal{E}$ via the unitary B-algebra automorphism of $B[a]$ defined by $a \mapsto a + \lambda b$.

3. So we have $\mathcal{E}_\lambda \otimes \mathcal{E}_\mu \simeq \mathcal{E}_{\lambda+\mu}$.

4. The tensor product of two simple poles (a,b)-module has a simple pole, because if $ax = bx'$ and $ay = by'$ we have $a(x \otimes y) = b(x' \otimes y + x \otimes y')$. Then the tensor product of two regular (a,b)-modules is regular.

5. The tensor product of two semi-simple (a,b)-modules is again semi-simple.

4.2 The semi-simple filtration

Definition 4.2.1 *Let $\mathcal{E}$ be a regular (a,b)-module and x an element in $\mathcal{E}$. We shall say that x is* **semi-simple** *if $B[a]x$ is a semi-simple (a,b)-module.*

It is clear that any element in a semi-simple (a,b)-module is semi-simple. The next lemma shows that the converse is true.

Lemma 4.2.2 *Let $\mathcal{E}$ be a regular (a,b)-module such that any $x \in \mathcal{E}$ is semi-simple. Then $\mathcal{E}$ is semi-simple.*

PROOF. Let $e_1, \ldots, e_k$ be a B-basis of $\mathcal{E}$. Then each $B[a]e_j$ is semi-simple, and the map $\oplus_{j=1}^{k} B[a]e_j \to \mathcal{E}$ is surjective. So $\mathcal{E}$ is semi-simple thanks to Corollary 4.1.3 and the comment following Definition 4.1.1. ■

Lemma 4.2.3 *Let $\mathcal{E}$ be a regular (a,b)-module. The subset $S_1(\mathcal{E})$ of semi-simple elements in $\mathcal{E}$ is a normal sub-module in $\mathcal{E}$.*

PROOF. As a finite direct sum of semi-simple (a,b)-modules and also a quotient of a semi-simple (a,b)-module by a normal sub-module is semi-simple, it is clear that for x and y semi-simple the sum $B[a]x + B[a]y$ is semi-simple. So $x + y$ is semi-simple. This implies that $S_1(\mathcal{E})$ is a sub-module of $\mathcal{E}$. If bx is in $S_1(\mathcal{E})$, then $B[a]bx$ is semi-simple. Then

$$B[a]bx \otimes_{a,b} \mathcal{E}_{-1} = bB[a]x \otimes_{a,b} \mathcal{E}_{-1} \simeq B[a]x \otimes_{a,b} b\mathcal{E}_{-1} \simeq B[a]x \otimes_{a,b} \mathcal{E}_0 \simeq B[a]x$$

is also semi-simple, and then $S_1(\mathcal{E})$ is normal in $\mathcal{E}$. ■

Definition 4.2.4 *Let $\mathcal{E}$ be a regular (a,b)-module. Then the sub-module $S_1(\mathcal{E})$ of semi-simple elements in $\mathcal{E}$ will be called the* ***semi-simple part*** *of $\mathcal{E}$.*
Defining inductively $S_j(\mathcal{E})$ as the pull-back in $\mathcal{E}$ of the semi-simple part of $\mathcal{E}/S_{j-1}(\mathcal{E})$ for $j \geq 1$ with the initial condition $S_0(\mathcal{E}) = \{0\}$, we obtain an increasing sequence of normal sub-modules in $\mathcal{E}$ such that $S_j(\mathcal{E})/S_{j-1}(\mathcal{E}) = S_1(\mathcal{E}/S_{j-1}(\mathcal{E}))$ is semi-simple. We shall call it the ***semi-simple filtration*** *of $\mathcal{E}$. The smallest integer d such we have $S_d(\mathcal{E}) = \mathcal{E}$ will be called the* ***nilpotent order*** *of $\mathcal{E}$ and we shall denote it $d(\mathcal{E})$.*

EXEMPLE. If T is a $[\alpha]$-primitive theme, then its semi-simple filtration coincides with the filtration giving its unique Jordan-Hölder sequence □

This filtration has the following properties

Lemma 4.2.5 *Let $\mathcal{E}$ be a regular (a,b)-module and let $\mathcal{F}$ be any sub-module. Then $S_j(\mathcal{F}) = S_j(\mathcal{E}) \cap \mathcal{F}$ for any $j \in \mathbb{N}^*$ with the convention that, for $j \geq d(\mathcal{E}) + 1$ we define $S_j(\mathcal{E}) = \mathcal{E}$. In particular $d(\mathcal{F}) \leq d(\mathcal{E})$.*

If $\mathcal{F}$ is normal in $\mathcal{E}$, the quotient map $\mathcal{E} \to \mathcal{E}/\mathcal{F}$ sends $S_j(\mathcal{E})$ into $S_j(\mathcal{E}/\mathcal{F})$. This implies that $d(\mathcal{E}/\mathcal{F}) \leq d(\mathcal{E})$.
For any subset $\mathscr{A} \in \mathbb{C}/\mathbb{Z}$ and any j we have $S_j(\mathcal{E})_{[\mathscr{A}]} = S_j(\mathcal{E}_{[\mathscr{A}]})$.

PROOF. The inlusion $S_1(\mathcal{F}) \subset S_1(\mathcal{E})$ is obvious, so $S_1(\mathcal{F}) = \mathcal{F} \cap S_1(\mathcal{E})$ holds true. Assume that we have already proved that $S_j(\mathcal{F}) = S_j(\mathcal{E}) \cap \mathcal{F}$ for some $j \geq 1$. Then we have

$$S_{j+1}(\mathcal{F})/S_j(\mathcal{F}) = S_1(\mathcal{F}/S_j(\mathcal{F})) = S_1(\mathcal{F}/(S_j(\mathcal{E}) \cap \mathcal{F})) =$$
$$\big(\mathcal{F}/(S_j(\mathcal{E}) \cap \mathcal{F})\big) \cap S_1(\mathcal{E}/S_j(\mathcal{E})) = \big(\mathcal{F}/(S_j(\mathcal{E}) \cap \mathcal{F})\big) \cap (S_{j+1}(\mathcal{E})/S_j(\mathcal{E}))$$

and this gives the equality $S_{j+1}(\mathcal{F}) = S_{j+1}(\mathcal{E}) \cap \mathcal{F}$ thanks to inductive assumption.
If $\mathcal{F}$ is normal the image of $S_1(\mathcal{E})$ by the quotient map $\mathcal{E} \to \mathcal{E}/\mathcal{F}$ is a semi-simple sub-module in $\mathcal{E}/\mathcal{F}$ so is contained in $S_1(\mathcal{E}/\mathcal{F})$.
Assume that we have already proved that for each $h \in [1, j]$ the quotient map sends $S_h(\mathcal{E}) \to S_h(\mathcal{E}/\mathcal{F})$. Then it sends $S_j(\mathcal{E})/S_{j-1}(\mathcal{E})$ into $S_j(\mathcal{E}/\mathcal{F})/S_{j-1}(\mathcal{E}/\mathcal{F})$ and so $S_1(\mathcal{E}/S_j(\mathcal{E})) = S_{j+1}(\mathcal{E})/S_j(\mathcal{E}))$ into $S_1(\mathcal{E}/\mathcal{F})/S_j(\mathcal{E}/\mathcal{F})) = S_{j+1}(\mathcal{E}/\mathcal{F})/S_j(\mathcal{E}/\mathcal{F})$. The conclusion follows.
Now if $d := d(\mathcal{E})$ we have $\mathcal{E} = S_d(\mathcal{E})$ and the quotient map sends $S_d(\mathcal{E}) = \mathcal{E}$ into $S_d(\mathcal{E}/\mathcal{F})$. Then $S_d(\mathcal{E}/\mathcal{F}) = \mathcal{E}/\mathcal{F}$.
To complete the proof, remark that the equalities, for any sub-module $\mathcal{F}$ of a regular (a,b)-module $\mathcal{E}$, $S_j(\mathcal{E}) \cap \mathcal{F} = S_j(\mathcal{F})$ and $\mathcal{E}_{\mathscr{A}} \cap \mathcal{F} = \mathcal{F}_{\mathscr{A}}$ we have

$$S_j(\mathcal{E}_{\mathscr{A}}) = S_j(\mathcal{E}) \cap \mathcal{E}_{\mathscr{A}} = S_j(\mathcal{E})_{\mathscr{A}}. \qquad \blacksquare$$

REMARKS.

i) As $S_1(\mathcal{E})$ is the maximal semi-simple sub-module of $\mathcal{E}$ it contains any rank 1 sub-module of $\mathcal{E}$. So $S_1(\mathcal{E}) = \{0\}$ happens if and only if $\mathcal{E} = \{0\}$.

ii) Let $\mathcal{F}$ be a sub-module of the regular (a,b)-module $\mathcal{E}$ such that $S_j(\mathcal{F}) = \mathcal{F}$. Then $\mathcal{F}$ is contained in $S_j(\mathcal{E})$ thanks to the previous lemma.

iii) The semi-simple filtration of $\mathcal{E}$ is strictly increasing for $j \in [0, d(\mathcal{E})]$.

The next lemma will help to compute the ranks of the various $S_j(\mathcal{E})$ and the nilpotent order in the case of geometric (a,b)-modules (see Section 5 for the definition of "geometric").

Lemma 4.2.6 *Let $\mathcal{E}'$ be a sub-module of a regular (a,b)-module $\mathcal{E}$ such that $\mathcal{E}/\mathcal{E}'$ is a finite dimensional complex space. Then, for each j the quotient $S_j(\mathcal{E})/S_j(\mathcal{E}')$ is also a finite dimensional complex vector space. So $S_j(\mathcal{E})$ and $S_j(\mathcal{E}')$ have the same rank on B and $\mathcal{E}'$ and $\mathcal{E}$ have the same nilpotent order.*

PROOF. As we know that $S_j(\mathcal{E}') = S_j(\mathcal{E}) \cap \mathcal{E}'$ the quotient $S_j(\mathcal{E})/S_j(\mathcal{E}')$ is a sub-vector space of $\mathcal{E}/\mathcal{E}'$ and then has finite dimension.
So for each j $S_j(\mathcal{E}')$ has finite co-dimension in $S_j(\mathcal{E})$ so has the same rank as a B-module. This implies that $S_j(\mathcal{E}') = S_j(\mathcal{E}) \cap \mathcal{E}'$ since $S_j(\mathcal{E}')$ is normal in $\mathcal{E}'$. In particular $d(\mathcal{E}') \leq d(\mathcal{E})$.
Conversely, if $S_\delta(\mathcal{E}') = \mathcal{E}'$, it has the same rank that $\mathcal{E}'$, then $S_\delta(\mathcal{E})$ has the same rank than $\mathcal{E}$ and must be equal to $\mathcal{E}$. ■

Corollary 4.2.7 *Let $\mathcal{E}$ be a regular (a,b)-module. If the (a,b)-module $\mathcal{E}$ is semi-simple, then $\mathcal{E}_{\mathscr{A}}$ is semi-simple for any subset $\mathscr{A} \subset \mathbb{C}/\mathbb{Z}$.*
Conversely, if for any $\alpha \in \mathbb{C}/\mathbb{Z}$, $\mathcal{E}_{[\alpha]}$ is semi-simple, then $\mathcal{E}$ is semi-simple.

PROOF. The direct part is clear.
Since $\mathcal{E}$ is semi-simple if and only if $\mathcal{E}^\sharp$ is semi-simple, the converse follows from the finite co-dimension in $\mathcal{E}^\sharp$ of the sub-module $\sum_{\alpha \in \mathscr{A}} \mathcal{E}_{[\alpha]}$ and the previous lemma. ■

The next result shows that the ranks of the successive quotients of the semi-simple filtration is non increasing.

Proposition 4.2.8 *Let $\mathcal{E}$ be a regular (a,b)-module and note $d := d(\mathcal{E})$ its nilpotent order. Then for each $j \in [1, d]$ we have*

$$rk(S_j(\mathcal{E})/S_{j-1}(\mathcal{E})) \geq rk(S_{j+1}(\mathcal{E})/S_j(\mathcal{E})).$$

PROOF. Remark that it is enough to prove the result when $\mathcal{E}$ has a simple pole, because $S_j(\mathcal{E}^\sharp) = S_j(\mathcal{E})^\sharp$ for each j and thanks to Lemma 4.2.6 above.
Note also that for $d(\mathcal{E}) = 1$ there is nothing to prove.
We shall begin by the case where $d(\mathcal{E}) = 2$. Then consider the exact sequence of (a,b)-modules $0 \to S_1(\mathcal{E}) \to \mathcal{E} \to \mathcal{E}/S_1(\mathcal{E}) \to 0$ which give the exact sequence of finite dimensional vector spaces

$$0 \to S_1(\mathcal{E})/bS_1(\mathcal{E}) \to \mathcal{E}/b\mathcal{E} \to \mathcal{E}/(S_1(\mathcal{E}) + b\mathcal{E}) \to 0$$

which is compatible with the respective actions of $b^{-1}a$ on these quotients. But as $S_1(\mathcal{E})$ and $\mathcal{E}/S_1(\mathcal{E})$ are semi-simple, the action of $b^{-1}a$ on $S_1(\mathcal{E})/bS_1(\mathcal{E})$ and on $\mathcal{E}/(S_1(\mathcal{E}) + b\mathcal{E})$ are semi-simple and the action of $b^{-1}a$ on $\mathcal{E}/b\mathcal{E}$ has a nilpotent part N which satisfies $N^2 = 0$. So we have $ImN \subset KerN$ and $\dim \mathcal{E}/b\mathcal{E} \leq 2 \dim KerN$. Since $KerN = S_1(\mathcal{E})/bS_1(\mathcal{E})$ because the (a,b)-module generated by a diagonal basis of $KerN$ for $b^{-1}a$ generates a semi-simple sub-module of $\mathcal{E}$ (see Lemma 4.2.2), so it is equal to $S_1(\mathcal{E})$, the conclusion follows for $d(\mathcal{E}) = 2$.
To prove the general case, consider a geometric (a,b)-module $\mathcal{E}$ with $d(\mathcal{E}) \geq 3$ and an integer $j \in [2, d(\mathcal{E}) - 1]$. Then let $\mathcal{F} := S_{j+1}(\mathcal{E})/S_{j-1}(\mathcal{E})$. We have $d(\mathcal{F}) = 2$ and so

$$rk(\mathcal{F}) \leq 2rk(S_1(\mathcal{F})).$$

Note $s_h := rk(S_h(\mathcal{E}))$ for each h. The inequality proved above gives $s_j + s_{j+1} \leq 2s_j$ because $S_1(\mathcal{F}) = S_j(\mathcal{E})/S_{j-1}(\mathcal{E})$ as $\mathcal{F}$ is a sub-module of $\mathcal{E}/S_{j-1}(\mathcal{E})$ which implies $S_1(\mathcal{F}) = \mathcal{F} \cap S_1(\mathcal{E}/S_{j-1}(\mathcal{E})) = S_j(\mathcal{E})/S_{j-1}(\mathcal{E})$. This concludes the proof of the inequality $s_{j+1} \leq s_j$ for each j. ■

The following corollary will be useful later on

Corollary 4.2.9 *Let $\mathcal{E}$ be a regular (a,b)-module which has a unique rank 1 normal sub-module. Then, for any $j \geq 2$, $\mathcal{E}$ cannot have two distinct normal rank j sub-modules.*

PROOF. We begin by the case $j = 2$. We shall argue by contradiction. So let G_1 and G_2 be two distinct normal sub-modules with rank 2. By uniqueness of the normal rank 1 sub-module H of $\mathcal{E}$ we have $H \subset G_1 \cap G_2$ since each G_i must contain $H := S_1(\mathcal{E})$, and moreover the intersection cannot be of rank 2 since it is also a normal sub-module. Then G_1/H and G_2/H are two distinct normal rank 1 sub-modules of $S_2(\mathcal{E})/H$. Then $S_2(\mathcal{E}/S_1(\mathcal{E}))$ has rank at least two while $S_1(\mathcal{E}) = H$ has rank 1. This contradicts Proposition 4.2.8.
Now we shall argue by induction on $j \geq 3$. So assume that $j \geq 2$ and the result proved for the rank at most equal $j-1$. We shall argue again by contradiction. Then the quotient $\mathcal{E}/F_{j-2}$, where F_{j-2} is the unique normal sub-module of rank $j-2$ in $\mathcal{E}$ thanks to our inductive assumption, has a unique rank 1 normal sub-module and then has a unique normal rank 2 normal submodule. If G_1 and G_2 are distinct normal sub-modules of rank j in $\mathcal{E}$ then $G_1 \cap G_2$ is equal to F_{j-1}, the unique normal rank $j-1$ sub-module of $\mathcal{E}$[6] and F_{j-1} contains F_{j-2}.
Then $G_i/F_{j-1}, i = 1, 2$ are two distinct rank 1 normal sub-modules of $\mathcal{E}/F_{j-1}$. Contradiction. ■

REMARK. Note that under the hypothesis of the previous corollary the rank of $S_1(\mathcal{E})$ is 1 (if $\mathcal{E} \neq \{0\}$). Then, the rank of $S_2(\mathcal{E})$ is 1 or 2. In the first case $S_2(\mathcal{E}) = S_1(\mathcal{E})$ and $\mathcal{E} = S_1(\mathcal{E})$ has rank 1. If the rank of $S_2(\mathcal{E})$ is 2, then the rank of $S_3(\mathcal{E})$ is 2 or 3. If the rank of $S_3(\mathcal{E})$ is 2, then $S_2(\mathcal{E}) = \mathcal{E}$ and the rank of $\mathcal{E}$ is 2. And so on .... So if k is the rank of $\mathcal{E}$ we have $S_k(\mathcal{E}) = \mathcal{E}$ and $d(\mathcal{E}) = k$.
Conversely, if $\mathcal{E}$ is regular with rank k and such that $d(\mathcal{E}) = k$, the inclusions $S_j(\mathcal{E}) \subset S_{j+1}(\mathcal{E})$ are strict for $j \in [0, k-1]$ and each quotient $S_{j+1}(\mathcal{E})/S_j(\mathcal{E})$ has rank 1. In particular $S_1(\mathcal{E})$ has rank 1 and so it is the unique normal rank 1 sub-module in $\mathcal{E}$.

We leave the proof of the following corollary as an exercise for the reader :

[6]The existence of J-H. sequence shows that any rank $j \geq 1$ normal sub-module of a regular (a,b)-module contains a rank $j-1$ normal sub-module.

Corollary 4.2.10 *For any non zero regular (a,b)-module $\mathcal{E}$ we have the equivalence between the conditions:*

- $rank(S_1(E)) = 1$ *and*
- $d(\mathcal{E}) = rank(\mathcal{E})$
- $\mathcal{E}$ *is a* $[\alpha]$*-primitive theme.*
- $\mathcal{E}$ *has a unique Jordan-Hölder sequence.* ■

5 Asymptotic expansions and geometric (a,b)-modules

5.1 The Embedding Theorem

Fix $\alpha \in]0,1] \cap \mathbb{Q}$ and $N \in \mathbb{N}$. We shall consider convergent asymptotic expansions of the type

$$\Xi_\alpha^{(N)} := \{\sum_{j=0}^{N} \sum_{m=0}^{\infty} c_m^j s^{\alpha+m-1} \frac{(Log\, s)^j}{j!}\} \tag{12}$$

where we ask that exists $R > 0$ and a positive constant C_R such that the following estimates hold

$$|c_m^j| \leq C_R R^m \quad \forall j \in [0,N] \quad \forall m \in \mathbb{N}. \tag{13}$$

Of course, $\Xi_\alpha^{(N)}$ is a free $\mathbb{C}\{s\}$-module with rank $(N+1)$ and basis

$$e_j := s^{\alpha-1} \frac{(Log\, s)^j}{j!},\ j \in [0,N]. \tag{14}$$

We shall consider the dual Fréchet topology on $\Xi_\alpha^{(N)}$ deduced from the natural topology of $\mathbb{C}\{s\}$ (which is independent of the choice of the basis) and we define on $\Xi_\alpha^{(N)}$ the $\mathbb{C}$-linear continuous endomorphism[7] defined inductively on $j \in [0,N]$ by the following rules:

1. For $j = 0$ and $m \in \mathbb{N}$ we put $b(s^{\alpha+m-1}) := s^{\alpha+m}/(\alpha+m)$.
2. For $j \geq 1$ and $m \in \mathbb{N}$ we put

$$b(s^m e_j) = \frac{s^{m+1}}{\alpha+m} e_j - \frac{b(s^m e_{j-1})}{\alpha+m}. \tag{E}$$

Of course this endomorphism corresponds to the term-wise integration (without constant) of these series.

[7]The continuity is an easy consequence of the formula

$$b(s^m e_j) = \sum_{h=0}^{j} \frac{(-1)^{j-h}}{(\alpha+m)^{j-h+1}} s^{m+1} e_h$$

since $j \in [0,N], m \in \mathbb{N}$ and $\alpha > 0$.

NOTATION (RAPPEL). We shall note $A := \mathbb{C}\{a\}$ and $B := \mathbb{C}\{\{b\}\}$, and a will be the multiplication by s on A-modules.

Proposition 5.1.1 *The $\mathbb{C}[b]$-action on $\Xi_\alpha^{(N)}$ defined above extends to a B-action and $\Xi_\alpha^{(N)}$ is a free B-module of rank $N+1$ with basis $e_j, j \in [0,N]$. Moreover, we have $a\Xi_\alpha^{(N)} = b\Xi_\alpha^{(N)}$, and the action of $b^{-1}a$ on $\Xi_\alpha^{(N)}$ may be written as*

$$b^{-1}a = \Delta + \mathcal{N} \quad \text{with} \quad \mathcal{N}(e_0) = 0 \quad \text{and} \quad \mathcal{N}(e_j) = e_{j-1} \quad \forall j \in [1,N]$$

where $\mathcal{N}$ is A-linear and B-linear, where $\Delta(s^m e_j) = (\alpha+m)s^m e_j$, $\forall m\ \forall j$ and where $b^{-1}a$, Δ and $\mathcal{N}$ commute.

PROOF. The formula (E) gives for each integers m and j

$$as^m e_j = (\alpha+m)bs^m e_j + bs^m e_{j-1}$$

so $\Delta(s^m e_j) = (\alpha+m)s^m e_j + \mathcal{N}(s^m e_j)$ if we define the A-linear map $\mathcal{N}$ by putting $\mathcal{N}(e_j) = e_{j-1}$ with the convention $\mathcal{N}(e_0) = 0$.
Then it is clear that satisfies $\mathcal{N}^{N+1} = 0$. The fact that Δ is $\mathbb{C}$-linear continuous and bijective gives $a\Xi_\alpha^{(N)} = b\Xi_\alpha^{(N)}$.
We want to check now that $\mathcal{N}$ is b-linear.
Then first remark that we have $b^{-1}ab - bb^{-1}a = b^{-1}b(a+b) - a = b$.
We shall prove by induction on $j \geq 0$ that $\Delta b - b\Delta = b$.
For $j = 0$ we have

$$(\alpha+m)\Delta bs^m e_0 = \Delta(s^{m+1}e_0) = (\alpha+m+1)s^{m+1}e_0$$

and

$$(\alpha+m)b\Delta s^m e_0 = (\alpha+m)^2 bs^m e_0 = (\alpha+m)s^{m+1}e_0$$

and so

$$(\alpha+m)(\Delta b - b\Delta)(s^m e_0) = s^{m+1}e_0 = (\alpha+m)b(s^m e_0).$$

Assume now $j \geq 1$ and the relation $\Delta b - b\Delta = b$ proved for $s^m e_{j-1}$.
Then we have, using the induction hypothesis:

$$\Delta(bs^m e_j) = \frac{1}{\alpha+m}\Delta\big(s^{m+1}e_j - b(s^m e_{j-1}\big) = \frac{\alpha+m+1}{\alpha+m}s^{m+1}e_j - \frac{1}{\alpha+m}\Delta(bs^m e_{j-1})$$
$$\Delta(bs^m e_j) = \frac{\alpha+m+1}{\alpha+m}s^{m+1}e_j - bs^m e_{j-1} - \frac{1}{\alpha+m}bs^m e_{j-1} \quad \text{and also}$$
$$b\Delta(s^m e_j) = b(\alpha+m)s^m e_j = s^{m+1}e_j - bs^m e_{j-1} \quad \text{so we obtain}$$
$$(\Delta b - b\Delta)(s^m e_j) = \frac{1}{\alpha+m}\big(s^{m+1}e_j - bs^m e_{j-1}\big) = bs^m e_j.$$

This gives the b-linearity of $\mathcal{N} = b^{-1}a - \Delta$, and then the B-linearity by continuity. Note that $[b^{-1}a, b] = a$ and the A-linearity of $\mathcal{N}$ gives $[\Delta, a] = a$ which is easy to check directly.

Using the fact that B acts on $\Xi_\alpha^{(N)}$[8] and that b is injective on $\Xi_\alpha^{(N)}$, we conclude, as $(e_j, j \in [0,N])$ induces a basis of the vector space $\Xi_\alpha^{(N)}/b\Xi_\alpha^{(N)} = \Xi_\alpha^{(N)}/a\Xi_\alpha^{(N)}$, and then $\Xi_\alpha^{(N)}$ is a free B-module with basis $(e_j, j \in [0,N])$. ■

Let $\mathscr{A}$ be a finite subset of $]0,1] \cap \mathbb{Q}$ and let V be a finite dimensional complex space, we define the free finite type B-module

$$\Xi_{\mathscr{A}}^{(N)} := \oplus_{\alpha\in\mathscr{A}} \Xi_\alpha^{(N)} \tag{15}$$

which is also a free finite type A-module. Then, defining the action of B and of A as the identity on V, the tensor product $\Xi_{\mathscr{A}}^{(N)} \otimes_{\mathbb{C}} V$ is again a free finite type B-module which is also a free finite type A-module. Of course this implies that it is a (simple pole) convergent (a,b)-module.

Definition 5.1.2 *We say that a left $B[a]$ sub-module $\mathcal{E}$ of $\Xi_{\mathscr{A}}^{(N)} \otimes_{\mathbb{C}} V$ for some $N \in \mathbb{N}$ and some finite dimensional complex vector space V is a* **geometric** *(a,b)-module.*
A geometric (a,b)-module $\mathcal{E}$ of the form $B[a]\varphi$ for some φ in some $\Xi_{\mathscr{A}}^{(N)} \otimes_{\mathbb{C}} V$ is called a **fresco**.
A fresco $\mathcal{E}$ will be called a **theme** *when it may be written $\mathcal{E} = B[a]\varphi$ with $\varphi \in \Xi_{\mathscr{A}}^{(N)}$ (so with $V = \mathbb{C}$).*

Remarks.

1. Of course a geometric (a,b)-module is a regular (a,b)-module because it is a free finite rank B-module, stable by the action of a which is continuous, and which is regular since any $\Xi_{\mathscr{A}}^{(N)} \otimes_{\mathbb{C}} V$ is a simple pole (a,b)-module.

2. The saturation of a geometric (a,b)-module $\mathcal{E}$ is again a geometric (a,b)-module since the stability of $\Xi_{\mathscr{A}}^{(N)} \otimes_{\mathbb{C}} V$ by $b^{-1}a$ implies the inclusion of $\mathcal{E}^\sharp$ in $\Xi_{\mathscr{A}}^{(N)} \otimes_{\mathbb{C}} V$ when $\mathcal{E}$ is in $\Xi_{\mathscr{A}}^{(N)} \otimes_{\mathbb{C}} V$.

3. It is easy to see that the Bernstein polynomial of $\Xi_\alpha^{(N)}$ is equal to $(x+\alpha)^{N+1}$ and then to deduce that the roots of the Bernstein polynomial of a geometric (a,b)-module are negative rational numbers.

4. For $\alpha = -1$ the free rank 1 A-module $A(1/s)$ is not stable by b, but the free rank 2 A-module $A(1/s) \oplus A(Log\, s)$ is stable by b if we defined $b(1/s) := Log\, s$ and $b(Log\, s) := sLog\, s - s$ which gives

$$ba(1/s) = ab(1/s) - b^2(1/s) = sLog\, s - (sLog\, s - s) = s = b(1).$$

   So it coincides with the free rank 2 B-module $\mathcal{E}$ with basis $1/s$ and 1.
   But its saturation by $b^{-1}a$ inside $\mathcal{E} \otimes_B B[b^{-1}]$ contains the non zero element $b^{-1}a(1/s) = b^{-1}[1]$ which satisfies $ab^{-1}[1] = 0$. So a is not injective on $\mathcal{E}^\sharp$ which is not a free A-module although it is a free B-module.

[8]The easy estimates corresponding to this assertion is left to the reader.

Theorem 5.1.3 [The Embedding Theorem] *Let $\mathcal{E}$ be a regular (a,b)-module such that its Bernstein polynomial has negative rational roots. Then there exists a finite subset $\mathscr{A}$ in $\mathbb{Q}\cap]0,1]$, a finite dimensional complex vector space V, an integer $N \in \mathbb{N}$ and a $B[a]$-linear injective map $f : \mathcal{E} \to \Xi_{\mathscr{A}}^{(N)} \otimes V$.*

Important remark. An obvious consequence of the previous theorem is the following equivalent definition of a geometric (a,b)-module:

- A regular (a,b)-module is **geometric** if and only if the roots of its Bernstein polynomial are negative rational numbers (compare with [19] and [18]).

In the section 5.2 we describe, for a given geometric (a,b)-module $\mathcal{E}$, what are the smallest N, V and $\mathscr{A}$ for which there exists an embedding of $\mathcal{E}$ in $\Xi_{\mathscr{A}}^{(N)} \otimes_{\mathbb{C}} V$.

The proof of the Embedding Theorem will use the following lemmas[9].

Lemma 5.1.4 *Let γ be a positive rational number. Then for any element y in $\Xi_{\mathscr{A}}^{(N)} \otimes V$ there exists $x \in \Xi_{\mathscr{A}}^{(N+1)} \otimes V$ such that $(a - \gamma b)x = by$.*

Proof. It is enough to consider the case $V = \mathbb{C}$ and $\mathscr{A} := \{\alpha\}$.
Assume first that $\gamma \notin \alpha + \mathbb{N}$. Since $\Delta - \gamma : \Xi_{\alpha}^{(N)} \to \Xi_{\alpha}^{(N)}$ is bijective and $\mathcal{N}^{N+1} = 0$ the map $\Delta - \gamma - \mathcal{N}$ is also bijective and the formula $a - \gamma b = b(\Delta - \gamma - \mathcal{N})$ implies the result.
If $\gamma = \alpha + m_0$ for some $m_0 \in \mathbb{N}$ let Z be the closed $\mathbb{C}$-linear span of the vectors $s^m e_j$ for $m \in \mathbb{N} \setminus \{m_0\}$ and $j \in [0, N]$. Then $\Delta - \gamma$ is continuous bijective on Z and $\mathcal{N}$ satisfies $\mathcal{N}(Z) \subset Z$ and $\mathcal{N}^{N+1} = 0$. Now the formula $a - \gamma b = b(\Delta - \gamma - \mathcal{N})$ implies the equality $(a - (\alpha + m_0)b)Z = b(\Delta - \gamma - \mathcal{N})Z$. So, to complete the proof, it is enough to shows that $(a-(\alpha+m_0)b)(\Xi_{\alpha}^{(N+1)})$ contains the vectors $s^{m_0+1}e_j, j \in [0, N]$. This is given by the formulas

$$(a - (\alpha + m_0)b)(s^{m_0} e_j) = b(s^{m_0} e_{j-1})$$

and an induction on j in $[1, N + 1]$. ■

Proof of the Embedding Theorem. Remark first that it is enough to prove the existence of an embedding in the case where $\mathcal{E}$ has simple pole since $\mathcal{E}$ and $\mathcal{E}^\sharp$ have the same Bernstein polynomial.
We shall make an induction on the rank of $\mathcal{E}$ assumed to have a simple pole.
In rank 1 we have $\mathcal{E} \simeq \mathcal{E}_\alpha$ with $\alpha \in \mathbb{Q}^{+*}$ and since $\mathcal{E}_\alpha \simeq B[a]s^{\alpha-1}$ is a sub-module of $\Xi_{[\alpha]}^{(0)}$, this gives the desired embedding, where $[\alpha]$ is the class of α in $\mathbb{Q}/\mathbb{Z} \simeq \mathbb{Q}\cap]0,1]$.
Assume that the existence of such an embedding has been proved when $\mathcal{E}$ has rank

[9] This easy lemma is missing in the proof of Theorem 4.2.12 in [9].

$(k-1)$ with $k \geq 2$. Then consider a rank k simple pole (a,b)-module $\mathcal{E}$ and let $\mathcal{F}$ be a rank $(k-1)$ normal sub-module in $\mathcal{E}$ (the existence of such an $\mathcal{F}$ is consequence of the existence of J-H sequence for $\mathcal{E}$; see section 3.4). Then $\mathcal{E}/\mathcal{F}$ is isomorphic to $\mathcal{E}_\alpha$ for some $\alpha \in \mathbb{Q}^{+*}$ because it has rank 1 and the only root $-\alpha$ of its Bernstein polynomial is a root of the Bernstein polynomial of $\mathcal{E}$ (see Remark 3 following Definition 3.2.1). Let e be an element in $\mathcal{E}$ such that its image in $\mathcal{E}/\mathcal{F}$ is the standard generator[10] of $\mathcal{E}_\alpha$. Then, since $\mathcal{E}$ has a simple pole and $\mathcal{F}$ is normal, we have $z := (a - \alpha b)e$ is in $\mathcal{F} \cap b\mathcal{E} = b\mathcal{F}$. Then we may write $z = by$ with $y \in \mathcal{F}$.
Now our induction hypothesis gives us an injective $B[a]$-linear embedding

$$g : \mathcal{F} \to \Xi^{(N)}_{\mathscr{A}} \otimes V.$$

Then $g(z)$ is in $b\Xi^{(N)}_{\mathscr{A}} \otimes V$ and the lemma 5.1.4 gives an $x \in \Xi^{(N+1)}_{\mathscr{A}} \otimes V$ such that $(a - \alpha b)x = g(z)$.
Define a B-linear map $f : \mathcal{E} \to \Xi^{(N+1)}_{\mathscr{A}} \otimes (V \oplus \mathbb{C}\varepsilon)$ by

$$f(t) = g(t) \quad \text{when } t \in \mathcal{F} \quad \text{and } f(e) = x \oplus s^{\alpha-1}\varepsilon,$$

using the direct sum decomposition (as B-module) $\mathcal{E} = \mathcal{F} \oplus Be$.
We shall verify the a-linearity of f and also its injectivity.
Let σ be in B and t in $\mathcal{F}$. We have $a(t \oplus \sigma e) = (at + \sigma(a - \alpha b)e) \oplus \alpha\sigma be$. Then

$$\begin{aligned} f(a(t \oplus \sigma e)) &= g(at + \sigma(a - \alpha b)e) + \alpha\sigma b(x + s^{\alpha-1}\varepsilon) \\ &= ag(t) + \sigma g(z) + \sigma s^\alpha \varepsilon - \sigma(a - \alpha b)x + \sigma a x \\ &= af(t \oplus \sigma e) \quad \text{since} \quad g(z) = (a - \alpha b)(x). \end{aligned}$$

So the a-linearity is proved.
As g is injective, $f(t \oplus \sigma e) = 0$ implies $t + \sigma x = 0$ and $\sigma s^{\alpha-1}\varepsilon = 0$. But since $W := V \oplus \mathbb{C}\varepsilon$, this implies $\sigma = 0$ and then $t = 0$, concluding the existence of an embedding for $\mathcal{E}$. ■

Corollary 5.1.5 *A $[\alpha]$-primitive theme of rank ≥ 2 is not semi-simple. In fact its semi-simple part has rank 1.*

Proof. Let $\mathcal{E} := B[a]\varphi \subset \Xi^{(N)}_{[\alpha]}$ a rank k $[\alpha]$-primitive theme which is semi-simple. Then $\mathcal{E}^\sharp$ is isomorphic to a direct sum of $\mathcal{E}_\beta$ for a finite set of β in $\alpha+\mathbb{N}$ (may be with repetitions), thanks to Proposition 4.1.4. Since $\Xi^{(N)}_{[\alpha]}$ has a simple pole, the inclusion of $\mathcal{E}$ in $\Xi^{(N)}_{[\alpha]}$ extends to a $B[a]$-linear map $j^\sharp : \mathcal{E}^\sharp \to \Xi^{(N)}_{[\alpha]}$. But for any $\beta \in \alpha + \mathbb{N}$ a $B[a]$-linear map $\mathcal{E}_\beta \to \Xi^{(N)}_{[\alpha]}$ has its image in $\Xi^{(0)}_{[\alpha]}$ because $\mathbb{C}s^\beta$ is the vector space of solutions of the equation $(a - \beta b)x = 0$ for $x \in \Xi^{(N)}_{[\alpha]}$. Then the image of $j^\sharp$ has rank at most 1 and $k \leq 1$. ■

Corollary 5.1.6 *Let $\mathcal{E}$ be an $[\alpha]$-primitive theme of rank k. Then $d(\mathcal{E}) = k$.*

[10] a non zero element ε in $\mathcal{E}_\alpha$ such that $a\varepsilon = \alpha b\varepsilon$.

PROOF. Remark first that we have an isomorphism $\Xi^{(N)}_{[\alpha]}/\Xi^{(N-p)}_{[\alpha]} \simeq \Xi^{(p)}_{[\alpha]}$ which is given by $e_j \mapsto e_{j-p}$ for $j \in [N-p, N-1]$ (and $e_j \mapsto 0$ for $j \in [0, N-p-1]$), where $e_j := s^{\alpha-1}(Log\, s)^j/j!$ for $j \in [0, N-1]$.
Let $\mathcal{E}$ be an $[\alpha]$-primitive theme of rank $k \geq 2$. Then $\mathcal{E}/S_1(\mathcal{E})$ is an $[\alpha]$-primitive theme of rank $k-1$ since the proof of the previous corollary shows that an embedding of $\mathcal{E}$ in $\Xi^{(N)}_{[\alpha]}$ gives an embedding of $\mathcal{E}/S_1(\mathcal{E})$ in $\Xi^{(N)}_{[\alpha]}/\Xi^{(0)}_{[\alpha]} \simeq \Xi^{(N-1)}_{[\alpha]}$. Then

$$\mathcal{E}/S_1(\mathcal{E})/S_1(\mathcal{E}/S_1(\mathcal{E})) \simeq \mathcal{E}/S_2(\mathcal{E})$$

is a rank $k-2$ $[\alpha]$-primitive theme. We obtain, continuing in this way, that for any $j \in [0, k-1]$ that $\mathcal{E}/S_j(\mathcal{E})$ is a rank $k-j$ $[\alpha]$-primitive theme and then that $\mathcal{E} = S_k(\mathcal{E})$ and $d(\mathcal{E}) = k$. ■

Note that the proof above shows that the semi-simple filtration of an $[\alpha]$-primitive theme $\mathcal{E}$ is the unique J-H. sequence for $\mathcal{E}$.

5.2 Complements to the Embedding Theorem

We give now some complements to the Embedding Theorem 5.1.3.

Proposition 5.2.1 *Let $\mathscr{A}$ be a finite subset in $\mathbb{Q}\cap]0,1]$, V be a non zero finite dimensional vector space and N be a non negative integer. Then $S_j(\Xi^{(N)}_{\mathscr{A}} \otimes V) = \Xi^{(j-1)}_{\mathscr{A}} \otimes V$ for each $j \in [1, N+1]$, and the nilpotent order of this geometric (a,b)-module is $N+1$.*

The proof of this proposition will use the following three lemmas.

Lemma 5.2.2 *Fix $\alpha \in \mathbb{Q}\cap]0,1]$ and $\beta := \alpha + m$ with $m \in \mathbb{N}$. Let P be a degree M polynomial. Then there exists a unique degree $M+1$ polynomial Q without constant term such that $(a - \beta b)(s^{\beta-1}Q(Log\, s)) = s^\beta P(Log\, s)$.*

PROOF. An elementary computation shows that Q is the primitive vanishing at 0 of the polynomial $\beta P + P'$ which has degree M since $\beta > 0$. So Q is a polynomial of degree $M+1$. ■

Lemma 5.2.3 *We keep the notations of the previous lemma and assume $M \geq 1$. Let $\varphi := s^{\beta-1}(Log\, s)^M + \psi$ where ψ is in $\Xi^{(M-1)}_{[\alpha]}$.*
Then the degree in $Log\, s$ of $(a - \gamma b)\varphi$ is equal to M for $\gamma \neq \beta$ and $M-1$ for $\gamma = \beta$.

PROOF. For $\gamma \neq \beta$ we have

$$(a - \gamma b)s^{\beta-1}(Log\, s)^M = (1 - \gamma/\beta)s^\beta(Log\, s)^M \text{ modulo } \Xi^{(M-1)}_{[\alpha]}$$

and the conclusion follows.
For $\gamma = \beta$ we have

$$(a - \beta b)\varphi = Mb(s^{\beta-1}(Log\, s)^{M-1}) + (a - \beta b)\Xi_{[\alpha]}^{(M-1)}.$$

But as $M \geq 1$ the Lemma 5.2.6 implies that $s^{\beta}(Log\, s)^{M-1}$ is not in $(a - \beta b)\Xi_{[\alpha]}^{(M-1)}$ since the kernel of $a - \beta b$ is $\mathbb{C}s^{\beta-1}$. Now the equality

$$b(s^{\beta-1}(Log\, s)^{M-1} = \frac{1}{\beta}s^{\beta}(Log\, s)^{M-1} - \frac{M-1}{\beta}b(s^{\beta-1}(Log\, s)^{M-2})$$

shows that the term $s^{\beta}(Log\, s)^{M-1}$ has still a non zero coefficient in $(a - \beta b)\varphi$ concluding the proof. ■

Lemma 5.2.4 *Fix $\alpha \in \mathbb{Q} \cap]0, 1]$ and $M \in \mathbb{N}$. Let φ be in $\Xi_{[\alpha]}^{(M)}$ of the form*

$$\varphi = s^{\beta-1}(Log\, s)^{M} + \psi$$

with $\beta > 0$ in $\alpha + \mathbb{N}$ and $\psi \in \Xi_{[\alpha]}^{(M-1)}$. Then $B[a]\varphi$ is a $[\alpha]$-primitive theme of rank $M + 1$.

Proof. By definition $\tilde{\mathcal{A}}_{conv.}\varphi$ is a $[\alpha]$-primitive theme. We shall argue by induction on $M \geq 0$. For $M = 0$ since $\Xi_{[\alpha]}^{(0)} \simeq \mathcal{E}_{\alpha}$ is rank 1, the point is to prove that $B[a]\varphi$ is not a finite dimensional complex space. As a is injective this is clear.
Assume that $M \geq 1$ and that the corollary is proved for $M-1$. Thanks to Corollary 5.2.3 and the induction hypothesis we know that $B[a](a - \beta b)\varphi$ is a rank M theme. Then the exact sequence

$$0 \to B[a](a - \beta b)\varphi \to B[a]\varphi \to Q \to 0$$

where Q is a quotient of $\mathcal{E}_{\beta}$, shows that the rank of $B[a]\varphi$ is either $M + 1$ or M. If the rank is M, then there exists $m \in \mathbb{N}$ such that $b^m\varphi$ is in $B[a](a - \beta b)\varphi$ which is contained in $\Xi_{[\alpha]}^{(M-1)}$. This is clearly impossible. Then the rank of $B[a]\varphi$ is $M+1$.■

Remark. If, in the situation of the previous lemma, $\varphi = S(b)s^{\beta-1}(Log\, s)^{M} + \psi$ for some S invertible in B, then φ also generates a $[\alpha]$-primitive theme.

Proof of Proposition 5.2.1. As $\Xi_{\mathscr{A}}^{(0)} \simeq \oplus_{\alpha \in \mathscr{A}}\mathcal{E}_{\alpha}$ it is clear that $\Xi_{\mathscr{A}}^{(0)} \otimes V$ is semi-simple as a finite direct sum of such $\mathcal{E}_{\alpha}$. So it is contained in $S_1(\mathcal{E})$.
Conversely, if $\varphi \in \Xi_{\mathscr{A}}^{(N)} \otimes V$ has degree at least equal to 1 in $Log\, s$, then Corollary 5.2.4 implies (by choosing a convenient linear form on V) that some $[\alpha]$-primitive sub-module $B[a]\varphi$ has a $[\alpha]$-primitive quotient theme of rank ≥ 2 for some $[\alpha] \in \mathscr{A}$,

so is not semi-simple (see Corollary 5.1.5). So $S_1(\mathcal{E}) = \Xi^{(0)}_{\mathscr{A}} \otimes V$.
Assume that $j \geq 1$ and that we have proved the equality $S_j(\Xi^{(N)}_{\mathscr{A}} \otimes V) = \Xi^{(j-1)}_{\mathscr{A}} \otimes V$. Now, as $\Xi^{(j)}_{\mathscr{A}} \otimes V / \Xi^{(j-1)}_{\mathscr{A}} \otimes V$ is semi-simple, since it is isomorphic to $\Xi^{(0)}_{\mathscr{A}} \otimes V$, we obtain that $\Xi^{(j)}_{\mathscr{A}} \otimes V \subset S_{j+1}(\Xi^{(N)}_{\mathscr{A}} \otimes V)$ by the definition of $S_{j+1}(\mathcal{E})$. To complete our induction it is enough to prove that if $\varphi \in \Xi^{(N)}_{\mathscr{A}} \otimes V$ has degree $j+1$ in $Log\, s$ then φ is not in $S_{j+1}(\Xi^{(N)}_{\mathscr{A}} \otimes V)$. But under this hypothesis $B[a]\varphi$, thanks to Lemma 5.2.4 admits a quotient which is a rank $j+2$ theme.
Then thanks to Corollary 5.1.6 $d(B[a]\varphi) = j+2$ and φ is not in $S_{j+1}(\Xi^{(N)}_{\mathscr{A}} \otimes V)$.■

As a consequence of the previous proposition, using Lemma 4.2.5, we obtain that for any sub-module $\mathcal{E} \subset (\Xi^{(N)}_{\alpha} \otimes V)$ the equality $S_j(\mathcal{E}) = \mathcal{E} \cap (\Xi^{(j-1)}_{[\alpha]} \otimes V)$ for each $j \in [1, d(\mathcal{E})]$.

The next proposition is also a complement to the Embedding Theorem 5.1.3.

Proposition 5.2.5 *Let $\mathcal{E}$ be a geometric (a,b)-module and assume that $S_1(\mathcal{E})$ may be embedded in $\Xi^{(0)}_{\mathscr{A}} \otimes V$. Then we can extend this embedding to an embedding of $\mathcal{E}$ in $\Xi^{(N)}_{\mathscr{A}} \otimes V$ with $N := d(\mathcal{E}) - 1$.*

Proof. To simplify the notation, we shall write $d := d(\mathcal{E})$ the nilpotent order of $\mathcal{E}$. Remark that it is enough to prove the result when $\mathcal{E}$ has a simple pole because any embedding of a geometric (a,b)-module in some $\Xi^{(N)}_{\mathscr{A}} \otimes V$ extends to an embedding of $\mathcal{E}^{\sharp}$, as any $\Xi^{(N)}_{\mathscr{A}} \otimes V$ has a simple pole.
Now as a geometric simple pole (a,b)-module decomposes as a direct sum of its $[\alpha]$-primitive parts when α describes $\mathbb{Q} \cap]0, 1]$, we may assume that $\mathcal{E}$ is $[\alpha]$-primitive.
We shall prove the result by induction on d, assuming that $\mathcal{E}$ has a simple pole and is $[\alpha]$-primitive (see Theorem 3.3.7). The case $d = 1$ being trivial, assume $d \geq 2$ and that we have already proved the case $d-1$. Then the inductive assumption gives that we have already extend our embedding $\varphi : S_{d-1}(\mathcal{E}) \to \Xi^{(d-2)}_{[\alpha]} \otimes V$. Then we shall now make an induction on the rank of $\mathcal{E}/S_{d-1}(\mathcal{E})$.
First assume that this rank is 1. Then let $e \in \mathcal{E}$ which is send to the the standard generator of $\mathcal{E}/S_{d-1}(\mathcal{E}) \simeq \mathcal{E}_{\beta}$ (so $(a - \beta b)e$ is in $S_{d-1}(\mathcal{E})$) where β is in $\alpha + \mathbb{N}$. Note that $[\alpha]$ is in $\mathscr{A}$ because $-\mathscr{A}$ contains the class modulo $\mathbb{Z}$ of any root of the Bernstein polynomial of $\mathcal{E}$. As we assume that $\mathcal{E}$ has simple pole, $(a - \beta b)e$ is in $S_{d-1}(\mathcal{E}) \cap b\mathcal{E} = bS_{d-1}(\mathcal{E})$ since $S_{d-1}(\mathcal{E})$ is normal in $\mathcal{E}$. Then $\varphi(a - \beta b)e)$ is in $b\Xi^{(d-2)}_{[\alpha]} \otimes V$ and applying Lemma 5.1.4 we may find $\varepsilon \in \Xi^{(d-1)}_{[\alpha]} \otimes V$ such that $(a - \beta b)\varepsilon = \varphi(e)$. Then we can extend φ to an embedding of $\mathcal{E}$ by sending e to ε as in the proof of the Embedding Theorem.
To complete our induction on the rank of $\mathcal{E}/S_{d-1}(\mathcal{E})$, we have to prove the case of the rank of $\mathcal{E}/S_{d-1}(\mathcal{E})$ is equal to $k \geq 2$, assuming that the case of rank $k-1$ is already proved,
Let $\mathcal{F}$ be a co-rank 1 normal sub-module containing $S_{d-1}(\mathcal{E})$. This is easily obtained by considering a J-H. sequence for $\mathcal{E}/S_{d-1}(\mathcal{E})$. Then we have $S_{d-1}(\mathcal{F}) = S_{d-1}(\mathcal{E})$

and $S_d(\mathcal{F}) = \mathcal{F}$. The rank of $\mathcal{F}/S_{d-1}(\mathcal{F})$ is $k-1$ so our inductive assumption gives an embedding $\varphi : \mathcal{F} \to \Xi_{[\alpha]}^{(d-1)} \otimes V$. Define $\gamma \in \mathbb{Q}^{*+}$ by $\mathcal{E}/\mathcal{F} \simeq \mathcal{E}_\gamma$ and as in the proof of the Embedding Theorem (note that $\mathcal{F}$ has a simple pole because it is normal in $\mathcal{E}$ which has a simple pole) let $e \in \mathcal{E}$ inducing the standard generator of $\mathcal{E}_\gamma$ via the quotient map $\mathcal{E} \to \mathcal{E}/\mathcal{F}$. Then $(a - \gamma b)e$ is in $\mathcal{F}$ and in fact in $b\mathcal{F}$ using the simple pole assumption, so that $\varphi(e)$ is in $b\Xi_{[\alpha]}^{(d-1)} \otimes V$ and, thanks to Lemma 5.1.4 we may find $\varepsilon \in \Xi_{[\alpha]}^{(d)} \otimes V)$ which satisfies $(a - \gamma b)\varepsilon = \varphi(e)$. Then, as in the proof of the Embedding Theorem, this allows to define an extension $\tilde{\varphi} : \mathcal{E} \to \Xi_{[\alpha]}^{(d)} \otimes V$ by putting $\tilde{\varphi}(e) = \varepsilon$. This extension is injective because its kernel has rank at most 1 so is contained in $S_1(\mathcal{E}) \subset \mathcal{F}$. Moreover, if ε is not in $\Xi_\alpha^{(d-1)} \otimes V$ then $B[a].\varepsilon$ is a rank $d+1$ theme, thanks to Lemma 5.2.4, and then $\mathcal{T} := B[a].e \subset \mathcal{E}$ is a $d+1$-theme in $\mathcal{E}$, so $d(\mathcal{T}) = d+1$ contradicting the fact that $S_d(\mathcal{E}) = \mathcal{E}$. So $\tilde{\varphi}$ is an embedding of $\mathcal{E}$ in $\Xi_\alpha^{(d-1)} \otimes V$ concluding the proof. ■

We conclude this sub-section by the following characterization of $[\alpha]$-primitive themes (compare with Corollary 5.1.6)..

Proposition 5.2.6 *A fresco $\mathcal{E}$ is an $[\alpha]$-primitive theme, for some $\alpha \in]0,1] \cap \mathbb{Q}$, if and only if it has at most one rank 1 normal sub-module.*
In this case, it has, for each $j \in [0,k]$, a unique normal rank j sub-module, where k is the rank of $\mathcal{E}$.

Proof. By definition an $[\alpha]$-primitive theme $\mathcal{E}$ is a sub-module of the type $\tilde{\mathcal{A}}_{conv.}\varphi$ where φ is in a module $\Xi_\alpha^{(N)}$ for some $N \in \mathbb{N}$. Then $S_1(\mathcal{E}) \subset S_1(\Xi_\alpha^{(N)}) \simeq \mathcal{E}_\alpha$. So if $\mathcal{E} \neq 0$ the rank of $S_1(\mathcal{E})$ is 1 and this implies the uniqueness of the normal rank 1 sub-module in $\mathcal{E}$.
The quotient $\mathcal{E}/S_1(\mathcal{E})$ is again a theme, so, using the equality

$$\Xi_\alpha^{(N)}/S_1(\Xi_\alpha^{(N)}) \ \simeq \ \Xi_\alpha^{(N-1)},$$

and an induction of the rank j of a normal sub-module of $\mathcal{E}$, we obtain the uniqueness of a normal rank j sub-module of $\mathcal{E}$ for each $j \in [0,k]$ where k is the rank of $\mathcal{E}$.
Conversely if $0 \neq \mathcal{E} \subset \Xi_{\mathscr{A}}^{(N)} \otimes V$ is a fresco which has a unique rank 1 sub-module. Then $S_1(\mathcal{E}) = \mathcal{E} \cap (\Xi_{\mathscr{A}}^{(0)} \otimes V)$ is rank 1 and we may embed $\mathcal{E}$ with an $\mathscr{A}$ which has only one element α and with a 1-dimensional V, thanks to Proposition 5.2.5.
So $\mathcal{E}$ is an $[\alpha]$-primitive theme. ■

6 Monodromy and the semi-simple filtration

6.1 Monodromy

The goal of the present sub-section is to define the action of the nilpotent part of the logarithm of the monodromy (logarithm given by $2i\pi b^{-1}a$) on a simple pole ge-

ometric (a,b)-module and to show that the semi-simple filtration of any geometric (a,b)-module $\mathcal{E}$ coincides with the filtration induced on $\mathcal{E}$ by the successive kernels of this nilpotent part acting on $\mathcal{E}^\sharp$.

The first remark is that in a simple pole (a,b)-module the $\mathbb{C}$-linear (bijective) endomorphism $u := b^{-1}a$ satisfies the following commutations relations:

$$ua - au = a \quad \text{and} \quad ub - bu = b.$$

Lemma 6.1.1 *Let X be a $\mathbb{C}$-algebra with unit and let u and x elements in X satisfying $ux - xu = x$. Then for each $n \in \mathbb{N}$ we have $u^n x - xu^n = x((1+u)^n - u^n)$.*

Proof. The result is clear for $n = 0, 1$ so assume it is already proved for $n-1$ with $n \geq 2$. Then we have:

$u^n x - xu^n = u(xu^{n-1} + x(1+u)^{n-1} - xu^{n-1}) - xu^n$
$u^n x - xu^n = ux(1+u)^{n-1} - xu^n = x(1+u)^n - xu^n$ ■

Note that if z is a complex number then, replacing x by $z^n x$ gives

$$(zu)^n x - x(zu)^n = x((z(u+1))^n - (zu)^n) \quad \forall n \in \mathbb{N}.$$

Assume now that the series $\sum_{j=0}^{\infty}(2i\pi u)^n/n!$ converges in the algebra X having a topology for which the product is continuous. Then we have

$$\begin{aligned}\exp(2i\pi u)x - x\exp(2i\pi u) &= \sum_{j=0}^{\infty}[(2i\pi u)^n/n!]x - x[(2i\pi u)^n/n!] \\ &= x\big[\sum_{j=0}^{\infty}(2i\pi(u+1))^n/n! - \sum_{j=0}^{\infty}(2i\pi u)^n/n!\big] \\ &= x\big[\exp(2i\pi(u+1)) - \exp(2i\pi u)\big] = 0,\end{aligned}$$

which gives that $\exp(2i\pi u)$ and x commute.

Lemma 6.1.2 *Let α be a rational number in $]0,1]$ and N a non negative integer. Then the $\mathbb{C}$-linear map $\mathcal{T} := \exp(2i\pi b^{-1}a) := \sum_{q=0}^{\infty}(2i\pi b^{-1}a)^q/q!$ is well defined on $\Xi^N_{[\alpha]}$ and is $B[a]$-linear (and also A-linear). It is induced by the standard monodromy around 0 given by $Log\, s \mapsto Log\, s + 2i\pi$.*

Proof. We have $\mathcal{N}^{N+1} = 0$ and $b^{-1}a = (\Delta + \mathscr{N})$ as a $\mathbb{C}$-linear endomorphisms of $\Xi^{(N)}_\alpha$ where

$$\Delta(s^\beta \frac{(Log\, s)^j}{j!}) = \beta s^\beta \frac{(Log\, s)^j}{j!}$$

for any $\beta \in \alpha + \mathbb{N}$. So $\exp(2i\pi\Delta) = \exp(2i\pi\alpha)$ on $\Xi^N_{[\alpha]}$ and since Δ commutes with $\mathscr{N}$, we obtain:

$$\exp(-2i\pi\alpha)\exp(2i\pi b^{-1}a) = \exp(2i\pi\mathscr{N}) = \sum_{p=0}^{N+1} \frac{(2i\pi\mathscr{N})^p}{p!}.$$

Evaluation at $s^\beta \frac{(Log\, s)^j}{j!}$ gives, since by definition $\mathcal{T} := \exp(2i\pi b^{-1}a)$:

$$\mathcal{T}(s^\beta \frac{(Log\, s)^j}{j!}) = \exp(2i\pi\alpha)\sum_{p=0}^{j}(2i\pi)^p\ s^\beta \frac{(Log\, s)^{j-p}}{(j-p)!p!} = \exp(2i\pi\alpha)s^\beta \frac{((Log\, s)+2i\pi)^j}{j!}$$

thanks to the binomial formula.
Then we obtain, since $\exp(2i\pi\mathscr{N}) = (\exp(-2i\pi\alpha))\mathcal{T}$, the equality:

$$\exp(2i\pi\mathscr{N})(s^\beta \frac{(Log\, s)^j}{j!}) = s^\beta \frac{(Log\, s + 2i\pi)^j}{j!}.$$

. ■

Corollary 6.1.3 *For any simple pole geometric (a,b)-module the automorphism $\mathcal{T} := \exp(2i\pi b^{-1}a)$ is well defined and is $B[a]$-linear (and $\tilde{\mathcal{A}}_{conv.}$-linear). It is compatible with any $\tilde{\mathcal{A}}_{conv.}$-linear map between geometric simple pole (a,b)-modules.*

Proof. The preceding lemma extends immediately to any $\Xi^{(N)}_{\mathscr{A}} \otimes V$ and then, using the Embedding Theorem 5.1.3 the result is clear. ■

Definition 6.1.4 *The $\tilde{\mathcal{A}}_{conv.}$-linear automorphism $\mathcal{T} := \exp(2i\pi b^{-1}a)$ of a simple pole geometric (a,b)-module $\mathcal{E}$ is called the* **monodromy automorphism** *of $\mathcal{E}$.*

6.2 A direct construction in the formal case

We give now a direct approach to the monodromy of a formal simple pole geometric (a,b)-module which does not use the Embedding Theorem. The convergent case seems more difficult to obtain in this way because the convergence of the series defining $\exp(2i\pi b^{-1}a)$ for a simple pole *convergent* (a,b)-module is not obvious.

Lemma 6.2.1 *Let E be a simple pole α-primitive formal (a,b)-module which is geometric, where α is in $\mathbb{Q}\cap]0,1]$. Define $N := \exp(-2i\pi(\alpha - b^{-1}a)) - 1$. Then for each $x \in E$ the series*

$$2i\pi\mathcal{N}(x) := \sum_{p=1}^{\infty}(-1)^p \frac{N^p}{p}(x)$$

converges in E and $\mathcal{N} : E \to E$ is a $\tilde{\mathcal{A}}$-linear endomorphism of E which satisfies $\mathcal{N}^k = 0$ where k is the rank of E.

PROOF. Since $\exp(-2i\pi(\alpha - b^{-1}a))$ is a unipotent automorphism of E/bE, N is nilpotent on E/bE and then $N^k(E) \subset bE$ where k is the rank of E. So the series converges for the b-adic filtration and $\mathcal{N}(x)$ is well defined for any $x \in E$. The commutation relations $[b^{-1}a, a] = a$ and $[b^{-1}a, b] = b$ in the $\mathbb{C}$-linear algebra endomorphisms of E implies the $\tilde{\mathcal{A}}$-linearity of N and then, of $\mathcal{N}$.
Then, since $\mathcal{N}$ is nilpotent on E/bE the b-linearity implies that $\mathcal{N}^k = 0$ on E. Moreover if $f : E \to F$ is a $\tilde{\mathcal{A}}$-linear map between two geometric simple poles formal (a,b)-modules which are α-primitive, the fact that

$$bf(b^{-1}ax) = f(ax) = af(x) = b(b^{-1}a)f(x)$$

implies that $f(\mathcal{N}(x)) = \mathcal{N}(f(x))$.

Definition 6.2.2 *Let E be a simple pole geometric formal (a,b)-module. We define the nilpotent endomorphism $\mathcal{N}$ on E using the direct sum decomposition*

$$E = \oplus_{\alpha \in \mathscr{A}} E_{[\alpha]}$$

where $\mathscr{A}$ is the subset of $\mathbb{Q} \cap]0, 1]$ of class modulo $\mathbb{Z}$ of the eigenvalues of the action of $b^{-1}a$ on E/bE. Then $\mathcal{N}$ is the direct sum of the various $\mathcal{N}$ for each $E_{[}\alpha]$. It satisfies $\mathcal{N}^k = 0$ where k is the rank of E.
Then we define the $\mathbb{C}$-linear automorphism Δ of E by the formula

$$\Delta := b^{-1}a - \mathcal{N}.$$

REMARKS.

1. Since $b^{-1}a$ and $\mathcal{N}$ commute, Δ commutes with $b^{-1}a$ and $\mathcal{N}$.
2. The commutation relations $[b^{-1}a, a] = a$, $[b^{-1}a, b] = b]$ and the $\tilde{\mathcal{A}}$-linearity of $\mathcal{N}$ implies the commutation relations
$$[\Delta, a] = a \quad \text{and} \quad [\Delta, b] = b.$$
3. For any $\tilde{\mathcal{A}}$-linear map $f : E \to F$ between simple poles geometric (a,b)-modules we have $f \circ \Delta = \Delta \circ f$.
4. Δ is bijective as a consequence of the bijectivity of $b^{-1}a$ and the nilpotence of $\mathcal{N}$ proved in the lemma above.

CONCLUSION. Let E a simple pole geometric formal (a,b)-module. Defining the monodromy automorphism on $E_{[\alpha]}$, the $[\alpha]$-primitive part of E by the formula

$$\exp(-2i\pi\alpha)\mathcal{T} = \exp(2i\pi\mathcal{N})$$

the Decomposition Theorem allows to define $\mathcal{T}$ on E as the direct sum of the monodromies of the $E_{[\alpha]}$ where α describes the image of the opposite of the roots of the Bernstein polynomial of E in $\mathbb{Q}/\mathbb{Z}$.

6.3 Nilpotent order

We come back to the study of *convergent* geometric (a,b)-modules.

Lemma 6.3.1 *Let $\mathcal{E}$ be a simple pole geometric (a,b)-module which is $[\alpha]$-primitive. Then $\mathscr{N}^d = 0$ if $d = d(\mathcal{E})$ is the nilpotent order of $\mathcal{E}$. Moreover $\mathscr{N}^{d-1} \neq 0$.*

PROOF. First of all remark that $\mathcal{T}$ is an automorphism of $\mathcal{E}$ so that it sends $S_j(\mathcal{E})$ bijectively on itself for any j. Moreover, for any normal sub-module $\mathcal{F}$ of $\mathcal{E}$ (so, in particular, for any $S_j(\mathcal{E})$) $\mathcal{T}$ induces the monodromy automorphism of $\mathcal{F}$ because $\mathcal{F}$ is stable by $b^{-1}a$.
Now when $\mathcal{E}$ is semi-simple, as the monodromy of any rank 1 simple pole (a,b)-module $\mathcal{E}_\beta$ for $\beta \in \alpha + \mathbb{Z}$ is the product by $\exp(2i\pi\alpha)$, $\mathcal{T}$ is also the mutiplication by $\exp(2i\pi\alpha)$ and the $\mathscr{N} = 0$.
When $\mathcal{E}$ is any simple pole geometric (a,b)-module which is $[\alpha]$-primitive, any quotient $S_j(\mathcal{E})/S_{j-1}(\mathcal{E})$ is also $[\alpha]$-primitive and this implies that $\mathscr{N}(S_j(\mathcal{E})) \subset S_{j-1}(\mathcal{E})$ for any j and then $\mathscr{N}^d = 0$ for $d = d(\mathcal{E})$.
To prove the second part of the lemma, remark that the endomorphism $\mathscr{N}$ of $\Xi_{[\alpha]}^{(N)} \otimes V$ has its image in $\Xi_{[\alpha]}^{(N-1)} \otimes V$. So, as an $B[a]$-linear embedding of a simple pole (a,b)-module $\mathcal{E}$ (then necessarily geometric and $[\alpha]$-primitive) in $\Xi_{[\alpha]}^{(N)} \otimes V$ commutes with the respective monodromies, using the case $N = d(\mathcal{E}) - 1$ thanks to Proposition 5.2.5, we see that $\mathscr{N}^{d-1}(\mathcal{E}) = 0$ implies that $\mathcal{E} \subset \Xi_{[\alpha]}^{(d-2)} \otimes V$ which forces $d(\mathcal{E}) \leq d-1$, contradicting our assumption that $d(\mathcal{E}) = d$. This conclude the proof.■

Let $\mathcal{E}$ be a simple pole geometric (a,b)-module. Then there exists a finite subset $\mathscr{A}$ in $\mathbb{Q} \cap]0, 1]$ such that $\mathcal{E}$ is $\mathscr{A}$-primitive. Then define the $B[a]$-linear endomorphism $\mathcal{N}$ of $\mathcal{E}$ by using the direct sum decomposition of $\mathcal{E}$ (see Theorem 3.3.7):

$$\mathcal{E} \simeq \oplus_{\alpha \in \mathscr{A}} \mathcal{E}_{[\alpha]}$$

Then define $\mathscr{N}$ as the direct sum of the endomorphism $\mathscr{N}$ on each $\mathcal{E}_{[\alpha]}, \alpha \in \mathscr{A}$.
Define also Δ on $\mathcal{E}$ as $b^{-1}a + \mathscr{N}$.
Then the following theorem is an easy consequence of the previous lemma using the fact that $S_d(\Xi_{[\alpha]}^{(N)} \otimes V) = \Xi_{[\alpha]}^{(d-1)} \otimes V$ and Lemma 4.2.5.

Theorem 6.3.2 *Let $\mathcal{E}$ be a geometric (a,b)-module and let $\mathcal{N}$ be the nilpotent part of the monodromy acting on $\mathcal{E}^\sharp$. Then intersection with $\mathcal{E}$ of the kernel of $\mathscr{N}^j$ is equal to $S_j(\mathcal{E})$ for all $j \in [0, d]$. So $d = d(\mathcal{E}) = d(\mathcal{E}^\sharp)$ is the nilpotent order of the action of the monodromy on $\mathcal{E}^\sharp$.* ■

In other terms the previous corollary explains that the **semi-simple filtration** of a geometric (a,b)-module coincides with the filtration induced on $\mathcal{E}$ by the successive kernels of the powers of nilpotent part of the monodromy of $\mathcal{E}^\sharp$.
Note that, in general, the inclusion $\mathscr{N}(S_j(\mathcal{E})) \subset S_{j-1}(\mathcal{E})$ is not an equality for a

simple pole geometric (a,b)-module. For instance if $\mathcal{E} = \mathcal{F} \oplus \mathcal{G}$ where $\mathcal{F}$ is semi-simple and $\mathcal{G}$ is not semi-simple with $d(\mathcal{G}) = 2$, $\mathscr{N}(\mathcal{E}) = \mathscr{N}(\mathcal{G}) \subset S_1(\mathcal{G})$ is strictly contained in $S_1(\mathcal{E}) = \mathcal{F} \oplus S_1(\mathcal{G})$ when $\mathcal{F} \neq \{0\}$.

We collect in the following proposition the main tools we have obtained to compute the nilpotent order of a geometric (a,b)-module.

Proposition 6.3.3 *Let $\mathcal{E}$ be a geometric (a,b)-module. Then we have the following properties :*

i) For each subset $\mathscr{A} \in \mathbb{Q} \cap]0,1]$ we have $S_j(\mathcal{E}_{[\mathscr{A}]}) = S_j(\mathcal{E})_{[\mathscr{A}]}$, for each $j \geq 1$.

ii) Any $[\alpha]-$primitive sub-theme $\mathcal{T}$ in $\mathcal{E}$ of rank j is contained in $S_j(\mathcal{E})$.

iii) Any $[\alpha]-$primitive quotient theme $\mathcal{T}$ of $S_j(\mathcal{E})$ has rank $\leq j$.

iv) The nilpotent order of $\mathcal{E} = \mathcal{E}_{[\mathscr{A}]}$ is equal to d if and only if d is the maximal rank of an $[\alpha]-$primitive quotient theme of $\mathcal{E}$ where α is in $\mathscr{A}$.

v) The nilpotent order of $\mathcal{E} = \mathcal{E}_{[\mathscr{A}]}$ is equal to d if and only if d is the maximal rank of an $[\alpha]-$primitive sub-theme of $\mathcal{E}$ where α is in $\mathscr{A}$. ■

Lemma 6.3.4 *Let $\mathcal{F}$ and $\mathcal{G}$ be two sub-modules of a geometric (a,b)-module. Assume that $d(\mathcal{F}) \leq p$ and $d(\mathcal{G}) \leq p$. Then $d(\mathcal{F} + \mathcal{G}) \leq p$.*

Proof. Assume that we have a surjective (a,b)-linear map $\pi : \mathcal{F} + \mathcal{G} \to T$ where T is an $[\alpha]$-primitive theme of rank $q > p$. Then let e be a generator of T as a $\tilde{\mathcal{A}}_{conv.}$-module and let $u \in \mathcal{F}$ and $v \in \mathcal{G}$ such that $e = \pi(u+v)$. Note $T_1 := \pi(\tilde{\mathcal{A}}_{conv.}u)$ and $T_2 := \pi(\tilde{\mathcal{A}}_{conv.}v)$. Let $\tilde{T}_i$ be the normalization of T_i in T for $i = 1, 2$. Then if $\tilde{T}_i \neq T$ for $i = 1, 2$ these two normal sub-module are contained in the co-rank 1 sub-module of T, and this is not possible since $T_1 + T_2 = T$.
So we have, for instance $\tilde{T}_1 = T$. But this means that there exists $n \in \mathbb{N}$ such that $b^n e$ is in T_1. Then $b^n T$ is contained in T_1 and so $\mathcal{F}$ contains the sub-module $\pi^{-1}(b^n T)$ which admits $b^n T$ as a quotient. But $b^n T$ is a rank q $[\alpha]$-primitive theme and then $d(\tilde{\mathcal{A}}_{conv.}u) \geq q$. This contradicts our hypothesis that $d(\mathcal{F}) \leq p$ since we assume that $q > p$.
So any quotient theme of $\mathcal{F} + \mathcal{G}$ has rank at most equal to p, concluding the proof, thanks to Proposition 6.3.3. ■

Corollary 6.3.5 *Let $\mathcal{E}$ be a geometric (a,b)-module. For each integer $j \geq 0$, $S_j(\mathcal{E})$ is the subset of all $x \in \mathcal{E}$ such that $d(\tilde{\mathcal{A}}_{conv.}x) \leq j$.*

PROOF. It is clear that $x \in S_j(\mathcal{E})$ implies $d(\tilde{\mathcal{A}}_{conv.}x) \leq j$. Conversely, let $x \in \mathcal{E}$ with $d(\tilde{\mathcal{A}}_{conv.}x) \leq j$. Then, thanks to the previous lemma, we have

$$d(\tilde{\mathcal{A}}_{conv.}x + S_j(\mathcal{E})) \leq j$$

and this implies that $\tilde{\mathcal{A}}_{conv.}x + S_j(\mathcal{E}) \subset S_j(\mathcal{E})$ because $S_j(\mathcal{E})$ is the maximal sub-module in $\mathcal{E}$ with nilpotent order equal to j (see remark ii) following Lemma 4.2.5). This is enough to conclude. ■

Lemma 6.3.6 *Let $\mathcal{E}$ be a geometric (a,b)-module. Then the nilpotent orders of $\mathcal{E}_{[\alpha]}$ and of $\mathcal{E}/\mathcal{E}_{[\neq\alpha]}$ are the same.*

PROOF. We have an obvious inclusion $\mathcal{E}_{[\alpha]} \subset \mathcal{E}/\mathcal{E}_{[\neq\alpha]}$ since the intersection of $\mathcal{E}_{[\alpha]}$ with $\mathcal{E}_{[\neq\alpha]}$ is $\{0\}$. Let us prove that the image of this injection has finite co-dimension. Thanks to the Decomposition Theorem we know that for $\mathcal{E}^\sharp$ we have

$$(\mathcal{E}^\sharp)_{[\alpha]} \oplus (\mathcal{E}^\sharp)_{[\neq\alpha]} = \mathcal{E}^\sharp.$$

So the result is clear for $\mathcal{E}^\sharp$.
An other consequence of the Decomposition Theorem is the fact that for each α we have equality of the ranks of $\mathcal{E}_{[\alpha]}$ and of $(\mathcal{E}^\sharp)_{[\alpha]}$:
Indeed the inequality

$$\sum_\alpha rk((\mathcal{E}^\sharp)_{[\alpha]}) = rk(\mathcal{E}^\sharp) = rk(\mathcal{E}) \geq \sum_\alpha rk(\mathcal{E}_{[\alpha]})$$

implies the equality of $rk(\mathcal{E}_{[\alpha]})$ with $rk((\mathcal{E}^\sharp)_{[\alpha]})$ for each α. So the inclusion of $\mathcal{E}_{[\alpha]} \subset (\mathcal{E}^\sharp)_{[\alpha]}$ implies for each α the equality

$$rk(\mathcal{E}_{[\alpha]}) = rk((\mathcal{E}_{[\alpha]})^\sharp) = rk((\mathcal{E}^\sharp)_{[\alpha]}).$$

This implies the equality of the ranks of $\mathcal{E}_{[\alpha]}$ and $\mathcal{E}/\mathcal{E}_{[\neq\alpha]}$.
So, since for each $j \geq 1$, $S_j(\mathcal{E}_{[\alpha]}) = S_j(\mathcal{E}/\mathcal{E}_{[\neq\alpha]}) \cap \mathcal{E}_{[\alpha]}$ (see Lemma 4.2.5) has finite co-dimension in $S_j(\mathcal{E}/\mathcal{E}_{[\neq\alpha]})$, they have the same rank. This is enough to conclude the proof since a normal sub-module of finite co-dimension in a rank k (a,b)-module $\mathcal{E}$ is equal to $\mathcal{E}$. ■

7 Higher Bernstein polynomials

7.1 Frescos

Recall that a fresco is a geometric (a,b)-module which has one generator as a $B[a]$-module (or as a $\tilde{\mathcal{A}}_{conv.}$-module). So a geometric (a,b)-module $\mathcal{E}$ is a fresco if and only if $\mathcal{E}/a\mathcal{E} + b\mathcal{E}$ is a complex vector space with dimension ≤ 1.
We begin by the convergent analog of the structure theorem for frescos given in [9] Theorem 3.4.1, in the formal case.

Theorem 7.1.1 *If $\mathcal{E}$ is a geometric (a,b)-module, for each x in $\mathcal{E}$ the left annihilator of x in $\tilde{\mathcal{A}}_{conv.}$ has a generator of the form*

$$P = (a - \lambda_1 b)S_1^{-1}(a - \lambda_2 b)\dots(a - \lambda_k b)S_k^{-1}$$

where $\lambda_1, \dots, \lambda_k$ are positive rational numbers satisfying $\lambda_j + j > k$, where $S_1, \dots, S_k$ are invertible elements in B and where k is the rank of $\tilde{\mathcal{A}}_{conv.}x \subset \mathcal{E}$ (as a B-module).

PROOF. Let $\mathcal{F} := \tilde{\mathcal{A}}_{conv.}x \subset \mathcal{E}$ and let k the rank of $\mathcal{F}$. We shall prove, by induction on $k \geq 1$ that the annihilator of x in $\mathcal{E}$ has the announced form. So assume that $k \geq 2$ and the case where $\mathcal{F}$ has rank $k-1$ is already proved (the case $k = 1$ is clear !).
Let F_{k-1} be a normal rank $(k-1)$ sub-module of $\mathcal{F}$. Such a sub-module exists thanks to Proposition 3.4.2 which gives the existence of a J-H. sequence for any regular (a,b)-module. Then there exists a positive rational number α such that $\mathcal{F}/F_{k-1}$ is isomorphic to $\mathcal{E}_\alpha$ since $-\alpha$ is a root of $B_{\mathcal{F}}$ and since we assume that $\mathcal{E}$ (and then $\mathcal{F}$) is geometric. Let e_α be the standard generator of $\mathcal{F}/F_{k-1}$ (so $(a - \alpha b)e_\alpha = 0$ and $\mathcal{E}_\alpha = Be_\alpha$). Then the image of x in $\mathcal{F}/F_{k-1}$ is equal to $S_k e_\alpha$ where S_k is an invertible element in B, since this image of x must generate $\mathcal{E}_\alpha$. Let $y := (a - \alpha b)S_k^{-1}x$. Then y is in F_{k-1}.

CLAIM. y is a generator of F_{k-1} as a $B[a]$-module.

PROOF OF THE CLAIM. Let z be an element in F_{k-1}. Since $S_k^{-1}x$ is a generator of $\mathcal{F}$ we may write $z = uS_k^{-1}x$ for some $u \in B[a]$. Now write $u = Q(a - \alpha b) + R$ where Q is in $B[a]$ and $R \in B$. This implies

$$z = (Q(a - \alpha b) + R)S_k^{-1}x = Qy + RS_k^{-1}x.$$

But the image of z in $\mathcal{F}/F_{k-1} \simeq \mathcal{E}_\alpha$ is zero and this implies $RS_k^{-1}e_\alpha = 0$ and so $R = 0$. Then we have $z = Qy$ proving the claim.
So F_{k-1} is a rank $k-1$ fresco and the annihilator in $\tilde{\mathcal{A}}_{conv.}$ of y in $\mathcal{E}$ is some P of the desired form (and rank $k-1$). Let $\Pi := P(a - \lambda_k b)S_k^{-1}$ where $\lambda_k := \alpha$. Then, to complete the proof, it is enough to show that Π generates the left ideal of $\tilde{\mathcal{A}}_{conv.}$ which is the annihilator of x in $\mathcal{E}$.
So let $v \in \tilde{\mathcal{A}}_{conv.}$ such that $vx = 0$. Then the image of vx in $\mathcal{E}_\alpha$ vanishes and so vS_k is in the ideal $\tilde{\mathcal{A}}_{conv.}(a - \lambda_k b)$ which is the annihilator of e_α. Then write $v = z(a - \lambda_k b)S_k^{-1}$ where z is in $\tilde{\mathcal{A}}_{conv.}$. Then $z(a - \lambda_k b)S_k^{-1}x = zy = 0$. So we may write $z = wP$ by our inductive assumption and this give $v = w\Pi$ and v is in the left ideal $\tilde{\mathcal{A}}_{conv.}\Pi$. ■

The following corollaries are direct consequences of the proof of Theorem 7.1.1. They are the convergent analogs of results of [9] proved in the formal setting for frescos.

Corollary 7.1.2 *Let $\mathcal{F}$ be a rank k fresco. Then there exists positive rational numbers $\lambda_1, \ldots, \lambda_k$ and invertible elements $S_1, \ldots, S_{k-1}$ in B such that $\mathcal{F}$ is isomorphic to $\tilde{\mathcal{A}}_{conv.}/\tilde{\mathcal{A}}_{conv.}P$ where $P := (a - \lambda_1 b)S_1^{-1}(a - \lambda_2 b)\ldots S_{k-1}^{-1}(a - \lambda_k b)$.*
Moreover we have $\lambda_j > k - j$ for each $j \in [1, k]$. ■

The following corollary of the proof above implies that the action of $\tilde{\mathcal{A}}_{conv.}$ on a geometric (a,b)-module may be reduced to the action of the sub-algebra $B[a]$ and even with a degree in a bounded by $k - 1$ where k is the rank of the geometric (a,b)-module we consider.

Corollary 7.1.3 *Let $\mathcal{F}$ be rank k fresco with generator x. Then each $y = ux$ in $\mathcal{F}$ may be written in unique way as $y = vx$ where v is a polynomial of degree $k - 1$ in $B[a]$.* ■

Note that if $e_1, \ldots, e_k$ is a B-basis of a geometric (a,b)-module $\mathcal{E}$, the corollary above shows that any $x \in \mathcal{E}$ may be written as $\sum_{j=1}^{k} u_j e_j$ where each u_j is a polynomial in $B[a]$ of degree at most $k - 1$.

It will be useful to note that the proof of the theorem above implies the following basic fact.

Corollary 7.1.4 *Let $\mathcal{E}$ be a fresco. If $\mathcal{F}$ is a normal sub-module of $\mathcal{E}$, $\mathcal{F}$ and $\mathcal{E}/\mathcal{F}$ are frescos.* ■

The next proposition is the analog in the convergent case of results in Section 3.4 of [9]. Its proof is the same.

Proposition 7.1.5 *Let $\mathcal{F} = \tilde{\mathcal{A}}_{conv.}/\tilde{\mathcal{A}}_{conv.}P$ be a rank k fresco as described in Corollary 7.1.2. The Bernstein polynomial of $\mathcal{F}$ is the* **characteristic** *polynomial of $-b^{-1}a$ acting on $\mathcal{F}^\sharp/b\mathcal{F}^\sharp$. And the* **Bernstein element** *$P_{\mathcal{F}}$ of $\mathcal{F}$, defined as the initial form in (a, b) of P, $P_{\mathcal{F}} := In(P) = (a - \lambda_1 b)(a - \lambda_2 b)\ldots(a - \lambda_k b)$ for any choice of an isomorphism $\mathcal{F} \simeq \tilde{\mathcal{A}}_{conv.}/\tilde{\mathcal{A}}_{conv.}P$, is the element in $\mathcal{A}$ defined by the Bernstein polynomial $B_{\mathcal{F}}$ of $\mathcal{F}$ by the following formula*

$$P_{\mathcal{F}} := (-b)^k B_{\mathcal{E}}(-b^{-1}a)$$

valid in the algebra $\mathcal{A}[b^{-1}a]$. ■

This implies that for any J-H. sequence of the fresco $\mathcal{F}$, if we define the numbers α_j by $F_j/F_{j-1} \simeq \mathcal{E}_{\alpha_j}$, the the roots of $B_{\mathcal{F}}$ are the numbers $-(\alpha_j + j - k)$ where k is the rank of $\mathcal{F}$.

Corollary 7.1.6 *Let $0 \to \mathcal{F} \to \mathcal{E} \to \mathcal{G} \to 0$ be an exact sequence of frescos.*
Then we have $P_{\mathcal{E}} = P_{\mathcal{F}} P_{\mathcal{G}}$ in the algebra $\mathcal{A}$ which implies $B_{\mathcal{E}}(x) = B_{\mathcal{F}}(x - r)B_{\mathcal{G}}(x)$ where r is the rank of $\mathcal{G}$. ■

A corollary of the previous result is the Decomposition Theorem for Bernstein polynomials of frescos, analog to Corollary 3.3.9 of the Decomposition Theorem for simple pole (a,b)-modules.

Theorem 7.1.7 *Let $\mathcal{F}$ be a fresco and let $-\mathscr{A}$ the image in $\mathbb{Q}/\mathbb{Z}$ of the set of roots of its Bernstein polynomial of $\mathcal{F}$. For $[\alpha] \in \mathscr{A}$ define $B_{\mathcal{F}}^{[\alpha]}$ the Bernstein polynomial of $\mathcal{F}/\mathcal{F}_{[\neq\alpha]}$. Then we have*

$$B_{\mathcal{F}} = \prod_{\alpha\in\mathscr{A}} B_{\mathcal{F}}^{[\alpha]}. \tag{B}$$

PROOF. For each $\alpha \in \mathscr{A}$ the exact sequence of frescos

$$0 \to \mathcal{F}_{[\neq\alpha]} \to \mathcal{F} \to \mathcal{F}/\mathcal{F}_{[\neq\alpha]} \to 0$$

which implies that the Bernstein polynomial $B_{\mathbb{F}}^{[\alpha]}$ of $\mathcal{F}/\mathcal{F}_{[\neq\alpha]}$ divides $B_{\mathcal{F}}$. Moreover, The Bernstein polynomial of $\mathcal{F}_{[\neq\alpha]}$ satisfies

$$B_{\mathcal{F}_{[\neq\alpha]}}(x-\delta^{\alpha}) = B_{\mathcal{F}}(x)/B_{\mathcal{F}/\mathcal{F}_{[\neq\alpha]}}(x)$$

where δ^{α} is the rank of $\mathcal{F}/\mathcal{F}_{[\neq\alpha]}$. Since $B_{\mathcal{F}_{[\neq\alpha]}}$ has not root in $-\alpha+\mathbb{Z}$ we conclude that $B_{\mathcal{F}}^{[\alpha]}$ is the greatest divisor of $B_{\mathcal{F}}$ having its roots in $-\alpha+\mathbb{Z}$. The conclusion follows. ■

REMARK. In the analog decomposition of the Bernstein polynomial for any simple pole (a,b)-module we have an isomorphism $\mathcal{E}/\mathcal{E}_{[\neq\alpha]} \simeq \mathcal{E}_{[\alpha]}$ for each $\alpha \in \mathscr{A}$. This is not true, in general, for a fresco.
For a $[\lambda]-$primitive formal fresco a more precise result is proved in [9] Proposition 3.5.2. The proof in the convergent case is the same.

Proposition 7.1.8 *When $\mathcal{F}$ is $[\alpha]-$primitive, there always exists a J-H. sequence such that the associated sequence $(\alpha_j + j)$ is non decreasing.* ■

Definition 7.1.9 *We say that a J-H. sequence of a $[\alpha]$-primitive fresco with characteristic numbers $(\alpha_1, \ldots, \alpha_k)$ is* **principal** *when the sequence $(\alpha_j + j)$ is non decreasing.*

The following uniqueness result is rather interesting since it shows that a fresco admits a canonical J-H. sequence (invariant by automorphisms).

Theorem 7.1.10 *Let $\mathcal{F}$ be a $[\alpha]$-primitive fresco. Then $\mathcal{F}$ admits a* **unique** *principal J-H. sequence*

REMARK. Of course, since a $[\alpha]$-primitive theme has a unique J-H. sequence, it has to be the principal one. So the successive quotients of the J-H. sequence of a $[\alpha]$-primitive theme satisfies $\alpha_j + j \leq \alpha_{j+1} + j + 1$. That means that the corresponding roots of the Bernstein polynomial are in a non increasing order since they are equal to the $-\alpha_j + k - j$ where k is the rank of the theme.

PROOF. We shall prove the uniqueness by induction on the rank k of the $[\alpha]$-primitive fresco $\mathcal{F}$.
We begin by the case of rank 2.

Lemma 7.1.11 *Let $\mathcal{F}$ be a rank 2 $[\alpha]$-primitive fresco and let (α_1, α_2) the numbers corresponding to a principal J-H. sequence of $\mathcal{F}$ (so $\alpha_1 + 1 \leq \alpha_2 + 2$). Then the normal rank 1 submodule of $\mathcal{F}$ isomorphic to $\mathcal{E}_{\alpha_1}$ is unique. Moreover, if there exists a $\beta \neq \alpha_1$ and a rank 1 normal sub-module isomorphic to $\mathcal{E}_\beta$ then $\beta = \alpha_2 + 1$. In this case there exists infinitely many different normal rank 1 sub-modules isomorphic to $\mathcal{E}_{\alpha_2+1}$.*

PROOF. The case $\alpha_1 + 1 = \alpha_2 + 2$ is obvious because then $\mathcal{F}$ is a $[\alpha]-$primitive theme (see [11] Corollary 2.1.7).
So we may assume that $\alpha_2 = \alpha_1 + p_1 - 1$ with $p_1 \geq 1$ and that $\mathcal{F}$ is the quotient $\tilde{\mathcal{A}}_{conv.}/\tilde{\mathcal{A}}_{conv.}(a - \alpha_1 b)(a - \alpha_2 b)$ (see the classification of regular rank 2 (a,b)-modules in [6]), because the result is clear when $\mathcal{F}$ is a theme. We shall use the B-basis e_1, e_2 of $\mathcal{F}$ where a is defined by the relations

$$(a - \alpha_2 b)e_2 = e_1 \quad \text{and} \quad (a - \alpha_1 b)e_1 = 0.$$

This basis comes from the isomorphism $\mathcal{F} \simeq \tilde{\mathcal{A}}_{conv.}/\tilde{\mathcal{A}}_{conv.}(a-\alpha_1 b)(a-\alpha_2 b)$ deduced from the classification of rank 2 frescos with $e_2 = [1]$ and $e_1 = (a - \alpha_2 b)e_2$.
Let look for $x := Ue_2 + Ve_1$ such that $(a - \beta b)x = 0$. Then we obtain

$$b^2U'e_2 + U(a - \alpha_2 b)e_2 + (\alpha_2 - \beta)bUe_2 + b^2V'e_1 + (\alpha_1 - \beta)bVe_1 = 0$$

which is equivalent to the two equations :

$$b^2U' + (\alpha_2 - \beta)bU = 0 \quad \text{and} \quad U + b^2V' + (\alpha_1 - \beta)bV = 0.$$

The first equation gives $U = 0$ for $\beta \notin \alpha_2 + \mathbb{N}$. As the case $U = 0$ will give (as we want also that $x \notin b\mathcal{F}$) that x is equal to e_1, up to a non zero multiplicative constant, we may assume that $\beta = \alpha_2 + q$ for some $q \in \mathbb{N}$. Moreover, as the second equation implies $U(0) = 0$, we may assume that $q \geq 1$. This already shows that $\beta \neq \alpha_1$ $(\alpha_2 + 1 = \alpha_1 + p_1 > \alpha_1)$ and this proves the first assertion. Now to finish our computation of normal rank 1 sub-modules, we have $U = \rho b^q$. Then the solutions in B of the equation

$$\rho b^{q-1} + bV' - (p_1 + q - 1)V = 0$$

are given by : $V = -(\rho/p)b^{q-1} + \sigma b^{p_1+q-1}$ and the condition $x \notin b\mathcal{F}$ implies now $q = 1$. So we obtain $\beta = \alpha_2 + 1$ and for each $\tau \in \mathbb{C}$ the element $x = (1 - p_1\tau b^{p_1})e_1 - p_1 be_2$ generates a normal rank 1 sub-module isomorphic to $\mathcal{E}_{\alpha_2+1}$. And with the unique sub-module isomorphic to $\mathcal{E}_{\alpha_1}$ they are all the normal rank 1 sub-modules in such an $\mathcal{F}$. ■

END OF THE PROOF OF THEOREM 7.1.10. As the result is obvious for $k = 1$, we may assume $k \geq 2$ and the result proved in rank $\leq k-1$. Let $F_j, j \in [1,k]$ and $G_j, j \in [1,k]$ two J-H. principal sequences for $\mathcal{F}$. As the sequences $\lambda_j + j$ and $\mu_j + j$ coïncide up to the order (they are of the form $-x_j + k$ where the $(x_j)_{j\in[1,k]}$ are the roots of the Bernstein polynomial, counting multiplicities) and are both not decreasing, they coïncide. Now let j_0 be the first integer in $[1,k]$ such that $F_{j_0} \neq G_{j_0}$. If $j_0 \geq 2$ applying the induction hypothesis to $\mathcal{F}/F_{j_0-1}$ gives $F_{j_0}/F_{j_0-1} = G_{j_0}/F_{j_0-1}$ and so $F_{j_0} = G_{j_0}$.
So we may assume that $j_0 = 1$. Let H be the normalization of $F_1 + G_1$. As F_1 and G_1 are normal rank 1 and distinct, then H is a rank 2 normal sub-module. It is a $[\alpha]$-primitive fresco of rank 2 with two different normal rank 1 sub-modules which are isomorphic as $\alpha_1 = \beta_1$. Moreover the principal J-H. sequence of H begins by a normal sub-module isomorphic to $\mathcal{E}_{\alpha_1}$. This contradicts the previous lemma and so $F_1 = G_1$. So for any $j \in [1,k]$ we have $F_j = G_j$. ■

Lemma 7.1.12 *Let $\mathcal{E}$ be a geometric (a,b)-module of rank k. There exists a fresco $\mathcal{F} \subset E$ with rank k, so such that the quotient $\mathcal{E}/\mathcal{F}$ is a finite dimensional complex vector space.*

PROOF. We shall prove this fact by induction on the rank k of $\mathcal{E}$. As the statement is obvious for $k \leq 1$, assume $k \geq 2$ and the result proved in rank $k-1$. As there exists a normal rank 1 submodule of $\mathcal{E}$, consider an exact sequence

$$0 \to \mathcal{E}_\lambda \to \mathcal{E} \xrightarrow{\pi} \mathcal{G} \to 0$$

where $\mathcal{G}$ is a rank $k-1$ geometric (a,b)-module. Let $x \in \mathcal{E}$ such that $\pi(x)$ generates a rank $k-1$ fresco in $\mathcal{G}$. Let $P \in B[a]$ be a monic degree $k-1$ polynomial in a with coefficients in B, which generates the annihilator of $\pi(x)$ in $\mathcal{G}$. Then Px is in $\mathcal{E}_\lambda$.
If $Px = 0$ we replace x by $x + b^m e_\lambda$ where $m \in \mathbb{N}$ is chosen in order that $Pb^m e_\lambda \neq 0$. Such an integer m exists because we have

$$Pb^m e_\lambda = (\lambda_1 - m - \lambda)\dots(\lambda_{k-1} - m - \lambda)b^{m+k-1}e_\lambda + b^{m+k}\mathcal{E}_\lambda$$

if the initial form of P is $(a - \lambda_1 b)\dots(a - \lambda_{k-1}b)$.
This allows us to assume that Px is a non zero element in $\mathcal{E}_\lambda \cap \tilde{\mathcal{A}}_{conv.}x$, which means that $\mathcal{E}_\lambda \cap \tilde{\mathcal{A}}_{conv.}x$ has rank 1 and has finite co-dimension in $\mathcal{E}_\lambda$.
Then the exact sequence

$$0 \to \mathcal{E}_\lambda/\mathcal{E}_\lambda \cap \tilde{\mathcal{A}}_{conv.}x \to \mathcal{E}/\tilde{\mathcal{A}}_{conv.}x \to \mathcal{G}/\tilde{\mathcal{A}}_{conv.}\pi(x) \to 0$$

gives the finiteness of the complex vector space $\mathcal{E}/\tilde{\mathcal{A}}_{conv.}x$, and the fresco $\tilde{\mathcal{A}}_{conv.}x$ has finite co-dimension in $\mathcal{E}$. ■

The next result shows that any root of the Bernstein polynomial of a geometric (a,b)-module $\mathcal{E}$ may be "realized" as by a fresco $\mathcal{F} \subset \mathcal{E}$.

Proposition 7.1.13 *Let $\mathcal{E}$ be a geometric (a,b)-module and let $-\alpha - m$ be a root of its Bernstein polynomial. Then there exists an element $x \in \mathcal{E}$ such that the Bernstein polynomial of the fresco $\tilde{\mathcal{A}}_{conv.}x \subset \mathcal{E}$ has $-\alpha - m$ has a root.*

The proof uses the following lemma.

Lemma 7.1.14 *Let $\mathcal{E}_1, \ldots, \mathcal{E}_p$ be sub-modules of a regular (a,b)-module $\mathcal{E}$ such that $\mathcal{E} = \sum_{j=1}^{p} \mathcal{E}_j$. Then we have the equality $\mathcal{E}^\sharp = \sum_{j=1}^{p} \mathcal{E}_j^\sharp$.*

Proof. First remark that we have, for any two regular (a,b)-modules $\mathcal{E}_1, \mathcal{E}_2$, the a natural isomorphism

$$(\mathcal{E}_1 \oplus \mathcal{E}_2)^\sharp \simeq \mathcal{E}_1^\sharp \oplus \mathcal{E}_2^\sharp.$$

The inclusion of $(\mathcal{E}_1 \oplus \mathcal{E}_2)^\sharp$ in $\mathcal{E}_1^\sharp \oplus \mathcal{E}_2^\sharp$ is clear. If $z = z_1 \oplus z_2$ is in $\mathcal{E}_1^\sharp \oplus \mathcal{E}_2^\sharp$, we may write, for an integer q large enough, $z_1 = \sum_{j=1}^{q} (b^{-1}a)^j x_j$ with $x_j \in \mathcal{E}_1$ and $z_2 = \sum_{j=1}^{q} (b^{-1}a)^j y_j$ with $y_j \in \mathcal{E}_2$. This gives $z = \sum_{j=1}^{q} (b^{-1}a)^j (x_j \oplus y_j)$ which shows that z is in $(\mathcal{E}_1 \oplus \mathcal{E}_2)^\sharp$.
Now assume that $\mathcal{E}_1, \mathcal{E}_2$ are sub-modules of the regular (a,b)-module $\mathcal{E}$ such that $\mathcal{E} = \mathcal{E}_1 + \mathcal{E}_2$. We have a surjective map $\mathcal{E}_1 \oplus \mathcal{E}_2 \to \mathcal{E}$ and then a surjective map $(\mathcal{E}_1 \oplus \mathcal{E}_2)^\sharp \to (\mathcal{E}_1 + \mathcal{E}_2)^\sharp = \mathcal{E}^\sharp$. The conclusion follows by an easy induction on p. ■

Remark. If $\pi : \mathcal{E} \to \mathcal{F}$ is a surjective (a,b)-linear map between regular (a,b)-modules, then the Bernstein polynomial of $\mathcal{F}$ divides the Bernstein polynomial of $\mathcal{E}$ because the map $\pi^\sharp : \mathcal{E}^\sharp \to \mathcal{F}^\sharp$ is surjective.

Thanks to the surjectivity of the map $\mathcal{E}_1^\sharp \oplus \mathcal{E}_2^\sharp \to \mathcal{E}^\sharp$, the following corollary of the previous lemma is obvious.

Corollary 7.1.15 *Assume that $\mathcal{E}_1, \mathcal{E}_2$ are sub-modules of the geometric (a,b)-module $\mathcal{E}$ such that $\mathcal{E} = \mathcal{E}_1 + \mathcal{E}_2$. Then each root of the Bernstein polynomial of $\mathcal{E}$ is also a root of the Bernstein polynomial either of $\mathcal{E}_1$ or of $\mathcal{E}_2$.* ■

Proof of Proposition 7.1.13. Let $x_1, \ldots, x_k$ be a B-basis of $\mathcal{E}$. Then we have $\mathcal{E} = \sum_{j=1}^{k} \tilde{\mathcal{A}}_{conv.}x_j$. Then Corollary 7.1.15 gives the conclusion by an easy induction on k. ■

Corollary 7.1.16 *Let $-\alpha - m$ be the biggest root of the Bernstein polynomial of a geometric (a,b)-module $\mathcal{E}$ which is in $-\alpha - \mathbb{N}$. Then there exists $x \in \mathcal{E}$ such that $-\alpha - m$ is the biggest root of the Bernstein polynomial of the fresco $\mathcal{F} := \tilde{\mathcal{A}}_{conv.}x$ in $-\alpha - \mathbb{N}$.*

PROOF. It is enough to prove, thanks to Proposition 7.1.13, that the Bernstein polynomial of $\mathcal{F}$ has no root in $-\alpha - \mathbb{N}$ which is strictly bigger than $-\alpha - m$. This is given by Lemma 3.2.8. ■

7.2 Semi-simple frescos

Proposition 7.2.1 *Let $\mathcal{F}$ be a semi-simple fresco with rank k and let $\lambda_1, \ldots, \lambda_k$ be the numbers associated to a J-H. sequence of $\mathcal{F}$. Let $\mu_1, \ldots, \mu_k$ be a twisted permutation*[11] *of $\lambda_1, \ldots, \lambda_k$. Then there exists a J-H. sequence for $\mathcal{F}$ with quotients corresponding to $\mu_1, \ldots, \mu_k$.*

PROOF. As the symmetric group $\mathfrak{S}_k$ is generated by the transpositions $t_{j,j+1}$ for $j \in [1, k-1]$, it is enough to show that, if $\mathcal{F}$ has a J-H. sequence with quotients given by the numbers $\lambda_1, \ldots, \lambda_k$, then there exists a J-H. sequence for $\mathcal{F}$ with quotients $\lambda_1, \ldots, \lambda_{j-1}, \lambda_{j+1}+1, \lambda_j - 1, \lambda_{j+2}, \ldots, \lambda_k$ for $j \in [1, k-1]$. Put $\mathcal{G} := F_{j+1}/F_{j-1}$; it is a rank 2 sub-quotient of $\mathcal{F}$ with an exact sequence

$$0 \to F_{\lambda_j} \to \mathcal{G} \to F_{\lambda_{j+1}} \to 0.$$

As $\mathcal{G}$ is a rank 2 semi-simple fresco, it admits also an exact sequence

$$0 \to G_1 \to \mathcal{G} \to \mathcal{G}/G_1 \to 0$$

with $G_1 \simeq \mathcal{E}_{\lambda_{j+1}+1}$ and $\mathcal{G}/G_1 \simeq \mathcal{G}_{\lambda_j - 1}$. Let $q : F_{j+1} \to \mathcal{G}$ be the quotient map. Now the J-H. sequence for $\mathcal{F}$ given by

$$F_1, \ldots, F_{j-1}, q^{-1}(G_1), F_{j+1}, \ldots, F_k = \mathcal{F}$$

satisfies our requirement. ■

REMARK. If $\mathcal{E}$ is a semi-simple geometric (a,b)-module, we may have $\mathcal{G} \simeq \mathcal{E}_{\lambda_j} \oplus \mathcal{E}_{\lambda_{j+1}}$ in the proof above, and then the conclusion does not hold. □

Proposition 7.2.2 *Let $\mathcal{F}$ be a $[\lambda]-$primitive fresco. A necessary and sufficient condition in order that $\mathcal{F}$ is semi-simple is that it admits a J-H. sequence with quotient corresponding to $\mu_1, \ldots, \mu_k$ such that the sequence $(\mu_j + j)$ is strictly decreasing.*

PROOF. Remark first that if we have, for a fresco $\mathcal{F}$, a J-H. sequence $F_j, j \in [1, k]$ such that $\lambda_j + j = \lambda_{j+1} + j + 1$ for some $j \in [1, k-1]$, then F_{j+1}/F_{j-1} is a sub-quotient of $\mathcal{F}$ which is a $[\lambda]-$primitive theme of rank 2. So $\mathcal{F}$ is not semi-simple. As a consequence, when a fresco $\mathcal{F}$ is semi-simple, thanks to Lemma 7.2.1,

[11] This means that the sequence $\mu_j + j, j \in [1, k]$ is a permutation (in the usual sense) of $\lambda_j + j, j \in [1, k]$.

we may choose a J-H. sequence with a strictly decreasing sequence $(\lambda_j + j)$.

Now let us prove the converse. So assume for some $k \geq 1$ we already know that a rank k (the case $k = 1$ is trivial) fresco admitting a J-H. sequence with characteristic numbers $\lambda_1, \ldots, \lambda_k$ such that the sequence $\lambda_j + j$ is strictly decreasing is semi-simple, and consider the case of a fresco $\mathcal{E}$ with rank $k+1$ and with a J-H. sequence $(F_j), j \in [1, k+1]$ with characteristic sequence $(\lambda_1, \ldots, \lambda_{k+1})$ such that $\lambda_j + j$ is strictly decreasing. So we have

$$\mu := \lambda_{k+1} < Inf_{j\in[1,k]}\{\lambda_j + j - k\} - 1. \qquad (@)$$

Note $\mathcal{F}$ the fresco F_k in the J-H. sequence of $\mathcal{E}$. Then we have an exact sequence of frescos

$$0 \to \mathcal{F} \to \mathcal{E} \to \mathcal{E}_\mu \to 0.$$

The induction hypothesis gives that $\mathcal{F}$ is semi-simple. Then assume that $\mathcal{E}$ is not semi-simple. Then we have $S_1(\mathcal{E}) = \mathcal{F}$ because the semi-simplicity of $\mathcal{F}$ gives $\mathcal{F} \subset S_1(\mathcal{E})$ and the rank of $S_1(\mathcal{E})$ is at most k, thanks to our hypothesis that $\mathcal{E}$ is not semi-simple. Then the equality is consequence of the normality of $\mathcal{F}$ and of $S_1(\mathcal{E})$.
Our hypothesis that $\mathcal{E}$ is not semi-simple implies, see Proposition 6.3.3 point *iv*), that there exists a surjective (a,b)-linear map $\varphi : \mathcal{E} \to T$ onto a rank 2 $[\lambda]$-primitive theme T. Let (ν_1, ν_2) be the characteristic pair of the unique J-H. sequence of T. So we have $\nu_1 + 1 \leq \nu_2 + 2$. Note $F_1(T) \simeq \mathcal{E}_{\nu_1}$ the semi-simple part of T.
Now the normal sub-module $\varphi^{-1}(F_1(T))$ is equal to $\mathcal{F}$ because the rank of $\varphi^{-1}(F_1(T))$ is k and the inclusion of $\varphi(\mathcal{F})$ in $F_1(T)$ is consequence of the semi-simplicity of $\mathcal{F}$. We obtain the equality using the normality of $\mathcal{F}$ and of $\varphi^{-1}(F_1(T))$ in $\mathcal{E}$.
Now this implies, since $F_1(T) \simeq \mathcal{E}_{\nu_1}$ is a quotient of $\mathcal{F}$, so there exists $j_0 \in [1, k]$ with $\lambda_{j_0} + j_0 - k = \nu_1$ Since we have an isomorphism

$$\mathcal{E}/\mathcal{F} \overset{\varphi}{\simeq} T/F_1(T) \simeq \mathcal{E}_{\nu_2} \simeq \mathcal{E}_\mu.$$

the inequality $\nu_1 - 1 \leq \nu_2$ implies

$$\lambda_{j_0} + j_0 - k - 1 \leq \nu_2 = \mu.$$

This contradicts the inequality (@). So $\mathcal{E}$ is semi-simple. ■

Remarks.

1. As a fresco is semi-simple if and only if for each $[\lambda]$ its $[\lambda]-$primitive part is semi-simple, this proposition gives also a criterium to semi-simplicity for any fresco.

2. This criterium is a very efficient tool to produce easily examples of semi-simple frescos.

3. Assume that the Bernstein polynomial of the semi-simple fresco

$$\mathcal{F} := \tilde{\mathcal{A}}_{conv.}/\tilde{\mathcal{A}}_{conv.}P,$$

where $P := (a - \lambda_1 b)S_1(a - \lambda_2 b)S_2 \dots (a - \lambda_k b)S_k$, where $S_1, \dots, S_k$ are invertible elements in B, has a root $-\alpha - m$. Then there is a unique $j \in [1, k]$ such that $\lambda_j + j - k$ is equal to $\alpha + m$, since the roots of the Bernstein polynomial of a semi-simple (a,b)-module are simple. $\square$

7.3 Higher Bernstein polynomials

Thanks to the previous sections we are able now to associate to any geometric (a,b)-module $\mathcal{E}$ many other geometric (a,b)-modules whose isomorphism classes depend only on the isomorphism class of $\mathcal{E}$:
For instance $S_j(\mathcal{E})$ for some $j \in [1, d(\mathcal{E})]$ and also $\mathcal{E}_{[\alpha]}$ for some $\alpha \in \mathbb{Q}/\mathbb{Z}$, but also quotients like $S_{j+h}(\mathcal{E})/S_j(\mathcal{E})$ or, for a fresco $\mathcal{F}$, the j-th term of the principal J-H. sequence of $\mathcal{F}_{[\alpha]}$ or of $\mathcal{F}^{[\alpha]}$, etc ....
Then the Bernstein polynomials of these geometric (a,b)-modules depend only on the isomorphism class of $\mathcal{E}$.
If $\mathcal{E}$ is $[\alpha]$-primitive for some $\alpha \in \mathbb{Q} \cap]0, 1]$, this gives us a collection of non negative integers, all the roots of these Bernstein polynomials are in $-\alpha - \mathbb{N}$, which are related to the monodromy of $\mathcal{E}^\sharp$.
But even in the case of an $[\alpha]$-primitive geometric (a,b)-module, the situation is quite complicate, and it is not clear how to use these numbers.
So in this section we shall only consider the case where $\mathcal{F}$ is a fresco to exploit some of these polynomials which only depend on the isomorphism class of the fresco $\mathcal{F}$.

Definition 7.3.1 *Let $\mathcal{F}$ be an $[\alpha]$-primitive fresco and let $d := d(\mathcal{F})$ its nilpotent order. Then for $j \in [1, d]$ we define the* **j-th Bernstein polynomial** *of $\mathcal{F}$ as the monic polynomial in the variable x, noted $B_j(\mathcal{F})$, which is equal to $\tilde{B}_j(x - \delta_j)$ where $\tilde{B}_j$ is the Bernstein polynomial of $S_j(\mathcal{F})/S_{j-1}(\mathcal{F})$ and where δ_j is the rank of $\mathcal{F}/S_j(\mathcal{F})$.*
It will be convenient to define $B_j(\mathcal{F}) = 1$ for $j > d(\mathcal{F})$.

Remarks.

1. The shift by $\delta_j = rk(\mathcal{F}/S_j(\mathcal{F}))$ in the definition above is motivated by the fact that we have a product decomposition of the Bernstein polynomial $B_{\mathcal{F}}$ of $\mathcal{F}$ in the product of the $B_j(\mathcal{F})$ for an $[\alpha]$-primitive fresco $\mathcal{F}$ (see Theorem 7.3.2 below).

2. The same definition as above may be given for any geometric $[\alpha]$-primitive (a,b)-module, but when $\mathcal{E}$ is not a fresco, the relation between the higher Bernstein polynomials and the usual Bernstein polynomial of $\mathcal{E}$ is not clear, at least for $1 \leq j < d(\mathcal{E})$ (see Remark 4 below).

3. Since each quotient $S_j(\mathcal{F})/S_{j-1}(\mathcal{F})$ is semi-simple, for each j the j-th Bernstein polynomial has always simple roots. Its degree if the rank of $S_j(\mathcal{F})/S_{j-1}(\mathcal{F})$. Thanks to Proposition 4.2.8 this degree is non increasing with j.

4. As $B_d(\mathcal{F})$ for $d := d(\mathcal{F})$ is the Bernstein polynomial of the quotient $\mathcal{F}/S_{d-1}(\mathcal{F})$ of $\mathcal{F}$ it is clear that the d-th Bernstein polynomial of $\mathcal{F}$ divides the Bernstein polynomial of $\mathcal{F}$ (also for a general $[\alpha]$-primitive geometric (a,b)-module).

The following theorem summarizes the principal properties of these higher order Bernstein polynomials for an $[\alpha]$-primitive fresco $\mathcal{F}$.

Theorem 7.3.2 *Let $\mathcal{F} := \tilde{\mathcal{A}}_{conv.}e$ be an $[\alpha]$-primitive fresco and let*

$$S_1(\mathcal{F}) \subset S_2(\mathcal{F}) \subset \cdots \subset S_d(\mathcal{F}) = \mathcal{F}$$

be its semi-simple filtration. Then the Bernstein element $P_{\mathcal{F}}$ of $\mathcal{F}$ has a natural product decomposition $P_{\mathcal{F}} = P_1 P_2 \dots P_d$ in the sub-algebra $\mathcal{A} := \mathbb{C}[b][a] \subset \tilde{\mathcal{A}}_{conv.}$ where P_j is the Bernstein element of the fresco $S_j(\mathcal{F})/S_{j-1}(\mathcal{F})$. We have the following properties, where B_j is the Bernstein polynomial of $S_j(\mathcal{F})/S_{j-1}(\mathcal{F})$ shifted by the rank δ_j of $\mathcal{F}/S_j(\mathcal{F})$:

1. *The Bernstein polynomial of the fresco $\mathcal{F}$ is given by $B_{\mathcal{F}} = B_1(\mathcal{F}) \dots B_d(\mathcal{F})$ where d is the nilpotent order of $\mathcal{F}$.*
2. *Each root of each $B_j(\mathcal{F})$ is simple. The degree of the polynomials B_j are non increasing and are equal to the ranks of the quotients $S_j(\mathcal{F})/S_{j-1}(\mathcal{F})$.*

Proof. The proof of point 1 is an immediate consequence of the existence of a J-H. sequence F_h for $\mathcal{F}$ which is compatible with the semi-simple filtration because in this situation the roots of B_j are the opposite of the numbers $-(\alpha_h + h - k + \delta_j)$ for $h \in [1, k - \delta_j]$ if k is the rank of $\mathcal{F}$, so $k - \delta_j$ is the rank of $S_j(\mathcal{F})$.
Point 2 is obvious because $S_j(\mathcal{F})/S_{j-1}(\mathcal{F})$ is a semi-simple fresco (see the remark following Proposition 4.1.4). ■
Note that if $-\alpha - m$ is a root of multiplicity p in the Bernstein polynomial $B_{\mathcal{F}}$ of a fresco $\mathcal{F}$ then $-\alpha - m$ is a root of each $B_j(\mathcal{F})$ for $j \in [1, p]$. But the interest of these higher order Bernstein polynomials appears when a simple root of $B_{\mathcal{F}}$ is a root of $B_j(\mathcal{F})$ for some $j \geq 2$; then this gives the existence of a Jordan bloc of size j for the monodromy $\exp(2i\pi b^{-1}a)$ of $\mathcal{F}^\sharp/b\mathcal{F}^\sharp$.

Now we shall give the definition of the higher order Bernstein polynomials for a general fresco (so not necessarily $[\alpha]$-primitive). In this case the relation with the semi-simple filtration is less simple because for a general fresco we may have $d(\mathcal{F})$ may be strictly bigger than $d(\mathcal{F}_{[\alpha]})$ for a given α, since

$$d(\mathcal{F}) = \sup_{\alpha \in \mathscr{A}} d(\mathcal{F}_{[\alpha]})$$

where $-\mathscr{A}$ is the image in $\mathbb{Q} \cap]0, 1]$ of the set of roots of $B_{\mathcal{F}}$.

Definition 7.3.3 *Let $\mathcal{F}$ be a fresco. For $\alpha \in \mathscr{A}$ we denote $\mathcal{F}^{[\alpha]}$ the $[\alpha]$-primitive fresco $\mathcal{F}/\mathcal{F}_{[\neq\alpha]}$. Then we define the j-th Bernstein polynomial of $\mathcal{F}$ by the formula*

$$B_j(\mathcal{F}) := \prod_{\alpha \in \mathscr{A}} B_j(\mathcal{F}^{[\alpha]}) \qquad (B_j)$$

Then an obvious consequence of the Decomposition Theorem for the Bernstein polynomial of a fresco (see 7.1.7) gives:

Corollary 7.3.4 *For any fresco $\mathcal{F}$ we have*

$$B_{\mathcal{F}} = \prod_{j=1}^{d(\mathcal{F})} B_j(\mathcal{F}). \tag{BB}$$

thanks to formulas (B) and (B_j). ■

REMARK. Beware that for a fresco the injective map $\mathcal{F}_{[\alpha]} \to \mathcal{F}^{[\alpha]} = \mathcal{F}/\mathcal{F}_{[\neq\alpha]}$ is not, in general, surjective. So the Bernstein polynomials are in general different. Since these two $[\alpha]$-primitive frescos have the same rank, the sum of shifts between corresponding roots of their respective polynomials[12] is equal to the dimension of the quotient $\mathcal{F}^{[\alpha]}/\mathcal{F}_{[\alpha]}$ (see Proposition 3.4.2).

7.4 Somme complements

We give now some useful complements which allow to make explicit computations in some cases of fresco associated to hyper-surface singularities (see Section 9) and so to apply in concrete cases our main results given in Section 8.

Proposition 7.4.1 *Let $\mathcal{F}$ be a $[\alpha]$-primitive fresco and assume that $-\alpha - m$ is a root of its k-th Bernstein polynomial. Then for each $j \in [1, k]$ there exists an integer $m_j \in [0, m]$ such that $-\alpha - m_j$ is a root of the j-th Bernstein polynomial of $\mathcal{F}$.*

PROOF. By definition, if the nilpotent order for $\mathcal{F}$ is strictly bigger than k then the Bernstein polynomial of $S_k(\mathcal{F})$ has a root which is strictly bigger than $-\alpha - m$. So it is enough to prove the lemme when k is the nilpotent order of $\mathcal{F}$. Then, by a descendant induction on $j \in [1, k-1]$ it is enough to prove the case $j = k - 1$.
Taking the quotient by $S_{k-2}(\mathcal{F})$ we reduce the question in the case $k = 2$.
In this case, there exists a quotient theme T with rank 2 whose Bernstein polynomial has the root $-\alpha - m$ as its minimal root. Then the other root $-\alpha - m'$ of the Bernstein polynomial of T satisfies $-\alpha - m' \geq -\alpha - m$. Since T is a quotient of $\mathcal{F}$, $-\alpha - m'$ is a root of the Bernstein polynomial of $\mathcal{F}$. If it is a root of the first Bernstein polynomial of $\mathcal{F}$ we are done. If this is not the case, $-\alpha - m'$ is a root of the second Bernstein polynomial of $\mathcal{F}$. But in this case $m' < m$ since the roots of the second Bernstein polynomial of $\mathcal{F}$ are simple. Then we can play the same game as before, but with the root $-\alpha - m'$. Since there is only finitely integer in $[0, m]$ we finally reach a root $-\alpha - m''$ of the first Bernstein polynomial of $\mathcal{F}$ such that m'' is in $[0, m]$. ■

[12]The consideration of a J-H. sequence of $\mathcal{F}^{[\alpha]}$ induces a J-H. sequence of $\mathcal{F}_{[\alpha]}$ defining such a correspondence.

REMARK. Let T be a $[\alpha]$-primitive theme with rank k. Then its j-th Bernstein polynomial has degree 1 for $j \in [1,k]$ and is equal to $(x+\alpha+m_j)$ where $-\alpha-m_1,\ldots,-\alpha-m_k$ are the roots of its Bernstein polynomial in decreasing order. So $-\alpha-m_k$ is the smallest root of its Bernstein polynomial.

Note that, for any $[\alpha]$-primitive fresco, the smallest root of the Bernstein polynomial is always a root of the k-th Bernstein polynomial where $k=d(\mathcal{F})$ is the nilpotent order of the fresco $\mathcal{F}$. But for a "general" $[\alpha]$-primitive fresco, we do not know other relation between the order of the roots of the Bernstein polynomial of $\mathcal{F}$ and the roots of the j-th Bernstein polynomial of $\mathcal{F}$ than the fact, given by the Lemma above.

Proposition 7.4.2 *Let $\mathcal{F}$ be a semi-simple fresco. Then $-\lambda$ is a root of the Bernstein polynomial of $\mathcal{F}$ if and only if there exists a $\tilde{\mathcal{A}}$-linear surjective map*

$$\pi : \mathcal{F} \to \mathcal{E}_\lambda \simeq \tilde{\mathcal{A}}/\tilde{\mathcal{A}}(a-\lambda b).$$

PROOF. The existence of π is sufficient because the Bernstein polynomial of a quotient of $\mathcal{F}$ divides the the Bernstein polynomial of $\mathcal{F}$.
Conversely, if λ is a root of the Bernstein polynomial of $\mathcal{F}$, since $\mathcal{F}$ is semi-simple, there exists a Jordan-Hölder sequence for $\mathcal{F}$ such its last quotient is $\mathcal{E}_\lambda$, thanks to Proposition 7.2.1 in [13]. So the proof is complete. ■.

Corollary 7.4.3 *Let $\mathcal{F}$ be be a $[\alpha]$-primitive fresco with nilpotent order k. Assume that $\mathcal{F}=\tilde{\mathcal{A}}e \subset \Xi_\alpha^{(k-1)}\otimes V$[13]. Let p be the rank of $\mathcal{F}/S_{k-1}(\mathcal{F})$. Then there exists p linearly independent vectors $v_1,\ldots,v_p$ in V such that e may be written*

$$e=\sum_{j=1}^{p} S_j(b)s^{\alpha+m_j-1}(Log\, s)^{k-1}\otimes v_j+\psi.$$

where ψ is in $\Xi_\alpha^{(k-2)}\otimes V$, and where the S_j are invertible elements in $\mathbb{C}[[b]]$. Moreover we may choose the vectors $v_1,\ldots,v_p$ such that $m_1<\cdots<m_p$.
When this condition is fulfilled the k-th Bernstein polynomial of $\mathcal{F}$ is equal to $\prod_{j=1}^{p}(x+\alpha+m_j)$.

For $\alpha=1$ it is convenient to replace $\Xi_1^{(k-1)}$ by $\Xi_1^{(k)}/\Xi_1^{(0)}$ to consider only the singular part of the asymptotic expansions. This is the case in the examples computed in Section 4.

PROOF. Since $S_{k-1}(\mathcal{F})=\mathcal{F}\cap(\Xi_\alpha^{(k-2)}\otimes V)$, it is enough to treat the semi-simple case. In this case, since each $\mathcal{E}_{\alpha+m}$ is embedded in $\Xi_\alpha^{(0)}$ we may assume that $e=\sum_{j=1}^{q} S_j(b)s^{\alpha+\mu_j-1}\otimes v_j$ where $v_1,\ldots,v_q$ is a basis of V (by definition of semi-simplicity), where $m_1,\ldots,m_q$ are non negative integers and where S_j are invertible

[13]This is not restrictive thanks to the results of Section 5 .

elements in $\mathbb{C}[[b]]$ or vanish identically. Moreover, since the saturation $\mathcal{F}^\sharp$ is a direct sum of $\mathcal{E}_{\alpha+m}$ and has the same rank than $\mathcal{F}$, we may assume that the vector v_j for which $S_j \neq 0$ generate a subspace W of dimension p in V, where p is the rank of $\mathcal{F}$. If the integer $\mu_1, \ldots, \mu_p$ are pairwise distinct we may order the $v_1, \ldots, v_p$ such that $\mu_1 < \cdots < \mu_p$ and put $m_j := \mu_j$. If this is not the case, consider m_1 the infimum of the μ_j and when $\mu_j = m_1$ let $w_1 = v_1 + \sum c_j v_j$ where the sum is on each $j \geq 2$ such that $\mu_j = m_1$ and where $c_j = S_j(0)S_1(0)^{-1}$ with $\mu_j = m_1$. Now we obtain a new expression for e in the basis $w_1, v_2, \ldots, v_p$ of W, where m_1 is strictly less than all μ'_j which appear for $j \geq 2$. Continuing in this way we obtain that $(w_1, \ldots, w_p)$ is a new basis of W and $m_1 < \cdots < m_p$.
Then consider the $\tilde{\mathcal{A}}$-linear maps given by the linear forms $l_j \in V^*$ defined by $l_j(w_h) = \delta_{j,h}, h \in [1, p]$. The $\tilde{\mathcal{A}}$-linear map $\mathrm{id} \otimes l_j$ for $j \in [1, p]$ sends surjectively $\mathcal{F}$ to $\mathcal{E}_{\alpha+m_j}$ and this implies that $-(\alpha + m_j)$ is a root of the Bernstein polynomial of $\mathcal{F}$ for each $j \in [1, p]$. But since $\mathcal{F}$ has rank p we obtain all the roots of its Bernstein polynomial since the m_j are pair-wise distinct[14]. This completes the proof. ■

Remark. As a consequence of the previous corollary we have the following characterization of the roots of the k-th Bernstein polynomial of a $[\alpha]$-primitive fresco with nilpotent order k:

- $-\alpha - m$ is a root of the $[\alpha]$-primitive fresco with nilpotent order k if and only if there exists a $\tilde{\mathcal{A}}$-linear surjective map of $\mathcal{F}$ to a rank k theme T_k such its k-th Bernstein polynomial is $(x + \alpha + m)$.

7.5 Some examples

Consider $\alpha \in [0, 1[$ and $m \in \mathbb{N}^*$. Then define

$$e := s^{\alpha+m-1} Log\, s + s^{\alpha-1}$$

Let F be the fresco generated by e in Ξ^1_α. We have

$$\begin{aligned}
ae &= s^{\alpha+m} Log\, s + s^\alpha \\
(\alpha+m)be &= s^{\alpha+m} Log\, s - \frac{s^{\alpha+m}}{\alpha+m} + (\alpha+m)\frac{s^\alpha}{\alpha} \\
(a-(\alpha+m)b)e &= \frac{s^{\alpha+m}}{\alpha+m} - m\frac{s^\alpha}{\alpha} \\
(a-(\alpha+m)b)e &= -\frac{\alpha}{m}(1+\gamma b^m)s^\alpha \quad \text{with} \quad \gamma = -\frac{\alpha}{m(\alpha+1)\ldots(\alpha+m-1)}
\end{aligned}$$

So we have

$$(a-(\alpha+1)b)(1+\gamma b^m)^{-1}(a-(\alpha+m)b)e = 0$$

[14]Note that the initial μ_j gives also roots of the k-th Bernstein polynomial of $\mathcal{F}$ but they may not give all the roots.

and this equality shows that the Bernstein polynomial of the fresco F is equal to $B_F(x) = (x+\alpha)(x+(\alpha+m))$.
But since F has finite co-dimension in the rank 2 $[\alpha]$-primitive theme generated by $\varepsilon := s^{\alpha+m} Log\, s$ in Ξ^1_α, it is a rank 2 theme (the fact that $\gamma \neq 0$ is also a way to prove this, looking at the classification of rank 2 regular (a,b)-modules).
Then the first Bernstein polynomial of F is equal to $x+\alpha$ corresponding to the semi-simple part $S_1(F)$ which generated by s^α (beware to the shift by the rank of $S_2(F)/S_1(F)$ which is 1 in the definition of B^1_F) and the second Bernstein polynomial of F is equal to $x+\alpha+m$.

More generally, using the fact that a $[\alpha]$-primitive theme T has a unique Jordan-Hölder sequence (see Corollary 5.1.6) and the fact that the corresponding simple quotients $E_{\lambda_j}, j \in [1,k]$ satisfy $\lambda_j \geq \lambda_{j+1} - 1$ (see the remark following Theorem 7.1.10), we obtain that, in such a situation, the j-th Bernstein polynomial of T is equal to $x+\lambda_j+j-k$. In fact $\lambda_1, \cdots, \lambda_k$ are necessarily in non decreasing order so if the roots $\mu_1, \ldots, \mu_k$ of the Bernstein polynomial of T are in non increasing order then the root of the j-th Bernstein polynomial of T is $\mu_j = -\lambda_j + k - j$ with the notation above.

The following easy lemma shows that the interaction between several roots of the Bernstein polynomial of a fresco may be maximal, even when their differences are integers arbitrarily big compare to the rank.

Lemma 7.5.1 *Let T be a rank $k \geq 2$ $[\alpha]$-primitive theme. Assume that its Bernstein polynomial is equal to $(x+\alpha)^k$. Then there exists a finite co-dimension sub-theme Θ in T (so it has also rank k and is $[\alpha]$-primitive) such that its Bernstein polynomial is equal to $(x+\alpha+p_1)\ldots(x+\alpha+p_k)$ for any given increasing sequence of integers $0 < p_1 < p_2 < \cdots < p_k$.*

Note that in this situation the i-th Bernstein polynomial of Θ is equal to $(x+\alpha+p_i)$.

Proof. We leave to the reader to convince itself that it is enough to treat the case where $T = \Xi^{(k-1)}_\alpha$ and then that, in this case, the sub-theme generated by

$$e := \sum_{i=1}^{k} s^{\alpha+p_i-1} (Log\, s)^{i-1}$$

satisfies the requirements. ■

The aim of the next paragraph is to find back in any $[\alpha]$-primitive fresco $\mathcal{F}$ whose nilpotent order is at least equal to k, a rank k theme T with Bernstein polynomial $(x+\alpha+m)^k$ such that its saturation $T^\sharp$ is also contained in $\mathcal{F}$. The saturation of such a theme is called a **Jordan bloc** (of size k and eigenvalue $\alpha+m$).

7.6 Jordan blocs

Lemma 7.6.1 *Let $\alpha \in]0,1] \cap \mathbb{Q}$ and let φ be an element in $\Xi_\alpha^{(N)}$ which is of degree N in $Log\, s$. Then inside the rank $N+1$ theme $T := \tilde{\mathcal{A}}_{conv.}.\varphi \subset \Xi_\alpha^{(N)}$, there exists an element*

$$\psi_N := s^{\alpha+m-1}\frac{(Log\, s)^N}{N!}$$

where m is an integer.

PROOF. Note first that $T := \tilde{\mathcal{A}}_{conv.}\varphi$ is a rank $(N+1)$ theme thanks to Lemma 5.2.4. We shall prove the lemma by induction on $N \geq 0$. Since the case $N = 0$ is clear, assume that the lemma is proved for $N-1$ and let $T \subset \Xi_\alpha^{(N)}$ a rank $N+1$ theme. Then $S_1(T)$ is equal to $T \cap \Xi_\alpha^{(0)}$ and we may embed the rank N theme $T/S_1(T)$ in

$$\Xi_\alpha^{(N-1)} \simeq \Xi_\alpha^{(N)}/\Xi_\alpha^{(0)}.$$

Thanks to our inductive hypothesis there exists an integer m' such that

$$s^{\alpha+m'-1}(Log\, s)^N/N! \qquad modulo\ \Xi_\alpha^{(0)}$$

is in $T/S_1(T)$ and, since $S_1(T) \subset \Xi_\alpha^{(0)}$, we may find an invertible element S in B such that $\varphi := s^{\alpha+m'-1}(Log\, s)^N/N! + S(b)s^{\alpha+M-1}$ is in T. Since $S_1(T)$ is isomorphic to $\mathcal{E}_{\alpha+q}$ for some positive integer q, for an integer m'' large enough, $s^{\alpha+m'+m''-1}(Log\, s)^N/N!$ will be in T, since $S(b)s^{\alpha+M+m''-1}$ will be in $S_1(T)$, concluding the proof. ■

Corollary 7.6.2 *Let $\mathcal{F}$ be a fresco and assume that the p-th Bernstein polynomial of $\mathcal{F}$ has a root in $-\alpha - \mathbb{N}$, where α is in $]0,1] \cap \mathbb{Q}$. Then there exists $w_1, \ldots, w_p$ in $\mathcal{F}$ (in fact in $\mathcal{F}_{[\alpha]}$) and an integer $m \in \mathbb{N}$ satisfying the relations:*

$$aw_j = (\alpha+m)bw_j + bw_{j-1} \quad \forall j \in [1,p] \quad \text{with the convention} \quad w_0 \equiv 0 \qquad (\star)$$

and which are B-linearly independent in $\mathcal{F}$.

PROOF. Since $S_p(\mathcal{F}_{[\alpha]}) = S_p(\mathcal{F})_{[\alpha]}$, thanks to Lemma 4.2.5, and has finite codimension in $S_p(\mathcal{F}^{[\alpha]})$ we may find an $[\alpha]$-primitive theme T_1 of rank p in $\mathcal{F}^{[\alpha]}$ thanks to Proposition 6.3.3. Then $T := q^{-1}(T_1) \cap \mathcal{F}_{[\alpha]}$, where $q : \mathcal{F} \to \mathcal{F}^\alpha$ is the quotient map, is a rank p theme in $\mathcal{F}_{[\alpha]}$. As we may assume that T is embedded in $\Xi_\alpha^{(p-1)}$ the previous lemma shows that there exists an integer m such that $s^{\alpha+m-1}(Log\, s)^p/p!$ is an element in T.
Define $w_j = s^{\alpha+m-1}(Log\, s)^j/j!$ for $j \in [1,p]$. Then the relations $(\star)$ are satisfied and imply that $w_1, \ldots, w_p$ are elements in T.
To shows that $w_1, \ldots, w_p$ are B-linearly independent, note J the B-sub-module generated by $w_1, \ldots, w_p$. Then it has rank at most p. But the relation $(\star)$ shows

that J is an (a,b)-sub-module of T with a simple pole. Moreover as $w_1, \ldots, w_p$ are clearly linearly independent over $A = \mathbb{C}\{s\}$, we have

$$\dim_{\mathbb{C}} J/aJ = \dim_{\mathbb{C}} J/bJ \geq p$$

so J has rank p as a B-module. ■

REMARK. Let $J := \sum_{j=1}^{p} Bw_j$ the sub-B-module generated by $w_1, \ldots, w_p$. Then J is a sub-(a,b)-module of $\mathcal{F}$ which has a simple pole and is $[\alpha]$-primitive; it is equal to $\mathcal{E}(J_{\alpha+m,p})$ where $J_{\alpha+m,p}$ is the matrix of the standard Jordan bloc with rank p and eigenvalue $\alpha + m$ (see the end of Section 2.3). The action of $b^{-1}a$ on J/bJ is given by $J_{\alpha+m,p}$. So the Bernstein polynomial of J is equal to $(x + \alpha + m)^p$.

It is interesting to compare this result with Corollary 3.2.6. Here we do not assume that the action of $b^{-1}a$ on $\mathcal{F}^\sharp/b\mathcal{F}^\sharp$ has a Jordan block of size p for some λ in $\alpha + \mathbb{N}$ but, in a way, that this happens for the eigenvalue $\exp(2i\pi b^{-1}a)$ acting of $\mathcal{F}^\sharp/b\mathcal{F}^\sharp$. And this hypothesis is precisely formulated by the existence of a root in $-\alpha - \mathbb{N}$ for the p-th Bernstein polynomial of the fresco $\mathcal{F}$.
Note that contrary to the result in Corollary 3.2.6. we have no control here on the integral shift between the root of the p-th Bernstein polynomial and the (multiple) root of the Bernstein polynomial of the Jordan block obtained.

8 Existence of poles

8.1 The complex of sheaves $(Ker\, df^\bullet, d^\bullet)$

THE STANDARD SITUATION . We consider now the following situation:
Let $f : U \to \mathbb{C}$ be a holomorphic function on an open polydisc U with center 0 in $\mathbb{C}^{n+1}$. We shall assume that U is small enough in order that the inclusion $\{df = 0\} \subset \{f = 0\}$ holds in U.
We denote Y the hypersurface $\{f = 0\}$ in U and we assume that Y is reduced. For each point $y \in Y$ we denote $f_y : X_y \to D_y$ a Milnor representative of the germ of f at y. So X_y is constructed by cutting a small ball, with center y and with radius $\varepsilon > 0$ very small, with $f^{-1}(D_\delta)$ where D_δ is an open disc with center 0 and radius $\delta \ll \varepsilon$.
For $y = 0$ we simply write $f : X \to D$ such a Milnor representative of the germ of f at the origin.
Let $\pi : H \to D^*$ be the universal cover of the punctured disc $D^* := D \setminus \{0\}$ and choose a base point $\tilde{s}_0$ in H over the base point s_0 in D^*. Fix a point $y \in Y$ and take for D the disc of a Milnor representative of f_y. Then we identify the Milnor fiber F_y of f at y with $f^{-1}(s_0)$.
For any p-cycle γ in $H_p(F_y, \mathbb{C})$ let $(\gamma_{\tilde{s}})_{\tilde{s} \in H}$ be the horizontal family of p-cycles in the

fibers of $f \times_{D^*} \pi$ over H taking the value γ at the point $\tilde{s}_0$. Then the regularity of the Gauss-Manin connection of f at y insures that for any $\omega \in \Omega_y^{p+1}$ which satisfies $d\omega = 0$ and $df \wedge \omega = 0$ the (multi-valued) function $s \mapsto \int_{\gamma_s} \omega/df$ has a convergent asymptotic expansion when s goes to 0, which is in $\Xi_{\mathscr{A}}^{(p-1)}$ where $\exp(2i\pi\mathscr{A})$ contains the eigenvalues of the monodromy of f at the point y.
We define on Y the following sheaves for each integer $p \in [1, n]$:
First let $Ker\, df^{p+1} \subset \Omega^{p+1}$ be the kernel of the map $\wedge df : \Omega^{p+1} \to \Omega^{p+2}$ of coherent sheaves on U and $Ker\, d^{p+1}$ be the kernel of the ($\mathbb{C}$-linear) de Rham differential

$$d^{p+1} : \Omega^{p+1} \to \Omega^{p+2}.$$

Then for $p \in [1, n]$ define the sheaf $\mathcal{H}^{p+1}$ as the (topological) restriction on Y of the sheaf $Ker\, df^{p+1} \cap Ker\, d^{p+1}/d(Ker\, df^p)$.
By convention we put $\mathcal{H}^{p+1} = 0$ for $p \notin [1, n]$.
Then we have a natural structure of A-modules on the sheaves $\mathcal{H}^{p+1}$ for each p induced by the natural action of A on $\Omega_{|Y}$ given by $(g, \omega) \mapsto f^*(g)\omega$ where g is in $A := \mathbb{C}\{s\}$ and ω is in Ω_y^{p+1}, for each $y \in Y$.
We have also an action of $\mathbb{C}[b]$ on $\mathcal{H}^{p+1}$ for each $p \in [1, n]$ which is defined as follows:

- For $\omega_y \in Ker\, d^{p+1} \cap Ker\, df^{p+1}$ write $\omega_y := du_y$ for some $u_y \in \Omega_y^p$ (holomorphic de Rham Lemma) and put $b[\omega_y] := [df \wedge u_y]$.
  Then clearly $d(df \wedge u_y) = 0$ and $df \wedge (df \wedge u_y) = 0$.

- Of course, if we change the choice of u_y (for $p \in [1, n]$) in $u_y + dv_y, v_y \in \Omega_y^{p-1}$, the class of $b[\omega_y] \in \mathcal{H}^{p+1}$ is the same since $df \wedge dv_y = -d(df \wedge v_y)$ is in $d(Ker\, df^p)$.

The sheaf $\mathcal{H}^{p+1}$ modulo its a-torsion, noted H^{p+1}, is the (a,b)-module version of the Gauss-Manin connection in degree p. As we assume f reduced, the 0-th cohomology of the Milnor fiber is $\mathbb{C}$ and the corresponding monodromy is trivial.

Lemma 8.1.1 *The actions of a and b on $\mathcal{H}^{p+1}$ satisfy the commutation relation $ab - ba = b^2$.*

Proof. For $\omega_y = du_y \in Ker\, df^{p+1} \cap Ker\, d^{p+1}$ we have

$$b(a[\omega_y] + b[\omega_y]) = b[fdu_y + df \wedge du_y] = b[d(fu_y)] = [df \wedge fu_y] = ab[\omega_y]$$

which gives the relation $b(a + b) = ab$ concluding the proof. ■

Note that the action of a is well defined on $Ker\, df^{p+1}$ but the action of b is only well defined on the cohomology $\mathcal{H}_y^{p+1}$ for each $p \in [1, n]$ and each $y \in Y$.

Theorem 8.1.2 *We keep the notations introduced above and let p be an integer in $[1, n]$. Let $\omega \in \Omega_y^{p+1}$ be in $Ker\, df^{p+1}$ such that $d\omega = 0$. Then for each $\gamma \in H_p(F_y, \mathbb{C})$*

define $\Phi(\gamma,\omega)$ *as the element in* $\Xi_{\mathscr{A}}^{(p-1)}$ *given by the asymptotic expansion*[15] *of the period-integral* $\int_{\gamma_s}\omega/df$*, where* $\mathscr{A}$ *is the image in* $]0,1]$ *of the opposite of the roots of the reduced Bernstein polynomial of* f *at the point* $y\in Y$ *and where* $(\gamma_s)_{s\in H}$ *is the horizontal family of* p*-cycles taking the value* γ *at the base point* $\tilde{s}_0$*. So we have:*

$$\Phi(\omega,\gamma):=\int_{\gamma_s}\omega/df\in\Xi_{\mathscr{A}}^{(p-1)}.$$

Then using the fact that $H^p(F_y,\mathbb{C})$ *is the dual of* $H_p(F_y,\mathbb{C})$ *and the linarity of* Φ *in* γ*,* Φ *defines a map*

$$\Psi:\mathcal{H}_y^{p+1}\to\Xi_{\mathscr{A}}^{(p)}\otimes_{\mathbb{C}}H^p(F_y,\mathbb{C}),\quad \Psi(\omega):=[\gamma\mapsto\Phi(\omega,\gamma)]$$

which is A-linear and b-linear and whose kernel is equal to the a-torsion of $\mathcal{H}^{p+1}$*.*

Proof. The A-linearity of Ψ is obvious. The b-linearity is an easy consequence of the derivation formula

$$\partial_s(\int_{\gamma_s}u)=\int_{\gamma_s}du/df$$

when u is in Ω_y^p satisfies $df\wedge du=0$.
Consider now $\omega\in Ker\, df^{p+1}$ such that $d\omega=0$ and assume that ω is in the Kernel of Ψ. Then for each γ the corresponding period-integral vanishes because the asymptotic expansion is convergent (thanks to the regularity of the Gauss-Manin connection). So the class induced by ω/df in $H^p(F_y,\mathbb{C})$ vanishes which implies that the class defined by ω in the f-relative de Rham cohomology vanishes and so we may find a meromorphic form $v\in\Omega_y^{p-1}[f^{-1}]$ such that $\omega=df\wedge dv$ (see [3] and [4] for $\alpha=1$). This implies that $a^N[\omega]=0$ in $\mathcal{H}_y^{p+1}$. ■

Remark. The map Φ satisfies also the relation $\Phi(\omega,T\gamma)=\mathscr{T}(\Phi(\omega,\gamma))$ where T is the monodromy acting on $H_p(F_y,\mathbb{C})$ and where $\mathscr{T}$ is the monodromy acting on $\Xi_{\mathscr{A}}^{(p-1)}$ via $Log\, s\mapsto Log\, s+2i\pi$. So the image of Ψ is contained in the sub-(a,b)-module of $\Xi_{\mathscr{A}}^{(p-1)}\otimes_{\mathbb{C}}H^p(F_y,\mathbb{C})$ which is invariant by $\mathscr{T}\otimes T^*$ where T^* is the action of the monodromy on $H^p(F_y,\mathbb{C})$.

Corollary 8.1.3 *For* $y\in Y$ *let* H_y^{p+1} *be the quotient of* $\mathcal{H}_y^{p+1}$ *by its a-torsion. Then* H_y^{p+1} *is a geometric (convergent) (a,b)-module.*

[15] for the eigenvalue 1 we consider only the *singular part* of the asymptotic expansion, so we replace $\Xi_1^{(p-1)}$ by $\Xi_1^{(p)}/\Xi_1^{(0)}$ which is isomorphic to $\Xi_1^{(p-1)}$; see [4].
This shift is related to the Γ-factor that we introduce below in the complex Mellin transform (see [5]) $F_h^{\omega,\omega'}(\lambda)$ of hermitian periods $\int_{f=s}\rho\omega\wedge\bar{\omega}'/df\wedge d\bar{f}$.

PROOF. The point is that H_y^{p+1} is a finite type A-module since A is noetherian and $\Xi_{\mathscr{A}}^{(p)}$ is a finite type (free) A-module. Then H_y^{p+1} is closed for the natural dual Fréchet topology induced by $\Xi_{\mathscr{A}}^{p} \otimes H^p(F_y, \mathbb{C})$. As it is b-stable it is also stable by the action of B and even of $\tilde{\mathcal{A}}_{conv.}$. So it is a geometric (a,b)-module. ■

Note that it is not simple to show directly that B acts on H_y^{p+1} (and a fortiori that $\tilde{\mathcal{A}}_{conv.}$ acts on it) contrary to the formal case.

Definition 8.1.4 *In the situation above, let ω be a germ at $y \in Y$ of the sheaf $Ker\, df^{p+1}$ which is d-closed. Then we define the* **fresco** $\mathcal{F}_{f,\omega,y}$ *associated to these data as the fresco $\tilde{\mathcal{A}}_{conv.}[\omega] \subset H_y^{p+1}$ which is generated in the geometric (a,b)-module H_y^{p+1} by the class of ω.*

Note that for $p = n$ each germ ω at a point y of Ω_y^{n+1} satisfies $df \wedge \omega = 0$ and $d\omega = 0$. In the sequel we shall mainly use the case $p = n$ with $y = 0$. So we simplify the notation to $\mathcal{F}_\omega$ when we consider the fresco $\mathcal{F}_{f,\omega,0}$ in H_0^{n+1}.

8.2 The use of frescos

We begin by the definition of the main hypothesis on the holomorphic germ f which is assumed in the sequel.

Definition 8.2.1 *In the standard situation, fix a rational number $\alpha \in]0,1]$. We say that the germ f has* **an isolated singularity for the eigenvalue** $\exp(2i\pi\alpha)$ **of its monodromy** *when the local monodromy of f at each point $y \neq 0$ in the reduced hypersurface $Y = f^{-1}(0)$, acting on the* reduced *cohomology of the Milnor fiber at the point y does not admit this eigenvalue. This hypothesis is denoted $H(\alpha, 1)$ in the sequel.*

Let us recall some known facts.

1. The hypothesis $H(\alpha, 1)$ is equivalent to the fact that, in open neighborhood of the origin, the local *reduced* b-function of f at any point $x \neq 0$ has no root in $-\alpha - \mathbb{N}$.

2. The hypothesis $H(\alpha, 1)$ is equivalent to the fact that, in an open neighborhood of the origin, the polar parts of the meromorphic extension of the distributions

$$\frac{1}{\Gamma(\lambda)} |f|^{2\lambda} \bar{f}^{-h}, \quad \forall h \in \mathbb{Z},$$

at points in $-\alpha - \mathbb{N}$ are distributions with support $\{0\}$.

3. The hypothesis $H(\alpha,1)$ is equivalent to the fact that, for any test form φ in $\mathscr{C}_c^\infty(\mathbb{C}^{n+1})^{n+1,n+1}$ with compact support in $X\setminus\{0\}$ the meromorphic extension of the functions
$$\frac{1}{\Gamma(\lambda)}\int_X |f|^{2\lambda}\bar{f}^{-h}\varphi$$
has no pole in $-\alpha-\mathbb{N}$ for each $h\in\mathbb{Z}$.

4. Since the monodromy of f is defined on $H^p(F_y,\mathbb{Z})$, for $\alpha\in]0,1[$ the hypothesis $H(\alpha,1)$ is equivalent to the hypothesis $H(1-\alpha,1)$.
So the hypotheses of isolated singularity at the origin for the eigenvalues $\exp(\pm 2i\pi\alpha)$ of the monodromy for a holomorphic germ f are equivalent.

Assume that we are in the standard situation and that f satisfies the hypothesis $H(\alpha,1)$, that is to say that f has an isolated singularity for the eigenvalue $\exp(2i\pi\alpha)$ of its monodromy.
Let $\omega,\omega'\in\Omega_0^{n+1}$ and let $\rho\in\mathscr{C}_c^\infty(\mathbb{C}^{n+1})$ such that $\rho\equiv 1$ near 0 and its support is small enough in order that $\rho\omega\wedge\bar{\omega}'$ is a well defined and $\mathscr{C}_c^\infty$ differential form of type $(n+1,n+1)$ on $\mathbb{C}^{n+1}$. Then for any $h\in\mathbb{Z}$ the holomorphic function, defined for $2\Re(\lambda)>\sup\{0,h\}$ by the formula
$$F_h^{\omega,\omega'}(\lambda):=\frac{1}{\Gamma(\lambda)}\int_X |f|^{2\lambda}\bar{f}^{-h}\rho\omega\wedge\bar{\omega}' \qquad (\mathrm{F})$$
has a meromorphic continuation to the all complex plane with poles in $-\mathscr{A}-\mathbb{N}$ where $-\mathscr{A}$ is the finite subset of $\mathbb{Q}^-$ which is the set of the roots of the reduced Bernstein polynomial $\tilde{b}_{f,0}$ of f at the origin.
Moreover, thanks to our hypothesis $H(\alpha,1)$ we have the following properties (see [12] for a proof) :

1. The polar parts of $F_h^{\omega,\omega'}(\lambda)$ on the points in $-\alpha-\mathbb{N}$ do not depend on the choice (with the conditions specified above) of the function ρ.

2. The polar parts of $F_h^{\omega,\omega'}(\lambda)$ at points in $-\alpha-\mathbb{N}$ depend, for given ω' and h, only on the image of ω in the formal (a,b)-module $\widehat{H}_0^{n+1}$, which is the formal completion of the geometric (a,b)-module H_0^{n+1} defined in section 8.1.

The following result is proved in [12] Proposition 3.1.1.

Proposition 8.2.2 *In the standard situation assume that the hypothesis $H(\alpha,1)$ is satisfied. Let ω and ω' be holomorphic $(n+1)$-differential forms on X_0 and let ρ be a $\mathscr{C}^\infty$ function with compact support in X_0 which satisfies $\rho\equiv 1$ near 0. We have the following properties:*

i) If there exists $v\in\Omega^n(X_0)$ satisfying $df\wedge v\equiv 0$ and $dv=\omega$ on X_0, then $F_h^{\omega,\omega'}(\lambda)$ has no pole in $-\alpha-\mathbb{N}$ for any $h\in\mathbb{Z}$ and any $\omega'\in\Omega_0^{n+1}$.

ii) $F_h^{a\omega,\omega'}(\lambda)-(\lambda+1)F_{h-1}^{\omega,\omega}(\lambda+1)$ has no pole in $-\alpha-\mathbb{N}$ for any $h\in\mathbb{Z}$ and any $\omega'\in\Omega_0^{n+1}$.

iii) $F_h^{b\omega,\omega'}(\lambda)+F_{h-1}^{\omega,\omega'}(\lambda+1)$ *has no pole in* $-\alpha-\mathbb{N}$ *for any* $h\in\mathbb{Z}$ *and any* $\omega'\in\Omega_0^{n+1}$.

iv) For any complex number μ, $F_h^{(a-\mu b)\omega,\omega'}(\lambda)-(\lambda+\mu+1)F_{h-1}^{\omega,\omega'}(\lambda+1)$ *has no pole in* $-\alpha-\mathbb{N}$ *for any* $h\in\mathbb{Z}$ *and any* $\omega'\in\Omega_0^{n+1}$. ■

An easy consequence of the proposition above is the following:

Corollary 8.2.3 *Under the hypothesis* $H(\alpha,1)$ *assume that the meromorphic extension of the holomorphic function* $F_h^{\omega,\omega'}(\lambda)$ *has never a pole of order* $\geq p$ *at each point in* $-\alpha-\mathbb{N}$ *for some given* $\omega'\in\Omega_0^{n+1}$ *but for each* $h\in\mathbb{Z}$. *Then the same is true for any* $w\in\Omega_0^{n+1}$ *such that* $[w]$ *is in the fresco* $\mathcal{F}_\omega=\tilde{\mathcal{A}}_{conv.}[\omega]\subset H_0^{n+1}$.

Proof of Corollary 8.2.3. Assume that the result is not true. So we have a $P\in B[a]$, an integer $m\in\mathbb{N}$ and some $h\in\mathbb{Z}$ such that $F_h^{P\omega,\omega'}(\lambda)$ has a pole of order at least equal to p at the point $-\alpha-m$. First remark that if $p+q\geq m+1$ the points *ii*) and *iii*) of Proposition 8.2.2 show that $F_h^{a^pb^q\omega,\omega'}(\lambda)$ has no pole at the point in $-\alpha-m$. So we may assume that the total degree of P in (a,b) is bounded by $m+1$ and the previous proposition gives a contradiction with our assumption.■

The following important tool for the sequel is also a consequence of Proposition 8.2.2, using the Structure Theorem for frescos of [9] extended in Corollary 7.1.2 to the convergent case.

The following terminology will be convenient:

The property $P(\omega,\omega',p)$. In the standard situation with the hypothesis $H(\alpha,1)$ fix two holomorphic germs ω,ω' in Ω_0^{n+1}. Let $p\geq 1$ be an integer and assume that there exists $h\in\mathbb{Z}$ such that $F_h^{\omega,\omega'}(\lambda)$ has a pole of order at least equal to p at a point in $-\alpha-\mathbb{N}$. Then we shall say that the integer m has the property $P(\omega,\omega',p)$ when m is the smallest integer such that there exists an integer $h\in\mathbb{Z}$ with a pole of order $\geq p$ at the point $\lambda=-\alpha-m$ for $F_h^{\omega,\omega'}(\lambda)$.

Proposition 8.2.4 *In the situation described above, assume that, for some* $h\in\mathbb{Z}$, *there exists a pole of order* $\geq p$ *at the point* $-\alpha-m$ *for* $F_h^{\omega,\omega'}(\lambda)$. *Then the following properties hold true:*

1. *Assume that the integer* m *satisfies the property* $P(\omega,\omega',p)$. *For each* $S\in\widehat{B}$ *such that* $S(0)\neq 0$ *there exists a pole of order at least equal to* p *for* $F_{h+1}^{S(b)\omega,\omega'}(\lambda)$ *at the point* $-\alpha-m$. *Moreover, the integer* m *satisfies also the property* $P(S\omega,\omega',p)$.

2. *If* $\mu\neq\alpha+m$ *there exists a pole of order at least equal to* p *for* $F_{h+1}^{(a-\mu b)\omega,\omega'}(\lambda)$ *at the point* $-\alpha-m-1$. *Moreover, if the integer* m *satisfies the property* $P(\omega,\omega',p)$, *the integer* $m+1$ *satisfies the property* $P((a-\mu b)\omega,\omega',p)$.

3. *For $\mu = \alpha + m$, there exists a pole of order at least equal to $p-1$ for $F_{h+1}^{(a-\mu b)\omega,\omega'}(\lambda)$ at the point $-\alpha - m - 1$.*

PROOF. Assume that X is a sufficiently small open neighborhood of 0 in $\mathbb{C}^{n+1}$ such that the germs ω and ω' are holomorphic on X and that there exists $u \in \Omega^n(X)$ satisfying $du = \omega$ on X.
Thanks to Stokes Formula and hypothesis $H(\alpha, 1)$ (see Proposition 3.1.1 in [12] or Proposition 8.2.2 above) the meromorphic function

$$F_h^{b\omega,\omega'}(\lambda) + (\lambda+1)F_{h-1}^{\omega,\omega'}(\lambda+1) = -\frac{1}{\Gamma(\lambda)}\int_X |f|^{2\lambda}\bar{f}^{-h}d\rho \wedge u \wedge \bar{\omega}'$$

has no poles at points in $-\alpha - \mathbb{N}$ for any choice of ω', h and $\rho \in \mathscr{C}_c^\infty(X)$ which is identically 1 near the origin.
Since m satisfies Property $P(\omega, \omega', p)$, it is clear that for any positive integer q, $F_{h'}^{b^q\omega,\omega'}(\lambda)$ has no pole of order $\geq p$ at $-\alpha - m'$ for each $m' \leq m - q$. Since we have never a pole for $F_h^{\omega,\omega'}(\lambda)$ at points where $\Re(\lambda) \geq 0$, we conclude that for any $S \in \widehat{B}$ with $S(0) \neq 0$ we have a pole of order p for $F_{h+1}^{S(b)\omega,\omega'}(\lambda)$ at the point $-\alpha - m$. Moreover m satisfies the property $P(S\omega, \omega', p)$.
With the same arguments (and the same Proposition 3.1.1 in [12] or Proposition 8.2.2) the meromorphic function

$$F_h^{(a-\mu b)\omega,\omega'}(\lambda) - (\lambda+\mu+1)F_{h-1}^{\omega,\omega'}(\lambda+1)$$

has no pole at points in $-\alpha - \mathbb{N}$ for any choice of ω', h and $\rho \in \mathscr{C}_c^\infty(X)$ which is identically 0 near the origin.
Now the same line of proof gives the assertions 2 and 3 of the proposition using point *iv*) in Proposition 8.2.2. ■

Here appears the main strategy of proof to locate the bigger order p pole in $-\alpha - \mathbb{N}$ for a given pair ω, ω'.

Corollary 8.2.5 *Assume that there exists a pole of order at least equal to p at the point $-\alpha - m$ for $F_h^{\omega,\omega'}(\lambda)$ for some integer $h \in \mathbb{Z}$ and assume that the integer m satisfies Property $P(\omega, \omega', p)$.*
Let $\Pi := (a - \mu_1 b)S_1(a - \mu_2 b)S_2 \ldots (a - \mu_k b)S_k$ where $S_1, \ldots, S_k$ are invertible elements in B and $\mu_1, \ldots, \mu_k$ are positive rational numbers.

1. *Assume that $\mu_j + j - k \neq \alpha + m$ for each $j \in [1, k]$. Then $F_{h+k}^{\Pi\omega,\omega'}(\lambda)$ has a pole of order at least equal to p at the point $-\alpha - m - k$. Moreover the integer $m + k$ satisfies the property $P(\Pi\omega, \omega', p)$.*

2. *If μ_1 is the only value of $j \in [1, k]$ such that $\mu + j - k = \alpha + m$ then $F_{h+k}^{\Pi\omega,\omega'}(\lambda)$ has a pole of order $p - 1$ at the point $-\alpha - m - k$.*

PROOF. Using inductively the assertions $1, 2, 3$ of the previous proposition gives this corollary. ■

Corollary 8.2.6 *Assume $H(\alpha, 1)$ and that the nilpotent order of $(\tilde{\mathcal{A}}_{conv.}\omega)_{[\alpha]}$ (the $[\alpha]$-primitive part of the fresco $\tilde{\mathcal{A}}_{conv.}\omega$) is at most $p-1$. Then for any choice of ω' and h, the meromorphic extension of $F_h^{\omega,\omega'}(\lambda)$ has no pole of order $\geq p$ at each point in $-\alpha - \mathbb{N}$.*

PROOF. We shall prove the result by induction on $p \geq 1$. For $p = 1$ our hypothesis means that $(\tilde{\mathcal{A}}_{conv.}\omega)_{[\alpha]} = \{0\}$ so if $\Pi := (a - \mu_1 b)S_1(a - \mu_2 b)S_2 \dots (a - \mu_k b)S_k$ where $S_1, \dots, S_k$ are invertible elements in B, generates the annihilator of $[\omega]$ in the geometric (a,b)-module H_0^{n+1} (see Corollary 7.1.2), we may assume that $\mu_1, \dots, \mu_k$ are not in $-\alpha - \mathbb{N}$. Then, since $F_h^{\Pi\omega,\omega'}(\lambda)$ has no poles in $-\alpha - \mathbb{N}$ (see Proposition 8.2.2 i)), we obtain immediately a contradiction with the assertion of Corollary 8.2.5 if we assume that for some choice of ω' and h the meromorphic function has a pole at some point $-\alpha - m$.
Thanks to the case proved above, we may replace ω by a generator of the fresco $\tilde{\mathcal{A}}_{conv.}\omega / (\tilde{\mathcal{A}}_{conv.}\omega)_{\neq[\alpha]}$, which means that we may assume now that $\tilde{\mathcal{A}}_{conv.}\omega$ is an $[\alpha]$-primitive fresco with nilpotent order at most $p-1$ with $p \geq 2$ (note that we use here Lemma 6.3.6).
Define $\mathcal{F} := S_{p-1}(\tilde{\mathcal{A}}_{conv.}\omega)/S_{p-2}(\tilde{\mathcal{A}}_{conv.}\omega)$. This fresco is $[\alpha]$-primitive, semi-simple and generated by $[\omega]$. So the generator $\Pi := (a - \mu_1 b)S_1(a - \mu_2 b)S_2 \dots (a - \mu_k b)S_k$ where $S_1, \dots, S_k$ are invertible elements in B, of the annihilator of the class $[\omega]$ in this semi-simple fresco may be chosen (see Proposition 7.2.1) such that we may choose any order for the sequence $\mu_j + j$. Since these numbers are pairwise distinct there exists at most one $j \in [1, k]$ such that $\mu_j + j - k = \alpha + m$. We have two cases : either there is no such $j \in [1, k]$ or there exists a unique $j \in [1, k]$ such that $\mu_{j_0} + j_0 - k = \alpha + m$ and in this case we may choose $j_0 = 1$.
So using inductively Corollary 8.2.5 we see that if we assume that $F_h^{\omega,\omega'}(\lambda)$ has a pole of order $\geq p$ at the point $-\alpha - m$, we shall find a pole of order $\geq p-1$ for $F_{h+k}^{\Pi\omega,\omega'}(\lambda)$ at the point $-\alpha - m - k$. Since the fresco $\mathcal{G}$ generated by the class $\Pi[\omega]$ satisfies $\mathcal{G} = S_{p-2}(\tilde{\mathcal{A}}_{conv.}[\omega])$, its nilpotent order is at most equal to $p-2$. This contradicts our induction hypothesis.
The case where there is no $j \in [1, k]$ such that $\mu_j + j - k = \alpha + m$ leads to a pole of order $\geq p$ at the point $-\alpha - m - k$, so gives also a contradiction. ■

8.3 The final key

Note that in Section 8.2 we always assume the existence of poles at some point in $-\alpha - \mathbb{N}$ for $F_h^{\omega,\omega'}(\lambda)$ (under our hypothesis $H(\alpha, 1)$) and obtain consequences on the Bernstein polynomial of the fresco $\mathcal{F}_\omega$. These results go in the same direction than

the results in [12]. To go in the other direction, that is to say to prove the existence of such poles as consequence of informations on the Bernstein polynomial of $\mathcal{F}_\omega$, we shall use now the main idea of [3] (and also [4] in the case $\alpha = 1$). This is the point where the use of convergent (a,b)-modules is essential. It allows to show that the non vanishing of the class induced by ω in the $[\alpha]$-primitive part of H_0^{n+1} implies that the cohomology class induces by ω/df in the spectral part for the eigenvalue $\exp(-2i\pi\alpha)$ of the monodromy of f acting on $H^n(F_0, \mathbb{C})$ does not vanish.

Theorem 8.3.1 *Assume that $H(\alpha, 1)$ is satisfied by $f : X \to D$, a Milnor representative of a holomorphic germ near the origin in $\mathbb{C}^{n+1}$. Let $u \in \Omega^n(X)$ such that there exists $m \in \mathbb{N}$ with $fdu = (\alpha+m)df \wedge u$ on X and assume that the class induced by u in $H^n(F_0, \mathbb{C})$ is not 0. Then there exists a germ $\omega' \in \Omega_0^{n+1}$, and an integer $h \in \mathbb{N}$ such that for any $\rho \in \mathscr{C}_c^\infty(X)$ which is identically 1 near 0 and with support small enough in order that $\rho\omega'$ is in $\mathscr{C}_c^\infty(X)$, the meromorphic extension of*

$$\frac{1}{\Gamma(\lambda)} \int_X |f|^{2\lambda} \bar{f}^{-h} \rho \frac{df}{f} \wedge u \wedge \bar{\omega'} \tag{16}$$

has a pole at $-\alpha - m$.

Proof. Define, for $j \in \mathbb{N}$, the $(n, 0)-$current on X by the formula[16]

$$\langle T_j, \psi \rangle := Pf(\lambda = -\alpha - m,\ \frac{1}{\Gamma(\lambda)} \int_X |f|^{2\lambda} \bar{f}^{-j} u \wedge \psi)$$

where ψ is a test form of type $(1, n+1)$ which is $\mathscr{C}_c^\infty$ in X.

Claim. Then we have the following properties for each $j \in \mathbb{N}$

1. $\bar{f} T_{j+1} = T_j$ on X
2. The support of the current $d'T_j$ is contained in $\{0\}$.
3. The support of the current $d''T_j + (\alpha + m + j)\bar{df} \wedge T_{j+1}$ is contained in $\{0\}$.

proof of the claim. The first assertion is clear.
Let us compute $d'T_j$. Let φ be a $\mathscr{C}_c^\infty(X)$ test form of type $(0, n+1)$. We have

$$\langle d'T_j, \varphi \rangle := (-1)^n \langle T_j, d'\varphi \rangle = (-1)^n \langle T_j, d\varphi \rangle$$

But for $\Re(\lambda) \gg 1$ the form $|f|^{2\lambda} \bar{f}^{-j} \cdot u \wedge \varphi$ is in $\mathscr{C}_c^1(X)$ and Stokes Formula and the meromorphic continuation give

$$0 = \frac{1}{\Gamma(\lambda)} \int_X d\big(|f|^{2\lambda} \bar{f}^{-j} u \wedge \varphi\big) = \frac{(\lambda + \alpha + m)}{\Gamma(\lambda)} \int_X |f|^{2\lambda} \bar{f}^{-j} \frac{df}{f} \wedge u \wedge \varphi + (-1)^n \langle T_j, d'\varphi \rangle$$

[16]Here $Pf(\lambda = \lambda_0, F(\lambda))$ denote the constant term in the Laurent expansion at $\lambda = \lambda_0$ of the meromorphic function $F(\lambda)$.

because $du = (\alpha+m)\frac{df}{f}\wedge u$ and $d\bar f\wedge\varphi\equiv 0$. Then we obtain[17]

$$\langle d'T_j,\varphi\rangle = Res(\lambda=-\alpha-m,\ \frac{1}{\Gamma(\lambda)}\int_X |f|^{2\lambda}\bar f^{-j}\frac{df}{f}\wedge u\wedge\varphi) \tag{17}$$

This gives our assertion 2) because we know that the poles of the meromorphic extension of $\frac{1}{\Gamma(\lambda)}\int_X |f|^{2\lambda}\bar f^{-j}\Box$ at points in $-\alpha+\mathbb{Z}$ are supported by the origin, thanks to our hypothesis $H(\alpha,1)$.
In an analogous way let us compute $d''T_j$; let ψ be a $\mathscr{C}^\infty_c(X)$ test form of type $(1,n)$. We have:

$$\langle d''T_j,\psi\rangle := (-1)^n\langle T_j, d''\psi\rangle = (-1)^n\langle T_j, d\psi\rangle$$

But for $\Re(\lambda)\gg 1$ the form $|f|^{2\lambda}\bar f^{-j}u\wedge\psi$ is in $\mathscr{C}^1_c(X)$ and so:

$$d\big(|f|^{2\lambda}\bar f^{-j}u\wedge\psi\big) = (\lambda-j)|f|^{2\lambda}\bar f^{-j-1}\bar{df}\wedge u\wedge\psi + (-1)^n|f|^{2\lambda}\bar f^{-j}u\wedge d\psi$$

because the type of du as well as the type of $df\wedge u$ is $(0,n+1)$. Then Stokes Formula and the meromorphic continuation give

$$\langle d''T_j+(\alpha+m+j)\bar{df}\wedge T_{j+1},\psi\rangle = Res(\lambda=-\alpha-m,\ \frac{1}{\Gamma(\lambda)}\int_X |f|^{2\lambda}\bar f^{-j-1}d\bar f\wedge u\wedge\psi).$$

This proves the assertion 3), again thanks to our hypothesis $H(\alpha,1)$.
Now we shall argue by contradiction and we shall assume that for each $j_0\in\mathbb{N}$, the current $d'T_{j_0}$ induces the class zero in the conjugate of the space $H^{n+1}_{[0]}(X,\mathcal{O}_X)$ which means that there exists a $(n,0)-$current Θ_{j_0} with support $\{0\}$ satisfying $d'\Theta_{j_0}=d'T_{j_0}$ on X. Then, as we have $\bar f^k T_{j_0}=T_{j_0-k}$ for any $k\in\mathbb{N}$ thanks to 1), we obtain that $d'\bar f^k\Theta_{j_0}=\bar f^k d'\Theta_{j_0}=\bar f^k T_{j_0}=T_{j_0-k}$. Now we fix $j_0\gg 1$ and define, for each $j\le j_0$, $\Theta_j := \bar f^{j_0-j}\Theta_{j_0}$. So for any such $j\le j_0$ this gives $d'\Theta_j=d'T_j$.
Now we shall use Lemma C_1, C_2 and Lemma D in [3] and Lemma C_1', C_2' in [4] in the case $\alpha=1$, for the family of currents $\tilde T_j := T_j-\Theta_j$ for $j\le j_0$. They satisfy

1. $d'\tilde T_j = 0$ on X.
2. $d''\tilde T_j + (\alpha+m+j)\bar{df}\wedge\tilde T_{j+1}$ has its support in $\{0\}$.
3. The current $\tilde T_j$ coincides with $|f|^{-2(\alpha+m)}\bar f^{-j}u$ on the Milnor fiber $F_0=f^{-1}(s_0)$ (these currents are smooth outside Y).

Note that we have $H^p(X\setminus\{0\},\mathcal{O}_X)=0$ for $1\le p\le n-1$ which is used for checking the hypothesis of Lemmas C_1', C_2' in the case $\alpha=1$.
Then we contradict our assumption that the class induced by u in $H^n(F_0,\mathbb{C})$ does not vanish.
So we obtain that there exists $j_0\in\mathbb{N}$ such that the class induced by $d'T_{j_0}$ does not vanish in the dual of the space $\overline{\Omega^{n+1}_0}$ of the germs at the origin of anti-holomorphic volume forms on $\mathbb{C}^{n+1}$ (and then for any j_0+k for $k\in\mathbb{N}$ also). So there exists

[17]Here $Res(\lambda=\lambda_0,F(\lambda))$ denotes the residue at $\lambda=\lambda_0$ of the meromorphic function $F(\lambda)$.

$\omega' \in \Omega_0^{n+1}$ and $\rho \in \mathscr{C}_c^\infty(X)$ which is identically 1 near 0 and with support small enough in order that $\rho\omega'$ is in $\mathscr{C}_c^\infty(X)$ such that

$$\langle d'T_{j_0}, \rho\overline{\omega}'\rangle = Res\big(\lambda = -\alpha - m,\ \frac{1}{\Gamma(\lambda)}\int_X |f|^{2\lambda}\bar{f}^{-j_0}\frac{df}{f}\wedge u\wedge\rho\overline{\omega}'\big) \neq 0$$

concluding the proof of the theorem. ■

REMARK. To apply the results in degree n of [3] and [4] used in the previous proof, it is enough to assume that $j_0 = n+1$ to conlude. That is to say that if there is no pole in the range $[-\alpha - n - 1, -\alpha] \cap \{-\alpha - \mathbb{N}\}$ for $F_h^{\omega,\omega'}(\lambda)$, there is no pole in $-\alpha - \mathbb{N}$.

Corollary 8.3.2 *Assume that we have holomorphic forms $u_j \in \Omega^n(X)$ for each integer j in $[-N, p]$ such that*

$$f du_j = (\alpha + m)df \wedge u_j = df \wedge u_{j-1}$$

with the hypothesis that $[du_j] = 0$ in H_0^{n+1} for each $j \in [-N, 0]$ and with $u_1 = u$ as in the previous theorem (so the class induced by u in $H^n(F_0, \mathbb{C})$ is not 0).
Then there exists $h \in \mathbb{N}$ and $\omega' \in \Omega_0^{n+1}$ such that the meromorphic extension of

$$\frac{1}{\Gamma(\lambda)}\int_X |f|^{2\lambda}\bar{f}^{-h}\rho\frac{df}{f}\wedge u_p\wedge\bar{\omega}' \tag{18}$$

has a pole of order at least equal to p at the point $\lambda = -\alpha - m$.

PROOF. For $\Re(\lambda) \gg 1$ the differential form $|f|^{2\lambda}\bar{f}^{-h}\rho u_j \wedge \bar{\omega}'$ is of class $\mathscr{C}^1$ and satisfies

$$\begin{aligned} d(|f|^{2\lambda}\bar{f}^{-h}\rho u_j\wedge\bar{\omega}') &= (\lambda+\alpha+m)|f|^{2\lambda}\bar{f}^{-h}\rho(df/f)\wedge u_j\wedge\bar{\omega}' + \\ &+ |f|^{2\lambda}\bar{f}^{-h}\rho(df/f)\wedge u_{j-1}\wedge\bar{\omega}' + |f|^{2\lambda}\bar{f}^{-h}d\rho\wedge u_j\wedge\bar{\omega}'. \end{aligned}$$

Then Stokes Formula and the meromorphic extension gives, where $P_q(\lambda = \lambda_0, F(\lambda))$ means the coefficient of $(\lambda - \lambda_0)^{-q}$ in the Laurent expansion of the meromorphic function F at the point $\lambda = \lambda_0$, that for each $q \geq 0$ we have:

$$\begin{aligned} &P_{q+1}(\lambda = -\alpha - m, \frac{1}{\Gamma(\lambda)}\int_X |f|^{2\lambda}\bar{f}^{-h}\rho(df/f)\wedge u_j\wedge\bar{\omega}') = \\ &- P_q(\lambda = -\alpha - m, \frac{1}{\Gamma(\lambda)}\int_X |f|^{2\lambda}\bar{f}^{-h}\rho(df/f)\wedge u_{j-1}\wedge\bar{\omega}'). \end{aligned}$$

Then the fact that there exists $h \in \mathbb{N}$ and $\omega' \in \Omega_0^{n+1}$ with (here $P_1 = Res$!)

$$P_1(\lambda = -\alpha - m, \frac{1}{\Gamma(\lambda)}\int_X |f|^{2\lambda}\bar{f}^{-h}\rho(df/f)\wedge u_1\wedge\bar{\omega}') \neq 0$$

which is the content of the previous theorem since we assume thet $u_1 = u$, implies

$$P_p(\lambda = -\alpha - m, \frac{1}{\Gamma(\lambda)} \int_X |f|^{2\lambda} \bar{f}^{-h} \rho(df/f) \wedge u_p \wedge \bar{\omega}') \neq 0$$

concluding the proof. ■

To be able to use the previous corollary, the following lemma, combined with Corollary 7.6.2 will be useful.

Lemma 8.3.3 *Let $w_1, \ldots, w_p$ be in Ω_0^{n+1} such that the induced class in H_0^{n+1} satisfy the relations:*

$$a[w_j] = (\alpha + m)b[w_j] + b[w_{j-1}] \quad \forall j \in [1, p] \quad \text{with the convention} \quad [w_0] = 0 \quad (\star)$$

Then there exists an integer N and $u_1 \ldots, u_p$ in Ω_0^n such that

$$f du_j = (\alpha + m + N) df \wedge u_j + df \wedge u_{j-1} \quad \text{with the convention} \quad u_0 = 0 \qquad (\star\star)$$

and such that we have $[du_j] = (a + b)^N [w_j]$ in H_0^{n+1}.

Proof. Choose for each $j \in [1, p]$ a $v_j \in \Omega_0^n$ such that $dv_j = w_j$. Then for each $j \in [1, p]$ the class induced in H_0^{n+1} by the form

$$f dv_j - (\alpha + m) df \wedge v_j - df \wedge v_{j-1} \quad \text{with the convention } v_0 = 0$$

is of a-torsion $\mathcal{H}_0^{n+1}$. So there exists an integer N and $t_j \in (Ker\, df)_0^n$ such that, for $j \in [1, p]$, we have

$$f^{N+1} dv_j - (\alpha + m) df \wedge f^N v_j - df \wedge f^N v_{j-1} = f dt_j.$$

This equality may be written

$$f d(f^N v_j + t_j) - (\alpha + m + N) df \wedge (f^N v_j + t_j) - df \wedge (f^N v_{j-1} + t_{j-1}) = 0$$

with the convention $t_0 = 0$, using the fact that $df \wedge t_j = 0$ for each j. Then defining $u_j := f^N v_j + t_j$ for $j \in [1, p]$ concludes the proof since the class induced in H_0^{n+1} by du_j is equal to $a^N[w_j] + N a^{N-1} b[w_j]$, thanks to the equality $a^N + N a^{N-1} b = (a+b)^N$ (see the exercise below). ■

Exercise. Show that the commutation relation $ab - ba = b^2$ implies the relation[18]

$$(a + b)^q = a^{q-1}(a + qb) \quad \forall q \in \mathbb{N}^*$$

[18] Compare with Corollary 2.4.2.

8.4 Statements and proofs

Our first result gives an improvement of the result in [12] but is also a precise converse of this statement. It shows the interest in considering the higher order Bernstein polynomials introduced in section 7.

Theorem 8.4.1 *In the standard situation described above, assume that the hypothesis $H(\alpha,1)$ is satisfied. Consider a germ $\omega \in \Omega_0^{n+1}$ such that the p-th Bernstein polynomial of the fresco $\mathcal{F}_\omega := \tilde{\mathcal{A}}_{conv.}\omega$ in H_0^{n+1} has a root in $-\alpha - \mathbb{N}$. Then there exists $\omega' \in \Omega_0^{n+1}$ and an integer h such that the meromorphic extension of the integral*

$$F_{\omega,h}^{\omega'}(\lambda) := \frac{1}{\Gamma(\lambda)} \int_X |f|^{2\lambda} \bar{f}^{-h} \rho\omega \wedge \bar{\omega}' \qquad \text{(A)}$$

has a pole of order at least equal to p at $\lambda = -\alpha - m$ for m a large enough integer, where $\rho \in \mathscr{C}_c^\infty(X)$ is identically 1 near zero.

Remark. The converse of this result, that is to say the fact that, for a germ $\omega \in \Omega_0^{n+1}$, the existence of such ω', h, m giving a pole of order p at a point in $-\alpha - \mathbb{N}$ for (A) implies that the p-th Bernstein polynomial of the fresco $\mathcal{F}_\omega = \tilde{\mathcal{A}}_{conv.}\omega$ has a root in $-\alpha - \mathbb{N}$, will be a consequence of the Theorem 8.5.3 which is more precise, using the following consequence of Proposition 4.2.8 :

- If the q-th Bernstein polynomial of the fresco $\mathcal{F}$ has a root in $-\alpha - \mathbb{N}$ then for each $p \in [1,q]$ the p-th Bernstein polynomial of $\mathcal{F}$ has also a root in $-\alpha - \mathbb{N}$.

For the proof of Theorem 8.4.1 we shall need the following result.

Proposition 8.4.2 *Assume that the hypothesis $H(\alpha,1)$. Suppose that $u_1 \in \Omega_0^n$ satisfies the relation $f du_1 = (\alpha + m) df \wedge u_1$ for some integer m. If $[du_1]$ is not zero in H_0^{n+1} then the cohomology class induced by u_1 in $H^n(F_0, \mathbb{C})$ is not zero. So $u_{|F_0}$ induces a class which is an eigenvector of the monodromy for the eigenvalue $\exp(-2i\pi\alpha)$.*

Proof. Thanks to Grothendieck (see [16]), the meromorphic relative de Rham complex of f computes the cohomology of $X \setminus f^{-1}(0)$ and under the hypothesis $H(\alpha,1)$ the spectral sub-space $H^n(F_0,\mathbb{C})_{\exp(-2i\pi\alpha)}$ of the monodromy is isomorphic to the n-th cohomology group of the complex

$$\left(\Omega_0^\bullet[f^{-1}] / df \wedge \Omega_0^{\bullet-1}[f^{-1}], (d - \alpha \frac{df}{f} \wedge)^\bullet \right).$$

If we assume that u_1 induces 0 in $H^n(F_0,\mathbb{C})$, since we have

$$d(f^{-m}u_1) - \alpha \frac{df}{f} \wedge f^{-m}u_1 = 0,$$

there exists $v, w \in \Omega_0^{n-1}[f^{-1}]$ such that

$$dv - \alpha \frac{df}{f} \wedge v = f^{-m} u_1 + df \wedge w.$$

This gives

$$d(f^{-m}u_1) = -df \wedge dw + \alpha df \wedge v/f \quad \text{and then}$$
$$f^{-m}du_1 - m\frac{df}{f} \wedge f^{-m}u_1 = (1 - m/(m+\alpha))f^{-m}du_1 = df \wedge d(-w + \alpha v/f)$$

and this implies, since α is in $]0,1]$, that $[du_1]$ is of a-torsion in $\mathcal{H}_0^{n+1}$ and then 0 in H_0^{n+1}. Contradiction. ■

Proof of Theorem 8.4.1. Using Corollary 7.6.2 there exist $[w_1], \ldots, [w_p]$ in $\mathcal{F}_\omega$ and an integer $m \in \mathbb{N}$ satisfying the relations:

$$a[w_j] = (\alpha + m)b[w_j] + b[w_{j-1}] \quad \forall j \in [1,p] \quad \text{with the convention} \quad [w_0] = 0 \quad (*)$$

and which are B-linearly independent in $\mathcal{F}_\omega$. Assuming that the Theorem does not hold would imply, thanks to Corollary 8.2.3 and to Corollary 8.3.2, that writing $w_1 = du_1$ with $u_1 \in \Omega_0^n$, the class induced by u_1 in $H^n(F_0, \mathbb{C})$ vanishes.
But this contradicts the hypothesis that $[w_1]$ is not zero in $\mathcal{F}_\omega \subset H_0^{n+1}$, thanks to Proposition 8.4.2. ■

The following corollary of Theorem 8.4.1 is clear since we may use a Bernstein identity at the origin to describe the poles of the meromorphic extension of the distribution $\frac{1}{\Gamma(\lambda)}|f|^{2\lambda}\bar{f}^{-h}$ for any $h \in \mathbb{Z}$ (see [2] or [15]).

Corollary 8.4.3 *In the situation of the previous theorem, the existence of a germ $\omega \in \Omega_0^{n+1}$ such that the p-th Bernstein polynomial of the fresco $\tilde{\mathcal{A}}_{conv.}\omega \subset H_0^{n+1}$ has a root in $-\alpha - \mathbb{N}$ implies the existence of at least p roots of the reduced b-function $b_{f,0}$ of f at the origin in $-\alpha - \mathbb{N}$ counting multiplicities.* ■

Remark. The interest of this corollary lies in the fact that the existence of p roots in $-\alpha - \mathbb{N}$ for the reduced Bernstein polynomial $b_{f,0}$ does not implies, in general under our hypothesis, the existence of a pole of order p at some point $\lambda = -\alpha - m$ with $m \in \mathbb{N}$ large, for the meromorphic extension of $\frac{1}{\Gamma(\lambda)}\int_X |f|^{2\lambda}\bar{f}^{-h}\varphi$ for some test $(n+1, n+1)$-form φ.
The consideration of higher order Bernstein polynomials of frescos associated to germs $\omega \in \Omega_0^{n+1}$ is then a tool which may help to determine the nilpotency order of the monodromy of f at the origin in the case of an isolated singularity for the eigenvalue $\exp(2i\pi\alpha)$.

Our next result is an improvement of Theorem 8.4.1.

Theorem 8.4.4 *In the standard situation, we assume that the hypothesis $H(\alpha,1)$ is satisfied. Assume that there exists $\omega \in \Omega_0^{n+1}$ such that $B_p(\mathcal{F}^{[\alpha]})$ has a root in $-\alpha-\mathbb{N}$ where $\mathcal{F}^{[\alpha]} := \mathcal{F}_\omega/(\mathcal{F}_\omega)_{[\neq\alpha]}$ and where $p = d(\mathcal{F}^{[\alpha]})$ is the nilpotent order of the fresco $\mathcal{F}^{[\alpha]}$. Let $-\alpha-m$ the biggest root of $B_p(\mathcal{F}^{[\alpha]})$ in $-\alpha-\mathbb{N}$. Then there exists $\omega' \in \Omega_0^{n+1}$ and $h \in \mathbb{Z}$ such that $F_h^{\omega,\omega'}(\lambda)$ has a pole of order p at the point $-\alpha-m$.*

Recall that, of course, in the previous statement $B_p(\mathcal{F}$ denotes the p-th Bernstein polynomial of the fresco $\mathcal{F}$.

PROOF. First recall that, thanks to Lemma 6.3.6, we have, for any geometric (a,b)-module $\mathcal{E}$, the equality $d(\mathcal{E}_{[\alpha]}) = d(\mathcal{E}/\mathcal{E}_{[\neq\alpha]}) = d(\mathcal{E}^{[\alpha]})$.
Let $-\alpha-m$ be the biggest root of $B_p(\mathcal{F}^{[\alpha]})$. Then we may choose a J-H. sequence of $\mathcal{F}^{[\alpha]}/S_{p-1}(\mathcal{F}^{[\alpha]})$ such that its last quotient is isomorphic to $\mathcal{E}_{\alpha+m}$. This possible because the fresco $\mathcal{F}^{[\alpha]}/S_{p-1}(\mathcal{F}^{[\alpha]})$ is semi-simple and has $-\alpha-m$ as a root of its Bernstein polynomial (remind that the Bernstein polynomial fo $\mathcal{F}^{[\alpha]}/S_{p-1}(\mathcal{F}^{[\alpha]})$ divides $B_p(\mathcal{F}^{[\alpha]})$ which also divides $B_p(\mathcal{F})$). Then if Π_0 is the generator of the annihilator of $[\omega]$ in $\mathcal{F}^{[\alpha]}/S_{p-1}(\mathcal{F}^{[\alpha]})$, it may be written $\Pi_0 = (a-(\alpha+m+1-k)b)\Pi_0'$ where k is the rank of $\mathcal{F}^{[\alpha]}/S_{p-1}(\mathcal{F}^{[\alpha]})$. Then, choosing a J-H. sequence of $\mathcal{F}$ which begins by a J-H. sequence of $\mathcal{F}_{[\neq\alpha]}$ and ending by the J-H. sequence of $\mathcal{F}^{[\alpha]}$ chosen above, we see that the annihilator of ω in $\mathcal{F}$ may be written as $\Pi = \Pi_2\Pi_1(a-(\alpha+m+1-k)b)\Pi_0'$ with $\tilde{\mathcal{A}}_{conv.}/\tilde{\mathcal{A}}_{conv.}\Pi_0'$ semi-simple $[\alpha]$-primitive with a Bernstein polynomial having roots strictly less than $-\alpha-m$, with $d(\tilde{\mathcal{A}}_{conv.}/\tilde{\mathcal{A}}_{conv.}\Pi_1) \leq p-1$, since this fresco is isomorphic to $S_{p-1}(\mathcal{F}^{[\alpha]})$ and with $(\tilde{\mathcal{A}}_{conv.}/\tilde{\mathcal{A}}_{conv.}\Pi_2)_{[\alpha]} = \{0\}$ since $\tilde{\mathcal{A}}_{conv.}/\tilde{\mathcal{A}}_{conv.}\Pi_2$ is isomorphic to $\mathcal{F}_{[\neq\alpha]}$.
Now, applying Theorem 8.4.1 we find $\omega' \in \Omega_0^{n+1}$, $h \in \mathbb{Z}$ and $m_1 \in \mathbb{N}$ such that $F_h^{\omega,\omega'}(\lambda)$ has an order p pole at the point $-\alpha-m_1$ and such that the integer m_1 satisfies the property $P(\omega,\omega',p)$.
Using then Proposition 8.2.2 we see that if $m \neq m_1$ we contradict Corollary 8.2.3 because we find a pole of order p at a point $-\alpha-m_1-k$ for $F_{h-q-1}^{\Pi_0\omega,\omega'}(\lambda)$ where k is the degree in a of Π_0. So we obtain $m = m_1$. ■

The following corollaries are obvious consequences of the previous result.

Corollary 8.4.5 *In the standard situation described above, under the assumption $H(\alpha,1)$, consider a germ $\omega \in \Omega_0$ and assume that $-\alpha-m$ is the biggest possible pole in $-\alpha-\mathbb{N}$ for any choices of $\omega' \in \Omega_0^{n+1}$ and any $h \in \mathbb{Z}$ for the meromorphic functions $F_h^{\omega,\omega'}(\lambda)$. Then $-\alpha-m$ is the biggest root in $-\alpha-\mathbb{N}$ of the Bernstein polynomial of the fresco $\mathcal{F}_\omega := (\tilde{\mathcal{A}}_{conv.}\omega) \subset H_0^{n+1}$.* ■

Corollary 8.4.6 *In the standard situation described above, under the assumption $H(\alpha,1)$, assume that $-\alpha-m$ is the biggest root of the Bernstein polynomial in $-\alpha-\mathbb{N}$ of the geometric (a,b)-module H_0^{n+1}. Then there exists $h \in \mathbb{Z}$ such the*

meromorphic extension of the distribution $\frac{1}{\Gamma(\lambda)}|f|^{2\lambda}\bar{f}^{-h}$ *has a pole at* $-\alpha-m$. ■

Note that it is enough to consider the integers $h \in [m+1, n-m]$ in the previous statement since the exponent of $\bar{f}$ has to be negative and thanks to the remark following Theorem 8.3.1 (which also implies that $\alpha + m \leq n+1$).

The following consequence of the previous corollary is obvious, since in the case of an isolated singularity at 0 for f it is known (see [20]) that the Brieskorn module coincides with H_0^{n+1} and that its Bernstein polynomial coincides with the reduced Bernstein polynomial $\tilde{b}_f$ of f.

Corollary 8.4.7 *Assume that the germ* $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ *of holomorphic function has an isolated singularity at the origin. For* $\alpha \in]0,1] \cap \mathbb{Q}$ *let* $-\alpha - m$ *be the biggest root of the reduced Bernstein polynomial of* f *in* $-\alpha - \mathbb{N}$. *Then there exists* $h \in \mathbb{Z}$ *such the meromorphic extension of the distribution* $|f|^{2\lambda}\bar{f}^{-h}/\Gamma(\lambda)$ *has a pole at* $-\alpha - m$. ■

Question. In the case of an isolated singularity for the eigenvalue $\exp(2i\pi\alpha)$ of the monodromy (so with our hypothesis $H(\alpha, 1)$), is the Bernstein polynomial of the geometric (a,b)-module $H_0^{n+1}/(H_0^{n+1})_{\neq[\alpha]}$ (which is the biggest polynomial having its root in $-\alpha-\mathbb{N}$ and dividing the Bernstein polynomial of the (a,b)-module H_0^{n+1}) coincides with the biggest polynomial having its root in $-\alpha - \mathbb{N}$ and dividing the reduced Bernstein polynomial of f at the origin ?

8.5 Some improvements of Theorem 3.1.2 in [12]

The goal of this paragraph is to show that, using the higher Bernstein polynomials of the fresco $\mathcal{F}_{f,\omega}$ generated by the class of ω in H_0^{n+1}, and the tools introduced above, we can improve the main result in [12] (Theorem 3.1.2). The converse of this result which is proved in Section 8.4 is in fact a converse of the improvement obtained below in Theorem 8.5.1.

We begin by some remarks to make clear the correspondence between our present notations with these used in [12].

Remarks.

1. We use here the notation $H(\alpha, 1)$ with $\alpha \in]0,1] \cap \mathbb{Q}$ instead of the notation $H(\xi, 1)$ with $\xi \in \mathbb{Q}$.

2. To consider a form $\psi \in \mathscr{C}_c^\infty(\mathbb{C}^{n+1})^{0,n+1}$ with small enough support and such that $d\psi = 0$ in a neighborhood of 0 is equivalent to consider $\rho.\bar{\omega}'$ where ω' is in Ω_0^{n+1} and ρ is a function in $\mathscr{C}_c^\infty(\mathbb{C}^{n+1})$ with small enough support which is

identically 1 near the origin.
Indeed any such ψ may be written as $\psi = \bar{\omega}'$ for some $\omega' \in \Omega^{n+1}$ near the origin thanks to Dolbeault' Lemma, and then $\psi - \rho\bar{\omega}'$ is identically 0 near the origin, so replacing ψ by $\rho\bar{\omega}'$ do not change the poles which may appear in $-\alpha - \mathbb{N}$ for the functions we are looking at (what ever is the choice of $h \in \mathbb{Z}$ thanks to our hypothesis $H(\alpha, 1)$.
Then, for ω, ω' in Ω_0^{n+1} we use the notation $F_h^{\omega,\omega'}(\lambda)$ where the function $\rho \in \mathscr{C}^\infty{}_c(\mathbb{C}^{n+1})$ which is identically 1 near the origin and has a sufficiently small support in order that $\rho\omega \wedge \bar{\omega}'$ is smooth, does not appear in this notation because the poles at points in $-\alpha - \mathbb{N}$ do not depend on the choice of this ρ. This corresponds to the notation $F_h^{\psi}(\lambda)$ where ψ is in $\mathscr{C}_c^\infty(X)^{0,n+1}$ is d-closed near the origin (where ω is given in $\Omega^{n+1}(X)$) and with $\psi = \rho\bar{\omega}'$.

3. Note also that we change the sign of the integer $h \in \mathbb{Z}$ between these two articles.

Theorem 8.5.1 *Let $\alpha \in]0, 1]$ and assume the hypothesis $H(\alpha, 1)$ for the germ at the origin in $\mathbb{C}^{n+1}$ of holomorphic function $\tilde{f} : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$. Assume that ω in Ω_0^{n+1} is such that there exists an integer $h \in \mathbb{Z}$ and a form $\omega' \in \Omega_0^{n+1}$ for which the function $F_h^{\omega,\omega'}(\lambda)$ has a pole of order $p \geq 1$ at some point ξ in $\{-\alpha - \mathbb{N}\}$. Note $\xi_p = -\alpha - m$ be the biggest such number ξ in $\{-\alpha - \mathbb{N}\}$ for any choice of ω' and $h \in \mathbb{Z}$. Then the p-th Bernstein polynomial of the fresco $\mathcal{F}_{f,\omega}$ has a root in $[-\alpha - m, -\alpha] \cap \mathbb{Z}$.*

Proof. Note $P := P_1P_2$ the annihilator of the class of $[\omega]$ in the $[\alpha]$-primitive quotient

$$\mathcal{F}^\alpha := \mathcal{F}_{f,\omega} \big/ \big(\mathcal{F}_{f,\omega}\big)_{\neq\alpha}$$

of the fresco $\mathcal{F}_{f,\omega} := \tilde{\mathcal{A}}[\omega]$ inside the (a,b)-module H_0^{n+1} associated to f, where P_2 is the annihilator of $[\omega]$ in $\mathcal{F}_{f,\omega}/S_{p-1}\big(\mathcal{F}^\alpha\big)$. If $F_h^{\omega,\omega'}(\lambda)$ has a pole of order at least equal to p at the point $-\alpha - m$ and if $-\alpha - m$ is not a root of the p-th Bernstein polynomial of $\mathcal{F}_{f,\omega}^\alpha$, then $-\alpha - m$ is not a root of the (usual) Bernstein polynomial of the fresco $\tilde{\mathcal{A}}/\tilde{\mathcal{A}}P_2$ which is isomorphic to $S_{p-1}(\mathcal{F}^\alpha)$. In this situation, using Corollary 8.2.5 we see $F_{h+p_2}^{P_2\omega,\omega'}(\lambda)$ has a pole of order at least equal to p at $-\alpha - m$, where k is the rank of the fresco $\mathcal{F}^\alpha/S_{p-1}\big(\mathcal{F}^\alpha\big)$. But this is impossible, according to Corollary 8.2.6 since the nilpotent order of $S_{p-1}\big(\mathcal{F}^\alpha\big)$ is $p - 1$. So $-\alpha - m$ is a root of some $(p + j)$-th Bernstein polynomial of $\mathcal{F}_{f,\omega}^\alpha$ for some integer j ≥ 0. ■

The end of Theorem 3.1.2 in [12] is also improved as follows:

Corollary 8.5.2 *In the situation of the previous theorem, let, for each integer s in $[1, p]$, ξ_s be the biggest element in $-\alpha - \mathbb{N}$ for which there exists $h \in \mathbb{Z}$ and $\omega' \in \Omega_0^{n+1}$ such that $F_h^{\omega,\omega'}(\lambda)$ has a pole of order at least equal to s at ξ_s. Then ξ_s is a root of*

some $(s+j)$-th Bernstein polynomial of the fresco $\mathcal{F}^{\alpha}_{f,\omega}$ for some $j \in \mathbb{N}$.
Moreover, if $\xi_s = \xi_{s+1} = \cdots = \xi_{s+p}$ then there exists at least p distinct values of $j \in \mathbb{N}$ such that ξ_s is root of the $(s+j)$-th Bernstein polynomial of the fresco $\mathcal{F}^{\alpha}_{f,\omega}$.

PROOF. The proof of the first assertion is analogous to the proof of the theorem above.
The second assertion is an immediate consequence of the fact that the roots of the Bernstein polynomial of a semi-simple fresco are simple, applied to the successive semi-simple quotients

$$S_d(\mathcal{F}^{\alpha})/S_{d-1}(\mathcal{F}^{\alpha})$$

for $d = s+1, s+2, \ldots, s+p$. ■

Now we conclude by a result which combine the results in both direction to precise the link between the first pole of order $\geq p$ for a given pair (ω, ω') with the roots of the Bernstein polynomials of order /*geqp* of the fresco $\mathcal{F}_{f,\omega}$ associated to (f, ω).

Theorem 8.5.3 *In the standart situation, assume that the hypothesis $H(\alpha, 1)$ is satisfied. Let ω be in Ω_0^{n+1} and define the fresco $\mathcal{F}_{\omega} := \tilde{\mathcal{A}}_{conv.}\omega$. Assume that $p := d(\mathcal{F}^{\alpha}_{\omega})$ is at least equal to 1 and choose*[19] *$\omega' \in \Omega_0^{n+1}$ such that there exists $h \in \mathbb{Z}$ such that $F_h^{\omega,\omega'}(\lambda)$ has a pole of order p at some point in $-\alpha - \mathbb{N}$. For each $j \in [1, p]$, let m_j be the integer which has the property $P(\omega, \omega', j)$. Then $-\alpha - m_j$ is a root of at least one of the polynomials $B_{j+q}(\mathcal{F}_{\omega})$, for some integer q in $\mathbb{N}$.*

PROOF. Assuming that for some $j \geq 1$ no root of the polynomials B_{j+q} for $q \geq 0$ is equal to $-\alpha - m_j$ allows to find a J-H. sequence of $\mathcal{F}_{\omega}$ such the corresponding generator of the annihilator of ω is of the form $\Pi := \Pi_2\Pi_1$ where Π_1 has no factor $(a - \lambda_h b)$ with $\lambda_h + h - k$ equal to $\alpha + m_j$ and where the nilpotent order of the fresco $d(\tilde{\mathcal{A}}_{conv.}/\tilde{\mathcal{A}}_{conv.}\Pi_2)^{\alpha}) \leq j - 1$ and where k is the rank of $\mathcal{F}_{\omega}$. Then we conclude as in the previous Theorem using Corollary 8.2.3. ■

9 Examples

It is, in general, rather difficult to compute the Bernstein of the fresco associated to a given pair (f, ω), even in the case where f has an isolated singularity.
Nevertheless, in the case where f is a polynomial in $\mathbb{C}[x_0, \ldots, x_n]$ having $(n+2)$ monomials, we describe in the article [12], a rather elementary method to obtain an estimation for the Bernstein polynomial of the fresco $\mathcal{F}_{f,\,\omega}$ associated to a monomial $(n+1)$-form ω.
Of course, when the full Bernstein polynomial has a root of multiplicity $k \geq 2$ then this root is also a root of the j-th Bernstein polynomial for each $j \in [1, k]$ but when

[19] such a ω' exists thanks to Theorem 8.4.1.

the Bernstein polynomial has only simple roots, the computation of the higher order Bernstein polynomials, even in the special situation of [12], is not easy. We present in below some examples where we show that the second Bernstein polynomial is not trivial but where the full Bernstein polynomial has no multiple root.

Proposition 9.0.1 *Let $f(x,y,z) := xy^3 + yz^3 + zx^3 + \lambda xyz$ where $\lambda \neq 0$ is any complex number which is a parameter, and consider the holomorphic forms*

$$\omega_1 := dx \wedge dy \wedge dz,\ \omega_2 := y^3z^2\omega_1,\ \omega_3 = y^7\omega_1,\ \text{and } \omega_4 := xy^3\omega_1.$$

Then, in each of these cases, the fresco $\mathcal{F}_{f,\omega_i}$ is a rank 2 theme and the second Bernstein polynomial is equal respectively to $x+1$, $x+4$, $x+5$ and $x+3$.
Moreover, for $i = 3,4$ the corresponding (full) Bernstein polynomial of the corresponding frescos has only simple roots.

Note that this proposition allows to apply Theorem 8.5.1 in [13] to conclude that for each $i \in \{1,2,3,4\}$, there exists some integer h and some germ $\omega_i' \in \Omega_0^3$ such that the meromorphic extension of

$$F_h^{\omega_i,\omega_i'}(\lambda) = \frac{1}{\Gamma(\lambda)} \int_X |f|^{2\lambda} \bar{f}^{-h} \rho\omega_i \wedge \bar{\omega}_i'$$

has a double pole at the point λ_i equal to the root of the second Bernstein polynomial of the fresco $\mathcal{F}_{f,\omega_i}$.

The proof of this proposition uses several lemmas and the technic of computation described in [12] (see paragraph 4.3.2 in *loc. cit.*).

Lemma 9.0.2 *Let e be a generator of the rank 2 theme $T := \tilde{\mathcal{A}}/\tilde{\mathcal{A}}(a-2b)(a-b)$ (which is the unique fresco with Bernstein polynomial $(x+1)^2$). Assume that we have three homogeneous polynomials P, Q and R in $\mathcal{A}$ of respective degrees $3,4$ and k with the following conditions*

1. *P, Q and R are monic in a.*
2. *Then exists a non zero constant c such that $P + cQ$ kills e in T.*
3. *The Bernstein polynomial of Q[20] is not a multiple of $(x+1)$ or of $(x+2)$.*
4. *The Bernstein polynomial of R is not a multiple of $(x+3)(x+2)(x+1)$*

Then Re generates a rank two sub-theme in T.

[20] By definition B_P is defined by the formula

$$(-b)^p B_P(-b^{-1}a) = P$$

where P is in $\mathcal{A}$, is homogeneous in (a,b) of degree p and monic in a. This is the Bernstein polynomial of the fresco $\tilde{\mathcal{A}}/\tilde{\mathcal{A}}P$.

PROOF. First, remark that our hypothesis implies that $P = (a-\nu b)(a-2b)(a-b)$ for some $\nu \in \mathbb{C}$ since T is isomorphic to $\tilde{\mathcal{A}}/\tilde{\mathcal{A}}(a-2b)(a-b)$. We may realize T in the simple pole asymptotic expansion module with rank 2 which is isomorphic to $T^\sharp$

$$\Theta := \Xi_1^2/\Xi_1^0 \simeq \mathbb{C}[[s]](Log\, s)^2 \oplus \mathbb{C}[[s]Log\, s$$

where a is the multiplication by s and b is defined by $ab - ba = b^2$ and

$$b(Log\, s) = sLog\, s \quad \text{and} \quad b((Log\, s)^2) = s(Log\, s)^2 - 2sLog\, s.$$

Then let us prove that image of e in Ξ_1^2/Ξ_1^0 may be written

$$e = u(Log\, s)^2 + vs(Log\, s)^2 + ws^3(Log\, s)^2 + s^4\mathbb{C}[[s]](Log\, s)^2 + \mathbb{C}[[s]](Log\, s) \quad (@)$$

where φ is in Ξ_1^2/Ξ_1^0 and where $uvw \neq 0$ are complex numbers.
Remark that the only restrictive condition for writing e as in (@) is the condition $uvw \neq 0$. The condition $u \neq 0$ is easy because we assume that e is a generator of T with Bernstein polynomial $(x+1)^2$, so writing e as a $\mathbb{C}[[b]]$-linear combination of the $\mathbb{C}[[b]]$-basis $e_1 = (Log\, s)^2$ and $e_2 = Log\, s$ of T we see that the coefficient of e_1 must be invertible in $\mathbb{C}[[b]]$.
But the condition $(P + cQ)(e) = 0$ implies, since the Bernstein element of T is $(a-2b)(a-b)$, that we may write[21] $P = (a-\nu b)(a-2b)(a-b)$.
The annihilator of $(Log\, s)^2$ in Ξ_1^2/Ξ_1^0 is the ideal $\tilde{\mathcal{A}}(a-2b)(a-b)$ so we have $P((Log\, s)^2) = 0$ in T. Since $Q((Log\, s)^2)$ has a non zero term in $s^4(Log\, s)^2$, because -1 is not a root of B_Q, only the term coming from

$$P(s(Log\, s)^2) = \frac{4-\nu}{24}s^4(Log\, s)^2 \quad \text{modulo } \mathbb{C}[[s]]Log\, s$$

can compensate for this term, in order to obtain the equality $(P+cQ)(e) = 0$. Then $u \neq 0$. implies $v \neq 0$.
But now, the only term which can kill the non zero term in $s^5(Log\, s)^2$ coming from $Q(vs(Log\, s)^2)$ (using that B_Q is not a multiple of $(x+2)$) can only comes from $P(ws^2(Log\, s)^2)$ and this proves that $w \neq 0$. So the assertion (@) holds true.
Now if R is homogeneous of degree k in (a,b) a necessary condition on R such that $R(e)$ has no term in $s^{k+i}(Log\, s)^2$, for $i = 0, 1, 2$, is that B_R divides $(x+1)(x+2)(x+3)$. So, when it is not the case, Lemma 5.2.4 implies that $R(e)$ is a rank 2 theme and that its second Bernstein polynomial has a (unique) root equal to $-(k+j)$ where $-j$ is the smallest integer among $\{-1,-2,-3\}$ which is not a root of B_R (see Corollary 7.4.3). ■

Note that the Lemma above may be easily generalized to many $[\alpha]$-primitive frescos provided that the nilpotent order is known and that it has a generator which admits a enough simple element in $\mathcal{A}$ belonging to its annihilator.

[21] In our choice of f and ω_1, $\mu = 3$.

Lemma 9.0.3 *In the situation of Proposition 9.0.1, the frescos generated by the forms*

$$\omega_1 := dx \wedge dy \wedge dz,\ \omega_2 := y^3z^2\omega_1,\ \omega_3 := y^7\omega_1,\ \text{and}\ \omega_4 := xy^3\omega_1$$

generate rank 2 [1]*-primitive themes. Their Bernstein polynomials are respectively equal to*

$$(x+1)^2, \quad (x+3)^2 \text{ or } (x+2)(x+3), \quad (x+3)(x+5) \quad \text{and} \quad (x+2)(x+3)$$

and their respective 2*-Bernstein polynomials are* $(x+1)$*,* $(x+3)$*,* $(x+5)$ *and* $(x+3)$*. In the cases* $i = 3, 4$ *there is no double root for the Bernstein polynomial of* $\mathcal{F}_{f,\omega_i}$*.*

Proof. The first point is to show that $\mathcal{F}_{f,\omega_1}$ has rank 2. Since f has an isolated singularity at the origin, we have $Ker df^n = df \wedge \Omega^{n-1}$ and then $H^{n+1}/bH^{n+1} \simeq \mathcal{O}_0/J(f)$ and H^{n+1} has no b-torsion and no a-torsion. Since f is not[22] in $J(f)$ the image of ω_1 and $a\omega_1 = f\omega_1$ in H^{n+1} are linearly independent (over $\mathbb{C}$) and then the rank of $\tilde{\mathcal{A}}\omega_1$ is at least equal to 2. Now the computation in [12] (see 4.3.2) shows that the Bernstein polynomial of this fresco divides $(x+1)^3$ (see also the detailed computation below). So it is a theme of rank 2 or 3. But using our main result, the rank 3 would imply that there exists a pole of order 3 for some $F_h^{\omega_1,\omega'}(\lambda)$ which is impossible[23] in $\mathbb{C}^3$. So $\mathcal{F}_{f,\omega_1}$ is a rank 2 theme with Bernstein polynomial $(x+1)^2$. The computation in [12] gives that $P_3 + c\lambda^{-4}P_4$ kills ω_1 in H^{n+1} where

$$P_3 := (a-3b)(a-2b)(a-b), P_4 = (a-(13/4)b)(a-(5/2)b)(a-(7/4)b)a, \text{and } c = 4^4$$

This is easily obtained by using the technic of the computation of *loc.cit.* (see the detailed computation in the Appendix of [13]). Then we may apply Lemma 9.0.2 to see that $\lambda m_1 m_2 \omega_1 = \lambda(a-2b)(a-b)\omega_1$ generates rank 2 themes in H^{n+1}. But the identity $\lambda m_1 m_2 = m_4 y^3 z^2$ shows that ω_2 generates also rank 2 in H^{n+1} since $m_4\omega_2 = \lambda m_1 m_2 \omega_1 = \lambda(a-2b)(a-b)\omega_1$ applying Lemma 9.0.2 with $R = (a-2b)(a-b)$ whose Bernstein polynomial is $(x+1)^2$. Moreover we see that Re has a non zero term in $s^3(Log\, s)^2$.
Since $m_4\omega_2$ generates a rank 2 theme, then ω_2 generates a rank 2 theme also (the rank 3 is again excluded because it would imply that $f^2 \notin J(f)$ which is impossible as explained above).
The technic of computation in [12] applied to ω_2 gives now that the Bernstein polynomial of the rank 2 theme $\tilde{\mathcal{A}}\omega_2$ has to divide[24] the polynomial $(x+2)(x+3)^2$. But the fact that $m_4\omega_2$ has a non zero term in $s^3(Log\, s)^2$ (and no term in $(Log\, s)^2$ or in $s(Log\, s)^2$) implies, since we have

$$m_4\omega_2 = 4(a-2b)\omega_2$$

[22]This point is not so easy to check directly. But the rank is not 1 since this would implies that this fresco has a simple pole and the argument used in Lemma 9.0.2 gives then a contradiction.
[23]This would give an order 4 pole for the meromorphic continuation of $|f|^{2\lambda}$!
[24]This computation gives that $Q_3 + d\lambda^{-4}Q_4$ kills ω_2 in H^{n+1} with $Q_3 := (a-4b)(a-4b)(a-3b)$.

ω_2 has a non zero term in $s^2(Log\, s)^2$ and then -3 is a root of the second Bernstein polynomial of the fresco $\mathcal{F}_{f,\omega_2}$. So the Bernstein polynomial is either $(x+2)(x+3)$ or $(x+3)^2$.
We know that the Bernstein polynomial of $\mathcal{F}_{f,\omega_3}$ divides $(x+5)(x+3)(x+2)$ by using the technic of [12].
We know also that $m_1^2 m_4\omega_1 = \lambda m_3\omega_3$ has a non zero term in $s^5(Log\, s)^2$ (as a consequence of Lemma 9.0.2) and, since $-m_3\omega_3 = (a-2b)\omega_3$ implies that ω_3 has a non zero term in $s^4(Log\, s)^2$, the second Bernstein polynomial of $\mathcal{F}_{f,\omega_3}$ is $x+5$.

Note that the Bernstein polynomial of the fresco $\mathcal{F}_{f,\omega_3}$ has two simple roots.

The last case is similar, since we know that $m_1\omega_1$ has a non zero term in $s^2(Log\, s)^2$. So our assertion is consequence of the estimation of the Bernstein polynomial. ■

The reader may find more details on the previous computations in [13].

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