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# ABOUT UNIVERSALITY OF LARGE DEVIATION PRINCIPLES FOR CONJUGACY INVARIANT PERMUTATIONS

ALICE GUIONNET AND MOHAMMED SLIM KAMMOUN

ABSTRACT. We prove the universality of the large deviations for conjugacy invariant permutations with few cycles. As an application, we establish the universality of large deviation at speeds  $n$  and  $\sqrt{n}$  for the length of monotone subsequences in conjugacy invariant permutations, with a sharp control over the total number of cycles. This universality class includes the well-known Ewens measures.

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## 1. INTRODUCTION AND MAIN RESULTS

Recent advances in random matrix theory have shown a surprising universality feature in large deviations principles [4, 29, 18, 7, 17, 2]. It was shown in [17] that the large deviations for the largest eigenvalues of Wigner matrices are universal, and the same as those of matrices with Gaussian entries derived in [4], within a class of matrices with subGaussian entries whose distribution is called sharp subGaussian because their Laplace transform is uniformly bounded by the Gaussian one with the same variance. [3, 11] show that the probability of deviating towards a sufficiently small value stay universal for all subGaussian entries, but are different for deviations towards large enough value if the entries are not

sharp subGaussian. The goal of this article is to study a similar universality phenomenon for random permutations.

In the last few years, universality for random permutations have been investigated thoroughly, including various aspects like global convergence and fluctuations. In particular, for conjugacy invariant permutations (see Definition 1), many results suggest that for a large family of functions on permutations, the asymptotic behaviour depends only on the number of fixed points [16, 25], or only on number of fixed points and two cycles [9, 19, 24]. For general functions, some results have been proven for example in [15, 23].

Nevertheless, relatively few results are known for large deviations for general conjugacy invariant permutations. In some sense, this question is related with large deviations for random matrices since [20] shows that the law of the longest subsequence in a permutation chosen at random has the same distribution as the largest eigenvalue of a Gaussian Wishart matrices. We will show that large deviations for random permutations can easily be seen to be universal in a rather wide class of conjugacy invariant permutations with a sharp control over the total number of cycles. The main technique that we will introduce in this article is based on coupling of random permutations and exponential approximations [12, Section 4.2.2], see section 1.2.

Hereafter, we will denote by  $\mathfrak{S}_n$  the set of permutations of  $\{1, \dots, n\} = [n]$ . To state more precisely our result, let us remind the definition of conjugacy invariant permutations.

**Definition 1.** A random permutation  $\sigma_n$  of size  $n$  is said to be conjugacy invariant if for every  $\rho \in \mathfrak{S}_n$ ,

$$\rho \sigma_n \rho^{-1} \stackrel{d}{=} \sigma_n.$$

In other words,  $\sigma_n$  is conjugacy invariant if and only if the map  $\sigma \in \mathfrak{S}_n \mapsto \mathbb{P}(\sigma_n = \sigma)$  depends only on the cycle structure of  $\sigma$ . This class of permutations first emerged in Biology. It includes well-known measures like the Ewens measures (see Definition 1), along with various generalizations, such as the Kingman virtual permutations [14, 26, 37]. In this article, we are interested in the following class of conjugacy-invariant permutations with a sharp control over the total number of cycles described as follows.

**Definition 2.** Let  $0 < \alpha, \beta \leq 1$ . We say that a sequence of random permutations  $(\sigma_n)_{n \geq 1}$  satisfies  $(\mathcal{CI}_{\alpha, \beta})$  if

- for any  $n$ ,  $\sigma_n$  is a conjugacy invariant permutation of size  $n$ ,
- and, for any  $\varepsilon > 0$ ,

$$(\mathcal{CI}_{\alpha, \beta}) \quad \lim_{n \rightarrow \infty} \frac{\ln \mathbb{P}\left(\frac{\#\sigma_n}{n^\alpha} > \varepsilon\right)}{n^\beta} = -\infty,$$

where  $\#(\sigma)$  is the total number of cycles of  $\sigma$ .

The main goal of this article will be to show that the sets  $(\mathcal{CI}_{\alpha, \beta})$  provide natural universality classes for the large deviation of the uniform law on permutations for some appropriate choices of  $\alpha$  and  $\beta$ .

We observe that for any  $\beta \leq \alpha$ ,  $\mathcal{CI}_{\alpha, \beta} \subset \mathcal{CI}_{\alpha, \alpha}$ . A classical example of random permutations satisfying  $(\mathcal{CI}_{\alpha, \alpha})$  (for any  $\alpha$ ) is the uniform permutation. More generally, we prove in section 2.2, see Corollary 16, that the Ewens measures satisfy  $(\mathcal{CI}_{\alpha, \alpha})$  for any positive real number  $\alpha$ . Let  $\theta$  be a non-negative real number. We say that a random permutation  $\sigma_{\theta, n}^{\text{Ew}}$  follows the

Ewens distribution with parameter  $\theta$  if for every  $\sigma \in \mathfrak{S}_n$ ,

$$(1) \quad \mathbb{P}(\sigma_{\theta,n}^{\text{Ew}} = \sigma) = \frac{\theta^{\#(\sigma)-1}}{\prod_{i=1}^{n-1} (\theta + i)}.$$

In particular,  $\sigma_{1,n}^{\text{Ew}} = \sigma_n^{\text{Unif}}$  is the uniform permutation and  $\sigma_{0,n}^{\text{Ew}}$  is a uniform cyclic permutation. Clearly, Ewens measures are conjugacy invariant since by definition  $\mathbb{P}(\sigma_{\theta,n}^{\text{Ew}} = \sigma)$  depends only on the number of cycles of  $\sigma$ .

Our primary emphasis is the application of our universality results to monotone subsequences of random permutations.

**1.1. Monotone subsequences.** Let  $\mathfrak{S}_n$  be the set of permutations of size  $n$ . Given  $\sigma \in \mathfrak{S}_n$ , a subsequence  $(\sigma(i_1), \dots, \sigma(i_k))$  is an increasing (resp. decreasing) subsequence of  $\sigma$  of length  $k$  if  $i_1 < \dots < i_k$  and  $\sigma(i_1) < \dots < \sigma(i_k)$  (resp.  $\sigma(i_1) > \dots > \sigma(i_k)$ ). We denote by  $\text{LIS}(\sigma)$  (resp.  $\text{LDS}(\sigma)$ ) the length of the longest increasing (resp. decreasing) subsequence of  $\sigma$ . For example,

$$\text{if } \sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 3 & 2 & 4 & 1 \end{pmatrix}, \text{ LIS}(\sigma) = 2 \text{ and } \text{LDS}(\sigma) = 4.$$

The study of the asymptotic behavior of monotone subsequences within random permutations is famously known as Ulam's problem. In his seminal work [38], Ulam posed the conjecture that the limit, as  $n$  tends towards infinity, of  $\frac{\mathbb{E}(\text{LIS}(\sigma_n^{\text{Unif}}))}{\sqrt{n}}$  exists. Vershik, Kerov, Logan, and Shepp [39, 27] proved that this limit is equal to 2. For historical details and full proofs, we refer to [34]. Large deviations of speed  $\sqrt{n}$  and  $n$  were also proved when  $\sigma_n = \sigma_n^{\text{Unif}}$  follows the uniform law on  $\mathfrak{S}_n$ :

**Theorem 3.** [36, Theorem 2][13]

For any  $x \geq 2$ ,

$$\lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}} \ln \mathbb{P}(\text{LIS}(\sigma_n^{\text{Unif}}) \geq x\sqrt{n}) = -I_{\text{LIS}, \frac{1}{2}}(x)$$

and for any  $0 < x < 2$ ,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \ln \mathbb{P}(\text{LIS}(\sigma_n^{\text{Unif}}) \leq x\sqrt{n}) = -I_{\text{LIS}, 1}(x)$$

where

$$I_{\text{LIS}, \frac{1}{2}}(x) = 2x \cosh^{-1}\left(\frac{x}{2}\right) \quad \text{and} \quad I_{\text{LIS}, 1}(x) = -1 + \frac{x^2}{4} + 2 \ln\left(\frac{x}{2}\right) - \left(2 + \frac{x^2}{2}\right) \ln\left(\frac{2x^2}{4+x^2}\right).$$

Note that the large deviations to the right (for  $x \geq 2$ ) and to the left (for  $x < 2$ ) have different speed. This is very similar to the large deviations principles for the largest eigenvalue of the Gaussian matrix ensembles [30, 4, 5]. The large deviations to the left necessitate to move all the spectrum and in fact are related to the large deviations of the empirical measure of the eigenvalues : their speed is  $n^2$ , typically the number of random entries of the matrices [5]. Whereas large deviations to the right are easier as they only require to move one eigenvalue, namely the largest, which can be achieved for instance by making one very large entry, which corresponds to a speed of order  $n$ . Therefore, these two results are in fact rather different in nature.

Our first result, proven in Section 2.4, is the following.

**Theorem 4.** *If the sequence of random permutations  $(\sigma_n)_{n \geq 1}$  satisfies  $(\mathcal{CI}_{\frac{1}{2}, \frac{1}{2}})$ , then the law of  $\frac{\text{LIS}(\sigma_n)}{\sqrt{n}}$  satisfies a large deviation principle with speed  $\sqrt{n}$  and good rate function  $J_{\text{LIS}, \frac{1}{2}}$  which is equal to  $I_{\text{LIS}, \frac{1}{2}}$  on  $[2, +\infty)$  and  $+\infty$  on  $(-\infty, 2)$ . In other words, for any measurable subset  $E$  of  $\mathbb{R}$ , we have :*

$$(2) \quad - \inf_{x \in E^\circ} J_{\text{LIS}, \frac{1}{2}}(x) \leq \liminf_{n \rightarrow \infty} \frac{\ln \mathbb{P}\left(\frac{\text{LIS}(\sigma_n)}{\sqrt{n}} \in E\right)}{\sqrt{n}} \leq \limsup_{n \rightarrow \infty} \frac{\ln \mathbb{P}\left(\frac{\text{LIS}(\sigma_n)}{\sqrt{n}} \in E\right)}{\sqrt{n}} \leq - \inf_{x \in \bar{E}} J_{\text{LIS}, \frac{1}{2}}(x).$$

Moreover, if the sequence of random permutations  $(\sigma_n)_{n \geq 1}$  satisfies  $(\mathcal{CI}_{\frac{1}{2}, 1})$ , then the law of  $\frac{\text{LIS}(\sigma_n)}{\sqrt{n}}$  satisfies a large deviation principle with speed  $n$  and good rate function  $J_{\text{LIS}, 1}$  where

$$J_{\text{LIS}, 1}(x) = \begin{cases} I_{\text{LIS}, 1}(x) & 0 < x \leq 2 \\ 0 & x > 2 \\ +\infty & x \leq 0 \end{cases}.$$

In other words, for every measurable subset  $E$  of  $\mathbb{R}$ ,

$$(3) \quad - \inf_{x \in E^\circ} J_{\text{LIS}, 1}(x) \leq \liminf_{n \rightarrow \infty} \frac{\ln \mathbb{P}\left(\frac{\text{LIS}(\sigma_n)}{\sqrt{n}} \in E\right)}{n} \leq \limsup_{n \rightarrow \infty} \frac{\ln \mathbb{P}\left(\frac{\text{LIS}(\sigma_n)}{\sqrt{n}} \in E\right)}{n} \leq - \inf_{x \in \bar{E}} J_{\text{LIS}, 1}(x),$$

The same result holds if we replace LIS par LDS, the length of the longest decreasing subsequence.

Even if Ewens measures do not satisfy  $(\mathcal{CI}_{\frac{1}{2}, 1})$  (except for  $\theta = 0$ ) so that (3) is not guaranteed, we will show by a separate argument that universality still holds in the sense that the Ewens distribution still satisfies the same large deviation principle, see section 2.2:

**Proposition 5.** *For any  $\theta \geq 0$ , the sequence of random permutations following the Ewens distribution  $\sigma_n \stackrel{d}{=} \sigma_{\theta, n}^{\text{Ew}}$ ,  $n \geq 1$ , satisfy the large deviation principle (3).*

**1.2. Main Technical trick.** In this section, we introduce our main technical trick, namely Theorem 6, which can be thought as an exponential approximation argument as presented in [12, Section 4.2.2]. To this end, we see the statistic under study, such as LIS, as a function  $f$  of the permutation. Our point is then that if this function  $f$  depends sufficiently smoothly on the permutation in the sense of Hypothesis  $\mathcal{H}_1$  and the law of  $n^{-\alpha} f(\sigma_n^{\text{Unif}})$  satisfies a large deviation principle with speed  $n^\beta$ , then this large deviation principle remains true for all the random permutations satisfying  $(\mathcal{CI}_{\alpha, \beta})$ . We then show that many well known statistics verify hypothesis  $\mathcal{H}_1$  and then apply our result to various examples.

The goal of this section is to introduce a tool to prove universality for the large and moderate deviations bounds for some statistics. We first define for a permutation  $\sigma \in \mathfrak{S}_n$ , the set

$$A_\sigma = \begin{cases} \{\sigma\} & \text{if } \#(\sigma) = 1 \\ \{\rho \in \mathfrak{S}_n, \rho = \sigma \circ (i_1, i_2) \circ (i_1, i_3) \cdots \circ (i_1, i_{\#(\sigma)}) \text{ and } \#(\rho) = 1\} & \text{if } \#(\sigma) > 1 \end{cases}.$$

Above,  $\{i_1, \dots, i_{\#(\sigma)}\}$  are distinct elements of  $\{1, \dots, n\}$ .

For example:  $A_{(3,4)} = \{(1, 2, 3, 4), (1, 2, 4, 3), (1, 3, 4, 2), (1, 4, 3, 2)\}$ . Indeed,  $(1, 2, 3, 4) = (3, 4)(4, 2)(4, 1)$ ,  $(1, 2, 4, 3) = (3, 4)(3, 2)(3, 1)$ , etc.

Our main theorem states that large deviations are universal within the class  $(\mathcal{CI}_{\alpha, \beta})$ .

**Theorem 6.** Let  $f$  be a function defined on  $\bigcup_{n \geq 1} \mathfrak{S}_n$  and having values in  $\mathbb{R}^d$  for some fixed integer number  $d$ . We denote  $|\cdot|$  the Euclidean distance in  $\mathbb{R}^d$ . Suppose that

- the following holds true

$$(\mathcal{H}_1) \quad \sup_{\sigma \in \bigcup_{n \geq 1} \mathfrak{S}_n} \sup_{\rho \in A_\sigma} \frac{|f(\sigma) - f(\rho)|}{\#(\sigma)} < +\infty.$$

- there exists some function  $J$  **continuous**, some  $0 < \beta \leq \alpha < 1$  and  $\mathcal{I}$  an open set of  $\mathbb{R}^d$  such that for any  $x \in \mathcal{I}$

$$(\mathcal{H}_2) \quad \lim_{n \rightarrow \infty} \frac{1}{n^\beta} \ln \mathbb{P} \left( f \left( \sigma_n^{Unif} \right) \geq xn^\alpha \right) = -J(x).$$

The relation  $a = (a_1, \dots, a_d) \geq b = (b_1, \dots, b_d)$  means  $a_i \geq b_i$  for all  $i \in [d]$ .

- Moreover, the sequence  $(\sigma_n)_{n \geq 1}$  of random permutations satisfies  $(\mathcal{CI}_{\alpha, \beta})$

Then, for any  $x \in \mathcal{I}$

$$\lim_{n \rightarrow \infty} \frac{1}{n^\beta} \ln \mathbb{P}(f(\sigma_n) \geq xn^\alpha) = -J(x).$$

It may appear challenging to verify  $(\mathcal{H}_1)$  for an arbitrary statistic. However, in some cases, it is more straightforward to prove the following condition:

$$(\mathcal{H}'1) \quad \sup_{\sigma \in \bigcup_{n \geq 1} \mathfrak{S}_n} \sup_{i, j} |f(\sigma) - f(\sigma \circ (i, j))| < +\infty.$$

A noteworthy observation is that  $(\mathcal{H}'1)$  implies  $(\mathcal{H}_1)$ , as it directly follows from the triangle inequality.

*Remark 7.* The set of functions satisfying  $(\mathcal{H}_1)$  (resp.  $(\mathcal{H}'1)$ ) is a vector space.

We next show that Hypothesis  $(\mathcal{H}'1)$  is satisfied for a large class of well-known statistics. Hence, Theorem 6 implies the universality of the large deviations for these statistics as soon as they are known for the uniform measure. We prove the following Property in subsection 2.1.

**Property 8.** The following functions satisfy  $(\mathcal{H}'1)$  :

- i) The longest increasing subsequence LIS.
- ii) The longest decreasing subsequence LDS.
- iii) The vector  $(\lambda_1, \dots, \lambda_d)$  if  $\lambda_i$  denotes the length of the  $i^{\text{th}}$  row of the RSK image for  $i \in [d]$ .  $d$  is a fixed integer number.
- iv) The normalized Inversions count :  $\frac{\text{Inv}(\sigma)}{n} = \frac{\text{card}(\{(i, j) : i < j, \sigma(i) > \sigma(j)\})}{n}$ .
- v) The descents count :  $D(\sigma) := \text{card}\{i : \sigma(i+1) < \sigma(i)\}$ .
- vi) The ascents count :  $A(\sigma) := \text{card}\{i : \sigma(i+1) > \sigma(i)\}$ .
- vii) The peaks count :  $\text{Peaks}(\sigma) = \text{card}\{i : \sigma(i-1) < \sigma(i) > \sigma(i+1)\}$ .
- viii) The valleys count :  $\text{Valleys}(\sigma) = \text{card}\{i : \sigma(i-1) > \sigma(i) < \sigma(i+1)\}$ .
- ix) The exceedance count :  $\text{Exc}(\sigma) := \text{card}\{i : \sigma(i) > i\}$ <sup>1</sup>
- x) The normalized major index:  $\frac{\text{Maj}(\sigma)}{n} = \frac{\sum_{\sigma(i+1) > \sigma(i)} i}{n}$ .

<sup>1</sup>For the specialists, in general, the normalized number of occurrences of any set of classical (inversions for example), consecutive (descents, ascents, double descents, peaks, valleys, etc. ) or more generally (bi-)vincular pattern.

xi) *The longest alternating subsequence LAS:* Let  $1 \leq i_1 < i_2 < \dots < i_k \leq n$ . We say that  $\sigma(i_1), \dots, \sigma(i_k)$  is an alternating subsequence of length  $k$  if  $\sigma(i_1) < \sigma(i_2) > \sigma(i_3) < \sigma(i_4) \dots$  and let  $\text{LAS}$  be the length of the longest alternating subsequence.

In (iii),  $\lambda(\sigma) = (\lambda_i(\sigma))_{i \geq 1}$  denotes the RSK shape associated with the permutation  $\sigma$ , see a definition in [34, Section 1.6]

### 1.3. Applications.

1.3.1. *Applications to the longest increasing subsequence.* A direct application of Theorem 6 gives the upper large deviations (2) of Theorem 4.

**Corollary 9.** *If the sequence  $(\sigma_n)_{n \geq 1}$  of random permutations satisfies  $(\mathcal{CI}_{\frac{1}{2}, \frac{1}{2}})$ , then for any  $x > 2$*

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}} \ln \mathbb{P}(\text{LIS}(\sigma_n) \geq x\sqrt{n}) &= -I_{\text{LIS}, \frac{1}{2}}(x) \\ \lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}} \ln \mathbb{P}(\text{LDS}(\sigma_n) \geq x\sqrt{n}) &= -I_{\text{LIS}, \frac{1}{2}}(x). \end{aligned}$$

*Proof.* The first point is a direct application of Theorem 6 by taking  $f = \text{LIS}$  and  $\alpha = \beta = \frac{1}{2}$ . The hypotheses  $(\mathcal{H}_1)$  is satisfied thanks to Property 8 and  $(\mathcal{H}_2)$  is satisfied thanks to Theorem 3. Moreover, when  $\sigma_n$  follows the uniform law, the LDS has the same distribution than the LIS. This can be seen for instance by replacing the collection  $\{i\}$  into  $\{n - i\}$ . The proof of the large deviation principle for the function LDS is therefore the same than for the function LIS. □

For the upper moderate deviation, we find the following universal result:

**Corollary 10.** *Let  $\frac{1}{6} < \nu < \frac{1}{2}$ . If the sequence  $(\sigma_n)_{n \geq 1}$  of random permutations satisfies  $(\mathcal{CI}_{\nu, \frac{3\nu}{2} - \frac{1}{4}})$ , then for any  $x > 0$ ,*

$$\lim_{n \rightarrow \infty} \frac{1}{n^{\frac{3\nu}{2} - \frac{1}{4}}} \ln \mathbb{P}(\text{LIS}(\sigma_n) \geq 2\sqrt{n} + xn^\nu) = -\frac{4}{3}x^{\frac{3}{2}}.$$

This result was proven in the uniform case in [28]. Our universality result therefore follows from Theorem 6 since the coefficients  $(\alpha, \beta) = (\nu, \frac{3\nu}{2} - \frac{1}{4})$  satisfy  $\beta = \frac{3\nu}{2} - \frac{1}{4} \leq \nu = \alpha$  for every  $\nu \leq 1/2$  so that  $(\mathcal{H}_2)$  is satisfied.  $(\mathcal{H}_1)$  follows from Property 8.

1.3.2. *Applications to large deviations for the Eulerian statistics.* Let  $D$  be the number of descents in  $\sigma$ .

**Definition 11.** *A function  $f$  is called an Eulerian statistic if  $f(\sigma_n^{\text{Unif}}) \stackrel{d}{=} D(\sigma_n^{\text{Unif}})$  for every integer number  $n$ .*

It is known for example that the exceedance count  $\text{Exc}$  and the ascents count  $A$  are Eulerian. Note that the equality in distribution is not true for general conjugacy invariant permutations.

**Corollary 12.** *If the sequence of random permutations  $(\sigma_n)_{n \geq 1}$  satisfies  $(\mathcal{CI}_{1,1})$ , then for  $f \in \{D, A, \text{Exc}\}$ , for any  $\frac{1}{2} < x < 1$ ,*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \ln \mathbb{P}(f(\sigma_n) \geq xn) = -I_D(x).$$

Where  $I_D(x) = \sup_t \{xt - \ln(\frac{\exp(t)-1}{t})\}$ .

*Proof.* For the descents, this LDP is already known in the uniform case [6] which implies the same large deviation principle for the other Eulerian statistics readily as they have the same distribution. Hence,  $(\mathcal{H}_2)$  is satisfied with  $\alpha = \beta = 1$ . Universality follows again from Theorem 6, since Property 8 implies that  $D, A, \text{Exc}$  satisfy  $(\mathcal{H}_1)$ .  $\square$

1.3.3. *Edge of RSK.* In this subsection, we give an additional result, which is not a direct application of Theorem 6 but which proof uses the same techniques.

We are interested in the lower tail of LIS, and more precisely the length of the first rows of the RSK image.

**Proposition 13.** *If the sequence  $(\sigma_n)_{n \geq 1}$  of random permutations satisfies  $(CI_{\frac{1}{2}, 1})$ , then for any  $0 < x_d < \dots < x_1 < 2$ ,*

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \ln \mathbb{P}(\forall i \in [d], \lambda_i(\sigma_n) \leq x_i \sqrt{n}) = -I_{\text{LIS}, 1}(x_d).$$

To the best of our knowledge, this result is not stated in the uniform case for general integer number  $d$  but it is immediate to adapt the proof of [13] to get the uniform case, see Section 2.6.

1.4. **Some comments.** Except for the special case of the uniform permutation, there is a lack of existing results regarding the large deviations of general conjugacy-invariant permutations. Even for the uniform case, the large deviation theory of many statistics has not been studied yet. For example, we are not aware of existing results for the large deviations for LAS, the permutations patterns counts, and the upper edge of RSK in the uniform case. Even if the hypotheses  $(\mathcal{H}'1)$  is satisfied for a large family of functions, since we are using comparison techniques, the application of Theorem 6 is not possible without knowing the large deviations in the case of uniform permutations.

In the literature, some non-universality results have been established for certain statistics but for other families of random permutations. For instance, [13] studied the longest increasing subsequence of i.i.d. points sampled from a measure on the unit square. Additionally, [32] explores the large deviations of LIS and LAS for permutations uniformly chosen among those that avoid a pattern of length 3. Moreover, in [31, Theorem B], a large deviations principle for the major index has been proven for a distinct family of random permutations.

## 2. PROOF OF THE RESULTS

2.1. **Proof of Property 8.** In this section, we prove that many natural statistics satisfy  $(\mathcal{H}'1)$ .

- *Monotone subsequences* (LIS, LDS, and  $\lambda_i$ ): The cases of LIS and LDS have already been demonstrated in [21, Lemma 3.1], and the proof for  $\lambda_i$  is presented in [21, Lemma 3.4].
- *Inversions:*

Let  $\sigma \in \mathfrak{S}_n$  be a fixed permutation, and let  $1 \leq i_1 < j_1 \leq n$ . Define  $\rho = \sigma \circ (i_1, j_1)$ .

The key observation is that for all  $i \notin \{i_1, j_1\}$ , we have  $\rho(i) = \sigma(i)$ . The remaining of the proof varies depending on the specific statistic considered, but the underlying idea remains the same. We will provide detailed explanations for a few of these statistics.

Inversions can be expressed as follows:

$$\text{Inv}(\sigma) = |\{(i, j) : i < j, \sigma(i) > \sigma(j)\}|.$$



We make the following decomposition:

$$\underbrace{\{(i, j) : i < j, \sigma(i) > \sigma(j)\}}_s = \underbrace{\{(i, j) : i < j, \sigma(i) > \sigma(j), \{i, j\} \cap \{i_1, j_1\} \neq \emptyset\}}_{s_1} \cup \underbrace{\{(i, j) : i < j, \sigma(i) > \sigma(j), \{i, j\} \cap \{i_1, j_1\} = \emptyset\}}_{s_2}.$$

Furthermore, for inversions of  $\rho$ , we have:

$$\underbrace{\{(i, j) : i < j, \rho(i) > \rho(j)\}}_{s'} = \underbrace{\{(i, j) : i < j, \rho(i) > \rho(j), \{i, j\} \cap \{i_1, j_1\} \neq \emptyset\}}_{s'_1} \cup \underbrace{\{(i, j) : i < j, \rho(i) > \rho(j), \{i, j\} \cap \{i_1, j_1\} = \emptyset\}}_{s'_2}.$$

Using the key observation, one can see that  $s'_2 = s_2$ , which allows us to write:

$$\begin{aligned} |\text{Inv}(\sigma) - \text{Inv}(\rho)| &= |\text{card}(s) - \text{card}(s')| = |\text{card}(s'_1) - \text{card}(s_1)| \\ &\leq \max(\text{card}(s'_1), \text{card}(s_1)) \\ &\leq 2n. \end{aligned}$$

As a consequence,

$$\frac{1}{n} |\text{Inv}(\sigma) - \text{Inv}(\rho)| \leq 2$$

which proves that  $n^{-1}\text{Inv}$  satisfies  $(\mathcal{H}'1)$ .

- *Descents and Major index:* The same idea applies here.

Recall that  $D(\sigma) = |\{i : \sigma(i+1) < \sigma(i)\}|$ . Let

$$\underbrace{\{i : \sigma(i+1) < \sigma(i)\}}_s = \underbrace{\{i : \sigma(i+1) < \sigma(i), \{i, i+1\} \cap \{i_1, j_1\} \neq \emptyset\}}_{s_1} \cup \underbrace{\{i : \sigma(i+1) < \sigma(i), \{i, i+1\} \cap \{i_1, j_1\} = \emptyset\}}_{s_2}$$

and

$$\underbrace{\{i : \rho(i+1) < \rho(i)\}}_{s'} = \underbrace{\{i : \rho(i+1) < \rho(i), \{i, i+1\} \cap \{i_1, j_1\} \neq \emptyset\}}_{s'_1} \cup \underbrace{\{i : \rho(i+1) < \rho(i), \{i, i+1\} \cap \{i_1, j_1\} = \emptyset\}}_{s'_2}.$$

Then, we have

$$|D(\sigma) - D(\rho)| \leq \max(\text{card}(s_1), \text{card}(s'_1)) \leq 4.$$

Similarly, with  $\text{Maj}(\sigma) = \sum_{\sigma(i+1) > \sigma(i)} i$ , we find

$$|\text{Maj}(\sigma) - \text{Maj}(\rho)| = \left| \sum_{i \in s_1} i - \sum_{i \in s'_1} i \right| \leq \max\left(\sum_{i \in s_1} i, \sum_{i \in s'_1} i\right) \leq n \max(\text{card}(s_1), \text{card}(s'_1)) \leq 4n.$$

– For *Peaks and Valleys*, the proof is similar to that of descents.

- For longest alternating subsequence LAS, we can use the following characterization (see [33, Corollary 2])

$$\text{LAS}(\sigma) = 1 + \sum_{i=1}^{n-1} M_i(\sigma),$$

where  $M_1(\sigma) = 1_{\sigma(1) > \sigma(2)}$  and for  $1 < k < n$ ,

$$M_k(\sigma) = 1_{\sigma(k-1) > \sigma(k) < \sigma(k+1)} + 1_{\sigma(k-1) < \sigma(k) > \sigma(k+1)}.$$

We have that  $\text{LAS} = M_1 + \text{Valleys} + \text{Peaks}$ .  $M_1$  satisfies  $(\mathcal{H}'1)$  since it is a bounded function. Therefore, by using Remark 7, LAS also satisfies  $(\mathcal{H}'1)$ .

**2.2. Proof of Property  $(\mathcal{CI}_{\alpha, \alpha})$  for Ewens distributions.** For the Ewens distribution, it is known that the law of the total number of cycles has a nice description as a sum of independent Bernoulli variables.

**Proposition 14.** *The number of cycles of  $\sigma_{\theta, n}^{\text{Ew}}$  is equal in distribution to  $\sum_{i=1}^n a_{\theta, i}$  where  $(a_{\theta, i})_i$  are independent Bernoulli variables with  $\mathbb{P}(a_{\theta, i} = 1) = \frac{\theta}{i + \theta - 1}$ .*

This property can be proved independently by the Chinese restaurant process description of Ewens permutations or by the Feller coupling. It is a classical result, we can cite for example [10, equation (1.1)] and [1]. Many concentration inequalities are known for the sum of independent variables. For our purpose, we use a special form of the Bennett's inequality.

**Proposition 15.** [8, Theorem 2.9]<sup>2</sup>

*Let  $X_1, X_2, \dots, X_n$  be independent random variables such that almost surely  $X_i \leq 1$  then*

$$(4) \quad \ln \left( \mathbb{P} \left( \sum_{i=1}^n X_i - \mathbb{E}(X_i) > t \right) \right) \leq -(v + t) \ln \left( 1 + \frac{t}{v} \right).$$

where  $v = \sum_{i=1}^n \mathbb{E}X_i^2$ .

**Corollary 16.** *The sequence  $(\sigma_{\theta, n}^{\text{Ew}})_{n \geq 1}$  satisfies  $(\mathcal{CI}_{\alpha, \alpha})$ , for every  $\theta \geq 0$ , and every  $\alpha > 0$ ,*

*Proof.* Using Proposition 14, we have  $\#(\sigma_{\theta, n}^{\text{Ew}}) \stackrel{d}{=} \sum_{i=1}^n a_{\theta, i}$ , and

$$\sum_{i=1}^n \mathbb{E}(a_{\theta, i}) = \sum_{i=1}^n \mathbb{E}(a_{\theta, i}^2) = \theta \ln(n) + O(1).$$

We want to prove that for every  $\varepsilon > 0$ , every  $\alpha > 0$ ,

$$(5) \quad \lim_{n \rightarrow \infty} \frac{\ln \mathbb{P} \left( \frac{\# \sigma_{\theta, n}^{\text{Ew}}}{n^\alpha} > \varepsilon \right)}{n^\alpha} = -\infty,$$

This is a direct consequence of Proposition 14 by setting  $X_i = a_{\theta, i}$ ,  $t = \varepsilon n^\alpha - v$  in Proposition 15.  $\square$

<sup>2</sup>The version we use is [8, Theorem 2.9] by setting  $b = 1$ .

**2.3. Proof of Theorem 6.** In order to prove Theorem 6, we need to introduce a one step Markov chain  $T$ . It is the same as in [22]. It maps a conjugacy invariant random permutations  $\sigma_n$  to a permutation having the same law as  $\sigma_{0,n}^{\text{Ew}}$ .

This Markov chain does not change a lot statistics satisfying  $(\mathcal{H}_1)$ . Let  $T$  be the Markov chain defined on  $\mathfrak{S}_n$  and associated to the stochastic matrix  $\left[ \frac{1_{A_\sigma(\rho)}}{\text{card}(A_\sigma)} \right]_{\sigma, \rho \in \mathfrak{S}_n}$  where we recall that

$$A_\sigma = \begin{cases} \{\sigma\} & \text{if } \#(\sigma) = 1 \\ \{\rho \in \mathfrak{S}_n, \sigma^{-1} \circ \rho = (i_1, i_2) \circ (i_1, i_3) \cdots \circ (i_1, i_{\#(\sigma)}) \text{ and } \#(\rho) = 1\} & \text{if } \#(\sigma) > 1 \end{cases}.$$

$T$  is then the Markov operator mapping a permutation  $\sigma$  to a permutation uniformly chosen at random among the permutations obtained by merging the cycles of  $\sigma$  using transpositions having all a common point.

**Lemma 17.** [22, Lemma 6] *For any conjugacy invariant random permutation  $\sigma_n$  on  $\mathfrak{S}_n$ ,*

$$T(\sigma_n) \stackrel{d}{=} \sigma_{0,n}^{\text{Ew}}.$$

The proof is detailed in [22, Lemma 6] but the idea is rather simple. By construction,  $T(\sigma_n)$  has almost surely one cycle. Since permutations with one cycle belong to the same conjugacy class, it is sufficient to prove that  $T(\sigma_n)$  is conjugacy invariant as soon as  $\sigma_n$  is conjugacy invariant.

We move now to the proof of Theorem 6

*Proof of Theorem 6.* We consider a function  $f$  on  $\cup_{n \geq 1} \mathfrak{S}_n$  with values in  $\mathbb{R}^d$ . One can suppose for simplicity that

$$(6) \quad \sup_{\sigma \in \cup_{n \geq 1} \mathfrak{S}_n} \sup_{\rho \in A_\sigma} \frac{|f(\sigma) - f(\rho)|}{\#(\sigma)} \leq 1.$$

Otherwise one can apply the theorem to  $\frac{f}{\sup_{\sigma \in \cup_{n \geq 1} \mathfrak{S}_n} \sup_{\rho \in A_\sigma} \frac{|f(\sigma) - f(\rho)|}{\#(\sigma)}}$  since by hypothesis the denominator is finite. Let  $\sigma_n$  be conjugacy invariant. By Lemma 17, we know that

$$(7) \quad \mathbb{P}(f(\sigma_{0,n}^{\text{Ew}}) \geq xn^\alpha) = \mathbb{P}(f(T(\sigma_n)) \geq xn^\alpha).$$

Moreover, by (6), we have

$$\mathbb{P}(f(\sigma_n) \geq xn^\alpha + \#\sigma_n \mathbf{1}) \leq \mathbb{P}(f(T(\sigma_n)) \geq xn^\alpha) \leq \mathbb{P}(f(\sigma_n) \geq xn^\alpha - \#\sigma_n \mathbf{1}).$$

Here,  $\mathbf{1}$  is the vector of  $\mathbb{R}^d$  with all components equal to 1. Let  $\varepsilon > 0$ . We write the following decomposition

$$\begin{aligned} p^\pm := \mathbb{P}(f(\sigma_n) \geq xn^\alpha \pm \#\sigma_n \mathbf{1}) &= \underbrace{\mathbb{P}(f(\sigma_n) \geq xn^\alpha \pm \#\sigma_n \mathbf{1} | \#(\sigma_n) < \varepsilon n^\alpha)}_{p_1^\pm} \underbrace{\mathbb{P}(\#(\sigma_n) < \varepsilon n^\alpha)}_{p_2} \\ &+ \underbrace{\mathbb{P}(f(\sigma_n) \geq xn^\alpha \pm \#\sigma_n \mathbf{1} | \#(\sigma_n) \geq \varepsilon n^\alpha)}_{p_3^\pm} \underbrace{\mathbb{P}(\#(\sigma_n) \geq \varepsilon n^\alpha)}_{p_4}. \end{aligned}$$

Moreover,

$$p_1^- \leq \mathbb{P}(f(\sigma_n) \geq (x - \varepsilon \mathbf{1})n^\alpha | \#(\sigma_n) < \varepsilon n^\alpha)$$

gives readily that

$$p_1^- p_2 \leq \mathbb{P}(f(\sigma_n) \geq (x - \varepsilon \mathbf{1})n^\alpha \text{ and } \#(\sigma_n) < \varepsilon n^\alpha) \leq \mathbb{P}(f(\sigma_n) \geq (x - \varepsilon \mathbf{1})n^\alpha).$$

Consequently, we find that

$$(8) \quad p^- \leq \mathbb{P}(f(\sigma_n) \geq (x - \varepsilon \mathbf{1})n^\alpha) + p_4.$$

Similarly,

$$p_1^+ \geq \mathbb{P}(f(\sigma_n) \geq (x + \varepsilon \mathbf{1})n^\alpha | \#(\sigma_n) < \varepsilon n^\alpha)$$

and then

$$(9) \quad \begin{aligned} p^+ &\geq p_1^+ p_2 \geq \mathbb{P}(f(\sigma_n) \geq (x + \varepsilon \mathbf{1})n^\alpha \text{ and } \#(\sigma_n) < \varepsilon n^\alpha) \\ &= \mathbb{P}(f(\sigma_n) \geq (x + \varepsilon \mathbf{1})n^\alpha) - \mathbb{P}(f(\sigma_n) \geq (x + \varepsilon \mathbf{1})n^\alpha \text{ and } \#(\sigma_n) \geq \varepsilon n^\alpha) \\ &\geq \mathbb{P}(f(\sigma_n) \geq (x + \varepsilon \mathbf{1})n^\alpha) - p_4. \end{aligned}$$

To sum-up, for any conjugacy invariant permutation  $\sigma_n$ , for any  $\varepsilon > 0$ , (6),(7),(8),(9) imply

$$(10) \quad \begin{aligned} \mathbb{P}(f(\sigma_n) \geq (x + \varepsilon \mathbf{1})n^\alpha) - \mathbb{P}(\#(\sigma_n) \geq \varepsilon n^\alpha) &\leq \mathbb{P}(f(\sigma_{0,n}^{\text{Ew}}) \geq xn^\alpha) \\ &\leq \mathbb{P}(f(\sigma_n) \geq (x - \varepsilon \mathbf{1})n^\alpha) + \mathbb{P}(\#(\sigma_n) \geq \varepsilon n^\alpha). \end{aligned}$$

We next choose  $\sigma_n = \sigma_n^{\text{Unif}}$  to be the uniform permutation. Because the Ewens distribution with  $\theta = 1$  is the uniform distribution, Corollary 16 implies that  $\sigma_n^{\text{Unif}}$  is  $(\mathcal{CI}_{\alpha,\alpha})$ . Therefore, for any  $\varepsilon > 0$ , for  $n$  large enough we have under the hypothesis of Proposition 6 and because  $\beta \leq \alpha$ , we find for all  $M > 0$

$$\mathbb{P}(f(\sigma_n^{\text{Unif}}) \geq (x + \varepsilon \mathbf{1})n^\alpha) - \mathbb{P}(\#(\sigma_n^{\text{Unif}}) \geq \varepsilon n^\alpha) = \exp(-n^\beta(J(x + \varepsilon \mathbf{1}) + o(1))) - o(\exp(-n^\beta M))$$

If  $J(x + \varepsilon \mathbf{1})$  is infinite, then the right hand side will also be smaller than  $\exp(-n^\beta M)$  for  $M$  as large as wished, whereas if it is finite, taking  $M > J(x + \varepsilon \mathbf{1})$  also gives

$$\mathbb{P}(f(\sigma_n^{\text{Unif}}) \geq (x + \varepsilon \mathbf{1})n^\alpha) - \mathbb{P}(\#(\sigma_n^{\text{Unif}}) \geq \varepsilon n^\alpha) = \exp(-n^\beta(J(x + \varepsilon \mathbf{1}) + o(1))).$$

Similarly

$$\mathbb{P}(f(\sigma_n^{\text{Unif}}) \geq (x - \varepsilon \mathbf{1})n^\alpha) + \mathbb{P}(\#(\sigma_n^{\text{Unif}}) \geq \varepsilon n^\alpha) = \exp(-n^\beta J(x - \varepsilon \mathbf{1}) + o(n^\beta))(1 + o(1)).$$

We therefore conclude that, for every  $\varepsilon > 0$ ,

$$-J(x + \varepsilon \mathbf{1}) \leq \liminf_{n \rightarrow \infty} \frac{1}{n^\beta} \ln \mathbb{P}(f(\sigma_{0,n}^{\text{Ew}}) \geq xn^\alpha) \leq \limsup_{n \rightarrow \infty} \frac{1}{n^\beta} \ln \mathbb{P}(f(\sigma_{0,n}^{\text{Ew}}) \geq xn^\alpha) \leq -J(x - \varepsilon \mathbf{1}).$$

Consequently, since we assumed that  $J$  is continuous, we find by letting  $\varepsilon$  going to zero

$$(11) \quad \liminf_{n \rightarrow \infty} \frac{1}{n^\beta} \ln \mathbb{P}(f(\sigma_{0,n}^{\text{Ew}}) \geq xn^\alpha) = \limsup_{n \rightarrow \infty} \frac{1}{n^\beta} \ln \mathbb{P}(f(\sigma_{0,n}^{\text{Ew}}) \geq xn^\alpha) = -J(x).$$

Now let  $\sigma_n$  be a conjugacy invariant permutation. Equation (10) implies (by choosing first to replace  $x$  by  $x + \varepsilon \mathbf{1}$  then by  $x - \varepsilon \mathbf{1}$ ) that

$$(12) \quad \begin{aligned} \mathbb{P}(f(\sigma_{0,n}^{\text{Ew}}) \geq (x + \varepsilon \mathbf{1})n^\alpha) - \mathbb{P}(\#(\sigma_n) \geq \varepsilon n^\alpha) &\leq \mathbb{P}(f(\sigma_n) \geq xn^\alpha) \\ &\leq \mathbb{P}(f(\sigma_{0,n}^{\text{Ew}}) \geq (x - \varepsilon \mathbf{1})n^\alpha) + \mathbb{P}(\#(\sigma_n) \geq \varepsilon n^\alpha). \end{aligned}$$

Under hypothesis  $(\mathcal{CI}_{\alpha,\beta})$ , (11) implies that

$$(13) \quad \mathbb{P}(f(\sigma_{0,n}^{\text{Ew}}) \geq (x - \varepsilon \mathbf{1})n^\alpha) + \mathbb{P}(\#(\sigma_n) \geq \varepsilon n^\alpha) = \exp(-n^\beta J(x - \varepsilon \mathbf{1}) + o(n^\beta))(1 + o(1)),$$

and

$$(14) \quad \mathbb{P}(f(\sigma_{0,n}^{\text{Ew}}) \geq (x + \varepsilon \mathbf{1})n^\alpha) - \mathbb{P}(\#(\sigma_n) \geq \varepsilon n^\alpha) = \exp(-n^\beta J(x + \varepsilon \mathbf{1}) + o(n^\beta))(1 + o(1)).$$

Plugging these estimates in (12), letting  $n$  going to infinity and then  $\varepsilon$  going to zero (while using the continuity of  $J$ ) completes the proof of Theorem 6 for permutations satisfying  $(\mathcal{CI}_{\alpha,\beta})$ .  $\square$

**2.4. Proof of Theorem 4.** We first remark that it is enough to prove the Theorem with  $E = [x, \infty)$  for  $x \geq 2$  or  $E = (0, x)$  if  $x \leq 2$ . Indeed, for  $x > 2$ , it is not hard to see that  $I_{\text{LIS}, \frac{1}{2}}$  is strictly increasing so that the probability that LIS belongs to  $[x, +\infty)$  is equivalent to the probability that it belongs to  $[x, x + \delta]$  for any  $\delta > 0$ . Hence, (2) for  $E = [x, \infty)$  and every  $x \geq 2$  yields the weak large deviation principle. Exponential tightness is as well clear as  $I_{\text{LIS}, \frac{1}{2}}$  goes to  $+\infty$  at infinity. Hence, Corollary 9 implies the full large deviation principle above 2. Similarly, it is easy to see that  $I_{\text{LIS}, 1}$  is strictly decreasing on  $(0, 2)$  so that proving (3) for  $E = (0, x]$  yields the full large deviation principle below 2. However, proving (3) for  $E = (0, x]$  for  $x < 2$  is more complicated because it is not possible to use Theorem 6 since  $1 = \beta > \alpha = 1/2$ . We need a proof specific to the longest increasing subsequence that we detail below. Before proving this result we prove a key lemma.

**Lemma 18.** *For every given permutation  $\sigma$ , for every integer number  $k$ ,*

$$\mathbb{P}(\text{LIS}(T(\sigma)) \leq \text{LIS}(\sigma) + k) \geq \inf_{\tau \in \mathcal{S}_{\#(\sigma)}} \mathbb{P}(\text{LIS}(\tau \circ \sigma_{0, \#(\sigma)}^{\text{Ew}}) < k)$$

*Proof.* The idea is to observe first that  $\rho \in A_\sigma$  if and only if  $\sigma^{-1} \circ \rho = (i_1, i_2, \dots, i_{\#(\sigma)})$  where  $i_j$  and  $i_k$  are in different cycles of  $\sigma$  as soon as  $j \neq k$ . This implies that one way to construct  $T(\sigma)$  is to choose first uniformly,  $j_1 < j_2 < \dots < j_{\#(\sigma)}$  each from one cycle of  $\sigma$ ,  $\pi$  a uniform permutation of size  $\#(\sigma)$ . Then, it is easy to see that  $T(\sigma)$  has the same law as

$$\sigma \circ (j_1, j_{\pi(1)}, j_{\pi^2(1)} \dots, j_{\pi^{\#(\sigma)-1}(1)}).$$

Fix now  $j_1, j_2, \dots, j_{\#(\sigma)}$  each on a cycle of  $\sigma$  and  $\pi$  a cyclic permutation and let  $\ell_1 < \dots < \ell_{\text{LIS}(T(\sigma))}$  be such that  $T(\sigma)(\ell_1) < \dots < T(\sigma)(\ell_{\text{LIS}(T(\sigma))})$  be a maximal increasing subsequence of  $T(\sigma)$ . Let

$$E = \{j_1, j_2, \dots, j_{\#(\sigma)}\} \cap \{\ell_1, \ell_2, \dots, \ell_{\text{LIS}(T(\sigma))}\} = \{j_{a_1}, j_{a_2}, j_{a_{\text{card}(E)}}\}$$

with  $a_1 < a_2 \dots < a_{\text{card}(E)}$  and let  $F = \{\ell_1, \ell_2, \dots, \ell_{\text{LIS}(T(\sigma))}\} \setminus E$ . For any  $\ell_k \in F$ ,  $T(\sigma)(\ell_k) = \sigma(\ell_k)$  and then  $\text{card}(F) \leq \text{LIS}(\sigma)$ . Let  $\tau$  be the unique permutation of  $\{1, \dots, \#(\sigma)\}$  such that

$$\sigma(j_{\tau^{-1}(1)}) < \sigma(j_{\tau^{-1}(2)}) < \dots < \sigma(j_{\tau^{-1}(\#(\sigma))}).$$

Moreover,  $T(\sigma)(j_{a_k}) = \sigma(j_{\pi(a_k)})$  and then

$$\sigma(j_{\pi(a_1)}) < \dots < T(\sigma)(j_{\pi(a_{\text{card}(E)})})$$

In particular, the following holds true:

$$\text{card}(E) \leq \text{LIS}(\tau\pi).$$

Consequently, we find that

$$\text{LIS}(T(\sigma)) = \text{card}(E) + \text{card}(F) \leq \text{LIS}(\sigma) + \text{LIS}(\tau \circ \pi).$$

Now, applying this inequality we find that for any integer number  $k$ ,

$$\begin{aligned} \mathbb{P}(\text{LIS}(T(\sigma)) - \text{LIS}(\sigma) \leq k | \tau) &\geq \mathbb{P}(\text{LIS}(\tau \circ \pi) < k) \\ &= \mathbb{P}(\text{LIS}(\tau \circ \sigma_{0, \#(\sigma)}^{\text{Ew}}) < k) \geq \inf_{\tau' \in \mathfrak{S}_{\#(\sigma)}} \mathbb{P}(\text{LIS}(\tau' \circ \sigma_{0, \#(\sigma)}^{\text{Ew}}) < k) \end{aligned}$$

where in the second line we used that  $\pi$  follows the uniform cyclic permutation of length  $\#(\sigma)$ .  $\square$

To prove the second part of Theorem 4, we first show the result for the Ewens distribution with  $\theta = 0$ , namely that for every  $x < 2$ ,

$$(15) \quad \liminf_{n \rightarrow \infty} \frac{1}{n} \ln \mathbb{P}(\text{LIS}(\sigma_{0,n}^{\text{Ew}}) \leq x\sqrt{n}) = \limsup_{n \rightarrow \infty} \frac{1}{n} \ln \mathbb{P}(\text{LIS}(\sigma_{0,n}^{\text{Ew}}) \leq x\sqrt{n}) = -I_{\text{LIS},1}(x).$$

It is then straightforward to generalize this result to  $\sigma_n \in (\mathcal{CI}_{\frac{1}{2},1})$  as in the proof of Theorem 6 by taking  $f = -\text{LIS}$ ,  $\alpha = \frac{1}{2}$  and  $\beta = 1$ .

The upper bound is trivial since for any  $k \in \mathbb{N}$ ,

$$\mathbb{P}(\text{LIS}(\sigma_{0,n}^{\text{Ew}}) = k) = \frac{\text{card}\{\sigma : \text{LIS}(\sigma) = k, \#(\sigma) = 1\}}{(n-1)!} \leq \frac{\text{card}\{\sigma : \text{LIS}(\sigma) = k\}}{(n-1)!} = n\mathbb{P}(\text{LIS}(\sigma_n^{\text{Unif}}) = k).$$

Consequently, by Theorem 3, we deduce

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \ln \mathbb{P}(\text{LIS}(\sigma_{0,n}^{\text{Ew}}) \leq x\sqrt{n}) \leq \limsup_{n \rightarrow \infty} \frac{1}{n} (\ln \mathbb{P}(\text{LIS}(\sigma_n^{\text{Unif}}) \leq x\sqrt{n}) + \ln(n)) = -I_{\text{LIS},1}(x).$$

The lower bound is more sophisticated. Fix  $x < 2$ ,  $0 < \varepsilon < x$ , and  $\sigma \in \mathfrak{S}_n$ . We assume that  $\sigma$  is such that  $\text{LIS}(\sigma) < (x - 3\varepsilon)\sqrt{n}$  and  $\#(\sigma) < \varepsilon^2 n$ . Then, we find that

$$(16) \quad \begin{aligned} \mathbb{P}(\text{LIS}(T(\sigma)) \leq x\sqrt{n}) &= \mathbb{P}(\text{LIS}(T(\sigma)) \leq (x - 3\varepsilon)\sqrt{n} + 3\varepsilon\sqrt{n}) \\ &\geq \mathbb{P}(\text{LIS}(T(\sigma)) \leq \text{LIS}(\sigma) + 3\varepsilon\sqrt{n}) \end{aligned}$$

By Lemma 18, (16) gives

$$(17) \quad \mathbb{P}(\text{LIS}(T(\sigma)) \leq x\sqrt{n}) \geq \inf_{\tau \in \mathfrak{S}_{\#(\sigma)}} \mathbb{P}(\text{LIS}(\tau \circ \sigma_{0, \#(\sigma)}^{\text{Ew}}) < 3\varepsilon\sqrt{n}).$$

To conclude, we need the following easy lemma.

**Lemma 19.** *For any  $x > 2$ ,*

$$\lim_{n \rightarrow \infty} \inf_{\tau \in \mathfrak{S}_n} \mathbb{P}(\text{LIS}(\tau \circ \sigma_{0,n}^{\text{Ew}}) < x\sqrt{n}) = 1$$

*Proof.* The distribution of  $\tau \circ \sigma_{0,n}^{\text{Ew}}$  is uniform over a subset of size exactly  $(n-1)!$  of  $\mathfrak{S}_n$  (the image  $S_\tau$  by  $\tau$  of the cyclic permutations). This set has probability  $1/n$  for  $\sigma_n^{\text{Unif}}$ . We have then

$$\mathbb{P}(\text{LIS}(\tau \circ \sigma_{0,n}^{\text{Ew}}) \geq x\sqrt{n}) = n\mathbb{P}(\{\text{LIS}(\sigma_n^{\text{Unif}}) \geq x\sqrt{n}\} \cap \{\sigma_n^{\text{Unif}} \in S_\tau\}) \leq n\mathbb{P}(\{\text{LIS}(\sigma_n^{\text{Unif}}) \geq x\sqrt{n}\}) = o(1)$$

because of Theorem 3 and  $x > 2$ . Note that these bounds do not depend on  $\tau$ . Therefore

$$\mathbb{P}(\text{LIS}(\tau \circ \sigma_{0,n}^{\text{Ew}}) < x\sqrt{n}) = 1 - \mathbb{P}(\text{LIS}(\tau \circ \sigma_{0,n}^{\text{Ew}}) \geq x\sqrt{n}) = 1 - o(1).$$

$\square$

Consequently, for any  $x < 2$  and  $\varepsilon > 0$ , and  $\sigma_n$  conjugacy invariant, we have

$$\begin{aligned}
\mathbb{P}(\text{LIS}(\sigma_{0,n}^{\text{Ew}}) \leq x\sqrt{n}) &= \sum_{\sigma \in \mathfrak{S}_n} \mathbb{P}(\text{LIS}(T(\sigma)) \leq x\sqrt{n})\mathbb{P}(\sigma_n = \sigma) \\
&\geq \sum_{\text{LIS}(\sigma) \leq (x-3\varepsilon)\sqrt{n}, \#(\sigma) < \varepsilon^2 n} \inf_{\tau \in \mathfrak{S}_{\#(\sigma)}} \mathbb{P}(\text{LIS}(\tau \circ \sigma_{0,k}^{\text{Ew}}) < 3\varepsilon\sqrt{n})\mathbb{P}(\sigma_n = \sigma) \\
&\geq \inf_{k \leq \varepsilon^2 n} \inf_{\tau \in \mathfrak{S}_k} \mathbb{P}(\text{LIS}(\tau \circ \sigma_{0,k}^{\text{Ew}}) < 3\varepsilon\sqrt{n})\mathbb{P}(\text{LIS}(\sigma_n) \leq (x-3\varepsilon)\sqrt{n}, \#(\sigma_n) < \varepsilon^2 n) \\
(18) \quad &\geq (1+o(1))\mathbb{P}(\text{LIS}(\sigma_n) \leq (x-3\varepsilon)\sqrt{n}, \#(\sigma_n) < \varepsilon^2 n)
\end{aligned}$$

In the first line we used Lemma 17, in the second we restricted the summation over  $\sigma$  and bounded uniformly from below the first probability by using (17), in the third line we summed the second probability over the remaining permutations, and in the last line we used Lemma 19 when  $k$  goes to infinity (noticing that  $\varepsilon\sqrt{n}/\sqrt{k} \geq 1$  for  $n$  large enough), while the bound is clear when  $k$  is finite. Now choose  $\sigma_n$  to be uniform. We deduce from (16) that

$$\begin{aligned}
\mathbb{P}(\text{LIS}(\sigma_{0,n}^{\text{Ew}}) \leq x\sqrt{n}) &\geq (1+o(1))\mathbb{P}(\text{LIS}(\sigma_n^{\text{Unif}}) \leq (x-3\varepsilon)\sqrt{n}, \#(\sigma_n^{\text{Unif}}) < \varepsilon^2 n) \\
&= (1+o(1))(\mathbb{P}(\text{LIS}(\sigma_n^{\text{Unif}}) \leq (x-3\varepsilon)\sqrt{n}) \\
&\quad - \mathbb{P}(\text{LIS}(\sigma_n^{\text{Unif}}) \leq (x-3\varepsilon)\sqrt{n}, \#(\sigma_n^{\text{Unif}}) \geq \varepsilon^2 n)) \\
&\geq (1+o(1))\mathbb{P}(\text{LIS}(\sigma_n^{\text{Unif}}) \leq (x-3\varepsilon)\sqrt{n}) = e^{-nI_{\text{LIS}, \frac{1}{2}}(x-\varepsilon)}(1+o(1))
\end{aligned}$$

where we used Theorem 3 and that  $\sigma_n^{\text{Unif}}$  is  $(\mathcal{CI}_{\alpha, \alpha})$ . We finally can let  $n$  going to infinity and  $\varepsilon$  going to zero to get (15). To complete the proof of Theorem 4, one only needs to check that

- If the sequence  $(\sigma_n)_{n \geq 1}$  of random permutations satisfies  $(\mathcal{CI}_{\frac{1}{2}, 1})$ , for any  $\varepsilon > 0$ , for any  $x \geq 2$ .

$$\lim_{n \rightarrow \infty} \frac{1}{n} \ln \left( \mathbb{P} \left( \text{LIS}(\sigma_n) \in \left( (x-\varepsilon)\sqrt{n}, (x+\varepsilon)\sqrt{n} \right) \right) \right) = 0.$$

Since  $(\mathcal{CI}_{\frac{1}{2}, 1})$  implies  $(\mathcal{CI}_{\frac{1}{2}, \frac{1}{2}})$ , this is a direct consequence of Corollary 9.

- If the sequence  $(\sigma_n)_{n \geq 1}$  satisfies  $(\mathcal{CI}_{\frac{1}{2}, \frac{1}{2}})$ , for any  $0 < y < 2$ .

$$\lim_{n \rightarrow \infty} \frac{\ln(\mathbb{P}(\text{LIS}(\sigma_n) < y\sqrt{n}))}{\sqrt{n}} = -\infty$$

Indeed, by taking  $f = -\text{LIS}$ ,  $\varepsilon = (2-y)/2$ ,  $x = -(y+\varepsilon)$  and  $\alpha = \beta = \frac{1}{2}$  in (10), the first inequality becomes

$$\mathbb{P}(\text{LIS}(\sigma_n) \leq y\sqrt{n}) \leq \mathbb{P}(\#(\sigma_n) \geq \varepsilon\sqrt{n}) + \mathbb{P}(\text{LIS}(\sigma_{0,n}^{\text{Ew}}) \leq x\sqrt{n}).$$

The first term goes to zero faster than  $e^{-M\sqrt{n}}$  for any  $M > 0$ , since the sequence  $(\sigma_n)_{n \geq 1}$  satisfies  $(\mathcal{CI}_{\frac{1}{2}, \frac{1}{2}})$ , whereas the second term goes to zero

**2.5. Proof of Proposition 5.** For the upper tail, one can apply directly Corollary 9.

For the lower bound of the lower tail, let  $\theta > 0$ .

$$\begin{aligned} \mathbb{P}(\text{LIS}(\sigma_{\theta,n}^{\text{Ew}}) = k) &\geq \mathbb{P}(\text{LIS}(\sigma_{\theta,n}^{\text{Ew}}) = k, \#\sigma_{\theta,n}^{\text{Ew}} = 1) \\ &= \frac{\text{card}\{\sigma : \text{LIS}(\sigma) = k, \#(\sigma) = 1\} \Gamma(\theta)}{\Gamma(n + \theta)} \\ &= \frac{\Gamma(\theta)\Gamma(n)}{\Gamma(n + \theta)} \mathbb{P}(\text{LIS}(\sigma_{0,n}^{\text{Ew}}) = k) \geq \frac{\Gamma(\theta)}{(n + \theta)^\theta} \mathbb{P}(\text{LIS}(\sigma_{0,n}^{\text{Ew}}) = k). \end{aligned}$$

This implies by taking  $k \leq x\sqrt{n}$ ,

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \ln \mathbb{P}(\text{LIS}(\sigma_{\theta,n}^{\text{Ew}}) \leq x\sqrt{n}) \geq \liminf_{n \rightarrow \infty} \frac{1}{n} \ln \mathbb{P}(\text{LIS}(\sigma_{0,n}^{\text{Ew}}) \leq x\sqrt{n}) = -I_{\text{LIS}, \frac{1}{2}}(x).$$

For the upper bound, one can conclude directly by (18), by choosing  $\sigma_n$  to be  $\sigma_{\theta,n}^{\text{Ew}}$ .

**2.6. Proof of Proposition 13.** We will adapt the proof of the lower tail of Theorem 4. We choose to give two different separate proofs for readability reasons.

First remark that Proposition 13 is equivalent to that

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \ln \mathbb{P}\left(\forall j \in [d], \sum_{i=1}^j \lambda_i(\sigma_n) \leq \sum_{i=1}^j x_i \sqrt{n}\right) = -I_{\text{LIS}, 1}(x_d).$$

Indeed, because the right hand side only depends on  $x_d$ , we see that mostly the deviations of  $\lambda_d(\sigma_n)$  matters. The same phenomenon appears for random matrices: the probability fix the  $d$  largest eigenvalues to make a deviation below 2 is equivalent to the probability that the spectrum stays below the smallest one, namely  $x_d$ .

We recall that according to Green's theorem ([35, Theorem 3.5.3])  $\lambda_1 + \dots, \lambda_k$  is the maximum sum of lengths of  $k$  disjoint increasing subsequences.

The counterpart of Lemma 18 is that for every  $\sigma$  deterministic, for every integer numbers  $d, k$ ,

$$(19) \quad \mathbb{P}\left(\sum_{i=1}^d \lambda_i(T(\sigma)) \leq \left(\sum_{i=1}^d \lambda_i(\sigma)\right) + k\right) \geq \inf_{\tau \in \mathfrak{S}_{\#(\sigma)}} \mathbb{P}\left(\sum_{i=1}^d \lambda_i(\tau \circ \sigma_{0, \#(\sigma)}^{\text{Ew}}) < k\right).$$

As in the proof of Theorem 4, we want to prove that

$$(20) \quad \begin{aligned} \liminf_{n \rightarrow \infty} \frac{1}{n} \ln \mathbb{P}\left(\forall j \in [d], \sum_{i=1}^j \lambda_i(\sigma_{\theta,n}^{\text{Ew}}) \leq \sum_{i=1}^j x_i \sqrt{n}\right) &= \limsup_{n \rightarrow \infty} \frac{1}{n} \ln \mathbb{P}\left(\forall j \in [d], \sum_{i=1}^j \lambda_i(\sigma_{0,n}^{\text{Ew}}) \leq \sum_{i=1}^j x_i \sqrt{n}\right) \\ &= -I_{\text{LIS}, 1}(x_d). \end{aligned}$$

It is then straightforward to generalize this result to  $\sigma_n \in (\mathcal{CI}_{\frac{1}{2}, 1})$  as in the proof of Theorem 6 by taking  $f = (\lambda_1, \lambda_1 + \lambda_2, \dots, \sum_{j=1}^d \lambda_j)$ ,  $\alpha = \frac{1}{2}$  and  $\beta = 1$ .

The upper bound is trivial for the same reason that in the proof of Theorem 4 (the probability of any event under Ewens with parameter 0 is at most  $n$  times its probability under the uniform permutation).



The lower bound is more sophisticated. Fix  $0 < x_d < \dots < x_1 < 2$ ,  $0 < \varepsilon < x_d$ , and  $\sigma \in S_n$ . We assume that  $\sigma$  is such that for any  $j$ ,  $\sum_{i=1}^j \lambda_i(\sigma) < ((\sum_{i=1}^j x_i) - 3\varepsilon)\sqrt{n}$  and  $\#(\sigma) < \frac{\varepsilon^2 n}{d^2}$ . Then, we find that

$$(21) \quad \begin{aligned} \mathbb{P}\left(\forall j \in [d], \sum_{i=1}^j \lambda_i(T(\sigma)) \leq \sum_{i=1}^j x_i \sqrt{n}\right) &= \mathbb{P}\left(\forall i \in [d], \sum_{i=1}^j \lambda_i(T(\sigma)) \leq \left(\left(\sum_{i=1}^j x_i\right) - 3\varepsilon\right)\sqrt{n} + 3\varepsilon\sqrt{n}\right) \\ &\geq \mathbb{P}\left(\forall j \in [d], \sum_{i=1}^j \lambda_i(T(\sigma)) \leq \left(\sum_{i=1}^j \lambda_i(\sigma)\right) + 3\varepsilon\sqrt{n}\right) \end{aligned}$$

By (19), (21) gives

$$(22) \quad \mathbb{P}\left(\forall j \in [d], \sum_{i=1}^j \lambda_i(T(\sigma)) \leq \sum_{i=1}^j x_i \sqrt{n}\right) \geq \inf_{\tau \in \mathfrak{S}_{\#(\sigma)}} \mathbb{P}\left(\forall j \in [d], \sum_{i=1}^j \lambda_i(\tau \circ \sigma_{0, \#(\sigma)}^{\text{Ew}}) < 3\varepsilon\sqrt{n}\right).$$

To conclude, we need the following easy lemma.

**Lemma 20.** *For any  $x > 2$  and any integer number  $j$ ,*

$$\lim_{n \rightarrow \infty} \inf_{\tau \in \mathfrak{S}_n} \mathbb{P}\left(\sum_{i=1}^j \lambda_i(\tau \circ \sigma_{0, n}^{\text{Ew}}) < jx\sqrt{n}\right) = 1$$

*Proof.* First we have,

$$\lim_{n \rightarrow \infty} \inf_{\tau \in \mathfrak{S}_n} \mathbb{P}\left(\sum_{i=1}^j \lambda_i(\tau \circ \sigma_{0, n}^{\text{Ew}}) < jx\sqrt{n}\right) \geq \lim_{n \rightarrow \infty} \inf_{\tau \in \mathfrak{S}_n} \mathbb{P}(j \text{LIS}(\tau \circ \sigma_{0, n}^{\text{Ew}}) < jx\sqrt{n})$$

because of the non increasing of  $\lambda_i$ . Therefore, one can conclude by Lemma 19 that

$$\lim_{n \rightarrow \infty} \inf_{\tau \in \mathfrak{S}_n} \mathbb{P}(j \text{LIS}(\tau \circ \sigma_{0, n}^{\text{Ew}}) < jx\sqrt{n}) = \mathbb{P}(\text{LIS}(\tau \circ \sigma_{0, n}^{\text{Ew}}) < x\sqrt{n}) = 1 + o(1).$$

□

Consequently, for any  $\sigma_n$  conjugacy invariant, similarly to (18), we obtain

$$(23) \quad \begin{aligned} \mathbb{P}\left(\forall j \in [d], \sum_{i=1}^j \lambda_i(\sigma_{0, n}^{\text{Ew}}) \leq \sum_{i=1}^j x_i \sqrt{n}\right) \\ \geq (1 + o(1)) \mathbb{P}\left(\forall j \in [d], \sum_{i=1}^j \lambda_i(\sigma_n) \leq \left(\left(\sum_{i=1}^j x_i\right) - 3\varepsilon\right)\sqrt{n}, \#(\sigma_n) < \frac{\varepsilon^2 n}{d^2}\right). \end{aligned}$$

The remaining of the proof is identical to that of the lower tail of Theorem 4.

### 3. FURTHER DISCUSSIONS

We start by discussing the optimality of our conditions. The condition  $(\mathcal{CI}_{\frac{1}{2}, \frac{1}{2}})$  for LIS is optimal as a condition on cycles. Indeed, let us construct a sequence of random permutations  $\sigma_n$  such that  $\#(\sigma_n)/\sqrt{n}$  is of order  $x \neq 0$  with probability of order  $e^{\sqrt{n}C}$  for some  $C$  finite and show that our result does not apply to  $\sigma_n$ . In fact let  $\sigma_n$  be a permutation constructed as

follows. With probability  $e^{-(I_{\text{LIS},1/2}(x)-\varepsilon)\sqrt{n}}$ ,  $\sigma_n$  has  $\lfloor x\sqrt{n} \rfloor$  fixed points, the other points belonging to a cycle of length  $n - \lfloor x\sqrt{n} \rfloor$ . With probability  $1 - e^{-(I_{\text{LIS},1/2}(x)-\varepsilon)\sqrt{n}}$ ,  $\sigma_n$  is a uniform cyclic permutation. In this case, it is not difficult to check that since the fixed points furnish an increasing subsequence

$$\lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}} \ln \mathbb{P}(\text{LIS}(\sigma_n) \geq x\sqrt{n}) \geq -I_{\text{LIS},1/2}(x) + \varepsilon > -I_{\text{LIS},1/2}(x),$$

showing that  $\text{LIS}(\sigma_n)$  can not follow the same large deviation principle than  $\text{LIS}(\sigma_n^{\text{Unif}})$  as stated in Theorem 3.

When the number of cycles is not sufficiently controlled, it is only possible to obtain one bound.

**Proposition 21.** *If  $(\sigma_n)_{n \geq 1}$  satisfies  $(\mathcal{CI}_{1,1})$ , then for any  $0 < x < 2$ ,*

$$(24) \quad \limsup_{n \rightarrow \infty} \frac{1}{n} \ln \mathbb{P}(\text{LIS}(\sigma_n) \leq x\sqrt{n}) \leq -I_{\text{LIS},1}(x).$$

Moreover if  $(\sigma_n)_{n \geq 1}$  satisfies that for any  $\varepsilon > 0$

$$\liminf_{n \rightarrow \infty} \mathbb{P}\left(\frac{\#(\sigma_n)}{n} < \varepsilon\right) > 0,$$

then for any  $x > 2$ .

$$\liminf_{n \rightarrow \infty} \frac{1}{\sqrt{n}} \ln \mathbb{P}(\text{LIS}(\sigma_n) \geq x\sqrt{n}) \geq -I_{\text{LIS},\frac{1}{2}}(x).$$

*Sketch of the proof of Proposition 21.* We will compare directly  $\text{LIS}(\sigma_n)$  to  $\text{LIS}(\sigma_{0,n}^{\text{Ew}})$ . For the first inequality, (18) and (3) imply that

$$(25) \quad e^{-nI_{\text{LIS},1}(x)(1+o(1))} \geq \mathbb{P}(\text{LIS}(\sigma_{0,n}^{\text{Ew}}) \leq x\sqrt{n}) \\ \geq (1+o(1))\mathbb{P}(\text{LIS}(\sigma_n) \leq (x-3\varepsilon)\sqrt{n}, \#(\sigma_n) < \varepsilon^2 n).$$

Moreover, if  $\sigma_n$  satisfies  $(\mathcal{CI}_{1,1})$ , for every  $M > 0$  and  $n$  large enough,

$$\mathbb{P}(\text{LIS}(\sigma_n) \leq (x-3\varepsilon)\sqrt{n}, \#(\sigma_n) < \varepsilon^2 n) \geq \mathbb{P}(\text{LIS}(\sigma_n) \leq (x-3\varepsilon)\sqrt{n}) - e^{-nM}.$$

which gives (24) with (25).

The second inequality is an application of [23, Lemma 21]. First remark that a conjugacy invariant permutation conditioned on  $\#(\sigma_n) \leq k$  is still conjugacy invariant. One needs only to prove this result for permutations where the number of cycles is less than  $\varepsilon n$  for  $n \geq n_0$  almost surely.

By choosing  $\rho$  to be  $\sigma_{0,n}^{\text{Ew}}$  and  $k = 1$  in [23, Lemma 21], one can reformulate the lemma to obtain that for any conjugacy invariant permutation  $\sigma_n$ , there exists  $\widehat{\sigma}_n \stackrel{d}{=} \sigma_n$  such that, for any  $1 \leq a < b$  and for any  $c > 0$

$$\mathbb{E}\left(\left(\text{LIS}(\widehat{\sigma}_n) - \text{LIS}(\sigma_{0,n}^{\text{Ew}})\right)_- \mid \#\widehat{\sigma}_n < c, \text{LIS}(\sigma_{0,n}^{\text{Ew}}) \in [a, b]\right) \leq \frac{cb}{n}.$$

By Markov inequality, and by taking  $a = x\sqrt{n} + 4x\varepsilon\sqrt{n}$  and  $b = 2x\sqrt{n}$  and  $c = \varepsilon n$  we obtain that

$$\begin{aligned}
\mathbb{P}(\text{LIS}(\sigma_n) \geq x\sqrt{n}) &= \mathbb{P}(\text{LIS}(\widehat{\sigma}_n) \geq x\sqrt{n}) \\
&\geq \mathbb{P}(\text{LIS}(\sigma_{0,n}^{\text{Ew}} \in [a, b], (\text{LIS}(\widehat{\sigma}_n) - \text{LIS}(\sigma_{0,n}^{\text{Ew}}))_- \leq 4x\varepsilon\sqrt{n}) \\
&= \mathbb{P}((\text{LIS}(\widehat{\sigma}_n) - \text{LIS}(\sigma_{0,n}^{\text{Ew}}))_- \leq 4x\varepsilon\sqrt{n} | \text{LIS}(\sigma_{0,n}^{\text{Ew}} \in [a, b]), ) \mathbb{P}(\text{LIS}(\sigma_{0,n}^{\text{Ew}} \in [a, b]) \\
&\geq \left(1 - \frac{\frac{cb}{n}}{4x\varepsilon\sqrt{n}}\right) I_{\text{LIS}, \frac{1}{2}} \mathbb{P}(\text{LIS}(\sigma_{0,n}^{\text{Ew}} \in [a, b]) = \frac{1}{2} \mathbb{P}(\text{LIS}(\sigma_{0,n}^{\text{Ew}} \in [a, b])
\end{aligned}$$

which concludes the proof.  $\square$

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