

Symposium: Between Discrete and Continuous Time

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Foundations 2023
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Discrete Lagrangian Formalism

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Introduction

Two representations of time

- Representation with a continuous time symbol

$$t \in \mathbb{R}$$



- Representation with a discrete time symbol

$$t_k \in \mathbb{R}$$

$$k \in \mathbb{N}$$

The received view on the continuous time symbol

- It is generally admitted that it is not possible to formulate classical mechanics without a continuous representation of time.

“a physical theory whose fundamental laws take the form of difference equations” is “at best, a gleam in the eyes of hopeful speculative theoreticians” (Grünbaum 1968, in Salmon 2001, p. 246).

- Today, classical mechanics can be formulated with a discrete representation of time

Lee's project

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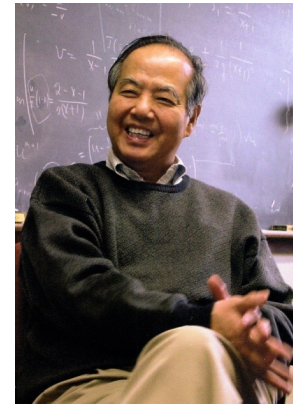
10 March 1983

CAN TIME BE A DISCRETE DYNAMICAL VARIABLE? ☆

T.D. LEE

Columbia University, New York, NY 10027, USA

Received 19 November 1982



“Abstract. The possibility that time can be regarded as a discrete dynamical variable is examined through all phases of mechanics from classical mechanics to nonrelativistic quantum mechanics, and to relativistic quantum field theories.”

Claims

- **Claim 1:** the continuous representation of time is **not mandatory**.

The continuous and discrete representations of time are two conventional theoretical frameworks.

- **Claim 2:** Lee's discrete mechanics fits well with a **relationalist** conception of time.

The time symbol is no longer the parameter responsible for the evolution of physical systems.

Outline

- 1. Lee's discrete classical mechanics
- 2. Claim 1: The dispensability of the continuous representation of time
- 3. Claim 2: Lee's mechanics and relationalism
- 4. Discrete quantum mechanics

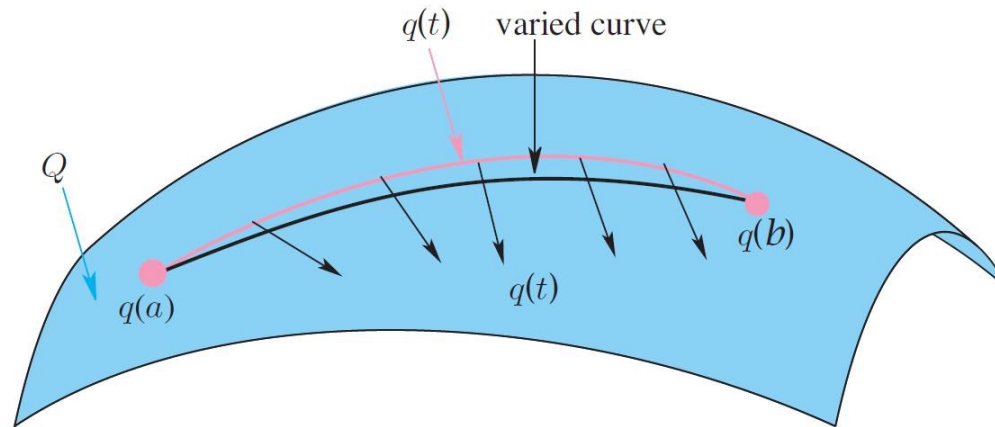
1. Lee's discrete classical mechanics

The preferable discrete mechanics

- There are various discrete classical mechanics (e.g., Greenspan 1973)
- Lee's formulation satisfies all the usual conservation laws (energy, momenta, symplectic form)
- It is based on a variational approach of classical mechanics

(Continuous) variational mechanics

- Based on the least action principle



$$\mathcal{S}(q) = \int_0^T L(q, q') dt \quad \delta\mathcal{S} = 0$$

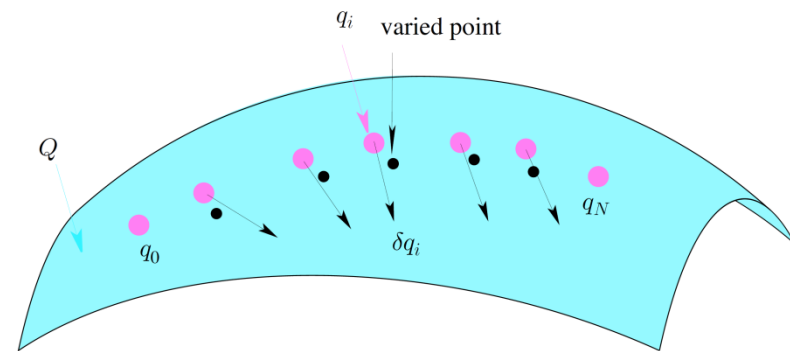
Discrete variational mechanics

$$\mathcal{S}_d = \sum_{k=0}^{N-1} \mathcal{L}_d(q_k, t_k, q_{k+1}, t_{k+1})$$

- A discrete least action principle:

$$\delta \mathcal{S}_d = 0$$

- Discrete Euler-Lagrange equations
- Discrete conservation of energy



(Marsden and West 2001)

see e.g., Ardourel 2015, Ardourel & Barberousse 2017

In practice

- Solving some systems

- Free particle:
$$q_k = \left(\frac{q_1 - q_0}{t_1 - t_0} \right) t_k + q_0$$

- Harmonic oscillator:
$$q_k = A \cos(\omega_d t_k) + B \sin(\omega_d t_k)$$

2. The dispensability of the continuous representation of time

Empirical equivalence

- Discrete and continuous mechanics are empirically equivalent for sufficiently large N (the number of steps).

$$\mathcal{S}_d = \sum_{k=0}^{N-1} \mathcal{L}_d(q_k, t_k, q_{k+1}, t_{k+1}) \quad \xrightarrow{N \rightarrow \infty} \quad \mathcal{S}(q) = \int_0^T L(q, q') dt$$

- The equations of discrete mechanics (and their solutions) tend to the equations of continuous mechanics (and their solutions) when the N tends to infinity

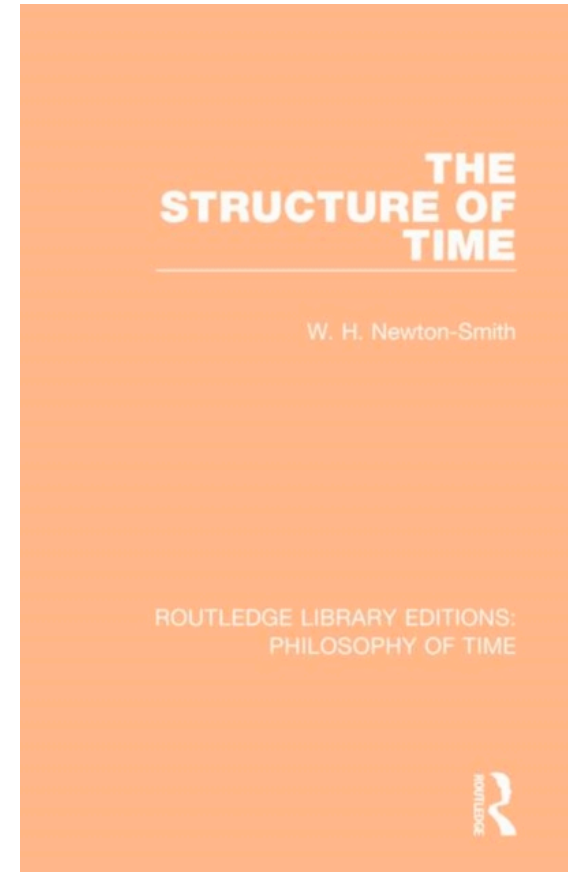
see e.g., Ardourel 2018

The consequences of empirical equivalence



William H. Newton-Smith

“if our best theories should in the end turn out to involve representing time by a discrete time variable, we would have at least as much reason to regard time as discrete as we now have for regarding it as continuous”(p. 118)



“Undecidability” thesis

- The empirical equivalence leads to an undecidability thesis.

“How ought we to respond to this undecidability result?”
(1980, p. 126)

1) either, “time is continuous” and “time is discrete” are **inaccessible empirical statements**.

2) or, “time is continuous” and “time is discrete” are **non-empirical statements describing theoretical frameworks**

- **Amendments**: restricting the consequences to the **representation** of time.

A twofold representation of time

- Lee's (1983) treats time “as a discrete dynamical variable”
 - I have discussed what he means by **discrete**
 - But what does he mean by **dynamical variable**?

3. Lee's mechanics and relationalism

Discretization of the extended Lagrangian mechanics

- Lee's mechanics (1983) is the **discretization** of the **extended theory of Lagrangian mechanics**
- But, what is **the extended theory of Lagrangian mechanics**?

“If one wishes, one can treat time in a manner more similar to the position of a particle. In both the Lagrangian and Hamiltonian formalism, t is then treated as one of the configuration coordinates q , all of which are functions of a temporal parameter, τ say.” (Butterfield 2013, p. 226)

$$q = q(t)$$



$$q = q(\tau)$$

$$t = t(\tau)$$

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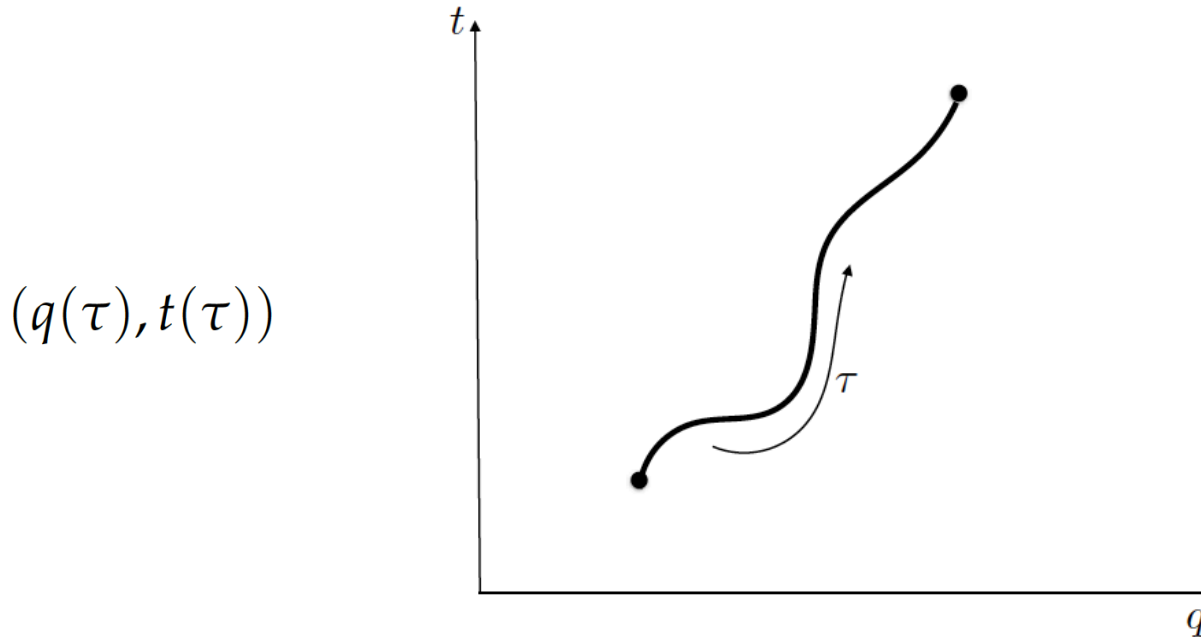
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$$q = q(\tau)$$

$$t = t(\tau)$$

What is the extended Lagrangian mechanics?



- The extended Lagrangian mechanics is invariant under parametrization

$$\tau \longrightarrow \tau' = \tau'(\tau)$$

- τ is a free gauge

What is the extended Lagrangian mechanics?

- Traditional Lagrangian theory $\mathcal{S}(q) = \int_0^{t_f} L(q, \dot{q}) dt$

- Extended Lagrangian theory $\mathcal{S}(q, t) = \int_0^{\tau_f} \mathcal{L}(q, t, \dot{q}, \dot{t}) d\tau$

Leibnizian viewpoint of time

- The extended Lagrangian theory is “a modification of the theory (...) that allows more naturally for a Leibnizian viewpoint”(Thébault 2021, p. 389)
- Time t is no longer an evolution parameter, but a coordinate. It is a “dynamical variable”.

$$\mathcal{S}(q, t) = \int_0^{\tau_f} \mathcal{L}(q, t, \dot{q}, \dot{t}) d\tau$$

$$\dot{q} = \frac{dq}{d\tau}$$

$$\dot{t} = \frac{dt}{d\tau}$$

Lee's mechanics: discretizing 'tau'

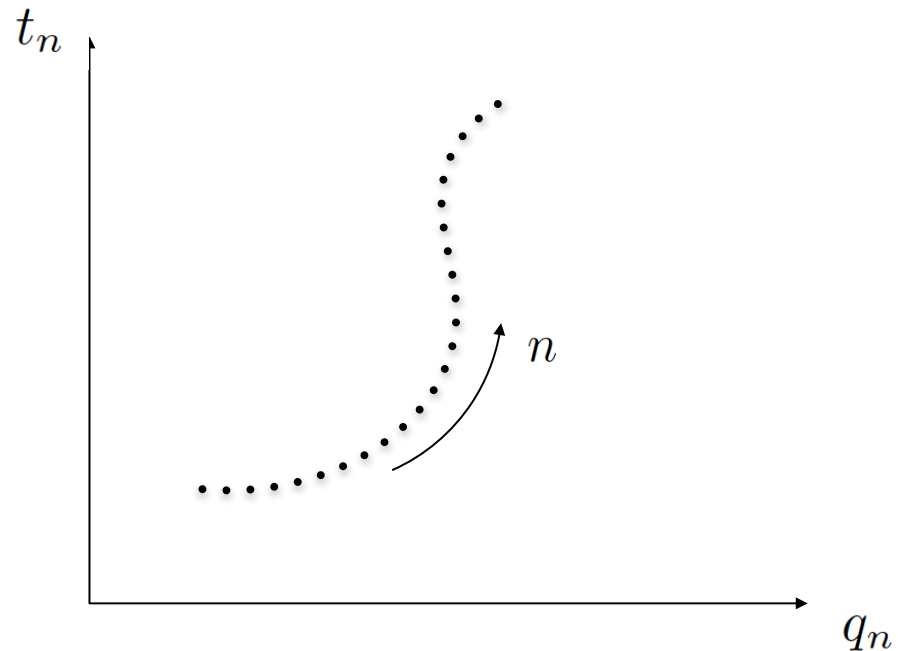
- Lee's mechanics (1983) is the **discretization** of the **extended theory of Lagrangian mechanics**

$$\mathcal{T} \longrightarrow n$$

$$q_n = q(n)$$

$$t_n = t(n)$$

$$\mathcal{S}_d = \sum_{k=0}^{N-1} \mathcal{L}_d(q_k, t_k, q_{k+1}, t_{k+1})$$



Leibnizian viewpoint of time

Lee's discrete mechanics goes well with a Leibnizian viewpoint of time.

The relationist account is more constrained in the discrete case than in the continuous case.

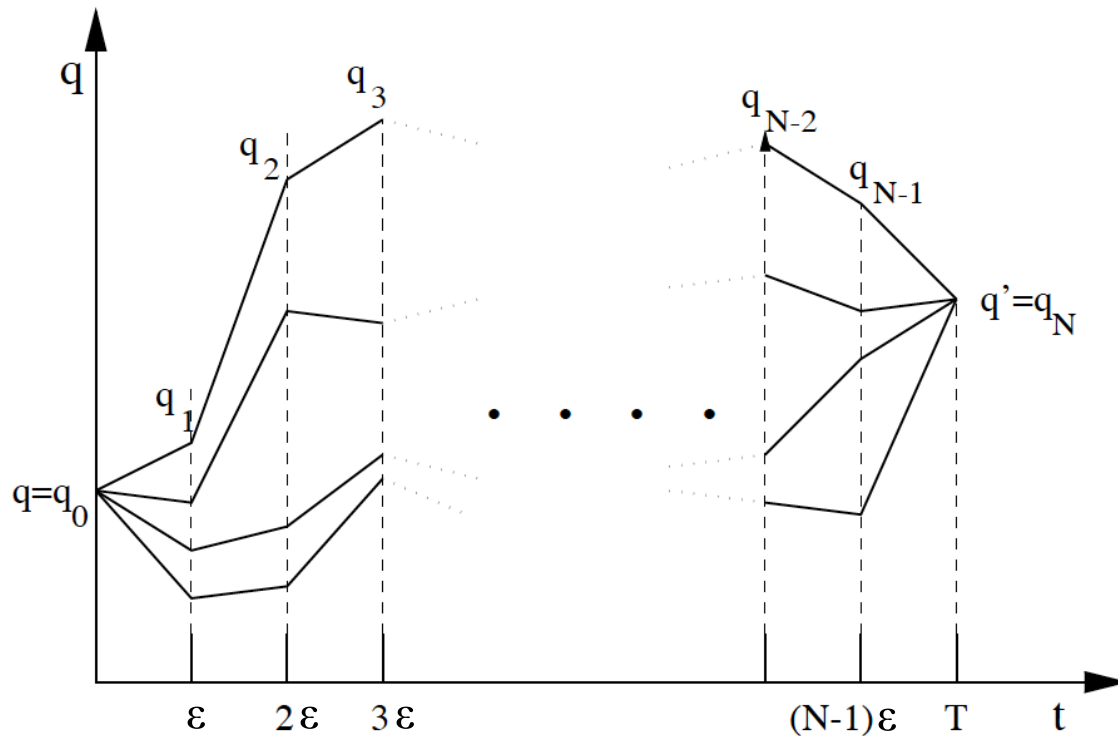
In the discrete case, we need to adopt the extended Lagrangian formulation to have all the conservation equations (Ge & Marsden 1988).

A non-trivial observation: the use of a discrete representation of time is not independent to the viewpoint on time.

The Leibnizian viewpoint follows from the discrete approach.

4. Discrete quantum mechanics

Feynman path integral formalism



Credit: MacKenzie 2000

“A path is first defined only by the positions x_i through which it goes at a sequence of equally spaced times, $t_i = t_{i-1} + \epsilon$. (...) The limit $\epsilon \rightarrow 0$ must be taken at the end of a calculation.” (Feynman 1948, p. 371)

With the traditional Lagrangian theory

- Feynman path integral is based on a discrete time step ϵ .

$$S(x_{i+1}, x_i) = \epsilon L\left(\frac{x_{i+1} - x_i}{\epsilon}, x_{i+1}\right)$$

- The discrete representation of time $t_n = n\epsilon$ is an *evolution parameter*, not a discrete *dynamical variable*.

With the extended Lagrangian theory

- However, in Feynman's paper (1948):

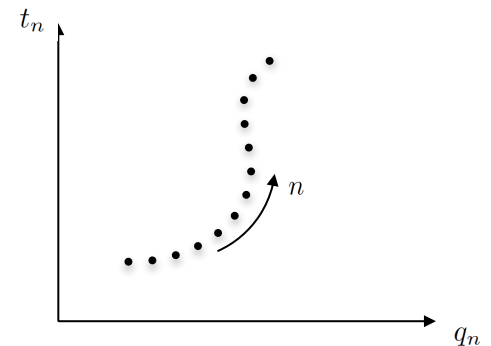
“the subdivision of time into equal intervals is not necessary. (...)
The total action S must be now represented as a sum:

$$S = \sum_i S(x_{i+1}, t_{i+1}; x_i, t_i)$$

- It is the discretization of the extended Lagrangian theory

$$q_n = q(n)$$

$$t_n = t(n)$$



With the extended Lagrangian theory

- No discrete time step ε , only the number N of steps

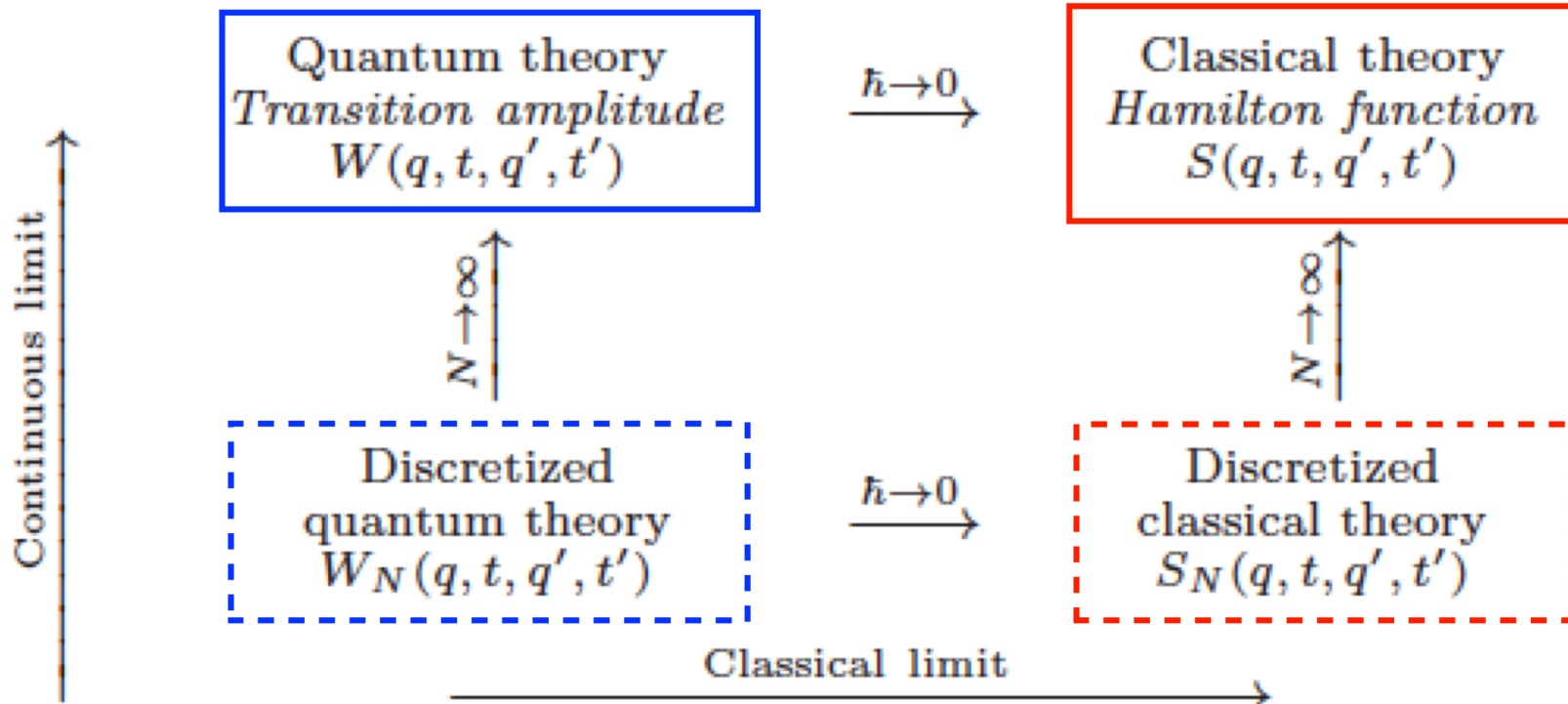
“At first this looks magic: the discretization of the Lagrangian does **not depend on the “lattice spacing”**! The number N appears only in the upper limit of the sum.” (Rovelli and Vidotto 2014, p. 87)

- The only limit required to recover the continuous case is $N \rightarrow \infty$
- Lee’s discrete quantum mechanics: take N large enough, but finite
- Empirical equivalence with finite N , if N is taken sufficiently large

Continuous and classical limits

Quantum mechanics

Classical mechanics



Discretization in Loop Quantum Gravity

- The discrete extended Lagrangian mechanics is connected to the continuum by only the number N of steps

“The reparametrization invariant systems are characterised by the fact that there is **no parameter to rescale** in the discrete action in the limit: the only place where N appears is in the number of steps.

This peculiar behavior of the discretization of reparametrization invariant systems can be found in the “good” discretizations of GR.”

Rovelli and Vidotto (2014, p. 87)

Conclusion

Summary

- Focus on Lee's project on physics with a **discrete time**
- **Claim 1:** the continuous representation of time is not mandatory
 - The representations of time are theoretical frameworks
- **Claim 2:** Lee's discrete mechanics fits well with a relationalist conception of time.
 - If we adopt a discrete representation of time, the relationalist viewpoint seems to follow.

Lee's project on discrete physics

“For more than three centuries we have been influenced by the precept that fundamental laws of physics should be expressed in terms of differential equations. Difference equations are always regarded as approximations. Here, I try to explore the opposite: Difference equations are more fundamental, and differential equations are regarded as approximations.” (Lee 1987, p. 859)

Thank you !

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