

Asymptotic expansions as modelling methods in physics

Vincent Ardourel

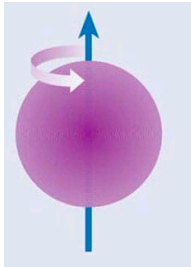


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Introduction

A prediction in physics

- One of the most accurate predictions in physics is the anomalous magnetic moment of the electron : $a_e = (g-2)/2$



- a_e (theory) = 0,001 159 652 180 (73)
- a_e (experiment) = 0,001 159 652 181 (78)

- a_e (theory) = $c_1\alpha + c_2\alpha^2 + c_3\alpha^3 + c_4\alpha^4 + \dots$
- a_e (theory) is asymptotic series, which is divergent

Aim of the talk

- Asymptotic methods are pervasive in physics
- Several issues in the philosophical literature:
 - 1. Asymptotic methods are “explanatory” (Batterman 1997, 2007)
 - 2. Asymptotic methods are divergent series (Batterman 1997, Miller 2021)
- Claim₁: asymptotic methods are modeling methods, which allows us to understand their “explanatory power”
- Claim₂: approaches for tackling divergences series

Outline

1. Batterman and Berry on rainbows
2. Asymptotic methods as modeling methods
3. Divergence

Batterman and Berry on rainbows



Supernumerary bows

Batterman (1997, 2007)

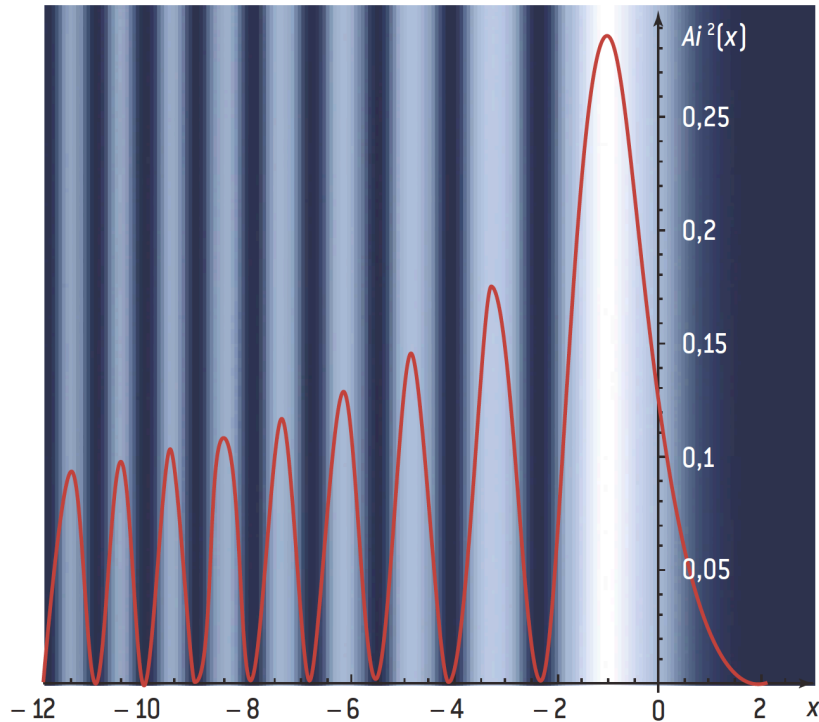
Berry (1989, 1994...)



Both **geometrical optics** and **wave optics** are required to explain supernumerary bows.

Predicting fringes

“On the Intensity of Light in the Neighborhood of a Caustic” (Airy, 1838)

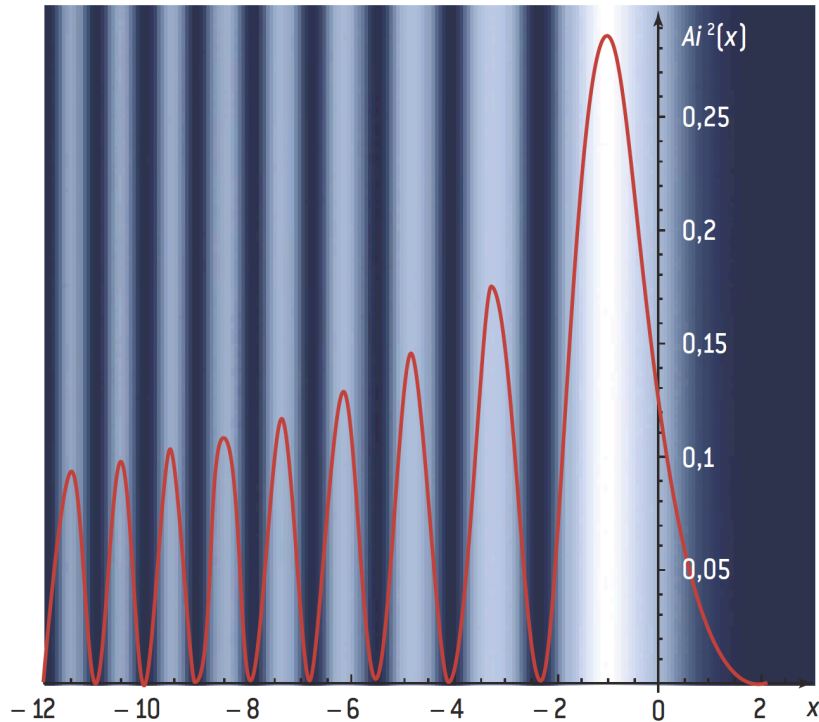


Airy function

$$Ai(x) \equiv \frac{1}{\pi} \int_0^{\infty} \cos\left(\frac{t^3}{3} + xt\right) dt.$$

Solution of $y'' - xy = 0$

Calculating fringes



Airy's method (1838)

Computing $Ai(x)$ with **convergent series**, allows him to correctly find the **first fringe**

Stokes's method (1850)

Computing $Ai(x)$ with **divergent series**, allows him to correctly find **thirty fringes**

Airy vs. Stokes

$$Ai(x) = A \left\{ 1 + \frac{9x+3}{2 \cdot 3} + \frac{9^2 x^6}{2 \cdot 3 \cdot 5 \cdot 6} + \frac{9^3 x^9}{2 \cdot 3 \cdot 5 \cdot 6 \cdot 8 \cdot 9} + \dots \right\} \\ + B \left\{ x + \frac{9x^4}{3 \cdot 4} + \frac{9^2 x^7}{3 \cdot 4 \cdot 6 \cdot 7} + \frac{9^3 x^{10}}{3 \cdot 4 \cdot 6 \cdot 7 \cdot 9 \cdot 10} + \dots \right\}.$$

$$Ai(x) \equiv Cx^{-\frac{1}{4}}e^{-2x^{\frac{3}{2}}} \left\{ 1 - \frac{1 \cdot 5}{1 \cdot 144x^{\frac{3}{2}}} + \frac{1 \cdot 5 \cdot 7 \cdot 11}{1 \cdot 2 \cdot 144^2 x^3} - \dots \right\} \\ + Dx^{-\frac{1}{4}}e^{2x^{\frac{3}{2}}} \left\{ 1 + \frac{1 \cdot 5}{1 \cdot 144x^{\frac{3}{2}}} + \frac{1 \cdot 5 \cdot 7 \cdot 11}{1 \cdot 2 \cdot 144^2 x^3} + \dots \right\}$$

Batterman's claims

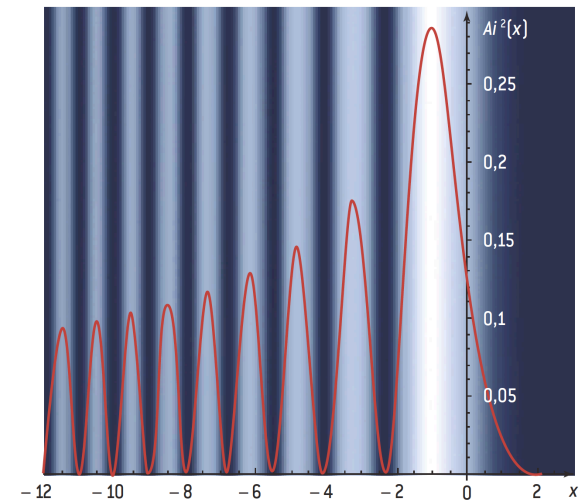
- Stokes' method is **more** than just an approximation method
- Stokes' method “brings out **the essential structures** underlying the phenomenon” (1997, p. 405)
- Stokes' method provides “an understanding and, hence, an explanation of **the most important physical features** of the phenomenon” (p. 406)

Stokes' method

- The first terms of the asymptotic (Stokes) series indeed provide :

- oscillatory behavior for $x < 0$, and $|x| \gg 1$

- exponential decay for $x > 0$, and $|x| \gg 1$



Stokes' method

- Stokes series is an **asymptotic expansion**
 - It is **divergent**, but truncated after a few terms
- Stokes methods is a particular case of the **asymptotic WKB method**.

The series is obtained by studying the equation :

$$\varepsilon y'' - xy = 0$$

and finding its solution as an infinite series that depend on ε

Batterman's account of explanation

Batterman (1997)

“the asymptotic representations make salient mathematical properties of the solutions of many differential equations arising in mathematical physics.”

“the analytical form of the terms in the asymptotic divergent series ‘mirrors’ the appropriate mathematical structures. Such mirroring is completely absent in the convergent series definition of the Airy integral.”

Batterman's account of explanation

- Comparing a Deductive-Nomological account of explanation

“A D-N type account (provided by the calculation of the convergent series) does, in some sense, contain all of the information about the phenomenon; nevertheless, it is so deeply encoded as to be useless for explanatory purposes. In fact, focusing on the details (the partial sums of terms in the convergent series (...)) simply obfuscates the salient explanatory facts.”

Asymptotic methods as modeling methods

Two questions

- 1. Why do asymptotic methods “provide dominant features”?
- 2. How can we argue for this *general* claim about asymptotic methods?

Perturbative methods

I claim that asymptotic methods provide dominant features of physical systems because they are **perturbation theory methods**

The solution is found by considering an equation with a **small perturbation ϵ**

The solution at the zero order already contains the structure of the unperturbed equation

It is a common feature of all asymptotic expansions, making sense for a general claim

Let's see a few examples

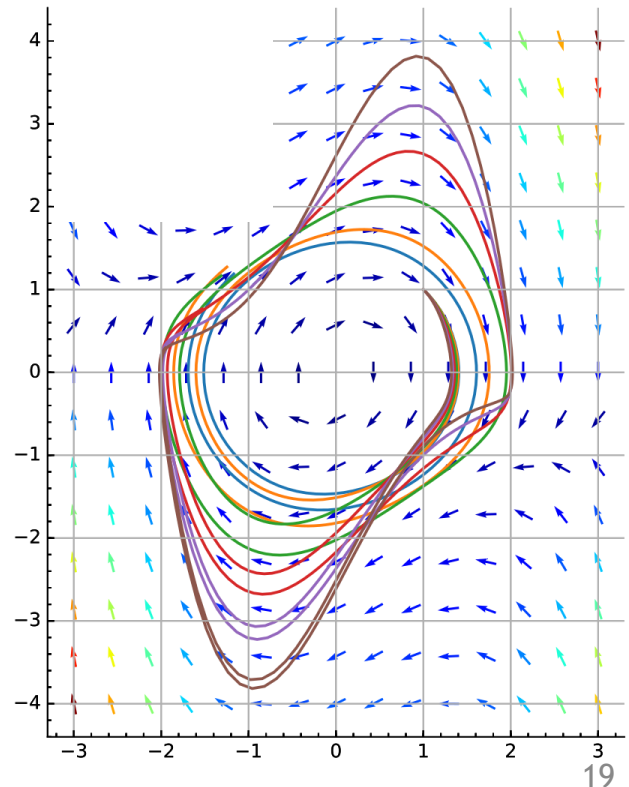
Van der Pol oscillator

A model proposed by for a beating heart

Self-excited oscillator, whose energy increases when $|y| < 1$ and decreases when $|y| > 1$.

$$y'' + \varepsilon (y^2 - 1) y' + y = 0$$

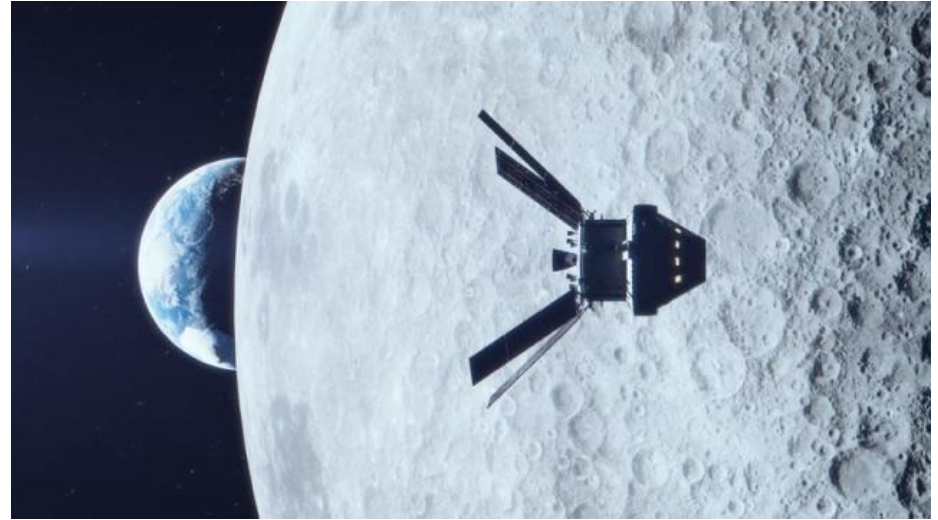
The solution obtained with asymptotic methods is the sum of the solution of unperturbed system, plus corrections (depending on the orders)



Three-body problem

Earth, moon, and a satellite

A restricted model is considered:



The mass of the satellite is much smaller than the masses of the Earth and the moon

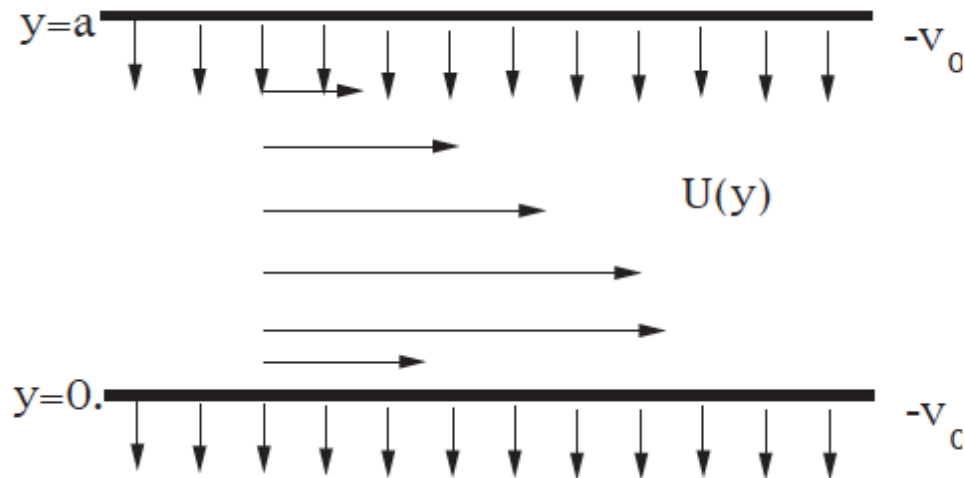
Asymptotic methods used to calculate the trajectory of the satellite

The solution obtained as corrections to the unperturbed system (a two-body problem)

Hydrodynamics

Steady Poiseuille flow with uniform blowing at one wall and aspiration at the opposite wall.

Small perturbation: $\varepsilon = \nu / (av_0)$



Change to a perturbative problem

When a problem is tackled with asymptotic methods, the representation of the system is changed to a perturbative problem.

- harmonic oscillator with a perturbation
- a two-body problem with a perturbation
- a fluid flow with a perturbation

These changes in the representation of the system correspond to a modification of the model

Asymptotic expansions are used within a modeling strategy

“Asymptotic explanations”

“Asymptotic explanations” put together physical problems solved with asymptotic expansions, problems with singular limits, problems exhibiting universal behaviors, exemplifying anti-reductionism and emergence ... (e.g., Batterman 2002)

- Distinction between:
 - asymptotic explanations
 - explanations with asymptotic methods
- For example, an anti-reductionistic account does not follow from the Van der Pol oscillator (see Wayne 2012 vs. Rueger 2000)
- Focus on “explanations with asymptotic methods”

Divergence

The problem of divergent series

“It seems that an obvious desideratum of an explanatory theory is that it not contain within its mathematical formulation the sort of vagueness and potential inexactness which has (...) been associated with the use of asymptotic expansions.”(Batterman 1997)

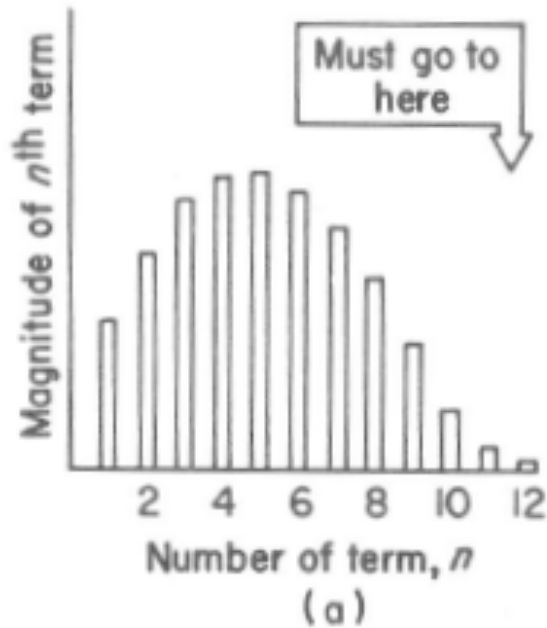
“divergent series are the invention of the devil, and it is shameful to base on them any demonstration whatsoever”(Abel 1828)

How does any information whatsoever about a function can be extracted from a divergent series?

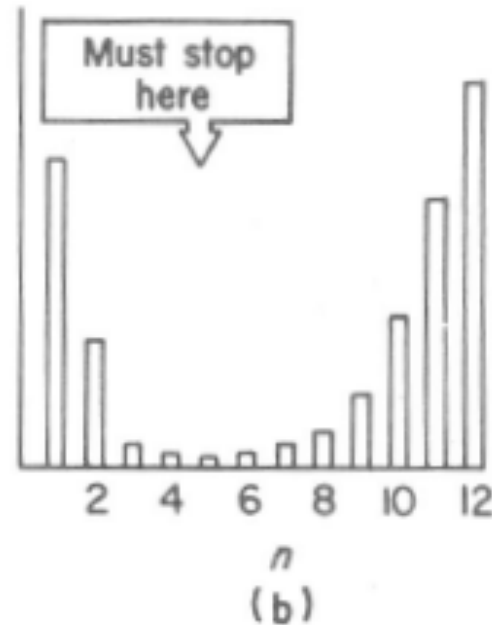
Different ways to tackle this problem.

1. Instrumental approach

Physicists follow a recipe, guided by conjectures and case-by-case analysis: “stop here the truncation”



convergent series

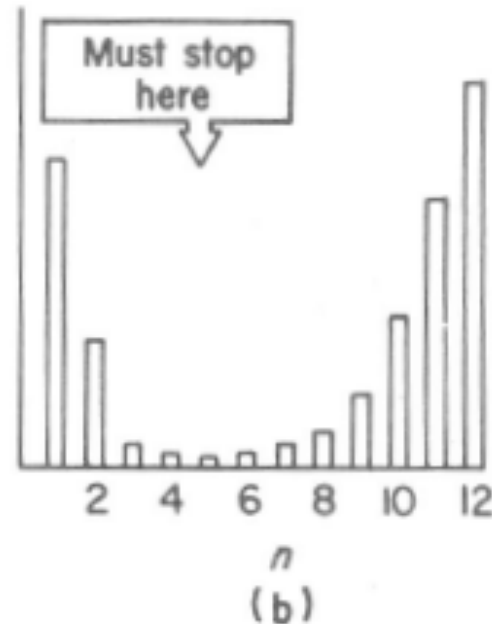
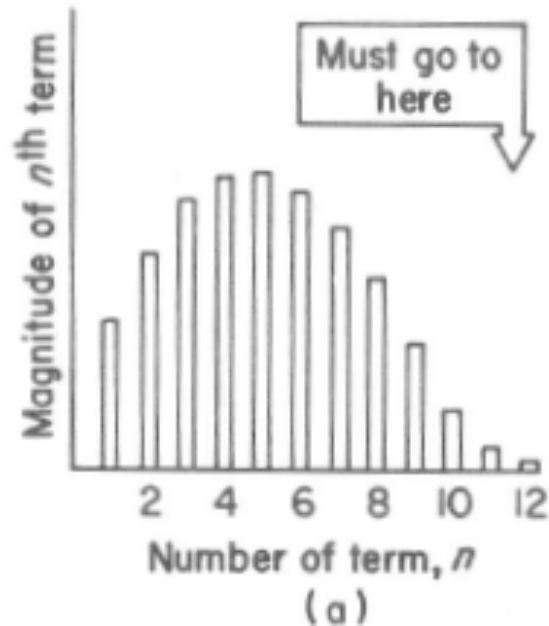


divergent series

1. Instrumental approach

used by mathematicians

used by physicist



Poincaré, H. (1892). *Les Nouvelles méthodes de la mécanique céleste*

Asymptoticity

$$f(x) \underset{x \rightarrow 0}{\sim} \sum_{n=0}^{\infty} a_n x^n$$

- (Poincaré's) definition: a series is asymptotic to a function $f(x)$ close to zero if, for any fixed N :

$$\lim_{x \rightarrow 0} \frac{f(x) - \sum_{n=0}^N a_n x^n}{x^N} = 0$$

- Physicists use asymptotic divergent series by truncating them after a few terms

What does the series represent?

An issue with Poincaré's definition of asymptotic expansion

$f(x)$ is not unique

The series is also the asymptotic expansion of $g(x)$, $h(x)$...

The asymptotic expansion fails to represent a unique function, even if the truncation of its first terms is useful

2. Representational account

Batterman's solution:

“part of the worry can be assuaged by eschewing Poincare's definition of an asymptotic power series. In fact, this is precisely the approach taken by Dingle in his book *Asymptotic Expansions: Their Derivation and Interpretation*” (Batterman 1997)

Turning to a new account of asymptotic series (Dingle's definition)

Asymptotic expansions are « *formally* » exact

2. Representational account

Batterman's solution:

“Dingle showed that for an extremely large class of functions, the formulas representing the late terms in the series all have the same basic form. So each individual asymptotic series would be summed to the least term and then ‘terminated’ in a manner that was *universal*. In other words, the formula for the asymptotic late terms of a wide variety of functions have the same basic form.”(Batterman 1997)

Function = first n terms of asymptotic series + n^{th} \times terminant

A research program, which deserves to be investigated!

2. Representational account

Miller's solution : turning to to another account of asymptotic series (based on Borel summation)

“Fortunately, there is a condition stronger than that the series is asymptotic to the function, but weaker than that the series is convergent, that **does determine the function uniquely**. This is the strong asymptotic condition.”(Miller 2021)

“Strong asymptoticity”
$$\left| f(z) - \sum_{n=0}^N a_n z^n \right| \leq C \sigma^{N+1} (N+1)! |z|^{N+1}$$

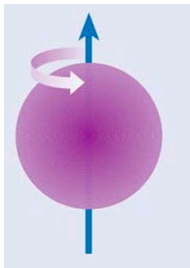
For example, Stark effet requires asymptotic expansions that can be reframed as strong asymptotic (Miller 2021)

2. Representational account

Miller's solution

Limitations: **not all** asymptotic expansions are “strongly asymptotic”

For example, the anomalous magnetic moment of the electron



- a_e (theory) = $c_1\alpha + c_2\alpha^2 + c_3\alpha^3 + c_4\alpha^4 + \dots$



- the three-body problem

2. Representational account

Miller's solution

If the strong asymptotic condition is not satisfied:

“This means that the calculations that establish the empirical adequacy of quantum electrodynamics, such as the determination of a_e , **cannot be treated as approximations.**” (Miller 2021)

Conclusion

To sum up

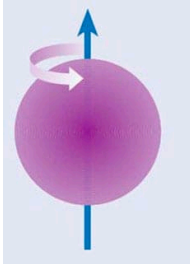
- A work in progress!
- Aim of the talk : clarifying why “the asymptotic representations allow one to exhibit the crucial characteristics”(Batterman 1997)
- Claim: asymptotic methods are **perturbative** modeling methods, which allows us to understand why they make salients the dominant features of a system.
- Future work on the problem of the reference
 - Can we solve it?
 - What is its extent? (investigating other case studies)

Thank you!

A few references

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- Miller M. (2021). Mathematical structure and empirical content, *British Journal for the Philosophy of Science*
- Rueger, A. (2000). Physical emergence, diachronic and synchronic. *Synthese*, 124: 297–322.
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Example of accurate predictions



- a_e (theory) = 0,001159652180(73)

$$a = c_1\alpha_1 + c_2\alpha_1^2 + c_3\alpha_1^3 + c_4\alpha_1^4 + \dots$$

$$\alpha_1 := \alpha/\pi$$

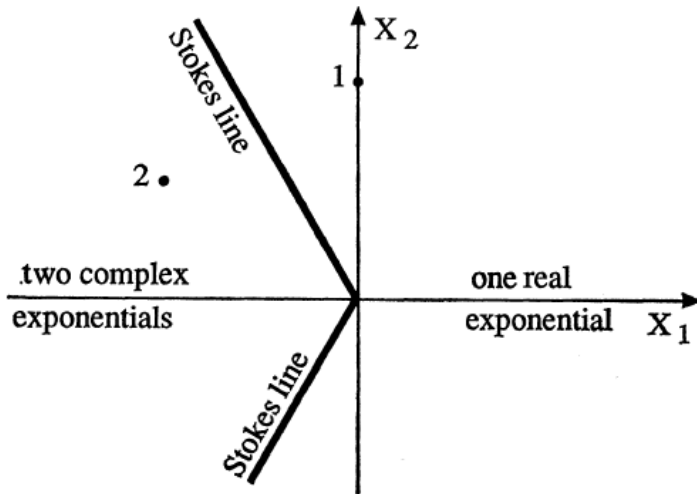
$$c_1 = 1/2 \quad (\text{Schwinger 1948})$$

$$c_2 = \frac{197}{144} + \left(\frac{1}{2} - (3 - \log 2)\right)\zeta(2) + \frac{3}{4}\zeta(3)$$

$$c_3 = \frac{82}{72}\pi^2\zeta(3) - \frac{215}{24}\zeta(5) + \frac{100}{3}\left(\text{Li}_4\left(\frac{1}{2}\right) + \frac{1}{24}\pi^2(\log 2)^2\right) \\ - \frac{239}{2160}\pi^4 + \frac{139}{18}\zeta(3) - \frac{298}{9}\pi^2 \log 2 + \frac{17101}{810}\pi^2 + \frac{28259}{5184}$$

$$c_4 \approx -1,7283(35).$$

Berry on Stokes' method



$$Ai(z) \equiv \frac{1}{2\pi} \int_{-\infty}^{\infty} ds \exp \{ i(s^3/3 + zs) \}$$

$$(z = X_1 + iX_2)$$

Stokes solved this problem [11] by inventing what we now call the WKB method, applied to the differential equation satisfied by Ai , to calculate asymptotic (divergent) series enabling Ai to be computed to high accuracy for large $|X_1|$. He also, almost in passing, invented what we now call the method of stationary phase.

The integral (13) has two stationary points, at $s = \pm (-z)^{1/2}$. When z is real and $X_1 \gg 0$ only one of these contributes to the integral, which is exponentially small. When z is real and $X_1 \ll 0$ both stationary points contribute and Ai oscillates (figure 3). This was the source of the difficulty which occupied Stokes for so long. How could one function have two asymptotic expansions (for $X_1 > 0$ and $X_1 < 0$)? The resolution of course lay in studying Ai for complex z . Somewhere between the positive and negative real axes, a second exponential must be born. This happens near the Stokes lines, which for Ai lie at $\arg(z) = 120^\circ$ and 240° .