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STABLE HOMOLOGY OF LIE ALGEBRAS OF DERIVATIONS AND HOMOTOPY INVARIANTS OF WHEELED OPERADS

VLADIMIR DOTSENKO

To Boris Feigin, with gratitude and admiration

ABSTRACT. We present a new equivalent definition of wheeled operads, and a related way to think of divergence of derivations of free operadic algebras, and use our viewpoint to study stable homology of Lie algebras of derivations of free \mathcal{O} -algebras, for any augmented operad \mathcal{O} . First, we relate stable homology of the positive part of the Lie algebra of all derivations to the homology of wheeled bar construction of \mathcal{O} with its trivial wheeled operad structure. This generalizes both the Loday–Quillen–Tsygan theorem on the homology of the Lie algebra of infinite matrices and the Fuchs stability theorem for the homology of Lie algebra of vector fields with tensor coefficients. Second, we prove a “nicer” version of the latter stability theorem for the positive part of the Lie algebra of all derivations with zero divergence, relating its homology to the homology of the wheeled bar construction of the wheeled completion of the operad \mathcal{O} .

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1. INTRODUCTION

The goal of this paper is to prove an ultimate common generalization of the following two results on homology of infinite-dimensional Lie algebras. The first one is the well known result proved independently by Loday and Quillen [50] and Tsygan [72], concerns the homology of the Lie algebra $\mathfrak{gl}(A) = \varinjlim \mathfrak{gl}_n(A)$ for a unital associative algebra over \mathbb{Q} : it asserts that this homology is isomorphic to the free (super)commutative algebra on the shifted cyclic homology $HC_{\bullet-1}(A)$; there is also a similar result proved independently by Feigin and Tsygan [24] and Hanlon [34] for non-unital algebras. The second result is the Fuchs stability theorem [26] that concerns the homology of the Lie algebra $L_1(n)$ of polynomial vector fields on the n -dimensional affine space vanishing of order two at the origin. It asserts that, for the fixed homological degree k and internal degree d and for all sufficiently large n , the d -homogeneous part of $H_k(L_1(n), \mathbb{C})$ vanishes for $k \neq d$. In modern language, the Lie algebra $L_1(n)$ is stably Koszul.

While in classical terms the Lie algebra $\mathfrak{gl}_n(A)$ is the Lie algebra of endomorphisms of the free A -module of rank n , and the Lie algebra $L_1(n)$ consists of certain derivations of the polynomial algebra in n variables, a reader who is sufficiently used to operads can make the connection between them much more precise. If we view the algebra A as an operad with only unary operations, the free n -generated algebra over that operad is the free A -module of rank n , and endomorphisms of that module are precisely derivations of that free algebra; also, the polynomial algebra is a free algebra for the operad Com of commutative associative algebras, and the Lie algebra $L_1(n)$ consists of certain derivations of that algebra. Thus, the right common context is that of derivations of free algebras with the sufficiently large number of generators. In that direction, it is tempting to suggest that a common generalization of the two theorems mentioned above is given by the celebrated work of Kontsevich [44, 45], who even characterizes the spirit of his argument as “somewhere in between Gelfand–Fuchs computations and cyclic homology”. However, work of Kontsevich studies derivations that preserve some version of symplectic structure; operadically, this suggests that the underlying algebraic structure is controlled by a cyclic operad. Most of the subsequent work in this direction did indeed focus on the cyclic operad situation; as a very non-exhaustive list of examples we refer the reader to Conant and Vogtmann [14], Ginzburg [30], Hamilton and Lazarev [33], and Mahajan [55, 56]. In the final stages of preparation of the paper, we became aware of the beautiful paper of Kapranov [36], where a non-cyclic version of the result of Kontsevich is briefly mentioned; since studying derivations of free operadic algebras was not the main goal of that work, the result is stated without a proof, and is not correct as stated. However, a remarkable contribution of Kapranov’s remark is the definition of an object that one would now call the wheeled completion of an operad, several years before wheeled operads and PROPs were formally introduced [57, 60], and a hint that wheeled operads are related to stable homology of Lie algebras of derivations (for the computation of Fuchs mentioned above, this observation was also made in

[52]). Making this philosophy into a precise mathematical statement is the main goal of this paper.

One of the main results that we prove concerns the Lie algebra $\text{Der}(\mathcal{O}(V))$ of all derivations of the free algebra $\mathcal{O}(V)$. This Lie algebra contains a subalgebra $\mathfrak{gl}(V)$ of derivations sending every generator to a linear combination of generators; moreover, for an augmented operad \mathcal{O} , the augmentation induces a Lie algebra augmentation $\text{Der}(\mathcal{O}(V)) \twoheadrightarrow \mathfrak{gl}(V)$. The kernel of that augmentation is denoted $\text{Der}^+(\mathcal{O}(V))$. If one considers the full homology $C_\bullet(\text{Der}(\mathcal{O}(V)), \mathbb{k})$, the action of its Lie subalgebra $\mathfrak{gl}(V)$ on the homology is trivial, which immediately reduces the calculation of the homology to the Loday–Quillen–Tsygan theorem in the case $A = \mathcal{O}(1)$, as remarked in [56]. By contrast, the Lie algebra $\text{Der}^+(\mathcal{O}(V))$ has many “interesting” homology classes. To keep track of all of them, we shall consider $H_\bullet(\text{Der}^+(\mathcal{O}(V)), \mathbb{k})$ with its natural $\mathfrak{gl}(V)$ -module structure. The structure of that module is fully captured by the “matrix elements”

$$(H_\bullet(\text{Der}^+(\mathcal{O}(V)), \mathbb{k}) \otimes \text{Hom}(V^{\otimes p}, V^{\otimes q}))^{\mathfrak{gl}(V)}$$

for various values of p and q . Thus, the following theorem essentially computes $H_\bullet(\text{Der}^+(\mathcal{O}(V)), \mathbb{k})$, albeit in an implicit way.

Theorem (Th. 4.1). *Let \mathcal{O} be an augmented operad, and V a finite-dimensional vector space. The space of matrix elements*

$$(H_\bullet(\text{Der}^+(\mathcal{O}(V)), \mathbb{k}) \otimes \text{Hom}(V^{\otimes p}, V^{\otimes q}))^{\mathfrak{gl}(V)}$$

stabilize as $\dim(V) \rightarrow \infty$ and, after stabilization, is isomorphic to the (p, q) th component of the coPROP completion of the wheeled cooperad $H_\bullet(\mathbb{B}^\circ(\mathcal{O}))$, where \mathcal{O} is viewed as a trivial wheeled operad (with all wheels equal to zero). This isomorphism is natural with respect to operad morphisms.

This theorem is a common generalization of the Loday–Quillen–Tsygan theorem (as well as its non-unital version of Feigin–Tsygan and Hanlon) and the Fuchs stability theorem mentioned above. We also prove Theorem 4.5, allowing one, in the case where the species derivative $\partial(\mathcal{O})$ is free as a right \mathcal{O} -module, to compute $H_\bullet(\mathbb{B}^\circ(\mathcal{O}))$ explicitly via a calculation of cyclic homology of twisted associative algebras, and use it for several examples of computations.

The appearance of the trivial wheeled operad should catch an unprepared reader by a surprise. In the paper of Kapranov [36] mentioned above, it is claimed that the coPROP in question is obtained from the wheeled completion of the operadic bar construction $\mathbb{B}(\mathcal{O})$. The problem here is that, while the underlying graded vector space is the same, the differential induced on it needs to be handled with care; in particular, Kapranov’s conclusion that, for a Koszul operad \mathcal{O} , the homology of that coPROP is the wheeled completion of the Koszul dual cooperad \mathcal{O}^i is only true for operads sufficiently similar to Com , and is not true for most “interesting” operads (for instance, it is definitely not true for the operad Lie). In fact, if we consider an associative algebra A as an operad concentrated in arity one, the theorem

of Loday–Quillen–Tsygan shows that the commutator quotient $A/[A, A]$ appears in the result, and if one hopes for some statement with a flavour of Koszul duality, one should at least find a way to remove that homotopically badly behaved piece. In the framework of Loday–Quillen–Tsygan, this corresponds to replacing $\mathfrak{gl}(A)$ with $\mathfrak{sl}(A)$, the Lie algebra of matrices whose trace vanishes in the commutator quotient $A/[A, A]$. As we shall see, this gives us a very useful hint on how to proceed.

We have already seen that wheeled operads are relevant in this context, and, in the theory of wheeled operads, the wheeled completion \mathcal{O}° of an operad \mathcal{O} is relevant. What does the bar construction of that wheeled operad tell us about derivations of free \mathcal{O} -algebras? It turns out to be possible to find a meaningful answer to this question involving the notion of divergence of a derivation of the free algebra. For polynomial algebras, derivations are polynomial vector fields, so the notion of divergence one naturally encounters is recognizable from differential geometry, while for $\mathfrak{gl}_n(A)$ it is natural to calculate the divergence as the image of the trace in the commutator quotient. A general notion that is suggested by these two particular cases was recently defined by Powell [64] for free algebras over arbitrary operads; previously, it was known some years in the cases of free Lie algebras (see, for example, Enomoto and Satoh [21] and Satoh [67]) and in the case of free associative algebras (see, for example, Alekseev, Kawazumi, Kuno and Naef [1]). Below, we give an alternative presentation of that definition which we find much easier to digest and to use in our context.

Using the notion of divergence, we define more infinite-dimensional Lie algebras that will be relevant for our second main result. The Lie algebra $\mathrm{SDer}^\wedge(\mathcal{O}(V))$ consists of all derivations of the free algebra $\mathcal{O}(V)$ with constant divergence, and the Lie algebra $\mathrm{SDer}(\mathcal{O}(V))$ consists of all derivations of the free algebra $\mathcal{O}(V)$ with zero divergence. As above, for an augmented operad \mathcal{O} , the augmentation induces a Lie algebra augmentation $\mathrm{SDer}^\wedge(\mathcal{O}(V)) \rightarrow \mathfrak{gl}(V)$. The kernel of that augmentation is denoted $\mathrm{SDer}^+(\mathcal{O}(V))$.

Theorem (Th. 4.10). *Let \mathcal{O} be an augmented operad, and V a finite-dimensional vector space. The space of matrix elements*

$$(H_\bullet(\mathrm{SDer}^+(\mathcal{O}(V)), \mathbb{k}) \otimes \mathrm{Hom}(V^{\otimes p}, V^{\otimes q}))^{\mathfrak{gl}(V)}$$

stabilizes as $\dim(V) \rightarrow \infty$ and, after stabilization, is isomorphic to the (p, q) th component to the coPROP completion of the wheeled cooperad $H_\bullet(\mathcal{B}^\circ(\mathcal{O}^\circ))$. This isomorphism is natural with respect to operad morphisms.

The advantage of this result is in the fact that, unlike the cases of all traces equal to zero, the wheeled completion \mathcal{O}° of a Koszul operad \mathcal{O} does often (though not always) happen to be Koszul as a wheeled operad. Our theorem implies that if the wheeled operad \mathcal{O}° is Koszul, then the Lie algebra $\mathrm{SDer}^+(\mathcal{O}(V))$ is Koszul in all weights less than $\frac{1}{3} \dim(V)$; this is reminiscent of a similar result for Torelli Lie algebras proved independently by Felder, Naef and Willwacher [22] and by

Kupers and Randal-Williams [47]. It follows from the results of Powell [64] that the stable homology of $\mathrm{SDer}^+(\mathcal{O}(V))$ in degree one is concentrated in weight one for any binary operad \mathcal{O} . It seems that the idea of [64] to use functors on the category $\mathcal{S}(R)$ of split monomorphisms between finite rank free R -modules may be useful in our context: for instance, the integer parameter k in the notion of k -torsion in that category corresponds to the precise number of extra variables one needs to adjoin for various isomorphisms to hold. This will be addressed elsewhere.

Let us remark that our results can be converted into homological stability theorems for the homology of the Lie algebra $\mathrm{Der}(\mathcal{O}(V))$ and $\mathrm{SDer}^\wedge(\mathcal{O}(V))$ with coefficients in $\mathrm{Hom}(V^{\otimes p}, V^{\otimes q})$, where both algebras act via the augmentation map to $\mathfrak{gl}(V)$. This parallels calculations of

$$H^\bullet(\mathrm{Aut}(F_n), \mathrm{Hom}_{\mathbb{Q}}(H(n)^{\otimes p}, H(n)^{\otimes q})),$$

where $H(n) = H_1(F_n, \mathbb{Q})$ is the standard representation of $\mathrm{Aut}(F_n)$; here homological stability is a consequence of general results of Randal-Williams and Wahl [66] and Randal-Williams [65]. A formula for that stable cohomology was conjectured by Djament [18] and recently proved by Lindell [48]. In fact, in work of Kawazumi and Vespa [38] the conjecture of Djament was already reformulated in the language of wheeled operads: it asserts that if we consider the stable cohomology for all possible values of p, q , the cohomology groups assemble into a wheeled PROP completion of the operadic suspension of the operad of commutative associative algebras. It would be interesting to determine a precise relationship between that result and our work; note that according to a theorem proved by Granåker [31], the wheeled operad completion of the operadic suspension of the operad of commutative associative algebras is the homology of the wheeled bar construction of the wheeled operad of *unimodular* Lie algebras. (Note that the fact that we work over a field ensures that the notion of a derivation is available at all, contrary, for example, to the cartesian context of Lawvere theories, where some recent significant breakthroughs have been made in studying stable (co)homology of automorphism groups as well as in computing the K -groups, see e.g. the work of Szymik and Wahl [71] and Bohmann and Szymik [12] for some compelling examples.)

Let us mention that the notion of divergence of a derivation of a free algebra has been implicit in the literature on automorphisms of free algebras for several decades. This goes back to the remarkable work of Bryant and Drensky [13], see also [6, 62]. The central notion motivating that work is that of a tame automorphism. Recall that an automorphism of a free algebra $\mathcal{O}(x_1, \dots, x_n)$ is called tame if it is obtained by compositions of automorphisms from $GL_n(\mathbb{k})$ and obvious “triangular” automorphisms that send one of the generators x_i to an element of the subalgebra generated by $x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n$ and fixes all those other generators. Already in the case of free commutative associative algebras and free associative algebras non-tame automorphisms exist [70, 73], which is one of the factors

which make studying automorphisms of free algebras hard. However, for commutative associative algebras, it was proved independently by Shafarevich [69] and Anick [2] that non-tame automorphisms can be topologically approximated by tame ones; in fact, Shafarevich deduces that from the fact that the Lie algebra of the group of all automorphisms consists of all derivations of constant divergence. For some operads, it is possible to show that all tangent derivations of automorphisms must have constant divergence, which identifies the class of situations where an analogue of Anick–Shafarevich theorem may hold.

A somewhat more conceptual relationship of our work to understanding the structure of automorphism groups of free algebras is as follows. Homology of the Lie algebra $\mathfrak{gl}(A)$ should be thought of as the “additive K -theory” of the algebra A , an approximation to the actual K -theory of A , which, over \mathbb{Q} , can be identified with the primitive elements for the canonical Hopf algebra structure on the homology of the group $GL(A) = \varinjlim GL_n(A)$. Similarly, it is not unreasonable to expect that our results contain information about the additive K -theory of the operad \mathcal{O} , which approximates the actual K -theory of \mathcal{O} , the primitive elements for the canonical Hopf algebra structure on the homology of the direct limit of $\text{Aut}(\mathcal{O}(V))$ as $\dim(V)$ goes to infinity. Though these K -groups can be described in the general context of algebraic K -theory [7], automorphism groups of free algebras are considerably more complicated than general linear groups, so this kind of K -theory does not seem to have been studied much. As one notable exception, one should mention the work of Connell concerning the K -theory of projective commutative associative algebras [15, 16, 17] which is as exciting as it is demoralizing, for one quickly realises that studying this subject instantly brings one to notoriously difficult open questions, such as the Zariski Cancellation Problem [68] and the Jacobian Conjecture [75]. If one considers only the group K_0 , there are more results available for various operads \mathcal{O} , for instance, in the work of Artamonov [5] on metabelian Lie algebras, as well as Artamonov [4] and Pirashvili [63] on nilpotent algebras in general. One possible way to build a connection between our work and the operadic K -theory computed via the automorphism groups would go along the lines of the beautiful theorem of Henion [35] asserting that the cyclic homology of an algebra is the tangent version of its K -theory in a very precise sense. It is also very natural to attempt a generalization of remarkable recent results on derived representation schemes and cyclic homology [8, 9, 10], where one can speculate that such a generalization would replace the derived representation scheme of representations of an algebra A on a vector space W by the derived moduli stack of \mathcal{O} -algebra structures on W .

2. RECOLLECTIONS

In this section, we recall some standard background information on twisted associative algebras, operads, and operadic algebras, as well as universal multiplicative envelopes and Kähler differentials of operadic algebras; the reader is

invited to consult the monograph of Loday and Vallette [51] for more details on operads and operadic algebras, and the monograph of Fresse [25] for more details on modules over operads, and a functorial view of universal multiplicative envelopes and Kähler differentials.

2.1. Conventions. All vector spaces in this paper are defined over a field \mathbb{k} of characteristic zero. For a finite set I and a family of vector spaces $\{V_i\}_{i \in I}$, the unordered tensor product of these vector spaces is defined by

$$\bigotimes_{i \in I} V_i := \left(\bigoplus_{(i_1, \dots, i_n) \text{ a total order on } I} V_{i_1} \otimes \cdots \otimes V_{i_n} \right)_{\text{Aut}(I)}.$$

All chain complexes are graded homologically, with the differential of degree -1 . To handle suspensions of chain complexes, we introduce a formal symbol s of degree 1, and define, for a graded vector space V , its suspension sV as $\mathbb{k}s \otimes V$.

2.2. Twisted associative algebras and operads. A *linear species* is a contravariant functor from the groupoid Fin of finite sets (the category whose objects are finite sets and whose morphisms are bijections) to the category of vector spaces. Sometimes, a “skeletal definition” is preferable: the category of linear species is equivalent to the category of symmetric sequences $\{\mathcal{S}(n)\}_{n \geq 0}$, where each $\mathcal{S}(n)$ is a right S_n -module, a morphism between the sequences \mathcal{S}_1 and \mathcal{S}_2 in this category is a sequence of S_n -equivariant maps $f_n: \mathcal{S}_1(n) \rightarrow \mathcal{S}_2(n)$. While this definition may seem more appealing, the functorial definition simplifies many definitions and proofs in a way that totally justifies the level of abstraction.

The *derivative* $\partial(\mathcal{S})$ of a species \mathcal{S} is defined by the formula

$$\partial(\mathcal{S})(I) := \mathcal{S}(I \sqcup \{\star\}).$$

The *Cauchy product* of two linear species \mathcal{S}_1 and \mathcal{S}_2 is defined by the formula

$$(\mathcal{S}_1 \otimes \mathcal{S}_2)(I) := \bigoplus_{I=I_1 \sqcup I_2} \mathcal{S}_1(I_1) \otimes \mathcal{S}_2(I_2).$$

Equipped with this product, the category of linear species becomes a symmetric monoidal category, with the monoidal unit being the species $\mathbf{1}$ supported at the empty set with the value $\mathbf{1}(\emptyset) = \mathbb{k}$. A *twisted associative algebra* is a monoid in that monoidal category.

The *composition product* of two linear species \mathcal{S}_1 and \mathcal{S}_2 is compactly expressed via the Cauchy product as

$$\mathcal{S}_1 \circ \mathcal{S}_2 := \bigoplus_{n \geq 0} \mathcal{S}_1(\{1, \dots, n\}) \otimes_{\mathbb{k}S_n} \mathcal{S}_2^{\otimes n},$$

that is, if one unwraps the definitions,

$$(\mathcal{S}_1 \circ \mathcal{S}_2)(I) = \bigoplus_{n \geq 0} \mathcal{S}_1(\{1, \dots, n\}) \otimes_{\mathbb{k}S_n} \left(\bigoplus_{I=I_1 \sqcup \cdots \sqcup I_n} \mathcal{S}_2(I_1) \otimes \cdots \otimes \mathcal{S}_2(I_n) \right).$$

Equipped with this product, the category of linear species becomes a (very non-symmetric) monoidal category, with the monoidal unit being the species $\mathbb{1}$ which vanishes on a finite set I unless $|I| = 1$, and whose value on $I = \{a\}$ is given by the one-dimensional vector space $\mathbb{k}a$. A (*symmetric*) *operad* is a monoid in that monoidal category, that is a triple $(\mathcal{O}, \gamma, \eta)$, where $\gamma: \mathcal{O} \circ \mathcal{O} \rightarrow \mathcal{O}$ is the product and $\eta: \mathbb{1} \rightarrow \mathcal{O}$ is the unit, which satisfy the usual axioms of a monoid in a monoidal category [54]. This defines algebraic (that is, \mathbb{k} -linear) operads, an object first introduced in the paper of Artamonov [3] under the slightly obscure name “clones of multilinear operations”, shortly before the appearance of the term “operad” in its topological glory [39, 59]. The prototypical example of an operad is given by the *endomorphism operad* End_V of a vector space V . It is given by $\text{End}_V(I) = \text{Hom}_{\mathbb{k}}(V^{\otimes I}, V)$, with the composition given by the usual composition of multilinear maps. In general, an operad corresponds to a \mathbb{k} -linear algebraic structure: for a class \mathcal{C} of \mathbb{k} -linear algebras, one can define an operad $\mathcal{O}_{\mathcal{C}}$ where $\mathcal{O}_{\mathcal{C}}(I)$ consists of all multilinear operations with inputs indexed by I that one can define in terms of the structure operations of \mathcal{C} .

Unless otherwise specified, all operads \mathcal{O} we consider will be *reduced* (that is, $\mathcal{O}(\emptyset) = 0$) and *augmented* (that is, equipped with a map of operads $\epsilon: \mathcal{O} \rightarrow \mathbb{1}$ which is the left inverse to the monoid unit η); we shall denote by $\overline{\mathcal{O}}$ the augmentation ideal of \mathcal{O} (the kernel of the augmentation). The reader may decide to only think of operads generated by finitely many binary operations, though our results hold in more general situations; in particular, allowing oneself unary operations is useful to recover the Loday–Quillen–Tsygan theorem mentioned in the introduction.

Thinking of twisted associative algebras and of operads as of monoids automatically leads to definitions of (left, right, and two-sided) modules using the corresponding monoidal structure. One can also use operadic module to re-define twisted associative algebras: a twisted associative algebra is the same as a left module over the operad of associative algebras. Moreover, one can check that the Cauchy product makes the category of right modules over an operad \mathcal{O} into a symmetric monoidal category.

It is also possible to encode operads using the *partial* (or *infinitesimal*) compositions

$$\circ_*: \partial(\mathcal{O})(I) \otimes \mathcal{O}(J) \rightarrow \mathcal{O}(I \sqcup J)$$

satisfying the “consecutive” and the “parallel” axioms [51]. Functorially, the consecutive axiom is given by the commutative diagram

$$\begin{array}{ccc} \partial(\mathcal{O})(I) \otimes \partial(\mathcal{O})(J) \otimes \mathcal{O}(K) & \xrightarrow{\text{id} \otimes \circ_*} & \partial(\mathcal{O})(I) \otimes \mathcal{O}(J \sqcup K) \\ \circ_* \otimes \text{id} \downarrow & & \downarrow \circ_* \\ \partial(\mathcal{O})(I \sqcup J) \otimes \mathcal{O}(K) & \xrightarrow{\circ_*} & \mathcal{O}(I \sqcup J \sqcup K) \end{array}$$

and the parallel axiom is given by the commutative diagram

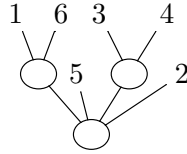
$$\begin{array}{ccc}
 \partial^2(\mathcal{O})(I) \otimes \mathcal{O}(J) \otimes \mathcal{O}(K) & \xrightarrow{(\circ_{\star_2} \otimes \text{id})(\text{id} \otimes \sigma)} & \partial(\mathcal{O})(I \sqcup K) \otimes \mathcal{O}(J) \\
 \circ_{\star_1} \otimes \text{id} \downarrow & & \downarrow \circ_{\star_1} \\
 \partial(\mathcal{O})(I \sqcup J) \otimes \mathcal{O}(K) & \xrightarrow{\circ_{\star_2}} & \mathcal{O}(I \sqcup J \sqcup K)
 \end{array}$$

where σ is the symmetry isomorphism $\mathcal{O}(J) \otimes \mathcal{O}(K) \cong \mathcal{O}(K) \otimes \mathcal{O}(J)$. Examining these diagrams immediately implies that if \mathcal{O} is an operad, the derivative species $\partial(\mathcal{O})$ has two structures: it is a right \mathcal{O} -module (via substitutions into inputs other than the special input \star) and a twisted associative algebra (via substitutions \circ_{\star}). These two structures commute: in fact, $\partial(\mathcal{O})$ is a twisted associative algebra in the symmetric monoidal category of right \mathcal{O} -modules.

It is also important that we can view operads as algebras over a monad. Let us recall that viewpoint which will be useful for us when we generalize to wheeled operads later. We start by defining an endofunctor \mathcal{T} of the category of species by the following formula :

$$\mathcal{T}(\mathcal{F})(I) := \bigoplus_{\substack{\tau \text{ a rooted tree, } \\ \text{Leaves}(\tau)=I}} \bigotimes_{v \in \text{Vertices}(\tau)} \mathcal{F}(\text{in}(v)),$$

so in plain words, we decorate each internal vertex v of τ with a label from \mathcal{F} , taking the component of \mathcal{F} indexed by the set of incoming edges of v . For example,



is a typical rooted tree with the set of leaves $\{1, 2, \dots, 6\}$, and the summand corresponding to that tree is $\mathcal{F}(\{1, 6\}) \otimes \mathcal{F}(\{3, 4\}) \otimes \mathcal{F}(\{2, 5, \star_1, \star_2\})$, where \star_1 and \star_2 correspond to the points of grafting of the two subtrees. It is easy to see that \mathcal{T} is in fact a monad: the natural transformation $\mathcal{T}\mathcal{T} \rightarrow \mathcal{T}$ comes from the fact that a rooted tree with tiny rooted trees inserted in its internal vertices can be viewed as a rooted tree if we forget this nested structure of insertions, and the natural transformation $1 \rightarrow \mathcal{T}$ comes from considering rooted trees with one internal vertex (“corollas”). We call this monad the *monad of rooted trees*. An operad is an algebra over \mathcal{T} , that is a species \mathcal{O} equipped with a structure map $\mathcal{T}(\mathcal{O}) \rightarrow \mathcal{O}$. If we keep the intuition of multilinear operations, the three viewpoints we discussed are as follows: the monoid viewpoint corresponds to composing operations along two-level rooted trees, the partial composition viewpoint corresponds to composing operations along rooted trees with two internal vertices, and the monad viewpoint corresponds to composing operations along arbitrary rooted trees.

Given an operad \mathcal{O} , its bar construction $B(\mathcal{O})$ is the chain complex structure on the rooted trees endofunctor $\mathcal{T}(s\overline{\mathcal{O}})$ with the differential obtained as a sum of

all ways of collapsing edges of the tree and using the operad structure maps on the labels of the vertices that are identified by such collapses. Moreover, one can talk about cooperads by reversing all arrows in the respective definitions; in that language, the bar construction of an operad \mathcal{O} is the cofree conilpotent cooperad on $s\overline{\mathcal{O}}$ equipped with the unique coderivation extending the maps induced by the structure maps of the operad \mathcal{O} .

2.3. Free algebras, universal enveloping algebras, and Kähler differentials.

To a vector space V , one can associate the species that we shall define by the same letter: $V(I)$ vanishes unless $I = \emptyset$, and $V(\emptyset) = V$. If \mathcal{S} is another species, we have

$$\mathcal{S} \circ V := \bigoplus_{n \geq 0} \mathcal{S}(\{1, \dots, n\}) \otimes_{\mathbb{k}\mathcal{S}_n} V^{\otimes n}.$$

Classically, the right hand side of this formula is called the Schur functor associated to \mathcal{S} and is denoted $\mathcal{S}(V)$.

The main reason to study operads is because of their algebras. For an operad \mathcal{O} , an algebra over \mathcal{O} is a vector space A together with a structure of a left \mathcal{O} -module on the species A . This way we obtain a class of \mathbb{k} -linear algebras \mathcal{C} , and our operad can be reconstructed as $\mathcal{O}_{\mathcal{C}}$. One can also check that a structure of an \mathcal{O} -algebra on A is the same as a morphism of operads $\mathcal{O} \rightarrow \text{End}_A$; in this sense, \mathcal{O} -algebras are representations of \mathcal{O} . In this paper, we shall only consider free operadic algebras; the free \mathcal{O} -algebra over \mathcal{O} generated by a vector space V is the vector space $\mathcal{O}(V)$, with the left module on the species $\mathcal{O}(V)$ coming from the fact that it is exactly the free left module generated by the species V .

Let A be a \mathcal{O} -algebra. Informally, the *universal multiplicative enveloping algebra* $U_{\mathcal{O}}(A)$ encodes all “actions of elements of A on a general (bi)module”. Formally, we define it via a relative composite product construction [25]:

$$U_{\mathcal{O}}(A) \cong \partial(\mathcal{O}) \circ_{\mathcal{O}} A.$$

Here we used the right \mathcal{O} -module structure, so the commuting structure of an associative algebra survives, and thus $U_{\mathcal{O}}(A)$ is indeed an associative algebra. For the universal multiplicative enveloping algebra of a free \mathcal{O} -algebra, we have

$$U_{\mathcal{O}}(\mathcal{O}(V)) \cong \partial(\mathcal{O}) \circ_{\mathcal{O}} \mathcal{O}(V) \cong \partial(\mathcal{O})(V).$$

Example 2.1. Let Ass denote the operad of associative algebras, and uAss denote the operad of unital associative algebras (a rare example of a non-reduced operad that emerges). The species $\partial(\text{Ass})$ is naturally identified with the Cauchy product $\text{uAss} \otimes \text{uAss}$. Indeed, if we have an element $a_{i_1} \cdots a_{i_n} \in \text{Ass}(I \sqcup \{\star\})$, then the position of a_{\star} in this product splits it into two possibly empty parts which is precisely the Cauchy product decomposition. The first factor corresponds to left multiplication, and the second factor to the right ones. In particular, if we consider the free non-unital associative algebra $\overline{T}(V)$, its universal multiplicative envelope is isomorphic to $T(V) \otimes T(V)$.

Generalizing the classical definition for commutative algebras, one may define the $U_{\mathcal{O}}(A)$ -module of Kähler differentials Ω_A^1 for any given operad \mathcal{O} and any

given \mathcal{O} -algebra A [25, Sec. 4.4]. The intrinsic definition states that $\Omega_{\mathcal{O}}^1(A)$ is the $U_{\mathcal{O}}(A)$ -module that represents the functor of derivations $\text{Der}(A, E)$ with values in a $U_{\mathcal{O}}(A)$ -module E . Explicitly this module is spanned by formal expressions $p(a_1, \dots, da_i, \dots, a_m)$, where $p \in \mathcal{O}(m)$, and $a_1, \dots, a_m \in A$, modulo the relations

$$\begin{aligned} p(a_1, \dots, q(a_i, \dots, a_{i+n-1}), \dots, da_j, \dots, a_{m+n-1}) = \\ (p \circ_i q)(a_1, \dots, a_i, \dots, a_{i+n-1}, \dots, da_j, \dots, a_{m+n-1}), \end{aligned}$$

$$\begin{aligned} p(a_1, \dots, dq(a_i, \dots, a_{i+n-1}), \dots, a_{m+n-1}) = \\ \sum_{j=i}^{i+n-1} (p \circ_i q)(a_1, \dots, a_i, \dots, da_j, \dots, a_{i+n-1}, \dots, a_{m+n-1}). \end{aligned}$$

We have the “universal derivation”

$$d: A \rightarrow \Omega_{\mathcal{O}}^1(A)$$

sending $a \in A$ to $\text{id}(da)$.

It is easy to see that for a free algebra $\mathcal{O}(V)$, we have a left $U_{\mathcal{O}}(\mathcal{O}(V))$ -module isomorphism

$$\Omega_{\mathcal{O}}^1(\mathcal{O}(V)) \cong U_{\mathcal{O}}(\mathcal{O}(V)) \otimes V \cong \partial(\mathcal{O})(V) \otimes V.$$

Since every derivation of a free algebra is defined on generators, the universal derivation of the free algebra can be regarded as a map

$$V \rightarrow \partial(\mathcal{O})(V) \otimes V.$$

Example 2.2. We continue considering the operad Ass of associative algebras. The module $\Omega_{\text{Ass}}^1(\overline{T}(V))$ is isomorphic to $T(V) \otimes T(V) \otimes V$. Let us give an illustration of how that isomorphism is implemented. Let $V = \text{Vect}(x, y)$, and consider $xyx \in \overline{T}(V)$. We have

$$\begin{aligned} d(xyx) &= d(x)yx + xd(y)x + xyd(x) = \\ & r_{yx}d(x) + r_x l_x d(y) + l_{xy}d(x) = (l_{xy} + r_{yx})d(x) + r_x l_x d(y). \end{aligned}$$

Here l_a denotes $a \otimes 1 \in T(V) \otimes T(V)$, and r_a denotes $1 \otimes a \in T(V) \otimes T(V)$.

2.4. PROPs. We shall also use PROPs, which are generalizations of operads with several inputs and several outputs, so that one should think of bi-species: functors from $\text{Fin}^{op} \times \text{Fin}$ to the category of vector spaces. However, the definition can be given more quickly: a PROP is a \mathbb{k} -linear symmetric monoidal category \mathcal{P} whose objects are natural numbers, and the monoidal structure is addition. Then the symmetry isomorphisms act on $n = 1 + 1 + \dots + 1$, and the hom-spaces $\mathcal{P}(m, n)$ naturally have a left action of S_n and a commuting right action of S_m , which is essentially the same as a bi-species. Moreover, there are “vertical” compositions $\mathcal{P}(m, n) \otimes \mathcal{P}(k, m) \rightarrow \mathcal{P}(k, n)$ that are just compositions in \mathcal{P} and “horizontal” compositions $\mathcal{P}(m, n) \otimes \mathcal{P}(m', n') \rightarrow \mathcal{P}(m + m', n + n')$ corresponding to the

juxtaposition of morphisms. These are associative and satisfy the “interchange law” between them.

To each operad \mathcal{O} , one can associate a PROP $\mathcal{P}(\mathcal{O})$ whose components are given by the formula

$$(\mathcal{P}(\mathcal{O}))(m, n) := (S^n(\mathcal{O}))(m),$$

where $S^n(\mathcal{O})$ is the n -th symmetric power of the species \mathcal{O} with respect to the Cauchy product, the vertical composition comes from the composition in \mathcal{O} , and the horizontal composition is the product in the symmetric algebra $S(\mathcal{O})$. This PROP is often referred to as the PROP completion of the operad \mathcal{O} .

2.5. Cyclic homology. Let us briefly recall how the cyclic homology of an associative algebra is defined; we refer the reader to the monographs of Loday [49] and Feigin and Tsygan [24] for details.

Given an associative algebra A , one can consider a chain complex structure on the (non-counital) cofree conilpotent coalgebra $\overline{T}^c(sA)$ where the differential d is the unique coderivation extending the map $\overline{T}^c(sA) \rightarrow sA \otimes sA \rightarrow sA$ obtained as the composition of the obvious projection with the product in A . That chain complex is often referred to as the bar construction of A ; we however choose to keep the notation of [51] and define the bar construction for augmented algebras, applying the above recipe to the augmentation ideal. The coalgebra $\overline{T}^c(sA)$ also happens to be an algebra with respect to the concatenation product, and we may consider its commutator quotient

$$\overline{T}^c(sA)/[\overline{T}^c(sA), \overline{T}^c(sA)],$$

where the commutator takes into account the signs arising from suspensions; it is nothing but the space of cyclic words $\text{Cyc}(sA)$.

The differential d induces a differential on $\text{Cyc}(sA)$, and one of the equivalent definitions the cyclic homology is defined via the degree one shift of that complex:

$$HC_\bullet(A) := H_\bullet(s^{-1} \text{Cyc}(sA)).$$

In particular, $HC_0(A) = A/[A, A]$, and another definition of cyclic homology interprets it as higher nonabelian derived functors of the commutator quotient $A/[A, A]$: concretely, if $R_\bullet = (R, d_R) \simeq A$ is a quasi-free resolution of A , that is a differential graded associative algebra whose underlying associative algebra R is free and whose homology is isomorphic to A , then

$$HC_\bullet(A) \cong H_\bullet(R_\bullet/[R_\bullet, R_\bullet]).$$

One of our results, Theorem 4.5 below, uses cyclic homology of twisted associative algebras; over a field of characteristic zero, all the results about cyclic homology of associative algebras extent *mutatis mutandis* to twisted associative algebras, if one replaces the tensor product of vector spaces by the Cauchy product. Alternatively, for a twisted associative algebra \mathcal{A} and any vector space V , the vector space $\mathcal{A}(V)$ has a canonical structure of an associative algebra induced by the

twisted associative algebra structure of \mathcal{A} , and one can view cyclic homology calculations for \mathcal{A} as cyclic homology calculations for all $\mathcal{A}(V)$ at the same time, keeping the answer functorial in V .

3. DERIVATIONS, DIVERGENCE, WHEELED OPERADS

In this section, we discuss derivations of free operadic algebras, their divergence, and wheeled operads. While formally no material of this section is new (an equivalent definition of divergence is due to Powell [64], and an equivalent definition of a wheeled operad is given in the foundational paper of Markl, Merkulov, and Shadrin [57]), our viewpoints on divergence and on wheeled operads are new, and we believe that these viewpoints fill a notable gap in the existing literature, clarifying these objects in a significant way.

3.1. Derivations of free algebras. For every operadic algebra, one has a well defined classical notion of a derivation. The set of derivations of A , denoted $\text{Der}(A)$, has an obvious structure of a vector space, and moreover of a Lie algebra with respect to the Lie bracket $[D, D'] = D \circ D' - D' \circ D$.

A derivation of a free algebra is completely determined by images of generators, so we have a canonical vector space isomorphism

$$\text{Der}(\mathcal{O}(V)) \cong \text{Hom}(V, \mathcal{O}(V)).$$

and we shall denote by ι_D the element of $\text{Hom}(V, \mathcal{O}(V))$ corresponding to D . This isomorphism allows us to see that for free algebras, the Lie bracket of derivations arises from a pre-Lie algebra structure on $\text{Der}(\mathcal{O}(V))$: we can define, for two derivations D, D' , the pre-Lie product $D \triangleleft D'$ to be given by

$$D \triangleleft D'(v) = D(\iota_{D'}(v)),$$

The Lie algebra $\text{Der}(\mathcal{O}(V))$ has an obvious non-negative grading: a derivation has weight w if it sends each generator $v \in V$ into a combination of elements obtained by applying elements of $\mathcal{O}(w+1)$ to elements of V . The Lie algebra $\text{Der}(\mathcal{O}(V))$ has a Lie subalgebra $\mathfrak{gl}(V)$ consisting of derivations for which the image of every element of V is in V . For an augmented operad, one can use the augmentation to define a surjective Lie algebra morphism $\text{Der}(\mathcal{O}(V)) \rightarrow \mathfrak{gl}(V)$; we shall denote its kernel $\text{Der}^+(\mathcal{O}(V))$. If our operad is connected (all unary operations are proportional to the identity), $\text{Der}^+(\mathcal{O}(V))$ consists of derivations of positive weight, but in general it also contains some elements of weight zero.

3.2. Divergence of derivations. We shall now introduce the notion of divergence of a derivation of $\mathcal{O}(V)$. This is equivalent to the definition recently given by Powell [64]. Though in [64, Appendix B] a viewpoint using universal multiplicative enveloping algebras is indicated, it is not the approach that is pursued in that paper; we believe that our viewpoint has the advantage of hinting at the role of wheeled operads in this context.

Let D be a derivation of $\mathcal{O}(V)$. The *divergence* of the derivation D , denoted by $\text{div}(D)$, is defined as follows. We compose the map $\iota_D \in \text{Hom}(V, \mathcal{O}(V))$ with

the universal derivation, obtaining a map

$$d \circ \iota_D: V \rightarrow \partial(\mathcal{O})(V) \otimes V,$$

which we identify, using the hom-tensor adjunction and symmetry isomorphisms of tensor products, with an element $j(d \circ \iota_D)$ of $V^* \otimes V \otimes \partial(\mathcal{O})(V)$. This latter is an associative algebra $A = \text{Hom}(V, V) \otimes \partial(\mathcal{O})(V)$, and we can take the universal trace on it, which is the canonical projection to commutator quotient $A/[A, A]$. It is well known that the commutator quotient of a tensor product of two associative algebras is the tensor product of commutator quotients, so the universal trace on $\text{Hom}(V, V) \otimes \partial(\mathcal{O})(V)$ takes values in

$$\begin{aligned} & \text{Hom}(V, V)/[\text{Hom}(V, V), \text{Hom}(V, V)] \otimes \partial(\mathcal{O})(V)/[\partial(\mathcal{O})(V), \partial(\mathcal{O})(V)] \cong \\ & \mathbb{k} \otimes \partial(\mathcal{O})(V)/[\partial(\mathcal{O})(V), \partial(\mathcal{O})(V)] \cong \partial(\mathcal{O})(V)/[\partial(\mathcal{O})(V), \partial(\mathcal{O})(V)]. \end{aligned}$$

The *divergence* of a derivation $D \in \text{Der}(\mathcal{O}(V))$, denoted $\text{div}(D)$, is the element of the commutator quotient of $\partial(\mathcal{O})(V)$ given by the universal trace of the element $j(d \circ \iota_D) \in \text{Hom}(V, V) \otimes \partial(\mathcal{O})(V)$.

Let us consider a simple yet useful example.

Example 3.1. Let A be an augmented associative algebra, and consider it as an operad \mathcal{O} that only has operations in arity one given by A . The free \mathcal{O} -algebra $\mathcal{O}(V)$ generated by a vector space V of dimension n is just the free left module $A \otimes V$, and a derivation of $\mathcal{O}(V)$ is just an A -module endomorphism. Thus, the Lie algebra $\text{Der}(\mathcal{O}(V))$ is the Lie algebra $\mathfrak{gl}_n(A^{\text{op}}) \cong \mathfrak{gl}_n(A)$. Moreover, the universal multiplicative enveloping algebra of $\mathcal{O}(V)$ is the algebra A itself, and the divergence is given by the composition $\mathfrak{gl}_n(A) \rightarrow A \rightarrow A/[A, A]$, where the first map computes the trace of the matrix, and the second map is the canonical projection onto the commutator quotient.

Let us give another example that is a bit more concrete.

Example 3.2. We continue considering the operad Ass of associative algebras. Let $V = \text{Vect}(x, y)$, and consider the derivation $D \in \text{Der}(\text{Ass}(V))$ that sends the generator x to $[x, y]^2$ and the generator y to zero. (It is the tangent derivation of the Bergman automorphism of the free algebra generated by two generic 2×2 -matrices [11].) We have $[x, y]^2 = xyxy - xyyx - yxyx + yxyx$, so

$$\begin{aligned} d([x, y]^2) &= d(x)yxy + xd(y)xy + xyd(x)y + xyxd(y) - \\ & \quad d(x)yyx - xd(y)yx - xyd(y)x - xyd(x) - \\ & \quad d(y)xyx - yd(x)xy - yxd(x)y - yxxd(y) + \\ & \quad d(y)xyx + yd(x)yx + yxd(y)x + yxyd(x), \end{aligned}$$

and computing the trace in this case means looking at the coefficient of $d(x)$ only, getting

$$r_{yxy} + l_{xy}r_y - r_{yyx} - l_{xyy} - l_yr_{xy} - l_{yx}r_y + l_yr_{yx} + l_{yxy}$$

which simplifies as

$$r_{[y,x,y]} + l_{[x,y]}r_y - l_{[x,y]} - l_yr_{[x,y]} = [r_{yx}, r_y] + [l_xr_y, l_y] - [l_{xy}, l_y] - [r_xl_y, r_y]$$

and so vanishes in the commutator quotient of the universal enveloping algebra. Thus, $\operatorname{div}(D) = 0$.

A fundamental property of divergence is the following cocycle equation.

Proposition 3.3. *Divergence is a 1-cocycle of $\operatorname{Der}(\mathcal{O}(V))$ with values in the commutator quotient of $\partial(\mathcal{O})(V)$. Concretely, for two derivations D, D' of $\mathcal{O}(V)$, we have*

$$\operatorname{div}([D, D']) = D_*(\operatorname{div}(D')) - D'_*(\operatorname{div}(D)).$$

Here D_* is the endomorphism of $\partial(\mathcal{O})(V)$ arising from the unique derivation of $\partial(\mathcal{O})(V) \cong U_{\mathcal{O}}(\mathcal{O}(V))$ induced by D .

Proof. Our definitions immediately imply

$$j(d \circ \iota_{D \triangleleft D'}) = D_*(j(d \circ \iota_{D'})) + j(d \circ \iota_{D'})j(d \circ \iota_D),$$

implying that

$$\begin{aligned} j(d \circ \iota_{[D, D']}) &= j(d \circ \iota_{D \triangleleft D' - D' \triangleleft D}) = \\ &= D_*(j(d \circ \iota_{D'})) - D'_*(j(d \circ \iota_D)) + [j(d \circ \iota_{D'}), j(d \circ \iota_D)], \end{aligned}$$

so in the commutator quotient we have

$$\operatorname{div}([D, D']) = D_*(\operatorname{div}(D')) - D'_*(\operatorname{div}(D)),$$

as required. \square

As an immediate consequence, all derivations with constant divergence (that is, proportional to the coset of the unit element in the commutator quotient) form a Lie subalgebra of the Lie algebra $\operatorname{Der}(\mathcal{O}(V))$, and all derivations with zero divergence form an ideal of codimension one of that latter subalgebra. We shall denote the former subalgebra $\operatorname{SDer}^{\wedge}(\mathcal{O}(V))$, and the latter one $\operatorname{SDer}(\mathcal{O}(V))$. For an augmented operad, the augmentation of \mathcal{O} induces surjective Lie algebra morphisms $\operatorname{SDer}^{\wedge}(\mathcal{O}(V)) \rightarrow \mathfrak{gl}(V)$ and $\operatorname{SDer}(\mathcal{O}(V)) \rightarrow \mathfrak{sl}(V)$ with the same kernel which we shall denote $\operatorname{SDer}^+(\mathcal{O}(V))$.

3.3. Wheeled operads. The notion of a wheeled operad introduced in [57] is designed to encode \mathcal{P} -algebras for which, together with structure operations, we also keep track of “universal traces” of elements of the universal multiplicative envelope. In [57], the notion of a wheeled operad is obtained as a specialization of the notion of a wheeled properad, which results in some definitions being less straightforward than they could be. In this section, we present these definitions in the way which will be more useful for our purposes. (We also choose a different convention for gradings in the bar construction to make everything agree with the conventions of [51] in the operadic case.)

We shall now consider two-coloured linear species, that of contravariant functors \mathcal{F} from Fin to the category of vector spaces decomposed into a direct sum of two subspaces. It will be convenient for us to think of the value $\mathcal{F}(I)$ as $\mathcal{F}_1(I) \oplus \mathcal{F}_0(I)$, the subspaces of “operations” with one output and no outputs, respectively. A *wheeled operad* is a two-coloured linear species $\mathcal{O} \oplus \mathcal{W}$, where:

- \mathcal{O} is an operad,
- \mathcal{W} is a right \mathcal{O} -module,
- $\partial(\mathcal{O})$ is given a *trace map*

$$\mathrm{tr}: \partial(\mathcal{O}) \rightarrow \mathcal{W},$$

which is a morphism of right \mathcal{O} -modules and vanishes on the commutators in the twisted associative algebra $\partial(\mathcal{O})$.

A morphism of wheeled operads is a map of two-coloured linear species that agrees with all the three structures.

Just as the prototypical example of an operad is given by the endomorphism operad, one may consider the *wheeled endomorphism operad* End_V° of a finite-dimensional vector space V . It is given by

$$\mathrm{Hom}_{\mathbb{k}}(V^{\otimes I}, V) \oplus \mathrm{Hom}_{\mathbb{k}}((V^{\otimes I})^{C_I}, \mathbb{k}),$$

where C_I is the cyclic group acting on the tensor product, where the operad and the right module structures are obvious, and where the map

$$\mathrm{tr}: \mathrm{Hom}(V^{\otimes(I \sqcup \{\star\})}, V) \rightarrow \mathrm{Hom}((V^{\otimes I})^{C_I}, \mathbb{k})$$

is obtained as the composite

$$\mathrm{Hom}(V^{\otimes(I \sqcup \{\star\})}, V) \rightarrow \mathrm{Hom}(V^{\otimes I}, \mathbb{k}) \rightarrow \mathrm{Hom}((V^{\otimes I})^{C_I}, \mathbb{k})$$

of the contraction of a multilinear map with respect to the position \star and the restriction to invariants of the cyclic group.

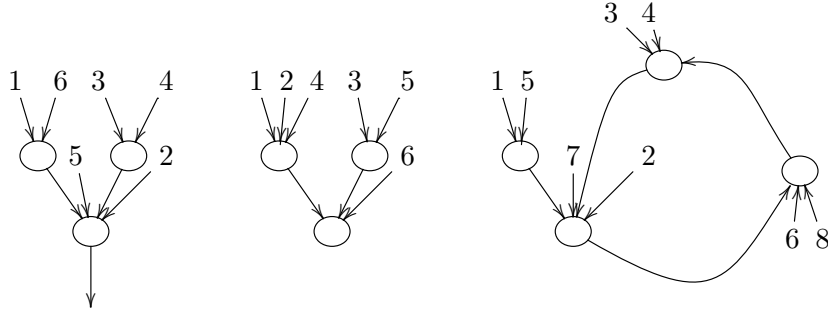
Let us also give a monadic definition of wheeled operads. Recall that we have an endofunctor of the category of species given by the monad of rooted trees. Existence of the root gives each edge of the rooted tree a canonical direction towards the root. Let us consider a slightly more general class of (isomorphism classes of) graphs with directed edges by only postulating that each internal vertex has at most one output edge, and not imposing any other restrictions.

Lemma 3.4 ([26, Lemma 1.4]). *If such a graph Γ is connected, it has at most one outgoing edge.*

Proof. If there are at least two outgoing edges, we may consider a (non-directed) path connecting them in Γ ; one of the vertices of the path will have two outgoing edges, which is a contradiction. \square

More precisely, there are three possible types of connected graphs that we allow: trees with one output edge, sinks (trees without output edges), and wheels

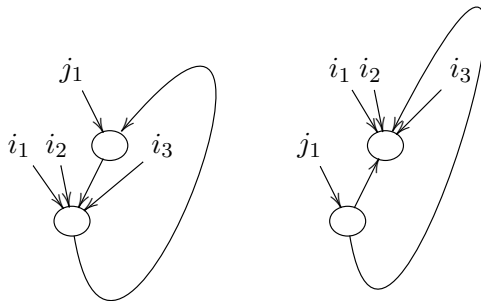
without output edges, as in the following picture:



Using our three kinds of graphs, we define an endofunctor \mathcal{T}° of the category of two-coloured linear species by setting $\mathcal{T}^\circ(\mathcal{F})_1(I) = \mathcal{T}(\mathcal{F}_1)(I)$ and

$$\mathcal{T}^\circ(\mathcal{F})_0(I) := \bigoplus_{\substack{\tau \text{ a sink,} \\ \text{Leaves}(\tau)=I}} \bigotimes_{v \in \text{Vertices}(\tau)} \mathcal{F}_{|\text{out}(v)|}(\text{in}(v)) \oplus \bigoplus_{\substack{\tau \text{ a wheel,} \\ \text{Leaves}(\tau)=I}} \bigotimes_{v \in \text{Vertices}(\tau)} \mathcal{F}_1(\text{in}(v)).$$

This endofunctor can be given a monad structure that is completely analogous to the rooted tree monad, via insertions of graphs into graphs for the monad product, and via considering one-vertex trees for the monad unit. We call this monad the monad of *rooted trees and wheels*. A wheeled operad is an algebra over that monad. The trace map corresponds to creating a wheel out of a tree (gluing the output edge to one of the input edges), and vanishing on the commutators corresponds to the fact that the following two wheel graphs are the same:



To each operad \mathcal{O} , one can associate canonically two different wheeled operads. The first option is to consider \mathcal{O} as a trivial wheeled operad $\mathcal{O} \oplus 0$ postulating that all traces are equal to zero. The second option is to consider the *wheeled completion*

$$\mathcal{O}^\circ := \mathcal{O} \oplus \partial(\mathcal{O}) / [\partial(\mathcal{O}), \partial(\mathcal{O})]$$

of an operad \mathcal{O} , where the trace map $\partial(\mathcal{O}) \rightarrow \partial(\mathcal{O})/[\partial(\mathcal{O}), \partial(\mathcal{O})]$ is the canonical projection. This notion was first introduced (some years before introducing wheeled operads) by Kapranov in [36, Sec. 3.3], where he interpreted what we identify as $\partial(\mathcal{O})/[\partial(\mathcal{O}), \partial(\mathcal{O})]$ as the “module of natural forms” on \mathcal{O} -algebras. If we consider the category of wheeled operads \mathcal{U} with the given operad \mathcal{O} as \mathcal{U}_1 , the wheeled operads \mathcal{O} and \mathcal{O}° are, respectively, the terminal and the initial objects of that category.

Given a wheeled operad $\mathcal{U} = \mathcal{O} \oplus \mathcal{W}$, its bar construction $B^\circ(\mathcal{U})$ is the chain complex structure on the trees and wheels endofunctor $\mathcal{T}^\circ(s\overline{\mathcal{O}} \oplus \mathcal{W})$ with the differential obtained as a sum of all ways of collapsing edges of a graph and using the wheeled operad structure maps on the labels of the vertices that are identified by such collapses. Explicitly, one can describe that chain complex using various operadic constructions as follows:

$$B^\circ(\mathcal{U}) = B(\mathcal{O}) \oplus (B(\mathcal{W}, \mathcal{O}) \oplus \partial(B(\mathcal{O}))/[\partial(B(\mathcal{O})), \partial(B(\mathcal{O}))]).$$

Moreover, one can talk about wheeled cooperads by reversing all arrows in the respective definitions; in that language, the bar construction of a wheeled operad $\mathcal{U} = \mathcal{O} \oplus \mathcal{W}$ is the cofree conilpotent wheeled cooperad on $s\overline{\mathcal{O}} \oplus \mathcal{W}$ equipped with the unique coderivation extending the maps induced by the structure maps of the wheeled operad \mathcal{U} .

As an illustration, let us discuss these notions in one very particular case which will be very informative for understanding the main results of this paper.

Example 3.5. Let A be an augmented associative algebra with the augmentation ideal I , and consider it as an operad that only has operations in arity one given by A . Then the zero completion of A is just the algebra A itself, and the bar construction $B^\circ(A)$ is $B(A) \oplus \text{Cyc}(sI)$. The differential of the bar construction is induced by the differential of the usual bar construction of A on both terms, and $\text{Cyc}(sI)$ computes (up to a degree shift by one) the cyclic homology of I , so its homology is $H_\bullet(B(A)) \oplus HC_{\bullet-1}(I)$. (Note that we use the convention of [51] for which the bar construction of an augmented algebra A is the cofree coalgebra on its augmentation ideal I .)

Similarly to the PROP completion of an operad, to each wheeled operad \mathcal{U} one can associate a PROP $\mathcal{P}(\mathcal{U})$ whose components are given by

$$(\mathcal{P}(\mathcal{U}))(m, n) := (S^n(\mathcal{U}))(m),$$

where the vertical composition comes from the composition in \mathcal{U}_1 and the right \mathcal{U}_1 -module structure on \mathcal{U}_0 , and the horizontal composition is the product in the symmetric algebra $S(\mathcal{U})$. This PROP is also referred to as the PROP completion of the wheeled operad \mathcal{U} .

4. STABLE HOMOLOGY AND BAR CONSTRUCTIONS OF WHEELED OPERADS

In this section, we shall prove the two theorems about stable homology of Lie algebras that we announced in the introduction. The computation is quite straightforward: the main tool is the use of invariant theory of $\mathfrak{gl}(V)$ in the spirit

of the classical Gelfand–Fuchs calculations [29] which inspired a great number of calculations of homology of infinite-dimensional Lie algebras, see, for example, the papers [24, 28, 32, 40, 41, 44].

The main tool for computing $\mathfrak{gl}(V)$ -invariants is provided by the First Fundamental Theorem for $\mathfrak{gl}(V)$ that asserts that the subspace of $\mathfrak{gl}(V)$ -invariants in $V^{\otimes r} \otimes (V^*)^{\otimes s}$ is nonzero only for $r = s$, and in that case, if we use the canonical identification of that space with its linear dual, is spanned by invariants of the form

$$f_\sigma: v_1 \otimes \cdots \otimes v_r \otimes \xi_1 \otimes \xi_r \mapsto \xi_1(v_{\sigma(1)}) \cdots \xi_r(v_{\sigma(r)}), \quad \sigma \in S_r.$$

Moreover, the Second Fundamental Theorem for $\mathfrak{gl}(V)$ asserts that those invariants are linearly independent as long as $\dim(V) \geq r$. Proofs of both of these theorems are contained in [74], and are at the core of the Schur–Weyl duality between representations of symmetric groups and general linear groups. For an exposition of these results that is very close in spirit to the present work, the reader is also invited to consult [58]. Besides the direct use of the two fundamental theorem, it will also important for us that they give meaning to the object that we study below: for a representation M of $\mathfrak{gl}(V)$ that is contained in sums of tensor products of V and V^* , one can completely determine the structure of M from the spaces of “matrix elements” $(M \otimes (V^*)^{\otimes p} \otimes V^{\otimes q})^{\mathfrak{gl}(V)}$ for various p, q .

4.1. Stable homology for the Lie algebra of all derivations. The first of the two theorems we would like to prove is the following one.

Theorem 4.1. *Let \mathcal{O} be an augmented operad, and V a finite-dimensional vector space. The space of matrix elements*

$$(H_\bullet(\mathrm{Der}^+(\mathcal{O}(V)), \mathbb{k}) \otimes \mathrm{Hom}(V^{\otimes p}, V^{\otimes q}))^{\mathfrak{gl}(V)}$$

stabilize as $\dim(V) \rightarrow \infty$ and, after stabilization, is isomorphic to the (p, q) th component of the coPROP completion of the wheeled cooperad $H_\bullet(\mathbb{B}^\circ(\mathcal{O}))$. This isomorphism is natural with respect to operad morphisms.

Proof. Since

$$\mathrm{Der}(\mathcal{O}(V)) \cong \bigoplus_{k \geq 1} \mathrm{Hom}(V, \mathcal{O}(k) \otimes_{\mathbb{k}S_k} V^{\otimes k}) \cong \bigoplus_{k \geq 1} V^* \otimes \mathcal{O}(k) \otimes_{\mathbb{k}S_k} V^{\otimes k},$$

we have

$$\begin{aligned} C_\bullet(\mathrm{Der}^+(\mathcal{O}(V)), \mathbb{k}) \otimes \mathrm{Hom}(V^{\otimes p}, V^{\otimes q}) &\cong \\ &\bigoplus_{\substack{k \geq 2, \\ n_1, \dots, n_k \geq 2 \\ p_1, \dots, p_k \geq 0}} \bigotimes_{i=1}^k S^{p_i}(sV^* \otimes \mathcal{O}(n_i) \otimes_{\mathbb{k}S_{n_i}} V^{\otimes n_i}) \otimes (V^*)^{\otimes p} \otimes V^{\otimes q}. \end{aligned}$$

Moreover, the part

$$\bigotimes_{i=1}^k S^{p_i}(sV^* \otimes \mathcal{O}(n_i) \otimes_{\mathbb{k}S_{n_i}} V^{\otimes n_i}) \otimes (V^*)^{\otimes p} \otimes V^{\otimes q}$$

is, as a $\mathfrak{gl}(V)$ -module, a submodule of several copies of $V^{\otimes N} \otimes V^{*\otimes M}$ where

$$N = \sum_{i=1}^k n_i p_i + q, \quad M = \sum_{i=1}^k p_i + p.$$

Taking $\mathfrak{gl}(V)$ -invariants and computing homology commute, and we shall make full use of that, describing the chain complex of matrix elements

$$(C_\bullet(\mathrm{Der}^+(\mathcal{O}(V)), \mathbb{k}) \otimes \mathrm{Hom}(V^{\otimes p}, V^{\otimes q}))^{\mathfrak{gl}(V)}$$

for $\dim(V)$ sufficiently large, and then computing its homology.

The following is a direct generalization of the argument of Fuchs [26]. It is advantageous to view

$$\bigoplus_{\substack{k \geq 2, \\ n_1, \dots, n_k \geq 2 \\ p_1, \dots, p_k \geq 0}} \bigotimes_{i=1}^k S^{p_i}(sV^* \otimes \mathcal{O}(n_i) \otimes_{\mathbb{k}S_{n_i}} V^{\otimes n_i}) \otimes (V^*)^{\otimes p} \otimes V^{\otimes q},$$

that is our chain complex before taking invariants, as the vector space spanned by all linear combinations of sets of ‘‘LEGO pieces’’ of the following three types:

- (1) a ‘‘corolla’’, that is a vertex v with one incoming half-edge and $n_v \geq 2$ outgoing half-edges, where v carries a label from $\mathcal{O}(\mathrm{out}(v))$, the incoming half-edge is labelled by an element $\xi \in V^*$, and each outgoing half-edge e is labelled by an element $u_e \in V$ (here we implicitly impose an equivalence relation saying that if we simultaneously act on α and on labels of outgoing half-edges by the same permutation, the corolla does not change);
- (2) a ‘‘source’’, that is a vertex v with one outgoing half-edge and no incoming half-edges, where v uniquely corresponds to an element of $\{1, \dots, q\}$, and the outgoing half-edge is labelled by an element $u \in V$;
- (3) a ‘‘sink’’, that is a vertex v with one incoming half-edge and no outgoing half-edges, where v uniquely corresponds to an element of $\{1, \dots, p\}$, and the incoming half-edge is labelled by an element $\xi \in V^*$.

Corollas with n_v outgoing half-edges correspond to elements of $sV^* \otimes \mathcal{O}(n_v) \otimes_{\mathbb{k}S_{n_v}} V^{\otimes n_v}$ of homological degree 1, and so reordering them creates a sign; the sources and the sinks appear in the natural order of their vertex labels, reproducing the vector space $(V^*)^{\otimes p} \otimes V^{\otimes q}$.

For the given M, N our chain complex is bigraded:

$$(C_d^{(w)}(\mathrm{Der}^+(\mathcal{O}(V)), \mathbb{k}) \otimes \mathrm{Hom}(V^{\otimes p}, V^{\otimes q}))^{\mathfrak{gl}(V)}$$

consists of elements of weight w (computed out of the weight on the Lie algebra $\mathrm{Der}^+(\mathcal{O}(V))$) and of homological degree d . In the notation above, we have $w =$

$\sum_{i=1}^k (n_i - 1)p_i$ and $d = \sum_{i=1}^k p_i$, so in particular $w + d = \sum_{i=1}^k n_i p_i$. Thus, our above formulas only depend on the bigrading and can be written as

$$N = w + d + q, \quad M = d + p.$$

From the First and the Second Fundamental Theorems for $\mathfrak{gl}(V)$, it follows that to have non-zero invariants at all we must have $N = M$, and that for the “stable range” $\dim(V) \geq M = d + p$ the $\mathfrak{gl}(V)$ -invariants in this module are isomorphic to a vector space with a basis of (equivalence classes of) graphs obtained by assembling the LEGO pieces together, that is directed decorated graphs Γ obtained by matching all the incoming half-edges with all the outgoing half-edges. Specifically, we obtain that each vertex v of Γ is one of the following three types:

- (1) a corolla v with one incoming half-edge and $n_v \geq 2$ outgoing half-edges labelled by an element from $\mathcal{O}(\text{out}(v))$;
- (2) a source with one outgoing half-edge uniquely corresponding to an element of $\{1, \dots, q\}$;
- (3) a sink with one incoming half-edge uniquely corresponding to an element of $\{1, \dots, p\}$.

Two such graphs are equivalent if there is an isomorphism of them that agrees with all labellings. Such a graph corresponds to an invariant that pairs V and V^* according to the edges between the vertices in the graph.

Moreover, the differential

$$\begin{aligned} \left(C_{d+1}^{(w)}(\text{Der}^+(\mathcal{O}(V)), \mathbb{k}) \otimes \text{Hom}(V^{\otimes p}, V^{\otimes q}) \right)^{\mathfrak{gl}(V)} \rightarrow \\ \left(C_d^{(w)}(\text{Der}^+(\mathcal{O}(V)), \mathbb{k}) \otimes \text{Hom}(V^{\otimes p}, V^{\otimes q}) \right)^{\mathfrak{gl}(V)} \end{aligned}$$

admits an elegant formula in the stable range $\dim(V) > d + p$, where the strict inequality comes from the fact that we need to know basis elements in homological degree $d + 1$ as well. For that, we recall that the Lie algebra of derivations comes from a pre-Lie structure, and, if we think of derivations in terms of linear combinations of corollas representing them, the pre-Lie product $D \triangleleft D'$ is equal to the sum of elements corresponding to pairing one of the outputs of D with the input of D' and computing the corresponding contraction $V^* \otimes V \rightarrow \mathbb{k}$ simultaneously with the partial composition in \mathcal{O} . Since our graphs encode contractions, and corollas are labelled by elements of \mathcal{O} , this means that on the level of decorated graphs, we have the “usual” graph complex differential which collapses edges of the graph and computes the composition of labels of the vertices at the same time.

Similarly to Lemma 3.4, we see that connected graphs that appear as basis elements in the stable range are precisely the connected graphs appearing as basis elements of the wheeled bar construction $B^\circ(\mathcal{O})$. Indeed, we have just established that the combinatorics of graphs is precisely correct: trees with one incoming edge and wheels with no incoming edges (note that the operadic bar construction appears with its counit, which is the graph obtained by matching a source with a sink, without any corollas involved). Moreover, the differential is exactly the differential of $B^\circ(\mathcal{O})$: it is made of edge contractions, and if an edge is a loop going

from a vertex to itself, contracting that edge does not correspond to computing the Lie bracket and hence does not contribute to the differential, while in $B^\circ(\mathcal{O})$ all such contributions vanish precisely because we consider \mathcal{O} as a wheeled operad with the zero trace map.

Passing to disconnected graphs corresponds to computing symmetric powers, which is precisely the coPROP completion of the wheeled bar construction viewed as a wheeled cooperad. Finally, the Künneth formula implies the main statement of the theorem (the statement on naturality is obvious). \square

Let us start by showing that this result does indeed generalize the theorems discussed in the introduction. Let us start with explaining how the Loday–Quillen–Tsygan theorem arises in our language.

Corollary 4.2 (Loday–Quillen [50], Tsygan [72]). *Let A be a unital \mathbb{k} -algebra. We have a Hopf algebra isomorphism*

$$H_\bullet(\mathfrak{gl}(A)) \cong S^c(HC_{\bullet-1}(A)).$$

Proof. Let us consider the algebra A_+ which is obtained from A by adjoining a unit (even though A is already unital). This algebra has the canonical augmentation for which the augmentation ideal I_+ is A . If we consider A_+ as an operad \mathcal{O} concentrated in arity one, and use what we learned in Example 3.1 for a vector space V of dimension n , we have

$$\begin{aligned} (C_\bullet(\mathrm{Der}^+(\mathcal{O}(V)), \mathbb{k}) \otimes \mathrm{Hom}(V^{\otimes p}, V^{\otimes q}))^{\mathfrak{gl}(V)} &\cong \\ &(C_\bullet(\mathfrak{gl}_n(A), \mathbb{k}) \otimes \mathrm{Hom}(V^{\otimes p}, V^{\otimes q}))^{\mathfrak{gl}(V)}. \end{aligned}$$

Theorem 4.1 implies that the stable limit of

$$(C_\bullet(\mathfrak{gl}_n(A), \mathbb{k}) \otimes \mathrm{Hom}(V^{\otimes p}, V^{\otimes q}))^{\mathfrak{gl}(V)}$$

for $\dim(V) \rightarrow \infty$ is the coPROP completion of

$$H_\bullet(B^\circ(A_+)).$$

According to Example 3.5,

$$H_\bullet(B^\circ(A_+)) \cong H_\bullet(B(A_+)) \oplus HC_{\bullet-1}(I_+) \cong HC_{\bullet-1}(A),$$

since $I_+ \cong A$, and since the bar construction is acyclic if the augmentation ideal is a unital algebra. Finally, $HC_{\bullet-1}(A)$ is the $(0, 0)$ -component of our wheeled cooperad, so the coPROP completion is $S^c(HC_{\bullet-1}(A))$. This gives the coalgebra structure on $H_\bullet(\mathfrak{gl}(A))$, and the Hopf algebra isomorphism is easy to obtain by recalling that the product on $H_\bullet(\mathfrak{gl}(A))$ comes from the canonical embeddings $\mathfrak{gl}(V) \oplus \mathfrak{gl}(W) \hookrightarrow \mathfrak{gl}(V \oplus W)$. \square

Let us now show how the Fuchs stability theorem is contained in our formalism.

Corollary 4.3 (Fuchs [26]). *Suppose that $n > w + 2d$. Then*

$$H_d^{(w)}(L_1(n), \mathbb{k}) = 0 \text{ for } w \neq d.$$

Proof. A large part of the proof of Theorem 4.1 already mimics the argument of Fuchs, and the beginning of this proof continues to do so. We remark that taking matrix elements of representations immediately implies that $H_d^{(w)}(L_1(n), \mathbb{k})$ vanishes if and only if

$$\left(H_d^{(w)}(L_1(n), \mathbb{k}) \otimes \text{Hom}(V^{\otimes p}, V^{\otimes q}) \right)^{\mathfrak{gl}(V)} = 0 \text{ for all } p, q.$$

If we denote $V = \mathbb{k}^n$, this latter vector space is the same as

$$\left(H_d^{(w)}(\text{Der}^+(\text{Com}(V))\mathbb{k}) \otimes \text{Hom}(V^{\otimes p}, V^{\otimes q}) \right)^{\mathfrak{gl}(V)},$$

which, according to Theorem 4.1, is stably isomorphic to the (p, q) th component to the coPROP completion of $H_\bullet(\text{B}^\circ(\text{Com}))$.

There are two steps that remain. First, we saw in the proof of Theorem 4.1 that $H_d^{(w)}(L_1(n), \mathbb{k})$ is contained in the direct sum of several copies of

$$\text{Hom}(V^{\otimes d}, V^{\otimes(w+d)}),$$

so it is enough to show that

$$\left(H_d^{(w)}(L_1(n), \mathbb{k}) \otimes \text{Hom}(V^{\otimes(w+d)}, V^{\otimes d}) \right)^{\mathfrak{gl}(V)} = 0.$$

For these parameters, the stable range condition is given exactly by $n > w + 2d$. Second, in the particular case of the operad Com a very particular phenomenon presents itself: if we consider it as a wheeled operad, it has just one wheeled relation stating that the universal trace of the operator of multiplication by a from $\partial(\text{Com})$ vanishes, and this wheeled operad is Koszul [57], so the homology $\text{B}^\circ(\text{Com})$, as well as the coPROP completion of that homology, is concentrated on the diagonal $w = d$. \square

Let us remark that according to [57, Ex. 5.2.5], the Koszul dual wheeled operad of Com is $\text{Lie}^\circ = \text{Lie} \oplus \text{Cyc}(s\mathbb{1})$, which not only immediately explains the appearance of factorials in [26, Prop. 1.5, Prop. 1.6]), but also highlights the difference between those factorials on the level of species. It is also worth noting that the Koszulness of Com as a wheeled operad, which is the key ingredient of the proof of Corollary 4.3, is a very rare coincidence. Indeed, in the case of Com , vanishing of all traces follows from vanishing of traces of all generators, a result which is certainly false for most operads. A class of operads where this statement is true includes, in particular, all operads obtained from weight graded commutative associative algebras [19, 42], and it would be interesting to compute homology of wheeled bar constructions of some operads of that class, generalizing [19, Th. 5.2, Th. 5.3].

One can also use Theorem 4.1 to prove that the multiplicity of each given irreducible $\mathfrak{gl}(V)$ -component of the homology $H_\bullet(\text{Der}^+(\mathcal{O}(V)), \mathbb{k})$ stabilizes as $\dim(V)$ goes to infinity, a generalization of the non-unital version of the Loday–Quillen–Tsygan theorem (proved independently by Feigin and Tsygan [24, Chapter 4] and Hanlon [34]). Not having immediate applications of this result in mind,

we omit the details.

From the proof of Corollary 4.3, one can see that we actually proved more: from vanishing of the homology of $B^\circ(\mathcal{O})$ in a certain range, one can deduce vanishing of $H_\bullet(\text{Der}^+(\mathcal{O}(V)), \mathbb{k})$ in certain range. This is very useful for estimates of homology in degrees 1 and 2, which are particularly interesting for the following reason. If the operad \mathcal{O} has an extra weight grading for which the augmentation ideal consists of elements of strictly positive weight (this happens, for example, if \mathcal{O} has no unary operations other than multiples of the identity), one can use that grading on the algebra $\text{Der}^+(\mathcal{O}(V))$ to give the homology additional weight grading, and use that grading to identify the first homology with the minimal space of generators of that Lie algebra and the second homology with the minimal space of relations between those generators [27]. The Künneth formula implies that for the first homology of $H_\bullet(\text{Der}^+(\mathcal{O}(V)), \mathbb{k})$ we only need the first homology of $B^\circ(\mathcal{O})$, and for the second homology of $H_\bullet(\text{Der}^+(\mathcal{O}(V)), \mathbb{k})$, we only need the first and the second homology of $B^\circ(\mathcal{O})$. Once that computation is done, one may try to convert the information obtained into information for a given number of generators. For example, in the case $\mathcal{O} = \text{Com}$, Feigin and Fuchs [23] proved that the Lie algebra $L_1(n) = \text{Der}^+(\text{Com}(V))$ is generated by elements of weight 1 for $\dim(V) > 1$, and that the relations between these elements are all of weight 2 for $\dim(V) > 2$. However, the following example shows that in general the situation is much more complicated.

Example 4.4. It is well known that the Lie algebra $\text{Der}^+(\text{Lie}(V))$ has generators of all weights [37, 61]. It is possible to give an easy explanation of that fact using the notion of divergence, which also hints that the Lie algebra of derivations with zero divergence is very likely to be more suitable for generalizing the results of [23]. From the cocycle property

$$\text{div}([D, D']) = D_*(\text{div}(D')) - D'_*(\text{div}(D))$$

and the fact that for each derivation D of positive weight of the free Lie algebra, the derivation D_* sends $V \subset T(V) \cong U_{\text{Lie}}(\text{Lie}(V))$ to a linear combination of nontrivial Lie monomials, which, in particular, vanishes in the space of cyclic words, it follows by induction that every commutator of two derivations of positive degree has zero divergence, and so we need generators in each weight to account for derivations with nonzero divergence.

Let us now establish a new somewhat general result about the bar homology of \mathcal{O} as a wheeled operad.

Theorem 4.5. *Suppose that $\partial(\mathcal{O})$ is free as a right \mathcal{O} -module, and let*

$$\partial(\mathcal{O})_0 := \partial(\mathcal{O}) \circ_{\mathcal{O}} \mathbb{1},$$

where $\mathbb{1}$ is equipped with an left \mathcal{O} -module structure via the augmentation, be the minimal set of generators of this module, considered as a twisted associative algebra. Then we have

$$H_\bullet(B^\circ(\mathcal{O})) \cong H_\bullet(B(\mathcal{O})) \oplus HC_{\bullet-1}(\partial(\overline{\mathcal{O}})_0).$$

Proof. The operadic part of $B^\circ(\mathcal{O})$ is $B(\mathcal{O})$, so there is nothing to prove. Let us consider the wheeled part of $B^\circ(\mathcal{O})$. It has a basis of decorated directed graphs with one wheel. Note that if a vertex is a part of the wheel, one of its outgoing half-edges (the one involved with the wheel) is different from others, so it is convenient to think that the decorations of each vertex of the wheel is an element of $s\partial(\overline{\mathcal{O}})$, while the decorations of all other vertices are elements of $s\overline{\mathcal{O}}$. We shall denote by d_{\rightarrow} the differential that is a sum of contractions of edges not involved in the wheel, and by d_{\circlearrowleft} the differential that is a sum of contractions of edges involved in the wheel; the fact that the structures of a twisted associative algebra and a right \mathcal{O} -module on $\partial(\mathcal{O})$ commute imply that these are two anti-commuting differentials.

To prove the statement of the theorem, let us consider the increasing filtration of our complex by the number of edges involved in the wheel. The differential d_{\circlearrowleft} decreases this number, and the differential d_{\rightarrow} preserves it, so in the associated graded complex, only the differential d_{\rightarrow} survives, making our complex isomorphic to

$$B(\partial(\overline{\mathcal{O}}), \mathcal{O})/[B(\partial(\overline{\mathcal{O}}), \mathcal{O}), B(\partial(\overline{\mathcal{O}}), \mathcal{O})],$$

where we take the bar construction of \mathcal{O} with coefficients in $\partial(\overline{\mathcal{O}})$, view it as a differential graded twisted associative algebra using the associative algebra structure on $\partial(\mathcal{O})$, and form the commutator quotient of that differential graded twisted associative algebra. Since we assumed that $\partial(\mathcal{O})$ is free as a right \mathcal{O} -module, the homological criterion of freeness [20] shows that the homology of $B(\partial(\overline{\mathcal{O}}), \mathcal{O})$ is concentrated in degree zero and is identified with $\partial(\overline{\mathcal{O}})_0$. Because of the former, the spectral sequence abuts on the following page, where the differential induced by d_{\circlearrowleft} is the differential of the cyclic complex of the twisted associative algebra $\partial(\overline{\mathcal{O}})_0$, completing the proof. \square

The key aspect on which Theorem 4.5 relies is the freeness property of $\partial(\mathcal{O})$ as a right \mathcal{O} -module; this property previously appeared in work of Khoroshkin [43] who used our work with Tamaroff [20] to show that this condition is necessary and sufficient for the functorial PBW property of the universal enveloping algebras of \mathcal{O} -algebras. Note that in [43] it is shown that the operad of Poisson algebras does not have the PBW property, and the same operad exhibits nontrivial cycles in the wheeled completion of its bar construction, which is an unpublished result obtained independently by Bruinsma (bachelor thesis at University of Amsterdam, 2010) and Khoroshkin (private communication). This ‘‘coincidence’’ is precisely what made us discover Theorem 4.5. Let us also remark that if the operad \mathcal{O} is generated by unary operations, we have no slots for the right \mathcal{O} -action on $\partial(\mathcal{O})$, so that action factors through the augmentation, and $\partial(\mathcal{O})_0 \cong \partial(\mathcal{O})$, so we recover the result of Example 3.5.

Let us give several examples of how Theorem 4.5 can be applied. We start with a result which is essentially the wheeled Koszul property of Com considered as a Koszul operad, first proved in [57], which we used in Corollary 4.3 above.

Corollary 4.6. *We have*

$$H_{\bullet}(\mathbf{B}^{\circ}(\text{Com})) \cong \text{Com}^i \oplus \text{Cyc}(s\mathbb{1}).$$

Proof. Indeed, we have the twisted associative algebra isomorphism $\partial(\text{Com}) \cong \text{uCom}$, so $\partial(\overline{\text{Com}}) \cong \text{Com}$ and $\partial(\overline{\text{Com}})_0 \cong \mathbb{1}$ with zero multiplication, so its cyclic homology as a twisted associative algebra is the cyclic complex $s^{-1} \text{Cyc}(s\mathbb{1})$. \square

The following result is essentially a new proof of [57, Th. 6.1.1], which also corrects a minor indexing issue in that theorem.

Corollary 4.7. *We have*

$$H_{\bullet}(\mathbf{B}^{\circ}(\text{Ass})) \cong \text{Ass}^i \oplus (\text{Cyc}(s\mathbb{1}) \oplus \text{Cyc}(s\mathbb{1}) \oplus \text{Cyc}(s\mathbb{1}) \otimes \text{Cyc}(s\mathbb{1})).$$

Proof. Indeed, we know that $\partial(\text{Ass}) \cong \text{uAss} \otimes \text{uAss}$, and from that it is easy to see that we have $\partial(\text{Ass})_0 = (\partial(\text{Com})_0)^{\otimes 2}$. Now, we have, in the standard cyclic homology notations,

$$HC_{\bullet}(\partial(\text{Com})_0) \cong \mathbb{k}[u] \otimes \mathbb{k} \oplus s^{-1} \text{Cyc}(s\mathbb{1}),$$

so by an analogue of [49, Prop. 4.4.8] for twisted associative algebras, we have

$$\begin{aligned} HC_{\bullet}(\partial(\text{Com})_0^{\otimes 2}) &\cong HC_{\bullet}(\partial(\text{Com})_0) \otimes \mathbb{k} \oplus HH_{\bullet}(\partial(\text{Com})_0) \otimes s^{-1} \text{Cyc}(s\mathbb{1}) \cong \\ &HC_{\bullet}(\mathbb{k}) \oplus s^{-1} \text{Cyc}(s\mathbb{1}) \oplus (\mathbb{k} \oplus \text{Cyc}(s\mathbb{1})) \otimes s^{-1} \text{Cyc}(s\mathbb{1}), \end{aligned}$$

and

$$HC_{\bullet-1}(\partial(\overline{\text{Ass}})_0) \cong \text{Cyc}(s\mathbb{1}) \oplus \text{Cyc}(s\mathbb{1}) \oplus (\text{Cyc}(s\mathbb{1}) \otimes \text{Cyc}(s\mathbb{1})),$$

\square

The following result is, to the best of our knowledge, new.

Corollary 4.8. *We have*

$$H_{\bullet}(\mathbf{B}^{\circ}(\text{Lie})) \cong \text{Lie}^i \oplus \text{Hook},$$

where *Hook* is the linear species with the homological degree d part of $\text{Hook}(n)$ being the S_n -module corresponding to the Young diagram $(n-d+1, 1^{d-1})$, $d \geq 1$.

Proof. Since the universal multiplicative enveloping algebra of a Lie algebra is known to coincide with the left adjoint of the functor from associative algebras to Lie algebras that retains the Lie bracket only, we have $\partial(\text{Lie}) \cong \text{uAss}$, and hence $\partial(\text{Lie})_0 \cong \text{uCom}$, so computing cyclic homology is essentially computing the cyclic homology of polynomial algebras. The answer (for the reduced homology) is given by $\text{Com}(\mathbb{1} \oplus s\mathbb{1})/\text{Im}(d_{dR})$, where d_{dR} is the unique derivation extending $s: \mathbb{1} \rightarrow s\mathbb{1}$. Note that we can explicitly compute the Cauchy product

$$S^p(\mathbb{1}) \otimes S^q(s\mathbb{1}) \cong S^p \otimes S^{1^q} \cong S^{p,1^q} \oplus S^{p+1,1^{q-1}}$$

using the Pieri rule for Schur functions [53], and it is almost obvious that d_{dR} sends $S^{p,1^q} \subset S^p(\mathbb{1}) \otimes S^q(s\mathbb{1})$ to the same submodule of $S^{p-1}(\mathbb{1}) \otimes S^{q+1}(s\mathbb{1})$. Thus, as a linear species, we have

$$HC_d^{(w)}(\partial(\overline{\text{Lie}})_0) \cong S^{w-d,1^d}.$$

Since we need $HC_{\bullet-1}$, shifting d by one completes the proof. \square

Note that in particular for $d = 1$ the module $\text{Hook}_d(n)$ is the trivial module for each n , which is in agreement with the abovementioned result of [37, 61] stating that the algebra $\text{Der}(\text{Lie}^+(V))$ has generators of all weights. Our computation is a first natural step towards computing the minimal model of Lie as a wheeled operad; we shall address this question elsewhere.

Let us also record a new result of the same flavour concerning the operad of pre-Lie algebras.

Corollary 4.9. *We have*

$$H_{\bullet}(\mathbf{B}^{\circ}(\text{PreLie})) \cong \text{PreLie}^{\circ} \oplus (\text{Hook} \oplus \text{uCom}(\mathbb{1} \oplus s\mathbb{1}) \otimes \text{Cyc}(\mathbb{1})).$$

Proof. It is known [43, 46] that $\partial(\text{PreLie})$ is a free right PreLie -module with the species of generators $\text{uCom} \otimes \text{uAss}$, and $HC_{\bullet}(\text{uAss}) = \mathbb{k}[u] \otimes \mathbb{k} \oplus \text{Cyc}(\mathbb{1})$, so by an analogue of [49, Prop. 4.4.8], we have

$$\begin{aligned} HC_{\bullet}(\text{uCom} \otimes \text{uAss}) &\cong HC_{\bullet}(\text{uCom}) \otimes \mathbb{k} \oplus HH_{\bullet}(\text{uCom}) \otimes \text{Cyc}(\mathbb{1}) \cong \\ &HC_{\bullet}(\text{uCom}) \oplus (\text{uCom}(\mathbb{1} \oplus s\mathbb{1}) \otimes \text{Cyc}(\mathbb{1})), \end{aligned}$$

so

$$HC_{\bullet-1}(\partial(\overline{\text{PreLie}})_0) \cong \text{Hook} \oplus \text{uCom}(\mathbb{k} \oplus s\mathbb{1}) \otimes \text{Cyc}(\mathbb{1}),$$

as required. \square

An interesting natural question that we shall address elsewhere is to compute the homology of the Lie algebra $\text{Der}(\mathcal{O}(V))$ for various non-reduced operads \mathcal{O} ; note that for $\mathcal{O} = \text{uCom}$, that homology can be identified with homology of a natural topological space, see [29] for details.

4.2. Stable homology for the Lie algebra of derivations of zero divergence.

We shall now move on to proving an analogue of Theorem 4.1 involving the notion of divergence of derivations.

Theorem 4.10. *Let \mathcal{O} be an augmented operad, and V a finite-dimensional vector space. The space of matrix elements*

$$(H_{\bullet}(\text{SDer}^+(\mathcal{O}(V)), \mathbb{k}) \otimes \text{Hom}(V^{\otimes p}, V^{\otimes q}))^{\mathfrak{gl}(V)}$$

stabilizes as $\dim(V) \rightarrow \infty$ and, after stabilization, is isomorphic to the (p, q) th component to the coPROP completion of the wheeled cooperad $H_{\bullet}(\mathbf{B}^{\circ}(\mathcal{O}^{\circ}))$. This isomorphism is natural with respect to operad morphisms.

Proof. The argument is very close to that of the proof of Theorem 4.1; however, the argument is slightly less direct, since, unlike what we saw in that proof, on the chain level the corresponding differential graded coPROPs are different. Let us start with a short argument that replaces $B^\diamond(\mathcal{O}^\diamond)$ by a complex with the same homology which is more relevant for our purposes. For that, we shall examine in some detail the wheeled part

$$B(\partial(\mathcal{O})/[\partial(\mathcal{O}), \partial(\mathcal{O})], \mathcal{O}) \oplus \partial(B(\mathcal{O}))/[\partial(B(\mathcal{O})), \partial(B(\mathcal{O}))]$$

of the wheeled bar construction $B^\diamond(\mathcal{O}^\diamond)$. The differential of the second summand is made of two parts, d_{edge} with values in $\partial(B(\mathcal{O}))/[\partial(B(\mathcal{O})), \partial(B(\mathcal{O}))]$ contracting edges between two different vertices in all possible ways, and d_{loop} with values in $B(\partial(\mathcal{O})/[\partial(\mathcal{O}), \partial(\mathcal{O})], \mathcal{O})$ contracting the only loop going from a vertex to itself (if it exists). Since the right module structure and the twisted associative algebra structure on $\partial(\mathcal{O})$ commute, it is easy to see that $\ker(d_{\text{loop}})$ is a subcomplex of the wheeled part of the bar construction (this is a huge complex: in particular, it contains all graphs without a loop). We shall see below that this is really the relevant complex that gives the stable limit

$$C_\bullet(\text{SDer}^\wedge(\mathcal{O}(V)), \mathfrak{gl}(V), \text{Hom}(V^{\otimes p}, V^{\otimes q})),$$

so we just need to check that $\ker(d_{\text{loop}})$ has the same homology as the whole wheeled part of the bar complex. This is quite easy: the quotient complex is identified with

$$B(s\partial(\mathcal{O})/[\partial(\mathcal{O}), \partial(\mathcal{O})], \mathcal{O}) \rightarrow B(\partial(\mathcal{O})/[\partial(\mathcal{O}), \partial(\mathcal{O})], \mathcal{O})$$

with the differential $d_{\text{edge}} + d_{\text{loop}}$; and if we design a spectral sequence for which we start calculation with the homology of d_{loop} , the corresponding complex is completely acyclic, proving the necessary result.

Let us now convince ourselves that nothing else really changes. We still have

$$(C_\bullet(\text{SDer}^+(\mathcal{O}(V)), \mathbb{k}) \otimes \text{Hom}(V^{\otimes p}, V^{\otimes q}))^{\mathfrak{gl}(V)} \subset \left(\bigoplus_{\substack{k \geq 2, \\ n_1, \dots, n_k \geq 2 \\ p_1, \dots, p_k \geq 0}} \bigotimes_{i=1}^k S^{p_i}(sV^* \otimes \mathcal{O}(n_i) \otimes_{\mathbb{k}S_{n_i}} V^{\otimes n_i}) \otimes (V^*)^{\otimes p} \otimes V^{\otimes q} \right)^{\mathfrak{gl}(V)}.$$

Each given bigraded component

$$C_d^{(w)}(\text{SDer}^+(\mathcal{O}(V)), \mathbb{k}) \otimes \text{Hom}(V^{\otimes p}, V^{\otimes q})$$

is still contained in the sum of several copies of $V^{\otimes N} \otimes V^{*\otimes M}$ with

$$N = w + d + q, \quad M = d + p.$$

From the First and the Second Fundamental Theorems for $\mathfrak{gl}(V)$, it still follows that to have non-zero invariants at all we must have $N = M$, and that for the “stable range” $\dim(V) \geq M = d + p$ the $\mathfrak{gl}(V)$ -invariants in this module are

isomorphic to a vector space with a basis of (equivalence classes of) some graphs obtained by assembling the LEGO pieces together. Moreover, the differential

$$\begin{aligned} \left(C_{d+1}^{(w)}(\text{SDer}^+(\mathcal{O}(V)), \mathbb{k}) \otimes \text{Hom}(V^{\otimes p}, V^{\otimes q}) \right)^{\mathfrak{gl}(V)} &\rightarrow \\ \left(C_d^{(w)}(\text{SDer}^+(\mathcal{O}(V)), \mathbb{k}) \otimes \text{Hom}(V^{\otimes p}, V^{\otimes q}) \right)^{\mathfrak{gl}(V)} & \end{aligned}$$

is still given, in the stable range $\dim(V) > d + p$, by the “usual” graph complex differential which collapses edges of the graph and computes the composition of labels of the vertices at the same time. As before, if the graph Γ is connected, it has at most one source. The only difference is that we now want to be sure that if our graph has a corolla with a loop, that corolla vanishes in the commutator quotient, which is precisely why we restricted ourselves to $\ker(d_{\text{loop}})$.

Passing to disconnected graphs corresponds to computing symmetric powers, which is precisely the coPROP completion of the wheeled bar construction viewed as a wheeled cooperad, so the Künneth formula implies the statement of the theorem (the naturality result is obvious). \square

The following result is well known, see, for example [49, Exercice 10.2.1]. It was one of the inspirations for Theorem 4.10, but we can also use that theorem to re-prove it.

Corollary 4.11. *Let A be a unital \mathbb{k} -algebra, and let $\mathfrak{sl}_n(A)$ to be the kernel of the map $\mathfrak{gl}_n(A) \rightarrow A/[A, A]$ sending f to the image of $\text{tr}(f)$ in the commutator quotient. If one defines $\mathfrak{sl}(A) = \varinjlim \mathfrak{sl}_n(A)$, we have a Hopf algebra isomorphism*

$$H_\bullet(\mathfrak{gl}(A)) \cong S^c(\overline{HC}_{\bullet-1}(A)),$$

where \overline{HC} is the reduced cyclic homology (without $HC_0(A) = A/[A, A]$).

Proof. Like in Corollary 4.2, we consider the algebra A_+ which is obtained from A by adjoining a unit (even though A is already unital), and view A_+ as an operad \mathcal{O} concentrated in arity one. According to Example 3.1, for a vector space V of dimension n , the Lie algebra $\mathfrak{sl}_n(A)$ is exactly $\text{SDer}^+(\mathcal{O}(V))$, so

$$\begin{aligned} C_\bullet(\text{SDer}^+(\mathcal{O}(V)), \mathfrak{gl}(V), \text{Hom}(V^{\otimes p}, V^{\otimes q})) &\cong \\ (C_\bullet(\text{SDer}^+(\mathcal{O}(V)), \mathbb{k}) \otimes \text{Hom}(V^{\otimes p}, V^{\otimes q}))^{\mathfrak{gl}(V)} &\cong \\ (C_\bullet(\mathfrak{sl}_n(A), \mathbb{k}) \otimes \text{Hom}(V^{\otimes p}, V^{\otimes q}))^{\mathfrak{gl}(V)}. & \end{aligned}$$

Theorem 4.10 implies that the stable limit of

$$(C_\bullet(\mathfrak{sl}_n(A), \mathbb{k}) \otimes \text{Hom}(V^{\otimes p}, V^{\otimes q}))^{\mathfrak{gl}(V)}$$

for $\dim(V) \rightarrow \infty$ is the coPROP completion of $B^\circ((A_+)^\circ)$. As we saw in the proof of Theorem 4.10, the operadic part of the latter complex is the bar construction $B(A_+)$, which is acyclic since the augmentation ideal of A_+ is a unital algebra, and the wheeled part is the loopless part of the cyclic bar complex, which precisely computes the (shifted) reduced cyclic homology. Since once again the homology is concentrated in the $(0, 0)$ -component of our wheeled cooperad, the

coPROP completion is just the cofree cocommutative coalgebra $S^c(\overline{HC}_{\bullet-1}(A))$, which essentially completes the proof. \square

Theorem 4.10 allows us to prove one general statement for which one generally has no hope if considering all derivations, see Example 4.4. To state it, recall that a weight graded Lie algebra \mathfrak{g} generated by elements of weight 1 is said to be Koszul in weight w if its homology in weight w is concentrated in homological degree w .

Corollary 4.12. *Suppose that the wheeled operad \mathcal{O}° is Koszul. If we equip the Lie algebra $\text{Der}(V)$ with the weight grading coming from the Koszul weight grading of the operad \mathcal{O} , the Lie algebra $\text{SDer}^+(\mathcal{O}(V))$ is Koszul in all weights $w < \frac{1}{3} \dim(V)$.*

Proof. Like in Corollary 4.3, one sees that if $\dim(V) > w + 2d$, then

$$H_d^{(w)}(\text{SDer}^+(\mathcal{O}(V)), \mathbb{k}) = 0 \text{ for } w \neq d.$$

Moreover, we always have $w \geq d$, so if $\dim(V) > 3w$, we automatically have $\dim(V) > w + 2d$, completing the proof. \square

Note that results of Powell [64, Th. 12.1] can be used to deduce that the stable homology of $\text{SDer}^+(\mathcal{O}(V))$ in degree one is concentrated in weight one for any binary operad \mathcal{O} .

Let us recall some examples where the homology of $\text{B}^\circ(\mathcal{O}^\circ)$ can be explicitly computed.

Example 4.13. The first two of the following results are proved in [57], and the last one is established in [60].

- (1) For $\mathcal{O} = \text{Com}$, we have

$$H_\bullet(\text{B}^\circ(\text{Com}^\circ)) \cong \text{Com}^i \oplus \overline{\text{Cyc}}(s\mathbb{1}),$$

where $\overline{\text{Cyc}}$ corresponds to removing elements of degree one,

- (2) for $\mathcal{O} = \text{Ass}$, we have

$$H_\bullet(\text{B}^\circ(\text{Ass}^\circ)) \cong \text{Ass}^i \oplus (\overline{\text{Cyc}}(s\mathbb{1}) \oplus \overline{\text{Cyc}}(s\mathbb{1}) \oplus \text{Cyc}(s\mathbb{1}) \otimes \text{Cyc}(s\mathbb{1})),$$

- (3) for $\mathcal{O} = \text{Lie}$, we have $H_\bullet(\text{B}^\circ(\text{Lie}^\circ)) \cong \text{Lie}^i$.

Two remarks are in order. First, one can note that the first two statements here bear a striking resemblance with Corollary 4.6 and Corollary 4.7 respectively, while the third one is drastically different from what Corollary 4.8 may suggest. One natural speculation here is to suggest that this relates to bad behaviour of the operadic module of Kähler differentials for $\text{Lie}^! \cong \text{Com}$, as opposed to two other examples. Second, we should warn the reader that it is not enough to require that an operad \mathcal{O} is Koszul to conclude that the wheeled operad \mathcal{O}° is Koszul: the abovementioned unpublished result of Bruinsma and Khoroshkin implies that it is not true for the operad of Poisson algebras.

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