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# Polynomial stabilization of the wave equation with a time varying delay term in the dynamical control.

Désiré Saba  
Université Joseph KI-Zerbo  
Ecole Doctorale Sciences et Technologies  
Laboratoire de Mathématiques et d'Informatique (LAMI)  
Burkina Faso\*

Gilbert Bayili  
Université Joseph KI-Zerbo  
Ecole Doctorale Sciences et Technologies  
Laboratoire de Mathématiques et d'Informatique (LAMI)  
Burkina Faso

Serge Nicaise  
Université Polytechnique Hauts-de-France  
CÉRAMATHS/DMATHS and FR CNRS 2037  
59313 Valenciennes Cedex 9, France

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## Abstract

We consider the one-dimensional wave equation with a time-varying delay term in the dynamical control. Under suitable assumptions, we show the well posedness of the problem. These results are obtained by using semi-group theory. Combining the multiplier method with a non linear integral inequality, a rational energy decay result of the system is established. A fundamental aspect of this paper is that our involved operator is time dependent, therefore the standard frequential method cannot be invoked.

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**Key words and phrases.** Dynamical control, stability, time varying delay.

## 1 Introduction

In this paper, we are interested in the effect of a time-varying delay term in boundary stabilization of the wave equation in one-dimensional case with dynamical control. Delay effects arise in many physical, electrical engineering systems or modelling of biological problems, refer [3, 8, 18, 17, 6, 14, 19] and it is well known that they can induce some unstabilities, and that sometimes, an arbitrarily small delay in the feedback can destabilise or improve the stability of the system, see [7, 1, 12].

There is a huge number of papers that deal with stability properties of the damped wave equation and many others dealing wave propagation with delay, see for instance [2, 7, 12, 17, 14, 4, 13]. Time delay, in a system or evolution equation, is a property of the system for which the response to an applied action is delayed. For some systems, the delay is assumed to be constant over the course of the evolution, but in practice this assumption may seem unrealistic. The goal of this paper is to consider the case when a varying delay term appears in a dynamical control. More precisely, on the interval  $(0, 1)$ , we look for the

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displacement  $u$ , solution of the next wave equation and the dynamical control  $\eta$  solution of the next linear differential equation:

$$u_{tt}(x, t) - u_{xx}(x, t) = 0 \quad \text{on } (0, 1) \times (0, \infty), \quad (1)$$

with boundary conditions

$$u(0, t) = 0 \quad ; \quad u_x(1, t) + \eta(t) = 0 \quad \forall t \in (0, \infty), \quad (2)$$

$$\eta_t - u_t(1, t) + \beta_1 \eta(t) + \beta_2 \eta(t - \tau(t)) = 0 \quad \forall t \in (0, \infty), \quad (3)$$

where, as usual the subindex  $t$  (resp.  $x$ ) means the time (resp. space) derivative of a function.

In (3),  $\tau(t)$  is the time varying delay that is supposed to be non-negative (and smooth enough, see below),  $\beta_1 > 0$  is a positive real number and  $\beta_2 \in \mathbb{R}$  is a real number.

In (3), the term  $\eta(t - \tau(t))$  is not defined if  $t - \tau(t) < 0$ , therefore we impose the relation

$$\eta(t - \tau(0)) = f_0(t - \tau(0)), \quad \forall t \in (0, \tau(0)). \quad (4)$$

Assuming that  $\tau'(t) < 1$ , we will always have  $\tau(t) - \tau(0) < t$  and if moreover  $t - \tau(t) < 0$ , then  $t - \tau(t)$  will be in  $(-\tau(0), 0)$ , and there exists a unique  $s_t \in (0, \tau(0))$  such that  $t - \tau(t) = s_t - \tau(0)$ . Therefore (4) allows to give a meaning to (3) for  $t - \tau(t) < 0$ .

Finally the system is closed with an initial condition on  $(u, u_t, \eta) = (u_0, u_1, \eta_0)$ .

Alltogether we obtain the system.

$$\left\{ \begin{array}{l} u_{tt}(x, t) - u_{xx}(x, t) = 0 \quad \text{on } (0, 1) \times (0, \infty), \\ u(0, t) = 0 \quad \forall t \in (0, \infty), \\ u_x(1, t) + \eta(t) = 0 \quad \forall t \in (0, \infty), \\ \eta_t - u_t(1, t) + \beta_1 \eta(t) + \beta_2 \eta(t - \tau(t)) = 0 \quad \forall t \in (0, \infty), \\ u(\cdot, 0) = u_0, \quad u_t(\cdot, 0) = u_1 \quad \text{on } (0, 1) ; \eta(0) = \eta_0, \\ \eta(t - \tau(0)) = f_0(t - \tau(0)) \quad \forall t \in (0, \tau(0)). \end{array} \right. \quad (5)$$

Notice that in [21], A. Wehbe has established an optimal polynomial decay rate in  $\frac{1}{t}$  of the system (5) without delay (namely  $\beta_2 = 0$ ). More recently in [4], assuming  $0 \leq \beta_2 < \beta_1$ , in general space dimension with constant delay, G. Bayili and all, have obtained on one hand, a rational decay of energy and have established on the other hand in [20] a polynomial stability of the system with distributive delay. From the previous results, we cannot expect to have an exponential decay of the energy of our system. Furthermore, since our operator is time-dependent, we cannot use a frequency domain approach to obtain some polynomial decay results (see for instance [5]), therefore we use an argument combining the multiplier method with a nonlinear integral inequality.

The present paper is organized as follows. In section 2, we prove the well posedness of the system (5) for any coefficients  $\beta_1 > 0$  and  $\beta_2 \in \mathbb{R}$  and with strictly positive time varying delay. Moreover for smooth enough initial data, we establish in section 3 the stability of the system with an energy decay rate of order  $\frac{1}{t}$ .

## 2 Well posedness results of the problem

We aim to show that problem (5) is well-posed. For that purpose, we use semigroup theory with the usual introduction of the auxilliary variable (see for example [12, 15, 13]):

$$w(r, t) = \eta(t - r\tau(t)), \quad r \in (0, 1). \quad (6)$$

Then the initial system (5) is equivalent to the following one

$$\left\{ \begin{array}{l} u_{tt}(x, t) - u_{xx}(x, t) = 0 \quad \text{on } (0, 1) \times (0, \infty), \\ u(0, t) = 0 \quad \forall t \in (0, \infty), \\ u_x(1, t) + \eta(t) = 0 \quad \forall t \in (0, \infty), \\ \eta_t - u_t(1, t) + \beta_1 \eta(t) + \beta_2 w(1, t) = 0 \quad \forall t \in (0, \infty), \\ u(\cdot, 0) = u_0, \quad u_t(\cdot, 0) = u_1 \quad \text{on } (0, 1); \eta(0) = \eta_0, \\ (1 - r\tau'(t))w_r + \tau(t)w_t = 0 \quad \forall (r, t) \in (0, 1) \times (0, \infty), \\ w(0, t) = \eta(t), \quad \forall t \in (0, \infty), \\ w(r, 0) = f_0(-r\tau(0)) \quad \forall r \in (0, 1). \end{array} \right. \quad (7)$$

Now, by the reduction of order method, setting  $U = (u, v, \eta, w)^T$ , the system (5) is equivalent to the following Cauchy problem

$$\left\{ \begin{array}{l} U'(t) = A(t)U(t) \quad \text{in } \mathcal{H}, \quad \forall t > 0, \\ U(0) = U_0 = (u_0, u_1, \eta_0, f_0(-\cdot\tau(0)))^T \quad \text{in } \mathcal{H}, \end{array} \right. \quad (8)$$

where

$$V = \{u \in H^1(0, 1) : u(0) = 0\},$$

and  $\mathcal{H}$  is defined by

$$\mathcal{H} = V \times L^2(0, 1) \times \mathbb{R} \times L^2(0, 1), \quad (9)$$

which is a Hilbert space with the inner product

$$\langle U, \tilde{U} \rangle_{\mathcal{H}} = \int_0^1 (u_x \tilde{u}_x + v \tilde{v}) dx + \int_0^1 w(r) \tilde{w}(r) dr + \eta \tilde{\eta}, \quad \forall U = (u, v, \eta, w)^T, \tilde{U} = (\tilde{u}, \tilde{v}, \tilde{\eta}, \tilde{w})^T \in \mathcal{H}. \quad (10)$$

In (8), the linear operator  $A(t)$  (depending on the time parameter  $t$ ) is defined by

$$A(t) \begin{pmatrix} u \\ v \\ \eta \\ w \end{pmatrix} = \begin{pmatrix} v \\ u_{xx} \\ v(1) - \beta_1 w(0) - \beta_2 w(1) \\ \frac{r\tau'(t) - 1}{\tau(t)} w_r \end{pmatrix}, \quad (11)$$

with domain  $D(A(t))$  given by

$$D(A(t)) = \left\{ (u, v, \eta, w) \in (V \cap H^2(0, 1)) \times V \times \mathbb{R} \times H^1(0, 1) : \begin{array}{l} \eta + u_x(1) = 0; \\ w(0) = \eta. \end{array} \right\}. \quad (12)$$

Let us directly notice that the domain  $D(A(t))$  is time independent, i.e.,

$$\forall t > 0, D(A(t)) = D(A(0)), \quad (13)$$

and that  $A(t)$  is well-defined provided  $\tau \in W^{1, \infty}([0, \infty))$  satisfies

$$\left\{ \begin{array}{l} \tau'(t) \leq d < 1, \quad \forall t \geq 0, \\ 0 < \tau_0 \leq \tau(t) \leq R, \quad \forall t \geq 0, \end{array} \right. \quad (14)$$

for two positive real numbers  $\tau_0, R$  and a non negative real number  $d$  such that  $d < 1$ . For future purposes, let us set

$$q = \sup_{t \geq 0} |\tau'(t)|. \quad (15)$$

A general existence theory for equations of type (8) was developed using semigroup theory in [10, 9]. Indeed, in reference to [10, 9], (precisely, Theorem 1.9 of [10] or Theorem 1.2 of [9]) if the triplet  $\{\mathcal{A}, \mathcal{H}, Y\}$  (with  $Y = D(A(0))$ ,  $\mathcal{A} = \{A(t), t \in [0, T]\}$  for some fixed  $T > 0$ ) forms a CD-system (or constant domain system), then an existence and uniqueness result follows. For completeness, let us recall this result.

**Theorem 2.1.** *Let  $T$  be a fixed real number. Let  $\mathcal{H}$  be a Hilbert space and  $A(t) : D(A(t)) \subset \mathcal{H} \rightarrow \mathcal{H}$  a linear operator satisfying the following hypotheses:*

- ( $H_1$ ) :  $D(A(t)) = D(A(0)) \forall t \in [0, T]$ ;
- ( $H_2$ ) :  $D(A(0))$  is a dense subset of  $\mathcal{H}$ ;
- ( $H_3$ ) :  $\forall t \in [0, T]$ ,  $A(t)$  generates a strongly continuous semigroup on  $\mathcal{H}$  and the family  $\mathcal{A} = \{A(t) : t \in [0, T]\}$  is stable with stability constants  $C$  and  $m$  independent of  $t$   
(i.e. the semigroup  $(S_t(s))_{s \geq 0}$  generated by  $A(t)$  satisfies  $\| \prod_{j=1}^k S_{t_j}(s_j) u \|_{\mathcal{H}} \leq C e^{ms} \| u \|_{\mathcal{H}}$ ,  
for all  $u \in \mathcal{H}, s_j \geq 0, 0 \leq t_1 \leq \dots \leq t_k \leq T, k \in \mathbb{N}$ );
- ( $H_4$ ) :  $\frac{\partial}{\partial t} A \in L_*^\infty([0, T]; B(D(A(0)), \mathcal{H}))$  which is the space of equivalent classes of essentially bounded, strongly measurable functions from  $[0, T]$  into the set  $B(D(A(0)), \mathcal{H})$  of bounded operators from  $D(A(0))$  into  $\mathcal{H}$ .

Then problem (8) has a unique solution  $U \in C([0, T]; \mathcal{H})$  for any initial datum  $U_0 \in \mathcal{H}$ .

Moreover if  $U_0 \in D(A(0))$ , then  $U \in C([0, T]; D(A(0))) \cap C^1([0, T]; \mathcal{H})$ .

Note that this Theorem also furnish a unique solution to our problem on  $(0, \infty)$  if we check that ( $H_1$ )-( $H_4$ ) are valid for all  $T > 0$ . As mentionned before the assumption ( $H_1$ ) always holds. To prove that ( $H_2$ ) and ( $H_3$ ) hold, we use a time variable-norm technique, namely for all  $t \geq 0$ , we introduce the following time dependent inner product

$$\langle U, \tilde{U} \rangle_t = \int_0^1 (u_x \tilde{u}_x + v \tilde{v}) dx + \beta_1 \tau(t) \int_0^1 w(r) \tilde{w}(r) dr + \eta \tilde{\eta}, \forall U = (u, v, \eta, w)^\top \in \mathcal{H}, \tilde{U} = (\tilde{u}, \tilde{v}, \tilde{\eta}, \tilde{w})^\top \in \mathcal{H}. \quad (16)$$

By our assumption on  $\tau$ , the norm  $\|\cdot\|_t$  induced by this new inner product is equivalent to the use one  $\|\cdot\|_{\mathcal{H}}$ <sup>1</sup> induced by (10), namely for all  $t \geq 0$ ,

$$\min\{1, \sqrt{\beta_1 \tau_0}\} \|U\|_{\mathcal{H}} \leq \|U\|_t \leq \max\{1, \sqrt{\beta_1 R}\} \|U\|_{\mathcal{H}}, \forall U \in \mathcal{H}. \quad (17)$$

For all  $t \geq 0$  let us now set

$$\varphi(t) = \frac{\sqrt{1 + \tau'^2(t)}}{2\tau(t)} + \frac{\beta_2^2}{2\beta_1(1-d)}, \quad (18)$$

and

$$K(t) = A(t) - \varphi(t)\mathbb{I}, \quad (19)$$

where  $\mathbb{I}$  means here the identity operator from  $\mathcal{H}$  into itself.

**Theorem 2.2.** *For all fixed  $t \geq 0$ , the operator  $K(t)$  is maximal dissipative in  $\mathcal{H}$  for the inner product defined by (16).*

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<sup>1</sup>i.e.,  $\|U\|_t^2 = \langle U, U \rangle_t$ , and  $\|U\|_{\mathcal{H}}^2 = \langle U, U \rangle_{\mathcal{H}}$ , for all  $U \in \mathcal{H}$

*Proof.* We first prove that for all fixed  $t \geq 0$ ,  $K(t)$  is dissipative in  $\mathcal{H}$  endowed with inner product  $\langle \cdot, \cdot \rangle_t$ . Fix  $U = (u, v, \eta, w)^\top \in D(A(0))$  (that is clearly the domain of  $K(t)$ ), then by the definition of  $A(t)$ , we have

$$\begin{aligned} \langle A(t)U, U \rangle_t &= \left\langle \begin{pmatrix} v \\ u_{xx} \\ v(1) - \beta_1 w(0) - \beta_2 w(1) \\ \frac{r\tau'(t) - 1}{\tau(t)} w_r \end{pmatrix}, \begin{pmatrix} u \\ v \\ \eta \\ w \end{pmatrix} \right\rangle_t \\ &= \int_0^1 (v_x u_x + u_{xx} v) dx + \beta_1 \int_0^1 (r\tau'(t) - 1) w w_r dr + \eta (v(1) - \beta_1 w(0) - \beta_2 w(1)). \end{aligned}$$

Integrating by parts in the first and second terms of this right-hand side, it follows

$$\langle A(t)U, U \rangle_t = [u_x v]_0^1 + \left[ \frac{\beta_1}{2} (r\tau'(t) - 1) w^2 \right]_0^1 - \frac{\beta_1}{2} \tau'(t) \int_0^1 w^2 dr + \eta (v(1) - \beta_1 w(0) - \beta_2 w(1)).$$

Since  $u_x(1) = -\eta$ ,  $v(0) = 0$ ,  $w(0) = \eta$ , we get

$$\langle A(t)U, U \rangle_t = -\frac{\beta_1}{2} \eta^2 - \beta_2 \eta w(1) + \frac{\beta_1}{2} (\tau'(t) - 1) w^2(1) - \frac{\beta_1 \tau'(t)}{2} \int_0^1 w^2 dr. \quad (20)$$

Since here we have no smallness assumption on  $\beta_2$ , we cannot absorb the second term of this right-hand side by the first and third ones, that is the reason of the use of the operator  $K(t)$ . Indeed recalling that

$$\langle K(t)U, U \rangle_t = \langle A(t)U, U \rangle_t - \varphi(t) \langle U, U \rangle_t,$$

we then have

$$\begin{aligned} \langle K(t)U, U \rangle_t &= \left( -\frac{\beta_1}{2} - \varphi(t) \right) \eta^2 - \beta_2 \eta w(1) + \frac{\beta_1}{2} (\tau'(t) - 1) w^2(1) + \\ &\quad - \frac{\beta_1}{2} \left( \sqrt{1 + \tau'^2(t)} + \tau'(t) + \frac{\beta_2^2 \tau(t)}{\beta_1(1-d)} \right) \int_0^1 w^2 dr - \varphi(t) \int_0^1 (u_x^2 + v^2) dx. \end{aligned}$$

By Young's inequality, for all  $\alpha > 0$ , we get

$$\begin{aligned} \langle K(t)U, U \rangle_t &\leq \left( -\frac{\beta_1}{2} - \varphi(t) + \frac{|\beta_2|}{2\alpha} \right) \eta^2 + \left( \alpha \frac{|\beta_2|}{2} + \frac{\beta_1}{2} (\tau'(t) - 1) \right) w^2(1) + \\ &\quad - \varphi(t) \int_0^1 (u_x^2 + v^2) dx - \frac{\beta_1}{2} \left( \sqrt{1 + \tau'^2(t)} + \tau'(t) + \frac{\beta_2^2 \tau(t)}{\beta_1(1-d)} \right) \int_0^1 w^2 dr. \\ &\leq \left( -\frac{\beta_1}{2} - \varphi(t) + \frac{|\beta_2|}{2\alpha} \right) \eta^2 + \left( \alpha \frac{|\beta_2|}{2} + \frac{\beta_1}{2} (\tau'(t) - 1) \right) w^2(1). \end{aligned}$$

Note that if  $\beta_2 = 0$  (that is no delay occurs in the system), then  $\langle K(t)U, U \rangle_t \leq 0$ . But for  $\beta_2 \neq 0$ , choosing  $\alpha = \frac{(1-d)\beta_1}{|\beta_2|}$ , we obtain

$$\langle K(t)U, U \rangle_t \leq \left( -\frac{\beta_1}{2} - \varphi(t) + \frac{\beta_2^2}{2\beta_1(1-d)} \right) \eta^2 + \frac{\beta_1}{2} (\tau'(t) - d) w^2(1).$$

But the definition (18) of  $\varphi(t)$  and the estimate  $\tau'(t) \leq d < 1$  in (14) imply that

$$\left( -\frac{\beta_1}{2} - \varphi(t) + \frac{\beta_2^2}{2\beta_1(1-d)} \right) \leq -\left( \frac{\beta_1}{2} + \frac{\sqrt{1 + \tau'^2(t)}}{2\tau(t)} \right) < 0, \quad \tau'(t) - d \leq 0,$$

therefore

$$\langle K(t)U, U \rangle_t \leq 0, \quad \forall U \in D(A(0)). \quad (21)$$

Now we prove that there exists  $\lambda > 0$  such that  $\lambda I - A(t) : D \rightarrow \mathcal{H}$  is surjective. In other words, for  $F = (f, g, h, y)^\top \in \mathcal{H}$ , we look for  $U = (u, v, \eta, w)^\top \in D(A(0))$  such that  $(\lambda I - A(t))U = F$ , or equivalently

$$\begin{cases} \lambda u - v = f, \\ \lambda v - u_{xx} = g, \\ \lambda \eta - v(1) + \beta_1 w(0) + \beta_2 w(1) = h, \\ \lambda w + \frac{1-r\tau'(t)}{\tau(t)} w_r = y, \\ w(0) = \eta, \\ u(0) = \eta + u_x(1) = 0. \end{cases} \quad (22)$$

Assuming that a solution exists, then eliminating

$$v = \lambda u - f, \quad (23)$$

by the first identity, replacing  $w(0)$  by  $\eta$ , and noticing that

$$1 - r\tau'(t) \geq 1 - d > 0, \forall t \geq 0, r \in (0, 1), \quad (24)$$

we find that  $u$  is solution of the boundary value problem

$$\begin{cases} \lambda^2 u - u_{xx} = \lambda f + g, \\ u(0) = \eta + u_x(1) = 0, \end{cases} \quad (25)$$

while  $w$  is solution of the following first order differential equation with initial condition at 0:

$$\begin{cases} \frac{\lambda\tau(t)}{1-r\tau'(t)} w + w_r = y \frac{\tau(t)}{1-r\tau'(t)}, \\ w(0) = \eta, \end{cases} \quad (26)$$

where  $\eta$  is given by

$$(\lambda + \beta_1)\eta + \beta_2 w(1) = h - f(1) + \lambda u(1). \quad (27)$$

Since  $w(1)$  depends on  $\eta$  via (26), we then first solve this differential system. As  $1 - r\tau'(t)$  is independent of  $r$  if  $\tau'(t) = 0$ , its resolution is made according to the case  $\tau'(t) = 0$  and  $\tau'(t) \neq 0$ . In both cases, by the method of variation of the constant, the solution of (26) is

$$w(r) = e^{-\lambda\tau(t)r} \left( \eta + \tau(t) \int_0^r e^{\lambda\tau(t)s} y(s) ds \right), \forall r \in (0, 1), \quad (28)$$

if  $\tau'(t) = 0$ , while

$$w(r) = \exp \left( \frac{\lambda\tau(t)}{\tau'(t)} \ln(1 - r\tau'(t)) \right) \left( \eta + \int_0^r \left[ \frac{\tau(t)y(s)}{1-s\tau'(t)} \exp \left( -\frac{\lambda\tau(t)}{\tau'(t)} \ln(1 - s\tau'(t)) \right) \right] ds \right), \forall r \in (0, 1) \quad (29)$$

if  $\tau'(t) \neq 0$ . Note that the expression (28) can be recovered from (29) with the convention that

$$\frac{\ln(1 - r\tau'(t))}{r\tau'(t)} = -1, \forall r \in (0, 1] \text{ if } \tau'(t) = 0.$$

From now on, using this convention, using (29) we find

$$w(1) = b(t)\eta + a(t), \quad (30)$$

where

$$\begin{aligned} a(t) &= b(t) \int_0^1 \left[ \frac{\tau(t)y(s)}{1 - s\tau'(t)} \exp \left( -\frac{\lambda\tau(t)}{\tau'(t)} \ln(1 - s\tau'(t)) \right) \right] ds, \\ b(t) &= \exp \left( \frac{\lambda\tau(t)}{\tau'(t)} \ln(1 - \tau'(t)) \right). \end{aligned}$$

Inserting the expression (30) in (27), we obtain

$$\eta = \frac{h + \lambda u(1) - f(1) - \beta_2 a(t)}{\lambda + \beta_1 + \beta_2 b(t)}, \quad (31)$$

as soon as  $\lambda$  is large enough so that the denominator is positive, in other words for

$$\lambda > -\beta_1 - \beta_2 b(t). \quad (32)$$

Coming back to (25) and using (31), we deduce that  $u$  is solution of

$$\begin{cases} \lambda^2 u - u_{xx} = \lambda f + g \text{ in } (0, 1), \\ u(0) = 0, \\ \frac{\lambda}{\lambda + \beta_1 + \beta_2 b(t)} u(1) + u_x(1) = -\frac{h - f(1) - \beta_2 a(t)}{\lambda + \beta_1 + \beta_2 b(t)}. \end{cases}$$

Note that the boundary condition at 1 is of Robin type and that  $a(t)$  depends only on the datum  $y$  (and not on  $u$ ). This boundary value problem has a unique solution in  $H^2(0, 1)$ , because the coefficient  $\frac{\lambda}{\lambda + \beta_1 + \beta_2 b(t)}$  is non-negative. With this solution  $u$ , we define  $v$  by (23),  $\eta$  by (31) and  $w$  by (29), that by construction satisfies (22). In other words, for  $\lambda \geq 0$  satisfying (32),  $\lambda \mathbb{I} - A(t)$  is surjective.

Finally as  $\lambda \mathbb{I} - K(t) = (\lambda + \varphi(t)) \mathbb{I} - A(t)$ , we deduce that  $\lambda \mathbb{I} - K(t)$  is also surjective if  $\lambda \geq 0$  satisfies (32) (reminding that  $\varphi(t) \geq 0$ ). This surjectivity results and (21) complete the proof of Theorem 2.2.  $\blacksquare$

Before going on, let us notice that since we are in a Hilbert setting, the above Theorem implies that  $D(K(t))$  is dense in  $\mathcal{H}$ , consequently assumption  $(H_2)$  holds for  $A(t)$  recalling that  $D(K(t)) = D(A(t)) = D(A(0))$ .

We are further ready to check that the operators  $A(t)$  satisfy the assumption  $(H_3)$  from Theorem 2.1.

**Proposition 2.1.** *For any  $T > 0$ ,  $\mathcal{A} = \{A(t) : t \in [0, T]\}$  is a stable family of  $C_0$ -semigroup generators with stability constants  $C$  and  $m > 0$  independent of  $T$ .*

*Proof.* First by Lumer-Phillips Theorem, for all  $t \geq 0$ ,  $K(t)$  is the infinitesimal generator of a semigroup of contractions on  $\mathcal{H}$ , equipped with the norm  $\|\cdot\|_t$ . This means that

$$\|e^{sK(t)}U\|_t \leq \|U\|_t, \forall U \in \mathcal{H}, s \geq 0,$$

for any  $t \geq 0$ . As  $A(t)$  is a bounded perturbation of  $K(t)$ , by a standard perturbation argument (see [16, Theorem 3.1.1]),  $A(t)$  is the infinitesimal generator of  $C_0$  semigroup on  $\mathcal{H}$ , equipped with the norm  $\|\cdot\|_t$ , with

$$\|e^{sA(t)}U\|_t \leq e^{s\varphi(t)}\|U\|_t, \forall U \in \mathcal{H}, s \geq 0,$$

for any  $t \geq 0$ . As by the assumption (14), we have

$$0 \leq \varphi(t) \leq m := \frac{\sqrt{1+q^2}}{2\tau_0} + \frac{\beta_2^2}{2\beta_1(1-d)},$$

we deduce that

$$\|e^{sA(t)}U\|_t \leq e^{ms}\|U\|_t, \forall U \in \mathcal{H}, s \geq 0,$$

for any  $t \geq 0$ . Finally in order to have the stability properties, we use the smoothness property of the inner product (16) with respect to  $t$ . Indeed for  $U = (u, v, \eta, w)^\top \in \mathcal{H}$ , we see that

$$\frac{d}{dt}\|U\|_t^2 = \beta_1 \tau'(t) \int_0^1 w(r)^2 dr,$$



hence using the assumption (14), we obtain

$$\frac{d}{dt} \|U\|_t^2 \leq \beta_1 \int_0^1 w(r)^2 dr \leq \frac{1}{\tau_0} \|U\|_t^2.$$

This estimate and Gronwall's inequality imply that

$$\|U\|_t \leq e^{\frac{1}{2\tau_0}|t-s|} \|U\|_s, \forall s, t \in [0, \infty).$$

This proves the Proposition with the help of Proposition 1.1 of [10]. ■

Finally, under the assumption

$$\tau \in W^{2,\infty}([0, T]) \quad \forall T > 0, \quad (33)$$

we readily check that the hypothesis  $(H_4)$  holds. Indeed under this assumption for  $U = (u, v, \eta, w)^\top \in D(A(0))$  we have

$$\frac{\partial A(t)}{\partial t} U = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{r\tau''(t)\tau(t) - \tau'(t)(r\tau'(t) - 1)}{\tau^2(t)} w_r \end{pmatrix}. \quad (34)$$

Since  $0 < \tau_0 \leq \tau$ , and by (33), we see that

$$S := \sup_{t \in [0, T]} \left| \frac{r\tau''(t)\tau(t) - \tau'(t)(r\tau'(t) - 1)}{\tau(t)} \right| < \infty,$$

which directly leads to

$$\left\| \frac{\partial A(t)}{\partial t} U \right\|_{\mathcal{H}} \leq S \left\| \frac{1}{\tau(t)} w_r \right\|_{L^2(0,1)} \leq \frac{S}{1-d} \|A(0)U\|_{\mathcal{H}},$$

which shows that  $\frac{\partial A(t)}{\partial t} \in L_*^\infty([0, T]; L(D(A(0)), \mathcal{H}))$ .

Owing to Theorem 2.1, we can state the following existence and uniqueness theorem.

**Theorem 2.3.** *Assume that (14) and (33) hold, then for any initial datum  $U_0 \in \mathcal{H}$ , there exists a unique solution  $U \in C([0, \infty); D(A(0)))$  of the system (8). Furthermore, if  $U_0 \in D(A(0))$ , then  $U \in C([0, \infty); D(A(0))) \cap C^1([0, \infty); \mathcal{H})$ .*

In the setting of the previous theorem, since  $w$  satisfies the transport equation

$$\begin{cases} (1 - r\tau'(t))w_r + \tau(t)w_t = 0 \quad \forall (r, t) \in (0, 1) \times (0, \infty), \\ w(0, t) = \eta(t), \quad \forall t \in (0, \infty), \\ w(r, 0) = f_0(-r\tau(0)) \quad \forall r \in (0, 1), \end{cases}$$

we have

$$w(r, t) = \begin{cases} \eta(t - r\tau(t)), & \forall t - r\tau(t) > 0, \\ f_0(t - r\tau(t)) & \forall t - r\tau(t) < 0. \end{cases}$$

Hence if we set

$$T_0 = \inf_{t \geq 0} \{t : t - \tau(t) \geq 0\}, \quad (35)$$

then

$$w(r, t) = \eta(t - r\tau(t)), \quad \forall t \geq T_0, r \in (0, 1). \quad (36)$$

Let us finally give an improved regularity result for smoother initial data.

**Theorem 2.4.** Assume (14) hold with the additional regularity

$$\tau \in W^{3,\infty}([0, T]) \quad \forall T > 0, \quad (37)$$

then for any initial datum  $U_0 \in D(A(0)^2)$ , the unique solution  $U \in C([0, \infty); D(A(0)))$  of the system (8) satisfies  $U \in C^1([0, \infty); D(A(0))) \cap C^2([0, \infty); \mathcal{H})$ .

*Proof.* We apply Theorem 2.13 of [10] with  $n = 2$  and  $X_j = Y_j = D(A(0))$ . In such a case, the assumption (P1) is satisfied due to Proposition 2.1, (P2) follows from the assumption (37), while (P3) is here automatically satisfied. Finally, we notice that  $D_2(0) = D(A(0)^2)$ . Indeed by section 2.4 of [10], we have  $D_1(t) = D(A(0))$  and

$$D_2(t) = \{ \phi \in D_1(t) : S^k(t)\phi \in D(A(0)), \forall 0 \leq k \leq 1 \},$$

where  $S^0(t) = \mathbb{I}$ , and  $S^1(t) = A(t)$ . Therefore

$$D_2(t) = \{ \phi \in D_1(t) : A(t)\phi \in D(A(0)) \},$$

which gives  $D_2(0) = D(A(0)^2)$ . ■

From the previous theorem, if  $U_0 \in D(A(0)^2)$ , we deduce that  $U = (u, u_t, \eta, w)$  satisfies

$$u \in C^3([0, \infty), L^2(0, 1)), u_t \in C^1([0, \infty), H^2(0, 1)), \quad \eta \in C^2([0, \infty)).$$

In particular, it holds

$$u_{ttt} = u_{xxt} \text{ in } L^2(0, 1), \forall t \geq 0.$$

### 3 Rational decay of Energy

We have shown in the previous section that the system (5) is well posed for any  $\beta_1 > 0, \beta_2 \in \mathbb{R}$  under the assumptions (14) and (33). Here, we aim to establish a rational energy decay result of the system under the additional assumption

$$|\beta_2| < \rho\beta_1. \quad (38)$$

where

$$\rho = \min \left\{ \sqrt{1-d} ; \frac{1}{\sqrt{1+q}} \right\}$$

with  $d \geq 0$  from the assumption (14) and  $q$  was defined in (15).

Let  $U$  be the unique solution of the system (5) with an initial datum  $U_0 \in \mathcal{H}$ , then we define its associated energy  $E(t)$  by

$$E(t) = \frac{1}{2} \|U\|_t^2 = \frac{1}{2} \left\{ \int_0^1 (u_x^2 + u_t^2) dx + \beta_1 \tau(t) \int_0^1 \eta^2(t - r\tau(t)) dr + \eta^2(t) \right\}, \quad \forall t \geq 0. \quad (39)$$

If moreover  $U_0 \in D(A(0))$ , then we can define its energy  $E_1(t)$  of higher order for  $t \geq T_0$  by

$$E_1(t) = \frac{1}{2} \left\{ \int_0^1 (u_{tx}^2 + u_{tt}^2) dx + \beta_1 \tau(t) \int_0^1 \eta_t^2(t - r\tau(t)) dr + \eta_t^2 \right\}, \quad \forall t \geq T_0. \quad (40)$$

Note these two energies are well-defined owing to Theorem 2.3 due to (24) and (36).

#### 3.1 Energy decay

**Theorem 3.1.** Assuming that (14) and (38) hold, then for any  $U_0 \in D(A(0))$ , the energy  $E$  of the unique solution  $U \in C([0, \infty); D(A(0))) \cap C^1([0, \infty); \mathcal{H})$  of (5) is nonincreasing and there exists  $C_1 > 0$  such that

$$\frac{d}{dt} E(t) \leq -C_1 (\eta^2(t) + \eta^2(t - \tau(t))) \leq 0, \quad \forall t \geq 0. \quad (41)$$

If additionally we assume that (37) holds, then the energy  $E_1$  is nonincreasing and there exists  $C_2 > 0$  such that

$$\frac{d}{dt} E_1(t) \leq -C_2 (\eta_t^2 + \eta_t^2(t - \tau(t))) \leq 0, \quad \forall t \geq T_0. \quad (42)$$

*Proof.* Let us start with the first assertion. By its definition, we have

$$\frac{d}{dt}E(t) = \langle U_t, U \rangle_t + \frac{\beta_1 \tau'(t)}{2} \int_0^1 w(r, t)^2 dr = \langle A(t)U, U \rangle_t + \frac{\beta_1 \tau'(t)}{2} \int_0^1 w(r, t)^2 dr$$

where  $U = (u, u_t, \eta, w)^\top$ . Due to (20), we then have

$$\frac{d}{dt}E(t) = -\frac{\beta_1}{2}\eta^2(t) - \beta_2\eta(t)\eta(t - \tau(t)) + \frac{\beta_1}{2}(\tau'(t) - 1)\eta^2(t - \tau(t)).$$

Young's inequality then yields

$$\begin{aligned} \frac{d}{dt}E(t) &\leq \left(-\frac{\beta_1}{2} + \frac{|\beta_2|}{2\alpha}\right)\eta^2(t) + \left(\frac{\alpha|\beta_2|}{2} + \frac{\beta_1}{2}(d-1)\right)\eta^2(t - \tau(t)) \quad \forall \alpha > 0. \\ &\leq \left(-\frac{\beta_1}{2} + \frac{|\beta_2|}{2\sqrt{1-d}}\right)\eta^2(t) + \left(\frac{\sqrt{1-d}|\beta_2|}{2} + \frac{\beta_1}{2}(d-1)\right)\eta^2(t - \tau(t)) \quad \text{for } \alpha = \sqrt{1-d} \\ &\leq \left(\frac{|\beta_2| - \beta_1\sqrt{1-d}}{2\sqrt{1-d}}\right)\eta^2(t) + \sqrt{1-d}\left(\frac{|\beta_2| - \beta_1\sqrt{1-d}}{2}\right)\eta^2(t - \tau(t)) \end{aligned}$$

Thanks to the hypothesis (38),  $\frac{|\beta_2| - \beta_1\sqrt{1-d}}{2} < 0$  and taking  $C_1 = -\sqrt{1-d}\left(\frac{|\beta_2| - \beta_1\sqrt{1-d}}{2}\right)$  we get (41). On the other hand for  $U_0 \in D(A(0)^2)$ , using the additional regularity  $U \in C^1([0, \infty); D(A(0))) \cap C^2([0, \infty); \mathcal{H})$  from Theorem 2.4, using the same argument than in the proof of Theorem 2.2 and noticing that  $\eta_t(t - r\tau(t))$  satisfies the same transport equation than  $\eta(t - r\tau(t))$ , we have for any  $t \geq T_0$

$$\begin{aligned} \frac{d}{dt}E_1(t) &= \int_0^1 (u_{ttx}u_{tx} + u_{ttt}u_{tt}) dx + \eta_t\eta_{tt} + \beta_1\tau(t) \int_0^1 \eta_t(t - r\tau(t))\eta_{tt}(t - r\tau(t))(1 - r\tau'(t))dr + \\ &\quad \frac{\beta_1\tau'(t)}{2} \int_0^1 \eta_t^2(t - r\tau(t)) dr. \\ &= \int_0^1 (u_{ttx}u_{tx} + u_{txx}u_{tt}) dx + \eta_t\eta_{tt} - \beta_1 \int_0^1 \eta_t(t - r\tau(t)) \frac{\partial}{\partial r}(\eta_t(t - r\tau(t)))(1 - r\tau'(t))dr + \\ &\quad \frac{\beta_1\tau'(t)}{2} \int_0^1 \eta_t^2(t - r\tau(t)) dr, \text{ because } u_{ttt} = u_{txx} \\ &= [u_{tt}u_{tx}]_0^1 + \eta_t\eta_{tt} - \left[\frac{\beta_1}{2}(1 - r\tau'(t))\eta_t^2(t - r\tau(t))\right]_0^1. \\ &= u_{tt}(1, t)u_{tx}(1, t) + \eta_t\eta_{tt} + \frac{\beta_1}{2}\eta_t^2 + \frac{\beta_1}{2}(\tau'(t) - 1)\eta_t^2(t - \tau(t)). \\ &= -\frac{\beta_1}{2}\eta_t^2 - \beta_2(1 - \tau'(t))\eta_t\eta_t(t - \tau(t)) + \frac{\beta_1}{2}(\tau'(t) - 1)\eta_t^2(t - \tau(t)). \end{aligned}$$

Again thanks to Young's inequality, we obtain

$$\begin{aligned} \frac{d}{dt}E_1(t) &= -\frac{\beta_1}{2}\eta_t^2 - \beta_2(1 - \tau'(t))\eta_t\eta_t(t - \tau(t)) + \frac{\beta_1}{2}(\tau'(t) - 1)\eta_t^2(t - \tau(t)). \\ &\leq \left(-\frac{\beta_1}{2} + \frac{|\beta_2|(1 - \tau'(t))}{2\alpha}\right)\eta_t^2 + (1 - \tau'(t))\left(\frac{\alpha|\beta_2|}{2} - \frac{\beta_1}{2}\right)\eta_t^2(t - \tau(t)) \quad \forall \alpha > 0. \\ &\leq \left(-\frac{\beta_1}{2} + \frac{|\beta_2|(1+q)}{2\alpha}\right)\eta_t^2 + (1 - \tau'(t))\left(\frac{\alpha|\beta_2|}{2} - \frac{\beta_1}{2}\right)\eta_t^2(t - \tau(t)) \quad \forall \alpha > 0. \\ &\leq \left(-\frac{\beta_1}{2} + \frac{|\beta_2|\sqrt{1+q}}{2}\right)\eta_t^2 + (1 - \tau'(t))\left(\frac{|\beta_2|\sqrt{1+q}}{2} - \frac{\beta_1}{2}\right)\eta_t^2(t - \tau(t)), \end{aligned}$$

for  $\alpha = \sqrt{1+q}$ . Thanks to (38),  $-\frac{\beta_1}{2} + \frac{|\beta_2|\sqrt{1+q}}{2} < 0$ , hence, by (24), taking

$$C_2 = \left(\frac{\beta_1}{2} - \frac{|\beta_2|\sqrt{1+q}}{2}\right)(1-d), \text{ we obtain (42).} \quad \blacksquare$$

### 3.2 Polynomial stability of the system

In order to establish the rational decay result of the energy, we first prove the following lemmas.

**Lemma 3.1.** *Assume that (14) and (33) hold. With  $T_0 \geq 0$  defined by (35), we have*

$$\int_S^T \int_0^1 \eta^2(t - r\tau(t)) dr dt \leq \frac{E(S - \tau(S))}{(1-d)C_1}, \forall T_0 \leq S \leq T. \quad (43)$$

*Proof.* Fix  $S$  and  $T$  such that  $T_0 \leq S \leq T$ . By Fubini's theorem, we may write

$$\int_S^T \int_0^1 (1 - r\tau'(t)) \eta^2(t - r\tau(t)) dr dt = \int_0^1 \int_S^T (1 - r\tau'(t)) \eta^2(t - r\tau(t)) dt dr.$$

By the change of variable  $s = t - r\tau(t)$  it holds

$$\int_0^1 \int_S^T (1 - r\tau'(t)) \eta^2(t - r\tau(t)) dt dr = \int_0^1 \int_{S-r\tau(S)}^{T-r\tau(T)} \eta^2(s) ds dr. \quad (44)$$

Thanks to Theorem 3.1, we have  $\eta^2(t) \leq -\frac{E'(t)}{C_1}$  for all  $t \geq 0$ . Therefore,

$$\int_{S-r\tau(S)}^{T-r\tau(T)} \eta^2(s) ds \leq - \int_{S-r\tau(S)}^{T-r\tau(T)} \frac{E'(s)}{C_1} ds \leq \frac{E(S - r\tau(S))}{C_1} \leq \frac{E(S - \tau(S))}{C_1}. \quad (45)$$

Since by (24), we have

$$(1-d) \int_0^1 \int_S^T \eta^2(t - r\tau(t)) dt dr \leq \int_0^1 \int_S^T (1 - r\tau'(t)) \eta^2(t - r\tau(t)) dt dr,$$

we directly get (43). ■

**Lemma 3.2.** *Assuming that (14) and (37) hold. Then*

$$\int_S^T E(t) dt \leq \left(2 + \frac{2 + 3\beta_1^2 + 3\beta_2^2}{2C_1}\right) E(S) + \frac{3}{2C_2} E_1(S) + \frac{\beta_1 R}{2(1-d)C_1} E(S - \tau(S)), \forall T_0 \leq S < T. \quad (46)$$

*Proof.* Integrating by parts, we have

$$\begin{aligned} \int_0^1 (u_x^2 + u_t^2) dx &= [x(u_x^2 + u_t^2)]_0^1 - 2 \int_0^1 x(u_x u_{xx} + u_t u_{tx}) dx \\ &= u_x^2(1, t) + u_t^2(1, t) - 2 \int_0^1 x(u_x u_{tt} + u_t u_{tx}) dx \\ &= u_x^2(1, t) + u_t^2(1, t) - 2 \frac{d}{dt} \left\{ \int_0^1 x u_x u_t dx \right\}. \end{aligned} \quad (47)$$

The boundary conditions give

$$u_x^2(1, t) + u_t^2(1, t) = \eta^2(t) + (\eta_t(t) + \beta_1 \eta(t) + \beta_2 \eta(t - \tau(t)))^2 \leq (1 + 3\beta_1^2) \eta^2(t) + 3\beta_2^2 \eta^2(t - \tau(t)) + 3\eta_t^2(t). \quad (48)$$

From (47) and (48), it follows

$$\int_0^1 (u_x^2 + u_t^2) dx \leq (1 + 3\beta_1^2) \eta^2(t) + 3\beta_2^2 \eta^2(t - \tau(t)) + 3\eta_t^2(t) - 2 \frac{d}{dt} \left\{ \int_0^1 x u_x u_t dx \right\}.$$

This estimate and the definition of the energy  $E(t)$  (and recalling that  $\tau(t) \leq R$ ) yield

$$E(t) \leq \left( \frac{2 + 3\beta_1^2 + 3\beta_2^2}{2} \right) (\eta^2 + \eta^2(t - \tau(t))) + \frac{3}{2} \eta_t^2 - \frac{d}{dt} \left\{ \int_0^1 x u_x u_t dx \right\} + \frac{\beta_1 R}{2} \int_0^1 \eta^2(t - r\tau(t)) dr.$$

Thanks to Theorem 3.1, it follows that

$$E(t) \leq - \left( \frac{2 + 3\beta_1^2 + 3\beta_2^2}{2C_1} \right) E'(t) - \frac{3}{2C_2} E_1'(t) - \frac{d}{dt} \left\{ \int_0^1 x u_x u_t dx \right\} + \frac{\beta_1 R}{2} \int_0^1 \eta^2(t - r\tau(t)) dr, \forall t \geq T_0. \quad (49)$$

Integrating the left-hand side between  $S$  and  $T$ , with  $T_0 \leq S < T$ , we get

$$\begin{aligned} \int_S^T E(t) dt &\leq \left( \frac{2 + 3\beta_1^2 + 3\beta_2^2}{2C_1} \right) E(S) + \frac{3}{2C_2} E_1(S) + \frac{\beta_1 R}{2} \int_S^T \int_0^1 \eta^2(t - r\tau(t)) dr dt + \\ &\quad \int_0^1 x u_x(x, S) u_t(x, S) dx - \int_0^1 x u_x(x, T) u_t(x, T) dx. \end{aligned} \quad (50)$$

By Young's inequality and thanks to the non increasing property of the energy  $E$ , we obtain

$$\int_0^1 x u_x(x, S) u_t(x, S) dx - \int_0^1 x u_x(x, T) u_t(x, T) dx \leq 2E(S). \quad (51)$$

This estimate and (43) in (50) imply that (46) holds.  $\blacksquare$

We conclude this section by the following rational energy decay result of the initial value problem.

**Theorem 3.2.** Assume that (14) and (37), and let  $U \in C([0, \infty); D(A(0))) \cap C^1([0, \infty); \mathcal{H})$  be the unique solution of (5) with  $U_0 \in D(A(0))$  different from 0. Then there exists a positive constant  $M(E(0), E_1(T_0))$  depending on  $E(0)$  and  $E_1(T_0)$  such that

$$E(t) \leq E(0) \frac{2M}{t + M}, \quad \forall t \geq 0. \quad (52)$$

*Proof.* In a first step, we assume that  $U_0 \in D(A(0)^2)$  (and is different from 0). Then thanks to Lemma 3.2 and to the decrease of the energy  $E$ , for all  $T_0 \leq S < T$  we have

$$\begin{aligned} \int_S^T E^2(t) dt &\leq E(S) \int_S^T E(t) dt \\ &\leq \left( 2 + \frac{2 + 3\beta_1^2 + 3\beta_2^2}{2C_1} \right) E^2(S) + \frac{3}{2C_2} E(S) E_1(S) + \frac{\beta_1 R}{2(1-d)C_1} E(S) E(S - \tau(S)). \end{aligned} \quad (53)$$

Again with the decrease of the energies  $E$  and  $E_1$ , it follows that

$$\int_S^T E^2(t) dt \leq \left( 2 + \frac{2 + 3\beta_1^2 + 3\beta_2^2}{2C_1} \right) E(S) E(0) + \frac{3}{2C_2} E(S) E_1(T_0) + \frac{\beta_1 k R}{2} E(S) E(0), \quad \text{for all } T_0 \leq S < T.$$

By letting  $T$  tend to infinity in this estimate, we get

$$\int_S^\infty E^2(t) dt \leq m E(S) E(0), \quad \forall S \geq T_0, \quad (54)$$

with

$$m = 2 + \frac{(2 + 3\beta_1^2 + 3\beta_2^2) C_2 + 3 \frac{E_1(T_0)}{E(0)} C_1}{2C_1 C_2} + \frac{\beta_1 R}{2(1-d)C_1}. \quad (55)$$

Taking  $M = \max\{m; T_0\}$ , the estimate (54) implies that

$$\int_S^\infty E^2(t) dt \leq M E(S) E(0), \quad \forall S \geq M. \quad (56)$$

In application of a classical result of Komornik [11, Theorem 9.1], we conclude from (56) that (52) holds.

For  $U_0 \in D(A(0))$  different from 0, we use a density argument. Namely, as  $D(A(0)^2)$  is dense in  $D(A(0))$ , there exists a sequence of  $U_0^{(n)} \in D(A(0)^2)$  (that, without loss of generality, we can suppose to be different from 0) such that

$$U_0^{(n)} \rightarrow U_0 \text{ in } D(A(0)).$$

Therefore if we denote by  $U^{(n)}$  the solution of (5) with initial datum  $U_0^{(n)}$ , it holds

$$U^{(n)}(t) \rightarrow U(t) \text{ in } D(A(0)), \forall t \geq 0.$$

This implies that the energy  $E^{(n)}(t)$  (resp.  $E_1^{(n)}(t)$ ) associated with  $U^{(n)}(t)$  converges to  $E^{(n)}(t)$ , for all  $t \geq 0$  (resp.  $E_1^{(n)}(t)$ , for all  $t \geq T_0$ ). As the estimate (52) is valid for  $U^{(n)}$ , namely

$$E^{(n)}(t) \leq E^{(n)}(0) \frac{2M^{(n)}}{t + M^{(n)}}, \quad \forall t \geq 0.$$

with  $M^{(n)} = \max\{m^{(n)}; T_0\}$ , and  $m^{(n)}$  given by (55) replacing  $E(0)$  (resp.  $E_1(T_0)$ ) by  $E^{(n)}(0)$  (resp. by  $E_1^{(n)}(T_0)$ ), we can pass to the limit in  $n$ , and conclude that (52) remains valid. ■

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