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# Optimal control of the temperature by the laser path and the thermal treatment time in selective laser melting process

Hiba HMEDE\*, Serge NICAISE†, Luc PAQUET‡, Ali WEHBE§

## Abstract

Additive manufacturing by laser fusion on metal oxides powder bed such as e.g. alumina ( $Al_2O_3$ ) or aluminium titanate ( $Al_2TiO_5$ ) has developed considerably in the last few years and allows today the production of a wide range of complex objects. The mathematical problem considered is to control the temperature inside some part  $\Omega$  of a powder layer. This phenomenon is governed by a parabolic initial boundary value problem with a heat source corresponding to the laser trajectory on some part of the boundary  $\partial\Omega$ . The main questions concern the optimization of the trajectories scanned by the laser on the boundary  $\partial\Omega$  according to given criteria: imposing that during the thermal process the temperature reaches a melting value in the structure to be built, a desired temperature distribution at the end of the thermal process, minimizing the thermal gradients, all this in the shortest possible thermal treatment time. To achieve this goal, we start by proving the existence of an optimal control, followed by first order necessary optimality conditions. Finally, we establish a second order sufficient optimality condition.

**Keywords.** Optimal control problems, parabolic equations, heat equations with moving source, trajectories, time of thermal treatment, cost functionals, existence of an optimal control, adjoint problem, first order necessary optimality conditions, second order sufficient optimality conditions.

## 1 Introduction

Additive manufacturing is a method used to produce 3D objects from metal oxides powder, layer by layer, by applying a Gaussian laser beam to each slice as a heat source to glue together the powder. However, the main problem is that repeated and rapid heating and cooling cycles of the metal oxide powder during this process cause high temperature gradients that contribute to inner stresses and sometimes cracks in the fabricated part.

Our purpose in this current article is to study a laser path optimization model aiming to design laser paths such that during the thermal process the temperature reaches a melting value everywhere in the structure to be built. We also seek to perform the thermal process with a minimum of thermal gradients and temperature distribution adjustment time. Related problems but with fewer requirements have been considered in [1, 2, 5]. Requirement about the shortest treatment time has been considered in the context of tumor treatment by cytotoxic drugs in [3].

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The necessity of a preheating phase for a ceramic powder during the process of SLM has been motivated by [17] where the importance of this first phase is highlighted to reduce cracks and achieve higher density, inspiring the formulation of an appropriate cost functional, see (1.1) below.

In practical applications, the powder bed layer  $\Omega = \Gamma_1 \times (-h, 0)$ , where  $h > 0$  is its thickness and  $\Gamma_1 \subset \mathbb{R}^2$  is its upper surface. The industrial goal is to solidify  $\Gamma_S \times (-h, 0)$ , by using a laser path  $\gamma : t \in [0, T] \rightarrow \gamma(t) \in \Gamma_S$ , where  $\Gamma_S$  is a closed subset of  $\Gamma_1$  and  $T > 0$  is a fixed maximal global time for the laser thermal process. We aim to reach all these purposes by using the following control optimization problem described below.

In mathematical sense, we are interested in proving the existence of optimal controls and give necessary first order and sufficient second order optimality conditions for the following optimal control problem with a parabolic PDE constraint on the state variable  $y$  and a box of constraints on the control variable  $\gamma$  defined by:

(OCP) Minimize the cost functional  $J_r : W(0, T) \times H^1(0, T; \mathbb{R}^2) \times [3r, T] \mapsto \mathbb{R}$  defined by

$$J_r(y, \gamma, \tau) = \frac{\lambda_Q}{2} \frac{1}{r} \int_{\frac{\tau}{2} - \frac{r}{2}}^{\frac{\tau}{2} + \frac{r}{2}} \int_{\Omega} |y(x, t) - y_Q(x, t)|^2 dx dt + \frac{\lambda_{\Omega}}{2} \frac{1}{r} \int_{\tau-r}^{\tau} \int_{\Omega} |y(x, t) - y_{\Omega}(x, t)|^2 dx dt + \frac{1}{2} \int_0^T \int_{\Omega} |\nabla y(x, t)|^2 dx dt + \frac{\lambda_{\gamma}}{2} \|\gamma\|_{H^1(0, T; \mathbb{R}^2)}^2 + \beta_T \tau \quad (1.1)$$

where  $r \in ]0, \frac{T}{3}[$  is fixed,  $\lambda_Q, \lambda_{\Omega}$  are positive constants and  $\lambda_{\gamma}, \beta_T$  are non-negative constants (thus allowed to be 0). subject to:

- the parabolic linear initial boundary value problem with Robin boundary conditions:

$$\begin{cases} \rho c \partial_t y - \kappa \Delta y = 0 & \text{in } Q = \Omega \times ]0, T[, \\ -\kappa \partial_{\nu} y = h y - g_{\gamma} & \text{on } \Sigma_1 = \Gamma_1 \times ]0, T[, \\ -\kappa \partial_{\nu} y = h y & \text{on } \Sigma_2 = \Gamma_2 \times ]0, T[, \\ -\kappa \partial_{\nu} y = h(y - y_B) & \text{on } \Sigma_3 = \Gamma_3 \times ]0, T[, \\ y(x, 0) = y_0(x) & \text{for } x \in \Omega, \end{cases} \quad (1.2)$$

- $\gamma$  to belong to the set of admissible trajectories  $U_{ad}$ , where

$$U_{ad} := \{\gamma \in H^1(0, T; \mathbb{R}^2); R(\gamma) \subset \Gamma_S, |\gamma'(t)| \leq c_{ad} \text{ for a.e. } t \in [0, T]\}, \quad (1.3)$$

and  $\Gamma_S$  is a closed subset of  $\Gamma_1$ . This set takes into account the constraint on the laser path range:  $R(\gamma) := \gamma([0, T]) \subset \Gamma_S$ . This constraint physically describes that  $\gamma$  must ensure that the region outside  $\Gamma_S$  is not affected by the fusion process. Here,  $\Gamma_3$  is the bottom of the powder layer and  $\Gamma_2$  its lateral surface. As well as the other constraint,  $c_{ad} > 0$  is the given fixed upper bound on the velocity of the trajectories. In addition, the positive constants  $\rho, c, \kappa$  and  $h$  represent respectively the mass density, the heat capacity, the thermal conductivity and the heat transfer coefficient. Also,  $y_0(x)$  denotes the initial temperature at  $x \in \Omega$ ,  $y_B$  represents the temperature of the built platform or of the upper surface of the previous layer and  $g_{\gamma}$  denotes the heat source on  $\Gamma_1$  given by

$$g_{\gamma}(x, t) = \alpha \frac{2P}{\pi R^2} \exp\left(-2 \frac{|x - \gamma(t)|^2}{R^2}\right), \quad \text{for all } (x, t) \in \Sigma_1 := \Gamma_1 \times ]0, T[, \quad (1.4)$$

where  $P$  is the laser power,  $R$  is the radius of the laser spot,  $\alpha$  the absorbance of the powder.

Given a fixed  $r \in ]0, \frac{T}{3}[$ , our goal is to find, using the theory of optimization with parabolic partial differential equations constraints [24, Chapter 3], an optimal trajectory  $\gamma$  inducing in  $\Omega$  the given target temperature distribution  $y_Q$  in the mid time interval  $]\frac{\tau}{2} - \frac{r}{2}, \frac{\tau}{2} + \frac{r}{2}[$  of length  $r$  and also the given target temperature distribution  $y_{\Omega}$  in the disjoint end time interval  $]\tau - r, \tau[$  of length  $r$  as well, minimizing the thermal gradients in the whole time interval  $[0, T]$  and the minimized time  $\tau \in [3r, T]$ . In other words, we look for  $(\gamma, \tau)$  in

$$W_{ad} := U_{ad} \times [3r, T] = \{(\gamma, \tau); \gamma \in U_{ad} \text{ and } \tau \in [3r, T]\}, \quad (1.5)$$

solution of

$$\min_{(\gamma, \tau) \in W_{ad}} J_r(y(\gamma), \gamma, \tau) \quad (1.6)$$

where  $y(\gamma) \in W(0, T)$  denotes the temperature distribution, weak solution of the initial boundary value problem (1.2) with heat source  $g_\gamma$  given by (1.4).

**Remark 1.1.** Let us observe that our cost functional (1.1)  $J_r$  is different from that one studied in [1]. One possible interpretation of our actual cost functional is the following. The first two terms in the r.h.s. of (1.1) correspond to a two steps heating process, the first one being interpreted as a preheating of time length  $r$  around the time  $\frac{\tau}{2}$  by choosing  $y_Q$  less than the melting temperature in  $\Gamma_S \times (-h, 0)$  and the second as the fusion one of time length  $r$  before time  $\tau$  by choosing  $y_\Omega$  larger than the melting temperature in  $\Gamma_S \times (-h, 0)$ , so that the temperature adjustment process is finished at time  $\tau$ . In this context, choosing  $\lambda_\Omega$  larger than  $\lambda_Q$  would seem a reasonable choice in order to guarantee that the powder inside  $\Gamma_S \times (-h, 0)$  has been fused together.

**Remark 1.2.** In the definition of the cost functional, we can replace the term  $\frac{\lambda_\gamma}{2} \|\gamma\|_{H^1(0, T; \mathbb{R}^2)}^2$  by  $\frac{\lambda_\gamma}{2} \|\gamma - \gamma_d\|_{H^1(0, T; \mathbb{R}^2)}^2$ , where  $\gamma_d$  is an estimation of the trajectory given by the engineer based on his physical intuition. During discretization, it would be more likely to converge.

## 1.1 General Assumptions

In its full generality, we now assume  $\Omega \subset \mathbb{R}^3$  to be a bounded lipschitz domain with boundary  $\Gamma$  decomposed into  $\Gamma_i, i = 1, 2, 3$ , disjoint open subsets of  $\Gamma$  such that  $\bigcup_{i=1}^3 \bar{\Gamma}_i = \Gamma$ , and let  $T > 0$  a fixed final time. We assume that  $\rho, c, \kappa$  and  $h$  are all positive constants with respect to  $(x, t)$ . We consider that the initial temperature  $y_0 \in L^2(\Omega)$ ,  $y_Q, y_\Omega$  and  $y_B$  are functions of  $(x, t)$ . We assume  $y_Q, y_\Omega \in L^2(Q)$  and that  $y_B$  the distribution of temperatures on the top of the previous layer or on the built platform belongs to  $L^2(\Sigma_3)$ . In (1.3), we assume that  $\Gamma_S \neq \emptyset$ .

**Remark 1.3.** In (1.2), in the Robin boundary conditions has been implicitly assumed that the ambient temperature is supposed to be constant  $y_a = 0$  (the temperature of the exterior of  $\Omega$ ). It is always possible to reduce the problem to that setting by considering as new unknown  $y - y_a$ .

## 2 Existence and Uniqueness of a Weak Solution to (1.2)

For further purposes, we define

$$W(0, T) := \{u \in L^2(0, T; H^1(\Omega)); \frac{du}{dt} \in L^2(0, T; H^1(\Omega)^*)\}. \quad (2.1)$$

Firstly, let us define the variational formulation of Problem (1.2). Multiplying the first equation of (1.2) by  $\xi \in H^1(\Omega)$ , formally integrating by parts over  $\Omega$  by Green's formula and using the boundary conditions in (1.2), this suggests to define the variational problem associated to our initial boundary value problem (1.2) as follows:

**Definition 2.1.** Let  $y \in W(0, T)$ . We shall say that  $y$  is a weak solution of Problem (1.2), if and only if for all  $\xi \in H^1(\Omega)$  :

$$\rho c < \frac{dy}{dt}, \xi >_{H^1(\Omega)^*, H^1(\Omega)} + \kappa \int_{\Omega} \nabla y \cdot \nabla \xi dx + h \int_{\Gamma} y \cdot \xi dS(x) = \int_{\Gamma_1} g_\gamma \cdot \xi dS(x) + h \int_{\Gamma_3} y_B \cdot \xi dS(x), \quad (2.2)$$

in  $L^2([0, T])$ , i.e. (2.2) is true for a.e.  $t \in [0, T]$ .

We can then say that  $y$  is a weak solution of Problem (1.2) with initial condition  $y(0) = y_0 \in L^2(\Omega)$ , which has sense as  $W(0, T) \hookrightarrow C([0, T]; L^2(\Omega))$ <sup>1</sup>

**Proposition 2.2.** Under our general assumptions, for every given initial datum  $y_0 \in L^2(\Omega)$ , Problem (1.2) has a unique weak solution with initial condition  $y(0) = y_0$ .

*Proof.* The existence and uniqueness of a weak solution to (1.2) with initial condition  $y(0) = y_0 \in L^2(\Omega)$ , follows easily from the general results for linear initial boundary value problems for parabolic equations [8, Theorem 11.7 p.192] or [10, pp.512-513] applied to the variational formulation (2.2).  $\square$

<sup>1</sup>Here and below,  $X \hookrightarrow Y$  means that  $X$  is continuously embedded into  $Y$ .

### 3 Existence of an Optimal Control

This section is devoted to prove the existence of at least one optimal control to (1.6), where the source term  $\gamma$  and the final adjustment time  $\tau$  will act as controls.

**Proposition 3.1.**  $U_{ad}$  is a weakly closed subset of  $H^1(0, T; \mathbb{R}^2)$ .

*Proof.* Essentially based on Mazur's theorem, see [1], [2] for further details.  $\square$

**Definition 3.2.** By the control to state mapping, we mean the operator associated with the state equation (1.2)

$$G : U_{ad} \rightarrow W(0, T) : \gamma \mapsto G(\gamma) = y(\gamma), \quad (3.1)$$

where  $y(\gamma)$  denotes the weak solution of the initial boundary value problem (1.2).

**Proposition 3.3.** The control to state operator  $G : U_{ad} \rightarrow W(0, T)$  is strongly continuous.

*Proof.* The continuity follows from the compact embedding of  $H^1(0, T; \mathbb{R}^2)$  into  $C([0, T]; \mathbb{R}^2)$  and from [24, Theorem 3.13, p.150].  $\square$

Concerning the weak topologies, we have the following

**Proposition 3.4.** The control-to-state mapping  $G : U_{ad} \rightarrow W(0, T)$  is weakly sequentially continuous.

*Proof.* Consider a sequence  $(\gamma_n)_{n \in \mathbb{N}}$  in  $U_{ad}$  weakly convergent to  $\gamma$  in  $H^1(0, T; \mathbb{R}^2)$ . We prove that  $(G(\gamma_n))_{n \in \mathbb{N}}$  is weakly convergent to  $G(\gamma)$  in  $W(0, T)$  by using the Lions-Aubin Compacity Theorem. See [1, 2] for the details.  $\square$

**Definition 3.5.** By the reduced cost functional, we mean the mapping:

$$\hat{J}_r : U_{ad} \times [3r, T] \rightarrow \mathbb{R} : (\gamma, \tau) \mapsto \hat{J}_r(\gamma, \tau) = J_r(G(\gamma), \gamma, \tau),$$

where

$$\begin{aligned} J_r(G(\gamma), \gamma, \tau) &= \frac{\lambda_Q}{2} \frac{1}{r} \int_{\frac{\tau}{2} - \frac{\tau}{2}}^{\frac{\tau}{2} + \frac{\tau}{2}} \int_{\Omega} |G(\gamma)(x, t) - y_Q(x, t)|^2 dx dt + \frac{\lambda_{\Omega}}{2} \frac{1}{r} \int_{\tau-r}^{\tau} \int_{\Omega} |G(\gamma)(x, t) - y_{\Omega}(x, t)|^2 dx dt \\ &\quad + \frac{1}{2} \int_0^T \int_{\Omega} |\nabla G(\gamma)(x, t)|^2 dx dt + \frac{\lambda_{\gamma}}{2} \|\gamma\|_{H^1(0, T; \mathbb{R}^2)}^2 + \beta_T \tau. \end{aligned} \quad (3.2)$$

**Theorem 3.6. (Existence of a global minimizer)**

There exists  $(\bar{\gamma}, \bar{\tau}) \in U_{ad} \times [3r, T]$  such that

$$\hat{J}_r(\bar{\gamma}, \bar{\tau}) \leq \hat{J}_r(\gamma, \tau) \quad \text{for every } (\gamma, \tau) \in U_{ad} \times [3r, T],$$

i.e. the optimal control problem admits at least one global minimizer.

*Proof.* First, we can observe that  $\hat{J}_r$  is non-negative in  $U_{ad} \times [3r, T]$ , hence  $\hat{J}_r$  is bounded from below and admits a finite infimum  $L$

$$L := \inf_{(\gamma, \tau) \in W_{ad}} \hat{J}_r(\gamma, \tau). \quad (3.3)$$

Thus, let us consider a minimizing sequence  $\{(\gamma_n, \tau_n)\}_{n \in \mathbb{N}}$  in  $U_{ad} \times [3r, T]$  such that

$$\lim_{n \rightarrow +\infty} \hat{J}_r(\gamma_n, \tau_n) = L \geq 0. \quad (3.4)$$

Since  $|\gamma_n'(t)| \leq c$  for a.e.  $t \in [0, T]$  and  $\Gamma_1$  is bounded, then the sequence  $(\gamma_n)_{n \in \mathbb{N}}$  is bounded in  $U_{ad} \subset H^1(0, T; \mathbb{R}^2)$  and by the compactness of the embedding  $H^1(0, T; \mathbb{R}^2) \subset C([0, T]; \mathbb{R}^2)$ , there exists a subsequence  $(\gamma_{n_j})_{j \in \mathbb{N}}$  such that

$$\gamma_{n_j} \xrightarrow{j \rightarrow \infty} \bar{\gamma} \text{ in } H^1(0, T; \mathbb{R}^2) \quad \text{and} \quad \gamma_{n_j} \xrightarrow{j \rightarrow \infty} \bar{\gamma} \text{ in } C([0, T]; \mathbb{R}^2). \quad (3.5)$$

120 The sequence  $(\tau_{n_j})_{j \in \mathbb{N}}$  is such that  $3r \leq \tau_{n_j} \leq T$ , so that  $(\tau_{n_j})_{j \in \mathbb{N}}$  is bounded in  $\mathbb{R}$ . Therefore, there exists a  
 121 subsequence  $(\tau_{n_{j_k}})_{k \in \mathbb{N}}$  that we rename  $(\tau_{n_j})_{j \in \mathbb{N}}$ , and  $\bar{\tau} \in [3r, T]$ , such that

$$\lim_{n \in \mathbb{N}} \tau_{n_j} = \bar{\tau} \in [3r, T]. \quad (3.6)$$

122 Using Proposition 3.4 and (3.5) so that  $y(\gamma_{n_j})$  weakly converges to  $y(\bar{\gamma})$  in  $W(0, T)$ . Thus, in particular:

$$y(\gamma_{n_j}) \xrightarrow{j \rightarrow \infty} y(\bar{\gamma}) \text{ in } L^2(0, T; H^1(\Omega)) \text{ and a fortiori in } L^2(0, T; L^2(\Omega)). \quad (3.7)$$

123 Also,  $\nabla : L^2(0, T; H^1(\Omega)) \rightarrow L^2(0, T; L^2(\Omega)^3)$  is a linear continuous operator, so that the sequence  $(\nabla y(\gamma_{n_j}))_{j \in \mathbb{N}}$   
 124 converges also weakly to  $\nabla y(\bar{\gamma})$  in  $L^2(0, T; L^2(\Omega)^3)$ . This implies:

$$\begin{cases} \|\bar{\gamma}\|_{H^1(0, T; \mathbb{R}^2)} \leq \liminf_j \|\gamma_{n_j}\|_{H^1(0, T; \mathbb{R}^2)} \leq \sqrt{\frac{2L}{\lambda_\gamma}} \\ \|y(\bar{\gamma})\|_{L^2(0, T; L^2(\Omega))} \leq \liminf_j \|y(\gamma_{n_j})\|_{L^2(0, T; L^2(\Omega))} \\ \|\nabla y(\bar{\gamma})\|_{L^2(0, T; L^2(\Omega)^3)} \leq \liminf_j \|\nabla y(\gamma_{n_j})\|_{L^2(0, T; L^2(\Omega)^3)}, \end{cases} \quad (3.8)$$

125 Now, let us prove by Fatou's lemma that:

$$\int_{\frac{\bar{\tau}}{2} - \frac{r}{2}}^{\frac{\bar{\tau}}{2} + \frac{r}{2}} \int_{\Omega} |y(\bar{\gamma}) - y_Q|^2 dx dt \leq \liminf_j \int_{\frac{\tau_{n_j}}{2} - \frac{r}{2}}^{\frac{\tau_{n_j}}{2} + \frac{r}{2}} \int_{\Omega} |y(\gamma_{n_j}) - y_Q|^2 dx dt. \quad (3.9)$$

126 Firstly,

$$\int_{\frac{\tau_{n_j}}{2} - \frac{r}{2}}^{\frac{\tau_{n_j}}{2} + \frac{r}{2}} \int_{\Omega} |y(\gamma_{n_j}) - y_Q|^2(x, t) dx dt = \int_0^T \int_{\Omega} |y(\gamma_{n_j}) - y_Q|^2(x, t) \mathbf{1}_{[\frac{\tau_{n_j}}{2} - \frac{r}{2}, \frac{\tau_{n_j}}{2} + \frac{r}{2}]}(t) dx dt. \quad (3.10)$$

Then, using (3.7), we get that  $(y(\gamma_{n_j}))_{j \in \mathbb{N}}$  admits a subsequence that we rename still  $(y(\gamma_{n_j}))_{j \in \mathbb{N}}$  such that

$$y(\gamma_{n_j})(x, t) \xrightarrow{j \rightarrow \infty} y(\bar{\gamma})(x, t) \text{ for almost every } (x, t) \in Q.$$

127 This implies that  $|y(\gamma_{n_j}) - y_Q|^2(x, t)$  converges to  $|y(\bar{\gamma}) - y_Q|^2(x, t)$  for a.e  $(x, t) \in Q$ . We also have for a.e  $t \in [0, T]$ ,

$$\mathbf{1}_{[\frac{\tau_{n_j}}{2} - \frac{r}{2}, \frac{\tau_{n_j}}{2} + \frac{r}{2}]}(t) \xrightarrow{j \rightarrow \infty} \mathbf{1}_{[\frac{\bar{\tau}}{2} - \frac{r}{2}, \frac{\bar{\tau}}{2} + \frac{r}{2}]}(t) \quad (3.11)$$

128 and thus also for a.e.  $(x, t) \in Q$  by Fubini's theorem. So, for almost every  $(x, t) \in Q$ ,

$$|y(\gamma_{n_j})(x, t) - y_Q(x, t)|^2 \mathbf{1}_{[\frac{\tau_{n_j}}{2} - \frac{r}{2}, \frac{\tau_{n_j}}{2} + \frac{r}{2}]}(t) \xrightarrow{j \rightarrow \infty} |y(\bar{\gamma})(x, t) - y_Q(x, t)|^2 \mathbf{1}_{[\frac{\bar{\tau}}{2} - \frac{r}{2}, \frac{\bar{\tau}}{2} + \frac{r}{2}]}(t) \quad (3.12)$$

129 Applying Fatou's Lemma [13, Theorem 2.10.5], we obtain (3.9). In the same way we also have

$$\int_{\bar{\tau}-r}^{\bar{\tau}} \int_{\Omega} |y(\bar{\gamma}) - y_\Omega|^2 dx dt \leq \liminf_j \int_{\tau_{n_j}-r}^{\tau_{n_j}} \int_{\Omega} |y(\gamma_{n_j}) - y_\Omega|^2 dx dt \quad (3.13)$$

130 After passing to the limit in  $\hat{J}_r(\gamma_{n_j}, \tau_{n_j})$  we get

$$\begin{aligned} L &\geq \frac{\lambda_Q}{2} \frac{1}{r} \liminf_j \int_{\frac{\tau_{n_j}}{2} - \frac{r}{2}}^{\frac{\tau_{n_j}}{2} + \frac{r}{2}} \int_{\Omega} |y(\gamma_{n_j})(x, t) - y_Q(x, t)|^2 dx dt \\ &\quad + \frac{\lambda_\Omega}{2} \frac{1}{r} \liminf_j \int_{\tau_{n_j}-r}^{\tau_{n_j}} \int_{\Omega} |y(\gamma_{n_j})(x, t) - y_\Omega(x, t)|^2 dx dt \\ &\quad + \frac{1}{2} \liminf_j \int_0^T \int_{\Omega} |\nabla y(\gamma_{n_j})(x, t)|^2 dx dt + \frac{\lambda_\Omega}{2} \liminf_j \|\gamma_{n_j}\|_{H^1(0, T; \mathbb{R}^2)}^2 + \beta_T \liminf_j \tau_{n_j} \geq \hat{J}_r(\bar{\gamma}, \bar{\tau}). \end{aligned} \quad (3.14)$$

131 Thus by the definition of  $L$  in (3.3), we get that  $L = \hat{J}_r(\bar{\gamma}, \bar{\tau})$ . □

## 4 Fréchet Differentiability of the Path-to-State Operator

Let  $U$  be the open neighbourhood of  $U_{ad}$  in  $H^1(0, T; \mathbb{R}^2)$  given by

$$U := \{\gamma \in H^1(0, T; \mathbb{R}^2); \gamma([0, T]) \subset \Gamma_1\}. \quad (4.1)$$

As there is no difference between our state operator  $G$  with that one considered in [1], we have the following result from [1]:

**Theorem 4.1.** *The control to state mapping  $G : U \rightarrow W(0, T)$  is a continuously Fréchet differentiable mapping, and for all  $\gamma \in U$  and all  $\delta\gamma \in H^1(0, T; \mathbb{R}^2)$ :*

$$v(\gamma, \delta\gamma) := DG(\gamma) \cdot \delta\gamma = 2ac_R \sigma(\exp(w(\gamma))\tilde{\gamma}(\gamma) \cdot \delta\gamma), \quad (4.2)$$

where  $a = \alpha \frac{2P}{\pi R^2}$ ,  $c_R = \frac{2}{R^2}$ ,  $\tilde{\gamma}(\gamma)(x, t) := x - \gamma(t)$  for all  $(x, t) \in \bar{\Gamma}_1 \times [0, T]$ ,  $w$  is defined by

$$w : U \rightarrow C(\bar{\Gamma}_1 \times [0, T]) : \gamma \mapsto -c_R |\tilde{\gamma}(\gamma)|^2, \quad (4.3)$$

and  $\sigma$  is the linear continuous mapping defined by

$$\sigma : L^2(\Sigma_1) \rightarrow W(0, T) : \delta v \mapsto \delta y_2, \quad (4.4)$$

where  $\delta y_2$  is the unique weak solution of

$$\begin{cases} \rho c \partial_t \delta y_2 - \kappa \Delta \delta y_2 = 0 & \text{in } Q, \\ -\kappa \partial_\nu \delta y_2 = h \delta y_2 - \delta v & \text{on } \Sigma_1, \\ -\kappa \partial_\nu \delta y_2 = h \delta y_2 & \text{on } \Sigma_2 \cup \Sigma_3, \\ \delta y_2(\cdot, 0) = 0 & \text{in } \Omega. \end{cases} \quad (4.5)$$

**Remark 4.2.** *The proof of Theorem 4.1 in [1] proceeds by writing  $G$  as the composition of the three continuously Fréchet differentiable mappings  $w$  (4.3),  $g$  and  $q$  where:*

$$g : C(\bar{\Gamma}_1 \times [0, T]) \rightarrow L^2(\Sigma_1) : u \mapsto a \exp(u) \quad (4.6)$$

with  $a = \alpha \frac{2P}{\pi R^2}$ , and

$$q : L^2(\Sigma_1) \rightarrow W(0, T) : v \mapsto y \quad (4.7)$$

where  $y$  denotes the weak solution of the initial boundary value problem:

$$\begin{cases} \rho c \partial_t y - \kappa \Delta y = 0 & \text{in } Q = \Omega \times ]0, T[ \\ -\kappa \partial_\nu y = h y - v & \text{in } \Sigma_1 = \Gamma_1 \times ]0, T[ \\ -\kappa \partial_\nu y = h y & \text{in } \Sigma_2 = \Gamma_2 \times ]0, T[ \\ -\kappa \partial_\nu y = h(y - y_B) & \text{in } \Sigma_3 = \Gamma_3 \times ]0, T[ \\ y(x, 0) = y_0(x) & \text{for } x \in \Omega. \end{cases} \quad (4.8)$$

by proving that:

$$Dw(\gamma) \cdot \delta\gamma = 2c_R \tilde{\gamma}(\gamma) \cdot \delta\gamma, \quad \text{for all } \delta\gamma \in H^1(0, T; \mathbb{R}^2) \quad (4.9)$$

$$Dg(u) \cdot \delta u = a \exp(u) \delta u \quad \text{for all } \delta u \in C(\bar{\Gamma}_1 \times [0, T]), \quad (4.10)$$

$$Dq(g) = \sigma \quad \text{for all } g \in L^2(\Sigma_1). \quad (4.11)$$

**Remark 4.3.** *If we suppose that the datum  $y_B \in L^s(\Sigma_3)$  for some  $s > 4$ , and the initial condition  $y_0 \in C(\bar{\Omega})$ , then the control to state mapping  $G$  is also continuously Fréchet differentiable from  $U \subset H^1(0, T; \mathbb{R}^2)$  into  $C(\bar{Q})$ . This can be proved by modifying somewhat the proof in [1]: we must rather consider  $g$  as a mapping from  $C(\bar{\Gamma}_1 \times [0, T])$  into itself, and  $q$  as an affine continuous mapping from  $L^s(\Sigma_1)$  into  $C(\bar{Q})$  as result from [15, Theorem 1.40 p.49] applied to (1.2).*

## 5 Fréchet Differentiability of the Reduced Cost Functional

In this section, using the fact that  $G$  is continuously Fréchet differentiable, we will begin by proving by composition, that the reduced cost functional  $\hat{J}_r : U \times [3r, T] \rightarrow \mathbb{R} : (\gamma, \tau) \mapsto J_r(G(\gamma), \gamma, \tau)$  is also continuously Fréchet differentiable in  $\gamma$  and  $\tau$  separately. Then, we will prove that  $\hat{J}_r$  is in fact continuously Fréchet differentiable.

### 5.1 Fréchet differentiability of the reduced cost functional with respect to the path

In this subsection, we consider  $\tau \in [3r, T]$  fixed in the expression of the reduced cost functional (3.2). Our aim is to prove that the mapping  $\gamma \mapsto J_r(G(\gamma), \gamma, \tau)$  from an open neighborhood  $U$  of  $U_{ad}$  into an appropriate state space  $\mathbb{R}$  is continuously Fréchet differentiable. We want to show that this mapping is continuously Fréchet differentiable and to compute  $D_\gamma \hat{J}_r(\gamma, \tau)$ , for every  $\gamma \in U$ .

**Proposition 5.1.** *The mapping  $\hat{J}_r(\cdot, \tau) : \gamma \mapsto \hat{J}_r(\gamma, \tau)$  from  $U$  into  $\mathbb{R}$  is continuously Fréchet differentiable and its derivative is given by the following expression*

$$\begin{aligned} D_\gamma \hat{J}_r(\gamma, \tau) \cdot \delta\gamma &= \frac{\lambda_Q}{r} \int_{\frac{\tau}{2} - \frac{\tau}{2}}^{\frac{\tau}{2} + \frac{\tau}{2}} \int_{\Omega} (G(\gamma)(x, t) - y_Q(x, t)) \cdot v(\gamma, \delta\gamma)(x, t) dx dt \\ &\quad + \frac{\lambda_\Omega}{r} \int_{\tau-r}^{\tau} \int_{\Omega} (G(\gamma)(x, t) - y_\Omega(x, t)) \cdot v(\gamma, \delta\gamma)(x, t) dx dt \\ &\quad + \int_0^T \int_{\Omega} \nabla G(\gamma)(x, t) \cdot \nabla v(\gamma, \delta\gamma)(x, t) dx dt + \lambda_\gamma(\gamma, \delta\gamma)_{H^1(0, T; \mathbb{R}^2)}, \end{aligned} \quad (5.1)$$

for all  $\delta\gamma \in H^1(0, T; \mathbb{R}^2)$ , where we recall that  $v(\gamma, \delta\gamma) := DG(\gamma) \cdot \delta\gamma$  is given by (4.2).

*Proof.*

**Step 1: First term in  $\hat{J}_r$ .**

Firstly, the mapping  $F : u \in L^2(0, T; L^2(\Omega)) \mapsto \frac{\lambda_Q}{2} \|u - y_Q\|_{L^2(\Omega \times ]\frac{\tau}{2} - \frac{\tau}{2}, \frac{\tau}{2} + \frac{\tau}{2}[)}^2 \in \mathbb{R}$  is continuously Fréchet differentiable, and

$$DF(u) \cdot h = \lambda_Q(u - y_Q, h)_{L^2(\Omega \times ]\frac{\tau}{2} - \frac{\tau}{2}, \frac{\tau}{2} + \frac{\tau}{2}[)}}, \quad \forall h \in L^2(0, T; L^2(\Omega)). \quad (5.2)$$

In fact, from this expression it is obvious that  $DF$  is a continuous mapping from  $L^2(0, T; L^2(\Omega))$  into  $L^2(0, T; L^2(\Omega))^*$ . By composition  $F \circ G$  is a continuously Fréchet differentiable mapping from  $U \subset H^1(0, T; \mathbb{R}^2)$  into  $\mathbb{R}$  and

$$D_\gamma \left( \frac{\lambda_Q}{2} \|G(\gamma) - y_Q\|_{L^2(\Omega \times ]\frac{\tau}{2} - \frac{\tau}{2}, \frac{\tau}{2} + \frac{\tau}{2}[)}^2 \right) \cdot \delta\gamma = \lambda_Q (G(\gamma) - y_Q, v(\gamma, \delta\gamma))_{L^2(\Omega \times ]\frac{\tau}{2} - \frac{\tau}{2}, \frac{\tau}{2} + \frac{\tau}{2}[)}. \quad (5.3)$$

**Step 2: Second term in  $\hat{J}_r$ .**

In a similar way, one proves that the mapping from  $U \subset H^1(0, T; \mathbb{R}^2)$  into  $\mathbb{R}$  which sends  $\gamma$  onto  $\frac{\lambda_\Omega}{2} \|G(\gamma) - y_\Omega\|_{L^2(\Omega \times ]\tau-r, \tau[)}^2$  is continuously Fréchet differentiable and that:

$$D_\gamma \left( \frac{\lambda_\Omega}{2} \|G(\gamma) - y_\Omega\|_{L^2(\Omega \times ]\tau-r, \tau[)}^2 \right) \cdot \delta\gamma = \lambda_\Omega (G(\gamma) - y_\Omega, v(\gamma, \delta\gamma))_{L^2(\Omega \times ]\tau-r, \tau[)}. \quad (5.4)$$

**Step 3: Third term in  $\hat{J}_r$ .**

Also, the mapping  $H$  from  $L^2(0, T; H^1(\Omega))$  to  $\mathbb{R}$  which sends  $u$  to  $\frac{1}{2} \|\nabla u\|_{L^2(0, T; L^2(\Omega)^3)}^2$  is Fréchet differentiable and

$$DH(u) \cdot h = (\nabla u, \nabla h)_{L^2(0, T; L^2(\Omega)^3)}, \quad \forall h \in L^2(0, T; H^1(\Omega)).$$

It is continuously Fréchet differentiable as, for every  $u_1, u_2$  belonging to  $L^2(0, T; H^1(\Omega))$ ,

$$\|DH(u_2) - DH(u_1)\|_{L^2(0, T; H^1(\Omega))^*} \leq \|\nabla u_2 - \nabla u_1\|_{L^2(0, T; L^2(\Omega)^3)} \leq \|u_2 - u_1\|_{L^2(0, T; H^1(\Omega))}$$

so that  $DH : L^2(0, T; H^1(\Omega)) \rightarrow L^2(0, T; H^1(\Omega))^*$  is continuous. So by composition, the mapping from  $U \subset H^1(0, T; \mathbb{R}^2)$  into  $\mathbb{R}$ , which sends  $\gamma$  onto  $\frac{1}{2} \|\nabla G(\gamma)\|_{L^2(Q)^3}^2$  is continuously Fréchet differentiable and

$$D_\gamma \left( \frac{1}{2} \|\nabla G(\gamma)\|_{L^2(Q)^3}^2 \right) \cdot \delta\gamma = (\nabla G(\gamma), \nabla v(\gamma, \delta\gamma))_{L^2(Q)^3}. \quad (5.5)$$



**Step 4: Fourth term in  $\hat{J}_r$ .**

The mapping  $\gamma \mapsto \frac{\lambda_\gamma}{2} \|\gamma\|_{H^1(0,T;\mathbb{R}^2)}^2$  is continuously Fréchet differentiable from  $H^1(0,T;\mathbb{R}^2)$  to  $\mathbb{R}$  with derivative

$$D_\gamma \left( \frac{\lambda_\gamma}{2} \|\gamma\|_{H^1(0,T;\mathbb{R}^2)}^2 \right) = \lambda_\gamma(\gamma, \cdot)_{H^1(0,T;\mathbb{R}^2)}. \quad (5.6)$$

where from that formula, it is clear that the mapping  $\gamma \mapsto D_\gamma \left( \frac{\lambda_\gamma}{2} \|\gamma\|_{H^1(0,T;\mathbb{R}^2)}^2 \right)$  is continuous from  $H^1(0,T;\mathbb{R}^2)$  into  $H^1(0,T;\mathbb{R}^2)^*$ .

As the sum of Fréchet differentiable mappings is continuously differentiable,  $\hat{J}_r(\cdot, \tau)$  is Fréchet differentiable and its derivative is given by (5.1).  $\square$

**Proposition 5.2.** *The mapping  $D_\gamma \hat{J}_r : (\gamma, \tau) \mapsto D_\gamma \hat{J}_r(\gamma, \tau)$  from  $U \times [3r, T]$  into  $H^1(0,T;\mathbb{R}^2)^*$  is continuous.*

*Proof.* Let us prove that the first term in the right hand side of equation (5.1) depends continuously on  $(\gamma, \tau) \in U \times [3r, T]$ . Let us fix  $(\gamma', \tau') \in U \times [3r, T]$  and let us prove that

$$\int_{\frac{\tau}{2}-\frac{r}{2}}^{\frac{\tau}{2}+\frac{r}{2}} \int_{\Omega} (G(\gamma)(x, t) - y_Q(x, t)) v(\gamma, \delta\gamma)(x, t) dx dt \rightarrow \int_{\frac{\tau'}{2}-\frac{r}{2}}^{\frac{\tau'}{2}+\frac{r}{2}} \int_{\Omega} (G(\gamma')(x, t) - y_Q(x, t)) v(\gamma', \delta\gamma)(x, t) dx dt \quad (5.7)$$

as  $(\gamma, \tau)$  tends to  $(\gamma', \tau')$  uniformly in  $\delta\gamma$  for  $\|\delta\gamma\|_{H^1(0,T;\mathbb{R}^2)} \leq 1$ .

By the triangular inequality, we have

$$\begin{aligned} & \left| \int_{\frac{\tau}{2}-\frac{r}{2}}^{\frac{\tau}{2}+\frac{r}{2}} \int_{\Omega} (G(\gamma) - y_Q)(x, t) v(\gamma, \delta\gamma)(x, t) dx dt - \int_{\frac{\tau'}{2}-\frac{r}{2}}^{\frac{\tau'}{2}+\frac{r}{2}} \int_{\Omega} (G(\gamma') - y_Q)(x, t) v(\gamma', \delta\gamma)(x, t) dx dt \right| \\ & \leq \int_0^T \int_{\Omega} |(G(\gamma) - y_Q)(x, t) v(\gamma, \delta\gamma)(x, t) - (G(\gamma') - y_Q)(x, t) v(\gamma', \delta\gamma)(x, t)| dx dt \\ & \quad + \int_{\mathcal{T}} \int_{\Omega} |(G(\gamma') - y_Q)(x, t) v(\gamma', \delta\gamma)(x, t)| dx dt, \end{aligned} \quad (5.8)$$

where  $\mathcal{T} = [\frac{\tau}{2} - \frac{r}{2}, \frac{\tau}{2} + \frac{r}{2}] \Delta [\frac{\tau'}{2} - \frac{r}{2}, \frac{\tau'}{2} + \frac{r}{2}]$  and  $I \Delta J = (I \setminus J) \cup (J \setminus I)$ .

For the first term in the right-hand side of (5.8), again by the triangular inequality, we get

$$\begin{aligned} & \int_0^T \int_{\Omega} |(G(\gamma) - y_Q)(x, t) v(\gamma, \delta\gamma)(x, t) - (G(\gamma') - y_Q)(x, t) v(\gamma', \delta\gamma)(x, t)| dx dt \\ & \lesssim \|(G(\gamma) - y_Q) DG(\gamma) \delta\gamma - (G(\gamma') - y_Q) DG(\gamma') \delta\gamma\|_{C([0,T];L^1(\Omega))} \\ & \lesssim \|(G(\gamma) - G(\gamma')) DG(\gamma) \delta\gamma\|_{C([0,T];L^1(\Omega))} + \|(G(\gamma') - y_Q) (DG(\gamma) \delta\gamma - DG(\gamma') \delta\gamma)\|_{C([0,T];L^1(\Omega))} \\ & \lesssim \|G(\gamma) - G(\gamma')\|_{C([0,T];L^2(\Omega))} \|DG(\gamma)\|_{\mathcal{L}(H^1(0,T;\mathbb{R}^2);W(0,T))} \\ & \quad + \|G(\gamma') - y_Q\|_{C([0,T];L^2(\Omega))} \|DG(\gamma) - DG(\gamma')\|_{\mathcal{L}(H^1(0,T;\mathbb{R}^2);W(0,T))} \longrightarrow 0 \text{ as } \gamma \rightarrow \gamma' \end{aligned} \quad (5.9)$$

by Theorem 4.1. Now let us look at the second term in the right-hand side of (5.8). As  $v(\gamma, \delta\gamma) = DG(\gamma) \cdot \delta\gamma$ , we obtain

$$\begin{aligned} & \int_{\mathcal{T}} \int_{\Omega} |(G(\gamma') - y_Q)(x, t) v(\gamma', \delta\gamma)(x, t)| dx dt \\ & \lesssim \|(G(\gamma') - y_Q)\|_{C([0,T];L^2(\Omega))} \|DG(\gamma')\|_{\mathcal{L}(H^1(0,T;\mathbb{R}^2);W(0,T))} \text{meas}(\mathcal{T}) \rightarrow 0 \text{ as } \tau \rightarrow \tau', \text{ since } \text{meas}(\mathcal{T}) \rightarrow 0. \end{aligned} \quad (5.10)$$

The proof for the continuity of the second term is similar. Thus, we obtain (5.7).

That the third term in (5.1) depends continuously on  $(\gamma, \tau)$ , follows from the proof of the previous proposition as in fact this mapping does not depend on the variable  $\tau$ .  $\square$

## 5.2 Fréchet differentiability of the reduced cost functional with respect to time $\tau$

We suppose in the following that the mappings  $y_Q$  and  $y_\Omega$  belong to  $C([0, T]; L^2(\Omega))$ . Let us set:

$$\begin{aligned} f_Q(x, t) &:= \frac{\lambda_Q}{2} \frac{1}{r} |G(\gamma)(x, t) - y_Q(x, t)|^2, \\ g_1(\tau) &:= \int_0^{\frac{\tau}{2} + \frac{r}{2}} \left( \int_\Omega f_Q(x, t) dx \right) dt \quad \text{and} \quad g_2(\tau) := - \int_0^{\frac{\tau}{2} - \frac{r}{2}} \left( \int_\Omega f_Q(x, t) dx \right) dt. \end{aligned} \quad (5.11)$$

We know that  $G(\gamma) \in C([0, T]; L^2(\Omega))$  by the compact embedding of  $W(0, T)$  in  $C([0, T]; L^2(\Omega))$ , and also  $y_Q$  and  $y_\Omega$  by hypothesis, so that the mapping  $h_Q : t \in [0, T] \mapsto \int_\Omega f_Q(x, t) dx \in \mathbb{R}$  is continuous on  $[0, T]$ . Consequently,  $g_1$  and  $g_2$  are differentiable at every point  $\tau \in [3r, T]$ , with

$$\begin{aligned} D_\tau g_1(\tau) &= \frac{1}{2} h_Q\left(\frac{\tau}{2} + \frac{r}{2}\right) = \frac{1}{2} \int_\Omega f_Q(x, \frac{\tau}{2} + \frac{r}{2}) dx, \\ D_\tau g_2(\tau) &= -\frac{1}{2} h_Q\left(\frac{\tau}{2} - \frac{r}{2}\right) = -\frac{1}{2} \int_\Omega f_Q(x, \frac{\tau}{2} - \frac{r}{2}) dx. \end{aligned}$$

Thus  $D_\tau \int_{\frac{\tau}{2} - \frac{r}{2}}^{\frac{\tau}{2} + \frac{r}{2}} \int_\Omega f_Q(x, t) dx dt$  exists and

$$D_\tau \int_{\frac{\tau}{2} - \frac{r}{2}}^{\frac{\tau}{2} + \frac{r}{2}} \int_\Omega f_Q(x, t) dx dt = \frac{\lambda_Q}{4} \frac{1}{r} \left[ \int_\Omega |G(\gamma)(x, \frac{\tau}{2} + \frac{r}{2}) - y_Q(x, \frac{\tau}{2} + \frac{r}{2})|^2 dx - \int_\Omega |G(\gamma)(x, \frac{\tau}{2} - \frac{r}{2}) - y_Q(x, \frac{\tau}{2} - \frac{r}{2})|^2 dx \right]. \quad (5.12)$$

Using similar arguments with

$$f_\Omega(x, t) := \frac{\lambda_\Omega}{2} \frac{1}{r} |G(\gamma)(x, t) - y_\Omega(x, t)|^2,$$

it follows that  $D_\tau \int_{\tau-r}^\tau \int_\Omega f_\Omega(x, t) dx dt$  exists and at every point  $\tau \in [3r, T]$ :

$$D_\tau \int_{\tau-r}^\tau \int_\Omega f_\Omega(x, t) dx dt = \frac{\lambda_\Omega}{2} \frac{1}{r} \left[ \int_\Omega |G(\gamma)(x, \tau) - y_\Omega(x, \tau)|^2 dx - \int_\Omega |G(\gamma)(x, \tau - r) - y_\Omega(x, \tau - r)|^2 dx \right]. \quad (5.13)$$

Consequently,  $D_\tau \hat{J}_r(\gamma, \tau)$  exists and

$$\begin{aligned} D_\tau \hat{J}_r(\gamma, \tau) &= \frac{\lambda_Q}{4} \frac{1}{r} \left[ \int_\Omega |G(\gamma)(x, \frac{\tau}{2} + \frac{r}{2}) - y_Q(x, \frac{\tau}{2} + \frac{r}{2})|^2 dx - \int_\Omega |G(\gamma)(x, \frac{\tau}{2} - \frac{r}{2}) - y_Q(x, \frac{\tau}{2} - \frac{r}{2})|^2 dx \right] \\ &\quad + \frac{\lambda_\Omega}{2} \frac{1}{r} \left[ \int_\Omega |G(\gamma)(x, \tau) - y_\Omega(x, \tau)|^2 dx - \int_\Omega |G(\gamma)(x, \tau - r) - y_\Omega(x, \tau - r)|^2 dx \right] + \beta_T. \end{aligned} \quad (5.14)$$

Using  $L^2(\Omega)$  norms, this formula may be rewritten equivalently:

$$\begin{aligned} D_\tau \hat{J}_r(\gamma, \tau) &= \frac{\lambda_Q}{4} \frac{1}{r} \left[ \|G(\gamma)(\cdot, \frac{\tau}{2} + \frac{r}{2}) - y_Q(\cdot, \frac{\tau}{2} + \frac{r}{2})\|_{L^2(\Omega)}^2 - \|G(\gamma)(\cdot, \frac{\tau}{2} - \frac{r}{2}) - y_Q(\cdot, \frac{\tau}{2} - \frac{r}{2})\|_{L^2(\Omega)}^2 \right] \\ &\quad + \frac{\lambda_\Omega}{2} \frac{1}{r} \left[ \|G(\gamma)(\cdot, \tau) - y_\Omega(\cdot, \tau)\|_{L^2(\Omega)}^2 - \|G(\gamma)(\cdot, \tau - r) - y_\Omega(\cdot, \tau - r)\|_{L^2(\Omega)}^2 \right] + \beta_T. \end{aligned} \quad (5.15)$$

$G$  being in particular a continuous mapping from  $U$  to  $W(0, T)$  and thus a fortiori from  $U$  to  $C([0, T]; L^2(\Omega))$  and the mappings  $y_Q$  and  $y_\Omega$  belonging to  $C([0, T]; L^2(\Omega))$  by hypothesis, it follows from formula (5.15) that the mapping  $D_\tau \hat{J}_r$  is also continuous from  $U \times [3r, T]$  in  $\mathbb{R}$ . We have thus obtained the following proposition:

**Proposition 5.3.** *Supposing that the mappings  $y_Q$  and  $y_\Omega$  belong to  $C([0, T]; L^2(\Omega))$ , then the partial derivative  $D_\tau \hat{J}_r(\gamma, \tau)$  exists for every  $\gamma \in U$  and  $\tau \in [3r, T]$ . Moreover the mapping  $D_\tau \hat{J}_r : U \times [3r, T] \rightarrow \mathbb{R}$  is also continuous.*

### 5.3 Continuous Fréchet differentiability of the reduced cost functional

By Propositions 5.2 and 5.3, using [11, (8.9.1)] or [7, Theorem 3.7.1], we obtain the following corollary:

**Corollary 5.4.** *The reduced cost functional  $\hat{J}_r$  is continuously Fréchet differentiable and for every  $(\gamma, \tau) \in U \times [3r, T]$ , we have:*

$$D_{(\gamma, \tau)} \hat{J}_r(\gamma, \tau) \cdot (\delta\gamma, \delta\tau) = D_\gamma \hat{J}_r(\gamma, \tau) \cdot \delta\gamma + D_\tau \hat{J}_r(\gamma, \tau) \cdot \delta\tau$$

for every  $(\delta\gamma, \delta\tau) \in H^1(0, T; \mathbb{R}^2) \times \mathbb{R}_+([3r, T] - \tau)$  with  $D_\gamma \hat{J}_r(\gamma, \tau)$  given by (5.1) and  $D_\tau \hat{J}_r(\gamma, \tau)$  given by (5.15).

## 6 Adjoint System and Necessary Optimality Conditions

Our purpose is to derive first order necessary conditions as variational inequalities for an admissible control to be an optimal one.

If we suppose  $\Gamma_S$  convex, then  $U_{ad}$  is a closed convex subset of  $H^1(0, T; \mathbb{R}^2)$ . In this case, provided that  $(\bar{\gamma}, \bar{\tau})$  is a minimizer of Problem (1.6) (even local minimizer), the reduced functional  $\hat{J}_r$  should satisfy the variational inequalities

$$D_\gamma \hat{J}_r(\bar{\gamma}, \bar{\tau})(\gamma - \bar{\gamma}) \geq 0 \quad \text{for all } \gamma \in U_{ad} \quad (6.1)$$

and

$$D_\tau \hat{J}_r(\bar{\gamma}, \bar{\tau})(\tau - \bar{\tau}) \geq 0 \quad \text{for all } \tau \in [3r, T], \quad (6.2)$$

given that  $\hat{J}_r$  is continuously Fréchet differentiable.

But in our setting, in general, the condition of convexity of  $U_{ad}$  is not necessary verified. Then, (6.1) is no more true. Therefore, we must introduce at any point  $\gamma \in U_{ad}$  the cone of admissible directions and the related variational inequality (6.4). More precisely, we recall from [9, pp.211-212] the following definition and result. We start by introducing the cone of admissible directions:

**Definition 6.1.** *Let  $V$  be a normed vector space and  $U$  a non-empty subset of  $V$ . For every  $u \in U$ , the cone of admissible directions at  $u$  is*

$$C(u) := \{0_V\} \cup \{d \in V \setminus \{0\}; \exists (u_k)_{k \geq 0} \subset U, \text{ such that } u_k \neq u \ \forall k \geq 0, \\ \lim_{k \rightarrow +\infty} u_k = u \text{ and } \lim_{k \rightarrow +\infty} \frac{u_k - u}{\|u_k - u\|} = \frac{d}{\|d\|}\}$$

In the particular case of a closed convex subset of a normed space, we have [4, pp.107-108]:

**Proposition 6.2.** *Let  $V$  a normed vector space and  $U$  a convex closed subset of  $V$ . For every  $u \in U$ , the cone of admissible directions  $C(u)$  at  $u$  is  $\overline{\mathbb{R}_+(U - u)}$ .*

**Theorem 6.3.** [9, p.212] *Let  $V$  be a normed vector space and  $U$  a non-empty subset of  $V$ . Let  $f : O \subset V \rightarrow \mathbb{R}$ , a function defined on an open set  $O$  of  $V$  such that  $U \subset O$ . If  $f$  has at  $\bar{u} \in U$  a relative minimum compared to the subset  $U$  and is Fréchet differentiable at  $\bar{u}$ , then*

$$Df(\bar{u}) \cdot \delta\bar{u} \geq 0, \text{ for all } \delta\bar{u} \in C(\bar{u}). \quad (6.4)$$

Let us now come back to our present setting with  $V = H^1(0, T; \mathbb{R}^2)$  and  $U = U_{ad}$ . Since  $\hat{J}_r$  is Fréchet differentiable, Theorem 6.3 is applicable and (6.4) implies the necessary conditions  $D_\gamma \hat{J}_r(\bar{\gamma}, \bar{\tau})\delta\gamma \geq 0$  for all  $\delta\gamma \in C(\bar{\gamma})$  and  $D_\tau \hat{J}_r(\bar{\gamma}, \bar{\tau})(\tau - \bar{\tau}) \geq 0$  for all  $\tau \in [3r, T]$ , for an admissible control  $(\bar{\gamma}, \bar{\tau}) \in U_{ad} \times [3r, T]$  to be optimal. However, condition  $D_\gamma \hat{J}_r(\bar{\gamma}, \bar{\tau})\delta\gamma \geq 0$  for all  $\delta\gamma \in C(\bar{\gamma})$ , would not be practical due to the appearance of  $v(\bar{\gamma}, \delta\gamma)$  in (5.1). Indeed, to check this condition for each  $\delta\gamma \in C(\bar{\gamma})$ , we have to solve the initial boundary value problem (4.5) with  $v = \exp(w(\bar{\gamma}))\bar{\gamma}(\bar{\gamma}) \cdot \delta\gamma$ . To remedy this difficulty, it is classical to introduce the “adjoint system” whose solution  $p$  is called the adjoint state [24] of  $\bar{y}$ . We claim that the adjoint system of our problem is the following linear backward parabolic boundary value problem:

$$\begin{cases} \rho c p_t + \kappa \Delta p = \Delta \bar{y} - \frac{1}{r} (\lambda_Q \mathbb{1}_{[\frac{\bar{\tau}}{2} - \frac{r}{2}, \frac{\bar{\tau}}{2} + \frac{r}{2}]}(t)(\bar{y} - y_Q) + \lambda_\Omega \mathbb{1}_{[\bar{\tau} - r, \bar{\tau}]}(t)(\bar{y} - y_\Omega)) & \text{in } Q = \Omega \times ]0, T[, \\ \kappa \partial_\nu p + h p = \partial_\nu \bar{y} & \text{on } \Sigma = \Gamma \times ]0, T[, \\ p(\cdot, T) = 0 & \text{in } \Omega. \end{cases} \quad (6.5)$$

In fact, we have guessed the adjoint problem by using the formal Lagrange method described in [24, pp.120-122], which at first gives

$$D_y \mathcal{L}(\bar{y}, \bar{\gamma}, \bar{\tau}, p)(y - \bar{y}) \geq 0, \quad (6.6)$$

for all  $y$  smooth functions satisfying  $y(\cdot, 0) = y_0$ . Renaming  $y - \bar{y}$  by  $y$ , it leads to the following variational inequality:

$$D_y \mathcal{L}(\bar{y}, \bar{\gamma}, \bar{\tau}, p)y = 0, \quad (6.7)$$

for all smooth functions  $y$  such that  $y(\cdot, 0) = 0$ . We want the following necessary optimality conditions to be verified:

$$\begin{cases} D_y \mathcal{L}(\bar{y}, \bar{\gamma}, \bar{\tau}, p)y = 0 & \text{for all } y \text{ with } y(\cdot, 0) = 0, \\ D_\gamma \mathcal{L}(\bar{y}, \bar{\gamma}, \bar{\tau}, p)\delta\gamma \geq 0 & \text{for all } \delta\gamma \in C(\bar{\gamma}), \\ D_\tau \mathcal{L}(\bar{y}, \bar{\gamma}, \bar{\tau}, p)(s - \bar{\tau}) \geq 0 & \text{for all } s \in [3r, T]. \end{cases} \quad (6.8)$$

Note that we define  $\mathcal{L}$  as the Lagrangian function associated with the problem (1.1)-(1.2)-(1.5)-(1.6), with respective Lagrangian multipliers  $p_1$  and  $p_2$ :

$$\begin{aligned} \mathcal{L}(y, \gamma, \tau, p_1, p_2) &= J_r(y, \gamma, \tau) - \int \int_Q (\rho c y_t - \kappa \Delta y) \cdot p_1 dx dt \\ &\quad - \int \int_\Sigma (\kappa \partial_\nu y + h y - g_\gamma \mathbb{1}_{\Sigma_1} - h y_B \mathbb{1}_{\Sigma_3}) \cdot p_2 dS(x) dt. \end{aligned} \quad (6.9)$$

The first necessary optimality condition leads to the adjoint equation.

$$\begin{aligned} D_y \mathcal{L}(\bar{y}, \bar{\gamma}, \bar{\tau}, p_1, p_2)y &= \frac{\lambda_Q}{r} \int_{\frac{\bar{\tau}}{2} - \frac{r}{2}}^{\frac{\bar{\tau}}{2} + \frac{r}{2}} \int_\Omega (\bar{y} - y_Q) \cdot y dx dt + \frac{\lambda_\Omega}{r} \int_{\bar{\tau}-r}^{\bar{\tau}} \int_\Omega (\bar{y} - y_\Omega) \cdot y dx dt \\ &\quad + \int_0^T \int_\Omega \nabla \bar{y} \cdot \nabla y dx dt - \rho c \int \int_Q y_t \cdot p_1 dx dt + \kappa \int \int_Q \Delta y \cdot p_1 dx dt \\ &\quad - \kappa \int \int_\Sigma \partial_\nu y \cdot p_2 dS(x) dt - h \int \int_\Sigma y \cdot p_2 dS(x) dt. \end{aligned} \quad (6.10)$$

We integrate by parts using Green's formula in time and space with  $y(0) = 0$  to get

$$\begin{aligned} &\frac{\lambda_Q}{r} \int_{\frac{\bar{\tau}}{2} - \frac{r}{2}}^{\frac{\bar{\tau}}{2} + \frac{r}{2}} \int_\Omega (\bar{y} - y_Q) \cdot y dx dt + \frac{\lambda_\Omega}{r} \int_{\bar{\tau}-r}^{\bar{\tau}} \int_\Omega (\bar{y} - y_\Omega) \cdot y dx dt - \int \int_Q \Delta \bar{y} \cdot y dx dt + \int \int_\Sigma \partial_\nu \bar{y} \cdot y dS(x) dt \\ &\quad + \rho c \int \int_Q y \cdot p_{1,t} dx dt - \rho c \int_\Omega y(T) \cdot p_1(T) dx + \kappa \int \int_Q \Delta p_1 \cdot y dx dt - \kappa \int \int_\Sigma \partial_\nu p_1 \cdot y dS(x) dt \\ &\quad + \kappa \int \int_\Sigma \partial_\nu y \cdot p_1 dS(x) dt - \kappa \int \int_\Sigma \partial_\nu y \cdot p_2 dS(x) dt - h \int \int_\Sigma y \cdot p_2 dS(x) dt = 0. \end{aligned} \quad (6.11)$$

For all  $y \in C_0^\infty(Q)$  the terms with  $y(0)$ ,  $y(T)$ , and  $y$ ,  $\partial_\nu y$  vanishes in  $\Omega$  and on  $\Sigma$  respectively. Therefore, (6.11) implies

$$\begin{aligned} &\frac{\lambda_Q}{r} \int \int_Q \mathbb{1}_{[\frac{\bar{\tau}}{2} - \frac{r}{2}, \frac{\bar{\tau}}{2} + \frac{r}{2}]}(t) (\bar{y} - y_Q) \cdot y dx dt + \frac{\lambda_\Omega}{r} \int \int_Q \mathbb{1}_{[\bar{\tau}-r, \bar{\tau}]}(t) (\bar{y} - y_\Omega) \cdot y dx dt \\ &\quad - \int \int_Q \Delta \bar{y} \cdot y dx dt + \rho c \int \int_Q y \cdot p_{1,t} dx dt + \kappa \int \int_Q \Delta p_1 \cdot y dx dt = 0, \quad \forall y \in C_0^\infty(Q). \end{aligned} \quad (6.12)$$

$C_0^\infty(Q)$  being dense in  $L^2(Q)$ , we should have:

$$\rho c p_{1,t} + \kappa \Delta p_1 - \Delta \bar{y} + \frac{1}{r} \left( \lambda_Q \mathbb{1}_{[\frac{\bar{\tau}}{2} - \frac{r}{2}, \frac{\bar{\tau}}{2} + \frac{r}{2}]}(t) (\bar{y} - y_Q) + \lambda_\Omega \mathbb{1}_{[\bar{\tau}-r, \bar{\tau}]}(t) (\bar{y} - y_\Omega) \right) \in C_0^\infty(Q)^{\perp L^2(Q)} = \{0\}.$$

Thus,

$$\rho c p_{1,t} + \kappa \Delta p_1 = \Delta \bar{y} - \frac{1}{r} \left( \lambda_Q \mathbb{1}_{[\frac{\bar{\tau}}{2} - \frac{r}{2}, \frac{\bar{\tau}}{2} + \frac{r}{2}]}(t) (\bar{y} - y_Q) + \lambda_\Omega \mathbb{1}_{[\bar{\tau}-r, \bar{\tau}]}(t) (\bar{y} - y_\Omega) \right) \quad \text{in } Q. \quad (6.13)$$

252 In particular, the sum of the integrals over  $Q$  in equation (6.11) vanishes. Now, consider the set of all functions  
 253  $y \in C^1(\bar{Q})$  such that  $\frac{\partial y}{\partial \nu}\Big|_{\Sigma} = 0$ ,  $y|_{\Sigma} = 0$ , and  $y(\cdot, 0) = 0$ . For such functions, it follows from (6.11) that

$$\int_{\Omega} y(T)p_1(T)dx = 0 \quad \text{in } \Omega. \quad (6.14)$$

254 The possible values  $y(T)$  form a dense subset of  $L^2(\Omega)$ , so that (6.14) implies

$$p_1(T) = 0 \quad \text{in } \Omega. \quad (6.15)$$

255 Now, we do not need to have  $y|_{\Sigma} = 0$ ,  $\frac{\partial y}{\partial \nu}\Big|_{\Sigma} = 0$  any more, so we consider the functions  $y \in C^1(\bar{Q})$  such that  
 256  $y(\cdot, 0) = 0$  to obtain:

$$\int_{\Sigma} \int_{\Sigma} \kappa(p_1 - p_2)\partial_{\nu} y dS(x)dt + \int_{\Sigma} \int_{\Sigma} (\partial_{\nu} \bar{y} - \kappa\partial_{\nu} p_1 - hp_2) \cdot y dS(x)dt = 0. \quad (6.16)$$

257 The set  $\{(y|_{\Sigma}, \frac{\partial y}{\partial \nu}\Big|_{\Sigma}); y \in C^1(\bar{Q})\}$  being dense in  $L^2(\Sigma) \times L^2(\Sigma)$ , we conclude that

$$\begin{cases} \kappa\partial_{\nu} p_1 + hp_2 = \partial_{\nu} \bar{y} & \text{on } \Sigma, \\ p_1 = p_2 & \text{on } \Sigma. \end{cases} \quad (6.17)$$

258 Renaming  $p_1$  by  $p$ , as  $p_2|_{\Sigma} = p_1|_{\Sigma} = p|_{\Sigma}$ , we obtain (6.5) for the adjoint system.

259 **Definition 6.4.** Let  $(\bar{\gamma}, \bar{\tau})$  be an optimal control of the minimizing problem (1.2)-(1.5)-(1.1)-(1.6) with associated  
 260 state  $\bar{y} = G(\bar{\gamma})$ . A function  $p \in W(0, T)$  is said to be a weak solution to the adjoint problem (6.5), if  $p(\cdot, T) = 0$  in  
 261  $\Omega$  and

$$\begin{aligned} & -\rho c \int_Q p_t(x, t) \cdot v(x, t) dx dt + \kappa \int_Q \nabla p(x, t) \nabla v(x, t) dx dt + h \int_{\Sigma} p(x, t) \cdot v(x, t) dS(x) dt \\ & = \int_Q \nabla G(\bar{\gamma})(x, t) \cdot \nabla v(x, t) dx dt + \frac{\lambda_Q}{r} \int_{\frac{\bar{\tau}}{2} - \frac{r}{2}}^{\frac{\bar{\tau}}{2} + \frac{r}{2}} \int_{\Omega} (G(\bar{\gamma}) - y_Q)(x, t) \cdot v(x, t) dx dt \\ & \quad + \frac{\lambda_{\Omega}}{r} \int_{\bar{\tau}-r}^{\bar{\tau}} \int_{\Omega} (G(\bar{\gamma}) - y_{\Omega})(x, t) \cdot v(x, t) dx dt, \end{aligned} \quad (6.18)$$

262 for every  $v \in L^2(0, T; H^1(\Omega))$ .

263 **Theorem 6.5.** Let  $(\bar{\gamma}, \bar{\tau})$  be an optimal control and  $\bar{y} = G(\bar{\gamma})$  be the corresponding state. Then the backward  
 264 parabolic problem (6.5) has a unique weak solution  $p \in W(0, T)$ , i.e. a unique solution of the variational problem  
 265 (6.18) satisfying the terminal condition  $p(\cdot, T) = 0$ .

266 *Proof.* First we start by a change of variables in (6.5) to get a classical parabolic forward problem. Let us set  
 267  $\tilde{p} : \Omega \times [0, T] \rightarrow \mathbb{R} : (x, \xi) \mapsto p(x, T - \xi)$  and  $\tilde{v} : \Omega \times [0, T] \rightarrow \mathbb{R} : (x, \xi) \mapsto v(x, T - \xi)$ . Then,  $\tilde{p}(x, 0) =$   
 268  $p(x, T)$ ,  $\tilde{p}(x, T) = p(x, 0)$ ,  $\tilde{v}(x, 0) = v(x, T)$ ,  $\tilde{v}(x, T) = v(x, 0)$ ,  $\tilde{p}(x, \xi) = p(x, T - \xi)$ , and  $\tilde{v}(x, \xi) := v(x, T - \xi)$ . Also

$$\rho c \int_Q p_t(x, t) v(x, t) dx dt = \rho c \int_Q p_t(x, T - \xi) v(x, T - \xi) dx d\xi = -\rho c \int_Q \tilde{p}_{\xi}(x, \xi) \tilde{v}(x, \xi) dx d\xi.$$

269 Then, the variational formulation (6.18) is equivalent to the following classical forward one:  $\tilde{p}(\cdot, 0) = 0$  and

$$\begin{aligned} & \rho c \int_Q \tilde{p}_{\xi}(x, \xi) \tilde{v}(x, \xi) dx d\xi + \kappa \int_Q \nabla \tilde{p}(x, \xi) \nabla \tilde{v}(x, \xi) dx d\xi + h \int_{\Sigma} \tilde{p}(x, \xi) \cdot \tilde{v}(x, \xi) dS(x) d\xi \\ & = \int_Q \nabla \tilde{\bar{y}}(x, \xi) \cdot \nabla \tilde{v}(x, \xi) dx d\xi + \frac{\lambda_Q}{r} \int_{T - \frac{\bar{\tau}}{2} - \frac{r}{2}}^{T - \frac{\bar{\tau}}{2} + \frac{r}{2}} \int_{\Omega} (\tilde{\bar{y}} - \tilde{y}_Q)(x, \xi) \cdot \tilde{v}(x, \xi) dx d\xi \\ & \quad + \frac{\lambda_{\Omega}}{r} \int_{T - \bar{\tau}}^{T - \bar{\tau} + r} \int_{\Omega} (\tilde{\bar{y}} - \tilde{y}_{\Omega})(x, \xi) \cdot \tilde{v}(x, \xi) dx d\xi, \quad \forall \tilde{v} \in L^2(0, T; H^1(\Omega)), \end{aligned} \quad (6.19)$$

where we have set  $G(\bar{\gamma})(x, t) = \bar{y}(x, t)$  and  $\tilde{y}(x, \xi) = \bar{y}(x, T - \xi)$ . This variational formulation is still equivalent to (the proof of this equivalence is analogous to that of [15, Theorem 1.33]):  $\tilde{p}(\cdot, 0) = 0$  and

$$\begin{aligned} & \rho c \int_{\Omega} \tilde{p}_{\xi}(x, \xi) w(x) dx + \kappa \int_{\Omega} \nabla \tilde{p}(x, \xi) \nabla w(x) dx + h \int_{\partial\Omega} \tilde{p}(x, \xi) \cdot w(x) dS(x) \\ &= \int_{\Omega} \nabla \tilde{y}(x, \xi) \cdot \nabla w(x) dx + \frac{\lambda_Q}{r} \mathbb{1}_{[T-\frac{\bar{r}}{2}-\frac{r}{2}, T-\frac{\bar{r}}{2}+\frac{r}{2}]}(\xi) \int_{\Omega} (\tilde{y} - \tilde{y}_Q)(x, \xi) \cdot w(x) dx \\ &+ \frac{\lambda_{\Omega}}{r} \mathbb{1}_{[T-\bar{r}, T-\bar{r}+r]}(\xi) \int_{\Omega} (\tilde{y} - \tilde{y}_{\Omega})(x, \xi) \cdot w(x) dx, \forall w \in H^1(\Omega), \text{ for a.e. } \xi \in ]0, T[. \end{aligned} \quad (6.20)$$

We define the bilinear form:

$$a: H^1(\Omega) \times H^1(\Omega) \rightarrow \mathbb{R}: (q, w) \mapsto a(q, w),$$

where

$$a(q, w) = \kappa \int_{\Omega} \nabla q(x) \cdot \nabla w(x) dx + h \int_{\Gamma} q(x) \cdot w(x) dS(x).$$

$a(\cdot, \cdot)$  is a bilinear and continuous form on  $H^1(\Omega) \times H^1(\Omega)$  and coercive on  $V = H^1(\Omega)$  from [24, (2.14)]. Therefore, by the Lax-Milgram's Theorem,  $a(\cdot, \cdot)$  defines an isomorphism  $A$  from  $H^1(\Omega)$  onto  $H^1(\Omega)^*$  such that  $a(q, w) = \langle Aq, w \rangle_{H^1(\Omega)^*, H^1(\Omega)}$ , for all  $(q, w) \in H^1(\Omega) \times H^1(\Omega)$ . Now, let us prove that:

$$L: [0, T] \rightarrow H^1(\Omega)^*: \xi \mapsto L(\xi),$$

where

$$L(\xi): H^1(\Omega) \rightarrow \mathbb{R}: w \mapsto L(\xi)w,$$

with

$$\begin{aligned} L(\xi)w &= \frac{\lambda_Q}{r} \mathbb{1}_{[T-\frac{\bar{r}}{2}-\frac{r}{2}, T-\frac{\bar{r}}{2}+\frac{r}{2}]}(\xi) \int_{\Omega} (\tilde{y}(x, \xi) - \tilde{y}_Q(x, \xi)) \cdot w(x) dx + \frac{\lambda_{\Omega}}{r} \mathbb{1}_{[T-\bar{r}, T-\bar{r}+r]}(\xi) \int_{\Omega} (\tilde{y}(x, \xi) - \tilde{y}_{\Omega}(x, \xi)) \cdot w(x) dx \\ &+ \int_{\Omega} \nabla \tilde{y}(x, \xi) \cdot \nabla w(x) dx, \end{aligned}$$

belongs to  $L^2(0, T; H^1(\Omega)^*)$ . Indeed,  $L(\xi)$  is a linear and continuous form on  $H^1(\Omega)$  since by Cauchy-Schwarz inequality:

$$|L(\xi)(w)| \leq (\|\tilde{y}(\cdot, \xi)\|_{H^1(\Omega)} + \frac{\lambda_Q}{r} \|(\tilde{y} - \tilde{y}_Q)(\cdot, \xi)\|_{L^2(\Omega)} + \frac{\lambda_{\Omega}}{r} \|(\tilde{y} - \tilde{y}_{\Omega})(\cdot, \xi)\|_{L^2(\Omega)}) \|w\|_{H^1(\Omega)}, \quad (6.21)$$

so that

$$\|L(\xi)\|_{H^1(\Omega)^*} \leq \|\tilde{y}(\cdot, \xi)\|_{H^1(\Omega)} + \frac{\lambda_Q}{r} \|(\tilde{y} - \tilde{y}_Q)(\cdot, \xi)\|_{L^2(\Omega)} + \frac{\lambda_{\Omega}}{r} \|(\tilde{y} - \tilde{y}_{\Omega})(\cdot, \xi)\|_{L^2(\Omega)}.$$

Thus

$$\|L\|_{L^2(0, T; H^1(\Omega)^*)} \leq \|\tilde{y}\|_{L^2(0, T; H^1(\Omega))} + \frac{\lambda_Q}{r} \|\tilde{y} - \tilde{y}_Q\|_{L^2(Q)} + \frac{\lambda_{\Omega}}{r} \|\tilde{y} - \tilde{y}_{\Omega}\|_{L^2(Q)}. \quad (6.22)$$

Applying [8, Theorem 11.7, p.192] there exists a unique  $\tilde{p} \in W(0; T)$  such that

$$\rho c \frac{d}{d\xi} \int_{\Omega} \tilde{p}(\xi)(x) w(x) dx + a(\tilde{p}(\cdot, \xi), w) = L(\xi)(w) \text{ in } \mathcal{D}'(]0, T[), \text{ for all } w \in H^1(\Omega). \quad (6.23)$$

By [8, Theorem 11.6 p.191],

$$\frac{d}{d\xi} \int_{\Omega} \tilde{p}(\xi)(x) w(x) dx = \left\langle \frac{d\tilde{p}}{d\xi}(\xi), w \right\rangle_{H^1(\Omega)^*, H^1(\Omega)} \text{ in } \mathcal{D}'(]0, T[), \text{ for all } w \in H^1(\Omega),$$

so that by the previous definitions of the bilinear continuous form  $a(\cdot, \cdot)$  and of the linear continuous form  $L(\xi)$  on  $H^1(\Omega)$ , we obtain

$$\begin{aligned} & \rho c \left\langle \frac{d\tilde{p}}{d\xi}(\xi), w \right\rangle_{H^1(\Omega)^*, H^1(\Omega)} + \kappa \int_{\Omega} \nabla \tilde{p}(\xi) \cdot \nabla w(x) dx + h \int_{\partial\Omega} \tilde{p}(x, \xi) \cdot w(x) dS(x) \\ &= \int_{\Omega} \nabla \tilde{y}(x, \xi) \cdot \nabla w(x) dx + \frac{\lambda_Q}{r} \mathbb{1}_{[T-\frac{\bar{r}}{2}-\frac{r}{2}, T-\frac{\bar{r}}{2}+\frac{r}{2}]}(\xi) \int_{\Omega} (\tilde{y} - \tilde{y}_Q)(x, \xi) \cdot w(x) dx \\ &+ \frac{\lambda_{\Omega}}{r} \mathbb{1}_{[T-\bar{r}, T-\bar{r}+r]}(\xi) \int_{\Omega} (\tilde{y} - \tilde{y}_{\Omega})(x, \xi) \cdot w(x) dx \end{aligned} \quad (6.24)$$

for all  $w \in H^1(\Omega)$  and for a.e.  $\xi \in ]0, T[$ , i. e., (6.20).  $\square$

**Proposition 6.6.** *If  $(\bar{\gamma}, \bar{\tau}) \in U_{ad} \times [3r, T]$  is an optimal control to the minimizing problem (1.6) with associated state  $\bar{y}$  and  $p \in W(0, T)$  the corresponding adjoint state that solves (6.5), then  $(\bar{\gamma}, \bar{\tau})$  verifies the variational inequality:*

$$\begin{aligned} D_{\gamma} \hat{J}_r(\bar{\gamma}, \bar{\tau}) \cdot \delta \bar{\gamma} &= 2ac_R \int \int_{\Sigma_1} \exp(w(\bar{\gamma})(x, t)) \bar{\gamma}(\bar{\gamma})(x, t) \cdot (\delta \bar{\gamma})(t) p(x, t) dS(x) dt \\ &+ \lambda_{\gamma} \int_0^T \bar{\gamma}(t) \cdot \delta \bar{\gamma}(t) dt + \lambda_{\gamma} \int_0^T \bar{\gamma}'(t) \cdot \delta \bar{\gamma}'(t) dt \geq 0, \quad \text{for all } \delta \bar{\gamma} \in C(\bar{\gamma}), \end{aligned} \quad (6.25)$$

where  $C(\bar{\gamma})$  denotes the cone of admissible directions to  $U_{ad}$  at  $\bar{\gamma}$  defined by (6.3) with  $V = H^1(0, T; \mathbb{R}^2)$ .  $(\bar{\gamma}, \bar{\tau})$  verifies also the variational inequality:

$$\begin{aligned} D_{\tau} \hat{J}_r(\bar{\gamma}, \bar{\tau})(s - \bar{\tau}) &= \frac{\lambda_Q}{4} \frac{s - \bar{\tau}}{r} \left[ \|G(\bar{\gamma})(\cdot, \frac{\bar{\tau}}{2} + \frac{r}{2}) - y_Q(\cdot, \frac{\bar{\tau}}{2} + \frac{r}{2})\|_{L^2(\Omega)}^2 - \|G(\bar{\gamma})(\cdot, \frac{\bar{\tau}}{2} - \frac{r}{2}) - y_Q(\cdot, \frac{\bar{\tau}}{2} - \frac{r}{2})\|_{L^2(\Omega)}^2 \right] \\ &+ \frac{\lambda_{\Omega}}{2} \frac{s - \bar{\tau}}{r} \left[ \|G(\bar{\gamma})(\cdot, \bar{\tau}) - y_{\Omega}(\cdot, \bar{\tau})\|_{L^2(\Omega)}^2 - \|G(\bar{\gamma})(\cdot, \bar{\tau} - r) - y_{\Omega}(\cdot, \bar{\tau} - r)\|_{L^2(\Omega)}^2 \right] \\ &+ \beta_T(s - \bar{\tau}) \geq 0, \quad \text{for all } s \in [3r, T]. \end{aligned} \quad (6.26)$$

*Proof.* If  $(\bar{\gamma}, \bar{\tau})$  is an optimal control of the minimizing problem then from Theorem 6.3, the optimal control  $\bar{\gamma}$  must satisfy the following first order necessary optimality condition with respect to  $\gamma$  :

$$D_{\gamma} \hat{J}_r(\bar{\gamma}, \bar{\tau}) \cdot \delta \bar{\gamma} \geq 0, \quad \forall \delta \bar{\gamma} \in C(\bar{\gamma}). \quad (6.27)$$

By equation (5.1), we have:

$$\begin{aligned} D_{\gamma} \hat{J}_r(\bar{\gamma}, \bar{\tau}) \cdot \delta \bar{\gamma} &= \frac{\lambda_Q}{r} \int_{\frac{\bar{\tau}}{2}-\frac{r}{2}}^{\frac{\bar{\tau}}{2}+\frac{r}{2}} \int_{\Omega} G(\bar{\gamma}) v(\bar{\gamma}, \delta \bar{\gamma}) dx dt - \frac{\lambda_Q}{r} \int_{\frac{\bar{\tau}}{2}-\frac{r}{2}}^{\frac{\bar{\tau}}{2}+\frac{r}{2}} \int_{\Omega} y_Q v(\bar{\gamma}, \delta \bar{\gamma}) dx dt + \frac{\lambda_{\Omega}}{r} \int_{\bar{\tau}-r}^{\bar{\tau}} \int_{\Omega} G(\bar{\gamma}) v(\bar{\gamma}, \delta \bar{\gamma}) dx dt \\ &- \frac{\lambda_{\Omega}}{r} \int_{\bar{\tau}-r}^{\bar{\tau}} \int_{\Omega} y_{\Omega} v(\bar{\gamma}, \delta \bar{\gamma}) dx dt + \int_0^T \int_{\Omega} \nabla G(\bar{\gamma}) \cdot \nabla v(\bar{\gamma}, \delta \bar{\gamma}) dx dt + \lambda_{\gamma} \int_0^T \bar{\gamma}(t) \cdot \delta \bar{\gamma}(t) dt + \lambda_{\gamma} \int_0^T \bar{\gamma}'(t) \cdot \delta \bar{\gamma}'(t) dt \geq 0 \end{aligned} \quad (6.28)$$

Using equation (6.18) with  $v = v(\bar{\gamma}, \delta \bar{\gamma})$ , we obtain:

$$\begin{aligned} D_{\gamma} \hat{J}_r(\bar{\gamma}, \bar{\tau}) \cdot \delta \bar{\gamma} &= -\rho c \int \int_Q p_t(x, t) v(\bar{\gamma}, \delta \bar{\gamma})(x, t) dx dt + \kappa \int \int_Q \nabla p(x, t) \cdot \nabla v(\bar{\gamma}, \delta \bar{\gamma})(x, t) dx dt \\ &+ h \int \int_{\Sigma} p(x, t) v(\bar{\gamma}, \delta \bar{\gamma})(x, t) dS(x) dt + \lambda_{\gamma} \int_0^T \bar{\gamma}(t) \cdot \delta \bar{\gamma}(t) dt + \lambda_{\gamma} \int_0^T \bar{\gamma}'(t) \delta \bar{\gamma}'(t) dt \geq 0, \end{aligned} \quad (6.29)$$

for all  $\delta \bar{\gamma} \in C(\bar{\gamma})$ , where  $v(\bar{\gamma}, \delta \bar{\gamma})$  is the weak solution of the following system:

$$\begin{cases} \rho c \partial_t v(\bar{\gamma}, \delta \bar{\gamma}) - \kappa \Delta v(\bar{\gamma}, \delta \bar{\gamma}) = 0 & \text{in } Q, \\ \kappa \partial_{\nu} v(\bar{\gamma}, \delta \bar{\gamma}) + h v(\bar{\gamma}, \delta \bar{\gamma}) = 2ac_R \exp(w(\bar{\gamma})) \bar{\gamma}(\bar{\gamma}) \cdot \delta \bar{\gamma} & \text{on } \Sigma_1, \\ \kappa \partial_{\nu} v(\bar{\gamma}, \delta \bar{\gamma}) + h v(\bar{\gamma}, \delta \bar{\gamma}) = 0 & \text{on } \Sigma_2 \cup \Sigma_3, \\ v(\bar{\gamma}, \delta \bar{\gamma})(\cdot, 0) = 0 & \text{in } \Omega, \end{cases} \quad (6.30)$$

and  $p$  the weak solution of (6.5). We now use the fact that  $v(\bar{\gamma}, \delta\bar{\gamma})$  is the weak solution of (6.30), and considering  $p(\cdot, \cdot)$  as a test function of the weak formulation of (6.30), we integrate by parts in  $W(0, T)$  using [24, Theorem 3.11 p.148] noting that  $v(\bar{\gamma}, \delta\bar{\gamma})(\cdot, 0) = 0$  and  $p(\cdot, T) = 0$ , and obtain:

$$\begin{aligned} & -\rho c \int_Q p_t(x, t) \cdot v(\bar{\gamma}, \delta\bar{\gamma})(x, t) dx dt + \kappa \int_Q \nabla p(x, t) \cdot \nabla v(\bar{\gamma}, \delta\bar{\gamma})(x, t) dx dt \\ & + h \int_{\Sigma} p(x, t) \cdot v(\bar{\gamma}, \delta\bar{\gamma})(x, t) dS(x) dt = 2ac_R \int_{\Sigma_1} \exp(w(\bar{\gamma})(x, t)) \bar{\gamma}(\bar{\gamma})(t) \cdot (\delta\bar{\gamma})(t) p(x, t) dS(x) dt. \end{aligned} \quad (6.31)$$

Substituting the above identity in (6.29) we obtain (6.25).

From Theorem 6.3, the optimal control  $(\bar{\gamma}, \bar{\tau})$  must also satisfy the following first order necessary optimality condition with respect to time given by

$$D_{\tau} \hat{J}_r(\bar{\gamma}, \bar{\tau})(s - \bar{\tau}) \geq 0, \quad \forall s \in [3r, T], \quad (6.32)$$

as  $[3r, T]$  is a closed convex subset of  $\mathbb{R}$ . Owing to (5.14), we obtain (6.26).  $\square$

**Remark 6.7.** The condition (6.32) can be simplified by taking the following arguments on  $s$  :

- If  $\bar{\tau} \in ]3r, T[$ , we consider  $s = \bar{\tau} + h$  or  $s = \bar{\tau} - h$  where  $h > 0$ , and obtain  $D_{\tau} \hat{J}_r(\bar{\gamma}, \bar{\tau}) = 0$
- If  $\bar{\tau} = 3r$  then  $s - \bar{\tau} \geq 0$ , for any  $s \in [3r, T]$ , so we have  $D_{\tau} \hat{J}_r(\bar{\gamma}, \bar{\tau}) \geq 0$ .
- If  $\bar{\tau} = T$ , then  $s - \bar{\tau} \leq 0$ , and deduce that  $D_{\tau} \hat{J}_r(\bar{\gamma}, \bar{\tau}) \leq 0$ .

Corollary 5.4 and the previous proposition imply the following corollary:

**Corollary 6.8.** If  $(\bar{\gamma}, \bar{\tau}) \in U_{ad} \times [3r, T]$  is an optimal control to the minimizing problem (1.6), then

$$D_{(\gamma, \tau)} \hat{J}_r(\bar{\gamma}, \bar{\tau})(\delta\bar{\gamma}, \delta\bar{\tau}) \geq 0 \quad \text{for every } (\delta\bar{\gamma}, \delta\bar{\tau}) \in C_{(\bar{\gamma}, \bar{\tau})}. \quad (6.33)$$

We have introduced the Lagrangian functional previously, rather informally, in order to guess the adjoint problem. Let us now define it rigorously:

**Definition 6.9.** The Lagrangian functional associated with Problem (1.1)-(1.2)-(1.5)-(1.6) is the mapping

$$\mathcal{L} : W(0, T) \times H^1(0, T; \mathbb{R}^2) \times [3r, T] \times L^2(0, T; H^1(\Omega)) \rightarrow \mathbb{R}$$

which sends  $(y, \gamma, \tau, p) \in W(0, T) \times H^1(0, T; \mathbb{R}^2) \times [3r, T] \times L^2(0, T; H^1(\Omega))$  onto the real number:

$$\mathcal{L}(y, \gamma, \tau, p) = J_r(y, \gamma, \tau) - \int_Q (\rho c y_t p + \kappa \nabla y \cdot \nabla p) dx dt - \int_{\Sigma} (h y - g_{\gamma} \mathbf{1}_{\Sigma_1} - h y_B \mathbf{1}_{\Sigma_3}) \cdot p dS(x) dt. \quad (6.34)$$

This rigorous definition allows us to reformulate our first order necessary optimality conditions obtained in Proposition 6.6 in terms of the Lagrangian functional:

**Proposition 6.10.** If  $(\bar{\gamma}, \bar{\tau}) \in U_{ad} \times [3r, T]$  is an optimal control to the minimizing problem (1.6) with associated state  $\bar{y}$  and  $p \in W(0, T)$  the corresponding adjoint state to  $\bar{y}$  that solves (6.5), then  $(\bar{\gamma}, \bar{\tau})$  verifies the variational inequality:

$$D_{\gamma} \mathcal{L}(\bar{y}, \bar{\gamma}, \bar{\tau}, p) \delta\bar{\gamma} \geq 0 \quad \text{for all } \delta\bar{\gamma} \in C(\bar{\gamma}), \quad (6.35)$$

where  $C(\bar{\gamma})$  denotes the cone of admissible directions to  $U_{ad}$  at  $\bar{\gamma}$  defined by (6.3) with  $V = H^1(0, T; \mathbb{R}^2)$ .

$(\bar{\gamma}, \bar{\tau})$  verifies also the variational inequality:

$$D_{\tau} \mathcal{L}(\bar{y}, \bar{\gamma}, \bar{\tau}, p)(s - \bar{\tau}) \geq 0 \quad \text{for all } s \in [3r, T]. \quad (6.36)$$



*Proof.* In view of the definition of the Lagrangian function, we have to prove that the mapping  $F : H^1(0, T; \mathbb{R}^2) \rightarrow \mathbb{R} : \gamma \mapsto \iint_{\Sigma} g_{\gamma}(x, t) \mathbb{1}_{\Sigma_1}(x, t) p(x, t) dS(x) dt = \iint_{\Sigma_1} g_{\gamma}(x, t) p(x, t) dS(x) dt = \int_0^T \int_{\Gamma_1} g_{\gamma}(x, t) p(x, t) dS(x) dt$  is continuously Fréchet differentiable and that at every point  $\gamma \in H^1(0, T; \mathbb{R}^2)$ :

$$F'(\gamma) \cdot \delta\gamma = \frac{8\alpha P}{\pi R^4} \iint_{\Sigma_1} e^{-\frac{2}{R^2}|x-\gamma(t)|^2} (x - \gamma(t)|\delta\gamma(t))_{\mathbb{R}^2} p(x, t) dS(x) dt, \text{ for every } \delta\gamma \in H^1(0, T; \mathbb{R}^2).$$

Let us firstly compute its Gâteaux derivative at point  $\gamma \in H^1(0, T; \mathbb{R}^2)$ . Let us consider a sequence of positive real numbers  $(h_n)_{n \in \mathbb{N}}$  converging to zero.

$$\iint_{\Sigma_1} \frac{g_{\gamma+h_n\delta\gamma}(x, t) - g_{\gamma}(x, t)}{h_n} p(x, t) dS(x) dt = \frac{2\alpha P}{\pi R^2} \iint_{\Sigma_1} \frac{1}{h_n} (e^{-\frac{2}{R^2}|x-\gamma(t)-h_n\delta\gamma(t)|^2} - e^{-\frac{2}{R^2}|x-\gamma(t)|^2}) p(x, t) dS(x) dt.$$

We want to apply Lebesgue dominated convergence theorem. We have:

$$\frac{d}{dh} e^{-\frac{2}{R^2}|x-\gamma(t)-h\delta\gamma(t)|^2} = \frac{4}{R^2} e^{-\frac{2}{R^2}|x-\gamma(t)-h\delta\gamma(t)|^2} [(x - \gamma(t)|\delta\gamma(t))_{\mathbb{R}^2} + |\delta\gamma(t)|_{\mathbb{R}^2}^2 h],$$

so that:

$$\begin{aligned} \frac{1}{h_n} (e^{-\frac{2}{R^2}|x-\gamma(t)-h_n\delta\gamma(t)|^2} - e^{-\frac{2}{R^2}|x-\gamma(t)|^2}) &= \frac{1}{h_n} \int_0^{h_n} \frac{4}{R^2} e^{-\frac{2}{R^2}|x-\gamma(t)-h\delta\gamma(t)|^2} [(x - \gamma(t)|\delta\gamma(t))_{\mathbb{R}^2} + |\delta\gamma(t)|_{\mathbb{R}^2}^2 h] dh \\ &\rightarrow \frac{4}{R^2} e^{-\frac{2}{R^2}|x-\gamma(t)|^2} (x - \gamma(t)|\delta\gamma(t))_{\mathbb{R}^2} \text{ as } n \rightarrow +\infty. \end{aligned}$$

Also by  $\frac{1}{h_n} (e^{-\frac{2}{R^2}|x-\gamma(t)-h_n\delta\gamma(t)|^2} - e^{-\frac{2}{R^2}|x-\gamma(t)|^2}) = \frac{1}{h_n} \int_0^{h_n} \frac{4}{R^2} e^{-\frac{2}{R^2}|x-\gamma(t)-h\delta\gamma(t)|^2} [(x - \gamma(t)|\delta\gamma(t))_{\mathbb{R}^2} + |\delta\gamma(t)|_{\mathbb{R}^2}^2 h] dh$  follows that:

$$\left| \frac{e^{-\frac{2}{R^2}|x-\gamma(t)-h_n\delta\gamma(t)|^2} - e^{-\frac{2}{R^2}|x-\gamma(t)|^2}}{h_n} \right| \leq \frac{4}{R^2} \left[ |(x - \gamma(t)|\delta\gamma(t))_{\mathbb{R}^2}| + |\delta\gamma(t)|_{\mathbb{R}^2}^2 \frac{h_n}{2} \right].$$

Thus, the hypotheses of Lebesgue dominated convergence theorem are verified and

$$\begin{aligned} \iint_{\Sigma_1} \frac{g_{\gamma+h_n\delta\gamma}(x, t) - g_{\gamma}(x, t)}{h_n} p(x, t) dS(x) dt &\rightarrow \frac{2\alpha P}{\pi R^2} \iint_{\Sigma_1} \frac{4}{R^2} e^{-\frac{2}{R^2}|x-\gamma(t)|^2} (x - \gamma(t)|\delta\gamma(t))_{\mathbb{R}^2} p(x, t) dS(x) dt \\ &= \frac{8\alpha P}{\pi R^4} \iint_{\Sigma_1} e^{-\frac{2}{R^2}|x-\gamma(t)|^2} (x - \gamma(t)|\delta\gamma(t))_{\mathbb{R}^2} p(x, t) dS(x) dt. \end{aligned}$$

Consequently, the directional derivative  $F'(\gamma) \cdot \delta\gamma$  exists and:

$$F'(\gamma) \cdot \delta\gamma = \frac{8\alpha P}{\pi R^4} \iint_{\Sigma_1} e^{-\frac{2}{R^2}|x-\gamma(t)|^2} (x - \gamma(t)|\delta\gamma(t))_{\mathbb{R}^2} p(x, t) dS(x) dt. \quad (6.37)$$

Since  $p \in W(0, T)$ ,  $p|_{\Sigma_1} \in L^2(0, T; L^2(\Gamma_1))$  and from  $H^1(0, T; \mathbb{R}^2) \hookrightarrow C([0, T]; \mathbb{R}^2)$ , it follows that  $F'(\gamma)$  is a continuous linear form on  $H^1(0, T; \mathbb{R}^2)$ . Thus, the Gâteaux derivative of  $F$  at point  $\gamma \in H^1(0, T; \mathbb{R}^2)$  exists and is given by (6.37). Let us show that the mapping  $H^1(0, T; \mathbb{R}^2) \rightarrow H^1(0, T; \mathbb{R}^2)^* : \gamma \mapsto F'(\gamma)$  is continuous. This will imply that the Fréchet derivative exists and is equal to the Gâteaux derivative. Let  $\gamma_0 \in H^1(0, T; \mathbb{R}^2)$ . For  $\delta\gamma \in H^1(0, T; \mathbb{R}^2)$  such that  $\|\delta\gamma\|_{H^1(0, T; \mathbb{R}^2)} \leq 1$ , we have:

$$\begin{aligned} |F'(\gamma) \cdot \delta\gamma - F'(\gamma_0) \cdot \delta\gamma| &\lesssim \left| \iint_{\Sigma_1} e^{-\frac{2}{R^2}|x-\gamma(t)|^2} (\gamma_0(t) - \gamma(t)|\delta\gamma(t))_{\mathbb{R}^2} p(x, t) dS(x) dt \right. \\ &\quad \left. + \iint_{\Sigma_1} (e^{-\frac{2}{R^2}|x-\gamma(t)|^2} - e^{-\frac{2}{R^2}|x-\gamma_0(t)|^2}) (\gamma_0(t)|\delta\gamma(t))_{\mathbb{R}^2} p(x, t) dS(x) dt \right| \\ &\lesssim \iint_{\Sigma_1} \|\gamma_0 - \gamma\|_{C([0, T]; \mathbb{R}^2)} |p(x, t)| dS(x) dt + \iint_{\Sigma_1} |e^{-\frac{2}{R^2}|x-\gamma(t)|^2} - e^{-\frac{2}{R^2}|x-\gamma_0(t)|^2}| |p(x, t)| dS(x) dt \\ &\lesssim \|\gamma_0 - \gamma\|_{C([0, T]; \mathbb{R}^2)} \iint_{\Sigma_1} |p(x, t)| dS(x) dt \lesssim \|\gamma_0 - \gamma\|_{H^1(0, T; \mathbb{R}^2)}, \end{aligned}$$

showing that  $F'(\gamma)$  tends to  $F'(\gamma_0)$  in  $[H^1(0, T; \mathbb{R}^2)]^*$  as  $\gamma \rightarrow \gamma_0$  in  $H^1(0, T; \mathbb{R}^2)$ . Consequently,  $F$  is continuously Fréchet differentiable and  $F'(\gamma) \cdot \delta\gamma$  is given by formula (6.37). Thus by (6.9):

$$D_{\gamma} \mathcal{L}(\bar{y}, \bar{\gamma}, \bar{\tau}, p) \delta\bar{\gamma} = \lambda_{\gamma} \int_0^T \bar{\gamma}(t) \cdot \delta\bar{\gamma}(t) dt + \lambda_{\gamma} \int_0^T \bar{\gamma}'(t) \cdot \delta\bar{\gamma}'(t) dt + F'(\bar{\gamma}) \cdot \delta\bar{\gamma}, \quad (6.38)$$

which by formula (6.37) coincides with the r.h.s. of (6.25).

Let us now prove that  $(\bar{\gamma}, \bar{\tau})$  verifies also the variational inequality:

$$D_{\tau} \mathcal{L}(\bar{y}, \bar{\gamma}, \bar{\tau}, p)(s - \bar{\tau}) \geq 0 \quad \text{for all } s \in [3r, T]. \quad (6.39)$$

It follows immediately from the definition of the Lagrangian function that  $D_{\tau} \mathcal{L}(\bar{y}, \bar{\gamma}, \bar{\tau}, p) = D_{\tau} J_{\tau}(\bar{y}, \bar{\gamma}, \bar{\tau}, p) = D_{\tau} \hat{J}_{\tau}(\bar{\gamma}, \bar{\tau}, p)$ , so that the variational inequality (6.26) is equivalent to the variational inequality (6.39).  $\square$

## 7 Second order Fréchet differentiability of the control-to-state operator

In this section, we aim to find explicitly the second order differential of  $G$ . To do that we proceed in the following steps:

**Step 1:** Finding the second order differential in a formal way.

We already know the first order differential from (4.2). Giving an increment  $\delta\gamma_2$  to  $\gamma$  in  $G$ , we formally get the following by using the first order finite Taylor expansion,

$$\begin{aligned} e^{w(\gamma+\delta\gamma_2)}\tilde{\gamma}(\gamma+\delta\gamma_2) \cdot \delta\gamma_1 &= e^{w(\gamma)+2c_R\tilde{\gamma}(\gamma)\cdot\delta\gamma_2+o(\|\delta\gamma_2\|_{H^1(0,T;\mathbb{R}^2)})}[x-(\gamma+\delta\gamma_2)(t)] \cdot \delta\gamma_1 \quad \text{by (4.9)} \\ &= e^{w(\gamma)}[1+2c_R\tilde{\gamma}(\gamma) \cdot \delta\gamma_2+o(\|\delta\gamma_2\|)](\tilde{\gamma}(\gamma)-\delta\gamma_2) \cdot \delta\gamma_1 \\ &= e^{w(\gamma)}\tilde{\gamma}(\gamma) \cdot \delta\gamma_1 + e^{w(\gamma)}(-\delta\gamma_1 \cdot \delta\gamma_2 + 2c_R\tilde{\gamma}(\gamma) \cdot \delta\gamma_1\tilde{\gamma}(\gamma) \cdot \delta\gamma_2) + o(\|\delta\gamma_2\|) \cdot \delta\gamma_1 \end{aligned} \quad (7.1)$$

where  $\lim_{x \rightarrow 0} \frac{o(\|\delta\gamma_2\|)}{\|\delta\gamma_2\|} = 0$ , then

$$D^2G(\gamma)(\delta\gamma_1, \delta\gamma_2) = 2ac_R\sigma \left[ e^{w(\gamma)}(2c_R\tilde{\gamma}(\gamma) \cdot \delta\gamma_1\tilde{\gamma}(\gamma) \cdot \delta\gamma_2 - \delta\gamma_1 \cdot \delta\gamma_2) \right] \quad (7.2)$$

**Step 2:** Let us prove that the right-hand side of (7.2) is well defined.

Let  $\delta\gamma_1, \delta\gamma_2 \in H^1(0, T; \mathbb{R}^2)$ . As  $H^1(0, T; \mathbb{R}^2) \hookrightarrow C([0, T]; \mathbb{R}^2)$ , we get that  $\delta\gamma_1, \delta\gamma_2 \in C([0, T]; \mathbb{R}^2)$ . Therefore  $\delta\gamma_1 \cdot \delta\gamma_2 \in L^2(0, T; \mathbb{R})$  being continuous on a compact interval and thus bounded.

In addition, we also have that  $\tilde{\gamma} : (x, t) \mapsto \tilde{\gamma}(\gamma)(x, t) = x - \gamma(t) \in \mathbb{R}^2$  is continuous on  $\bar{\Sigma}_1 = \bar{\Gamma}_1 \times [0, T]$  and also  $\delta\gamma_i$  which can be viewed as defined on  $\bar{\Sigma}_1$  by  $(x, t) \mapsto t \mapsto \delta\gamma_i(t) \in \mathbb{R}^2$  for all  $(x, t) \in \bar{\Sigma}_1$  ( $i = 1, 2$ ). Thus,

$$2c_R\tilde{\gamma}(\gamma) \cdot \delta\gamma_1\tilde{\gamma}(\gamma) \cdot \delta\gamma_2 - \delta\gamma_1 \cdot \delta\gamma_2 \in C(\bar{\Sigma}_1; \mathbb{R}) \hookrightarrow L^2(\Sigma_1).$$

Further, as  $e^{w(\gamma)} := e^{-c_R|\tilde{\gamma}(\gamma)|^2}$  is bounded from above by 1 on  $\Sigma_1$ , it belongs to  $L^\infty(\Sigma_1)$ . Consequently

$$e^{w(\gamma)}(2c_R\tilde{\gamma}(\gamma) \cdot \delta\gamma_1\tilde{\gamma}(\gamma) \cdot \delta\gamma_2 - \delta\gamma_1 \cdot \delta\gamma_2) \in L^2(\Sigma_1).$$

Now  $\sigma$ , as defined in (4.4), being a linear and continuous operator from  $L^2(\Sigma_1)$  into  $W(0, T)$ , we get

$$\sigma(e^{w(\gamma)}(2c_R\tilde{\gamma}(\gamma) \cdot \delta\gamma_1\tilde{\gamma}(\gamma) \cdot \delta\gamma_2 - \delta\gamma_1 \cdot \delta\gamma_2)) \text{ has sense} \quad (7.3)$$

and belongs to  $W(0, T)$ , in particular to  $L^2(0, T; H^1(\Omega))$ .

**Step 3:** If the second derivative of  $G$  at point  $\gamma$ ,  $D^2G(\gamma)$ , exists then  $D^2G(\gamma)$  belongs to the space

$$\mathcal{L}(H^1(0, T; \mathbb{R}^2); \mathcal{L}(H^1(0, T; \mathbb{R}^2); W(0, T)))$$

which is identified to the space of bilinear continuous mappings from  $H^1(0, T; \mathbb{R}^2)^2$  into  $W(0, T)$

$$\mathcal{L}_2(H^1(0, T; \mathbb{R}^2); W(0, T)).$$

From this comes the idea that we must prove that the mapping

$$B_\gamma : (\delta\gamma_1, \delta\gamma_2) \mapsto 2ac_R\sigma \left[ e^{-c_R|\tilde{\gamma}(\gamma)|^2}(2c_R\tilde{\gamma}(\gamma) \cdot \delta\gamma_1\tilde{\gamma}(\gamma) \cdot \delta\gamma_2 - \delta\gamma_1 \cdot \delta\gamma_2) \right] \quad (7.4)$$

is bilinear and continuous from  $H^1(0, T; \mathbb{R}^2) \times H^1(0, T; \mathbb{R}^2)$  into  $W(0, T)$ . Firstly, let us prove that

$$\begin{aligned} F_\gamma : H^1(0, T; \mathbb{R}^2)^2 &\rightarrow L^2(\Sigma_1) \\ (\delta\gamma_1, \delta\gamma_2) &\mapsto e^{-c_R|\tilde{\gamma}(\gamma)|^2}(2c_R\tilde{\gamma}(\gamma) \cdot \delta\gamma_1\tilde{\gamma}(\gamma) \cdot \delta\gamma_2 - \delta\gamma_1 \cdot \delta\gamma_2) \end{aligned} \quad (7.5)$$

is bilinear and continuous from  $H^1(0, T; \mathbb{R}^2)^2$  into  $L^2(\Sigma_1)$ .

From its definition, we clearly see that  $F_\gamma$  is bilinear. Further,  $F_\gamma$  is continuous since

$$\begin{aligned} |F_\gamma(\delta\gamma_1, \delta\gamma_2)(x, t)| &= e^{-c_R|\tilde{\gamma}(\gamma)(x, t)|^2} |2c_R(x - \gamma(t)) \cdot \delta\gamma_1(t)(x - \gamma(t)) \cdot \delta\gamma_2(t) - \delta\gamma_1(t) \cdot \delta\gamma_2(t)| \\ &\leq 2c_R|x - \gamma(t)|^2 |\delta\gamma_1(t)| |\delta\gamma_2(t)| + |\delta\gamma_1(t)| |\delta\gamma_2(t)| \\ &\leq c(\|\gamma\|_\infty) \|\delta\gamma_1\|_\infty \|\delta\gamma_2\|_\infty, \end{aligned} \quad (7.6)$$

where  $\|\beta\|_\infty := \sup_{t \in [0, T]} |\beta(t)|$ ,  $\forall \beta \in H^1(0, T; \mathbb{R}^2)$ . Using that  $\delta\gamma_i \in H^1(0, T; \mathbb{R}^2) \hookrightarrow C([0, T]; \mathbb{R}^2)$  ( $i = 1, 2$ ), we have:

$$\begin{aligned} \|F_\gamma(\delta\gamma_1, \delta\gamma_2)\|_{L^2(\Sigma_1)}^2 &= \int_{\Sigma_1} |F_\gamma(\delta\gamma_1, \delta\gamma_2)(x, t)|^2 dx dt \\ &\leq c \int_{\Sigma_1} \|\delta\gamma_1\|_{H^1(0, T; \mathbb{R}^2)}^2 \|\delta\gamma_2\|_{H^1(0, T; \mathbb{R}^2)}^2 dx dt. \end{aligned} \quad (7.7)$$

Thus

$$\|F_\gamma(\delta\gamma_1, \delta\gamma_2)\|_{L^2(\Sigma_1)} \leq C(\Sigma_1, \|\gamma\|_\infty) \|\delta\gamma_1\|_{H^1(0, T; \mathbb{R}^2)} \|\delta\gamma_2\|_{H^1(0, T; \mathbb{R}^2)}. \quad (7.8)$$

Now, we know that  $\sigma$  is a linear continuous operator from  $L^2(\Sigma_1)$  into  $W(0, T)$ . It follows that the composite function  $\sigma \circ F_\gamma : (\delta\gamma_1, \delta\gamma_2) \mapsto \sigma \circ F(\delta\gamma_1, \delta\gamma_2)$  is linear with respect to  $\delta\gamma_1$ . The same can be done to prove the linearity of  $\sigma \circ F_\gamma$  with respect to  $\delta\gamma_2$ . Moreover, as a composite function,  $\sigma \circ F_\gamma$  is also continuous. We have thus proved that  $B_\gamma$  defined by (7.4) is a bilinear and continuous mapping from  $H^1(0, T; \mathbb{R}^2) \times H^1(0, T; \mathbb{R}^2)$  into  $W(0, T)$ .

**Step 4:** To prove that  $B_\gamma = 2ac_R \sigma \circ F_\gamma$  defined by equation (7.4) is the second order derivative  $D^2G(\gamma) \in \mathcal{L}_2(H^1(0, T; \mathbb{R}^2); W(0, T))$  of the function  $G : H^1(0, T; \mathbb{R}^2) \rightarrow W(0, T)$  at point  $\gamma$ , we need to prove that for every  $\delta\gamma_2 \in H^1(0, T; \mathbb{R}^2)$ :

$$\|DG(\gamma + \delta\gamma_2) - DG(\gamma) - B_\gamma(\cdot, \delta\gamma_2)\|_{\mathcal{L}(H^1(0, T; \mathbb{R}^2); W(0, T))} = o(\|\delta\gamma_2\|_{H^1(0, T; \mathbb{R}^2)}) \quad (7.9)$$

i.e.

$$\sup_{\|\delta\gamma_1\|_{H^1(0, T; \mathbb{R}^2)} \leq 1} \|DG(\gamma + \delta\gamma_2) \cdot \delta\gamma_1 - DG(\gamma) \cdot \delta\gamma_1 - B_\gamma(\delta\gamma_1, \delta\gamma_2)\|_{W(0, T)} = o(\|\delta\gamma_2\|_{H^1(0, T; \mathbb{R}^2)}). \quad (7.10)$$

So, let us compute  $DG(\gamma + \delta\gamma_2) \cdot \delta\gamma_1 - DG(\gamma) \cdot \delta\gamma_1 - 2ac_R \sigma \circ F_\gamma(\delta\gamma_1, \delta\gamma_2)$ . By (7.5), we have:

$$\begin{aligned} &DG(\gamma + \delta\gamma_2) \cdot \delta\gamma_1 - DG(\gamma) \cdot \delta\gamma_1 - 2ac_R \sigma \left[ e^{w(\gamma)} (2c_R \tilde{\gamma}(\gamma) \cdot \delta\gamma_1 \tilde{\gamma}(\gamma) \cdot \delta\gamma_2 - \delta\gamma_1 \cdot \delta\gamma_2) \right] \\ &= 2ac_R \sigma \left[ e^{-c_R |\tilde{\gamma}(\gamma + \delta\gamma_2)|^2} \tilde{\gamma}(\gamma + \delta\gamma_2) \cdot \delta\gamma_1 - e^{-c_R |\tilde{\gamma}(\gamma)|^2} \tilde{\gamma}(\gamma) \cdot \delta\gamma_1 \right. \\ &\quad \left. - e^{-c_R |\tilde{\gamma}(\gamma)|^2} (2c_R \tilde{\gamma}(\gamma) \cdot \delta\gamma_1 \tilde{\gamma}(\gamma) \cdot \delta\gamma_2 - \delta\gamma_1 \cdot \delta\gamma_2) \right]. \end{aligned} \quad (7.11)$$

It suffices to consider the expression between the square brackets as  $\sigma \in \mathcal{L}(L^2(\Sigma_1); W(0, T))$ . We have :

$$\begin{aligned} &\left[ e^{-c_R |\tilde{\gamma}(\gamma + \delta\gamma_2)|^2} \tilde{\gamma}(\gamma + \delta\gamma_2) \cdot \delta\gamma_1 - e^{-c_R |\tilde{\gamma}(\gamma)|^2} \tilde{\gamma}(\gamma) \cdot \delta\gamma_1 - e^{-c_R |\tilde{\gamma}(\gamma)|^2} (2c_R \tilde{\gamma}(\gamma) \cdot \delta\gamma_1 \tilde{\gamma}(\gamma) \cdot \delta\gamma_2 - \delta\gamma_1 \cdot \delta\gamma_2) \right] (x, t) \\ &= \left[ e^{-c_R |x - \gamma(t) - \delta\gamma_2(t)|^2} (x - \gamma(t) - \delta\gamma_2(t)) \cdot \delta\gamma_1(t) - e^{-c_R |x - \gamma(t)|^2} (x - \gamma(t)) \cdot \delta\gamma_1(t) \right. \\ &\quad \left. - e^{-c_R |x - \gamma(t)|^2} (2c_R (x - \gamma(t)) \cdot \delta\gamma_1(t) (x - \gamma(t)) \cdot \delta\gamma_2(t) - \delta\gamma_1(t) \cdot \delta\gamma_2(t)) \right]. \end{aligned} \quad (7.12)$$

Taking  $\delta\gamma_1(t)$  as factor in the three terms, (7.12) can be rewritten:

$$\begin{aligned} &\delta\gamma_1(t) \cdot \left[ e^{-c_R |x - \gamma(t) - \delta\gamma_2(t)|^2} (x - \gamma(t) - \delta\gamma_2(t)) - e^{-c_R |x - \gamma(t)|^2} (x - \gamma(t)) \right. \\ &\quad \left. - e^{-c_R |x - \gamma(t)|^2} (2c_R (x - \gamma(t)) \cdot \delta\gamma_2(t) (x - \gamma(t)) - \delta\gamma_2(t)) \right]. \end{aligned} \quad (7.13)$$

**Step 5:** To prove that  $D^2G(\gamma)$  exists and is equal to  $B_\gamma = 2ac_R \sigma \circ F_\gamma$ , we must prove that:

$$\|DG(\gamma + \delta\gamma_2) \cdot \delta\gamma_1 - DG(\gamma) \cdot \delta\gamma_1 - 2ac_R \sigma \circ F_\gamma(\delta\gamma_1, \delta\gamma_2)\|_{W(0, T)} = o(\|\delta\gamma_2\|_{H^1(0, T; \mathbb{R}^2)})$$

uniformly in  $\delta\gamma_1$  for  $\|\delta\gamma_1\|_{H^1(0, T; \mathbb{R}^2)}$  bounded. By (7.13), this will be proved, if we show that:

$$\begin{aligned} &\sup_{\|\delta\gamma_1\|_{H^1(0, T; \mathbb{R}^2)} \leq 1} \|\delta\gamma_1\|_\infty \left\| e^{-c_R |x - \gamma(t) - \delta\gamma_2(t)|^2} (x - \gamma(t) - \delta\gamma_2(t)) - e^{-c_R |x - \gamma(t)|^2} (x - \gamma(t)) \right. \\ &\quad \left. - e^{-c_R |x - \gamma(t)|^2} \left( 2c_R (x - \gamma(t)) \cdot \delta\gamma_2(t) (x - \gamma(t)) - \delta\gamma_2(t) \right) \right\|_{L^2(\Sigma_1)^2} = o(\|\delta\gamma_2\|_{H^1(0, T; \mathbb{R}^2)}). \end{aligned} \quad (7.14)$$

But  $\|\delta\gamma_1\|_{H^1(0,T;\mathbb{R}^2)} \leq 1$  so that a fortiori  $\|\delta\gamma_1\|_\infty \leq c$  for some positive constant  $c$ . Also due to the compact embedding  $H^1(0,T;\mathbb{R}^2)$  in  $C([0,T];\mathbb{R}^2)$ , a  $o(\|\delta\gamma_2\|_\infty)$  is a fortiori a  $o(\|\delta\gamma_2\|_{H^1(0,T;\mathbb{R}^2)})$ . Therefore it suffices to prove that

$$\begin{aligned} & \|e^{-c_R|x-\gamma(t)-\delta\gamma_2(t)|^2}(x-\gamma(t)-\delta\gamma_2(t)) - e^{-c_R|x-\gamma(t)|^2}(x-\gamma(t)) \\ & - e^{-c_R|x-\gamma(t)|^2}(2c_R(x-\gamma(t)).\delta\gamma_2(t)(x-\gamma(t)) - \delta\gamma_2(t))\|_{L^2(\Sigma_1)^2} = o(\|\delta\gamma_2\|_\infty). \end{aligned} \quad (7.15)$$

Substituting the direct equality

$$e^{-c_R|x-\gamma(t)-\delta\gamma_2(t)|^2} = e^{-c_R|x-\gamma(t)|^2} \cdot e^{-c_R[\|\delta\gamma_2(t)\|^2 - 2(x-\gamma(t)).\delta\gamma_2(t)]}$$

in (7.15), it reduces to prove that

$$\begin{aligned} & \|e^{-c_R|x-\gamma(t)|^2}[e^{-c_R[\|\delta\gamma_2(t)\|^2 - 2(x-\gamma(t)).\delta\gamma_2(t)]}(x-\gamma(t)-\delta\gamma_2(t)) \\ & - (x-\gamma(t)) - 2c_R(x-\gamma(t)).\delta\gamma_2(t)(x-\gamma(t)) + \delta\gamma_2(t)]\|_{L^2(\Sigma_1)^2} = o(\|\delta\gamma_2\|_\infty) \end{aligned} \quad (7.16)$$

i.e.

$$\begin{aligned} & \|e^{-c_R|x-\gamma(t)|^2}[(e^{-c_R[\|\delta\gamma_2(t)\|^2 - 2(x-\gamma(t)).\delta\gamma_2(t)]} - 1)(x-\gamma(t)-\delta\gamma_2(t)) \\ & - 2c_R(x-\gamma(t)).\delta\gamma_2(t)(x-\gamma(t))]\|_{L^2(\Sigma_1)^2} = o(\|\delta\gamma_2\|_\infty). \end{aligned} \quad (7.17)$$

Now, we may drop the factor  $e^{-c_R|x-\gamma(t)|^2}$  since it is bounded from above by 1, so that it suffices to prove that

$$\begin{aligned} & \|(e^{-c_R[\|\delta\gamma_2(t)\|^2 - 2(x-\gamma(t)).\delta\gamma_2(t)]} - 1)(x-\gamma(t)-\delta\gamma_2(t)) - 2c_R(x-\gamma(t)).\delta\gamma_2(t)(x-\gamma(t))\|_{L^2(\Sigma_1)^2} \\ & = o(\|\delta\gamma_2\|_\infty). \end{aligned} \quad (7.18)$$

Now,  $\delta\gamma_2(t)$  and  $x-\gamma(t)$  being bounded, it is sufficient to prove that

$$\begin{aligned} & \| -c_R[\|\delta\gamma_2(t)\|^2 - 2(x-\gamma(t)).\delta\gamma_2(t)](x-\gamma(t)-\delta\gamma_2(t)) - 2c_R(x-\gamma(t)).\delta\gamma_2(t)(x-\gamma(t)) \|_{L^2(\Sigma_1)^2} \\ & = o(\|\delta\gamma_2\|_\infty), \end{aligned} \quad (7.19)$$

i.e.

$$\| -c_R[\|\delta\gamma_2(t)\|^2 - 2(x-\gamma(t)).\delta\gamma_2(t)](x-\gamma(t)-\delta\gamma_2(t)) - 2c_R(x-\gamma(t)).\delta\gamma_2(t)(x-\gamma(t)) \|_{L^2(\Sigma_1)^2} = o(\|\delta\gamma_2\|_{H^1(0,T;\mathbb{R}^2)}) \quad (7.20)$$

which is obvious.

**Step 6:** It is also clear from formula (7.4), that  $D^2G$  is also continuous from  $U \subset H^1(0,T;\mathbb{R}^2)$  into  $\mathcal{L}_2(H^1(0,T;\mathbb{R}^2);W(0,T))$ . We have thus proved the following proposition:

**Proposition 7.1.** *The control to state mapping  $G$  is twice continuously Fréchet differentiable from  $U \subset H^1(0,T;\mathbb{R}^2)$  into  $W(0,T)$ . The second derivative  $D^2G(\gamma)$  at point  $\gamma \in U$ , is the bilinear continuous mapping which sends  $(\delta\gamma_1, \delta\gamma_2) \in [H^1(0,T;\mathbb{R}^2)]^2$  to  $2ac_R y$ , where  $y = y(\delta\gamma_1, \delta\gamma_2) \in W(0,T)$  is the weak solution of*

$$\begin{cases} \rho c \partial_t y - \kappa \Delta y = 0 & \text{in } Q, \\ \kappa \partial_\nu y + h y = e^{w(\gamma)}(2c_R \tilde{\gamma}(\gamma). \delta\gamma_1 \tilde{\gamma}(\gamma). \delta\gamma_2 - \delta\gamma_1. \delta\gamma_2) & \text{on } \Sigma_1, \\ \kappa \partial_\nu y + h y = 0 & \text{on } \Sigma_2 \cup \Sigma_3, \\ y(\cdot, 0) = 0 & \text{in } \Omega. \end{cases} \quad (7.21)$$

## 8 Second order differentiability of the reduced cost functional

**Proposition 8.1.** *For fixed  $\tau$ , the mapping  $\gamma \in U \mapsto \hat{J}_r(\gamma, \tau) \in \mathbb{R}$  is twice continuously differentiable and we have for all  $\delta\gamma_1, \delta\gamma_2$  in  $H^1(0, T; \mathbb{R}^2)$ :*

$$\begin{aligned} D_{\gamma\gamma} \hat{J}_r(\gamma, \tau) \cdot [\delta\gamma_1, \delta\gamma_2] = & \\ & \frac{\lambda_Q}{r} \int_{\frac{\tau}{2}-\frac{r}{2}}^{\frac{\tau}{2}+\frac{r}{2}} \int_{\Omega} \left( (G(\gamma) - y_Q)(x, t) D_{\gamma\gamma} G(\gamma) \cdot [\delta\gamma_1, \delta\gamma_2](x, t) + v(\gamma, \delta\gamma_2)(x, t) \cdot v(\gamma, \delta\gamma_1)(x, t) \right) dx dt \\ & + \frac{\lambda_{\Omega}}{r} \int_{\tau-r}^{\tau} \int_{\Omega} \left( (G(\gamma) - y_{\Omega})(x, t) D_{\gamma\gamma} G(\gamma) \cdot [\delta\gamma_1, \delta\gamma_2](x, t) + v(\gamma, \delta\gamma_2)(x, t) \cdot v(\gamma, \delta\gamma_1)(x, t) \right) dx dt \\ & + \int_0^T \int_{\Omega} \left( \nabla v(\gamma, \delta\gamma_2)(x, t) \cdot \nabla v(\gamma, \delta\gamma_1)(x, t) + \nabla G(\gamma)(x, t) \cdot \nabla D_{\gamma\gamma} G(\gamma) \cdot [\delta\gamma_1, \delta\gamma_2](x, t) \right) dx dt \\ & + \lambda_{\gamma}(\delta\gamma_2, \delta\gamma_1)_{H^1(0, T; \mathbb{R}^2)}. \end{aligned} \quad (8.1)$$

*Proof.* Let us consider the first term in the right-hand side of  $\hat{J}_r$  in (3.2),

$$\gamma \mapsto \int_{\frac{\tau}{2}-\frac{r}{2}}^{\frac{\tau}{2}+\frac{r}{2}} \int_{\Omega} |G(\gamma)(x, t) - y_Q(x, t)|^2 dx dt.$$

We may consider it as the composite function of

$$h : U \rightarrow L^2(Q)^2 : \gamma \mapsto (G(\gamma) - y_Q, G(\gamma) - y_Q)$$

followed by the continuous bilinear mapping

$$[\cdot, \cdot] : (f, g) \in L^2(Q)^2 \mapsto \int_{\frac{\tau}{2}-\frac{r}{2}}^{\frac{\tau}{2}+\frac{r}{2}} \int_{\Omega} f(x, t) g(x, t) dx dt \in \mathbb{R}.$$

As we know that  $G$  is two times continuously differentiable from  $U$  into  $W(0, T)$  and thus a fortiori into  $L^2(Q)$ , the mapping  $h$  is two times continuously differentiable. The expression of its first derivative follows immediately from (4.2) and of its second derivative from (7.4). The continuous bilinear mapping  $[\cdot, \cdot]$  is two times continuously differentiable. Its first derivative at point  $(f, g) \in L^2(Q)^2$  is the linear continuous mapping

$$(\delta f, \delta g) \in L^2(Q)^2 \mapsto [\delta f, g] + [f, \delta g] \in \mathbb{R}$$

and its second derivative at any point  $(f, g) \in L^2(Q) \times L^2(Q)$  is the bilinear continuous mapping

$$((\Delta f, \Delta g), (\delta f, \delta g)) \in (L^2(Q)^2 \times L^2(Q)^2) \mapsto [\delta f, \Delta g] + [\Delta f, \delta g] \in \mathbb{R}.$$

Thus by Theorem 5.4.2 of [7, p.72] the composite mapping corresponding to the first term of (3.2) is two times continuously differentiable and the expression of its second derivative follows from formula (7.5.1) of [7, p.95] which gives the expression of the second derivative of the composite of two functions.

The second term in (3.2) is treated similarly. For the third term,

$$\begin{aligned} D_{\gamma\gamma} \left( \frac{1}{2} \int_0^T \int_{\Omega} |\nabla G(\gamma)(x, t)|^2 dx dt \right) [\delta\gamma_1, \delta\gamma_2] = & \int_0^T \int_{\Omega} \left( \nabla v(\gamma, \delta\gamma_2)(x, t) \cdot \nabla v(\gamma, \delta\gamma_1)(x, t) \right. \\ & \left. + \nabla G(\gamma)(x, t) \cdot \nabla D_{\gamma\gamma} G(\gamma) \cdot [\delta\gamma_1, \delta\gamma_2] \right) dx dt. \end{aligned}$$

The last but one term  $\gamma \mapsto \frac{\lambda_{\gamma}}{2} \|\gamma\|_{H^1(0, T; \mathbb{R}^2)}^2$  in (3.2) is obviously of class  $\mathbf{C}^2$  and the computation of its second derivative is immediate being the quadratic form associated with the continuous bilinear form of the scalar product in  $H^1(0, T; \mathbb{R}^2)$ . Therefore, we get (8.1).  $\square$

**Proposition 8.2.** *The mapping  $U \times [3r, T] \rightarrow \mathcal{L}_2(H^1(0, T; \mathbb{R}^2); \mathbb{R}) : (\gamma, \tau) \mapsto D_{\gamma\gamma} \hat{J}_r(\gamma, \tau)$  is continuous.*

398 *Proof.* Let us fix some  $\gamma' \in U \subset H^1(0, T; \mathbb{R}^2)$  and some  $\tau' \in [3r, T]$ . We suppose that  $\gamma$  in  $U$  moves to  $\gamma'$  and that  
 399  $\tau \in [3r, T]$  moves to  $\tau'$ . For the first part of the first term in the right hand side of equation (8.1), we have:

$$\begin{aligned}
 & \left| \int_{\frac{\tau}{2}-\frac{r}{2}}^{\frac{\tau}{2}+\frac{r}{2}} \int_{\Omega} (G(\gamma) - y_Q)(x, t) D_{\gamma\gamma} G(\gamma) \cdot [\delta\gamma_1, \delta\gamma_2](x, t) dx dt - \int_{\frac{\tau'}{2}-\frac{r}{2}}^{\frac{\tau'}{2}+\frac{r}{2}} \int_{\Omega} (G(\gamma') - y_Q)(x, t) D_{\gamma\gamma} G(\gamma') \cdot [\delta\gamma_1, \delta\gamma_2](x, t) dx dt \right| \\
 & \leq \left| \int_{\frac{\tau}{2}-\frac{r}{2}}^{\frac{\tau}{2}+\frac{r}{2}} \int_{\Omega} (G(\gamma) - G(\gamma'))(x, t) D_{\gamma\gamma} G(\gamma) \cdot [\delta\gamma_1, \delta\gamma_2](x, t) dx dt \right| \\
 & \quad + \left| \int_{\frac{\tau}{2}-\frac{r}{2}}^{\frac{\tau}{2}+\frac{r}{2}} \int_{\Omega} (G(\gamma') - y_Q)(x, t) [D_{\gamma\gamma} G(\gamma) - D_{\gamma\gamma} G(\gamma')] \cdot [\delta\gamma_1, \delta\gamma_2](x, t) dx dt \right| \\
 & \quad + \left| \int_{\frac{\tau}{2}-\frac{r}{2}}^{\frac{\tau}{2}+\frac{r}{2}} \int_{\Omega} (G(\gamma') - y_Q)(x, t) D_{\gamma\gamma} G(\gamma') \cdot [\delta\gamma_1, \delta\gamma_2](x, t) dx dt \right. \\
 & \quad \left. - \int_{\frac{\tau'}{2}-\frac{r}{2}}^{\frac{\tau'}{2}+\frac{r}{2}} \int_{\Omega} (G(\gamma') - y_Q)(x, t) D_{\gamma\gamma} G(\gamma') \cdot [\delta\gamma_1, \delta\gamma_2](x, t) dx dt \right|
 \end{aligned} \tag{8.2}$$

401 Let us bound the first term in the right hand side in inequality (8.2). For  $\delta\gamma_1, \delta\gamma_2 \in H^1(0, T; \mathbb{R}^2)$  with  $\|\delta\gamma_1\|_{H^1(0, T; \mathbb{R}^2)},$   
 402  $\|\delta\gamma_2\|_{H^1(0, T; \mathbb{R}^2)} \leq 1$ :

$$\begin{aligned}
 & \left| \int_{\frac{\tau}{2}-\frac{r}{2}}^{\frac{\tau}{2}+\frac{r}{2}} \int_{\Omega} [(G(\gamma) - G(\gamma'))(x, t)] D_{\gamma\gamma} G(\gamma) \cdot [\delta\gamma_1, \delta\gamma_2](x, t) dx dt \right| \\
 & \leq \int_0^T \|(G(\gamma) - G(\gamma'))(\cdot, t)\|_{L^2(\Omega)} \|D_{\gamma\gamma} G(\gamma) \cdot [\delta\gamma_1, \delta\gamma_2](\cdot, t)\|_{L^2(\Omega)} dt \\
 & \leq T \|G(\gamma) - G(\gamma')\|_{C([0, T]; L^2(\Omega))} \|D_{\gamma\gamma} G(\gamma) \cdot [\delta\gamma_1, \delta\gamma_2]\|_{C([0, T]; L^2(\Omega))} \\
 & \lesssim \|G(\gamma) - G(\gamma')\|_{C([0, T]; L^2(\Omega))} \|D_{\gamma\gamma} G(\gamma) \cdot [\delta\gamma_1, \delta\gamma_2]\|_{W(0, T)} \\
 & \lesssim \|G(\gamma) - G(\gamma')\|_{C([0, T]; L^2(\Omega))} \|D_{\gamma\gamma} G(\gamma)\|_{\mathcal{L}_2(H^1(0, T; \mathbb{R}^2); W(0, T))} \longrightarrow 0 \text{ as } \gamma \rightarrow \gamma'.
 \end{aligned} \tag{8.3}$$

403 as  $D_{\gamma\gamma} G(\gamma)$  is continuous from  $H^1(0, T; \mathbb{R}^2)$  into  $\mathcal{L}_2(H^1(0, T; \mathbb{R}^2); W(0, T))$  so that  $\|D_{\gamma\gamma} G(\gamma)\|_{\mathcal{L}_2(H^1(0, T; \mathbb{R}^2); W(0, T))}$   
 404 is bounded for  $\gamma$  near  $\gamma'$ .

405 Let us now bound the second term in the right hand side of inequality (8.2): for  $\delta\gamma_1, \delta\gamma_2 \in H^1(0, T; \mathbb{R}^2)$  with  
 406  $\|\delta\gamma_1\|_{H^1(0, T; \mathbb{R}^2)}, \|\delta\gamma_2\|_{H^1(0, T; \mathbb{R}^2)} \leq 1$ :

$$\begin{aligned}
 & \left| \int_{\frac{\tau}{2}-\frac{r}{2}}^{\frac{\tau}{2}+\frac{r}{2}} \int_{\Omega} (G(\gamma') - y_Q)(x, t) [D_{\gamma\gamma} G(\gamma) - D_{\gamma\gamma} G(\gamma')] \cdot [\delta\gamma_1, \delta\gamma_2](x, t) dx dt \right| \\
 & \leq \int_0^T \|(G(\gamma') - y_Q)(\cdot, t)\|_{L^2(\Omega)} \|[D_{\gamma\gamma} G(\gamma) - D_{\gamma\gamma} G(\gamma')] \cdot [\delta\gamma_1, \delta\gamma_2](\cdot, t)\|_{L^2(\Omega)} dt \\
 & \lesssim \|G(\gamma') - y_Q\|_{C([0, T]; L^2(\Omega))} \|[D_{\gamma\gamma} G(\gamma) - D_{\gamma\gamma} G(\gamma')] \cdot [\delta\gamma_1, \delta\gamma_2]\|_{C([0, T]; L^2(\Omega))} \\
 & \lesssim \|[D_{\gamma\gamma} G(\gamma) - D_{\gamma\gamma} G(\gamma')] \cdot [\delta\gamma_1, \delta\gamma_2]\|_{W(0, T)} \\
 & \lesssim \|D_{\gamma\gamma} G(\gamma) - D_{\gamma\gamma} G(\gamma')\|_{\mathcal{L}_2(H^1(0, T; \mathbb{R}^2); W(0, T))} \longrightarrow 0 \text{ as } \gamma \rightarrow \gamma'.
 \end{aligned} \tag{8.4}$$

Let us finally bound the third term in the right hand side of inequality (8.2):

$$\begin{aligned}
 & \left| \int_{\frac{\tau}{2}-\frac{r}{2}}^{\frac{\tau}{2}+\frac{r}{2}} \int_{\Omega} (G(\gamma') - y_Q)(x, t) D_{\gamma\gamma} G(\gamma') \cdot [\delta\gamma_1, \delta\gamma_2](x, t) dx dt - \int_{\frac{\tau'}{2}-\frac{r}{2}}^{\frac{\tau'}{2}+\frac{r}{2}} \int_{\Omega} (G(\gamma') - y_Q)(x, t) D_{\gamma\gamma} G(\gamma') \cdot [\delta\gamma_1, \delta\gamma_2](x, t) dx dt \right| \\
 & \lesssim \int_{I\Delta J} \int_{\Omega} |(G(\gamma') - y_Q)(x, t)| |D_{\gamma\gamma} G(\gamma') \cdot [\delta\gamma_1, \delta\gamma_2](x, t)| dx dt \\
 & \lesssim \|G(\gamma') - y_Q\|_{C([0, T]; L^2(\Omega))} \|D_{\gamma\gamma} G(\gamma') \cdot [\delta\gamma_1, \delta\gamma_2]\|_{C([0, T]; L^2(\Omega))} \text{meas}(I\Delta J) \\
 & \lesssim \|G(\gamma') - y_Q\|_{C([0, T]; L^2(\Omega))} \|D_{\gamma\gamma} G(\gamma')\|_{\mathcal{L}_2(H^1(0, T; \mathbb{R}^2); W(0, T))} \text{meas}(I\Delta J) \longrightarrow 0 \text{ as } \tau \rightarrow \tau'.
 \end{aligned} \tag{8.5}$$

408 Thus by inequality (8.2), as  $(\gamma, \tau)$  tends to  $(\gamma', \tau')$ , we have

$$\int_{\frac{\tau}{2}-\frac{r}{2}}^{\frac{\tau}{2}+\frac{r}{2}} \int_{\Omega} (G(\gamma) - y_Q) D_{\gamma\gamma} G(\gamma) [\delta\gamma_1, \delta\gamma_2](x, t) dx dt \rightarrow \int_{\frac{\tau'}{2}-\frac{r}{2}}^{\frac{\tau'}{2}+\frac{r}{2}} \int_{\Omega} (G(\gamma') - y_Q) D_{\gamma\gamma} G(\gamma') [\delta\gamma_1, \delta\gamma_2](x, t) dx dt \tag{8.6}$$

For the second part of the first term in the right hand side of equation (8.1), we have:

$$\begin{aligned} & \left| \int_{\frac{\tau}{2}-\frac{r}{2}}^{\frac{\tau}{2}+\frac{r}{2}} \int_{\Omega} v(\gamma, \delta\gamma_1) v(\gamma, \delta\gamma_2)(x, t) dx dt - \int_{\frac{\tau'}{2}-\frac{r}{2}}^{\frac{\tau'}{2}+\frac{r}{2}} \int_{\Omega} v(\gamma', \delta\gamma_1) v(\gamma', \delta\gamma_2)(x, t) dx dt \right| \\ & \leq \left| \int_{\frac{\tau}{2}-\frac{r}{2}}^{\frac{\tau}{2}+\frac{r}{2}} \int_{\Omega} (v(\gamma, \delta\gamma_1) v(\gamma, \delta\gamma_2) - v(\gamma', \delta\gamma_1) v(\gamma', \delta\gamma_2)) dx dt \right| + \left| \int_{I\Delta J} \int_{\Omega} v(\gamma', \delta\gamma_1) v(\gamma', \delta\gamma_2) dx dt \right|. \end{aligned} \quad (8.7)$$

For the second term in the right-hand side of the previous inequality :

$$\begin{aligned} & \left| \int_{I\Delta J} \int_{\Omega} v(\gamma', \delta\gamma_1) v(\gamma', \delta\gamma_2) dx dt \right| \leq \int_{I\Delta J} \|v(\gamma', \delta\gamma_1)(\cdot, t)\|_{L^2(\Omega)} \|v(\gamma', \delta\gamma_2)(\cdot, t)\|_{L^2(\Omega)} dt \\ & \lesssim \|v(\gamma', \delta\gamma_1)\|_{C([0, T]; L^2(\Omega))} \|v(\gamma', \delta\gamma_2)\|_{C([0, T]; L^2(\Omega))} \text{meas}(I\Delta J) \\ & \lesssim \|DG(\gamma')\|_{\mathcal{L}(H^1(0, T; \mathbb{R}^2); W(0, T))}^2 \text{meas}(I\Delta J) \end{aligned} \quad (8.8)$$

which tends to zero uniformly in  $\delta\gamma_1, \delta\gamma_2 \in \{\delta\gamma \in H^1(0, T; \mathbb{R}^2); \|\delta\gamma\|_{H^1(0, T; \mathbb{R}^2)} \leq 1\}$  as  $\tau \rightarrow \tau'$ .

For the first term in the right hand side of inequality (8.7):

$$\begin{aligned} & \left| \int_{\frac{\tau}{2}-\frac{r}{2}}^{\frac{\tau}{2}+\frac{r}{2}} \int_{\Omega} (v(\gamma, \delta\gamma_1) v(\gamma, \delta\gamma_2) - v(\gamma', \delta\gamma_1) v(\gamma', \delta\gamma_2)) dx dt \right| \\ & \leq \left| \int_{\frac{\tau}{2}-\frac{r}{2}}^{\frac{\tau}{2}+\frac{r}{2}} \int_{\Omega} (v(\gamma, \delta\gamma_1) - v(\gamma', \delta\gamma_1)) v(\gamma, \delta\gamma_2) dx dt \right| + \left| \int_{\frac{\tau}{2}-\frac{r}{2}}^{\frac{\tau}{2}+\frac{r}{2}} \int_{\Omega} v(\gamma', \delta\gamma_1) (v(\gamma, \delta\gamma_2) - v(\gamma', \delta\gamma_2)) dx dt \right| \\ & \lesssim \|(DG(\gamma) - DG(\gamma'))\delta\gamma_1\|_{C([0, T]; L^2(\Omega))} \|DG(\gamma)\delta\gamma_2\|_{C([0, T]; L^2(\Omega))} \\ & \quad + \|(DG(\gamma) - DG(\gamma'))\delta\gamma_2\|_{C([0, T]; L^2(\Omega))} \|DG(\gamma')\delta\gamma_1\|_{C([0, T]; L^2(\Omega))} \\ & \lesssim \|(DG(\gamma) - DG(\gamma'))\delta\gamma_1\|_{W(0, T)} \|DG(\gamma)\delta\gamma_2\|_{W(0, T)} \\ & \quad + \|(DG(\gamma) - DG(\gamma'))\delta\gamma_2\|_{W(0, T)} \|DG(\gamma')\delta\gamma_1\|_{W(0, T)} \\ & \lesssim \|DG(\gamma) - DG(\gamma')\|_{\mathcal{L}(H^1(0, T; \mathbb{R}^2); W(0, T))} \left( \|DG(\gamma)\|_{\mathcal{L}(H^1(0, T; \mathbb{R}^2); W(0, T))} + \|DG(\gamma')\|_{\mathcal{L}(H^1(0, T; \mathbb{R}^2); W(0, T))} \right), \end{aligned} \quad (8.9)$$

which tends to zero uniformly in  $\delta\gamma_1, \delta\gamma_2 \in \{\delta\gamma \in H^1(0, T; \mathbb{R}^2); \|\delta\gamma\|_{H^1(0, T; \mathbb{R}^2)} \leq 1\}$  and in  $\tau$  as  $\gamma \rightarrow \gamma'$ . Consequently by inequality (8.7):

$$\int_{\frac{\tau}{2}-\frac{r}{2}}^{\frac{\tau}{2}+\frac{r}{2}} \int_{\Omega} v(\gamma, \delta\gamma_1) v(\gamma, \delta\gamma_2)(x, t) dx dt \rightarrow \int_{\frac{\tau'}{2}-\frac{r}{2}}^{\frac{\tau'}{2}+\frac{r}{2}} \int_{\Omega} v(\gamma', \delta\gamma_1) v(\gamma', \delta\gamma_2)(x, t) dx dt, \quad (8.10)$$

uniformly in  $\delta\gamma_1, \delta\gamma_2 \in \{\delta\gamma \in H^1(0, T; \mathbb{R}^2); \|\delta\gamma\|_{H^1(0, T; \mathbb{R}^2)} \leq 1\}$  as  $(\gamma, \tau) \rightarrow (\gamma', \tau')$ .

By (8.6) and (8.10), the first term in the right-hand side of equation (8.1) is continuous in  $(\gamma, \tau)$ . The proof is similar for the second term in the right-hand side of equation (8.1), and the third term in the right-hand side of equation (8.1) depends only on  $\gamma$  and not on  $\tau$ . Its continuity with respect to  $\gamma$  follows from Proposition 3.3, Proposition 7.1 and Theorem 4.1.  $\square$

Let us investigate the partial derivative with respect to  $\tau$  of  $D_{\gamma} \hat{J}_r$  given (5.1).

**Proposition 8.3.** *The mapping  $[3r, T] \rightarrow H^1(0, T; \mathbb{R}^2)^{\star} : \tau \mapsto D_{\gamma} \hat{J}_r(\gamma, \tau)$  is differentiable and:*

$$\begin{aligned} & D_{\tau\gamma} \hat{J}_r(\gamma, \tau) \cdot [\delta\gamma_1, \delta\tau_2] = \\ & \frac{\lambda_Q \delta\tau_2}{2r} \int_{\Omega} \left( (G(\gamma) - y_Q)(x, \frac{\tau}{2} + \frac{r}{2}) v(\gamma, \delta\gamma_1)(x, \frac{\tau}{2} + \frac{r}{2}) - (G(\gamma) - y_Q)(x, \frac{\tau}{2} - \frac{r}{2}) v(\gamma, \delta\gamma_1)(x, \frac{\tau}{2} - \frac{r}{2}) \right) dx \\ & + \frac{\lambda_Q \delta\tau_2}{r} \int_{\Omega} \left( (G(\gamma) - y_Q)(x, \tau) v(\gamma, \delta\gamma_1)(x, \tau) - (G(\gamma) - y_Q)(x, \tau - r) v(\gamma, \delta\gamma_1)(x, \tau - r) \right) dx. \end{aligned} \quad (8.11)$$

*Proof.* We have formula (5.1) for  $D_{\gamma} \hat{J}_r(\gamma, \tau) \cdot \delta\gamma_1$ . We must prove that its derivative with respect to  $\tau$  exists uniformly in  $\delta\gamma_1$  for  $\|\delta\gamma_1\|_{H^1(0, T; \mathbb{R}^2)} \leq 1$  and prove that it is given by the right-hand side of (8.11). Fortunately,

due to the compact embedding of  $H^1(0, T; \mathbb{R}^2)$  into  $C([0, T]; \mathbb{R}^2)$ , and the continuity of  $\sigma$  in (4.2) from  $L^2(\Sigma_1)$  into  $W(0, T)$ , the set  $\{D_\gamma G(\gamma) \cdot \delta\gamma_1; \|\delta\gamma_1\|_{H^1(0, T; \mathbb{R}^2)} \leq 1\}$  is compact in  $W(0, T)$  and also in  $C([0, T]; L^2(\Omega))$  due to the continuous embedding from  $W(0, T)$  into  $C([0, T]; L^2(\Omega))$ . Thus for every  $\varepsilon > 0$ , there exists  $\delta > 0$  such that for every  $\delta\gamma_1 \in H^1(0, T; \mathbb{R}^2)$  such that  $\|\delta\gamma_1\|_{H^1(0, T; \mathbb{R}^2)} \leq 1$ ,  $0 \leq t - (\frac{\tau}{2} + \frac{r}{2}) \leq \delta$  implies

$$\|(D_\gamma G(\gamma) \cdot \delta\gamma_1)(\cdot, t) - D_\gamma G(\gamma) \cdot \delta\gamma_1(\cdot, \frac{\tau}{2} + \frac{r}{2})\|_{L^2(\Omega)} \leq \varepsilon.$$

422 This implies that

$$\begin{aligned} & \|(G(\gamma)(\cdot, \frac{\tau}{2} + \frac{r}{2}) - y_Q(\cdot, \frac{\tau}{2} + \frac{r}{2}))[(D_\gamma G(\gamma) \cdot \delta\gamma_1)(\cdot, t) - (D_\gamma G(\gamma) \cdot \delta\gamma_1)(\cdot, \frac{\tau}{2} + \frac{r}{2})]\|_{L^1(\Omega)} \\ & \leq \varepsilon \|G(\gamma)(\cdot, \frac{\tau}{2} + \frac{r}{2}) - y_Q(\cdot, \frac{\tau}{2} + \frac{r}{2})\|_{L^2(\Omega)}, \quad \text{if } \frac{\tau}{2} + \frac{r}{2} < t < \frac{\tau}{2} + \frac{r}{2} + \delta. \end{aligned} \quad (8.12)$$

423 As  $DG(\gamma)$  is a linear continuous operator from  $H^1(0, T; \mathbb{R}^2)$  into  $W(0, T)$ , we have  $\|DG(\gamma) \cdot \delta\gamma_1\|_{W(0, T)} \leq c$   
 424 for  $\|\delta\gamma_1\|_{H^1(0, T; \mathbb{R}^2)} \leq 1$  and by the continuous embedding from  $W(0, T)$  into  $C([0, T]; L^2(\Omega))$ , we have a fortiori  
 425  $\|DG(\gamma) \cdot \delta\gamma_1\|_{C([0, T]; L^2(\Omega))} \leq c$  for  $\|\delta\gamma_1\|_{H^1(0, T; \mathbb{R}^2)} \leq 1$  for some constant  $c > 0$ . Thus, we have also:

$$\begin{aligned} & \left\| [(G(\gamma)(\cdot, t) - y_Q(\cdot, t)) - (G(\gamma)(\cdot, \frac{\tau}{2} + \frac{r}{2}) - y_Q(\cdot, \frac{\tau}{2} + \frac{r}{2}))](D_\gamma G(\gamma) \cdot \delta\gamma_1)(\cdot, t) \right\|_{L^1(\Omega)} \\ & \leq \|D_\gamma G(\gamma) \cdot \delta\gamma_1\|_{C([0, T]; L^2(\Omega))} \|G(\gamma)(\cdot, t) - y_Q(\cdot, t) - (G(\gamma)(\cdot, \frac{\tau}{2} + \frac{r}{2}) - y_Q(\cdot, \frac{\tau}{2} + \frac{r}{2}))\|_{L^2(\Omega)} \leq \varepsilon \end{aligned} \quad (8.13)$$

for  $\frac{\tau}{2} + \frac{r}{2} < t < \frac{\tau}{2} + \frac{r}{2} + \delta$  if  $\delta > 0$  is choosen sufficiently small.

Now, let us set

$$f(t, \delta\gamma_1) := \frac{\lambda_Q}{r} \int_{\Omega} (G(\gamma)(x, t) - y_Q(x, t)) v(\gamma, \delta\gamma_1)(x, t) dx.$$

426 We have that:

$$\begin{aligned} & \frac{1}{h} \left( \int_0^{\frac{\tau+h}{2} + \frac{r}{2}} f(t, \delta\gamma_1) dt - \int_0^{\frac{\tau}{2} + \frac{r}{2}} f(t, \delta\gamma_1) dt \right) = \frac{1}{h} \int_{\frac{\tau}{2} + \frac{r}{2}}^{\frac{\tau+h}{2} + \frac{r}{2}} f(t, \delta\gamma_1) dt \\ & = \frac{1}{h} \int_{\frac{\tau}{2} + \frac{r}{2}}^{\frac{\tau}{2} + \frac{r}{2} + \frac{h}{2}} (f(t, \delta\gamma_1) - f(\frac{\tau}{2} + \frac{r}{2}, \delta\gamma_1)) dt + \frac{1}{2} f(\frac{\tau}{2} + \frac{r}{2}, \delta\gamma_1). \end{aligned} \quad (8.14)$$

427 By inequalities (8.12) and (8.13),  $\left\| \frac{1}{h} \int_{\frac{\tau}{2} + \frac{r}{2}}^{\frac{\tau+h}{2} + \frac{r}{2}} (f(t, \delta\gamma_1) - f(\frac{\tau}{2} + \frac{r}{2}, \delta\gamma_1)) dt \right\|$  will be smaller than  $\varepsilon$  for sufficiently  
 428 small  $h$  uniformly in  $\delta\gamma_1 \in H^1(0, T; \mathbb{R}^2)$  for  $\|\delta\gamma_1\|_{H^1(0, T; \mathbb{R}^2)} \leq 1$ . This proves that the right-derivative at point  $\tau$  of

$$\delta\gamma_1 \mapsto \frac{\lambda_Q}{r} \int_0^{\frac{\tau}{2} + \frac{r}{2}} \int_{\Omega} (G(\gamma)(x, t) - y_Q(x, t)) \cdot v(\gamma, \delta\gamma_1)(x, t) dx dt \quad (8.15)$$

429 is equal to

$$\delta\gamma_1 \mapsto \frac{\lambda_Q}{2r} \int_{\Omega} (G(\gamma)(x, \frac{\tau}{2} + \frac{r}{2}) - y_Q(x, \frac{\tau}{2} + \frac{r}{2})) \cdot v(\gamma, \delta\gamma_1)(x, \frac{\tau}{2} + \frac{r}{2}) dx. \quad (8.16)$$

430 Similar reasoning shows that this is also true for the left derivative at point  $\tau$ . Writing the integral

$$\frac{\lambda_Q}{r} \int_{\frac{\tau}{2} - \frac{r}{2}}^{\frac{\tau}{2} + \frac{r}{2}} \int_{\Omega} (G(\gamma)(x, t) - y_Q(x, t)) \cdot v(\gamma, \delta\gamma_1)(x, t) dx dt$$

431 as

$$\frac{\lambda_Q}{r} \int_0^{\frac{\tau}{2} + \frac{r}{2}} \int_{\Omega} (G(\gamma) - y_Q)(x, t) \cdot v(\gamma, \delta\gamma_1)(x, t) dx dt - \frac{\lambda_Q}{r} \int_0^{\frac{\tau}{2} - \frac{r}{2}} \int_{\Omega} (G(\gamma) - y_Q)(x, t) \cdot v(\gamma, \delta\gamma_1)(x, t) dx dt, \quad (8.17)$$

we obtain that the partial derivative with respect to the variable  $\tau$  of

$$\delta\tau_1 \in H^1(0, T; \mathbb{R}^2) \mapsto \frac{\lambda_Q}{r} \int_{\frac{\tau}{2} - \frac{r}{2}}^{\frac{\tau}{2} + \frac{r}{2}} \int_{\Omega} (G(\gamma)(x, t) - y_Q(x, t)) \cdot v(\gamma, \delta\gamma_1)(x, t) dx dt$$



exists and is given by the first term in the right-hand side of (8.11).

Similar reasonings show that the partial derivative with respect to the variable  $\tau$  of

$$\delta\tau_1 \in H^1(0, T; \mathbb{R}^2) \mapsto \frac{\lambda_\Omega}{r} \int_{\tau-r}^{\tau} \int_{\Omega} (G(\gamma)(x, t) - y_\Omega(x, t)) \cdot v(\gamma, \delta\gamma_1)(x, t) dx dt$$

exists and is given by the second term in the right-hand side of (8.11). □

**Proposition 8.4.** *The mapping  $D_{\tau\gamma}\hat{J}_r : U \times [3r, T] \rightarrow \mathbb{R} : (\gamma, \tau) \mapsto D_{\tau\gamma}\hat{J}_r(\gamma, \tau)$  is continuous from  $U \times [3r, T]$  into  $\mathcal{L}(H^1(0, T; \mathbb{R}^2), \mathbb{R}; \mathbb{R})$ .*

*Proof.* Let us prove that the first term in the right-hand side of equation (8.11) defines a continuous mapping from  $U \times [3r, T]$  into  $\mathcal{L}(H^1(0, T; \mathbb{R}^2) \times \mathbb{R}; \mathbb{R})$ , the proof being similar for the other terms. This amounts to prove that the mapping

$$U \times [3r, T] \rightarrow H^1(0, T; \mathbb{R}^2)^* \\ (\gamma, \tau) \mapsto \left[ \delta\gamma_1 \rightarrow \int_{\Omega} (G(\gamma) - y_Q)(x, \frac{\tau}{2} + \frac{r}{2}) v(\gamma, \delta\gamma_1)(x, \frac{\tau}{2} + \frac{r}{2}) dx \right]$$

is continuous, the proof being similar for the second part of this term. Let us fix some  $(\gamma', \tau')$  in  $U \times [3r, T]$ . We must prove that when  $(\gamma, \tau)$  in  $U \times [3r, T]$  tends to  $(\gamma', \tau')$ , that

$$\int_{\Omega} (G(\gamma) - y_Q)(x, \frac{\tau}{2} + \frac{r}{2}) v(\gamma, \delta\gamma_1)(x, \frac{\tau}{2} + \frac{r}{2}) dx \longrightarrow \int_{\Omega} (G(\gamma') - y_Q)(x, \frac{\tau'}{2} + \frac{r}{2}) v(\gamma', \delta\gamma_1)(x, \frac{\tau'}{2} + \frac{r}{2}) dx \quad (8.18)$$

uniformly in  $\delta\gamma_1$  for  $\|\delta\gamma_1\|_{H^1(0, T; \mathbb{R}^2)} \leq 1$ . Now

$$\begin{aligned} & \left| \int_{\Omega} (G(\gamma) - y_Q)(x, \frac{\tau}{2} + \frac{r}{2}) v(\gamma, \delta\gamma_1)(x, \frac{\tau}{2} + \frac{r}{2}) dx - \int_{\Omega} (G(\gamma') - y_Q)(x, \frac{\tau'}{2} + \frac{r}{2}) v(\gamma', \delta\gamma_1)(x, \frac{\tau'}{2} + \frac{r}{2}) dx \right| \\ & \leq \int_{\Omega} |(G(\gamma) - y_Q)(x, \frac{\tau}{2} + \frac{r}{2}) v(\gamma, \delta\gamma_1)(x, \frac{\tau}{2} + \frac{r}{2}) - (G(\gamma') - y_Q)(x, \frac{\tau}{2} + \frac{r}{2}) v(\gamma', \delta\gamma_1)(x, \frac{\tau}{2} + \frac{r}{2})| dx \\ & \quad + \int_{\Omega} |(G(\gamma') - y_Q)(x, \frac{\tau}{2} + \frac{r}{2}) v(\gamma', \delta\gamma_1)(x, \frac{\tau}{2} + \frac{r}{2}) - (G(\gamma') - y_Q)(x, \frac{\tau'}{2} + \frac{r}{2}) v(\gamma', \delta\gamma_1)(x, \frac{\tau'}{2} + \frac{r}{2})| dx. \end{aligned} \quad (8.19)$$

The second term in the right-hand side of (8.19) can still be decomposed as

$$\begin{aligned} & \int_{\Omega} |(G(\gamma') - y_Q)(x, \frac{\tau}{2} + \frac{r}{2}) v(\gamma', \delta\gamma_1)(x, \frac{\tau}{2} + \frac{r}{2}) - (G(\gamma') - y_Q)(x, \frac{\tau'}{2} + \frac{r}{2}) v(\gamma', \delta\gamma_1)(x, \frac{\tau'}{2} + \frac{r}{2})| dx \\ & \leq \int_{\Omega} |(G(\gamma') - y_Q)(x, \frac{\tau}{2} + \frac{r}{2}) v(\gamma', \delta\gamma_1)(x, \frac{\tau}{2} + \frac{r}{2}) - (G(\gamma') - y_Q)(x, \frac{\tau'}{2} + \frac{r}{2}) v(\gamma', \delta\gamma_1)(x, \frac{\tau}{2} + \frac{r}{2})| dx \\ & \quad + \int_{\Omega} |(G(\gamma') - y_Q)(x, \frac{\tau'}{2} + \frac{r}{2}) [v(\gamma', \delta\gamma_1)(x, \frac{\tau}{2} + \frac{r}{2}) - v(\gamma', \delta\gamma_1)(x, \frac{\tau'}{2} + \frac{r}{2})]| dx. \end{aligned} \quad (8.20)$$

Now  $\{v(\gamma', \delta\gamma_1); \delta\gamma_1 \in H^1(0, T; \mathbb{R}^2) \text{ and } \|\delta\gamma_1\|_{H^1(0, T; \mathbb{R}^2)} \leq 1\}$  is compact in  $C([0, T]; L^2(\Omega))$  as  $Dw(\gamma')$  defined by (4.9) is a compact mapping from  $H^1(0, T; \mathbb{R}^2)$  into  $C(\bar{\Gamma}_1 \times [0, T])$  and thus equicontinuous by Ascoli's theorem [11, (7.5.7)]. Consequently, given  $\varepsilon > 0$ :

$$\left\| v(\gamma', \delta\gamma_1)(\cdot, \frac{\tau}{2} + \frac{r}{2}) - v(\gamma', \delta\gamma_1)(\cdot, \frac{\tau'}{2} + \frac{r}{2}) \right\|_{L^2(\Omega)} \leq \frac{\varepsilon}{1 + \left\| (G(\gamma') - y_Q)(\cdot, \frac{\tau'}{2} + \frac{r}{2}) \right\|_{L^2(\Omega)}}$$

for  $\tau$  sufficiently close to  $\tau'$  and this for all  $\delta\gamma_1 \in H^1(0, T; \mathbb{R}^2)$  such that  $\|\delta\gamma_1\|_{H^1(0, T; \mathbb{R}^2)} \leq 1$ . This implies that:

$$\int_{\Omega} |(G(\gamma') - y_Q)(x, \frac{\tau'}{2} + \frac{r}{2}) [v(\gamma', \delta\gamma_1)(x, \frac{\tau}{2} + \frac{r}{2}) - v(\gamma', \delta\gamma_1)(x, \frac{\tau'}{2} + \frac{r}{2})]| dx \leq \frac{\varepsilon}{2}. \quad (8.21)$$

On the other hand

$$\begin{aligned}
& \int_{\Omega} |(G(\gamma') - y_Q)(x, \frac{\tau}{2} + \frac{r}{2})v(\gamma', \delta\gamma_1)(x, \frac{\tau}{2} + \frac{r}{2}) - (G(\gamma') - y_Q)(x, \frac{\tau'}{2} + \frac{r}{2})v(\gamma', \delta\gamma_1)(x, \frac{\tau}{2} + \frac{r}{2})|dx \\
& \leq \int_{\Omega} |(G(\gamma') - y_Q)(x, \frac{\tau}{2} + \frac{r}{2}) - (G(\gamma') - y_Q)(x, \frac{\tau'}{2} + \frac{r}{2})|v(\gamma', \delta\gamma_1)(x, \frac{\tau}{2} + \frac{r}{2})|dx \\
& \leq \left\| (G(\gamma') - y_Q)(\cdot, \frac{\tau}{2} + \frac{r}{2}) - (G(\gamma') - y_Q)(\cdot, \frac{\tau'}{2} + \frac{r}{2}) \right\|_{L^2(\Omega)} \|v(\gamma', \delta\gamma_1)(\cdot, \frac{\tau}{2} + \frac{r}{2})\|_{L^2(\Omega)} \\
& \leq \|DG(\gamma')\|_{\mathcal{L}(H^1(0,T;\mathbb{R}^2);C([0,T];L^2(\Omega)))} \left\| (G(\gamma') - y_Q)(\cdot, \frac{\tau}{2} + \frac{r}{2}) - (G(\gamma') - y_Q)(\cdot, \frac{\tau'}{2} + \frac{r}{2}) \right\|_{L^2(\Omega)} \leq \frac{\varepsilon}{2}
\end{aligned} \tag{8.22}$$

if  $\tau$  is sufficiently close to  $\tau'$ .

From (8.20), (8.21) and (8.22) follows that

$$\int_{\Omega} |(G(\gamma') - y_Q)(x, \frac{\tau}{2} + \frac{r}{2})v(\gamma', \delta\gamma_1)(x, \frac{\tau}{2} + \frac{r}{2}) - (G(\gamma') - y_Q)(x, \frac{\tau'}{2} + \frac{r}{2})v(\gamma', \delta\gamma_1)(x, \frac{\tau'}{2} + \frac{r}{2})|dx \leq \varepsilon \tag{8.23}$$

if  $\tau$  is sufficiently close to  $\tau'$  for all  $\delta\gamma_1 \in H^1(0, T; \mathbb{R}^2)$  such that  $\|\delta\gamma_1\|_{H^1(0,T;\mathbb{R}^2)} \leq 1$ .

Let us now work on the first term in the right-hand side of (8.19). We have  $v(\gamma, \delta\gamma_1) = DG(\gamma, \delta\gamma_1) \rightarrow DG(\gamma', \delta\gamma_1) = v(\gamma', \delta\gamma_1)$  in  $C([0, T]; L^2(\Omega))$  uniformly in  $\delta\gamma_1 \in H^1(0, T; \mathbb{R}^2)$  for  $\|\delta\gamma_1\|_{H^1(0,T;\mathbb{R}^2)} \leq 1$ , as  $DG(\gamma) \rightarrow DG(\gamma')$  in  $\mathcal{L}(H^1(0, T; \mathbb{R}^2); W(0, T))$  and thus a fortiori in  $\mathcal{L}(H^1(0, T; \mathbb{R}^2); C([0, T]; L^2(\Omega)))$  when  $\gamma$  tends to  $\gamma'$  in  $U \subset H^1(0, T; \mathbb{R}^2)$ . Consequently  $v(\gamma, \delta\gamma_1)(\cdot, \frac{\tau}{2} + \frac{r}{2})$  tends uniformly in  $\delta\gamma_1 \in H^1(0, T; \mathbb{R}^2)$  for  $\|\delta\gamma_1\|_{H^1(0,T;\mathbb{R}^2)} \leq 1$  and uniformly in  $\tau$  to  $v(\gamma', \delta\gamma_1)(\cdot, \frac{\tau}{2} + \frac{r}{2})$  in  $L^2(\Omega)$  when  $\gamma$  tends to  $\gamma'$ .

$G$  being continuous from  $U \subset H^1(0, T; \mathbb{R}^2)$  into  $W(0, T)$ , by the continuous injection  $W(0, T) \hookrightarrow C([0, T]; L^2(\Omega))$ ,  $G(\gamma) \rightarrow G(\gamma')$  in  $C([0, T]; L^2(\Omega))$  when  $\gamma$  tends to  $\gamma'$  in  $U \subset H^1(0, T; \mathbb{R}^2)$  so that  $(G(\gamma) - y_Q)(x, \frac{\tau}{2} + \frac{r}{2}) \rightarrow (G(\gamma') - y_Q)(x, \frac{\tau}{2} + \frac{r}{2})$  in  $L^2(\Omega)$  uniformly in  $\tau$  (let us recall that we have supposed that  $y_Q \in C([0, T]; L^2(\Omega))$ ).

These two facts imply that  $(G(\gamma) - y_Q)(\cdot, \frac{\tau}{2} + \frac{r}{2})v(\gamma, \delta\gamma_1)(\cdot, \frac{\tau}{2} + \frac{r}{2})$  tends to  $(G(\gamma') - y_Q)(\cdot, \frac{\tau}{2} + \frac{r}{2})v(\gamma', \delta\gamma_1)(\cdot, \frac{\tau}{2} + \frac{r}{2})$  in  $L^1(\Omega)$  uniformly in  $\delta\gamma_1 \in H^1(0, T; \mathbb{R}^2)$  for  $\|\delta\gamma_1\|_{H^1(0,T;\mathbb{R}^2)} \leq 1$  and uniformly in  $\tau$ , so that

$$\int_{\Omega} |(G(\gamma) - y_Q)(x, \frac{\tau}{2} + \frac{r}{2})v(\gamma, \delta\gamma_1)(x, \frac{\tau}{2} + \frac{r}{2}) - (G(\gamma') - y_Q)(x, \frac{\tau}{2} + \frac{r}{2})v(\gamma', \delta\gamma_1)(x, \frac{\tau}{2} + \frac{r}{2})|dx \tag{8.24}$$

tends to 0 uniformly in  $\tau$  and in  $\delta\gamma_1$  for  $\|\delta\gamma_1\|_{H^1(0,T;\mathbb{R}^2)} \leq 1$ , when  $\gamma$  tends to  $\gamma'$  in  $U \subset H^1(0, T; \mathbb{R}^2)$ .

By (8.23) and (8.24), (8.18) follows.  $\square$

Now we want to prove that:

**Proposition 8.5.** *The mapping*

$$U \rightarrow \mathbb{R} : \gamma \mapsto D_{\tau} \hat{J}_r(\gamma, \tau)$$

*is Fréchet differentiable and*

$$\begin{aligned}
& D_{\gamma\tau} \hat{J}_r(\gamma, \tau) \cdot [\delta\tau_1, \delta\gamma_2] = \\
& \frac{\lambda_Q}{2} \frac{\delta\tau_1}{r} \int_{\Omega} \left[ (G(\gamma) - y_Q)(x, \frac{\tau}{2} + \frac{r}{2}) \cdot v(\gamma, \delta\gamma_2)(x, \frac{\tau}{2} + \frac{r}{2}) - (G(\gamma) - y_Q)(x, \frac{\tau}{2} - \frac{r}{2}) \cdot v(\gamma, \delta\gamma_2)(x, \frac{\tau}{2} - \frac{r}{2}) \right] dx \\
& + \lambda_{\Omega} \frac{\delta\tau_1}{r} \int_{\Omega} \left[ (G(\gamma) - y_{\Omega})(x, \tau) \cdot v(\gamma, \delta\gamma_2)(x, \tau) - (G(\gamma) - y_{\Omega})(x, \tau - r) \cdot v(\gamma, \delta\gamma_2)(x, \tau - r) \right] dx.
\end{aligned} \tag{8.25}$$

*Proof.* We know by (5.14) that:

$$\begin{aligned}
D_{\tau} \hat{J}_r(\gamma, \tau) \cdot \delta\tau_1 &= \frac{\lambda_Q}{4} \frac{\delta\tau_1}{r} \left[ \int_{\Omega} |(G(\gamma) - y_Q)(x, \frac{\tau}{2} + \frac{r}{2})|^2 dx - \int_{\Omega} |(G(\gamma) - y_Q)(x, \frac{\tau}{2} - \frac{r}{2})|^2 dx \right] \\
&+ \frac{\lambda_{\Omega}}{2} \frac{\delta\tau_1}{r} \left[ \int_{\Omega} |(G(\gamma) - y_{\Omega})(x, \tau)|^2 dx - \int_{\Omega} |(G(\gamma) - y_{\Omega})(x, \tau - r)|^2 dx \right] + \beta_T \cdot \delta\tau_1.
\end{aligned} \tag{8.26}$$

We want to prove that its partial derivative with respect to  $\gamma$  exists and to compute it. Let us consider the first term  $\frac{\lambda_Q}{4} \frac{\delta \tau_1}{r} \left[ \int_{\Omega} |(G(\gamma) - y_Q)(x, \frac{\tau}{2} + \frac{r}{2})|^2 dx \right]$  in the right hand side of (8.26). We may consider the function

$$F : U \subset H^1(0, T; \mathbb{R}^2) \rightarrow \mathcal{L}(\mathbb{R}; \mathbb{R}) \simeq \mathbb{R} : \gamma \mapsto \int_{\Omega} |G(\gamma)(x, \frac{\tau}{2} + \frac{r}{2}) - y_Q(x, \frac{\tau}{2} + \frac{r}{2})|^2 dx$$

as the composite of the functions

$$G_2 : U \rightarrow C([0, T]; L^2(\Omega)) : \gamma \mapsto G(\gamma)$$

followed by the function

$$T_r : C([0, T]; L^2(\Omega)) \rightarrow C([0, T]; L^2(\Omega)) : f \mapsto f - y_Q$$

followed by the function

$$T_a : C([0, T]; L^2(\Omega)) \rightarrow L^2(\Omega) : g \mapsto g(\frac{\tau}{2} + \frac{r}{2})$$

followed by the function

$$diag : L^2(\Omega) \rightarrow L^2(\Omega) \times L^2(\Omega) : h \mapsto (h, h)$$

and still followed by the bilinear continuous mapping

$$[\cdot, \cdot] : L^2(\Omega) \times L^2(\Omega) \rightarrow \mathbb{R} : (\varphi, \psi) \mapsto (\varphi|\psi)_{L^2(\Omega)}$$

where  $(\cdot|\cdot)_{L^2(\Omega)}$  denotes the scalar product in  $L^2(\Omega)$ .

$G_2$  is two times continuously differentiable,  $T_r$  is infinitely continuously differentiable,  $T_a$  is a linear continuous function and thus also infinitely continuously differentiable,  $diag$  is a linear continuous function and thus also infinitely continuously differentiable, and  $[\cdot, \cdot]$  being a bilinear continuous mapping is also infinitely continuously differentiable. Thus  $F$  is certainly continuously differentiable and its derivative  $D_{\gamma}F$  is the composite of the derivatives of these functions, so that

$$DF(\gamma) \cdot \delta \gamma_2 = \int_{\Omega} \left[ (G(\gamma)(x, \frac{\tau}{2} + \frac{r}{2}) - y_Q(x, \frac{\tau}{2} + \frac{r}{2})) \cdot v(\gamma, \delta \gamma_2)(x, \frac{\tau}{2} + \frac{r}{2}) \right] dx.$$

The proof is similar for the other terms in the right hand side of (8.26).  $\square$

By inspection of formula (8.25), we see that  $D_{\gamma\tau}\hat{J}_r(\gamma, \tau)$  is equal to  $D_{\tau\gamma}\hat{J}_r(\gamma, \tau)$ , so that by Proposition 8.4, it follows immediately:

**Corollary 8.6.** *The mapping  $D_{\gamma\tau}\hat{J}_r : U \times [3r, T] \rightarrow \mathbb{R} : (\gamma, \tau) \mapsto D_{\gamma\tau}\hat{J}_r(\gamma, \tau)$  is continuous from  $U \times [3r, T]$  into  $\mathcal{L}(H^1(0, T; \mathbb{R}^2), \mathbb{R}; \mathbb{R})$ .*

This follows also from the proof of the previous proposition.

**Proposition 8.7.** *Let  $\gamma$  be fixed in  $U$ . Let us suppose that  $y_B \in L^2(\Sigma_3)$  and that the initial condition  $y_0$  of the initial boundary value problem (1.2) belongs to  $L^2(\Omega)$ . Then the time derivative  $\frac{dy}{dt}$  of its weak solution  $y = G(\gamma) \in W(0, T)$  belongs to  $C([\varepsilon, T]; L^2(\Omega))$  for every  $\varepsilon \in ]0, T[$ .*

*Proof.* 1. From  $\dot{g}_{\gamma}(x, t) = \frac{8\alpha P}{\pi R^4} \exp\left(-2\frac{|x-\gamma(t)|^2}{R^2}\right)(x - \gamma(t)|\dot{\gamma}(t)|)_{\mathbb{R}^2}$  and  $\gamma \in H^1(0, T; \mathbb{R}^2)$ , which implies  $\gamma \in C([0, T]; \mathbb{R}^2)$  and  $\dot{\gamma} \in L^2(0, T; \mathbb{R}^2)$ , follows that  $\dot{g}_{\gamma} \in L^2(\Sigma_1)$ .

2. Let us consider the “derivated problem with respect to time” of the initial boundary value problem (1.2) with initial condition  $v_0 \in L^2(\Omega)$ :

$$\begin{cases} \rho c \partial_t v - \kappa \Delta v = 0 & \text{in } Q = \Omega \times ]0, T[, \\ \kappa \partial_{\nu} v + hv = \dot{g}_{\gamma} & \text{on } \Sigma_1 = \Gamma_1 \times ]0, T[, \\ \kappa \partial_{\nu} v + hv = 0 & \text{in } \Sigma_2 = \Gamma_2 \times ]0, T[, \\ \kappa \partial_{\nu} v + hv = h \dot{y}_B & \text{in } \Sigma_3 = \Gamma_3 \times ]0, T[, \\ v(x, 0) = v_0(x) & \text{for } x \in \Omega, \end{cases} \quad (8.27)$$

whose variational formulation is:

$$\left\{ \begin{array}{l} \rho c < \frac{dv}{dt}(\cdot, t), \xi >_{H^1(\Omega)^*, H^1(\Omega)} + \kappa \int_{\Omega} \nabla v(x, t) \cdot \nabla \xi(x) \, dx + h \int_{\partial\Omega} v(x, t) \xi(x) \, dS(x) \\ \quad = \int_{\Gamma_1} \dot{g}_{\gamma}(x, t) \xi(x) \, dS(x) + h \int_{\Gamma_3} \dot{y}_B(x, t) \xi(x) \, dS(x), \, \forall \xi \in H^1(\Omega), \\ \text{with } v(\cdot, 0) = v_0. \end{array} \right. \quad (8.28)$$

We will precise later what is  $v_0$ . As  $\dot{g}_{\gamma} \in L^2(\Sigma_1)$ , for a.e.  $t \in ]0, T[$ , the integral  $\int_{\Gamma_1} \dot{g}_{\gamma}(x, t) \xi(x) \, dS(x)$  has sense for all  $\xi \in H^1(\Omega)$ . Due to our hypothesis that  $\dot{y}_B \in L^2(\Sigma_3)$ , the integral  $\int_{\Gamma_3} \dot{y}_B(x, t) \xi(x) \, dS(x)$  has sense  $\forall \xi \in H^1(\Omega)$ . Thus problem (8.27) admits one and only one weak solution  $v \in W(0, T)$  i.e. satisfying  $v(\cdot, 0) = v_0$  and (8.28)<sub>(i)</sub> for a.e.  $t \in ]0, T[$ . Setting

$$\tilde{y}(x, t) = \int_0^t v(x, \tau) d\tau + \tilde{y}_0, \quad (8.29)$$

with  $\tilde{y}_0 \in H^1(\Omega)$  specified later on, we have:

$$\tilde{y} \in C([0, T]; H^1(\Omega)) \quad (8.30)$$

and

$$\frac{d\tilde{y}}{dt} \in L^2(0, T; H^1(\Omega)), \quad (8.31)$$

so that a fortiori  $\tilde{y} \in W(0, T)$ . Integrating equation (8.28)<sub>(i)</sub> with respect to  $t$  from 0 to  $t$ , we have:

$$\begin{aligned} \rho c < v(\cdot, t), \xi >_{H^1(\Omega)^*, H^1(\Omega)} - \rho c < v(\cdot, 0), \xi >_{H^1(\Omega)^*, H^1(\Omega)} + \kappa \int_{\Omega} \nabla \tilde{y}(x, t) \cdot \nabla \xi(x) \, dx \\ & - \kappa \int_{\Omega} \nabla \tilde{y}_0(x) \cdot \nabla \xi(x) \, dx + h \int_{\partial\Omega} \tilde{y}(x, t) \xi(x) \, dS(x) - h \int_{\partial\Omega} \tilde{y}_0(x) \xi(x) \, dS(x) \\ & = \int_{\Gamma_1} g_{\gamma}(x, t) \xi(x) \, dS(x) - \int_{\Gamma_1} g_{\gamma}(x, 0) \xi(x) \, dS(x) + h \int_{\Gamma_3} y_B(x, t) \xi(x) \, dS(x) \\ & - h \int_{\Gamma_3} y_B(x, 0) \xi(x) \, dS(x). \end{aligned} \quad (8.32)$$

This equation may be rewritten:

$$\begin{aligned} \rho c < \frac{d\tilde{y}}{dt}(\cdot, t), \xi >_{H^1(\Omega)^*, H^1(\Omega)} + \kappa \int_{\Omega} \nabla \tilde{y}(x, t) \cdot \nabla \xi(x) \, dx + h \int_{\partial\Omega} \tilde{y}(x, t) \xi(x) \, dS(x) \\ & = \rho c < v(\cdot, 0), \xi >_{H^1(\Omega)^*, H^1(\Omega)} + \kappa \int_{\Omega} \nabla \tilde{y}_0(x) \cdot \nabla \xi(x) \, dx + h \int_{\partial\Omega} \tilde{y}_0(x) \xi(x) \, dS(x) - \int_{\Gamma_1} g_{\gamma}(x, 0) \xi(x) \, dS(x) \\ & - h \int_{\Gamma_3} y_B(x, 0) \xi(x) \, dS(x) + \int_{\Gamma_1} g_{\gamma}(x, t) \xi(x) \, dS(x) + h \int_{\Gamma_3} y_B(x, t) \xi(x) \, dS(x) \end{aligned} \quad (8.33)$$

Let us now define  $\tilde{y}_0 \in H^1(\Omega)$  as the weak solution of the stationary problem:

$$\left\{ \begin{array}{l} (-\kappa \Delta + \rho c) \tilde{y}_0 = 0 \text{ in } \Omega, \\ \kappa \frac{\partial \tilde{y}_0}{\partial \nu} + h \tilde{y}_0 = \psi \text{ on } \partial\Omega, \end{array} \right. \quad (8.34)$$

where  $\psi \in L^2(\Gamma)$  is defined by:

$$\psi(x) := \left\{ \begin{array}{l} g_{\gamma}(x, 0) \text{ on } \Gamma_1, \\ 0 \text{ on } \Gamma_2, \\ h y_B(x, 0) \text{ on } \Gamma_3. \end{array} \right. \quad (8.35)$$

By [24, Theorem 2.6, pp.35-36], problem (8.34)-(8.35) possesses one and only one weak solution  $\tilde{y}_0 \in H^1(\Omega)$  i.e. verifying

$$\begin{aligned} \kappa \int_{\Omega} \nabla \tilde{y}_0(x) \cdot \nabla \xi(x) \, dx + \rho c \int_{\Omega} \tilde{y}_0(x) \xi(x) \, dx + h \int_{\partial\Omega} \tilde{y}_0(x) \xi(x) \, dS(x) \\ - \int_{\Gamma_1} g_{\gamma}(x, 0) \xi(x) \, dS(x) - \int_{\Gamma_3} h y_B(x, 0) \xi(x) \, dS(x) = 0, \, \forall \xi \in H^1(\Omega). \end{aligned} \quad (8.36)$$

Moreover:

$$\|\tilde{y}_0\|_{H^1(\Omega)} \lesssim \|\psi\|_{L^2(\Gamma)} \lesssim (1 + \|y_B(\cdot, 0)\|_{L^2(\Gamma)}). \quad (8.37)$$

Let us observe however that  $\tilde{y}_0$  depends on  $\gamma(0)$ , see (1.4). By the equation  $(-\kappa\Delta + \rho c)\tilde{y}_0 = 0$  follows also that  $\Delta\tilde{y}_0 \in H^1(\Omega)$ . Choosing as initial condition in the “derivated problem with respect to time” (8.27):  $v(\cdot, 0) = \tilde{y}_0$ , using (8.36), equation (8.33) simplifies into

$$\begin{aligned} \rho c < \frac{d\tilde{y}}{dt}(\cdot, t), \xi >_{H^1(\Omega)^*, H^1(\Omega)} + \kappa \int_{\Omega} \nabla \tilde{y}(x, t) \cdot \nabla \xi(x) \, dx + h \int_{\partial\Omega} \tilde{y}(x, t) \xi(x) \, dS(x) \\ = \int_{\Gamma_1} g_{\gamma}(x, t) \xi(x) \, dS(x) + h \int_{\Gamma_3} y_B(x, t) \xi(x) \, dS(x), \end{aligned} \quad (8.38)$$

for every  $\xi \in H^1(\Omega)$ , which is the same variational equation (2.2). Also  $\tilde{y}$  and  $y$  both belong to the same functional space  $W(0, T)$ , but the initial conditions are different:  $\tilde{y}(\cdot, 0) = \tilde{y}_0$  by (8.29), whereas for problem (1.2):  $y(\cdot, 0) = y_0$ .

So we consider in  $W(0, T)$ , the weak solution of the auxiliary initial boundary value problem:

$$\begin{cases} \rho c \frac{dz}{dt} - \kappa \Delta z = 0, & \text{in } \Omega \times ]0, T[, \\ \kappa \frac{\partial z}{\partial \nu} + h z = 0, & \text{on } \Sigma := \Gamma \times ]0, T[, \\ z(\cdot, 0) = y_0 - \tilde{y}_0. \end{cases} \quad (8.39)$$

From (8.39) and (1.2), it is clear that  $y = \tilde{y} + z$  so that  $\frac{dy}{dt} = \frac{d\tilde{y}}{dt} + \frac{dz}{dt} = v + \frac{dz}{dt}$ . We know that  $v \in W(0, T)$  and thus a fortiori belongs to  $C([0, T]; L^2(\Omega))$ . It remains thus to prove that  $\frac{dz}{dt}$  belongs to  $C([\varepsilon, T]; L^2(\Omega))$  for every  $\varepsilon \in ]0, T[$ . We may consider problem (8.39) as an abstract parabolic problem in the space  $L^2(\Omega)$ . For this, we introduce the linear operator  $A$  in  $L^2(\Omega)$ :

$$D(A) = \{\zeta \in H^1(\Omega); \Delta \zeta \in L^2(\Omega) \text{ and } \kappa \frac{\partial \zeta}{\partial \nu} + h \zeta = 0 \text{ on } \Gamma\}, \quad (8.40)$$

$$A : D(A) \rightarrow L^2(\Omega) : \zeta \mapsto \frac{\kappa}{\rho c} \Delta \zeta,$$

where  $\Delta u$  must be taken in the sense of distributions and  $\frac{\partial u}{\partial \nu}$  in the boundary condition  $\kappa \frac{\partial u}{\partial \nu} + h u = 0$  on  $\partial\Omega$  in the sense of the normal trace in  $H^{-\frac{1}{2}}(\partial\Omega)$  of  $\nabla u \in H(\text{div}; \Omega)$  [22, p.6, pp.9-12].

Proceeding similarly as in the proof of [20, Theorem 2.7, Chapter 7] (see Appendix), it can be proved that  $A$  generates a holomorphic semigroup of contractions  $(P_t)_{t \geq 0}$  in  $L^2(\Omega)$ . A holomorphic semigroup is regularizing i.e. for every  $z_0 \in L^2(\Omega)$ , and every  $t > 0$ :  $P_t z_0 \in D(A)$ . Consequently, we may write for  $z$  solution of the initial boundary value problem (8.39):  $z(t) = P_t(y_0 - \tilde{y}_0) = P_{t-\varepsilon} P_{\varepsilon}(y_0 - \tilde{y}_0)$  for  $t \geq \varepsilon$ . Consequently:

$$\frac{dz}{dt}(t) = P_{t-\varepsilon} A P_{\varepsilon}(y_0 - \tilde{y}_0) \text{ for } t \geq \varepsilon \quad (8.41)$$

so that  $\frac{dz}{dt} \in C([\varepsilon, T]; L^2(\Omega))$ . From  $\frac{dy}{dt}(t) = v + \frac{dz}{dt}(t)$  and  $v \in C([0, T]; L^2(\Omega))$  follows that  $\frac{dy}{dt} \in C([\varepsilon, T]; L^2(\Omega))$ .

**Proposition 8.8.** *Supposing moreover that  $D_t y_Q \in C([r, \frac{2}{3}T]; L^2(\Omega))$  and  $D_t y_{\Omega} \in C([2r, T]; L^2(\Omega))$ , the mapping*

$$[3r, T] \rightarrow \mathbb{R} : \tau \mapsto D_{\tau} \hat{J}_r(\gamma, \tau)$$

*is differentiable, and we have:*

$$\begin{aligned} D_{\tau\tau} \hat{J}_r(\gamma, \tau) \cdot [\delta\tau_1, \delta\tau_2] &= \frac{\lambda_Q \delta\tau_1 \delta\tau_2}{4r} \left( \int_{\Omega} (G(\gamma) - y_Q)(x, \frac{\tau}{2} + \frac{r}{2}) D_t (G(\gamma) - y_Q)(x, \frac{\tau}{2} + \frac{r}{2}) dx \right. \\ &\quad \left. - \int_{\Omega} (G(\gamma) - y_Q)(x, \frac{\tau}{2} - \frac{r}{2}) D_t (G(\gamma) - y_Q)(x, \frac{\tau}{2} - \frac{r}{2}) dx \right) \\ &\quad + \frac{\lambda_{\Omega} \delta\tau_1 \delta\tau_2}{2r} \left( \int_{\Omega} (G(\gamma) - y_Q)(x, \tau) D_t (G(\gamma) - y_Q)(x, \tau) \right. \\ &\quad \left. - \int_{\Omega} (G(\gamma) - y_Q)(x, \tau - r) D_t (G(\gamma) - y_Q)(x, \tau - r) dx \right). \end{aligned} \quad (8.42)$$

*Proof.* Firstly, let us observe that due to the preceding proposition and our hypotheses on  $D_t y_Q$  and  $D_t y_\Omega$ , that the integrals in the right hand side of (8.42) have sense. Let us consider the first term in the right-hand side of (8.26) which up to multiplicative constants is:

$$\int_{\Omega} |G(\gamma)(x, \frac{\tau}{2} + \frac{r}{2}) - y_Q(x, \frac{\tau}{2} + \frac{r}{2})|^2 dx, \quad (8.43)$$

the other terms being similar. We may consider it as the composite of the functions

$$[3r, T] \rightarrow L^2(\Omega)^2: \tau \mapsto ((G(\gamma) - y_Q)(\cdot, \frac{\tau}{2} + \frac{r}{2}), (G(\gamma) - y_Q)(\cdot, \frac{\tau}{2} + \frac{r}{2}))$$

followed by the continuous bilinear mapping

$$[\cdot, \cdot]: L^2(\Omega) \times L^2(\Omega) \rightarrow \mathbb{R}: (f, g) \mapsto \int_{\Omega} f(x)g(x)dx.$$

We know that  $G(\gamma) \in W(0, T) \hookrightarrow C([0, T]; L^2(\Omega))$  and by the previous proposition that  $D_t G(\gamma)$  belongs to  $C([\varepsilon, T]; L^2(\Omega))$  for every  $\varepsilon \in ]0, T[$ . Thus the mapping

$$[3r, T] \rightarrow L^2(\Omega)^2: \tau \mapsto ((G(\gamma) - y_Q)(\cdot, \frac{\tau}{2} + \frac{r}{2}), (G(\gamma) - y_Q)(\cdot, \frac{\tau}{2} + \frac{r}{2}))$$

is continuously differentiable. Consequently the composite function

$$[3r, T] \rightarrow \mathbb{R}: \tau \mapsto \int_{\Omega} |G(\gamma)(x, \frac{\tau}{2} + \frac{r}{2}) - y_Q(x, \frac{\tau}{2} + \frac{r}{2})|^2 dx, \quad (8.44)$$

is of class  $C^1$  and its derivative given by

$$\tau \in [3r, T] \mapsto \int_{\Omega} [(G(\gamma)(x, \frac{\tau}{2} + \frac{r}{2}) - y_Q(x, \frac{\tau}{2} + \frac{r}{2}))(D_t G(\gamma)(x, \frac{\tau}{2} + \frac{r}{2}) - D_t y_Q(x, \frac{\tau}{2} + \frac{r}{2}))] dx \in \mathbb{R}. \quad (8.45)$$

□

**Lemma 8.9.** *For every  $\varepsilon \in ]0, T[$ , the mapping*

$$U \rightarrow C([\varepsilon, T]; L^2(\Omega)): \gamma \mapsto D_t G(\gamma) \quad (8.46)$$

*is continuous.*

*Proof.* The proof follows by inspection of the proof of Proposition 8.7:

By (1.4):

$$g_\gamma(x, t) = \alpha \frac{2P}{\pi R^2} \exp \left( -2 \frac{|x - \gamma(t)|^2}{R^2} \right), \quad (8.47)$$

from which it follows that

$$\dot{g}_\gamma(x, t) = \frac{8\alpha P}{\pi R^4} \exp \left( -2 \frac{|x - \gamma(t)|^2}{R^2} \right) (x - \gamma(t)|\dot{\gamma}(t))_{\mathbb{R}^2}. \quad (8.48)$$

Consequently, if  $\gamma_n \rightarrow \gamma$  in  $U \subset H^1(0, T; \mathbb{R}^2)$  follows that  $(\gamma_n|\dot{\gamma}_n)_{\mathbb{R}^2} \rightarrow (\gamma|\dot{\gamma})_{\mathbb{R}^2}$  in  $L^2(0, T)$  so that  $\dot{g}_{\gamma_n} \rightarrow \dot{g}_\gamma$  in  $L^2(\Sigma_1)$ . Thus the mapping  $U \subset H^1(0, T; \mathbb{R}^2) \rightarrow L^2(\Sigma_1): \gamma \mapsto \dot{g}_\gamma$  is continuous. Also, by (8.34) and (8.35),  $\tilde{y}_0 \in H^1(\Omega)$  depends continuously on  $\gamma(0)$ . Consequently  $v \in W(0, T) \hookrightarrow C([0, T]; L^2(\Omega))$  solution of (8.28) with initial condition  $v_0 = \tilde{y}_0$  depends continuously on  $\gamma \in H^1(0, T; \mathbb{R}^2)$ .  $z$  is solution of the initial boundary value problem (8.39) and it follows from (8.41) that  $\frac{dz}{dt} \in C([\varepsilon, T]; L^2(\Omega))$  depends continuously on  $\tilde{y}_0$  and thus on  $\gamma \in H^1(0, T; \mathbb{R}^2)$ . As  $\frac{dy}{dt} = v + \frac{dz}{dt}$  (see the end of the proof of Proposition 8.7), it follows that  $D_t G(\gamma) = \frac{dy}{dt} \in C([\varepsilon, T]; L^2(\Omega))$  depends continuously on  $\gamma \in H^1(0, T; \mathbb{R}^2)$ .

□

**Corollary 8.10.** *Under the assumptions of Proposition 8.8, the mapping  $D_{\tau\tau}\hat{J}_\tau: U \times [3r, T] \rightarrow \mathbb{R}: (\gamma, \tau) \mapsto D_{\tau\tau}\hat{J}_\tau(\gamma, \tau)$  is continuous.*

*Proof.* This follows by inspection of the right-hand side of equation (8.42) and the preceding lemma.  $\square$

**Corollary 8.11.** *Under the assumptions of Proposition 8.8,  $\hat{J}_r$  is of class  $C^2$ .*

*Proof.* This follows from the existence and continuity of the first and second partial derivatives of  $\hat{J}_r$  by applying [7, Prop.5.2.1, p.69].  $\square$

**Corollary 8.12.** *Under the assumptions of Proposition 8.8, supposing that  $\Gamma_S$  is a closed convex subset of  $\Gamma_1$ , let the control  $(\bar{\gamma}, \bar{\tau}) \in U_{ad} \times [3r, T]$  satisfy the first-order necessary condition  $\hat{J}_r'(\bar{\gamma}, \bar{\tau})(\gamma - \bar{\gamma}, \tau - \bar{\tau}) \geq 0, \forall (\gamma, \tau) \in U_{ad} \times [3r, T]$  and that there exists some  $\delta > 0$  such that for every  $(\gamma, \tau) \in U_{ad} \times [3r, T]$ :*

$$\hat{J}_r''(\bar{\gamma}, \bar{\tau})[(\gamma - \bar{\gamma}, \tau - \bar{\tau}), (\gamma - \bar{\gamma}, \tau - \bar{\tau})] \geq \delta \|(\gamma - \bar{\gamma}, \tau - \bar{\tau})\|_{H^1(0, T; \mathbb{R}^2) \times \mathbb{R}}^2. \quad (8.49)$$

*Then, there are constants  $\epsilon > 0$  and  $\sigma > 0$  such that we have the following quadratic growth property:*

$$\hat{J}_r(\gamma, \tau) \geq \hat{J}_r(\bar{\gamma}, \bar{\tau}) + \sigma \|(\gamma, \tau) - (\bar{\gamma}, \bar{\tau})\|_{H^1(0, T; \mathbb{R}^2) \times \mathbb{R}}^2, \quad (8.50)$$

*for every  $(\gamma, \tau)$  in  $U_{ad} \times [3r, T]$  such that  $\|(\gamma, \tau) - (\bar{\gamma}, \bar{\tau})\|_{H^1(0, T; \mathbb{R}^2) \times \mathbb{R}} \leq \epsilon$ . In particular,  $\hat{J}_r$  has a local minimum in  $U_{ad} \times [3r, T]$  at  $(\bar{\gamma}, \bar{\tau})$ .*

*Proof.* From the hypothesis on  $\Gamma_S$  follows that  $U_{ad}$  is a closed convex subset of  $H^1(0, T; \mathbb{R}^2)$ , so that the result follows from [24, Theorem 4.23 pp.231-232] by the previous corollary.  $\square$

## 9 Appendix

In this appendix we give the detailed proof that the operator  $A$  defined by (8.40) generates a holomorphic semigroup as in Theorem 2.7 from [20, Chapter 7] where the domain of the operator is defined by the Dirichlet homogeneous boundary condition and not by a homogeneous Robin boundary condition.

**Proposition 9.1.** *Let  $\Omega$  be a bounded Lipschitz domain in  $\mathbb{R}^3$ . Then the linear operator  $A$  in  $L^2(\Omega)$  defined by (8.40) is a linear closed operator in  $L^2(\Omega)$ .*

*Moreover  $A^{-1}$  exists and is a bounded linear operator in  $L^2(\Omega)$  (thus 0 belongs to the resolvent set  $\rho(A)$  of  $A$ ).*

*Proof.* 1°) Let us suppose that  $u \in D(A)$  and let us set  $f = \frac{\kappa}{\rho c} \Delta u$ . For every  $v$  in  $H^1(\Omega)$ :

$$\begin{aligned} \int_{\Omega} f(x) v(x) \, dx &= \frac{\kappa}{\rho c} \int_{\Omega} (\operatorname{div} \nabla u)(x) v(x) \, dx \\ &= \frac{\kappa}{\rho c} \langle \nabla u \cdot \nu, v \rangle_{H^{-1/2}(\partial\Omega), H^{1/2}(\partial\Omega)} - \frac{\kappa}{\rho c} \int_{\Omega} (\nabla u)(x) \cdot \nabla v(x) \, dx \\ &= -\frac{h}{\rho c} \int_{\partial\Omega} u(x) v(x) \, dS(x) - \frac{\kappa}{\rho c} \int_{\Omega} (\nabla u)(x) \cdot \nabla v(x) \, dx. \end{aligned}$$

Thus:

$$\frac{\kappa}{\rho c} \int_{\Omega} \nabla u(x) \cdot \nabla v(x) \, dx + \frac{h}{\rho c} \int_{\partial\Omega} u(x) v(x) \, dS(x) = - \int_{\Omega} f(x) v(x) \, dx, \quad \forall v \in H^1(\Omega). \quad (9.1)$$

2°) Reciprocally, let  $f$  be given in  $L^2(\Omega)$ . The bilinear form on  $H^1(\Omega) \times H^1(\Omega)$  which sends  $(u, v) \in H^1(\Omega)^2$  onto the real number

$$\frac{\kappa}{\rho c} \int_{\Omega} \nabla u(x) \cdot \nabla v(x) \, dx + \frac{h}{\rho c} \int_{\partial\Omega} u(x) v(x) \, dS(x)$$

is bilinear, continuous and coercive on  $H^1(\Omega)$  as

$$u \mapsto \left( \frac{\kappa}{\rho c} \int_{\Omega} |\nabla u(x)|^2 \, dx + \frac{h}{\rho c} \int_{\partial\Omega} u(x)^2 \, dS(x) \right)^{1/2}$$

is an equivalent norm on  $H^1(\Omega)$ . Thus by Lax-Milgram lemma, there exists a unique solution  $u \in H^1(\Omega)$  of (9.1) that satisfies  $\|u\|_{H^1(\Omega)} \lesssim \|f\|_{L^2(\Omega)}$ . Considering test functions  $v \in \mathcal{D}(\Omega) \subset H^1(\Omega)$ , (9.1) gives us  $\frac{\kappa}{\rho c} \int_{\Omega} \nabla u(x) \cdot \nabla v(x) \, dx = - \int_{\Omega} f(x) v(x) \, dx, \forall v \in \mathcal{D}(\Omega)$  i.e.  $\frac{\kappa}{\rho c} \Delta u = f$  in the sense of distributions.  $f$  belonging to  $L^2(\Omega)$ , this implies that

572  $\Delta u \in L^2(\Omega)$ . Thus  $\nabla u \in H(\text{div}; \Omega)$ . Consequently,  $\frac{\partial u}{\partial \nu} = \nabla u \cdot \nu$  has sense and belongs to  $H^{-\frac{1}{2}}(\Gamma)$  and by [22, (1.19)  
573 p.9]:

$$\begin{aligned} \frac{\kappa}{\rho c} \left\langle \frac{\partial u}{\partial \nu}, v \right\rangle_{H^{-\frac{1}{2}}(\Gamma), H^{\frac{1}{2}}(\Gamma)} &= \frac{\kappa}{\rho c} \langle \nabla u \cdot \nu, v \rangle_{H^{-\frac{1}{2}}(\Gamma), H^{\frac{1}{2}}(\Gamma)} = \frac{\kappa}{\rho c} \int_{\Omega} \nabla u(x) \cdot \nabla v(x) \, dx \\ &\quad + \frac{\kappa}{\rho c} \int_{\Omega} \text{div}(\nabla u)(x) v(x) \, dx \\ &= \frac{\kappa}{\rho c} \int_{\Omega} \nabla u(x) \cdot \nabla v(x) \, dx + \frac{\kappa}{\rho c} \int_{\Omega} (\Delta u)(x) v(x) \, dx = \frac{\kappa}{\rho c} \int_{\Omega} \nabla u(x) \cdot \nabla v(x) \, dx \\ &\quad + \int_{\Omega} f(x) v(x) \, dx. \end{aligned} \quad (9.2)$$

574 Consequently by (9.1) :

$$\frac{\kappa}{\rho c} \left\langle \frac{\partial u}{\partial \nu}, v \right\rangle_{H^{-\frac{1}{2}}(\Gamma), H^{\frac{1}{2}}(\Gamma)} = -\frac{h}{\rho c} \int_{\partial \Omega} u(x) v(x) \, dS(x), \quad \forall v \in H^1(\Omega), \quad (9.3)$$

575 and thus

$$\left\langle \kappa \frac{\partial u}{\partial \nu} + h u, v \right\rangle_{H^{-\frac{1}{2}}(\Gamma), H^{\frac{1}{2}}(\Gamma)} = 0, \quad \forall v \in H^1(\Omega). \quad (9.4)$$

576 Thus  $u \in D(A)$  and  $Au = f$ . Moreover

$$\|u\|_{H^1(\Omega)} \lesssim \|f\|_{L^2(\Omega)}, \quad (9.5)$$

577 implying that  $A^{-1}$  is a linear continuous operator in  $L^2(\Omega)$ , in particular a linear closed operator. Consequently,  $A$   
578 is also a linear closed operator in  $L^2(\Omega)$ . This shows also that 0 belongs to the resolvent set of  $A$ .  $\square$

579 **Proposition 9.2.** *The linear operator  $A$  in  $L^2(\Omega)$  defined by (8.40) generates a holomorphic semigroup in  $L^2(\Omega)$ .*

580 *Proof.* Firstly, it is obvious that we can extend our linear operator  $A$  in a linear operator  $\tilde{A}$  in  $L^2(\Omega; \mathbb{C}) :=$   
581  $L^2(\Omega) \oplus i L^2(\Omega)$  by setting:

$$D(\tilde{A}; \mathbb{C}) = \{\zeta \in H^1(\Omega; \mathbb{C}); \Delta \zeta \in L^2(\Omega; \mathbb{C}) \text{ and } \kappa \frac{\partial \zeta}{\partial \nu} + h \zeta = 0 \text{ on } \Gamma\},$$

582

$$\tilde{A} : D(\tilde{A}; \mathbb{C}) \rightarrow L^2(\Omega; \mathbb{C}) : \zeta \mapsto \frac{\kappa}{\rho c} \Delta \zeta.$$

583 For  $\zeta \in D(\tilde{A}; \mathbb{C})$ , setting  $\xi = \text{Re}(\zeta)$  and  $\eta = \text{Im}(\zeta)$ , we have:

$$\begin{aligned} \text{Re}(\tilde{A}\zeta, \zeta)_{L^2(\Omega; \mathbb{C})} &= \frac{\kappa}{\rho c} \text{Re}(\Delta \zeta, \zeta)_{L^2(\Omega; \mathbb{C})} = \frac{\kappa}{\rho c} \text{Re} \int_{\Omega} \Delta \zeta(x) \bar{\zeta}(x) \, dx \\ &= -\frac{\kappa}{\rho c} \int_{\Omega} |\nabla \xi(x)|^2 \, dx - \frac{\kappa}{\rho c} \int_{\Omega} |\nabla \eta(x)|^2 \, dx + \frac{\kappa}{\rho c} \int_{\partial \Omega} \frac{\partial \xi}{\partial \nu}(x) \xi(x) \, dS(x) \\ &\quad + \frac{\kappa}{\rho c} \int_{\partial \Omega} \frac{\partial \eta}{\partial \nu}(x) \eta(x) \, dS(x) \\ &= -\frac{\kappa}{\rho c} \int_{\Omega} |\nabla \xi(x)|^2 \, dx - \frac{\kappa}{\rho c} \int_{\Omega} |\nabla \eta(x)|^2 \, dx - \frac{h}{\rho c} \int_{\partial \Omega} \xi(x)^2 \, dS(x) \\ &\quad - \frac{h}{\rho c} \int_{\partial \Omega} \eta(x)^2 \, dS(x) \\ &\leq 0, \end{aligned} \quad (9.6)$$

584

$$\begin{aligned} \text{Im}(\tilde{A}\zeta, \zeta)_{L^2(\Omega; \mathbb{C})} &= \frac{\kappa}{\rho c} \text{Im}(\Delta \zeta, \zeta)_{L^2(\Omega; \mathbb{C})} = \frac{\kappa}{\rho c} \text{Im} \int_{\Omega} \Delta \zeta(x) \bar{\zeta}(x) \, dx \\ &= -\frac{\kappa}{\rho c} \int_{\Omega} \Delta \xi(x) \eta(x) \, dx + \frac{\kappa}{\rho c} \int_{\Omega} \Delta \eta(x) \xi(x) \, dx \\ &= \frac{\kappa}{\rho c} \int_{\Omega} \nabla \xi(x) \cdot \nabla \eta(x) \, dx - \frac{\kappa}{\rho c} \int_{\Omega} \nabla \eta(x) \cdot \nabla \xi(x) \, dx - \frac{\kappa}{\rho c} \int_{\Omega} \frac{\partial \xi}{\partial \nu}(x) \eta(x) \, dS(x) \\ &\quad + \frac{\kappa}{\rho c} \int_{\Omega} \frac{\partial \eta}{\partial \nu}(x) \xi(x) \, dS(x) \\ &= \frac{h}{\rho c} \int_{\Omega} \xi(x) \eta(x) \, dx - \frac{h}{\rho c} \int_{\Omega} \eta(x) \xi(x) \, dx = 0. \end{aligned} \quad (9.7)$$

585 It follows from (9.6) and (9.7), that  $W(\tilde{A}) \subset ]-\infty, -c_{N.R.}]$  for some strictly positive constant  $c_{N.R.}$  where the  
586 numerical range  $W(\tilde{A})$  of the operator  $\tilde{A}$  is defined by [20, p.12]:

$$W(\tilde{A}) := \{(\tilde{A}\zeta, \zeta)_{L^2(\Omega; \mathbb{C})}; \zeta \in D(\tilde{A}), \|\zeta\|_{L^2(\Omega; \mathbb{C})} = 1\}. \quad (9.8)$$

587 1°) Let us denote by  $\Sigma$  the complementary of  $\overline{W(\tilde{A})}$  in  $\mathbb{C}$  i.e.  $\Sigma = \mathbb{C} \setminus \overline{W(\tilde{A})}$ . From the definition of  $W(\tilde{A})$  follows  
588 that for every  $\lambda \in \Sigma$  and for every  $\zeta \in D(\tilde{A})$  such that  $\|\zeta\|_{L^2(\Omega; \mathbb{C})} = 1$  :

$$0 < d(\lambda, \overline{W(\tilde{A})}) \leq |\lambda - (\tilde{A}\zeta, \zeta)_{L^2(\Omega; \mathbb{C})}| = |((\lambda I - \tilde{A})\zeta, \zeta)_{L^2(\Omega; \mathbb{C})}| \leq \|\lambda \zeta - \tilde{A}\zeta\|. \quad (9.9)$$



This last inequality implies in particular that the linear operator  $\lambda I - \tilde{A}$  is injective, for every  $\lambda \in \Sigma$ .

2°) Let  $(z_n)_{n \in \mathbb{N}}$  be a sequence in the range of  $\lambda I - \tilde{A}$  i.e.  $z_n \in R(\lambda I - \tilde{A})$  such that  $z_n \neq z_m$  for  $n \neq m \in \mathbb{N}$  and converging to some  $z \in L^2(\Omega; \mathbb{C})$ . For every  $n \in \mathbb{N}$ , there exists  $\zeta_n \in D(\tilde{A})$  such that  $z_n = (\lambda I - \tilde{A})\zeta_n$ .  $\lambda I - \tilde{A}$  being injective for every  $\lambda \in \Sigma$ ,  $\zeta_n \neq \zeta_m$  for  $n \neq m \in \mathbb{N}$ . So for every  $m \neq n \in \mathbb{N}$ , we have:

$$\frac{\|z_n - z_m\|}{\|\zeta_n - \zeta_m\|} = \left\| (\lambda I - \tilde{A}) \left( \frac{\zeta_n - \zeta_m}{\|\zeta_n - \zeta_m\|} \right) \right\| \geq d(\lambda, \overline{W(\tilde{A})}) > 0,$$

by inequality (9.9), so that  $\|\zeta_n - \zeta_m\| \leq \frac{1}{d(\lambda, \overline{W(\tilde{A})})} \|z_n - z_m\|$ , for every  $(n, m) \in \mathbb{N}^2$ . Thus  $(\zeta_n)_{n \in \mathbb{N}}$  is also a Cauchy sequence in  $L^2(\Omega; \mathbb{C})$  and therefore converges to some  $\zeta \in L^2(\Omega; \mathbb{C})$ .  $\lambda I - \tilde{A}$  being a closed linear operator as  $\tilde{A}$  is a closed linear operator by Proposition 9.1,  $\zeta \in D(\tilde{A})$  and  $(\lambda I - \tilde{A})\zeta = z$ . Thus  $R(\lambda I - \tilde{A})$  is a closed subspace of  $L^2(\Omega; \mathbb{C})$ .

3°) By (9.6) and (9.7),  $|(\tilde{A}\zeta, \zeta)_{L^2(\Omega; \mathbb{C})}| \geq c_0 \|\zeta\|_{L^2(\Omega; \mathbb{C})}^2$  for some constant  $c_0 > 0$ . Thus by Lax-Milgram lemma [13, pp.207-208],  $\tilde{A}$  is also surjective. Consequently 0 which belongs to  $\Sigma$  belongs also to  $\rho(\tilde{A})$ .

4°) Let  $\lambda \in \Sigma \cap \rho(\tilde{A})$ . For every  $z \in L^2(\Omega; \mathbb{C})$ :

$$\|z\| = \|(\lambda I - \tilde{A})R(\lambda; \tilde{A})z\| \geq d(\lambda, \overline{W(\tilde{A})}) > 0$$

if  $\|R(\lambda; \tilde{A})z\| = 1$  by inequality (9.9). Thus  $\|R(\lambda; \tilde{A})z\| = 1 \Rightarrow \|z\| \geq d(\lambda, \overline{W(\tilde{A})})$ . Consequently, for every  $z \in L^2(\Omega; \mathbb{C}) \setminus \{0\}$ :

$$\left\| \frac{z}{\|R(\lambda; \tilde{A})z\|} \right\| \geq d(\lambda, \overline{W(\tilde{A})}) \quad (9.10)$$

so that:

$$\|R(\lambda; \tilde{A})z\| \leq \frac{1}{d(\lambda, \overline{W(\tilde{A})})} \|z\|, \forall z \in L^2(\Omega; \mathbb{C})$$

and thus

$$\|R(\lambda; \tilde{A})\| \leq \frac{1}{d(\lambda, \overline{W(\tilde{A})})}, \forall \lambda \in \Sigma \cap \rho(\tilde{A}). \quad (9.11)$$

5°) Let  $\Sigma_0$  be a connected component of  $\Sigma$ .  $\Sigma$  being an open subset in  $\mathbb{C}$ ,  $\Sigma_0$  is also an open subset in  $\mathbb{C}$ . Let us consider  $\Sigma_0 \cap \rho(\tilde{A})$  and let us suppose that  $\Sigma_0 \cap \rho(\tilde{A}) \neq \emptyset$ . Then the spectrum  $\sigma(\tilde{A})$  of  $\tilde{A}$  is contained in the complementary  $\mathbb{C} \setminus \Sigma_0$  i.e.  $\Sigma_0 \cap \rho(\tilde{A}) = \Sigma_0$  (for a proof of this see the last paragraph in the proof of Theorem 3.9. in [20, pp.12-13]: it consists in proving that  $\Sigma_0 \cap \rho(\tilde{A})$  which is obviously an open set of  $\Sigma_0$  is also closed in  $\Sigma_0$  by using (9.11) and then concluding by using the connectivity of  $\Sigma_0$ ). Consequently as 0 belongs to  $\mathbb{C} \setminus [-\infty, -c_{N.R.}] \cap \rho(\tilde{A})$  and  $\mathbb{C} \setminus [-\infty, -c_{N.R.}] \subset \Sigma$ ,  $\sigma(\tilde{A}) \subset [-\infty, -c_{N.R.}]$  and  $\mathbb{C} \setminus [-\infty, -c_{N.R.}] \subset \Sigma \cap \rho(\tilde{A})$ .

6°) By (9.11), the previous point and  $W(\tilde{A}) \subset [-\infty, -c_{N.R.}]$ :

$$\|R(\lambda; \tilde{A})\| \leq \frac{1}{d(\lambda, [-\infty, -c_{N.R.}])}, \forall \lambda \in \mathbb{C} \setminus [-\infty, -c_{N.R.}]. \quad (9.12)$$

Consequently, for every  $0 < \delta < \frac{\pi}{2}$ , there exists a constant  $M > 0$  such that

$$\rho(\tilde{A}) \supset \{\lambda \in \mathbb{C}; |\arg(\lambda)| < \frac{\pi}{2} + \delta\} \cup \{0\}$$

and

$$\|R(\lambda; \tilde{A})\| \leq \frac{M}{|\lambda|}, \text{ for } \{\lambda \in \mathbb{C} \setminus \{0\}; |\arg(\lambda)| < \frac{\pi}{2} + \delta\}. \quad (9.13)$$

By [20, Theorem 5.2. pp. 61-63],  $\tilde{A}$  is thus the generator of a holomorphic semigroup of angle  $\frac{\pi}{2}$  in  $L^2(\Omega; \mathbb{C})$ .  $\square$

**Corollary 9.3.** *A is the generator of a differentiable semigroup  $(P_t)_{t \geq 0}$  in  $L^2(\Omega)$ . Consequently, for every  $u \in L^2(\Omega)$ ,  $P_t u \in D(A)$  for  $t > 0$ .*

*Proof.* This follows from [20, Theorem 5.2.(d) p.63] (for the definition of a differentiable semigroup see [20, Definition 4.1 p.51]).  $\square$

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