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## Graphical Abstract

**Stability analysis of quasilinear systems on time scale based on a new estimation of the upper bound of the time scale matrix exponential function**

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## Highlights

### **Stability analysis of quasilinear systems on time scale based on a new estimation of the upper bound of the time scale matrix exponential function**

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- In this paper, we have studied the matrix exponential function  $e_A(t, t_0)$ , where  $A$  is an arbitrary matrix,  $t, t_0 \in \mathbb{T}$ ,  $\mathbb{T}$  is an arbitrary nonempty closed subset of  $\mathbb{R}$ , which is called a time scale. This function generalizes the usual matrix exponential  $e^{A(t-t_0)}$  as well as the  $m$ -th power  $(I + A)^m$  for an arbitrary time scale.
- Using the time scale theory, we have proposed new estimations of  $\|e_A(t, t_0)\|$  in the following form:

$$\|e_A(t, t_0)\| \leq K e_\alpha(t, t_0), \quad \forall t, t_0 \in \mathbb{T}(t \geq t_0),$$

where  $\mathbb{T}$  is an arbitrary time scale,  $K, \alpha$  are known parameters and  $e_\alpha(t, t_0)$  is a scalar exponential function. This estimation is more accurate than known ones.

- It is shown that the proposed estimation can be used for stability analysis of dynamic equations on a time scale.

# Stability analysis of quasilinear systems on time scale based on a new estimation of the upper bound of the time scale matrix exponential function

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## Abstract

In this paper, we analyze the stability of quasilinear systems on time scale based on a new estimation of the upper bound of the time scale matrix exponential function. First, some upper bounds for the norm of the matrix exponential function  $e_A(t, t_0)$  are derived, where  $A$  is a regressive square matrix,  $t, t_0 \in \mathbb{T}$ ,  $\mathbb{T}$  being an arbitrary nonempty closed subset of the set of real numbers  $\mathbb{R}$ , which is called a time scale. The matrix exponential function generalizes the usual matrix exponential as well as the integer power of a matrix: if  $\mathbb{T} = \mathbb{R}$  and  $A$  is a constant matrix, then  $e_A(t, t_0) = e^{A(t-t_0)}$ , while if  $\mathbb{T} = \mathbb{Z}$  and  $I + A$  is invertible, then  $e_A(t, t_0) = (I + A)^{(t-t_0)}$ . Using Putzer algorithm and estimating the maximum value of a function defined and bounded on a time scale  $\mathbb{T}$ , we derive an upper bound for  $e_A(t, t_0)$  in the form:  $\|e_A(t, t_0)\| \leq Ke_\alpha(t, t_0)$ , where  $t, t_0 \in \mathbb{T}$ ,  $K$  is a positive constant,  $\alpha(t)$  is a function defined on  $\mathbb{T}$  and  $e_\alpha(t, t_0)$  is a scalar exponential function. One of the upper bounds is used for stability investigation of quasilinear systems on a time scale.

*Keywords:* Matrix exponential function; time scale theory; stability analysis, quasilinear systems.

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## 1. Introduction

Time scale theory, initiated by Stefan Hilger in his PhD thesis in 1988, is a promising area of mathematical science whose topic of study is functions defined on an arbitrary nonempty closed subset of the set of real numbers  $\mathbb{R}$ . In this theory, this subset is called a time scale. The main tasks of this theory are to unify concepts and results from continuous and discrete analysis and generalize the ones for any time scale. Differential equations from continuous analysis and difference equations from discrete analysis can be unified by time scale dynamic equations using time scale theory.

Many famous results from the theory of ordinary differential equations, calculus of variations, theory of difference equations have been generalized using the time scale theory in [1, 2, 3, 4, 5]. Recently, many experts in the time scale theory are interested in the study of the stability of solutions of dynamic equations on a time scale as well as the stabilization problem on a time scale. In particular, in [6, 7, 8] various properties of solutions of dynamic equations on time scales, such as stability, positivity, controllability, are investigated. The monograph [5] has investigated different approaches for stability analysis of time scale dynamic equations: i) approaches based on dynamic integral inequalities; ii) approaches based on a generalization of the direct Lyapunov method for equations on time scales; iii) approaches which are an adaptation of the comparison method which is well-known for continuous-time and discrete-time systems. In [9, 10, 11, 12] the stability of switched systems on a time scale was studied. In [13, 14, 15], the control design for dynamic equations on time scales was studied. In the papers [16, 17, 18, 19, 20] the consensus problem for multi-agent systems using time scale theory was investigated.

When investigating the stability of solutions of dynamic equations on a time scale, it is often necessary to estimate the norm of the matrix exponential function  $e_A(t, t_0)$ , which unifies the usual matrix exponential  $e^{A(t-t_0)}$  from continuous analysis and the integer power of  $(I + A)$  from discrete analysis. In the literature, one can find solutions for the problem of estimating the norm  $e_A(t, t_0)$  for some particular cases. For instance, in [21], when  $A$  is diagonal, the upper bound of the norm of  $e_A(t, t_0)$  is obtained in the form:

$$\|e_A(t, t_0)\| \leq K e_\alpha(t, t_0), \quad \forall t, t_0 \in \mathbb{T}(t \geq t_0), \quad (1)$$

where  $\mathbb{T}$  is an arbitrary time scale,  $K, \alpha$  are known parameters and  $e_\alpha(t, t_0)$  is a scalar exponential function. If  $A$  is a Jordan block, a similar estimation

is also obtained. However, for this case, the formula for calculating  $K$  in explicit form is not given.

In [10], for an arbitrary matrix  $A$ , another estimation for  $\|e_A(t, t_0)\|$  of the form (1) is obtained and the formula for calculating the constant  $K$  is proposed. According to that formula, one has to solve a maximum problem on the unbounded subset of the time scale  $\mathbb{T}$ , which can be a very difficult task in practice. In the recent paper [22] the estimation (1) with  $K = 1$  and  $\alpha = m(A, t)$  is obtained, where  $m(A, t)$  denotes a specific matrix measure (in the paper, it is called "initial growth rate") of  $A$ .

In the present paper, a new upper bound for  $\|e_A(t, t_0)\|$  of the form (1) is obtained and new formulas for calculating constant  $K$  are proposed. The new bound is applied for stability investigation of quasilinear systems on a time scale. The novelties of this paper are as follows:

- The estimation holds for an arbitrary matrix  $A$  and any time scale  $\mathbb{T}$ .
- The proposed formulas allow to calculate  $K$  by solving a maximum problem on the bounded or even finite subset of  $\mathbb{T}$ .
- The proposed estimation of  $\|e_A(t, t_0)\|$  is more accurate than the one from the paper[22].
- It is shown that the proposed estimation can be used for stability analysis of dynamic equations on a time scale.

The paper is organized as follows. Section 2 contains mainly basic facts from the calculus on time scale. In Section 3, we deduce three upper bounds for the norm of the matrix exponential function. The first one has the form of a product of the exponential function and some  $\Delta$ -differentiable function. The two other bounds have the form of the right hand part of inequality (1). In Section 4, one of the bounds is used for stability investigation of quasilinear systems on a time scale.

Some mathematical notations are used throughout this paper. For a square matrix  $A$  of size  $n \times n$ , let  $\text{Sp } A$  be the spectrum of  $A$  and let  $\lambda_i(A)$ ,  $i = 1, \dots, n$  be the eigenvalues of  $A$  repeated according to its multiplicity.  $\|\cdot\|$  denotes the spectral norm of a square matrix, i.e.  $\|A\| = \sqrt{\lambda_{\max}(A^T A)}$ , where  $\lambda_{\max}(A^T A) = \max_{i=1, \dots, n} \lambda_i(A^T A)$ . As usual,  $I_n \in \mathbb{R}^{n \times n}$  is the identity matrix.

## 2. Preliminary results

In this section, we give some facts from calculus on time scales and propose few concepts which will be needed to state our main result. More information on this topic can be found in [2, 3, 5].

According to [2], a time scale  $\mathbb{T}$  is an arbitrary nonempty closed subset of the set of real numbers  $\mathbb{R}$ . For any  $t \in \mathbb{T}$ , the forward jump operator  $\sigma$  is defined by  $\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$ . For each  $t \in \mathbb{T}$ ,  $\mu(t) = \sigma(t) - t$  is called the graininess of time scale  $\mathbb{T}$ . Let  $\mu^*$  and  $\mu_*$  denote  $\sup_{t \in \mathbb{T}} \mu(t)$  and  $\inf_{t \in \mathbb{T}} \mu(t)$ , respectively. Further, let us recall the definitions of  $\Delta$ -derivative, regressivity and exponential function.

**Definition 1.** [5, Def. 1.2.1] *A function  $f : \mathbb{T} \rightarrow \mathbb{R}$  is said to be  $\Delta$ -differentiable at the point  $t \in \mathbb{T}$  if there exists  $\alpha \in \mathbb{R}$  such that for any  $\varepsilon > 0$ , there exists a neighborhood  $U$  of  $t$  such that the inequality*

$$| [f(\sigma(t)) - f(s)] - \alpha[\sigma(t) - s] | < \varepsilon | \sigma(t) - s |$$

*holds for all  $s \in U$ . In this case, we denote  $f^\Delta(t) = \alpha$ . If  $f : \mathbb{T} \rightarrow \mathbb{R}$  is  $\Delta$ -differentiable at any  $t \in \mathbb{T}$ , then it is called  $\Delta$ -differentiable.*

In [2, 5] and other monographs on time scales calculus one can find the properties of  $\Delta$ -derivative. Based on them, the following formula for calculating of  $\Delta$ -derivative can be obtained:

$$f^\Delta(t) = \lim_{\tau \rightarrow \mu(t)+0} \frac{f(t + \tau) - f(t)}{\tau}. \quad (2)$$

**Definition 2.** [5, Def. 1.4.1] *A rd-continuous function  $p : \mathbb{T} \rightarrow \mathbb{R}$  is said to be regressive and we write  $p \in \mathcal{R}$  if  $1 + \mu(t)p(t) \neq 0$  for all  $t \in \mathbb{T}$ . Moreover,  $p$  is called positively regressive and we write  $p \in \mathcal{R}^+$  if  $1 + \mu(t)p(t) > 0$  for all  $t \in \mathbb{T}$ .*

If a real number  $a$  satisfies the condition  $1 + \mu(t)a > 0$ , for all  $t \in \mathbb{T}$ , then we will write similarly:  $a \in \mathcal{R}^+$ .

Denote by  $\mathcal{R}(\mathbb{T}; \mathbb{C})$  the space of rd-continuous, regressive mappings from  $\mathbb{T}$  to  $\mathbb{C}$ , where  $\mathbb{C}$  denotes the set of complex numbers.

The generalized exponential function of scalar function  $p \in \mathcal{R}$  is defined hereafter.

**Definition 3.** [5, Def. 1.4.3] For  $p \in \mathcal{R}$  and  $s \in \mathbb{T}$ , the exponential function  $e_p(\cdot, s)$  is defined by

$$e_p(t, s) = \exp \left( \int_s^t \xi_{\mu(\tau)}(p(\tau)) \Delta\tau \right), \quad t \in \mathbb{T},$$

where  $\xi_h$  is a cylindric transformation defined by the formula

$$\xi_h(z) = \begin{cases} \frac{1}{h} \text{Log}(1 + zh), & \text{if } h > 0, \\ z, & \text{if } h = 0, \end{cases} \quad (3)$$

where  $\text{Log}$  is the principal logarithm function.

Here,  $\int_a^b f(t) \Delta t$  denotes the Cauchy integral of  $f$ , defined on a time scale. Further, the matrix exponential function is defined.

**Definition 4.** [5, Def. 1.5.2] A rd-continuous  $n \times n$ -matrix-valued function  $A(t)$  is called regressive if  $I_n + \mu(t)A(t)$  is invertible for all  $t \in \mathbb{T}$ . The class of such matrices  $A$  is denoted by  $\mathcal{R} = \mathcal{R}(\mathbb{T}) = \mathcal{R}(\mathbb{T}, \mathbb{R}^{n \times n})$ .

**Definition 5.** [2, Def. 5.18] Let  $t_0 \in \mathbb{T}$  and assume that  $A \in \mathcal{R}$  is  $n \times n$ -matrix-valued function. The unique matrix-valued solution of the initial value problem

$$Y^\Delta = A(t)Y, \quad Y(t_0) = I_n,$$

is called the matrix exponential function (at  $t_0$ ), and it is denoted by  $e_A(\cdot, t_0)$ .

**Example 1.** [2, Example 5.19] Assume that  $A$  is a constant  $n \times n$ -matrix. If  $\mathbb{T} = \mathbb{R}$ , then  $e_A(t, t_0) = e^{A(t-t_0)}$ , while if  $\mathbb{T} = \mathbb{Z}$  and  $I + A$  is invertible, then  $e_A(t, t_0) = (I + A)^{(t-t_0)}$ .

The next theorem gives us an algorithm to compute the matrix exponential function:

**Theorem 1.** [2, Thm 5.35] Let  $A \in \mathcal{R}$  be a constant  $n \times n$ -matrix. Suppose  $t_0 \in \mathbb{T}$ . If  $\lambda_1, \lambda_2, \dots, \lambda_n$  are the eigenvalues of  $A$ , then

$$e_A(t, t_0) = \sum_{i=0}^{n-1} r_{i+1}(t) P_i, \quad (4)$$

where  $r(t) := (r_1(t), r_2(t), \dots, r_n(t))^T$  is the solution of the initial value problem

$$r^\Delta = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 1 & \lambda_2 & 0 & \ddots & \vdots \\ 0 & 1 & \lambda_3 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 & \lambda_n \end{bmatrix} r, \quad r(t_0) = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad (5)$$

and the  $P$ -matrices  $P_0, P_1, \dots, P_n$  are recursively defined by  $P_0 = I_n$  and  $P_{k+1} = (A - \lambda_{k+1}I_n)P_k$ , for  $0 \leq k \leq n-1$ .

The following lemma gives us the estimation of the norm of  $e^{At}$ .

**Lemma 2.** [23, Lemma 10.2.1] *If  $\nu_i$  are the eigenvalues of a matrix  $A$  and  $\Lambda = \max_i \Re(\nu_i)$ , then*

$$\|e^{At}\| \leq e^{\Lambda t} \sum_{k=0}^{n-1} \frac{(2t\|A\|)^k}{k!}, \quad t \geq 0. \quad (6)$$

The following functions, called generalized polynomials, are defined in [2] recursively as follows:

$$h_0(t, s) \equiv 1, \quad \forall s, t \in \mathbb{T}, \quad (7)$$

and given  $h_k$  for  $k \in \mathbb{N}_0$ , the function  $h_{k+1}$  is

$$h_{k+1}(t, s) = \int_s^t h_k(\tau, s) \Delta\tau, \quad s, t \in \mathbb{T}. \quad (8)$$

In [2] the functions  $h_k(t, s)$  are explicitly given for the cases  $\mathbb{T} = \mathbb{R}$  and  $\mathbb{T} = \mathbb{Z}$ , namely:

- if  $\mathbb{T} = \mathbb{R}$ , then  $h_k(t, s) = \frac{(t-s)^k}{k!}$  for all  $s, t \in \mathbb{R}$ ;
- if  $\mathbb{T} = \mathbb{Z}$ , then  $h_k(t, s) = C_{t-s}^k$  for all  $s, t \in \mathbb{Z}$ , where  $C_m^k$  with  $k \in \mathbb{N}_0$ ,  $m \in \mathbb{R}$ , denotes the general binomial coefficient  $\frac{m(m-1)\cdots(m-k+1)}{k!}$ .

Following [2], let us introduce a notion, which plays a significant role to obtain our main result.

**Definition 6.** Given a function  $\lambda \in \mathcal{R}(\mathbb{T}; \mathbb{C})$  such that  $1 + \mu(t)\lambda(t) \neq 0$ , for all  $t \in \mathbb{T}$ , then

$$\mathfrak{R}_{\mu(t)}(\lambda) = \lim_{s \rightarrow \mu(t)+0} \frac{|1 + s\lambda(t)| - 1}{s} \quad (9)$$

for  $t \in \mathbb{T}$  is the Hilger real part of  $\lambda$ .

Note, that for any complex number  $z$ ,  $\mathfrak{R}_0(z) = \Re(z)$ , where  $\Re(z)$  denotes the real part of  $z$ .

### 3. Estimation of the matrix exponential function on a time scale

In this section, we deduce the upper bounds for the norm of the matrix exponential function. This will be done according to the following steps:

- 1) using formula (4) from Theorem 1 (Putzer algorithm), we estimate the norm of the matrix exponential function  $e_A(t, t_0)$  by the product  $e_{\eta_2}(t, t_0)f(t, t_0)$ , where  $\eta_2$  characterizes the spectrum of matrix  $A$  and  $f(\cdot, t_0)$  is a  $\Delta$ -differentiable function.
- 2) considering an arbitrary  $\varepsilon > 0$  and rewriting the obtained estimation in the form  $e_{\eta_2 \oplus \varepsilon}(t, t_0)e_{\ominus \varepsilon}(t, t_0)f(t, t_0)$ , we prove that for positive values of  $\varepsilon$  the function  $F(t) = e_{\ominus \varepsilon}(t, t_0)f(t, t_0)$  is bounded on  $[t_0, +\infty)$ . We estimate from above the maximum value of  $F(t)$  and, as a result, obtain an upper bound for  $\|e_A(t, t_0)\|$ .

The next theorem investigates the first point of this scheme. In Theorem 4, the second point is deduced. Theorem 5 gives a modification of the result obtained in Theorem 4.

**Theorem 3.** Let  $A \in \mathbb{R}^{n \times n}$  be a matrix. If there exist real numbers  $\eta_1, \eta_2 \in \mathcal{R}^+$  such that  $\eta_1 < \eta_2$  and the following conditions are satisfied:

- (i)  $\eta_1 \leq \mathfrak{R}_{\mu(t)}(\lambda_i(A)) \leq \eta_2, \forall t \in \mathbb{T}$ , for each eigenvalue  $\lambda_i(A)$ ;
- (ii)  $\eta := \min \{1 + \mu_*\eta_1, 1 + \mu^*\eta_1\} > 0$ .

Then the following estimation holds for all  $t_0, t \in \mathbb{T}$

$$\|e_A(t, t_0)\| \leq e_{\eta_2}(t, t_0) \sum_{k=0}^{n-1} \left( \frac{2\|A\|}{\eta} \right)^k h_k(t, t_0), \quad (10)$$

where  $t \geq t_0$  and the functions  $h_k$  are defined by formulas (7) and (8).

**Proof.** In order to prove Theorem 3, we propose to use the representation of the matrix exponential function from Theorem 1 and the idea of deducing the estimation (6) from Lemma 2. From formula (4) we get the estimation:

$$\|e_A(t, t_0)\| \leq \sum_{k=0}^{n-1} |r_{k+1}(t, t_0)| \|P_k\|, \quad (11)$$

where  $r_{k+1}(t, t_0)$  denotes  $r_{k+1}(t)$  from formula (4). To deduce (10) from (11) let us estimate  $|r_k(t, t_0)|$  and  $\|P_k\|$  from above. To estimate  $|r_k(t, t_0)|$ , let us prove the following estimate:

$$|e_{\lambda_k}(t, t_0)| \leq e_{\eta_2}(t, t_0), \quad \forall t \geq t_0, \quad k = 1, \dots, n. \quad (12)$$

According to Definition 3 of the exponential function, the following formula holds for all  $t \geq t_0$  and  $k = 1, 2, \dots, n$ :

$$\begin{aligned} |e_{\lambda_k}(t, t_0)| &= \left| \exp \left( \int_{t_0}^t \xi_{\mu(\tau)}(\lambda_k) \Delta \tau \right) \right| \\ &= \exp \left( \int_{t_0}^t \Re(\xi_{\mu(\tau)}(\lambda_k)) \Delta \tau \right), \end{aligned} \quad (13)$$

where  $\xi_{\mu(\tau)}(\lambda_k)$  is the cylindric transformation defined by (3). From (3) we get the formulas:

$$\begin{aligned} \Re(\xi_{\mu(\tau)}(\lambda_k)) &= \begin{cases} \Re\left(\frac{\text{Log}(1+\mu(\tau)\lambda_k)}{\mu(\tau)}\right), & \text{if } \mu(\tau) > 0, \\ \Re(\lambda_k), & \text{if } \mu(\tau) = 0. \end{cases} \\ &= \begin{cases} \frac{\ln|1+\mu(\tau)\lambda_k|}{\mu(\tau)}, & \text{if } \mu(\tau) > 0, \\ \Re(\lambda_k), & \text{if } \mu(\tau) = 0. \end{cases} \end{aligned} \quad (14)$$

which hold for all  $\tau \in \mathbb{T}$  and  $k = 1, 2, \dots, n$ . According to condition (i) of the present theorem  $\Re_{\mu(t)}(\lambda_k) \leq \eta_2$  for all  $t \in \mathbb{T}$  and for any  $k = 1, 2, \dots, n$ . Using this property, Definition 6 of the Hilger real part  $\Re_{\mu(t)}(\lambda)$  and the positive regressivity of  $\eta_2 \in \mathcal{R}^+$ , we obtain from (14) the following estimation:

$$\begin{aligned} \Re(\xi_{\mu(\tau)}(\lambda_k)) &\leq \begin{cases} \frac{\ln(1+\mu(\tau)\eta_2)}{\mu(\tau)}, & \text{if } \mu(\tau) > 0, \\ \eta_2, & \text{if } \mu(\tau) = 0. \end{cases} \\ &= \begin{cases} \frac{\text{Log}(1+\mu(\tau)\eta_2)}{\mu(\tau)}, & \text{if } \mu(\tau) > 0, \\ \eta_2, & \text{if } \mu(\tau) = 0. \end{cases} = \xi_{\mu(\tau)}(\eta_2), \end{aligned} \quad (15)$$

for all  $\tau \in \mathbb{T}$  and  $k = 1, 2, \dots, n$ . Consequently, using properties of an integral on time scale (see Theorem 1.3.3 in [5]), from (13) and (15) we get (12).

For the next step we show that (12) implies the following estimation of the function  $|r_k(t, t_0)|$ :

$$|r_k(t, t_0)| \leq e_{\eta_2}(t, t_0) \frac{h_{k-1}(t, t_0)}{\eta^{k-1}}, \quad (16)$$

which holds for all  $t \geq t_0$  from  $\mathbb{T}$  and  $k = 1, 2, \dots, n$ . The proof of (16) is made by induction on  $k$ . Let  $k = 1$ , then according to formula (5) the function  $r_1(t, t_0)$  is a solution of the initial value problem:

$$r_1^\Delta = \lambda_1 r_1, \quad r_1(t_0) = 1. \quad (17)$$

As it is known (see Theorem 2.33 from [2]), this solution can be presented in the form  $r_1(t, t_0) = e_{\lambda_1}(t, t_0)$ . Then, (12) implies the estimation:

$$|r_1(t, t_0)| = |e_{\lambda_1}(t, t_0)| \leq e_{\eta_2}(t, t_0) = e_{\eta_2}(t, t_0) h_0(t, t_0) \frac{1}{\eta^0},$$

which means that estimation (16) holds for  $k = 1$ .

Now, we assume that (16) holds for some arbitrary natural number  $l$  ( $1 \leq l \leq n - 1$ ), i.e.

$$|r_l(t, t_0)| \leq e_{\eta_2}(t, t_0) \frac{h_{l-1}(t, t_0)}{\eta^{l-1}}, \quad (18)$$

and prove that estimation (16) holds for  $l + 1$ . According to formula (5) the function  $r_{l+1}(t, t_0)$  is a solution of the initial value problem:

$$r_{l+1}^\Delta = r_l + \lambda_{l+1} r_{l+1}, \quad r_{l+1}(t_0) = 0. \quad (19)$$

The solution of problem (19) is found by the formula of variation of constants (see Theorem 2.77 from [2]):

$$r_{l+1}(t, t_0) = \int_{t_0}^t e_{\lambda_{l+1}}(t, \sigma(\tau)) r_l(\tau, t_0) \Delta\tau.$$

Taking into account property (vii) from Theorem 1.4.2 given in [5], the last identity can be written in the form:

$$r_{l+1}(t, t_0) = \int_{t_0}^t \frac{e_{\lambda_{l+1}}(t, \tau)}{1 + \mu(\tau)\lambda_{l+1}} r_l(\tau, t_0) \Delta\tau. \quad (20)$$

From condition (i) and Definition (6) of the Hilger real part, it follows that

$$|1 + \mu(\tau)\lambda_{l+1}| \geq 1 + \mu(\tau)\eta_1,$$

for all  $\tau \in \mathbb{T}$ . Using this estimation and condition (ii), we obtain the following estimation for the denominator of the integrand of the last integral in (20):

$$|1 + \mu(\tau)\lambda_{l+1}| \geq \eta > 0, \quad \forall \tau \in \mathbb{T}. \quad (21)$$

Then, using (12), (18) and (21) we deduce from (20) the following estimation:

$$\begin{aligned} |r_{l+1}(t, t_0)| &\leq \int_{t_0}^t \left| \frac{e_{\lambda_{l+1}}(t, \tau)}{1 + \mu(\tau)\lambda_{l+1}} r_l(\tau, t_0) \right| \Delta\tau \\ &\leq \int_{t_0}^t \frac{e_{\eta_2}(t, \tau)}{\eta} e_{\eta_2}(\tau, t_0) \frac{h_{l-1}(\tau, t_0)}{\eta^{l-1}} \Delta\tau \\ &= \frac{e_{\eta_2}(t, t_0)}{\eta^l} \int_{t_0}^t h_{l-1}(\tau, t_0) \Delta\tau = \frac{e_{\eta_2}(t, t_0)}{\eta^l} h_l(t, t_0), \end{aligned}$$

which proves that the estimation (16) holds for all  $t \geq t_0$  and  $k = 1, 2, \dots, n$ .

Now, let us estimate  $\|P_k\|$  from above. As it is known,  $\max_{i=1, n} |\lambda_i| \leq \|A\|$ .

Consequently,

$$\begin{aligned} \|P_k\| &= \|(A - \lambda_k I_n)P_{k-1}\| \leq \|A - \lambda_k I_n\| \|P_{k-1}\| \\ &\leq (\|A\| + |\lambda_k|) \|P_{k-1}\| \leq 2\|A\| \|P_{k-1}\|, \end{aligned} \quad (22)$$

for all  $k = 1, 2, \dots, n$ . Since  $\|P_0\| = \|I_n\| = 1$ , then (22) implies:

$$\|P_k\| \leq (2\|A\|)^k, \quad (23)$$

for all  $k = 1, 2, \dots, n$ . Thereby, using estimations (16) and (23) we obtain from inequality (11) the estimation (10). The proof is completed. ■

**Example 2.** *It is easy to check that Lemma 2 is a corollary of Theorem 3. Let us set*

$$\begin{aligned} \mathbb{T} &= \mathbb{R}, \quad \eta_1 = \min_i \Re(\lambda_i(A)), \\ \eta_2 &= \max_i \Re(\lambda_i(A)), \quad t_0 = 0. \end{aligned} \quad (24)$$

*Then, all conditions of Theorem 3 hold. Using Example 1, we get from these assumptions:*

$$\begin{aligned} e_A(t, t_0) &= e^{At}, \quad e_{\eta_2}(t, t_0) = e^{\eta_2 t}, \\ h_k(t, t_0) &= \frac{t^k}{k!}, \quad \eta = 1. \end{aligned} \quad (25)$$

*Finally, from (10), we get (6).*

**Example 3.** *Let us set*

$$\begin{aligned}\mathbb{T} &= \mathbb{Z}, \quad \eta_1 = \min_i |1 + \lambda_i(A)| - 1, \\ \eta_2 &= \max_i |1 + \lambda_i(A)| - 1,\end{aligned}\tag{26}$$

and assume that  $1 + \eta_1 > 0$ . Then, all conditions of Theorem 3 hold. Denoting  $t - t_0 = m$  and using Example 1, we get from these assumptions:

$$\begin{aligned}e_A(t, t_0) &= (I + A)^m, \\ e_{\eta_2}(t, t_0) &= (\max_i |1 + \lambda_i(A)|)^m, \\ \eta &= \min_i |1 + \lambda_i(A)|, \quad h_k(t, t_0) = C_m^k.\end{aligned}\tag{27}$$

Taking into account the last formulas, we get from (10) the following estimation:

$$\begin{aligned}\|(I + A)^m\| &\leq \left(\max_i |1 + \lambda_i(A)|\right)^m \\ &\times \sum_{k=0}^{n-1} C_m^k \left(\frac{2\|A\|}{\min_i |1 + \lambda_i(A)|}\right)^k, \quad \forall m \in \mathbb{N}.\end{aligned}\tag{28}$$

The next theorem gives another estimation for the norm of the matrix exponential function. This estimation has a form:  $\|e_A(t, t_0)\| \leq M_\varepsilon e_{\eta_2 \oplus \varepsilon}(t, t_0)$ , where  $\varepsilon$  and  $M_\varepsilon$  are positive constants. Before formulating the next theorem, let us introduce the function  $\omega_\varepsilon(\mu)$ :

$$\omega_\varepsilon(\mu) = \frac{1}{\varepsilon} \sum_{k=0}^{n-2} (1 + \mu\varepsilon)^{\frac{k}{n-1}}, \quad \forall \varepsilon > 0,\tag{29}$$

and the functions defined for all  $t_0 \in \mathbb{T}$  by the following formulas:

$$t_1(t_0) = \inf \left\{ s \in \mathbb{T} : s \geq t_0 + \frac{(n-1)\eta}{2\|A\|} \right\},\tag{30}$$

$$\begin{aligned}\underline{t}(t_0) &= \sup \{s \in \mathbb{T} : s \leq t_0 + \omega_\varepsilon(\mu_*)\}, \\ \bar{t}(t_0) &= \inf \{s \in \mathbb{T} : s \geq t_0 + \omega_\varepsilon(\mu^*)\},\end{aligned}\tag{31}$$

$$\varphi(t) = \sum_{k=0}^{n-1} \left(\frac{2\|A\|}{\eta}\right)^k h_k(t, t_0), \quad \forall t \in \mathbb{T}, t \geq t_0.\tag{32}$$

**Theorem 4.** Let  $A \in \mathbb{R}^{n \times n}$  be a matrix such that there exist real numbers  $\eta_1, \eta_2 \in \mathcal{R}^+$  with  $\eta_1 < \eta_2$  such that the conditions (i) and (ii) from Theorem 3 are satisfied. Then, for an arbitrary  $t_0 \in \mathbb{T}$  and a real number  $\varepsilon > 0$  such that:

$$\omega_\varepsilon(\mu_*) \geq t_1(t_0) - t_0, \quad (33)$$

and for any  $t \geq t_0$  from the scale  $\mathbb{T}$ , the following estimate holds:

$$\|e_A(t, t_0)\| \leq L(\varepsilon)e_{\eta_2 \oplus \varepsilon}(t, t_0), \quad (34)$$

where the positive constant  $L(\varepsilon)$  is given by:

$$\begin{aligned} L(\varepsilon) = & \max \left\{ \varphi(t_1(t_0)), \frac{n(2\|A\|/\eta)^{n-1}}{(n-1)!} \right. \\ & \left. \times \max_{s \in [t(t_0), \bar{t}(t_0)]} ((s - t_0)^{n-1} e_{\ominus \varepsilon}(s, t_0)) \right\}. \end{aligned} \quad (35)$$

**Proof.** As all conditions of Theorem 3 are satisfied, estimation (10) holds. Let us take an arbitrary  $t_0 \in \mathbb{T}$  and consider  $\varepsilon > 0$  which satisfies condition (33) (it is not difficult to see that such a number exists, indeed from (29) we see that for any  $\mu \geq 0$ ,  $\lim_{\varepsilon \rightarrow 0^+} \omega_\varepsilon(\mu) = +\infty$ , consequently, for an arbitrary  $t_1(t_0) - t_0$  there exists  $\varepsilon > 0$  such that  $\omega_\varepsilon(\mu) \geq t_1(t_0) - t_0$ ). Then, (10) can be written in the following form:

$$\begin{aligned} \|e_A(t, t_0)\| & \leq e_{\eta_2}(t, t_0)e_\varepsilon(t, t_0)e_{\ominus \varepsilon}(t, t_0)\varphi(t) \\ & = e_{\eta_2 \oplus \varepsilon}(t, t_0)e_{\ominus \varepsilon}(t, t_0)\varphi(t). \end{aligned} \quad (36)$$

The main idea is to estimate the function  $e_{\ominus \varepsilon}(t, t_0)\varphi(t)$  from the right-hand side of (36), by a constant from above. As we will show, this function is bounded, so such a constant exists. The best constant is clearly the maximum value of the mentioned function on the interval  $[t_0, +\infty)$ , which appears to be very difficult to compute. Indeed, for the specific time scale  $\mathbb{T} = \mathbb{R}$ ,  $e_{\ominus \varepsilon}(t, t_0)\varphi(t)$  is a real-valued function of a real variable. Hence, to find its maximum value one can apply the theory of extrema from classical analysis. But in practice, the maximum point of that function can be found only approximately by numerically solving the algebraic equation of the  $n$ -th degree.

In the case of an arbitrary time scale, we will be faced with more difficulties in finding the maximum value of  $e_{\ominus \varepsilon}(t, t_0)\varphi(t)$ . These difficulties are caused by the following facts. Firstly, Ferma's theorem (also known as

interior extremum theorem) does not hold for functions defined on a time scale. Namely, in general, if a function has a local extremum at some point of a time scale and is  $\Delta$ -differentiable there, then its  $\Delta$ -derivative at that point is not necessarily equal to zero<sup>1</sup>. Second, for an arbitrary time scale the function  $\varphi(t)$  is not, in general, a polynomial, but has a more complex structure.

In order to overcome these difficulties, we estimate the function  $\varphi(t)$  from above by the simplest power function of the form  $C(t - t_0)^{n-1}$ , where  $C$  is a constant. The advantage of this estimate is that the maximum problem of the function  $C(t - t_0)^{n-1}e_{\ominus\varepsilon}(t, t_0)$  on the interval  $[t_0, +\infty)$ , as it will be shown, can be reduced to the problem of maximizing the same function on a subset which is finite or at least bounded.

Following the steps above, let us show by induction that the functions  $h_k(t, t_0)$  satisfy the following estimation:

$$h_k(t, t_0) \leq \frac{(t - t_0)^k}{k!}, \quad \forall t \geq t_0, t \in \mathbb{T}, \quad (37)$$

for all  $k = 0, 1, 2, \dots, n - 1$ .

As  $h_0(t, t_0) \equiv 1 \leq \frac{(t - t_0)^0}{0!}$ , (37) clearly holds for  $k = 0$ . Now, we assume that (37) holds for some arbitrary natural number  $l$  ( $0 \leq l \leq n - 2$ ), i.e.:

$$h_l(t, t_0) \leq \frac{(t - t_0)^l}{l!}, \quad \forall t \geq t_0, t \in \mathbb{T}, \quad (38)$$

and prove that (37) holds for  $l + 1$ . According to this assumption and formula (8), we have:

$$h_{l+1}(t, t_0) = \int_{t_0}^t h_l(s, t_0) \Delta s \leq \int_{t_0}^t \frac{(s - t_0)^l}{l!} \Delta s. \quad (39)$$

It is easy to show that

$$\frac{(s - t_0)^l}{l!} \leq \left( \frac{(s - t_0)^{l+1}}{(l + 1)!} \right)^\Delta, \quad \forall s \in [t_0, t] \cap \mathbb{T}. \quad (40)$$

---

<sup>1</sup>To see this, consider the function  $F(t) = (t - 0.1)^2$  defined on a time scale  $\mathbb{T} = \mathbb{Z}$ . At the points  $t = 0$  and  $t = 1$  from the time scale the function has a minimum on  $\mathbb{T}$ , however  $f^\Delta(0) = f(0 + 1) - f(0) = 0.8 \neq 0$  as well as  $f^\Delta(1) = f(1 + 1) - f(1) = 2.8 \neq 0$ .

Indeed, using the formula (2) for calculating the  $\Delta$ -derivative, we obtain:

$$\begin{aligned} \left[ \frac{(s-t_0)^{l+1}}{(l+1)!} \right]^\Delta &= \frac{\lim_{\tau \rightarrow \mu(s)+0} \left[ \frac{(s+\tau-t_0)^{l+1} - (s-t_0)^{l+1}}{\tau} \right]}{(l+1)!} \\ &= \frac{\left[ (l+1)(s-t_0)^l + \sum_{k=2}^{l+1} C_{l+1}^k (s-t_0)^{l+1-k} \mu^{k-1}(s) \right]}{(l+1)!} \\ &= \frac{(s-t_0)^l}{l!} + \sum_{k=2}^{l+1} \frac{C_{l+1}^k}{(l+1)!} (s-t_0)^{l+1-k} \mu^{k-1}(s) \geq \frac{(s-t_0)^l}{l!}, \end{aligned}$$

for all  $s \in [t_0, t] \cap \mathbb{T}$ . Thus, using (40) in estimate (39) we get:

$$h_{l+1}(t, t_0) \leq \int_{t_0}^t \left( \frac{(s-t_0)^{l+1}}{(l+1)!} \right)^\Delta \Delta s = \frac{(t-t_0)^{l+1}}{(l+1)!}, \quad (41)$$

which is (37) for  $l+1$ . We then conclude that estimation (37) holds for all  $k = 0, 1, 2, \dots, n-1$ . Then (37) implies the following estimation for  $\varphi(t)$ :

$$\varphi(t) \leq \sum_{k=0}^{n-1} \left( \frac{2\|A\|}{\eta} \right)^k \frac{(t-t_0)^k}{k!} = \sum_{k=0}^{n-1} \left( \frac{2\|A\|(t-t_0)}{\eta} \right)^k \frac{1}{k!}. \quad (42)$$

Let us continue to estimate the function  $\varphi(t)$  from above. Obviously, for a fixed  $k \in \mathbb{N}$ , the inequality

$$\left( \frac{2\|A\|(t-t_0)}{\eta} \right)^k \frac{1}{k!} \leq \left( \frac{2\|A\|(t-t_0)}{\eta} \right)^{k+1} \frac{1}{(k+1)!}$$

holds for  $t \geq t_0 + \frac{(k+1)\eta}{2\|A\|}$ . Hence, for all  $t \geq t_0 + \frac{(n-1)\eta}{2\|A\|}$ , the inequality

$$\left( \frac{2\|A\|(t-t_0)}{\eta} \right)^k \frac{1}{k!} \leq \left( \frac{2\|A\|(t-t_0)}{\eta} \right)^{n-1} \frac{1}{(n-1)!}$$

holds for all  $k = 0, 1, \dots, n-1$ . Using this estimate, we obtain from (42):

$$\varphi(t) \leq \frac{n}{(n-1)!} \left( \frac{2\|A\|(t-t_0)}{\eta} \right)^{n-1}, \quad (43)$$

for all  $t \geq t_1(t_0)$ ,  $t \in \mathbb{T}$  (recalling that  $t_1(t_0)$  is defined by (30)).

At the same time, since

$$\varphi^\Delta(t) = \sum_{k=0}^{n-1} \left[ \frac{2\|A\|}{\eta} \right]^k h_k^\Delta(t, t_0) = \sum_{k=0}^{n-1} \left[ \frac{2\|A\|}{\eta} \right]^k h_{k-1}(t, t_0)$$

and  $h_k(t, t_0) > 0$  for all  $k \in \mathbb{N}_0$  and  $t \geq t_0$  from the time scale,  $\varphi(t)$  increases on the set  $[t_0, +\infty) \cap \mathbb{T}$ . It implies the following estimation of  $\varphi(t)$  on the interval  $[t_0, t_1(t_0)]$  of the time scale:

$$\varphi(t) \leq \varphi(t_1(t_0)). \quad (44)$$

Estimations (43) and (44) can be combined into one formula, in which the functions on the right are defined on  $\mathbb{T}$ :

$$\varphi(t) \leq \begin{cases} \varphi(t_1(t_0)), & t \in [t_0, t_1(t_0)], \\ \frac{n}{(n-1)!} \left( \frac{2\|A\|(t-t_0)}{\eta} \right)^{n-1}, & t \geq t_1(t_0). \end{cases} \quad (45)$$

This estimation will allow us to estimate the function  $f(t) = e_{\ominus\varepsilon}(t, t_0)\varphi(t)$  appropriately.

Since  $\varepsilon > 0$ , then according to the property (x) of the exponential function from Theorem 1.4.2 given in [5], we obtain  $e_\varepsilon(t, t_0) > 0$ . Next, according to the property (viii) from the same Theorem, we get that  $e_\varepsilon^\Delta(t, t_0) = \varepsilon e_\varepsilon(t, t_0) > 0$ , i.e. the function  $e_\varepsilon(t, t_0)$  increases with respect to variable  $t$  on the interval  $[t_0, +\infty)$  of the time scale  $\mathbb{T}$ . Therefore  $e_\varepsilon(t, t_0) \geq e_\varepsilon(t_0, t_0) = 1$  and  $e_{\ominus\varepsilon}(t, t_0) = \frac{1}{e_\varepsilon(t, t_0)} \leq 1$ , for all  $t \geq t_0$  from the time scale. Taking into account this estimation as well as the estimation (45), the function  $f(t)$  can be estimated from above as follows:

$$f(t) \leq \varphi(t_1(t_0)), \quad (46)$$

for all  $t \in [t_0, t_1(t_0)]$  from the time scale  $\mathbb{T}$ . Let us estimate  $f(t)$  from above on the interval  $[t_1(t_0), +\infty)$  of the time scale  $\mathbb{T}$ . From (45), we obtain:

$$f(t) \leq \frac{n}{(n-1)!} \left( \frac{2\|A\|(t-t_0)}{\eta} \right)^{n-1} \cdot e_{\ominus\varepsilon}(t, t_0), \quad (47)$$

for all  $t \in [t_1(t_0), +\infty)$  from the time scale  $\mathbb{T}$ . Consider the function  $f_1(t) = (t-t_0)^{n-1}e_{\ominus\varepsilon}(t, t_0)$ . Using formula (2), property (3) from Theorem 1.2.2 and properties (iii) and (viii) from Theorem 1.4.2 given in [5], we can find its  $\Delta$ -derivative:

$$\begin{aligned} f_1^\Delta(t) &= ((t-t_0)^{n-1})^\Delta e_{\ominus\varepsilon}(t, t_0) + (t-t_0)^{n-1} e_{\ominus\varepsilon}^\Delta(t, t_0) \\ &= \frac{e_{\ominus\varepsilon}(t, t_0)}{1+\mu(t)\varepsilon} \left( (t-t_0) \lim_{s \rightarrow \mu(t)+0} \frac{1 - \frac{n-1}{s} \sqrt{1+s\varepsilon}}{s} + 1 \right) \\ &\quad \times \sum_{k=0}^{n-2} (t-t_0 + \mu(t))^{n-2-k} \left( (t-t_0)^{n-1} \sqrt{1 + \mu(t)\varepsilon} \right)^k. \end{aligned}$$

One can calculate the limits from the last expression:  $\lim_{s \rightarrow \mu(t)+0} \frac{1 - \sqrt[n-1]{1+s\varepsilon}}{s} = -\frac{1}{\omega_\varepsilon(\mu(t))}$ . Consequently,

$$f_1^\Delta(t) = \frac{e_{\ominus\varepsilon}(t, t_0)}{1 + \mu(t)\varepsilon} \left( \frac{t_0 - t}{\omega_\varepsilon(\mu(t))} + 1 \right) \times \sum_{k=0}^{n-2} (t - t_0 + \mu(t))^{n-2-k} \left( (t - t_0)^{n-1} \sqrt[n-1]{1 + \mu(t)\varepsilon} \right)^k.$$

Let us investigate increasing and decreasing intervals of the function  $f_1$ . From formula (29), we obtain:

$$\omega_\varepsilon(\mu_*) \leq \omega_\varepsilon(\mu(t)) \leq \omega_\varepsilon(\mu^*), \quad \forall t \in \mathbb{T}. \quad (48)$$

If  $t \in [t_1(t_0), t_0 + \omega(\mu_*)] \cap \mathbb{T}$ , then:

$$-\frac{t-t_0}{\omega_\varepsilon(\mu(t))} \geq -\frac{t_0 + \omega_\varepsilon(\mu_*) - t_0}{\omega_\varepsilon(\mu(t))} = \frac{-\omega_\varepsilon(\mu_*)}{\omega_\varepsilon(\mu(t))} \geq \frac{-\omega_\varepsilon(\mu_*)}{\omega_\varepsilon(\mu_*)} = -1.$$

Hence,  $f_1^\Delta(t) \geq 0$ , for all  $t \in [t_1(t_0), t_0 + \omega_\varepsilon(\mu_*)] \cap \mathbb{T}$ , and:

$$f_1^\Delta(t) \geq 0, \quad \forall t \in [t_1(t_0), \underline{t}(t_0)] \cap \mathbb{T}.$$

Due to properties of  $\Delta$ -derivative and integral on the time scale, the last inequality means that the function  $f_1(t)$  is increasing on  $[t_1(t_0), \underline{t}(t_0)]$ :

$$f_1(t) \leq f_1(\underline{t}(t_0)), \quad \forall t \in [t_1(t_0), \underline{t}(t_0)] \cap \mathbb{T}. \quad (49)$$

It is similarly proved, that:

$$f_1^\Delta(t) \leq 0, \quad \forall t \in [\bar{t}(t_0), +\infty) \cap \mathbb{T},$$

which means that  $f_1(t)$  is decreasing on  $t \in [\bar{t}(t_0), +\infty)$ :

$$f_1(t) \leq f_1(\bar{t}(t_0)), \quad \forall t \in [\bar{t}(t_0), +\infty) \cap \mathbb{T}. \quad (50)$$

From (49) and (50) we obtain:

$$f_1(t) \leq \max_{s \in [\underline{t}(t_0), \bar{t}(t_0)]} f_1(s), \quad \forall t \in [t_1(t_0), +\infty) \cap \mathbb{T}. \quad (51)$$

Hence, taking into account (47), we get:

$$f(t) \leq \frac{n}{(n-1)!} \left( \frac{2\|A\|}{\eta} \right)^{n-1} \times \max_{s \in [\underline{t}(t_0), \bar{t}(t_0)]} ((s - t_0)^{n-1} e_{\ominus\varepsilon}(s, t_0)), \quad (52)$$

for all  $t \geq t_1(t_0)$  from the time scale  $\mathbb{T}$ . Combining estimations (46) and (52), from inequality (36) we get the estimation (34), where  $L(\varepsilon)$  is defined by (35). Theorem 4 is proved. ■

**Theorem 5.** *Let  $A \in \mathbb{R}^{n \times n}$  be a matrix. If there exist real numbers  $\eta_1, \eta_2 \in \mathcal{R}^+$  with  $\eta_1 < \eta_2$  such that conditions (i) and (ii) from Theorem 3 are satisfied. Then, for an arbitrary  $t_0 \in \mathbb{T}$  and a real number  $\varepsilon > 0$  such that (33) is fulfilled and for any  $t \geq t_0$  from the scale  $\mathbb{T}$ , the following estimate holds:*

$$\|e_A(t, t_0)\| \leq L'(\varepsilon)e_{\eta_2 \oplus \varepsilon}(t, t_0), \quad (53)$$

where the positive constant  $L'(\varepsilon)$  is given by:

$$\begin{aligned} L'(\varepsilon) = \max \left\{ \sum_{k=0}^{n-1} \left( \frac{2\|A\|(t_1(t_0) - t_0)}{\eta} \right)^k \frac{1}{k!}, \frac{n}{(n-1)!} \left( \frac{2\|A\|}{\eta} \right)^{n-1} \right. \\ \left. \times \max_{s \in [t(t_0), \bar{t}(t_0)]} ((s - t_0)^{n-1} e_{\ominus \varepsilon}(s, t_0)) \right\}. \end{aligned} \quad (54)$$

**Proof.** The proof of this theorem is the same as the one of Theorem 4. The only difference is to obtain the constant  $L'(\varepsilon)$ , instead of estimation (44) the following estimation is proposed:

$$\varphi(t) \leq \sum_{k=0}^{n-1} \left( \frac{2\|A\|(t_1(t_0) - t_0)}{\eta} \right)^k \frac{1}{k!},$$

for all  $t \in [t_0, t_1(t_0)]$  from  $\mathbb{T}$ , which holds due to (42) and (44). ■

The following proposition gives a specific way to find  $\eta_1, \eta_2$ , which satisfy conditions (i) and (ii) of Theorem 3.

**Proposition 1.** *For a matrix  $A \in \mathbb{R}^{n \times n}$  let us set:*

$$\eta_1 = \min_{i=1, n} \Re_{\mu_*}(\lambda_i(A)), \quad \eta_2 = \max_{i=1, n} \Re_{\mu^*}(\lambda_i(A)). \quad (55)$$

*If  $\eta_1, \eta_2 \in \mathcal{R}^+$  and, besides,  $\eta_1$  satisfies condition (ii) from Theorem 3, then for an arbitrary  $t_0 \in \mathbb{T}$  and real  $\varepsilon > 0$  such that (33) is fulfilled, estimates (34)-(35) and (53)-(54) hold for any  $t \geq t_0$  from  $\mathbb{T}$ .*

**Proof.** Let us check condition (i) of Theorem 3 for the values  $\eta_1, \eta_2$  defined by (55). For this purpose, let us prove for any real numbers  $\mu_1, \mu_2 \geq 0$  and  $z \in \mathbb{C}$ , the following statement is true:

$$\Re_{\mu_1}(z) \leq \Re_{\mu_2}(z) \quad \text{whenever} \quad \mu_1 < \mu_2. \quad (56)$$

Suppose  $0 \leq \mu_1 < \mu_2$  and consider arbitrary  $z \in \mathbb{Z}$  and real numbers  $s_1, s_2$  such that  $s_2 > s_1 > 0$ . Then, we obtain

$$\begin{aligned} \frac{|1+zs_1|-1}{s_1} &= \left| \frac{1}{s_1} + z \right| - \frac{1}{s_1} = \left| \frac{1}{s_1} + z + \frac{1}{s_2} - \frac{1}{s_2} \right| - \frac{1}{s_1} \\ &\leq \left| \frac{1}{s_2} + z \right| + \left( \frac{1}{s_1} - \frac{1}{s_2} \right) - \frac{1}{s_1} = \left| \frac{1}{s_2} + z \right| - \frac{1}{s_2} \\ &= \frac{|1+zs_2|-1}{s_2}. \end{aligned}$$

Taking in both sides of the inequality the limit as  $s_1 \rightarrow \mu_1 + 0$  and  $s_2 \rightarrow \mu_2 + 0$ , we obtain (56).

From inequality (56), we obtain the following inequalities:

$$\begin{aligned} \eta_1 &= \min_{i=\overline{1,n}} \mathfrak{R}_{\mu_*}(\lambda_i(A)) \leq \mathfrak{R}_{\mu_*}(\lambda_j(A)) \leq \mathfrak{R}_{\mu(t)}(\lambda_j(A)) \\ &\leq \mathfrak{R}_{\mu^*}(\lambda_j(A)) \leq \max_{i=\overline{1,n}} \mathfrak{R}_{\mu^*}(\lambda_i(A)) = \eta_2, \end{aligned}$$

for all  $j = \overline{1,n}$ . Consequently, condition (i) of Theorem 3 holds. Now, we can conclude that all conditions of Theorems 4 and 5 are satisfied, i.e. estimates (34)-(35) and (53)-(54) hold for any  $t \geq t_0$  from  $\mathbb{T}$ . Theorem is proved. ■

**Remark 1.** *It is easy to check that  $L(\varepsilon) \leq L'(\varepsilon)$ , for all  $\varepsilon > 0$ , i.e. the estimation (53) is rougher than (34). However, for specific time scales estimating  $L'(\varepsilon)$  is simpler than  $L(\varepsilon)$ , because in (54), we do not need to calculate the value of the function  $h_k(t, t_0)$ . In such cases, it is better to use the estimation (53).*

**Remark 2.** *In the cases  $\mathbb{T} = \mathbb{R}$  or  $\mathbb{T} = h\mathbb{Z}$  ( $h > 0$ ), the time scale interval  $[\underline{t}(t_0), \bar{t}(t_0)]$  from the formula (35) degenerates to one or two points. In such cases, the problem of calculating the maximum of this formula becomes trivial. Indeed, if  $\mathbb{T} = \mathbb{R}$ , then  $\mu(t) \equiv 0$  on  $\mathbb{T}$  and  $\mu_* = \mu^* = 0$ , so  $\omega_\varepsilon(\mu_*) = \omega_\varepsilon(\mu^*) = \omega_\varepsilon(0) = \frac{n-1}{\varepsilon}$ . Then  $\underline{t}(t_0) = \bar{t}(t_0) = t_0 + \omega_\varepsilon(0) = t_0 + \frac{n-1}{\varepsilon}$ . Hence, in the formula (35) we get:*

$$\begin{aligned} \max_{s \in [\underline{t}(t_0), \bar{t}(t_0)]} ((s - t_0)^{n-1} e_{\ominus\varepsilon}(s, t_0)) &= \\ (\underline{t}(t_0) - t_0)^{n-1} e_{\ominus\varepsilon}(\underline{t}(t_0), t_0) &= (\bar{t}(t_0) - t_0)^{n-1} e_{\ominus\varepsilon}(\bar{t}(t_0), t_0). \end{aligned}$$

*If  $\mathbb{T} = h\mathbb{Z}$ , then  $\mu(t) \equiv h$  on  $\mathbb{T}$  and  $\mu_* = \mu^* = h$ , so  $\omega_\varepsilon(\mu_*) = \omega_\varepsilon(\mu^*) = \omega_\varepsilon(h)$ . Further two cases are possible. If  $t_0 + \omega_\varepsilon(h) \in h\mathbb{Z}$ , then  $\underline{t}(t_0) = \bar{t}(t_0) = t_0 + \omega_\varepsilon(h)$ . If  $t_0 + \omega_\varepsilon(h) \notin h\mathbb{Z}$ , then  $\underline{t}(t_0) < \bar{t}(t_0)$ , and between  $\underline{t}(t_0)$  and  $\bar{t}(t_0)$*

there is no point from the scale  $h\mathbb{Z}$ . In the last case, in the formula (35) we get:

$$\begin{aligned} & \max_{s \in [\underline{t}(t_0), \bar{t}(t_0)]} ((s - t_0)^{n-1} e_{\ominus \varepsilon}(s, t_0)) \\ &= \max \left\{ (\underline{t}(t_0) - t_0)^{n-1} e_{\ominus \varepsilon}(\underline{t}(t_0), t_0), \right. \\ & \quad \left. (\bar{t}(t_0) - t_0)^{n-1} e_{\ominus \varepsilon}(\bar{t}(t_0), t_0) \right\}. \end{aligned}$$

**Remark 3.** If  $\mathbb{T}$  is a time scale, which has bounded graininess function  $\mu(t)$  and consists of isolated points only, then the segment  $[\underline{t}(t_0), \bar{t}(t_0)]$  consists of a finite set of points of the scale. Therefore, in this case the maximum problem from (35) is simple.

Based on Remark 2, we obtain the following corollaries from Theorem 5.

**Corollary 1.** For any matrix  $A \in \mathbb{R}^{n \times n}$  and positive real number  $\varepsilon > 0$  such that  $\varepsilon < 2\|A\|$ , the following estimation holds for all  $t \in \mathbb{R}^+$ :

$$\|e^{At}\| \leq \max \left\{ \sum_{k=0}^{n-1} \frac{(n-1)^k}{k!}, \frac{n}{(n-1)!} \left( \frac{2\|A\|(n-1)}{\varepsilon} \right)^{n-1} \right\} \times e^{(\max_i \Re(\lambda_i(A)) + \varepsilon)t}. \quad (57)$$

**Proof.** Making assumptions (24), we obtain (25) and, besides,

$$\eta = 1, \quad t_1 = \frac{(n-1)}{2\|A\|}, \quad \underline{t} = \bar{t} = \frac{n-1}{\varepsilon}, \quad \varphi(t) = \sum_{k=0}^{n-1} \frac{(2\|A\|t)^k}{k!}.$$

Then, the estimation (34) becomes (57). ■

**Corollary 2.** For any matrix  $A \in \mathbb{R}^{n \times n}$  and positive real number  $\varepsilon > 0$  such that  $\min_{i=1, \dots, n} |1 + \lambda_i(A)| > 0$  and  $\omega_1 < \omega_\varepsilon(1)$ , the following estimation holds for all  $m \in \mathbb{N}$ :

$$\|(I + A)^m\| \leq \max_{i=1, \dots, n} L_i \cdot \left[ (1 + \varepsilon) \max_i |1 + \lambda_i(A)| \right]^m, \quad (58)$$

where

$$\begin{aligned}
L_1 &= \sum_{k=0}^{n-1} C_{\omega_1}^k \left( \frac{2\|A\|}{\min_i |1+\lambda_i(A)|} \right)^k, \\
L_2 &= \frac{n}{(n-1)!(1+\varepsilon)^{\underline{\omega}}} \left( \frac{2\|A\|\underline{\omega}}{\min_i |1+\lambda_i(A)|} \right)^{n-1}, \\
L_3 &= \frac{n}{(n-1)!(1+\varepsilon)^{\bar{\omega}}} \left( \frac{2\|A\|\bar{\omega}}{\min_i |1+\lambda_i(A)|} \right)^{n-1}, \\
\omega_1 &= \inf \left\{ s \in \mathbb{Z} : s \geq \frac{(n-1) \min_i |1+\lambda_i(A)|}{2\|A\|} \right\}, \\
\underline{\omega} &= \sup \{ s \in \mathbb{Z} : s \leq \omega_\varepsilon(1) \}, \bar{\omega} = \inf \{ s \in \mathbb{Z} : s \geq \omega_\varepsilon(1) \}.
\end{aligned}$$

**Proof.** Making assumptions (26) and denoting  $t - t_0 = m$ , we obtain (27) and, besides,

$$t_1 = \inf \left\{ s \in \mathbb{Z} : s \geq t_0 + \frac{(n-1) \min_i |1+\lambda_i(A)|}{2\|A\|} \right\} = t_0 + \omega_1,$$

$$\underline{t} = \sup \{ s \in \mathbb{Z} : s \leq t_0 + \omega_\varepsilon(1) \} = t_0 + \underline{\omega},$$

$$\bar{t} = \inf \{ s \in \mathbb{Z} : s \geq t_0 + \omega_\varepsilon(1) \} = t_0 + \bar{\omega},$$

$$\varphi(t) = \sum_{k=0}^{n-1} C_m^k \left( \frac{2\|A\|}{\min_i |1 + \lambda_i(A)|} \right)^k.$$

Then the estimation (34) becomes (58). ■

The following example shows the advantage of the estimations we propose over the ones proposed in [22].

**Example 4** Let us consider the matrix  $A$ , which is a 4-dimensional Jordan block, whose diagonal is filled with 2. The objective is to derive estimations of  $\|e_A(t, t_0)\|$  for the cases  $\mathbb{T} = \mathbb{R}$  and  $\mathbb{T} = \mathbb{Z}$  using Corollaries 1 and 2.

It is easy to check, that  $\|A\| \approx 2.85$ . Setting  $\varepsilon = 0.2$ , one can check that conditions of Corollaries 1 and 2 are satisfied. Hence, calculating the constants in formulas (57) and (58), we obtain the following estimations for  $\|e_A(t, 0)\|$  for time scales  $\mathbb{T} = \mathbb{R}$  (formula (59)) and  $\mathbb{T} = \mathbb{Z}$  (formula (60)):

$$\|e_A(t, 0)\| = \|e^{At}\| \leq 20772 \cdot e^{2.2t}, \forall t \in \mathbb{R}_+, \quad (59)$$

$$\|e_A(t, 0)\| = \|(I + A)^t\| \leq 1015 \cdot 3.6^t, \forall t \in \mathbb{N}. \quad (60)$$

It should be emphasized that these estimations are particular cases of the same formula (34).

As it was noted in the Introduction, in the paper [22] another approach to estimate  $\|e_A(t, t_0)\|$  is proposed. Applying this approach, we obtain the following estimation of  $\|e_A(t, t_0)\|$  in case  $\mathbb{T} = \mathbb{R}$  (formula (61)) and  $\mathbb{T} = \mathbb{Z}$  (formula (62)):

$$\|e_A(t, 0)\| = \|e^{At}\| \leq e^{2.81t}, \quad \forall t \in \mathbb{R}_+. \quad (61)$$

$$\|e_A(t, 0)\| = \|(I + A)^t\| \leq \|I + A\|^t \leq 3.84^t, \quad (62)$$

for all  $t \in \mathbb{N} \cup \{0\}$ .

In Figure 1, the graphs of the common logarithms of the estimations from (59) and (61), as well as  $\|e^{At}\|$  are presented. Such a representation via logarithm is used in order to show the behaviour of these functions on a large time interval

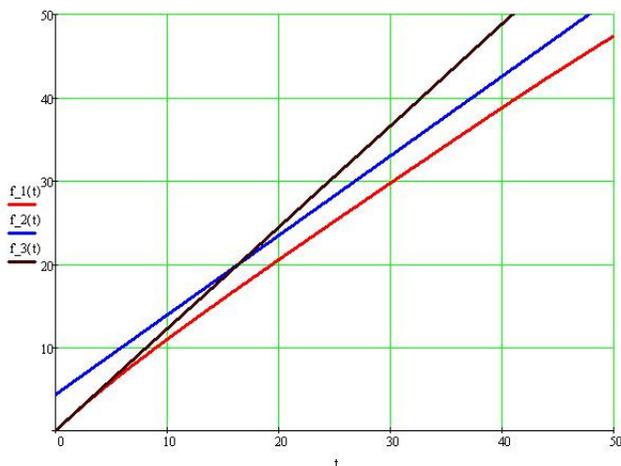


Figure 1: Logarithms of the function  $f_1(t) = \|e^{At}\|$  (red line) and its estimations  $f_2(t)$  from (59) (blue line) and  $f_3(t)$  from (61) (black line).

As we can see, for time scales  $\mathbb{T} = \mathbb{R}$  and  $\mathbb{T} = \mathbb{Z}$ , our estimations are more precise than estimations obtained from the approach proposed in [22].

#### 4. Stability analysis of quasilinear systems on a time scale

There are many publications devoted to stability investigation of quasilinear systems on a time scale. A review of the papers [24, 25, 5, 26, 22]

on the stability investigation of quasilinear systems by the method of integral inequalities shows that all stability conditions obtained in these papers contain so-called growth constants  $K$  and  $\alpha$  from estimation (1). This fact demonstrates the importance of the estimation (1). In this section we show how the estimations (34)-(35) and (53)-(54) that we have proposed, allow to investigate exponential stability of quasilinear systems on a time scale.

Let us consider the following system:

$$x^\Delta(t) = Ax(t) + f(t, x), \quad (63)$$

where  $x : \mathbb{T} \rightarrow \mathbb{R}^n$  is the unknown,  $A$  is  $n \times n$  regressive matrix, the vector function  $f : \mathbb{T} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  is an  $rd$ -continuous function on  $\mathbb{T}$ , satisfying  $f(t, x) = 0$  if and only if  $x = 0$  for all  $t \in \mathbb{T}$ .

According to Definition 2.2.1 from [5] the solution  $x = 0$  of system (63) is exponentially stable, uniformly with respect to  $t$  if there exist constants  $\gamma, \lambda > 0$  ( $-\lambda \in \mathcal{R}^+$ ) such that for any  $t_0 \in \mathbb{T}$  and  $x_0 \in \mathbb{R}^n$ , for the solutions  $x(t; t_0, x_0)$  of system (63) the following holds for all  $t \geq t_0 \in \mathbb{T}$ :  $\|x(t; t_0, x_0)\| \leq \|x(t_0)\| \gamma e_{-\lambda}(t, t_0)$ . In [5] it is proved (see Theorem 2.4.3) that as soon as system (63) satisfies the following conditions:

- 1) the state  $x = 0$  of the system  $x^\Delta(t) = Ax(t)$  is exponentially stable, i.e. there exist numbers  $L, \lambda > 0$  ( $-\lambda \in \mathcal{R}^+$ ) such that  $\|e_A(t, t_0)\| \leq L e_{-\lambda}(t, t_0)$ , for all  $t \geq t_0$ ;
- 2) there exists a constant  $\gamma > 0$  such that  $\|f(t, x)\| \leq \gamma \|x\|$ , for all  $(t, x) \in \mathbb{T} \times S$ , where  $S$  is defined as  $S = \{x \in \mathbb{R}^n : \|x\| < H, H = \text{const} > 0\}$ ;
- 3)  $\lambda - \gamma L > 0$ ,

then the solution  $x = 0$  of system (63) is exponentially stable on  $\mathbb{T}$ .

Since our Proposition 1 provides an appropriate procedure to check condition 1) and estimate the involved constants  $L$  and  $\lambda$ , it is a powerful tool to obtain some exponential stability results for system (63). Let us illustrate this tool by the following example.

**Example 5.** Let us consider system (63), where  $A = \begin{pmatrix} 0 & 2.65 & 1.5 \\ 0 & -1.165 & 0.85 \\ -0.2 & -1.2485 & -1.235 \end{pmatrix}$ ,  
 $f(t, x) = 5 \cdot 10^{-6} \cdot x \sin t$ ,  $x = (x_1, x_2, x_3)^T$ , with the time scale  $\mathbb{T} = \mathbb{P}_{\{t_{\sigma_k}, t_{k+1}\}} = \bigcup_{k=0}^{\infty} [k + \frac{k}{2k+1}, k + 1]$  (see Example 3.11 in [10]), and check if the solution

$x = 0$  is exponentially stable. For this aim let us first find estimation of  $\|e_A(t, t_0)\|$ , using Theorem 5 and Proposition 1. Indeed, let us set  $\varepsilon = 0.1$  and find some bounds on the constants  $\|A\|$ ,  $\mu_*$ ,  $\mu^*$ ,  $\eta$ ,  $\omega_\varepsilon(\mu_*)$ ,  $t_1(t_0)$ ,  $\underline{t}(t_0)$ ,  $\bar{t}(t_0)$ , which is necessary to get the estimation. Since

$$\mu(t) = \begin{cases} \frac{k}{2^{k+1}} & \text{if } t = k, \\ 0, & \text{if } t \in [k + \frac{k}{2^{k+1}}, k + 1), \end{cases}$$

then  $\mu_* = \inf_{t \in \mathbb{T}} \mu(t) = 0$ ,  $\mu^* = \sup_{t \in \mathbb{T}} \mu(t) = \frac{1}{2}$ . Besides,  $\|A\| \in [3.5366, 3.5368]$  and  $\text{Sp } A = \{-0.4, -1 - i, -1 + i\}$ . According to formulas (55):

$$\eta_1 = \min_{i=1,3} \Re(\lambda_i(A)) = -1, \eta_2 = \max_{i=1,3} \Re_{0.5}(\lambda_i(A)) = -0.4.$$

Using  $\eta_1$ , we obtain that  $\eta = 0.5 > 0$ . According to formula (29),  $\omega_\varepsilon(\mu) = \frac{1 + \sqrt{1 + \mu\varepsilon}}{\varepsilon}$ . Then:

$$\omega_\varepsilon(\mu_*) = 20, \quad \omega_\varepsilon(\mu^*) = 20 + \delta,$$

where  $\delta = 10(\sqrt{1.05} - 1)$ . Let us go on with  $t_1(t_0)$ . Due to formula (30)  $t_1(t_0) = \inf \{s \in \mathbb{T} : s \geq t_0 + \eta/\|A\|\}$ . If  $t_0 \in [k + 1 - \frac{\eta}{\|A\|}, k + 1]$  for  $k \in \mathbb{N}_0$ , then  $t_0 + \frac{\eta}{\|A\|} \in [k + 1, k + 1 + \frac{\eta}{\|A\|}]$ . Due to the fact that  $\frac{\eta}{\|A\|} \approx \frac{0.5}{3.5367} \approx 0.14$ , and the length of each "gap" of  $\mathbb{T}$  is larger than  $\frac{1}{3}$ , we conclude that  $t_0 + \frac{\eta}{\|A\|} \in [k + 1, k + 1 + \frac{k+1}{2^{k+3}}]$ , which implies that  $t_1(t_0) = k + 1 + \frac{k+1}{2^{k+3}}$ .

For the remaining values of  $t_0$  it can be seen that  $t_0 + \frac{\eta}{\|A\|}$  is an interior point for one of the segments  $[k + \frac{k}{2^{k+1}}, k + 1]$ , which implies that in this case  $t_1(t_0) = t_0 + \frac{\eta}{\|A\|}$ . Based on the analysis of  $t_1(t_0)$ , we obtain the following estimation:

$$t_1(t_0) - t_0 \leq \frac{1}{2} + \frac{\eta}{\|A\|}. \quad (64)$$

In order to calculate  $L$  we have also to estimate  $\underline{t}(t_0)$  as well as  $\bar{t}(t_0)$ . Since  $\omega_\varepsilon(\mu_*) = 20$  and  $\omega_\varepsilon(\mu^*) = 20 + \delta$ , then according to formula (31) we get:

$$\begin{aligned} \underline{t}(t_0) &= \sup \{s \in \mathbb{T} : s \leq t_0 + 20\}, \\ \bar{t}(t_0) &= \inf \{s \in \mathbb{T} : s \geq t_0 + 20 + \delta\}. \end{aligned} \quad (65)$$

Using formulas (65), we can calculate  $\underline{t}(t_0)$  and  $\bar{t}(t_0)$  and then use these numbers for calculating  $L'(\varepsilon)$  by formula (54). Since  $\underline{t}(t_0)$  and  $\bar{t}(t_0)$  depend on  $t_0$ ,  $L'(\varepsilon)$  also depends on  $t_0$ . This is an obstacle to the application of

conditions 1)-3), which guarantee the exponential stability of the solution  $x = 0$ , since in 1) and 3)  $L$  is a constant. To avoid this obstacle let us estimate  $L'(\varepsilon)$  from above. Firstly, we estimate the maximum from the formula (54):

$$\begin{aligned} & \max_{s \in [\underline{t}(t_0), \bar{t}(t_0)]} ((s - t_0)^2 e_{\ominus\varepsilon}(s, t_0)) \\ & \leq \max_{s \in [\underline{t}(t_0), \bar{t}(t_0)]} (s - t_0)^2 \cdot \max_{s \in [\underline{t}(t_0), \bar{t}(t_0)]} e_{\ominus\varepsilon}(s, t_0) \\ & \leq (\bar{t}(t_0) - t_0)^2 e_{\ominus\varepsilon}(\underline{t}(t_0), t_0). \end{aligned} \quad (66)$$

Here we used the fact that the function  $e_{\ominus\varepsilon}(\cdot, t_0)$  decreases on the set  $[t_0, +\infty) \cap \mathbb{T}$ . Now we estimate each of the multipliers from the last line of (66). For this aim we have to find estimations for  $\underline{t}(t_0)$  and  $\bar{t}(t_0)$ . Due to  $\delta < 0.25$  we obtain that if  $t_0 \in [k + \frac{k}{2k+1}, k + 1]$  for some  $k \in \mathbb{N}_0$ , then  $t_0 + 20 + \delta < k + 21 + \frac{k+21}{2k+43} \in \mathbb{T}$ , which allows us to conclude that  $\bar{t}(t_0) \leq k + 21 + \frac{k+21}{2k+43}$  in accordance with (65). Similarly, from the inequality  $t_0 + 20 > k + 20$  the following estimation is obtained:  $\underline{t}(t_0) \geq k + 20$ . Now we are able to estimate each multiplier in the inequality (66). For the first one we get:

$$\begin{aligned} & (\bar{t}(t_0) - t_0)^2 \\ & \leq \left( k + 21 + \frac{k+21}{2k+43} - \left( k + \frac{k}{2k+1} \right) \right)^2 < 484. \end{aligned} \quad (67)$$

Now let us estimate  $e_{\ominus\varepsilon}(\underline{t}(t_0), t_0)$  from above. Using Theorem 1.4.2 from [5] on properties of the exponential function, we obtain:  $e_{\ominus\varepsilon}(\underline{t}(t_0), t_0) = e_{\ominus\varepsilon}(\underline{t}(t_0), 0) e_{\ominus\varepsilon}(0, t_0)$ . Then, according to those properties and Remark 1.3.1 from [5], we conclude that function  $e_{\ominus\varepsilon}(t, s)$  increases with respect to the first argument  $t$  and decreases with respect to the second one  $s$ . Since  $t_0 \leq k + 1$  and  $\underline{t}(t_0) \geq k + 20$ , the following estimation holds:

$$e_{\ominus\varepsilon}(\underline{t}(t_0), t_0) \leq e_{\ominus\varepsilon}(k + 20, k + 1). \quad (68)$$

Now, let us estimate  $e_{\ominus\varepsilon}(k + 20, k + 1)$ . Due to properties of the exponential function and specifics of the time scale  $\mathbb{T}$ , we obtain the following estimation:

$$\begin{aligned} e_{\ominus\varepsilon}(k + 20, k + 1) &= \prod_{i=k+1}^{k+19} e_{\ominus\varepsilon}(i + 1, i) = \\ &= \prod_{i=k+1}^{k+19} \frac{1}{e_{\varepsilon}(i+1, i)} = \prod_{i=k+1}^{k+19} \frac{1}{e_{\varepsilon}(i+1, i + \frac{i}{2i+1}) e_{\varepsilon}(i + \frac{i}{2i+1}, i)}. \end{aligned}$$

Let us consider segments  $[i, i + \frac{i}{2i+1}]$  and estimate  $e_{\varepsilon}(i + \frac{i}{2i+1}, i)$ . Based on the structure of the time scale  $\mathbb{T}$  from the present example, we can conclude

that only ends of this segment belong to  $\mathbb{T}$  and besides  $i + \frac{i}{2i+1} = \sigma(i)$ . Hence,  $e_\varepsilon \left( i + \frac{i}{2i+1}, i \right) = e_\varepsilon \left( \sigma(i), i \right) = 1 + \mu(i)\varepsilon = 1 + \frac{i\varepsilon}{2i+1} > 1$ , for all  $i \in \mathbb{N}$ .

Similarly, let us consider segments  $\left[ i + \frac{i}{2i+1}, i + 1 \right]$  and estimate  $e_\varepsilon \left( i + 1, i + \frac{i}{2i+1} \right)$ . All points of that segment are the points from  $\mathbb{T}$ . Consequently,  $e_\varepsilon \left( i + 1, i + \frac{i}{2i+1} \right) = e^{\left(1 - \frac{i}{2i+1}\right)\varepsilon} = e^{\left(\frac{i+1}{2i+1}\right)\varepsilon} > e^{\frac{\varepsilon}{2}}$ , for all  $i \in \mathbb{N}$ .

From the estimate (68) and the calculations that follow it, we obtain the following estimation:

$$e_{\ominus\varepsilon}(\underline{t}(t_0), t_0) < \prod_{i=k+1}^{k+19} \frac{1}{1 \cdot e^{\frac{\varepsilon}{2}}} = e^{-\frac{19\varepsilon}{2}} = e^{-0.95}. \quad (69)$$

Finally, using the estimations (66), (67) and (69), from (53) and (54), we obtain the following estimation:

$$\|e_A(t, t_0)\| \leq 56192 \cdot e_{-0.3}(t, t_0).$$

From this inequality one can conclude that condition 1) from above holds with  $L = 56192$  and  $\lambda = -0.3$ . Since  $f(t, x) = 5 \cdot 10^{-6} \cdot x \sin t$ , then  $\|f(t, x)\| \leq 5 \cdot 10^{-6} \|x\|$ . It is now easy to verify that conditions 2)-3) also hold. Hence, the solution  $x = 0$  of system (63) is exponentially stable.

## 5. Conclusion

In this paper, we have studied the matrix exponential function on time scale. This function generalizes the usual matrix exponential  $e^{A(t-t_0)}$  as well as the  $m$ -th power  $(I + A)^m$  for an arbitrary time scale. Using the time scale theory, we have proposed new estimations of the norm of the matrix exponential function. It is shown that such an estimation can be used for stability investigation of dynamic equations on a time scale.

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