



HAL
open science

Reversibly greater downside risk aversion by a prudence-based measure

Donald Keenan, Arthur Snow

► **To cite this version:**

Donald Keenan, Arthur Snow. Reversibly greater downside risk aversion by a prudence-based measure. *Economics Letters*, 2022, 210, pp.110188. 10.1016/j.econlet.2021.110188 . hal-04315611

HAL Id: hal-04315611

<https://hal.science/hal-04315611>

Submitted on 22 Jul 2024

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



Distributed under a Creative Commons Attribution - NonCommercial 4.0 International License

Reversibly Greater Downside Risk Aversion by a Prudence-Based Measure

Donald C. Keenan¹

Department of Economics, THEMA
University of Cergy-Pontoise
Cergy-Pontoise, France, 95011
dkeenan@uga.edu

Arthur Snow

Department of Economics
University of Georgia
Athens, GA, USA 30602
snow@uga.edu

October 21, 2021

¹correspondence to dkeenan@uga.edu, 01 42 33 75 07

Abstract

We show that $p - 3r$ and r increasing, that is, both being greater for utility v than for u , implies greater downside risk aversion for v , where r is the Arrow-Pratt measure and p is the prudence measure. Moreover, this property is reversible, in that $p - 3r$ and r together decreasing implies less downside risk aversion.

keywords: prudence, risk aversion, downside risk aversion

JEL classification: D81

1 Introduction

In expected utility theory, the accepted definition of second-order risk aversion is $u''(y) < 0$ for utility function u over, say, income y , since this ensures that u dislikes any increase in income risk, that is, any mean preserving spread in the distribution for income y , as discussed in Rothschild & Stiglitz (1970). Correspondingly, the accepted definition of greater risk aversion for utility v relative to u is $\varphi''(u) < 0$ for $v = \varphi(u)$, since v dislikes any mean preserving spread in the distribution for utility u , as discussed by Diamond & Stiglitz (1974). The ranking of utility functions by the condition $\varphi'' < 0$ does, indeed, constitute a strict partial order, as one would want of a greater risk aversion relation. Notably, the same can also be said of less risk aversion, as defined by $\varphi'' > 0$, which yields a reverse strict partial ordering by less risk aversion. In this paper, we extend the ordering and reversibility properties to the third order, using prudence and risk measures.¹

The accepted definition of third-order downside risk aversion is $u''' > 0$, since this ensures that u dislikes any increase in downside risk, that is, any mean-and-variance preserving spread in the distribution for y , as shown by Menezes et al. (1980)). By analogy with the second order, one might expect that the definition of greater downside risk aversion would then be $\varphi''' > 0$. Indeed, a ranking of utility functions by this transformational criterion performs as one would wish in comparing the dislike of v for greater downside risk in the distribution of person u 's utility, to which that person would, instead, be indifferent, where the variance of the distribution is also being preserved (See Keenan & Snow (2009)). However, this ranking of utility functions by the condition $\varphi''' > 0$ is not at all a strict partial order: it can cycle to any order, even order two. (See Keenan & Snow (2012)).

A resolution to this ordering problem is effected by appending to the third-order condition $\varphi''' > 0$, the second-order condition $\varphi'' < 0$, as discussed by Keenan & Snow (2016), just as in comparative statics analyses, it often proves useful to accompany the downside risk aversion, $u'' > 0$, with risk aversion $u'' < 0$. As the pair of conditions $\varphi''' > 0, \varphi'' < 0$ constitutes a strict partial order, we shall define this pair to mean *greater downside risk aversion*. Furthermore, these transformational conditions between u and v characterize the changes in preferences assuring that v never likes

¹A strict partial order is one that is irreflexive, asymmetric and transitive. It is a useful property, in that the transitivity forbids cycles to any order, where a cycle would eliminate the possibility of any comparative static result signing the effect on an endogenous choice variable, since the latter cannot, in turn, cycle. There are, though, other means of preventing cycles, and, indeed, while not transitive, the less downside risk aversion relation, discussed below, does not, cycle, since it requires, in part, that there be less risk aversion, where that relation clearly does not permit cycles. Nonetheless, an economist is likely to become unhappy upon being told that, with a third-order increase in risk, a choice variable increases less for v than for u due to lessened downside risk aversion, with the same being true between w and v , but while it is then true that the variable also increases less for w than for u , it nonetheless, cannot necessarily be attributed to less downside risk aversion, when this might not hold between w and u . Liu & Wong (2019) conclude, rightly, that a ranking being a partial order is “necessary for the concept to be useful in comparative statics” (p. 114).

any change in income risk which induces a third-order stochastically dominated shift in the utility distribution for u , a necessarily larger class of distributional changes than those mentioned above when characterizing just the condition $\varphi''' > 0$.^{2,3}

Having solved the ordering problem, there remains the reversibility problem, namely that while $\varphi''' > 0, \varphi'' < 0$ is a partial order, the reverse condition $\varphi''' < 0, \varphi'' > 0$, giving the naturally corresponding requirement for less downside risk aversion, does not similarly form a partial order. One way of handling this additional problem, though, is to turn to risk measures - pairs of measures in this case - and identify a pair whose increase implies greater downside risk aversion, and whose decrease implies less downside risk aversion. The first of the measures will obviously be the Arrow-Pratt measure, $r_u(y) \equiv -u''(u)/u'(y)$, since its increase characterizes the second-order part, $\varphi'' < 0$, of these transformational conditions for comparative downside risk aversion. Equally clearly, the other measure will then have to be some third-order one. The consequence is that, while less downside risk aversion may not be a partial order, this will pose no problem for the refinement of this less downside relation induced by decreases in the pair of measures, which will necessarily constitute a strict partial suborder.⁴

We first observe that the Schwarzian measure, $S_u \equiv d_u - (3/2)r_u^2$, where $d_u \equiv u'''/u'$ is the downside risk aversion measure proposed by Crainich & Eeckhoudt (2008), serves well as a measure guaranteeing $\varphi''' > 0$, inasmuch as S_v is uniformly greater than S_u only if $\varphi''' > 0$. Thus, an increase in the pair of measures S and r suffices for assuring greater downside risk aversion (see Keenan & Snow (2002, 2009)). Unfortunately, S decreasing does not imply $\varphi''' < 0$, and, indeed, S and r both declining does not assure less downside risk aversion, either. There is, however, a measure, modelled on S , that does do the job, in concert with r : this measure is $D_u = d_u - 3r_u^2$, whose increase, together with r increasing, also implies greater downside risk aversion (Keenan & Snow (2020)), since D increasing is a stronger condition than S increasing, given r increasing. Then, when D decreases, it is again a stronger condition than S decreasing, given that r is now decreasing: indeed, it is seen to then be powerful

²See Keenan & Snow (2016) or Liu & Wong (2019) for the arguments proving these assertions. Keenan & Snow (2017) also discuss conditions sufficient for a change in income risk to constitute such a change in the utility distribution of u .

³Of course, just as $\varphi'' \geq 0$ proves useful, beyond just characterizing the reaction to mean-preserving second-order spreads in utility distributions, so greater downside risk aversion can be useful even in contexts where the just-described risk distributional changes characterizing it are not obviously involved. See the rent-seeking game below.

⁴Transformational conditions refer to the manner in which one utility function is globally altered to yield the other, through the transformation $v = \varphi(u)$, which thus treats both utility functions simultaneously. Being rather abstract, such a transformation can be somewhat difficult to calculate in practice. Measures are, instead, differential operators, which apply to a single utility function, and at any given income yield a real-valued measurement of the utility function's higher-order curvature. It is seldom very difficult to calculate the various derivatives that make up the measure of a given utility function. A measure is judged useful when a comparison between the separate measurements of the two different utility functions tells one about the transformation between them, since it is the latter that is most easily linked to their differing attitudes towards changes in risk distributions

enough to imply less downside risk aversion, whereas S decreasing does not, even given the fact of r decreasing.

2 The Prudence-Based Measure of Reversibly Greater Downside Risk Aversion

While the use of D and r may be considered a successful resolution of the reversibility problem, we recognize that, despite being assured that the Schwarzian $S_u = d_u - (3/2)r_u^2$ is the best-known of third-order measures among mathematicians, it still appears somewhat outlandish to many economists, and thus, the same may be said of $D = d_u - 3r_u^2$, which does not even enjoy the benefit of being well-known among mathematicians. Economists, it seems, are much more comfortable with the familiar second-order Arrow-Pratt measure r_u and its accompanying third-order prudence measure, $p_u \equiv -u'''/u''$ (Kimball (1990)), as well as in changes of these measures, whenever speaking of altered risk attitudes at the second and third orders. The hope, in this present note, is that this satisfaction with these measures should carry through to a certain additional acceptance of simple integer combinations of the two measures. This is because, rather than D_u , here, we propose working with the third-order measure $p_u - 3r_u$.

We note that the third-order transformational condition we seek to sign (in addition to the second-order one) can be conveniently expressed as

$$d_\varphi \equiv \frac{\varphi'''}{\varphi'} = \frac{1}{u'^2}[(d_v - d_u) - 3r_u(r_v - r_u)]. \quad (1)$$

theorem 1 *The measure p uniformly increasing more than three times the increase in r (with r increasing) in going from u to v implies greater downside risk aversion of v over u , and similarly, p uniformly decreasing more than three times r decreasing (with r decreasing) in going from u to v instead implies that v is less downside risk averse than u .*

proof:

$$\begin{aligned} (d_v - d_u) - 3r_u(r_v - r_u) &= p_v r_v - p_u r_u - 3r_u(r_v - r_u) \\ &= p_v(r_v - r_u) + r_u(p_v - p_u) + r_u(-3r_v + 3r_u) \\ &= p_v \Delta r + r_u \Delta(p - 3r). \quad QED \end{aligned} \quad (2)$$

Thus, rather than use changes in D and r to obtain reversibly greater or less downside greater risk aversion, one can equally well use changes in $p - 3r$ and r . Neither the use of $p - 3r$, nor the use of D , can be said to yield a sufficient condition

weaker than the other.⁵ Infinitesimally, though, parameterizing preferences by θ , with $\theta = 0$ signifying the original preferences, one has:

$$[u']^2 \frac{\partial d_\varphi}{\partial \theta} \Big|_{\theta=0} = \frac{\partial S}{\partial \theta} \Big|_{\theta=0} = \left(p \frac{\partial r}{\partial \theta} + r \frac{\partial(p-3r)}{\partial \theta} \right) \Big|_{\theta=0}. \quad (3)$$

On the other hand, the infinitesimal condition for a change in D is

$$\frac{\partial D}{\partial \theta} \Big|_{\theta=0} = \left(p \frac{\partial r}{\partial \theta} + r \frac{\partial(p-6r)}{\partial \theta} \right) \Big|_{\theta=0}. \quad (4)$$

Thus, in infinitesimal terms, D increasing is stronger than necessary (relative to $p-3r$, given r increasing) in order that one achieves d_φ positive, and similarly, D decreasing is stronger than necessary (again relative to $p-3r$, but now given r decreasing) in order that one achieve d_φ negative. Since the situation of small preference changes is typically of primary importance to economists, this may well tip matters in favor of $p-3r$ over the use of D . In any case, as said, we are confident that economists will generally prefer the form of $p_u - 3r_u$ over that of $D_u = d_u - 3r_u^2$.^{6,7}

Note that the sign of $p-3r$, if not its direction of change, already appears in numerous applied contexts, and plays a determinative role in the investigation of the relationship between changes in prudence p alone and the transformation condition $\varphi''' > 0$, as conducted by Keenan & Snow (2008).⁸

⁵It was argued in Keenan & Snow (2020) that when r_v approaches 0, then D decreasing becomes a necessary condition for less downside risk aversion. Similarly, when p_v approaches 0, then $p-3r$ decreasing becomes such a necessary condition, as seen from the last expression in the proof of the theorem.

⁶While it is true that we have achieved reversibility of d_φ using two different measures, D and $p-3r$, an examination of the expression for d_φ in Keenan & Snow (2020), and that of d_φ above, should convince the reader that there are not an unlimited number of similar measures, where one obviously, though, should not count measure conditions stronger than one or the other of the two pairs of measures being discussed. The fact that $D = r(p-3r)$ might leave one with the impression that one or the other results, here or in Keenan & Snow (2020), implies the other, but examination will show that this is not so.

⁷As well as it sufficing that the two terms, involving changes in $p-3r$ and r , on the r.h.s. of (3) be positive in order that our infinitesimal transformational conditions be, say, positive, so it is also necessary that they both be non-negative, whenever one is not sure about the magnitude of the r term.

⁸Applications involving the sign of $p-3r$ include, for example, Thiele & Wambach (1998), Sinclair-Desgagné (2001), Boucher & Guirkinger (2007), Fagart & Sinclair-Desgagné (2007), Kadan & Swinkels (2013), and Chade, & Vera de Serio (2014). The issue of simple greater or less risk aversion ($\varphi''' \geq 0$) for the agent u relative to the principal v does explicitly appear in such principal-agent problems, but since the principal is typically assumed to be risk-neutral, this reduces to the sign of $p_u - 3r_u$.

3 Applications

Reversible measures, like D or $p - 3r$ (together with r), can be quite useful, for automatically transferring results first proven using the greater downside risk aversion relation, which is a perfectly good partial order, to the case of less downside risk aversion, where the transformation is not a strict partial order. In the latter case, as well as possibly encountering paradoxes of interpretation, one might encounter difficulties in extending one's argument, when trying to directly encompass the somewhat problematic less downside risk aversion transformational relation. No such difficulties can appear, though, when using our measures, where the reversibility of results is then entirely straightforward. One also notes that the reason why one prefers to start with (greater) downside risk averse transformational reasoning is that experience shows that global arguments are generally more easily achieved using transformations, as opposed to already beginning one's reasoning with the use of measures.⁹

Numerous examples using greater downside risk aversion to obtain comparative static results, ones necessarily using transformations, can be found in the literature, covering such topics such as equilibrium precautionary saving (Keenan & Snow (2016), Liu & Wong (2019)), portfolio insurance (Keenan & Snow (2017)), and self-protection (Peter (2020, 2020a)). All of these can then be treated, instead, using our measures, with the reversal of the measures' directions then automatically yielding a result using less downside risk aversion. For example, Peter (2020) establishes conditions under which the optimal self-protection effort is lower for v than for u if v exhibits greater downside risk aversion. We then substitute in the sufficient measure conditions that r and $p - 3r$ are uniformly greater for v than for u . As these restrictions are reversible, we immediately get an opposite prediction, that optimal self-protection is greater for v than for u if both r and $p - 3r$ are uniformly less for v than for u , and that this may be attributed to less downside risk aversion.¹⁰

To further see how the program can be applied, we sketch the case of a rent-seeking game, originally introduced in Konrad & Schlesinger (1997) and further treated in Treisch (2010) and Liu & Wong (2019), this being a particularly straightforward example. Each of $N \geq 2$ entirely symmetric contestants, with $u'' < 0$ and $u''' > 0$, undertakes an effort e_i to obtain a fixed rent β , to be awarded to just one of them, by "lottery." There are, potentially, differences in each one's probability of success, $p_i(\mathbf{e})$,

⁹Of course, upon reversing the direction of the measures in the statement of any result, the two individuals, u and v , effectively switch their roles as to who goes before and who goes after. This then affects how the assumptions in the restatement of the result must read, at least when the two individuals are treated differently by the comparative static setup, beyond just the inevitable distinction arising from the individuals being ranked differently by their measures. One also sees the difficulty with trying to similarly switch, in such a facile fashion, results involving our third-order transformational conditions, since it is not the case that $\varphi''' > 0$, $\varphi'' < 0$ is the same as $\psi''' < 0$, $\psi'' > 0$, for $\psi = \varphi^{-1}$, as is true of just the second-order part.

¹⁰As discussed above, to automatically generate a result, one now has to move the standing assumptions on u to v , instead. As Peter demonstrates, this one automatic result does not prevent getting other non-automatic results involving less downside risk aversion, especially since one now has a strict partial order, given that $p - 3r$ and r are being used.

for winning the lottery, where $\mathbf{e} \equiv (e_1, \dots, e_N)$, but it is assumed that $p_i(e, \dots, e) = 1/N$, when all efforts are identical. The pure strategy Nash equilibrium will, of course, then be symmetric (and unique), and so, all efforts will, indeed, then be identical.¹¹ It can be shown that if all contestants become (still symmetrically) more downside risk averse, then the effort of each diminishes. The case of less downside risk aversion is not explicitly considered in the literature, but one can instantly extend matters to this case, as well as expressing their result in convenient measure terms, by inserting our measures $p - 3r$ and r in the already established greater downside risk aversion results. Obviously, then, decreasing $p - 3r$ and r gives the opposite result to increasing it, whereas the same sort of result is not so immediately obvious, if, instead, you choose to reverse the transformational conditions, even though reversing, instead, the directions of the measures allows the conclusion that one has, indeed, arrived at a case of lessened downside risk aversion.

4 Conclusion

It is shown that uniform changes in the measure $p - 3r$, where p is the prudence measure and r is the Arrow-Pratt measure, together with a corresponding change in r itself, are sufficient to obtain either greater or less downside risk aversion, depending on the direction of change of these second and third-order measures. Given that measures are generally favored by economists in applications, this suggests their use, rather than that of risk transformations, in stating and putting into practice comparative static results concerning changes in the degree of downside risk aversion.

¹¹Regularity assumptions include $\partial p_i(\mathbf{e})/\partial e_i > 0$, $\partial^2 p_i(\mathbf{e})/\partial e_i^2 < 0$, and $\partial p_i(\mathbf{e})/\partial e_j < 0$, $\partial^2 p_i(\mathbf{e})/\partial e_i \partial e_j \leq 0$, $i \neq j$.

References

- [1] S. Boucher, C. Guirkinger, Risk, Wealth, and Sectoral Choice in Rural Credit Markets, *Am. J. Agric. Econ.* 89 (2007), 991–1004.
- [2] H. Chade, V. Vera de Serio, Wealth Effects and Agency Costs, *Games Econ. Behav.* 86 (2014), 1-11.
- [3] D. Crainich, L. Eeckhoudt, On the Intensity of Downside Risk Aversion, *J. Risk Uncertain.* 36 (2008), 267-276.
- [4] P. Diamond, J. Stiglitz, Increases in Risk and in Risk Aversion, *J. Econ. Theory* 8 (1974), 337-360.
- [5] M.-C. Fagart, B. Sinclair-Desgagné, Ranking Contingent Monitoring Systems, *Manage. Sci.* 53 (2007), 1501-1509 .
- [6] O. Kadan, J. Swinkels, On The Moral Hazard Problem without the First-Order Approach, *J. Econ. Theory* 148 (2013), 2313-2343.
- [7] D. Keenan, A. Snow, Greater Downside Risk Aversion, *J. Risk Uncertain.* 24 (2002), 267-277.
- [8] D. Keenan, A. Snow, Greater Downside Risk Aversion in the Large, *J. Econ. Theory* 144 (2009), 1092-1101.
- [9] D. Keenan, A. Snow, Greater Prudence and Greater Downside Risk Aversion, *J. Econ. Theory* 145 (2010), 2018-2026.
- [10] D. Keenan, A. Snow, The Schwarzian Derivative as a Ranking of Downside Risk Aversion, *J. Risk Uncertainty* 44 (2012), 149-160.
- [11] D. Keenan, A. Snow, Strong Increases in Downside Risk Aversion, *Geneva Risk Insur. Rev.* 41 (2016), pp. 149-161.
- [12] D. Keenan, A. Snow, Greater Parametric Downside Risk Aversion, *J. Math. Econ.* 71 (2017), 119-128.
- [13] D. Keenan, A. Snow, Reversibly Greater Downside Risk Aversion, working paper (2020).
- [14] M. Kimball, Precautionary Saving in the Small and in the Large, *Econometrica* 58 (1990), 53-73.
- [15] K. Konrad, H. Schlesinger, Risk Aversion in Rent-Seeking and Rent-Augmenting Games, *Econ. J.* 107 (1997), 1671-1683.
- [16] L. Liu, K. Wong, Greater (Absolute) Arrow-Pratt Risk Aversion of Higher Orders, *J. Math. Econ.* 82 (2019), 122-124.

- [17] C. Menezes, C. Geiss, J. Tressler, Increasing Downside Risk, *Amer. Econ. Rev.* 70 (1980), 921-932.
- [18] R. Peter, Who Should Exert More Effort? Risk Aversion, Downside Risk Aversion and Optimal Prevention, *Econ. Theory* 141 (2020), 1259-1281.
- [19] R. Peter, A Fresh Look at Prevention for Health Risks, working paper (2020a).
- [20] M. Rothschild, J. Stiglitz, Increasing Risk I: A Definition, *J. Econ. Theory* 2 (1970), 225-243.
- [21] B. Sinclair-Desgagné, Incentives in Common Agency, *Cahiers de recherche, HEC Montréal, Institut d'économie appliquée* (2001).
- [22] H. Thiele, A. Wambach, Wealth Effects in the Principal Agent Model, *J. Econ. Theory* 89 (1999), 247-260.
- [23] N. Treich, Risk-aversion and Prudence in Rent-Seeking Games, *Public Choice* 145 (2010), 339-349.