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Jean Auriol

► **To cite this version:**

Jean Auriol. SOME INSIGHTS ON THE PRACTICAL CONTROL OF HYPERBOLIC SYSTEMS.
2023. hal-04315386

HAL Id: hal-04315386

<https://hal.science/hal-04315386v1>

Preprint submitted on 30 Nov 2023

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SOME INSIGHTS ON THE PRACTICAL CONTROL OF HYPERBOLIC SYSTEMS

JEAN AURIOL

Abstract. In this chapter, we present some recent developments in the practical control of hyperbolic systems. Although our primary focus lies on scalar balance laws, most of the results showcased herein can be extended to non-scalar systems or interconnected configurations. The control strategies proposed in this chapter are said to be practical as they are constructive, easily implementable, and offer degrees of freedom that can be tailored to satisfy specific performance requirements. We first design finite-time stabilizing controllers using the backstepping approach. Then, we show that finite-time stabilization can yield vanishing robustness margins. Therefore, we introduce tuning parameters in the design to allow for potential trade-offs between performance specifications (e.g., robustness and convergence rate). Finally, we consider possible approximations of the control law in view of practical implementation and give generic conditions to guarantee closed-loop stability.

1. Introduction

Distributed parameter systems offer a natural way to represent industrial processes that involve the evolution of quantities in both time and space. Hyperbolic Partial Differential Equations (PDEs), in particular, are essential for mathematically describing transport phenomena with finite propagation speeds, such as the movement of matter, sound waves, and information. These equations serve as a fundamental framework for describing a wide range of complex systems, including wave propagation, traffic network systems, electric transmission lines, hydraulic channels, drilling devices, communication networks, smart structures, as well as multiscale and multiphysics systems [22, 2, 95, 94]. Controlling and monitoring hyperbolic systems are **difficult control engineering problems** due to the **distributed** nature of the systems (time and space dependency) and the physical/economic **infeasibility of placing sensors and actuators everywhere** along the spatial domain. The stringent operating, environmental and economical requirements and the high mathematical complexity of these systems explain why traditional control methods exhibit a limited range of applicability and have not been successful at high technology readiness levels (TRLs) [15, 33]. Consequently, the theory of control for distributed parameter systems requires substantial advancements to achieve control and estimation objectives effectively. This explains the notable surge in scholarly publications dedicated to this subject over the past few decades.

In recent years, numerous theoretical approaches have emerged for designing boundary controllers and observers specifically tailored to one-dimensional linear balance law systems, a specific class of hyperbolic systems [22, Chap. 5]. Among these approaches, one can find flatness-based controllers [66, 85], optimization

controllers [63] or Lyapunov-based controllers [27, 26, 75] that have for instance enabled the design of dissipative boundary conditions [27, 26]. In this context, the **backstepping approach** [55] stands out as a constructive method with distinct characteristics. This technique employs an invertible change of variables, accomplished through an integral transformation, to map the original system onto a more amenable form known as the "target system." The backstepping approach allows for analysis, control, and observer design, resulting in explicit controllers expressed as functionals of the distributed states, akin to classical finite-dimensional counterparts. Initially introduced for a class of parabolic equations [87], the backstepping approach has been extended to wave equations [53] and linear or quasi-linear hyperbolic systems [31, 11, 28, 38]. A comprehensive historical account of the backstepping method for PDEs and its extensions can be found in [89]. For linear hyperbolic systems, one notable outcome of the backstepping controllers is their ability to partially address the finite-time stabilization and observability problems formulated in [83, 84] and further generalized by Tatsien Li in [61]. For the latest advancements in optimal finite-time stabilization of homogeneous quasilinear hyperbolic systems, we refer the reader to [29, 30].

The backstepping method encounters a major challenge in determining a suitable target system that strikes a balance between simplicity for control law design and the existence of a transformation mapping the original system to this target system. The choice of the target system significantly influences the closed-loop performance. However, the broader question of identifying reachable target systems remains an open problem. For balance laws systems, target systems are often selected to be finite-time stable [30], thereby neglecting the **robustness properties** of the corresponding closed-loop systems [64, 67]. These limitations in robustness can stem from various factors, including parameter uncertainties, disturbances acting on the system, measurement noise, neglected dynamics, or delays in the actuators. It has been observed that the introduction of even small time delays in the feedback loop can lead to instability for many feedback systems [34, 64]. Controllers designed to ensure finite-time stability are typically non-strictly proper and thus possess zero robustness margins. In [15], the authors introduced tuning parameters in the design to allow for potential trade-offs between different specifications, such as delay-robustness and convergence rate. The key idea behind these modifications was to avoid the complete cancellation of reflections at the spatial domain boundaries of the PDE, guaranteeing robustness margins. A general procedure for robustification was proposed in [7]. This robustification was achieved by the design of appropriate filters, generalizing the approaches in [24, 9]. To introduce additional natural tuning parameters (e.g., the dissipation rate) with clear energy interpretations, one could take advantage of **Port-Hamiltonian Systems (PHS) theory** [48]. PHS theory provides a comprehensive and modular energy-based representation for multi-physical systems, taking into account their inherent physical properties such as passivity, dissipativity, and reversibility. This theory establishes a physical framework through which the closed-loop properties of the target systems can be parameterized. Initially developed for finite-dimensional systems [40], the PHS framework has been extended to encompass Partial Differential Equations [60, 48]. The PHS framework proves particularly relevant in control design utilizing damping assignment or energy shaping methods. By leveraging

the natural physical properties of the system, it becomes possible to define well-posed candidates for the target systems. The degrees of freedom induced by such a general class of target systems can then be tuned to fulfill a given set of performance criteria. This strategy has been successfully applied to simple test cases, as demonstrated in previous studies [76, 78].

Finally, implementing these novel control strategies on real systems necessitates a thorough assessment of their computational costs and a comparative analysis against existing controllers, particularly PID controllers, in light of the advancements in computational systems, such as simple Arduino boards. In this context, it is essential to consider **complexity reduction strategies** to find the right compromise between performance and numerical complexity. This may imply approximating the controller by a finite-dimensional system (late lumping approach). The question of the convergence of late-lumping backstepping controllers has not been well-investigated, contrary to the approximation of the kernels themselves, e.g., in [49] using a trapezoidal rule or in [4] using a sum-of-squares approach. In [92], a method for computing the bounded part of the control operator is proposed. It relies on a finite-dimensional approximation of the state and enables efficient computing of the feedback law. Recently, Lyapunov methods were applied to assess the convergence of approximated backstepping controllers [17, 52]. Additionally, learning-based approaches have emerged as an intriguing avenue, with approximation guarantees (such as DeepONet algorithms [35, 23]).

In this chapter, we provide a comprehensive overview of recent advancements in the **practical control of hyperbolic systems**. While our focus is primarily on scalar balance laws, most presented results can be extended to more complex systems, including non-scalar and interconnected systems. The control strategies proposed in this study are said to be practical, as they are constructive, easily implementable, and offer degrees of freedom that can be tailored to satisfy specific performance requirements. Our methodology builds upon the backstepping approach and incorporates frequential analysis techniques. This chapter serves as a tutorial and is an introduction to the methods developed by the author for the control and stabilization of balance law systems.

This chapter is organized as follows. Section 2 presents the system under consideration and introduces the control objectives. We present in Section 3 a general introduction to the backstepping approach and show how this methodology can be used to design a stabilizing controller for the class of system under consideration. We analyze the robustness properties of the proposed control law in Section 4. We first show that finite-time stabilization can lead to vanishing robustness margins. We then propose some adjustments to obtain a robust control law. In Section 5, we show that it is possible to use the backstepping approach to map the original system to a large class of target systems. This allows introducing additional degrees of freedom in the design that can then be used to improve the closed-loop behavior. Section 6 focuses on model reduction aspects and how the proposed controllers can be approximated to be implemented on real systems while guaranteeing closed-loop stability. Some concluding remarks are finally given in Section 7.

Notations: We denote $L^2([0, 1], \mathbb{R})$ the space of real-valued square-integrable functions defined on $[0, 1]$ with the standard L^2 norm, i.e., for any $f \in L^2([0, 1], \mathbb{R})$, we

have

$$\|f\|_{L^2} = \left(\int_0^1 f^2(x) dx \right)^{\frac{1}{2}}.$$

By convention, for any $u \in L^2([0, 1], \mathbb{R})$ and $v \in L^2([0, 1], \mathbb{R})$, we denote $\|(u, v)\|_{L^2} = \sqrt{\|u\|_{L^2}^2 + \|v\|_{L^2}^2}$. We denote $H^1([0, 1], \mathbb{R})$ the one-dimensional Sobolev space, i.e., the subset of functions f in $L^2([0, 1], \mathbb{R})$ such that f and its weak derivative of order one have a finite L^2 norm. We denote $C^0([0, 1], \mathbb{R})$ the space of real-valued continuous functions defined on $[0, 1]$. We denote s , the Laplace variable. The partial derivative of a multivariable function f with respect to the variable x will be denoted $\frac{\partial f}{\partial x}$.

2. System under consideration

2.1. Balance law equations. In this chapter, we consider the following general class of linear hyperbolic systems, which appears in Saint-Venant equations, heat exchanger equations, and other linear hyperbolic balance laws [22]:

$$(1) \quad \frac{\partial u}{\partial t}(t, x) + \lambda \frac{\partial u}{\partial x}(t, x) = \sigma^{++}(x)u(t, x) + \sigma^{+-}(x)v(t, x),$$

$$(2) \quad \frac{\partial v}{\partial t}(t, x) - \mu \frac{\partial v}{\partial x}(t, x) = \sigma^{-+}(x)u(t, x) + \sigma^{--}(x)v(t, x),$$

with the linear boundary conditions

$$(3) \quad u(t, 0) = qv(t, 0), \quad v(t, 1) = \rho u(t, 1) + V(t),$$

where the velocities λ and μ are assumed to be strictly positive. The results we present in this chapter could be extended to spatially-varying velocities at the cost of technical computations. The boundary couplings q and ρ are constant and are respectively called **distal reflection** (reflection at the unactuated boundary) and **proximal reflection** (reflection at the actuated boundary). The in-domain couplings $\sigma^{\cdot\cdot}$ belong to $C^0([0, 1], \mathbb{R})$. The states u and v have values in \mathbb{R} . They are evolving in $\{(t, x) \mid t > 0, x \in [0, 1]\}$. The control input $V(t)$ belongs to \mathbb{R} . The initial condition is denoted $(u_0, v_0) \in (H^1([0, 1], \mathbb{R}))^2$. The following theorem assesses the well-posedness of the open-loop system

Theorem 1. *Consider that $V(t) = 0$ for all $t \geq 0$. For every initial condition $(u_0, v_0) \in (H^1([0, 1], \mathbb{R}))^2$ that verifies the compatibility conditions*

$$(4) \quad u_0(0) = qv_0(0), \quad v_0(1) = \rho u_0(1),$$

there exists one and one only

$$(u, v) \in \mathcal{C}^1([0, \infty), (L^2([0, 1], \mathbb{R}))^2) \cap \mathcal{C}^0([0, \infty), (H^1([0, 1], \mathbb{R}))^2),$$

which is a solution to the open-loop Cauchy problem (1)-(2). Moreover, there exists $\kappa_0 > 0$ such that for every $(u_0, v_0) \in (H^1([0, 1], \mathbb{R}))^2$ satisfying the compatibility conditions (4), the unique solution verifies

$$(5) \quad \|(u(t, \cdot), v(t, \cdot))\|_{L^2} \leq \kappa_0 e^{\kappa_0 t} \|(u_0, v_0)\|_{L^2}, \quad \forall t \in [0, \infty).$$

Proof. The proof can be adjusted from [22, Theorem A.1, Theorem A.6]. It relies on Lumer-Philippis theorem [65, 73]. \square

This theorem (and most of the results presented in the chapter) could be adjusted to deal with weak solutions to the Cauchy problem (1)-(2), as shown in [22]. However, this would be at the cost of involved technical computations. It is important to emphasize that Theorem 1 only shows the well-posedness in open-loop. The closed-loop well-posedness could be shown using the admissibility of the control operator [26] (the proposed control law will be time-continuous) or adjusting the proof of Theorem 1. Moreover, the different control strategies proposed throughout the manuscript rely mainly on the backstepping methodology [55]. Using invertible and bounded transformations, the closed-loop system can be mapped to a target system for which usual well-posedness results (as Theorem 1) can be straightforwardly applied. Consequently, the original and target systems share equivalent stability properties. This is why the well-posedness of the closed-loop system will not be discussed. The compatibility condition (4) may be changed in closed-loop to encompass the effect of the feedback law that has been designed. Thus, it may appear artificial and rather stringent, as it requires very specific values of the initial conditions. However, as shown in [31], it is possible to modify our control law in a way that, without losing the stabilizing properties, does not require any specific values in the initial values beyond the natural conditions.

2.2. Simplification of the system. Our first objective is to remove the growth terms $\sigma^{++}(x)$ and $\sigma^{--}(x)$ in equations (1)-(2) to simplify the equations. Let us consider the invertible exponential transformation defined for all $t > 0$ and all $x \in [0, 1]$ by

$$(6) \quad \bar{u}(t, x) = u(t, x)e^{-\int_0^x \frac{\sigma^{++}(\nu)}{\lambda} d\nu}, \quad \bar{v}(t, x) = v(t, x)e^{+\int_0^x \frac{\sigma^{--}(\nu)}{\mu} d\nu},$$

which is well-defined since the functions $\sigma^{--}(\cdot)$ and $\sigma^{++}(\cdot)$ are continuous. This yields

$$\begin{aligned} \frac{\partial \bar{u}}{\partial t}(t, x) + \lambda \frac{\partial \bar{u}}{\partial x}(t, x) &= \left(\frac{\partial u}{\partial t}(t, x) + \lambda \frac{\partial u}{\partial x}(t, x) - \sigma^{++}(x)u(t, x) \right) e^{-\int_0^x \frac{\sigma^{++}(\nu)}{\lambda} d\nu} \\ &= \sigma^{+-}(x)v(t, x)e^{-\int_0^x \frac{\sigma^{++}(\nu)}{\lambda} d\nu} \\ (7) \quad &= \sigma^{+-}(x)e^{-\int_0^x \frac{\sigma^{++}(\nu)}{\lambda} d\nu} e^{-\int_0^x \frac{\sigma^{--}(\nu)}{\mu} d\nu} \bar{v}(t, x) = \sigma^+(x)\bar{v}(t, x), \end{aligned}$$

where we have denoted $\sigma^+(x) = e^{-\int_0^x \frac{\sigma^{++}(\nu)}{\lambda} d\nu} e^{-\int_0^x \frac{\sigma^{--}(\nu)}{\mu} d\nu} \sigma^{+-}(x)$. Similarly, denoting $\sigma^-(x) = e^{+\int_0^x \frac{\sigma^{++}(\nu)}{\lambda} d\nu} e^{+\int_0^x \frac{\sigma^{--}(\nu)}{\mu} d\nu} \sigma^{+-}(x)$, we obtain

$$(8) \quad \frac{\partial \bar{v}}{\partial t}(t, x) - \mu \frac{\partial \bar{v}}{\partial x}(t, x) = \bar{\sigma}^-(x)\bar{u}(t, x).$$

The boundary conditions (3) rewrites

$$(9) \quad \bar{u}(t, 0) = q\bar{v}(t, 0), \quad \bar{v}(t, 1) = \bar{\rho}u(t, 1) + \bar{V}(t),$$

with $\bar{\rho} = \rho e^{+\int_0^1 \frac{\sigma^{++}(\nu)}{\lambda} d\nu} e^{+\int_0^1 \frac{\sigma^{--}(\nu)}{\mu} d\nu}$ and $\bar{V}(t) = e^{+\int_0^1 \frac{\sigma^{--}(\nu)}{\mu} d\nu} V(t)$. Consequently, the system (1)-(2) can be rewritten as

$$(10) \quad \frac{\partial u}{\partial t}(t, x) + \lambda \frac{\partial u}{\partial r}(t, x) = \sigma^+(x)v(t, x),$$

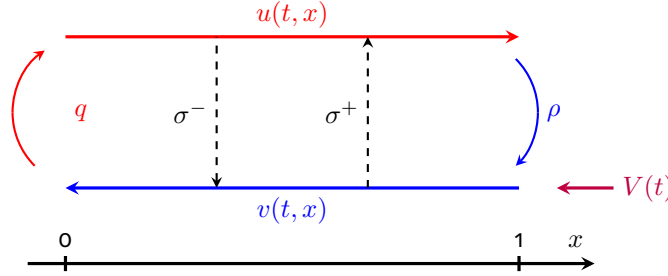
$$(11) \quad \frac{\partial v}{\partial t}(t, x) - \mu \frac{\partial v}{\partial x}(t, x) = \sigma^-(x)u(t, x),$$

with the boundary conditions

$$(12) \quad u(t, 0) = qv(t, 0), \quad v(t, 1) = \rho u(t, 1) + V(t).$$

Note that we have removed the bars in this new system for clarity. The initial conditions still satisfy the appropriate compatibility conditions. In this chapter, we consider that $q \neq 0$. Although the results we present can be extended in the case $q = 0$, choosing $q \neq 0$ simplifies the exposition of the different results. The system (10)-(12) is schematically pictured in Figure 1.

Figure 1. Schematic representation of the system (10)-(12)



2.3. **Control objectives.** Let us first recall the following definition

Definition 1 ([22]). *The hyperbolic system (10)-(12) is exponentially stable in the sense of the L^2 -norm if there exist $\nu > 0$, and $C > 0$ such that, for every initial condition $(u_0, v_0) \in (H^1([0, 1], \mathbb{R}))^2$ satisfying the compatibility conditions (4), the L^2 -solution of the problem (10)-(12) satisfies*

$$(13) \quad \|(u(t, \cdot), v(t, \cdot))\|_{L^2} \leq Ce^{-\nu t} \|(u_0(\cdot), v_0(\cdot))\|_{L^2}.$$

Note that although the solution of the system (10)-(12) lies in the H^1 -space, we consider the stability in the sense of the L^2 -norm that somehow relates to the energy of the system. One of the objectives of this chapter is to design a feedback control law $V = \mathcal{K}[(u, v)]$, where $\mathcal{K} : (H^1([0, 1], \mathbb{R}))^2 \rightarrow \mathbb{R}$ is a linear operator such that

- The state (u, v) of the resulting feedback system (10)-(12) is exponentially stable (**stabilization problem**).
- The resulting feedback system (10)-(12) is robustly stable with respect to small delays in the loop (**delay robustness**), i.e., there exists $\delta^* > 0$ such that for any $\delta \in [0, \delta^*]$, the control law $V(t - \delta)$ still stabilizes (10)-(12).

A control law that satisfies these two constraints is said to delay-robustly stabilize (in the sense of [64]) system (10)-(12). For system (10)-(12), it has been shown in [62] that there exists a state-feedback control law that ensures the finite time stabilization in the sense of the L^2 -norm. This finite time corresponds to $\tau = \frac{1}{\lambda} + \frac{1}{\mu}$. It has been proved [82, 81] that this minimum time is "critical" in the sense that it is, in general, impossible to satisfy the given initial and terminal conditions if less time is allowed. Moreover, for this specific case of two equations, it has been proved that the control law that ensures finite-time boundary stabilization in the minimum time is unique [83]. The proof of this result is based on the explicit evolution of the Riemann invariants along the characteristics (see [61] for details) and does not allow

for an explicit design of the control law. In the next section (Section 3), we will use the backstepping method to design such a finite-time controller. We will then show in Section 4 that such a controller may not ensure delay-robustness and present some adjustments to delay-robustly stabilize system (10)-(12).

3. Stabilization of system (10)-(12): an introduction to the backstepping approach

The **backstepping approach** is a control method initially developed for non-linear finite-dimensional systems (sometimes referred to as "backstepping by integrator") [54]. While such a method cannot be directly extended to infinite dimensional systems, it has been possible to propose an alternative version for systems of partial differential equations. It was first introduced to stabilize systems of parabolic equations [87] before being extended to wave equations [53] and hyperbolic linear (and quasilinear) systems [11, 28, 38, 46]. In this section, we present the main ideas inherent to this method to design a stabilizing controller for the scalar system of balance laws (10)-(12). The reader is referred to [55] for a deeper introduction to the backstepping method.

3.1. General ideas. The main idea behind the backstepping methodology is to perform a change of variables using a Volterra integral operator to transform the original system into a **target system**, amenable to analysis, control, and observer design. Such a transformation must be **boundedly invertible** to guarantee equivalent stability properties between the two systems (original and target). In particular, the stability of the target system must imply the stability of the original system. This results in explicit controllers, similar to classical finite-dimensional counterparts, expressed as functionals of the distributed states. In what follows, we will denote $(\alpha(t, x), \beta(t, x))$ the state of the target system. These new variables are defined from the variables $(u(t, x), v(t, x))$ using the following Volterra transformations

$$(14) \quad \alpha(t, x) = u(t, x) + \int_0^x K^{uu}(x, y)u(t, y) + K^{uv}(x, y)v(t, y)dy,$$

$$(15) \quad \beta(t, x) = v(t, x) + \int_0^x K^{vu}(x, y)u(t, y) + K^{vv}(x, y)v(t, y)dy,$$

where the functions $K^{\cdot\cdot}$ are called the **kernels** of the transformation. They are continuous functions defined on the triangular domain $\mathcal{T} = \{(x, y) \in [0, 1]^2, y \leq x\}$. As we will see later, these kernels must satisfy a set of partial differential equations (the so-called **kernel equations**) that depend on the chosen target system. If we show the existence of a solution for these kernel equations, this will imply the existence of a transformation mapping the original system to the desired target system.

3.2. Target system. One of the main difficulties with the backstepping method is finding a suitable target system. It should be simple enough to allow the design of the control law. Still, in the meantime, we must prove the existence of a transformation mapping the original system to this target system (which means showing the existence of a solution to the associated kernel equations). The choice of the target system directly impacts the closed-loop performance. The general question of reachable target systems is still an open problem. For hyperbolic equations,

they were usually chosen as finite-time stable [62]. In the case of the original system (10)-(12), the target system can be defined as [31]:

$$(16) \quad \frac{\partial \alpha}{\partial t}(t, x) + \lambda \frac{\partial \alpha}{\partial x}(t, x) = 0,$$

$$(17) \quad \frac{\partial \beta}{\partial t}(t, x) - \mu \frac{\partial \beta}{\partial x}(t, x) = 0,$$

with the boundary conditions

$$(18) \quad \alpha(t, 0) = q\beta(t, 0),$$

$$(19) \quad \beta(t, 1) = \rho\alpha(t, 1) + \int_0^1 (K^{vu}(1, y) - \rho K^{uu}(1, y))u(t, y)dy \\ + \int_0^1 (K^{vv}(1, y) - \rho K^{uv}(1, y))v(t, y)dy + V(t).$$

Note that the second boundary condition depends on the original set of variables. However, this target system is satisfying since the in-domain coupling terms (σ^+ and σ^-) have been moved to the actuated boundary. They now take the form of integral couplings. The PDEs now read as transport equations. Therefore, we can choose the control input $V(t)$ as

$$(20) \quad V(t) = -\rho\alpha(t, 1) - \int_0^1 (K^{vu}(1, y) - \rho K^{uu}(1, y))u(t, y)dy \\ - \int_0^1 (K^{vv}(1, y) - \rho K^{uv}(1, y))v(t, y)ds \\ = -\rho v(t, 1) - \int_0^1 K^{vu}(1, y)u(t, y) + K^{vv}(1, y)v(t, y)ds,$$

so that the system (16)-(19) rewrites as

$$(21) \quad \frac{\partial \alpha}{\partial t}(t, x) + \lambda \frac{\partial \alpha}{\partial x}(t, x) = 0, \quad \frac{\partial \beta}{\partial t}(t, x) - \mu \frac{\partial \beta}{\partial x}(t, x) = 0,$$

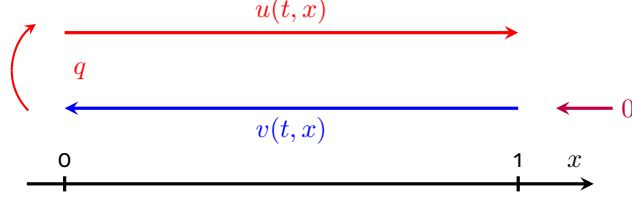
with the boundary conditions

$$(22) \quad \alpha(t, 0) = q\beta(t, 0), \quad \beta(t, 1) = 0.$$

This resulting system is exponentially stable [22]. Applying the method of characteristics, the states α and β converge to 0 in finite time equal to $\tau = \frac{1}{\lambda} + \frac{1}{\mu}$. Indeed, for all $x \in [0, 1]$ and all $t > \tau$, we have $\beta(t, x) = \beta(t - \frac{1-x}{\mu}, 1) = 0$ and $\alpha(t, x) = \alpha(t - \frac{x}{\lambda}, 0) = q\beta(t - \frac{x}{\lambda}, 0) = 0$. Of course, for the above expressions to make sense, it is first necessary to prove the existence of a transformation of the form (14)-(15) that maps the system (10)-(12) into the system (16)-(19). It is important to note that only the terms in the domain and the boundary condition at $x = 1$ have been modified. Transport velocities and the proximal reflection q remain unchanged. The target system (16)-(19) is pictured in Figure 2.

3.3. Kernel equations. We now show that it is possible to map the system (10)-(12) into the system (16)-(19) using the transformation (14)-(15). Let us first notice that the boundary condition of system (18)-(19) are always satisfied, independently of the choice of the kernels $K^{\cdot\cdot}$. Let us focus first on the β -transformation (equation (15)). The computations we present below can be easily adjusted for the α -transformation (equation (14)). Differentiating equation (15) with respect to space

Figure 2. Schematic representation of the target system (16)-(19)



and using Leibniz's formula, we obtain

$$\begin{aligned} \frac{\partial \beta}{\partial x}(t, x) &= \frac{\partial v}{\partial x}(t, x) + K^{vu}(x, x)u(t, x) + K^{vv}(x, x)v(t, x) \\ &+ \int_0^x \frac{\partial K^{vu}}{\partial x}(x, y)u(t, y) + \frac{\partial K^{vv}}{\partial x}(x, y)v(t, y)dy. \end{aligned}$$

Differentiating equation (15) with respect to t and using that (u, v) is a solution of (10)-(12), we obtain

$$\begin{aligned} \frac{\partial \beta}{\partial t}(t, x) &= \frac{\partial v}{\partial t}(t, x) + \int_0^x K^{vu}(x, y) \frac{\partial u}{\partial t}(t, y) + K^{vv}(x, y) \frac{\partial v}{\partial t}(t, y)dy \\ &= \mu \frac{\partial v}{\partial x}(t, x) + \sigma^- u(t, x) + \int_0^x \left(-\lambda K^{vu}(x, y) \frac{\partial u}{\partial y}(t, y) + K^{vu}(x, y) \right. \\ &\quad \left. \cdot \sigma^+(y)v(t, y) + \mu K^{vv}(x, y) \frac{\partial v}{\partial y}(t, y) + K^{vv}(x, y) \sigma^-(y)u(t, y)dy \right). \end{aligned}$$

Integrating by parts, and using that $u(t, 0) = qv(t, 0)$ (equation (12)), we get

$$\begin{aligned} \frac{\partial \beta}{\partial t}(t, x) &= \mu \frac{\partial v}{\partial x}(t, x) + \sigma^-(x)u(t, x) - \lambda K^{vu}(x, x)u(t, x) + \lambda q K^{vu}(x, 0) \\ &v(t, 0) + \mu K^{vv}(x, x)v(t, x) - K^{vv}(x, 0)v(t, 0) + \int_0^x \left(\lambda \frac{\partial K^{vu}}{\partial y}(x, y)u(t, y) \right. \\ &\quad \left. + K^{vu}(x, y)\sigma^+(y)v(t, y) - \mu \frac{\partial K^{vv}}{\partial y}(x, y)v(t, y) + K^{vv}(x, y)\sigma^-(y)u(t, y)dy \right). \end{aligned}$$

Consequently, we obtain

$$\begin{aligned} \frac{\partial \beta}{\partial t}(t, x) - \mu \frac{\partial \beta}{\partial x}(t, x) &= (\sigma^-(x) - (\lambda + \mu)K^{vu}(x, x)) u(t, x) + (\lambda q K^{vu}(x, 0) \\ &- \mu K^{vv}(x, 0)) v(t, 0) + \int_0^x \left(\lambda \frac{\partial K^{vu}}{\partial x}(x, y) - \mu \frac{\partial K^{vu}}{\partial y}(x, y) + K^{vv}(x, y)\sigma^-(y) \right) \\ &\cdot u(t, y) - \left(\mu \frac{\partial K^{vv}}{\partial x}(x, y) + \mu \frac{\partial K^{vv}}{\partial y}(x, y) - K^{vu}(x, y)\sigma^+(y) \right) v(t, y)ds. \end{aligned}$$

To obtain equation (17), we choose K^{vu} and K^{vv} such that

$$(23) \quad \mu \frac{\partial K^{vu}}{\partial x}(x, y) - \lambda \frac{\partial K^{vu}}{\partial y}(x, y) = \sigma^-(y)K^{vv}(x, y),$$

$$(24) \quad \mu \frac{\partial K^{vv}}{\partial x}(x, y) + \mu \frac{\partial K^{vv}}{\partial y}(x, y) = \sigma^+(y)K^{vu}(x, y),$$

along with the boundary conditions

$$(25) \quad K^{vu}(x, x) = \frac{\sigma^-(x)}{\lambda + \mu}, \quad K^{vv}(x, 0) = \frac{q\lambda}{\mu} K^{vu}(x, 0).$$

Similarly, we can differentiate equation (14) with respect to space and time to obtain the following set of equations that the kernels K^{vu} and K^{vv} should satisfy

$$(26) \quad \lambda \frac{\partial K^{uu}}{\partial x}(x, y) + \lambda \frac{\partial K^{uu}}{\partial y}(x, y) = -\sigma^-(y) K^{uv}(x, y),$$

$$(27) \quad \lambda \frac{\partial K^{uv}}{\partial x}(x, y) - \mu \frac{\partial K^{uv}}{\partial y}(x, y) = -\sigma^+(y) K^{uu}(x, y),$$

with the boundary conditions

$$(28) \quad K^{uv}(x, x) = -\frac{\sigma^+(x)}{\lambda + \mu}, \quad K^{uu}(x, 0) = \frac{\mu}{q\lambda} K^{uv}(x, 0).$$

We have the following theorem [31]

Theorem 2. [31, Theorem A.1] *The system (23)-(28) admits a unique continuous solution on \mathcal{T} .*

Proof. The proof is quite classical (see [46], [50] and [90]) and consists in writing the integral equations associated to equations (23)-(28) using the method of characteristics. These integral equations are then solved using the method of successive approximations. \square

Theorem 2 proves the existence of a unique solution to system (23)-(28) and, consequently, the existence of the transformation (14)-(15) that maps the original system (10)-(12) to the system (16)-(19). In the case of constant coefficients σ^+ and σ^- , it is possible to obtain an explicit solution to the kernel equations (23)-(28) using Bessel and Marcum Q -functions [88]. In the general case, only a numerical approximated solution is usually available.

3.4. Invertibility of the transformation. The transformation (14)-(15) is a Volterra transformation. Therefore, it is invertible [93]. More precisely, there exist continuous functions $L^{\alpha\alpha}, L^{\beta\alpha}, L^{\alpha\beta}, L^{\beta\beta}$ such that for all $t \geq 0$ and all $x \in [0, 1]$

$$(29) \quad u(t, x) = \alpha(t, x) + \int_0^x L^{\alpha\alpha}(x, y)\alpha(t, y) + L^{\alpha\beta}(x, y)\beta(t, y)dy,$$

$$(30) \quad v(t, x) = \beta(t, x) + \int_0^x L^{\beta\alpha}(x, y)\alpha(t, y) + L^{\beta\beta}(x, y)\beta(t, y)dy.$$

One can easily show that the functions $L^{\cdot\cdot}$ defined on $\mathcal{T} = \{(x, y) \in [0, 1]^2, y \leq x\}$ verify the set of PDEs

$$(31) \quad \lambda \frac{\partial L^{\alpha\alpha}}{\partial x}(x, y) + \lambda \frac{\partial L^{\alpha\alpha}}{\partial y}(x, y) = \sigma^+(x) L^{\beta\alpha}(x, y),$$

$$(32) \quad \lambda \frac{\partial L^{\alpha\beta}}{\partial x}(x, y) - \mu \frac{\partial L^{\alpha\beta}}{\partial y}(x, y) = \sigma^+(x) L^{\beta\beta}(x, y),$$

$$(33) \quad \mu \frac{\partial L^{\beta\alpha}}{\partial x}(x, y) - \lambda \frac{\partial L^{\beta\alpha}}{\partial y}(x, y) = -\sigma^-(x) L^{\alpha\alpha}(x, y),$$

$$(34) \quad \mu \frac{\partial L^{\beta\beta}}{\partial x}(x, y) + \mu \frac{\partial L^{\beta\beta}}{\partial y}(x, y) = -\sigma^-(x) L^{\alpha\beta}(x, y),$$

with the boundary conditions

$$(35) \quad L^{\alpha\alpha}(x, 0) = \frac{\mu}{q\lambda} L^{\alpha\beta}(x, 0), \quad L^{\beta\beta}(x, 0) = \frac{q\lambda}{\mu} L^{\beta\alpha}(x, 0)$$

$$(36) \quad L^{\alpha\beta}(x, x) = \frac{\sigma^+(x)}{\lambda + \mu}, \quad L^{\beta\alpha}(x, x) = -\frac{\sigma^-(x)}{\lambda + \mu}.$$

Remark 1. Using this inverse transformation, the boundary condition (19) can be rewritten as

$$(37) \quad \beta(t, 1) = \rho\alpha(t, 1) - \int_0^1 N^\alpha(\xi)\alpha(t, \xi) + N^\beta(\xi)\beta(t, \xi)d\xi + V(t),$$

where

$$(38) \quad N^\alpha(\xi) = L^{\beta\alpha}(1, \xi) - \rho L^{\alpha\alpha}(1, \xi), \quad N^\beta(\xi) = L^{\beta\beta}(1, \xi) - \rho L^{\alpha\beta}(1, \xi).$$

The functions K^\cdot and L^\cdot are continuous and defined on a compact set. Consequently, they are uniformly bounded by a constant $M > 0$. We have the following lemma

Lemma 1. There exist two constants $\kappa_0 > 0$ and $\kappa_1 > 0$ such that for all $t > 0$

$$(39) \quad \kappa_0 \|(\alpha(t, \cdot), \beta(t, \cdot))\|_{L^2} \leq \|(u(t, \cdot), v(t, \cdot))\|_{L^2} \leq \kappa_1 \|(\alpha(t, \cdot), \beta(t, \cdot))\|_{L^2}$$

Proof. The proof is straightforward and is omitted here. \square

3.5. Control law and stabilization.

Theorem 3. The closed-loop system (10)-(12) with the control input (20) is exponentially stable in the sense of the L^2 -norm. Moreover, for all $t > \frac{1}{\lambda} + \frac{1}{\mu}$, for all $x \in [0, 1]$, $u(t, x) = 0$ and $v(t, x) = 0$.

Proof. The complete proof can be found in [31]. Since the target system (16)-(19) with the control law (20) is L^2 -exponentially stable, there exists $\kappa > 0$ and $\nu > 0$ such that

$$\|(\alpha(t, \cdot), \beta(t, \cdot))\|_{L^2} \leq \kappa e^{-\nu t} \|(\alpha(0, \cdot), \beta(0, \cdot))\|_{L^2}.$$

Consequently, using equation (39), we obtain

$$\|(u(t, \cdot), v(t, \cdot))\|_{L^2} \leq \frac{\kappa_1}{\kappa_0} \kappa e^{-\nu t} \|(u(0, \cdot), v(0, \cdot))\|_{L^2},$$

which implies the exponential stability of the system (10)-(12). The finite-time convergence of the state (α, β) to zero is a direct consequence of the method of characteristics applied on the system (16)-(19). This directly implies the finite-time convergence of the state (u, v) to zero. \square

The control law (20) ensures the stabilization of the original system (10)-(12). However, the control law (20) is a full-state feedback as it requires the value of the state across the domain. Full-state distributed measurements are rarely available in practice, as only a finite number of sensors can be used. Backstepping-based observers can be designed to reconstruct the state and design an appropriate output-feedback controller [5]. This is not presented here.

Remark 2. *If the coefficient q is zero, the above method is invalid since the boundary condition (28) cannot be properly defined. However, it can be adjusted, slightly modifying the target system by*

$$(40) \quad \frac{\partial \alpha}{\partial t}(t, x) + \lambda \frac{\partial \alpha}{\partial x}(t, x) = g(x)\beta(t, 0), \quad \frac{\partial \beta}{\partial t}(t, x) - \mu \frac{\partial \beta}{\partial x}(t, x).$$

and choosing the control law $V(t)$ such that

$$(41) \quad \alpha(t, 0) = q\beta(t, 0), \quad \beta(t, 1) = 0.$$

Again, the system (40)-(41) is a cascade system which is L^2 exponentially stable and converges in finite time to zero. The kernel equations resulting from the backstepping transformation are identical to (23)-(28) except that the boundary condition for K^{uu} has been changed by $K^{uu}(x, 0) = h(x)$ where h is a continuous function that can be chosen as desired. Note that this has no impact on the resulting feedback law since it does not change the expression of K^{vu} and K^{vv} used in (20).

3.6. Simulation results and robustness. We illustrate below this result with simulations. The numerical parameters of the system are chosen as follows

$$(42) \quad \lambda = 1, \mu = 2, \sigma^{+-} = 1.7, \sigma^{-+} = 0.8, q = -0.8, \rho = 1.$$

These parameters are chosen to make the open-loop system unstable (as seen in Figure 3). The space domain $[0, 1]$ is discretized with a mesh of 100 points. The solution to the kernel equations (23)-(28) is obtained using a fixed-point algorithm [11]. The simulations use a classical finite volume method based on a Godunov scheme. The integral term in the control law (20) is computed using a trapezoidal method with a precision corresponding to the mesh size used to compute the kernels. We have pictured in Figure 3 the time-evolution of the L^2 -norm of the system (10)-(12) in open-loop and closed-loop using the control law (20). As expected, although the open-loop system is unstable, the closed-loop system is finite-time stable. The convergence time corresponds to the expected one: $\frac{1}{\lambda} + \frac{1}{\mu} = \frac{3}{2}s$.

Let us now introduce a small delay in the actuation path $\delta = 0.03s$. As seen in Figure 4, the L^2 -norm of the state (u, v) no longer converges to zero. This means that the control law does not delay-robustly stabilize the system (10)-(12). Focusing on finite-time stability, we have neglected the robustness aspects. In the next section, we will analyze the robustness properties of the control law (20) and modify it to guarantee delay-robust stabilization.

4. Design of robust control laws for hyperbolic systems

As seen in the previous section (Figure 4), the control law (20) may have vanishing robustness margins. This is not surprising since it has been observed that for many feedback systems, the introduction of arbitrarily small time delays in the loop may cause instability for any feedback law (see [34, 64]). In this section, we analyze the robustness properties of the control law (20) and propose some adjustments to guarantee the existence of robustness margins. The transformation (14)-(15) can still be applied, as it does not directly induce the design of the control law. Indeed, the system (16)-(19) is more amenable for analysis since it consists of two coupled transport equations. Using Remark 1, we can map the system (10)-(12) to the target

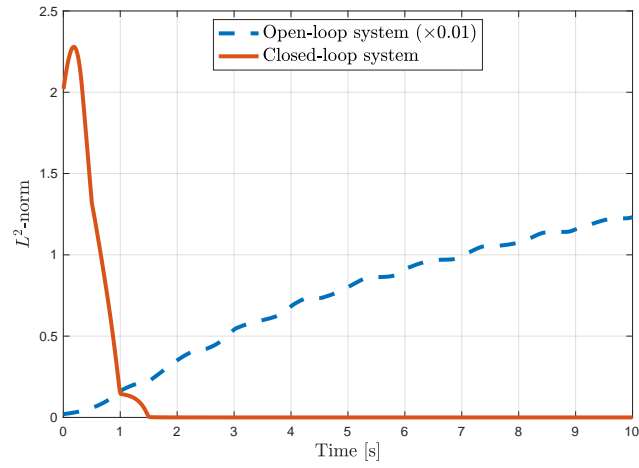


Figure 3. Time evolution of the L^2 -norm of the system (10)-(12) in open-loop (dashed blue) and in closed-loop using the control law (20) (red)

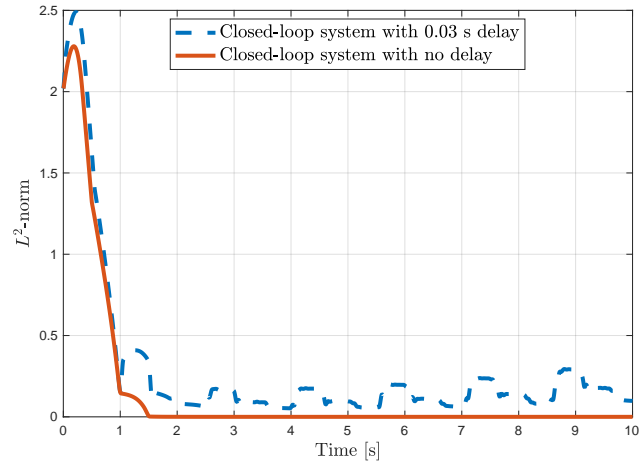


Figure 4. Time evolution of the L^2 -norm of the system (10)-(12) in closed-loop using the control law (20) with no delay (red) and with a $0.03s$ delay in the actuation path (dashed blue)

system

$$(43) \quad \frac{\partial \alpha}{\partial t}(t, x) + \lambda \frac{\partial \alpha}{\partial x}(t, x) = 0, \quad \frac{\partial \beta}{\partial t}(t, x) - \mu \frac{\partial \beta}{\partial x}(t, x) = 0,$$

$$(44) \quad \beta(t, 1) = \rho \alpha(t, 1) - \int_0^1 N^\alpha(\xi) \alpha(t, \xi) + N^\beta(\xi) \beta(t, \xi) d\xi + V(t),$$

Applying the method of characteristics on equations (43)-(44), we obtain for all $x \in [0, 1]$ and for all $t > 0$

$$\alpha(t, x) = q\beta\left(t - \frac{x}{\lambda} - \frac{1}{\mu}, 1\right), \quad \beta(t, x) = \beta\left(t - \frac{1-x}{\mu}, 1\right).$$

Consequently, combining these equations and equation (44), we get:

$$(45) \quad \beta(t, 1) = q\rho\beta(t - \tau, 1) - \int_0^\tau \tilde{N}(\nu)\beta(t - \nu, 1)d\nu + V(t),$$

where we recall that $\tau = \frac{1}{\lambda} + \frac{1}{\mu}$ and where \tilde{N} is defined by

$$(46) \quad \tilde{N}(\nu) = q\lambda N^\alpha\left(\lambda\nu - \frac{\lambda}{\mu}\right)\mathbb{1}_{[\frac{1}{\mu}, \tau]}(\nu) + \mu N^\beta(1 - \mu\nu)\mathbb{1}_{[0, \frac{1}{\mu}]}(\nu).$$

This invertible coordinate change enables us to rewrite β as the solution of a difference equation with distributed delays, more precisely, an **Integral Delay Equation** (IDE). For a single scalar equation, such connections between hyperbolic PDEs and IDEs were already established in [51]. Furthermore, in [51], the authors have stated that the stability analysis is easier when converting the PDEs to this time-delay form. These connections were then generalized for two balance law equations in [6, 15] and in [14] for non-scalar systems. Next, we show how this representation can be amenable to robustness analysis.

4.1. Open-loop analysis. It has been observed (see [34, 64]) that if the open-loop transfer function of a feedback system has an infinite number of poles in the complex right-half plane, then the introduction of arbitrarily small time delays in the loop may cause instability for any feedback control law. Therefore, it is first essential to analyze the open-loop behavior of the system (43)-(44), or equivalently of the system (45). Taking the Laplace transform of the open-loop system (45), we obtain

$$(47) \quad 1 - \rho q e^{-\tau s} + \int_0^\tau \tilde{N}(y)e^{-ys} dy = 0.$$

The next theorem shows how the properties of the characteristic equation (47) relate to the stability properties of equation (45)

Theorem 4. *The system (45) is exponentially stable if and only if there exists $\eta > 0$ such that all solution of the characteristic equation (47) satisfy $\text{Re}(s) < -\eta$.*

Proof. The proof is obtained with minor adjustments from [25]. \square

We then have the following lemma

Lemma 2. *If $|\rho q| > 1$, then the characteristic equation (47) has infinite zeros with a non-negative real part.*

Proof. Let us denote

$$F(s) = 1 - \rho q e^{-s\tau}, \quad H(s) = \int_0^\tau \tilde{N}(\xi)e^{-\xi s} d\xi.$$

Using Riemann-Lebesgue's lemma, we obtain

$$|H(s)| \rightarrow 0 \text{ as } |s| \rightarrow \infty, \quad \text{Re}(s) > 0.$$

The function F has an infinite number of zeros whose real parts are equal to $\frac{\ln(|\rho q|)}{2\tau}$. We can then apply [6, Lemma 3] to conclude that $F + H$ has infinite zeros with positive real parts. This concludes the proof. \square

We can actually get a slightly stronger result with the following lemma (whose proof is omitted)

Lemma 3. *If $|\rho q| \geq 1$, then, for any $\eta > 0$, the characteristic equation (47) has an infinite number of zeros whose real parts are greater than $-\eta$.*

We can finally state the following theorem

Theorem 5. *If $|\rho q| \geq 1$, system (10)-(12) cannot be exponentially delay-robustly stabilized.*

Proof. Due to Lemma 2, when $|\rho q| > 1$, the open-loop transfer function of system (10)-(12) has an infinite number of poles with a positive real part. Thus, system (10)-(12) cannot be delay-robustly stabilized (see [64, Theorem 1.2]). When $|\rho q| = 1$, due to Lemma 3, there is an asymptotic chain of zeros along the imaginary axis. Thus, the system cannot be exponentially robustly stabilized [45]. \square

This result shows that if the product $|\rho q|$ is larger than one, one cannot find a controller whose delay margin is non-zero. One should be aware that this condition does not hold for most practical applications. Models of the form (10)-(12) are simplistic and do not capture phenomena that would be susceptible to making the delay margins non-null. For instance, they may neglect the diffusivity stemming from Kelvin-Voigt damping. Therefore, although the delay-robustness margins for such systems would be poor, it remains theoretically possible to stabilize them robustly.

4.2. The end of finite-time stabilization? As seen in Figure 4, even when $|\rho q| < 1$, the introduction of a small delay in the control input (20) can induce zero robustness margins. Before designing a robust controller, we must understand the cause of non-robustness. The control law $V(t)$ defined by equation (20) can be expressed for all $t \geq \tau$ as

$$(48) \quad V(t) = -q\rho\beta(t - \tau, 1) + \int_0^\tau \tilde{N}(\nu)\beta(t - \nu, 1)d\nu,$$

such that equation (45) rewrites $\beta(t, 1) = 0$. In the presence of a small delay $\delta > 0$ in the actuation path, equation (45) rewrites

$$(49) \quad \begin{aligned} \beta(t, 1) = & q\rho(\beta(t - \tau, 1) - \beta(t - \tau - \delta, 1)) \\ & + \int_0^\tau \tilde{N}(\nu)(\beta(t - \nu, 1) - \beta(t - \nu - \delta, 1))d\nu, \end{aligned}$$

We then have the following theorem

Theorem 6. *If $|\rho q| > \frac{1}{2}$, then the system (10)-(12) with the delayed backstepping control law $V(t - \delta)$ defined by equation (48) is unstable for any $\delta > 0$.*

Proof. The characteristic equation associated with equation (49) rewrites

$$(50) \quad F(s) = H(s),$$

where

$$F(s) = 1 - \rho q e^{-\tau s} + \rho q e^{-(\tau + \delta)s}, \quad H(s) = \int_0^\tau \tilde{N}(\nu) e^{-\nu s} (1 - e^{-\delta s}) d\nu.$$

Since $2|\rho q| > 1$, the function $F(s)$ has an infinite number of zeros in the complex right half-plane [44]. The function $H(s)$ is strictly proper due Riemann-Lebesgues'

lemma, i.e. $|H(s)|$ converges to zero for $|s|$ large enough (with a positive real part). Due to Rouché's theorem, we have that $F - H$ has at least one root whose real part is strictly positive. This concludes the proof. \square

Note the proof can be adjusted when $|\rho q| = \frac{1}{2}$ to show that the delayed control law cannot exponentially stabilize the system (10)-(12) (the stabilization can at best be asymptotical). Theorem 6 implies that the backstepping controller (20) cannot be used in practical applications (for which there is always a small delay in the loop). Therefore, one must give up finite-time convergence to obtain a tractable implementation for the controller stabilizing system (10)-(12). These observations are consistent with reports by industrial practitioners on the limitations of the **impedance matching method**. The impedance matching method (see [3, 1, 43]) consists in matching the load impedance to the characteristic line impedance. For instance, in [56], the authors designed a controller preventing stick-slip oscillations, a class of undesired torsional oscillations characterized by a series of *stick* (a cessation of bit rotation) and *slip* (a sudden release of rotational energy) that often occur in drilling devices. They observed that canceling the proximal reflection coefficient can change the dynamics of the string in a way that makes the system unstable. All in all, controllers of the form (20) significantly trade off delay-robustness for performance, making them likely to be unusable.

4.3. Robustification of the control law. In this section, we present some methods to overcome the robustness limitation of the controller (20) while maintaining an analogous control structure. The control law (20) is composed of two parts:

- (1) the integral part whose objective is to remove the effect of in-domain couplings
- (2) the term $-\rho u(t, 1)$ whose objective is to cancel the proximal reflection and to ensure finite-time convergence.

As seen above, the robustness issue is due to the term $-\rho u(t, 1)$ in the control law. Indeed, the resulting feedback control operator may not be strictly proper [33, 32], thus inducing vanishing robustness margins. We present below two solutions to modify the control law and guarantee robustness margins, thereby giving up finite time stabilization.

4.3.1. Introduction of degrees of freedom. A first solution is to cancel only a part of the reflection terms in the PDE, using a convolutional procedure as performed, e.g., in [8, 91, 66]. More precisely, consider the control law

$$(51) \quad V_2(t) = -\tilde{\rho}u(t, 1) + (\rho - \tilde{\rho}) \int_0^1 K^{uu}(1, y)u(t, y) + K^{uv}(1, y)v(t, y)d\xi - \int_0^1 K^{vu}(1, y)u(t, y) + K^{vv}(1, y)v(t, y)dy,$$

where the coefficient $\tilde{\rho}$ is chosen such that

$$(52) \quad |\tilde{\rho}| < \frac{1 - |\rho q|}{|q|}.$$

The target system (43)-(44) now rewrites

$$(53) \quad \frac{\partial \alpha}{\partial t}(t, x) + \lambda \frac{\partial \alpha}{\partial x}(t, x) = 0, \quad \frac{\partial \beta}{\partial t}(t, x) - \mu \frac{\partial \beta}{\partial x}(t, x) = 0,$$

$$(54) \quad \alpha(t, 0) = q\beta(t, 0), \quad \beta(t, 1) = (\rho - \tilde{\rho})\alpha(t, 1),$$

Due to equation (52), we have $|q(\rho - \tilde{\rho})|$. Therefore, the target system (43)-(44) is exponentially stable. The coefficient $\tilde{\rho}$ can be interpreted as a tuning parameter, enabling a trade-off between performance (convergence rate) and robustness with respect to delays. We have the following theorem.

Theorem 7. *Consider the control law V_2 defined by (51) with $\tilde{\rho}$ satisfying (52). This control law delay-robustly stabilizes the system (10)-(12).*

Proof. Consider a positive delay δ . Slightly adjusting the method used to derive (45), we get the following equation satisfied by the output $\beta(t, 1)$.

$$(55) \quad \begin{aligned} \beta(t, 1) = & q\rho\beta(t - \tau, 1) - q\tilde{\rho}\beta(t - \tau - \delta, 1) \\ & - \int_0^\tau \tilde{N}(\nu)(\beta(t - \nu, 1) - \beta(t - \delta - \nu, 1))d\nu, \end{aligned}$$

where \tilde{N} is defined by (46). Taking the Laplace, we obtain the following characteristic equation

$$\mathbf{F}(s) = 1 - q\rho e^{-\tau s} + \tilde{\rho}q e^{-(\tau+\delta)s} - I(s, \delta) = 0,$$

where $I(s, \delta)$ is defined by

$$I(s, \delta) = \int_0^\tau \tilde{N}(\nu)(e^{-\nu s} - e^{-(\nu+\delta)s})d\nu$$

Let us now consider a complex number s such that $Re(s) \geq 0$. We then have

$$\begin{aligned} |\mathbf{F}(s)| & \geq |1 - q\rho e^{-\tau s} + \tilde{\rho}q e^{-(\tau+\delta)s}| - |I(s, \delta)| \\ & \geq 1 - |q\rho e^{-\tau s}| - |\tilde{\rho}q e^{-(\tau+\delta)s}| - |I(s, \delta)| \\ & \geq 1 - |q\rho| - |\tilde{\rho}q| - |I(s, \delta)|. \end{aligned}$$

Since $\tilde{\rho}$ satisfies (52), there exists $\epsilon_0 > 0$ such that

$$1 - |q\rho| - |\tilde{\rho}q| > \epsilon_0.$$

Let us now focus on the term $I(s, \delta)$. Due to Riemann-Lebesgues' lemma, we have

$$\forall |s| > M_0, \quad \left| \int_0^\tau \tilde{N}(\nu)e^{-\nu s}d\nu \right| < \frac{\epsilon_0}{2}$$

We can now choose δ_0 small enough such that for any $\delta \leq \delta_0$, for all complex s such that $|s| \leq M_0$, $|I(s, \delta)| < \epsilon_0$. With this choice of δ_0 , one can easily check that, $\forall \delta \leq \delta_0, \forall s \in \mathbb{C}$ such that $Re(s) \geq 0, |I(s, \delta)| < \epsilon_0$. Consequently, for $\delta \leq \delta_0$, we have

$$|\mathbf{F}(s)| > 0.$$

It means that for $0 < \delta \leq \delta_0$, the function $\mathbf{F}(s)$ does not have any root whose real part is positive. Moreover, there is no asymptotic chain of zeros in zero [44]. Consequently, equation (55) is exponentially stable. This concludes the proof. \square

Although such a procedure is now somehow standard, it presents the drawback of not distinguishing the effects of high and low frequencies in terms of stability and robustness. In addition, the implementation of such an approach can be challenging when considering networks of balance laws with many subsystems [9].

4.3.2. Filtering of the control law. An alternative robustifying approach has been proposed in [24, 9, 7]. It combines the proposed non-proper control law (20) with a well-tuned low-pass filter. The resulting control law then becomes strictly proper, which guarantees the existence of robustness margins. This filtering technique can simplify the design of stabilizing controllers for the proposed class of systems as it dissociates the stabilization problem from the robustness problem. More precisely, when applied conjointly with the existing controller, the low-pass filter will ensure the delay-robust stability of the closed-loop dynamics. The filters can be designed by leveraging the fact that robustness issues appear at high frequencies. We give below sufficient conditions that ensure robust stabilization provided the original control laws ensure exponential stabilization. The analysis will be carried out on the Laplace domain. Let us consider a positive constant M . Let us consider a low-pass filter $w(s)$ (i.e. $w(s) \rightarrow 1$ as $|s| \rightarrow 0$ and $|w(s)| \rightarrow 0$ as $|s| \rightarrow +\infty$) with relative degree 1, such that for all $s \in \mathbb{C}^+$, we have $|w(s)| < 1$, $|(1 - w(s))| < 1$ and if $|s| \leq M$

$$(56) \quad |1 - w(s)| < \frac{1}{|\rho q e^{-\tau s} + \int_0^\tau \tilde{N}(\nu) e^{-\nu s} ds| + 1}.$$

Note that it is always possible to find such a filter. One can for instance consider the low-pass filter defined for all $s \in \mathbb{C}^+$ by $w_{\nu_0}(s) = \frac{1}{(1 + \nu_0 s)}$. We immediately have $|w_{\nu_0}| < 1$ and $|1 - w_{\nu_0}| < 1$. Since the set $\mathcal{S} = \{s \in \mathbb{C}^+, |s| \leq M\}$ is compact, we can define $\tilde{M} = \inf_{s \in \mathcal{S}} \frac{1}{|\rho q e^{-\tau s} + \int_0^\tau \tilde{N}(\nu) e^{-\nu s} ds| + 1} > 0$. Choosing $\nu_0 = \frac{\tilde{M}}{M(1 + \tilde{M})}$, we directly obtain $|1 - w_{\nu_0}(s)| < \tilde{M}$. Indeed $|1 + \nu_0 s| \geq 1 - |\nu_0 s|$, which implies $|\frac{\nu_0 s}{1 + \nu_0 s}| < \tilde{M}$. We have the following theorem

Theorem 8. *Consider a low-pass filter $w(s)$ such that $|1 - w(s)| < 1$, $|w(s)| < 1$ and such that equation (56) is satisfied if $|s| \leq M$. Consider the control law V_f defined in the Laplace domain by $V_f(s) = w(s)V(s)$, where the control law $V(t)$ is defined by equation (20). Then, there exists $M > 0$ such that the control law delay-robustly stabilizes the system (10)-(12).*

Proof. Using equation (45), the characteristic equation associated with the closed-loop system rewrites

$$(57) \quad 1 = (1 - w(s))(\rho q e^{-\tau s} + \int_0^\tau \tilde{N}(\nu) e^{-\nu s} d\nu).$$

Due to equation (56), there exists $\eta > 0$ such that (57) does not admit any solution whose real part is greater than $-\eta$. Thus, the closed-loop system is exponentially stable. Finally, the control law $V_f(s)$ is strictly proper. Adjusting the proof from [33, Theorem 9.2.6], we can show that it is robust to small delays in the loop. \square

Note that having a strictly proper control law is sufficient to guarantee the existence of robustness margins. However, it is not a necessary condition. As seen

above, canceling a part of the reflection terms while guaranteeing robustness margins is possible. The proposed filtering technique considerably simplifies the design of delay-robust controllers since the filters can be plugged into the stabilizing control law. It extends the approach derived in [68, 69] for wave equations.

4.4. Simulation results. We illustrate below this result with simulations. The numerical parameters of the system are identical to the ones of Section 3.6. Since $|\rho q| = 0.8$, we must choose $|\tilde{\rho}| < 0.25$. We have pictured in Figure 5, the time evolution of the L^2 -norm of the system (10)-(12) in closed-loop using the control law (51) in the presence of a $0.05s$ delay for different values $\tilde{\rho}$ ($\tilde{\rho} = 0$, $\tilde{\rho} = 0.1$ and $\tilde{\rho} = 0.2$). In all cases, the control law guarantees delay-robust stabilization. However, this improvement in terms of delay margin comes at the cost of a diminution of the convergence rate as we have lost finite-time stabilization. The larger $\tilde{\rho}$ is, the smaller the corresponding delay margin will be, but in the meantime, the better will the convergence rate be. This illustrates the trade-off between performance and robustness. In Figure 6, we have pictured the evolution of the L^2 -norm of the closed-loop system in the presence of a $0.05s$ delay for the control law (51) ($\tilde{\rho} = 0.2$) and using the filtered control law introduced in Theorem 8. We considered two filters with different bandwidths: $125 \text{ rad}\cdot\text{s}^{-1}$ and $50 \text{ rad}\cdot\text{s}^{-1}$. The associated control efforts have been pictured in Figure 7. As expected, for the filtered controllers, the states of the system converge to zero despite the presence of the input delay. The convergence rate is better than the one obtained using the control law (51). This is because only high-frequency dynamics are affected by the filter. Moreover, the associated control effort is smaller. However, we have not analyzed the associated qualitative effect on the robustness margins or the impact of the different tuning parameters with respect to other performance specifications. This should be the purpose of future contributions.

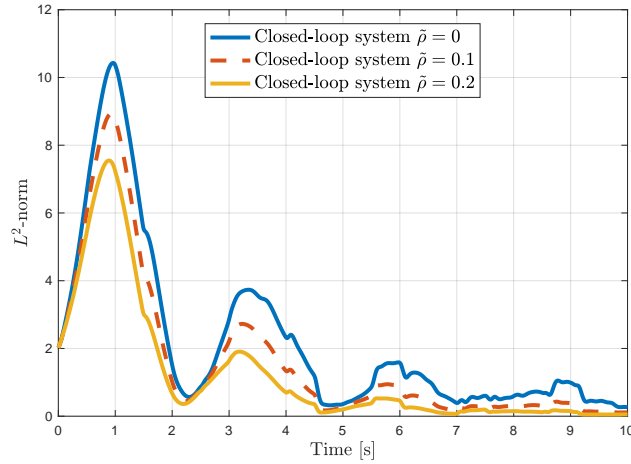


Figure 5. Time evolution of the L^2 -norm of the system (10)-(12) in closed-loop using the control law (51) in presence of a $0.05s$ delay in the actuation path for different values of $\tilde{\rho}$

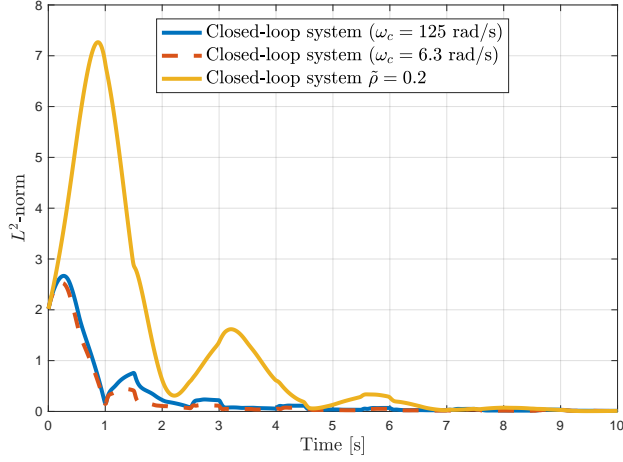


Figure 6. Time evolution of the L^2 -norm of the system (10)-(12) in closed-loop in presence of a $0.05s$ delay in the actuation path using the control law (51) with $\tilde{\rho} = 0.2$ and the filtered control law introduced in Theorem 8 for different filters

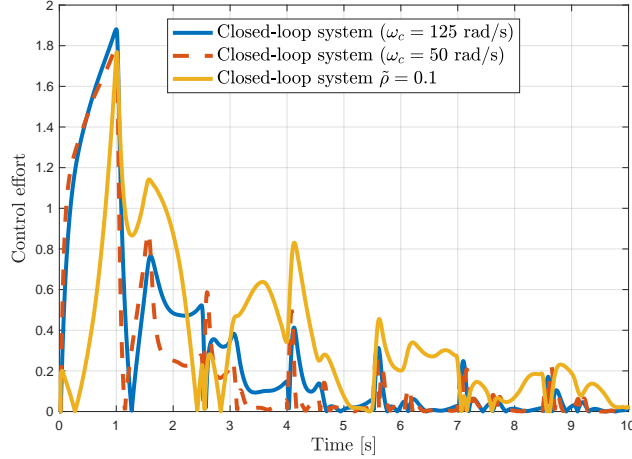


Figure 7. Time evolution of the control effort for system (10)-(12) in closed-loop in presence of a $0.05s$ delay in the actuation path using the control law (51) with $\tilde{\rho} = 0.2$ and the filtered control law introduced in Theorem 8 for different filters

4.5. Extensions. The proposed approaches can be easily extended to show the robustness with respect to additive, multiplicative or left-coprime-factor uncertainties [33]. Input-to-State Stability (ISS) has been shown in [15] for a general class of disturbances, thus extending the results from [36, 37] where the disturbance signal is generated by an exosystem of finite dimension or the results from [58, 57] where only smooth disturbances were considered. Moreover, in [15], the authors

introduced different tuning parameters in the design, thus guaranteeing potential trade-offs between different specifications (namely delay-robustness convergence rate, noise sensitivity, and disturbance rejection). The gained robustness comes at the price of degraded performance. It is important to emphasize that the resulting control laws introduce different degrees of freedom, but the choice of these tuning parameters and the underlying trade-offs still have to be qualitatively and quantitatively analyzed. Regarding the filtering approach, it can be extended to a large class of systems. In [7], the authors give general conditions under which it is possible to low-pass filter the (potentially non-strictly proper) control laws for a broad class of PDE systems coupled with ODEs. Finally, in [18], the authors showed the mean-square exponential robustness of the control law (51) in the presence of stochastic uncertainties. The proof relies on a specific Lyapunov analysis.

5. Towards parametrizable target systems.

One of the primary challenges associated with the backstepping method revolves around identifying a **suitable target system**. It should be simple enough to allow the design of the control law. Still, in the meantime, we must prove the existence of a transformation mapping the original system to this target system. The target system's choice significantly influences the closed-loop system's overall performance. The question of identifying general reachable target systems remains an unresolved issue. In the case of hyperbolic equations, these target systems have traditionally been selected as finite-time stable [30], thereby shadowing the robustness properties of the corresponding closed-loop systems [64, 67], as seen in the previous section. In [15], we introduced tuning parameters in the design, thus guaranteeing potential trade-offs between different specifications (namely delay-robustness and convergence rate). Nevertheless, the influence of these tuning parameters is limited, as they only impact the system's boundary conditions. General target systems (and thus additional degrees of freedom) could be obtained by preserving dissipative in-domain couplings while imposing a specific energy decay. This should imply a reduced control effort compared to traditional approaches.

5.1. Desired target system. In this section, we show how we can define an appropriate backstepping transformation to map the system (10)-(12) to any arbitrary target system with an analogous structure. More precisely, we will denote (\bar{u}, \bar{v}) as the state of the target system. It verifies

$$(58) \quad \frac{\partial \bar{u}}{\partial t}(t, x) + \lambda \frac{\partial \bar{u}}{\partial x}(t, x) = \bar{\sigma}^+(x) \bar{v}(t, x),$$

$$(59) \quad \frac{\partial \bar{v}}{\partial t}(t, x) - \mu \frac{\partial \bar{v}}{\partial x}(t, x) = \bar{\sigma}^-(x) \bar{u}(t, x),$$

with the boundary conditions

$$(60) \quad \bar{u}(t, 0) = q \bar{v}(t, 0), \quad \bar{v}(t, 1) = \bar{\rho} \bar{u}(t, 1),$$

where the parameters \bar{q} , $\bar{\rho}$, and the continuous functions $\bar{\sigma}^-$ and $\bar{\sigma}^+$ can be arbitrarily chosen. We emphasize that the parameters λ , μ , and q remain unchanged. This control objective somehow corresponds to in-domain damping assignment [78, 77]. We do not detail in this chapter how to adequately choose the parameters of the target system. However, in this context, the **Port-Hamiltonian approach** (PHS) [60] could be a path to follow. The PHS framework was initially developed

for finite-dimensional systems [40] and was then extended to PDEs [60, 48]. It is particularly relevant for control design using damping assignment or energy shaping methods. The natural physical properties of the system can be advantageously used to define well-posed, exponentially stable target system candidates. Such a strategy was successfully applied on simple test cases [76, 78]. Therefore, the PHS framework could help us to introduce **degrees of freedom** in the design and obtain a class of easily parametrizable (exponentially stable) closed-loop target systems. Other tuning parameters could also be added by following the robustness approach proposed in [15]. Then, we could use the backstepping method to map the original system (10)-(12) to this specific target system.

5.2. Backstepping transformation. We now map the system (10)-(12) to the target system (58)-(60). Consider the following backstepping transformation

$$(61) \quad \bar{u}(t, x) = u(t, x) + \int_0^x \bar{K}^{uu}(x, y)u(t, y) + \bar{K}^{uv}(x, y)v(t, y)dy,$$

$$(62) \quad \bar{v}(t, x) = v(t, x) + \int_0^x \bar{K}^{vu}(x, y)u(t, y) + \bar{K}^{vv}(x, y)v(t, y)dy,$$

where the kernels \bar{K}^{uu} , \bar{K}^{uv} , \bar{K}^{vu} and \bar{K}^{vv} are continuous functions defined on the triangular domain $\mathcal{T} = \{(x, y) \in [0, 1]^2, y \leq x\}$. Differentiating the transformation (61)-(62) with respect to time and space, we can show that the kernels verify the following set of equations

$$\begin{aligned} \lambda \frac{\partial \bar{K}^{uu}}{\partial x}(x, y) + \lambda \frac{\partial \bar{K}^{uv}}{\partial y}(x, y) &= \bar{\sigma}^+(z)\bar{K}^{vu}(x, y) - \sigma^-(y)\bar{K}^{uv}(x, y), \\ \lambda \frac{\partial \bar{K}^{uv}}{\partial x}(x, y) - \mu \frac{\partial \bar{K}^{uv}}{\partial y}(x, y) &= \bar{\sigma}^+(z)\bar{K}^{vv}(x, y) - \sigma^+(y)\bar{K}^{uu}(x, y), \\ \mu \frac{\partial \bar{K}^{vu}}{\partial x}(x, y) - \lambda \frac{\partial \bar{K}^{vu}}{\partial y}(x, y) &= \sigma^-(y)\bar{K}^{vv}(x, y) - \bar{\sigma}^-(x)\bar{K}^{uu}(x, y), \\ \mu \frac{\partial \bar{K}^{vv}}{\partial x}(x, y) + \mu \frac{\partial \bar{K}^{vv}}{\partial y}(x, y) &= \sigma^+(y)\bar{K}^{vu}(x, y) - \bar{\sigma}^-(x)\bar{K}^{uv}(x, y), \end{aligned}$$

with boundary conditions

$$\begin{aligned} \bar{K}^{uv}(x, x) &= \frac{\sigma^+(x) - \bar{\sigma}^+(x)}{\lambda + \mu}, \quad \bar{K}^{uu}(x, 0) = q^{-1}\bar{K}^{uv}(x, 0), \\ \bar{K}^{vu}(x, x) &= \frac{\bar{\sigma}^-(x) - \sigma^-(x)}{\lambda + \mu}, \quad \bar{K}^{vv}(x, 0) = q\bar{K}^{vu}(x, 0). \end{aligned}$$

The well-posedness of this set of kernel equations is proved in [39]. Therefore, a unique continuous solution exists on \mathcal{T} . The control input is then given by

$$(63) \quad \begin{aligned} V(t) &= (\bar{\rho} - \rho)u(t, 1) - \int_0^1 (\bar{K}^{vu}(1, y) - \bar{\rho}\bar{K}^{uu}(1, y))u(t, y)dy \\ &\quad - \int_0^1 (\bar{K}^{vv}(1, y) - \bar{\rho}\bar{K}^{uv}(1, y))v(t, y)dy. \end{aligned}$$

5.3. Simulation results. We illustrate below the results of this section with simulations. The numerical parameters of the system are identical to the ones of Section 3.6. We have pictured in Figure 5 the time evolution of the L^2 -norm of the system (10)-(12) in closed-loop using the control law (51) (with $\tilde{\rho} = 0$) and using the

control law (63). For this latter case, we have chosen $\bar{\rho} = \rho$, $\bar{\sigma}^+ = 0.7$, and $\bar{\sigma}^- = 0$, so the closed-loop system is exponentially stable. We have pictured in Figure 5 the associated control efforts. As can be seen, the two controllers guarantee the L^2 -stability in closed-loop. Moreover, the exponential decay rates are almost identical (even if the maximum value reached during the transient is larger when using the control law (63)). However, the control effort is smaller for the control law (63). Therefore, preserving some naturally dissipative in-domain coupling terms in the target system can guarantee a similar convergence rate while reducing the associated control effort. The next step would be to develop analytical techniques to quantify the performance of our output-feedback laws regarding a given set of performance specifications, such as disturbance rejection, robustness margins, noise sensitivity, or convergence rate. A thorough literature review and classification of existing criteria for finite-dimensional systems is needed to separate candidates for distributed parameter systems. Furthermore, these concepts must be tailored to the specific requirements of infinite-dimensional systems: the concept of phase margin is, for instance, an insufficient metric for infinite-dimensional systems since significant dynamics can be spread over a wide frequency range and with more esoteric behavior of systems that are not strictly proper [32, 64].

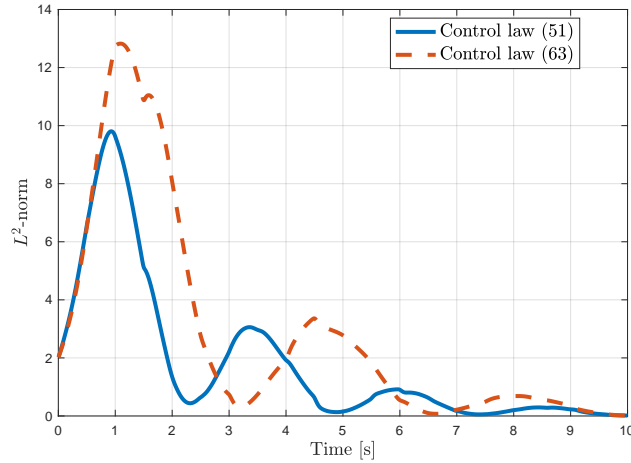


Figure 8. Time evolution of the L^2 -norm of the system (10)-(12) in closed-loop using the control law (51) (with $\bar{\rho} = 0$) and using the control law (63)

6. Integration, approximation, and model reduction.

The success of classical control design algorithms (such as P, PI, and PID controllers) in industrial engineering applications can be attributed to their simplicity and low computation burden. For instance, Soft-Speed and Soft-Torque, the industry-standard products that handle torsional vibrations in drilling devices, rely on PI regulators [41]. On the other hand, implementing infinite-dimensional output-feedback controllers (as the backstepping control law given in (20)) requires expertise and computational power. As explained in this chapter, the motivation for

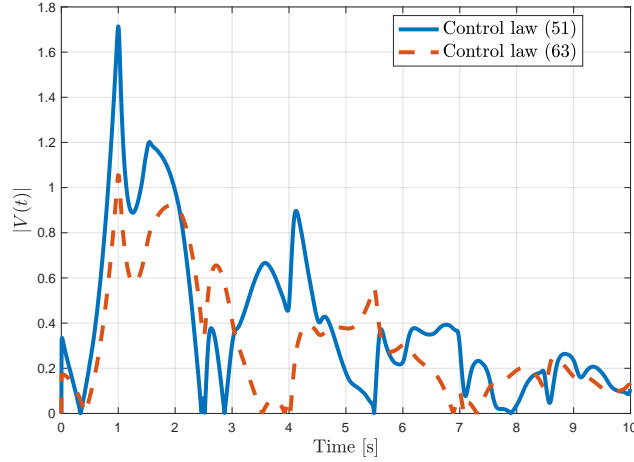


Figure 9. Time evolution of the control effort for the control law (51) (with $\tilde{\rho} = 0$) and the control law (63)

considering such complex control strategies stems from the desire for improved performance by explicitly considering delays and high-frequency content in the design. While advancements in electronic technology have enabled the use of computationally demanding techniques, evaluating the computational efforts inherent to these complex control strategies is crucial. Indeed, performing real-time state estimation for hyperbolic systems using rapidly converging observers, such as those based on backstepping, is computationally expensive and, in many cases, prohibitive. For instance, the method presented in [79] to stabilize interconnections of scalar hyperbolic systems needs to compute state predictions for each subsystem composing the interconnection, which is time-consuming. This **numerical burden** may explode when considering complex PDE networks. Consequently, evaluating the computational costs of the control methods we design appears crucial. To leverage the numerical effort induced by these controllers, it may be necessary to approximate them (e.g., by finite-dimensional systems). The approximation scheme should nonetheless guarantee satisfying closed-loop properties (and, in particular, closed-loop stability). As the last step in the design is to approximate the controller by a finite-dimensional or lumped parameter system, such kind of direct approach is sometimes referred to as late-lumping. This denomination emphasizes the distinction with early-lumping strategies where the control input is obtained using a finite-dimensional approximation of the PDE [20, 21, 71]. Different late-lumping approximation methods have been suggested in [42] or [17]. Recently, **machine-learning approximations** (based on the DeepONet algorithm) have been successfully tested in [86] on simple examples. Some stability guarantees have been given in [23] using Lyapunov functions. In this section, we briefly present some tools to obtain sufficient conditions guaranteeing the convergence of the backstepping-based late-lumping controller (51) when considering an approximation of the control law $V(t)$.

6.1. Approximation of the control input. Consider the system (10)-(12) with the sequence of control input $V_n(t)$ that corresponds to an approximation of the nominal control input $V(t)$ defined by equation (51). We assume that $|\rho q| < 1$, as explained in Section 4. To simplify the computations, we consider that $\tilde{\rho} = 0$. Applying the backstepping transformations (14)-(15), we obtain the target system

$$(64) \quad \frac{\partial \alpha}{\partial t}(t, x) + \lambda \frac{\partial \alpha}{\partial x}(t, x) = 0, \quad \frac{\partial \beta}{\partial t}(t, x) - \mu \frac{\partial \beta}{\partial x}(t, x) = 0,$$

with the boundary conditions

$$(65) \quad \alpha(t, 0) = q\beta(t, 0), \quad \beta(t, 1) = \rho\alpha(t, 1) + V_n(t) - V(t).$$

6.2. Lyapunov analysis. To analyze the stability of the target system (64)-(65), we consider the following Lyapunov function

$$(66) \quad W(t) = \int_0^1 \frac{e^{-\nu x}}{\lambda} \alpha^2(t, x) + \frac{q^2 e^{\nu x}}{\mu} \beta^2(t, x) dx,$$

where $\nu > 0$. The function W is equivalent to the square of L^2 -norm of the state (α, β) and consequently to the square of L^2 -norm of the state (u, v) . Taking the derivative with respect to time and integrating by parts, we obtain

$$\dot{W}(t) = -\nu W(t) - e^{-\nu} \alpha^2(t, 1) + q^2 e^{\nu} \beta^2(t, 1).$$

Using the boundary condition (65) and Young's inequality, we have

$$\dot{W}(t) \leq -\nu W(t) + e^{-\nu} \left(e^{2\nu} \frac{q^2 \rho^2}{2\epsilon} - 1 \right) \alpha^2(t, 1) + q^2 e^{\nu} \frac{\epsilon}{2} (V_n(t) - V(t))^2,$$

where ϵ is a positive constant. Let us now choose ν and ϵ such that $e^{2\nu} \frac{q^2 \rho^2}{2\epsilon} < 1$, which is always possible since $|\rho q| < 1$. Let us denote $C = q^2 e^{\nu} \frac{\epsilon}{2}$. We obtain

$$(67) \quad \dot{W}(t) \leq -\nu W(t) + C(V_n(t) - V(t))^2.$$

We then have the following approximation result

Theorem 9. *Assume that the approximating control law $V_n(t)$ uniformly converges to $V(t)$ when $n \rightarrow \infty$ such that there exists a sequence C_n such that*

$$(68) \quad |V_n(t) - V(t)| \leq C_n \|(u, v)\|_{L^2}^2,$$

$$(69) \quad C_n \rightarrow 0, \text{ when } n \rightarrow \infty.$$

Then, there exists $N > 0$ such that for all $n \geq N$, the system (10)-(12) with the control input $V_n(t)$ is exponentially stable.

Proof. For n large enough, equation (67) rewrites $\dot{W}(t) \leq -\eta W(t)$, where $\eta > 0$. Thus, the function W exponentially converges to zero, and so does the state (u, v) . \square

The properties (68)-(69) are verified for several approximation schemes such as Galerkin approximations and some machine-learning algorithms (e.g., DeepONet). However, generalizing the proposed approach to more general systems requires the design of appropriate Lyapunov functions. In this context, the methodology presented in [10] seems promising. This result is a first step toward practical applications of late-lumping controllers for hyperbolic systems. The question of the late-lumping backstepping controller-observer or the extension to systems of larger dimensions has not been considered so far.

6.3. Simulation results. We illustrate below the results of this section with simulations. The numerical parameters of the system are identical to the ones of Section 3.6. As this section is given for illustrating purposes, we only focus here on the control aspects, neglecting the design of any potential observer. However, to reflect that we do not have fully-distributed measurements, we assume that only an approximation of the state is available to design the control law. More precisely, we consider a Galerkin approximation of the system (10)-(12) (see [59, 70, 72] for details on the Galerkin approximation) and only assume that a finite number of modes n are available to design the control law. Consider the family ϕ_k defined for all $k = 1, 2 \dots$, by

$$(70) \quad \phi_k(x) = \begin{pmatrix} \phi_k^1(x) \\ \phi_k^2(x) \end{pmatrix} = \begin{pmatrix} \frac{1}{k\pi} \cos(k\pi x) \\ \cos(k\pi x) \end{pmatrix}$$

Define ϕ_0 and $\phi_{0,1}$ as

$$(71) \quad \phi_0(x) = (1 \ 0)^T, \quad \phi_{0,1}(x) = (0 \ 1)^T$$

For $n \in \mathbb{N}$, we now define $\chi_n = \text{span}\{\text{span}_{i=-n, \dots, n}\{\phi_i\}, \phi_{0,1}\}$ and denote P_n the orthogonal projection onto χ_n . The space χ_n is equipped with the \mathcal{H}^1 -norm. We consider that only the approximated state $(u_n, v_n) = P_n(u, v)$ is available for control design. The corresponding backstepping controller (defined by equation (51)) is denoted V_n . Due to Jackson's theorem [47],[74, Exercise 1.5.14], we can easily show that this approximation scheme satisfies the requirements of Theorem 9. We compare in Figure 10 the time evolution of the L^2 -norm of the system (10)-(12) in closed-loop using the control law (51) using the approximated states (u_n, v_n) for different values of n . The real system is still simulated using the previous finite volume method. When the number of modes used in the approximation scheme is too small, the system is not stabilized anymore. However, when this number of modes increases, we have closed-loop stability as stated in Theorem 9 (although the convergence rate can be impacted by the number of modes).

7. Concluding remarks

This chapter presents a comprehensive overview of recent developments for the practical control of scalar balance law systems using the backstepping approach. The main idea of the backstepping method is to map the original system to a target system with amenable properties. Although these target systems were originally chosen to be finite-time stable, we showed that finite-time stability could induce vanishing robustness margins. Therefore, we proposed a robustness analysis using a time-delay representation of the hyperbolic system under consideration. Such a representation allowed us to introduce degrees of freedom in the design to guarantee potential trade-offs between convergence rate and robustness margins. Interestingly, the proposed approach can be extended to more complex systems, including non-scalar systems [14], interconnected systems [80, 79] or systems actuated at both boundaries [12, 13]. We showed in this chapter that it was possible to map the original system to any arbitrary target system using appropriate backstepping transformations. Consequently, general target systems (and thus additional degrees of freedom) could be obtained by preserving dissipative in-domain couplings. This would require precise knowledge of their influence in terms of stability. The Port-Hamiltonian approach could be a path to follow. If the qualitative

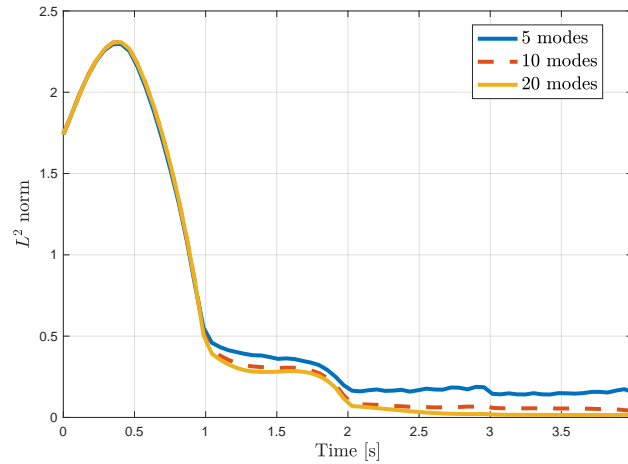


Figure 10. Time evolution of the L^2 -norm of the system (10)-(12) in closed-loop using the control law (51) and the approximation scheme P_n for different values of n .

effect of these tuning parameters can be understood in the PHS framework, it is crucial to assess and quantify the performance of the resulting controllers for a given set of specifications. This set of performance criteria should be defined in terms of practically relevant properties for industrial applications, e.g., sensitivity, robustness margins, smoothness of the state, or convergence rate. Such a complete performance analysis has to be developed. It will be the purpose of our future work. Finally, we focused on the questions related to implementing the proposed controllers in real test cases. Unlike simple PID control laws, the control strategies we develop explicitly consider the network structure and the distributed dynamics of each subsystem that compose it. Therefore, they may require some computational effort to be implemented. The incentive to do so is a performance objective: explicitly taking into account the delays and high-frequency content in the model should lead to overall increased performance. As a counterpart, the numerical burden may explode with the complexity of the system. Moreover, the implementation of the proposed control laws on micro-controllers requires numerical approximations whose effects in terms of stability and convergence should be verified. We gave generic conditions under which model reduction strategies (late-lumping approaches) can still guarantee closed-loop stability. The presented results are a first step toward practical applications of backstepping controllers. The question of the late-lumping backstepping controller-observer or the extension to systems of larger dimensions has yet to be considered. Although simulations against high-fidelity simulators have been performed for some applications cases, such as the control of torsional vibrations in drilling devices [16, 19], it is necessary to develop a systematic model-reduction methodology before deploying, demonstrating and validating the proposed techniques developed our experimental setups and real test cases.

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