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# COMPARISON PRINCIPLE FOR WALSH'S SPIDER HJB EQUATIONS WITH NON LINEAR LOCAL TIME KIRCHHOFF'S BOUNDARY TRANSMISSION

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ABSTRACT. The main purpose of this work is to obtain a comparison principle for viscosity solutions of a system of elliptic Walsh's spider Hamilton-Jacobi-Bellman equations, possessing a new boundary condition called *non linear local-time Kirchhoff's transmission*. The main idea is to build test functions at the neighborhood of the vertex solutions of ODE, with well-designed coefficients. The key point is to impose a 'local-time' derivative at the vertex absorbing the error term induced by - what we decide to call here - the *Kirchhoff's speed of the Hamiltonians*.

**Key words:** Discontinuous non degenerate Hamilton-Jacobi-Bellman equations, non linear local-time Kirchhoff's boundary transmission, comparison principle, speed of Hamiltonian, Kirchhoff's speed of Hamiltonians, stochastic optimal scattering control, Walsh's spider diffusion, local time, non-stickiness, speed measure of a diffusion.

## 1. INTRODUCTION

We are given  $R > 0$  and  $K > 0$  two boundaries of a 'space/local-time' domain

$$(0, R) \times (0, K) \ni (x, l),$$

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an integer number  $I$  (with  $I \geq 2$ ),  $I$  distinct angles  $(\theta_1, \dots, \theta_I) \in [0, 2\pi)^I$  in the plane  $\mathbb{R}^2$ , and a controlled spider measure  $\mathbb{S}$  with constant support  $(\theta_1, \dots, \theta_I)$ :

$$\mathbb{S} := \begin{cases} [0, K] \times \mathcal{O} \rightarrow \mathcal{M}([0, 2\pi)), \\ (l, \vartheta) \mapsto \sum_{i=1}^I \mathbb{S}(\theta_i, l, \vartheta) \delta_{\theta_i}(d\theta) \end{cases},$$

where  $\mathcal{O}$  is a compact of  $\mathbb{R}^I$  (and  $\mathcal{M}([0, 2\pi))$  denotes the set of the angle measures en the plan).

The main target of this work is to obtain a comparison theorem (thus uniqueness) for continuous viscosity solution of the following Walsh's spider Hamilton-Jacobi-Bellman system -  $\mathcal{W}_{alsh}(\mathbb{S})$  - having a new boundary condition at the vertex  $\mathbf{0}$ , called *non linear local-time Kirchhoff's boundary transmission*:

$$\mathcal{W}_{alsh}(\mathbb{S}) := \begin{cases} \lambda u_i(x, l) + \sup_{\beta_i \in \mathcal{B}_i} \left\{ -\sigma_i(x, l, \beta_i) \partial_x^2 u_i(x, l) + \right. \\ \left. b_i(x, l, \beta_i) \partial_x u_i(x, l) + h_i(x, l, \beta_i) \right\} = 0, \quad (x, l) \in (0, R) \times (0, K), \\ \text{Non linear local-time Kirchhoff's boundary transmission:} \\ \partial_l u(0, l) + \inf_{\vartheta \in \mathcal{O}} \left\{ \sum_{i=1}^I \mathbb{S}(\theta_i, l, \vartheta) \partial_x u_i(0, l) + h_0(l, \vartheta) \right\} = 0, \quad l \in (0, K) \\ u_i(R, l) = \chi_i(l), \quad l \in [0, K], \\ u_i(x, K) = \mathcal{T}_i(x), \quad x \in [0, R] \\ \forall (i, j) \in \llbracket 1, I \rrbracket^2, \quad \forall l \in [0, K], \quad u_i(0, l) = u_j(0, l). \end{cases} \quad (1)$$

In order to simplify our study, we have assumed in our framework that all the rays  $\mathcal{R}_i = [0, R] \times \{i\}$ ,  $i \in \llbracket 1, I \rrbracket$  have the same length  $R > 0$ , and that Dirichlet boundary condition  $\chi_i$  holds at  $x = R$  and  $\mathcal{T}_i$  at  $l = K$ . A more general setting could be treated with similar tools: one could for instance consider more general rays, and/or a mix of local-time Kirchoff's and Dirichlet boundary conditions at the  $x = R, l = K$ , etc.

On the other hand, it should be noted that the supremums and infimums appearing in the system do not play a preponderant role in the proof of our comparison theorem, and no convexity is required. The reader can easily obtain the same lines of proof by considering other supremums or infimums, such as Isaac's HJB equations. We wanted to formulate the non linearity using Hamiltonians that appear naturally in stochastic control theory.

The intuition is for the reader to keep an eye on the underlying process, in order to better understand the behavior of the process at the junction point and its non-stickiness. It is more precisely the understanding of the non-stickiness, in particular its proof in the case of a Walsh's spider diffusion (see few lines below for a brief introduction), that gave us some intuitions for the construction of the test functions.

Fully nonlinear Hamiltonians can be considered, as soon as they satisfy an ellipticity condition on each domain  $\mathcal{R}_i = [0, R] \times \{i\}$ ,  $i \in \llbracket 1, I \rrbracket$ ; quadratic growth with respect to the gradient; whereas the Kirchhoff's condition at the vertex  $\mathbf{0}$  must simply be strictly increasing with respect to the gradient. We aim in an upcoming work, to obtain existence and uniqueness of system (1) with more general Hamiltonians, and especially with a more general angular measure. Let us finally mention that our results can be also extended to time-dependent problems using classical arguments arising from the theory of viscosity solutions.

We start by explaining why we decided to call the system given in (1): *Walsh's spider Hamilton-Jacobi-Bellman system -  $\mathcal{W}_{alsh}(\mathbb{S})$*  - before describing more precisely the origins and the motivations which pushed us to consider a new condition called *non linear local-time Kirchhoff's boundary transmission*, at the vertex  $\mathbf{0}$ .

System (1) -  $\mathcal{W}_{alsh}(\mathbb{S})$  and its variants, whether in the linear or non-linear frameworks, are more often called in the literature: "system of PDE posed on a star-shaped network". See for instance amongst the vast works on this subject: [6], [12], [13], [14], [16], [17], [18], [19], [20]...

It is crucial to note that in the most works in literature dealing with the analysis of these PDE systems posed on networks, the angles appearing in the geometry of the network do not play any role in obtaining existence, regularity, comparison theorem... Indeed it appears that we can consider any star-shaped network with exactly the same rays in any Euclidean space  $\mathbb{R}^n$ ,  $n \geq 2$ ; spaced by the same number of angles, without modifying the mathematical analysis of these PDE systems.

However, PDE posed on networks, are naturally related to diffusions on graphs introduced in the seminal works of Freidlin and Wentzell [9] and Freidlin and Sheu [8]. More precisely, given  $I$  pairs  $(\sigma_i, b_i)_{i \in I}$  of mild coefficients of diffusion from  $[0, +\infty)$  to  $\mathbb{R}$  satisfying the following condition of ellipticity:  $\forall i \in \llbracket 1, I \rrbracket$ ,  $\sigma_i > 0$ , and given  $(\alpha_1, \dots, \alpha_I)$

positive constants satisfying  $\sum_{i=1}^I \alpha_i = 1$ , it is proved in [9] that there exists a continuous Feller Markov process  $(x(\cdot), i(\cdot))$  valued in the star-shaped network, whose generator is given by the following operator:

$$\mathcal{L} : \begin{cases} \mathcal{C}^2(\mathcal{N}_R) \rightarrow \mathcal{C}(\mathcal{N}_R), \\ f = f_i(x) \mapsto b_i(x) \partial_x f_i(x) + \frac{\sigma_i^2(x)}{2} \partial_x^2 f_i(x) \end{cases},$$

with domain

$$D(\mathcal{L}) := \left\{ f \in \mathcal{C}^2(\mathcal{N}_R), \sum_{i=1}^I \alpha_i \partial_x f_i(0) = 0 \right\}.$$

Recall that it is shown in [8] that there exists a one dimensional Wiener process  $W$  defined on a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  and adapted to the natural filtration of  $(x(\cdot), i(\cdot))$ , such that the process  $(x(\cdot))$  satisfies the following stochastic differential equality:

$$dx(t) = b_{i(t)}(x(t))dt + \sigma_{i(t)}(x(t))dW(t) + d\ell(t), \quad 0 \leq t \leq T.$$

In the above equality, the process  $\ell(\cdot)$  is the local time of the process  $(x(\cdot))$  at the vertex  $\mathbf{0}$ . Moreover, the process  $\ell(\cdot)$  has continuous increasing paths, starts from 0 and satisfies:

$$\forall t \in [0, T], \quad \int_0^t \mathbf{1}_{\{x(s) > 0\}} d\ell(s) = 0, \quad \mathbb{P} - \text{a.s.}$$

Recall also that the following Itô's formula was proved in [8]:

$$\begin{aligned} df_{i(t)}(x(t)) &= \left( b_{i(t)}(x(t)) \partial_x f_{i(t)}(x(t)) + \frac{1}{2} \sigma_{i(t)}^2(x(t)) \partial_x^2 f_{i(t)}(x(t)) \right) dt + \\ &\quad \partial_x f_{i(t)}(x(t)) \sigma_{i(t)}(x(t)) dW(t) + \sum_{i=1}^I \alpha_i \partial_x f_i(0) d\ell(t), \quad \mathbb{P} - \text{a.s.}, \end{aligned} \quad (2)$$

for any sufficiently regular  $f$ .

Diffusions on graphs can also be seen as part of the family of Walsh processes introduced in the epilogue [21]. Indeed, if one introduces a spider measure  $\mathbb{S}$ , such that:

$$\forall i \in \llbracket 1, I \rrbracket, \quad \mathbb{S}(\theta_i) = \alpha_i,$$

the Kirchhoff's term in the Itô's formula (2), can be rewritten as:

$$\sum_{i=1}^I \alpha_i \partial_x f_i(0) d\ell(t) = \left( \int_0^{2\pi} \partial_x f_\theta(0) \mathbb{S}(d\theta) \right) d\ell(t),$$

We now see under this formulation that the geometry of the system is uniquely determined by the support of the spider measure  $\mathbb{S}$ , namely the angles  $(\theta_1, \dots, \theta_I)$ . This is the main reason which pushed us to call the system (1): Walsh's spider Hamilton-Jacobi-Bellman system -  $\mathcal{W}_{alsh}(\mathbb{S})$ , keeping in mind the extension of our result to more general angular measure. The *non linear local-time Kirchhoff's* term appearing in (1) will be transformed in:

$$\partial_t u(0, l) + \inf_{\vartheta \in \mathcal{O}} \left\{ \int_0^{2\pi} \partial_x u_\theta(0, l) \mathbb{W}(l, \vartheta)(d\theta) + h_0(l, \vartheta) \right\},$$

with  $\mathbb{W} : [0, K] \times \mathcal{O} \mapsto \mathcal{M}([0, 2\pi])$ .

In the linear (parabolic) framework:

$$\left\{ \begin{array}{l} \partial_t u_i(t, x, l) - a_i(t, x, l) \partial_x^2 u_i(t, x, l) + b_i(t, x, l) \partial_x u_i(t, x, l) \\ \quad + c_i(t, x, l) u_i(t, x, l) = f_i(t, x, l), \quad (t, x, l) \in (0, T) \times (0, R) \times (0, K), \\ \text{Local time Kirchhoff's boundary condition:} \\ \partial_t u(t, 0, l) + \sum_{i=1}^I \alpha_i(t, l) \partial_x u_i(t, 0, l) = \phi(t, l), \quad (t, l) \in (0, T) \times (0, K), \\ \partial_x u_i(t, R, l) = 0, \quad (t, l) \in (0, T) \times (0, K), \\ \forall (i, j) \in \llbracket 1, I \rrbracket^2, \quad u_i(t, 0, l) = u_j(t, 0, l) = u(t, 0, l), \quad (t, l) \in [0, T] \times [0, K], \\ \forall i \in \llbracket 1, I \rrbracket, \quad u_i(t, x, K) = \psi_i(t, x), \quad (t, x) \in [0, T] \times [0, R], \\ \forall i \in \llbracket 1, I \rrbracket, \quad u_i(0, x, l) = g_i(x, l), \quad (x, l) \in [0, R] \times [0, K], \end{array} \right. \quad (3)$$

system (3) has been studied recently by Martinez-Ohavi in [16].

From a PDE technical aspect, since the variable  $l$  drives dynamically the system only at the junction point  $\mathbf{0}$  with the presence of the derivative  $\partial_t u(t, 0, l)$  in the local time Kirchhoff's boundary condition, the main challenge was to understand the regularity of the solution. Under mild assumptions, it is shown that classical solutions of the system (3) belong to the class  $\mathcal{C}^{1,2}$  in the interior of each edge and  $\mathcal{C}^{0,1}$  in the whole domain (with respect to the time-space variables  $(t, x)$ ). One can expect a regularity in the class  $\mathcal{C}^1$

for  $l \mapsto u(t, 0, l)$  and this is indeed the case (see Theorem 2.4 and point *iv*) in Definition 2.1 in [16]). Another technical aspect, was to obtain an Hölder continuity of the partial functions  $l \mapsto \left( \partial_t u_i(t, x, l), \partial_x u_i(t, x, l), \partial_x^2 u_i(t, x, l) \right)$  for any  $x > 0$ . It is shown using Schauder's estimates, that such regularity is guaranteed by the central assumption on the ellipticity of the diffusion coefficients on each rays, together with the mild dependency of the coefficients and free term with respect to the variable  $l$ . The comparison theorem holds true in the linear case under such regularity class; see Theorem 2.6 in [16]. The reader has to keep in mind that this class of regularity will be naturally used for test functions of system (1) -  $\mathcal{W}_{alsh}(\mathbb{S})$  - since if a viscosity solution is in this regularity class, then the formulation with test functions in this class remains compatible.

Keeping in mind that system (1) characterizes the value function of a problem of a stochastic scattering control of Walsh's spider diffusion, with optimal directions selected by its own local time, the same authors studied existence and uniqueness of Walsh's spider process with spinning measure selected from the local time in [15]. More precisely, it is shown that there exists a unique probability measure  $\mathbb{P}$ , such that the following process

$$\left( f_{i(s)}(s, x(s)) - f_i(0, x) - \int_0^s \left( \partial_t f_{i(u)}(u, x(u)) + \frac{1}{2} \sigma_{i(u)}^2(u, x(u), l(u)) \partial_{xx}^2 f_{i(u)}(u, x(u)) + b_{i(u)}(u, x(u), l(u)) \partial_x f_{i(u)}(u, x(u)) \right) du - \sum_{j=1}^I \int_0^s \alpha_j(u, l(u)) \partial_x f_j(u, 0) dl(u) \right)_{0 \leq s \leq T},$$

is a martingale under  $\mathbb{P}$ , for any  $f$  smooth enough. Up to our knowledge, this is the first result in literature dealing with a non constant spinning measure for a Walsh diffusion. Moreover, let us quote that uniqueness in distribution of such a process is completely determined by the uniqueness of the corresponding PDE system given in (3).

Now let us now describe the whys and wherefores of considering a *non linear local-time Kirchhoff's transmission*, in the non-linear framework for  $\mathcal{W}_{alsh}(\mathbb{S})$ .

Assume that one wants to find the optimal strategy, or the optimal scattering control at the vertex  $\mathbf{0}$ , using the following Kirchhoff's Hamiltonian (like it appears naturally in the Itô's rule (2)):

$$\inf_{\vartheta \in \mathcal{O}} \left\{ \sum_{i=1}^I \alpha_i(\vartheta) \partial_x u_i(0) \right\}.$$

As it appears, the optimal control  $\vartheta^*$  depends naturally only on the gradient of the value function at  $\mathbf{0}$ , namely:

$$\vartheta^*(u) = \vartheta^*(\partial_x u_1(0), \dots, \partial_x u_I(0)),$$

and behaves like a system of first order. This is quite intriguing, since the trajectories of the spider are chaotic at the junction point, whereas a first continuous order system does not present any irregularities induced by the behavior of a Brownian motion. Hence,  $\vartheta^*$  should depend in some way on the second order terms.

The local time  $\ell$  of the spider is related to the second order terms by the following quadratic approximation:

$$\lim_{\varepsilon \rightarrow 0} \mathbb{E}^{\mathbb{P}} \left[ \left| \left( \frac{1}{2\varepsilon} \sum_{i=1}^I \int_0^\cdot \sigma_i^2(0) \mathbf{1}_{\{0 \leq x(s) \leq \varepsilon, i(s)=i\}} ds \right) - \ell(\cdot) \right|_{(0,T)}^2 \right] = 0. \quad (4)$$

When the coefficients  $(\sigma_i^2(0))_{i \in \llbracket 1, I \rrbracket}$  vanish, the local time disappears and the system behaves more like a first order HJB system.

It appears therefore natural to consider the local time as an intrinsic variable to the stochastic control problem with the *non linear local-time Kirchhoff's boundary transmission* (induced by the Itô's rule extended now to the web/local-time space):

$$\partial_t u(0, \ell) + \inf_{\vartheta \in \mathcal{O}} \left\{ \sum_{i=1}^I \mathbb{S}(\theta_i, \ell, \vartheta) \partial_x u_i(0, \ell) \right\},$$

and to obtain an optimal control  $\vartheta^*$  at the vertex  $\mathbf{0}$  given by:

$$\vartheta^*(u, \ell) = \vartheta^*(\partial_x u_1(0, \ell), \dots, \partial_x u_I(0, \ell), \ell).$$

The dependence on the second order terms appears now with the quadratic approximation of the local time  $\ell$  given in (4).

Another key point when we study the behavior of a Walsh's spider motion, is to notice that all discontinuities generated by the coefficients  $(b_i(0) \neq b_j(0), \sigma_i(0) \neq \sigma_j(0))_{i \neq j}$ , disappear in Itô's rule (2) with the aid of what we call the *non-stickiness* condition satisfied by the Walsh's spider  $(x(t), i(t))_{t \geq 0}$  at  $\mathbf{0}$ :

$$\forall t \geq 0, \int_0^t \mathbf{1}_{\{x(s)=0\}} ds = 0, \quad \mathbb{P} \text{ a.s.}$$

To obtain this *non-stickiness* condition, the authors in [8] used an ordinary differential equation (ODE). This ODE involves coefficients depending on the speed measure  $s_i(x)dx$  of the Walsh's spider on each rays:

$$\forall x > 0, \quad \forall i \in \llbracket 1, I \rrbracket, \quad s_i(x)dx = \frac{2}{\sigma_i^2(x)} \exp\left(\int_0^x \frac{2b_i(z)}{\sigma_i^2(z)} dz\right) dx. \quad (5)$$

This fact has given us the intuitions to build test functions of system (1) at the vertex  $\mathbf{0}$ . If one considers the following Hamiltonians ( $H_i$ ,  $i \in \llbracket 1, I \rrbracket$ ):

$$H_i := \begin{cases} [0, R] \times [0, K] \times \mathbb{R}^3 \rightarrow \mathbb{R}, \\ (x, l, u, p, S) \mapsto \lambda u + \sup_{\beta_i \in \mathcal{B}_i} \left\{ -\sigma_i(x, l, \beta_i)S + b_i(x, l, \beta_i)p + h_i(x, l, \beta_i) \right\}, \end{cases}$$

appearing on each rays of system (1), we can define similarly to the speed measure  $s_i(x)dx$ , the *speed of the Hamiltonians* ( $\mathcal{S}_{peed}(H_i)$ ,  $i \in \llbracket 1, I \rrbracket$ ) as:

$$\mathcal{S}_{peed}(H_i) := \begin{cases} [0, R] \times [0, K] \times \mathbb{R} \rightarrow \mathbb{R}, \\ (x, l, p) \mapsto \sup_{\beta_i \in \mathcal{B}_i} \left\{ \frac{b_i(x, l, \beta_i)p + h_i(x, l, \beta_i)}{\sigma_i(x, l, \beta_i)} \right\} \end{cases}.$$

To prove the comparison theorem for system (1), we will build test functions at the junction point  $\mathbf{0}$  solutions of ODE with coefficients that may be viewed as a kind of envelope of all possible errors of the *speed of the Hamiltonians*  $(\mathcal{S}_{peed}(H_i))_{i \in \llbracket 1, I \rrbracket}$ . The key point in this construction is to impose a local-time derivative with respect to variable  $l$ , at  $\mathbf{0}$  -  $\partial_l \phi(0, l)$  - that will absorb all the error term induced by - that we have decide to call - the *Kirchhoff's speed of the Hamiltonians* -  $(\mathcal{K}_f \mathcal{S}_{peed}(H_i)$ ,  $i \in \llbracket 1, I \rrbracket$ ) - given by:

$$\mathcal{K}_f \mathcal{S}_{peed} := \begin{cases} [0, R] \times [0, K] \times \mathbb{R} \rightarrow \mathbb{R}, \\ (x, l, p) \mapsto \inf_{\vartheta \in \mathcal{O}} \left\{ \sum_{i=1}^I \mathbb{S}(\theta_i, l, \vartheta) \mathcal{S}_{peed}(H_i)(x, l, p) \right\} \end{cases}.$$

Up to our knowledge, this is the first result of uniqueness for HJB elliptic PDE system posed on a star shaped-network, having a nonlinear Kirchhoff's condition and non vanishing viscosity at the vertex.

Recall that in the theory of viscosity solutions, the formulation of upper and sub solutions for a second order nonlinear problem with Neumann  $N$  (or Kirchhoff's) boundary

condition:

$$\begin{cases} H(x, u(x), \nabla u(x), \Delta u(x)) = 0, & x \in \Omega, \\ N(x, u(x), \nabla u(x)) = 0, & x \in \partial\Omega \end{cases}, \quad (6)$$

(where  $\Omega$  denotes a smooth open set of  $\mathbb{R}^n$ ) can be stated in the strong sense namely using only the Neumann condition  $N(\cdot)$  at the boundary for both upper and sub solutions, or in the classical sense with both terms  $N(\cdot)$  and  $H(\cdot)$ , considering at the boundary  $\partial\Omega$  the term  $N(\cdot) \wedge H(\cdot)$  for sub solutions and  $N(\cdot) \vee H(\cdot)$  for supper solutions (see Section 7 in [5]). More precisely, in almost all cases the proof of the comparison theorem consists in introducing the doubling variable method, with the function:

$$\forall \varepsilon > 0, \quad w_\varepsilon(x, y) = u(x) - v(y) - \frac{|x - y|_{\mathbb{R}^n}^2}{2\varepsilon^2}, \quad (x, y) \in \mathbb{R}^n,$$

(where  $u$  is a supper solution, whereas  $v$  is a sub solution of (6)) and to obtain a contradiction, passing to the limit when  $\varepsilon \searrow 0$ , locally at any point  $x \in \overline{\Omega}$ . When  $x \in \partial\Omega$ , both conditions  $F$  and  $H$  are then considered, and the Hamiltonian  $H$  has to be continuous at the neighborhood of any point  $x \in \partial\Omega$ . Consequently, we will get that this method will fail for system (1), because of the discontinuities of the Hamiltonians at  $\mathbf{0}$ .

We will see that the construction of the test function for proving the comparison theorem in this work, allow us to formulate supper and sub viscosity solution in the strong sense at  $\mathbf{0}$ . Note that the *non linear local-time Kirchhoff's boundary transmission* appears at  $\mathbf{0}$ , without considering any values of the Hamiltonians at the vertex.

To finish this introduction, let us give an account of the main works that have been done in literature for similar systems close to (1). Note that the key fact in most all of them, is to consider a vanishing viscosity at the vertex  $\mathbf{0}$ , which is not the case here.

In [12], the authors introduce a notion of state-constraint viscosity solutions for one dimensional "junction"-type problems for first order Hamilton-Jacobi equations with non-convex coercive Hamiltonians and study its well-posedness and stability properties. Let us quote that in this work, the main results do not require any convexity conditions on the Hamiltonians, contrary to all the previous literature that is based on the control (deterministic) theoretical interpretation of the problem. Among the long list of references on this topic with convex Hamiltonians, we can cite for instance: [1], [2], [3], [4], [6],

[7]. The same authors in [13] have studied multi-dimensional junction problems for first and second-order PDE with Kirchhoff-type Neumann boundary conditions, showing that their generalized viscosity solutions are unique, but still with a vanishing viscosity at the vertex for the second order terms. Finally, let us cite the interesting approach studied in [14], where it is considered star-shaped tubular domains consisting of a number of non-intersecting, semi-infinite strips of small thickness that are connected by a central region. It is shown that classical regular solutions of uniformly elliptic partial differential equations converge in the thin-domain limit, to the unique solution of a second-order partial differential equation on the network satisfying an effective Kirchhoff-type transmission condition at the junction.

The paper is organized as follows: In Section 2, we introduce the main notations and we state the main Theorem 2.2 of this work. Section 3 is dedicated to give a sketch of proof in the linear case of our method, for simple elliptic PDE, in order to give to the reader some intuitions and ideas that will be used for the construction of tests functions for our central Theorem 2.2. Finally in Section 4, we prove our main Theorem, that is the comparison principle for system (1).

## 2. NOTATIONS AND DEFINITIONS

In this section, we introduce the main notations-definitions and we state our main Theorem 2.2. In all this work, we fix  $R > 0$  and  $K > 0$  the boundary of the 'space/local-time' domain

$$(0, R) \times (0, K) \ni (x, l)$$

where the system (1) will be studied.

We are given an integer number  $I$  (with  $I \geq 2$ ) and  $I$  distinct angles  $(\theta_1, \dots, \theta_I) \in [0, 2\pi)^I$  in the plane  $\mathbb{R}^2$ . We introduce the space of the spiders measures on the plan  $\mathbb{R}^2$ , denoted by:

$$\mathcal{Z}[(\theta_i)_{i \in [1, I]}].$$

consisting on the set of the discrete measures  $([0, 2\pi), \mathbb{B}([0, 2\pi)))$ , having  $(\theta_1, \dots, \theta_I)$  as **constant** support.

In other terms  $\mathbb{M} \in \mathcal{Z}[(\theta_i)_{i \in [1, I]}]$  if and only if there exists  $(\mathbb{M}(\theta_1), \dots, \mathbb{M}(\theta_I)) \in [0, +\infty)^I$  such that:

$$\mathbb{M}(d\theta) = \sum_{i=1}^I \mathbb{M}(\theta_i) \delta_{\theta_i}(d\theta).$$

To the angles  $(\theta_1, \dots, \theta_I)$ , we associate on the oriented euclidean plan  $(0_{\mathbb{R}^2}, \vec{u}, \vec{v})$ , a bounded star-shaped network  $\mathcal{N}_R$  defined by:

$$\mathcal{N}_R = \bigcup_{i=1}^I \mathcal{R}_i, \quad \text{with } \forall i \in [1, I], \quad (\vec{u}, \vec{\mathcal{R}}_i) = \theta_i,$$

$$\forall i \in [1, I] \quad \mathcal{R}_i := [0, R] \quad \text{and} \quad \forall (i, j) \in [1, I]^2, \quad i \neq j, \quad \mathcal{R}_i \cap \mathcal{R}_j = 0_{\mathbb{R}^2},$$

consisting of  $I$  compact rays  $\mathcal{R}_i \cong [0, R]$  ( $R > 0$ ) emanating from the origin  $0_{\mathbb{R}^2}$  of the plane  $\mathbb{R}^2$  - that we will often call the junction point or the vertex  $\mathbf{0}$  - and spaced successively to the abscissa axis by the angles  $(\theta_1, \dots, \theta_I)$ .

Let  $(\mathcal{B}_i)_{i \in [1, I]}$  be a collection of  $I$  compact sets of  $\mathbb{R}$  and  $\mathcal{O}$  a compact set of  $\mathbb{R}^I$ .

We introduce the following data:

$$\mathbf{Data:} \ (\mathcal{D}) \quad \left\{ \begin{array}{l} \left( \sigma_i \in \mathcal{C}([0, R] \times [0, K] \times \mathcal{B}_i, \mathbb{R}) \right)_{i \in [1, I]} \\ \left( b_i \in \mathcal{C}([0, R] \times [0, K] \times \mathcal{B}_i, \mathbb{R}) \right)_{i \in [1, I]} \\ \left( h_i \in \mathcal{C}([0, R] \times [0, K] \times \mathcal{B}_i, \mathbb{R}) \right)_{i \in [1, I]} \\ \mathbb{S} \in \mathcal{C}([0, K] \times \mathcal{O}, \mathcal{Z}[(\theta_i)_{i \in [1, I]}]) \\ h_0 \in \mathcal{C}([0, K] \times \mathcal{O}, \mathbb{R}), \\ \left( \chi_i \in \mathcal{C}([0, K], \mathbb{R}) \right)_{i \in [1, I]}, \\ \left( \mathcal{T}_i \in \mathcal{C}([0, R], \mathbb{R}) \right)_{i \in [1, I]}. \end{array} \right. .$$

We assume that the data  $(\mathcal{D})$  satisfy the following assumptions: (where  $(\mathbf{A})$  stands for the spider measure  $\mathbb{S}$ ,  $(\mathbf{E})$  for ellipticity, and  $(\mathbf{R})$  for Lipschitz regularity uniformly with respect to the control variables):

**Assumption  $(\mathcal{H})$** 

$$(\mathbf{A}) \quad \exists \underline{\zeta} > 0, \quad \forall i \in \llbracket 1, I \rrbracket, \quad \forall (l, \vartheta) \in [0, K] \times \mathcal{O}, \quad \mathbb{S}(\theta_i, l, \vartheta) \geq \underline{\zeta}.$$

$$(\mathbf{E}) \quad \exists \underline{\sigma} > 0, \quad \forall i \in \llbracket 1, I \rrbracket, \quad \forall (x, l, \beta_i) \in [0, R] \times [0, K] \times \mathcal{B}_i, \quad \sigma_i(x, l, \beta_i) \geq \underline{\sigma}.$$

$$(\mathbf{R}) \quad \exists (|b|, |h|, \bar{\zeta}, \bar{\sigma}) \in (0, +\infty)^4, \quad \forall i \in \llbracket 1, I \rrbracket,$$

$$\begin{aligned} (\mathbf{R} - i) \quad & \sup_{x, l, \beta_i} |b_i(x, l, \beta_i)| + \sup_{l, \beta_i} \sup_{(x, y), x \neq y} \frac{|b_i(x, l, \beta_i) - b_i(y, l, \beta_i)|}{|x - y|} \\ & + \sup_{x, \beta_i} \sup_{(l, l'), l \neq l'} \frac{|b_i(x, l, \beta_i) - b_i(x, l', \beta_i)|}{|l - l'|} \leq |b|, \end{aligned}$$

$$\begin{aligned} (\mathbf{R} - ii) \quad & \sup_{x, l, \beta_i} |\sigma_i(x, l, \beta_i)| + \sup_{l, \beta_i} \sup_{(x, y), x \neq y} \frac{|\sigma_i(x, l, \beta_i) - \sigma_i(y, l, \beta_i)|}{|x - y|} \\ & + \sup_{x, \beta_i} \sup_{(l, l'), l \neq l'} \frac{|\sigma_i(x, l, \beta_i) - \sigma_i(x, l', \beta_i)|}{|l - l'|} \leq \bar{\sigma}, \end{aligned}$$

$$\begin{aligned} (\mathbf{R} - iii) \quad & \sup_{x, l, \beta_i} |h_i(x, l, \beta_i)| + \sup_{l, \beta_i} \sup_{(x, y), x \neq y} \frac{|h_i(x, l, \beta_i) - h_i(y, l, \beta_i)|}{|x - y|} \\ & + \sup_{x, \beta_i} \sup_{(l, l'), l \neq l'} \frac{|h_i(x, l, \beta_i) - h_i(x, l', \beta_i)|}{|l - l'|} \leq |h|, \end{aligned}$$

$$(\mathbf{R} - iv) \quad \sup_{l, \vartheta} |\mathbb{S}(\theta_i, l, \vartheta)| + \sup_{\vartheta} \sup_{(l, l'), l \neq l'} \frac{|\mathbb{S}(\theta_i, l, \vartheta) - \mathbb{S}(\theta_i, l', \vartheta)|}{|l - l'|} \leq \bar{\zeta},$$

$$(\mathbf{R} - v) \quad \sup_{l, \vartheta} |h_0(l, \vartheta)| + \sup_{\vartheta} \sup_{(l, l'), l \neq l'} \frac{|h_0(l, \vartheta) - h_0(l', \vartheta)|}{|l - l'|} \leq |h|.$$

**Fix now all around this work:**  $\lambda > 0$ .

In this study, we will obtain a comparison theorem for the following Walsh's spider Hamilton-Jacobi-Bellman system  $\mathcal{W}_{alsh}(\mathbb{S})$ , having a *non linear local-time Kirchhoff's*

boundary transmission at  $\mathbf{0}$ :

$$\mathcal{W}_{alsh}(\mathbb{S}) := \begin{cases} \lambda u_i(x, l) + \sup_{\beta_i \in \mathcal{B}_i} \left\{ -\sigma_i(x, l, \beta_i) \partial_x^2 u_i(x, l) + \right. \\ \left. b_i(x, l, \beta_i) \partial_x u_i(x, l) + h_i(x, l, \beta_i) \right\} = 0, & (x, l) \in (0, R) \times (0, K), \\ \text{Non linear local-time Kirchhoff's boundary transmission:} \\ \partial_l u(0, l) + \inf_{\vartheta \in \mathcal{O}} \left\{ \sum_{i=1}^I \mathbb{S}(\theta_i, l, \vartheta) \partial_x u_i(0, l) + h_0(l, \vartheta) \right\} = 0, & l \in (0, K) \\ u_i(R, l) = \chi_i(l), & l \in [0, K], \\ u_i(x, K) = \mathcal{T}_i(x), & x \in [0, R] \\ \forall (i, j) \in \llbracket 1, I \rrbracket^2, \quad \forall l \in [0, K], \quad u_i(0, l) = u_j(0, l). \end{cases} \quad (7)$$

Given the following Hamiltonians  $(H_i, i \in \llbracket 1, I \rrbracket)$  defined by:

$$H_i := \begin{cases} [0, R] \times [0, K] \times \mathbb{R}^3 \rightarrow \mathbb{R}, \\ (x, l, u, p, S) \mapsto \sup_{\beta_i \in \mathcal{B}_i} \left\{ \lambda u - \sigma_i(x, l, \beta_i) S + b_i(x, l, \beta_i) p + h_i(x, l, \beta_i) \right\}. \end{cases}$$

we will often refer in this work to the *speed of the Hamiltonians*  $(\mathcal{S}_{peed}(H_i), i \in \llbracket 1, I \rrbracket)$  and the *Kirchhoff's speed of the Hamiltonians*  $(\mathcal{K}_f \mathcal{S}_{peed}(H_i), i \in \llbracket 1, I \rrbracket)$ , that are both defined by:

$$\mathcal{S}_{peed}(H_i) := \begin{cases} [0, R] \times [0, K] \times \mathbb{R} \rightarrow \mathbb{R}, \\ (x, l, p) \mapsto \sup_{\beta_i \in \mathcal{B}_i} \left\{ \frac{b_i(x, l, \beta_i) p + h_i(x, l, \beta_i)}{\sigma_i(x, l, \beta_i)} \right\} \end{cases},$$

and

$$\mathcal{K}_f \mathcal{S}_{peed} := \begin{cases} [0, R] \times [0, K] \times \mathbb{R} \rightarrow \mathbb{R}, \\ (x, l, p) \mapsto \inf_{\vartheta \in \mathcal{O}} \left\{ \sum_{i=1}^I \mathbb{S}(\theta_i, l, \vartheta) \mathcal{S}_{peed}(H_i)(x, l, p) \right\} \end{cases}.$$

In order to remain consistent with the results obtained in [16], more precisely with the class of regularity of the solutions in the linear framework (see Introduction in [16]), we introduce the following space of test functions for continuous viscosity solutions of the

system  $\mathcal{W}_{alsh}(\mathbb{S})$ :

$$\begin{aligned} \mathcal{C}_{0,1}^{2,0}(\mathcal{N}_R \times [0, K]) := & \left\{ f : \mathcal{N}_R \times [0, K], \quad ((x, i), l) \mapsto f_i(x, l) \mid \right. \\ & \forall i \in \llbracket 1, I \rrbracket, \quad f_i : [0, R] \times [0, K] \rightarrow \mathbb{R}, \quad (x, l) \mapsto f_i(x, l) \in \mathcal{C}^{2,0}([0, R] \times [0, K]), \\ & \forall (i, j, l) \in \llbracket 1, I \rrbracket^2 \times [0, K], \quad f_i(0, l) = f_j(0, l) = f(0, l), \\ & \left. f(0, \cdot) \in \mathcal{C}^1([0, K]) \right\}. \end{aligned}$$

We continue this section by giving the definition of continuous viscosity supper and sub solutions that belong to  $\mathcal{C}(\mathcal{N}_R \times [0, K])$ , defined by:

$$\begin{aligned} \mathcal{C}(\mathcal{N}_R \times [0, K]) := & \left\{ f : \mathcal{N}_R \times [0, K], \quad ((x, i), l) \mapsto f_i(x, l) \mid \right. \\ & \forall i \in \llbracket 1, I \rrbracket, \quad f_i : [0, R] \times [0, K] \rightarrow \mathbb{R}, \quad (x, l) \mapsto f_i(x, l) \in \mathcal{C}^0([0, R] \times [0, K]), \\ & \left. \forall (i, j, l) \in \llbracket 1, I \rrbracket^2 \times [0, K], \quad f_i(0, l) = f_j(0, l) = f(0, l) \right\}. \end{aligned}$$

for the Walsh's spider HJB system -  $\mathcal{W}_{alsh}(\mathbb{S})$  - given in (7).

**Definition 2.1.** *Let  $u \in \mathcal{C}(\mathcal{N}_R \times [0, K])$ .*

*a) We say that  $u$  is a continuous viscosity supper solution of the  $\mathcal{W}_{alsh}(\mathbb{S})$  system (7), if for all test function  $\phi \in \mathcal{C}_{0,1}^{2,0}(\mathcal{N}_R \times [0, K])$  and for all local minimum point  $(x_\star, i_\star, l_\star) \in [0, R] \times \llbracket 1, I \rrbracket \times [0, K]$  of  $u - \phi$ , we have:*

$$\left\{ \begin{aligned} & \lambda \phi_{i_\star}(x_\star, l_\star) + \sup_{\beta_{i_\star} \in \mathcal{B}_{i_\star}} \left\{ -\sigma_{i_\star}(x_\star, l_\star, \beta_i) \partial_x^2 \phi_{i_\star}(x_\star, l_\star) + \right. \\ & \left. b_{i_\star}(x_\star, l_\star, \beta_i) \partial_x \phi_{i_\star}(x_\star, l_\star) + h_{i_\star}(x_\star, l_\star, \beta_i) \right\} \geq 0, \quad \text{if } (x_\star, l_\star) \in (0, R) \times (0, K), \\ & \partial_l \phi(0, l_\star) + \inf_{\vartheta \in \mathcal{O}} \left\{ \sum_{i=1}^I \mathbb{S}(\theta_i, l_\star, \vartheta) \partial_x \phi_i(0, l_\star) + h_0(l_\star, \vartheta) \right\} \leq 0, \quad \text{if } x_\star = 0. \quad l_\star \in (0, K), \end{aligned} \right. .$$

*b) We say that  $u$  is a continuous viscosity sub solution of the  $\mathcal{W}_{alsh}(\mathbb{S})$  system (7), if for all test function  $\phi \in \mathcal{C}_{0,1}^{2,0}(\mathcal{N}_R \times [0, K])$  and for all local maximum point*

$(x_*, i_*, l_*) \in [0, R] \times \llbracket 1, I \rrbracket \times [0, K]$  of  $u - \phi$ , we have:

$$\begin{cases} \lambda \phi_{i_*}(x_*, l_*) + \sup_{\beta_{i_*} \in \mathcal{B}_{i_*}} \left\{ -\sigma_{i_*}(x_*, l_*, \beta_i) \partial_x^2 \phi_{i_*}(x_*, l_*) + \right. \\ \left. b_{i_*}(x_*, l_*, \beta_i) \partial_x \phi_{i_*}(x_*, l_*) + h_{i_*}(x_*, l_*, \beta_i) \right\} \leq 0, & \text{if } (x_*, l_*) \in (0, R) \times (0, K), \\ \partial_l \phi(0, l_*) + \inf_{\vartheta \in \mathcal{O}} \left\{ \sum_{i=1}^I \mathbb{S}(\theta_i, l_*, \vartheta) \partial_x \phi_i(0, l_*) + h_0(l_*, \vartheta) \right\} \geq 0, & \text{if } x_* = 0, l_* \in (0, K), \end{cases} .$$

c) We say that  $u$  is a continuous viscosity solution of the  $\mathcal{W}_{alsh}(\mathbb{S})$  system (7), if it is both a continuous viscosity supper and sub solution of the  $\mathcal{W}_{alsh}(\mathbb{S})$  system (7).

The main result of this work is the following Theorem:

**Theorem 2.2.** (Comparison Theorem.) Assume assumption  $(\mathcal{H})$ . Let  $v \in \mathcal{C}(\mathcal{N}_R \times [0, K])$  a continuous viscosity sub solution and  $u \in \mathcal{C}(\mathcal{N}_R \times [0, K])$  a continuous viscosity supper solution of the Walsh's spider HJB system -  $\mathcal{W}_{alsh}(\mathbb{S})$  - given in (7), satisfying the following boundary conditions:

$$\begin{aligned} \forall i \in \llbracket 1, I \rrbracket, \quad \forall l \in [0, K], \quad u_i(R, l) &\geq v_i(R, l), \\ \forall i \in \llbracket 1, I \rrbracket, \quad \forall x \in [0, R], \quad u_i(x, K) &\geq v_i(x, K). \end{aligned}$$

Then we have:

$$\forall (x, i, l) \in [0, R] \times \llbracket 1, I \rrbracket \times [0, K], \quad u_i(x, l) \geq v_i(x, l).$$

### 3. A SHORT EXAMPLE FOR THE CONSTRUCTION OF TEST FUNCTIONS IN THE LINEAR CASE

As a short introduction of the method that will be used in this work to prove our main Theorem 2.2, we propose in this section for the convenience of the reader, a simple example in the linear framework.

Consider the following elliptic linear PDE posed on the open set  $(0, R)$ , with Neumann boundary condition at  $x = 0$  and Dirichlet boundary condition at  $x = R$ :

$$\begin{cases} \lambda u(x) - \sigma(x) \partial_x^2 u(x) = 0, & x \in (0, R), \\ \partial_x u(0) = 0, \quad u(R) = z, \end{cases} \quad (8)$$

where  $\lambda > 0$ .  $\sigma \in \mathcal{C}[0, R]$  is strictly positive (elliptic) and  $z \in \mathbb{R}$ . We are going to give a simple sketch of proof for a comparison theorem.

Let  $f$  (and resp.  $g$ ) be a supper (resp. sub) continuous viscosity solution of (8) (namely in the class  $\mathcal{C}([0, R])$  in the sens of Definition 2.1 adapted to the simple example (8)). As explained in introduction, all our concentrations are focused on the behavior at the boundary  $x = 0$ , hence we will only show that the following assumption:

$$\sup_{x \in [0, R]} \{ g(x) - f(x) \} = g(0) - f(0) > 0, \quad (9)$$

will lead to a contradiction.

Let  $\eta > 0$  and  $\gamma > 0$  be two small parameters and  $\varepsilon \in (0, R)$  designed to drive the construction of the test functions at the neighborhood  $x = 0$ . Define  $\bar{\phi} = \bar{\phi}(\varepsilon, \eta, \gamma)$  the solution of the following ordinary differential equation:

$$\begin{cases} -\partial_x^2 \bar{\phi}(x) + \frac{\lambda f(x)}{\sigma(x)} = -\eta, & x \in (0, \varepsilon), \\ \bar{\phi}(0) = 0, \quad \bar{\phi}(\varepsilon) = f(\varepsilon) - f(0) - \gamma, \end{cases}, \quad (10)$$

The solution satisfies:

$$\forall x \in [0, \varepsilon], \quad \bar{\phi}(x) = f(\varepsilon) - f(0) - \gamma + \partial_x \bar{\phi}(0)(x - \varepsilon) + \int_\varepsilon^x \int_0^u \left( \eta + \frac{\lambda f(z)}{\sigma(z)} \right) dz du.$$

We are going to prove that  $\bar{\phi}$  is a test function of the supper solution  $f$  at  $x = 0$ . To obtain this fact, it is enough to show that the minimum of  $f - \bar{\phi}$  on the compact set  $[0, \varepsilon]$  is necessary reached at  $x = 0$ , and this is the case since:

-we have  $f(0) - \bar{\phi}(0) < f(\varepsilon) - \bar{\phi}(\varepsilon)$ ,

-and if the minimum of  $f - \bar{\phi}$  is reached at the interior of  $[0, \varepsilon]$ , for instance at  $y \in (0, \varepsilon)$ , because  $f$  is a supper solution we should have:

$$-\eta\sigma(y) = -\sigma(y)\partial_x^2 \bar{\phi}(y) + \lambda f(y) \geq 0,$$

which is a contradiction (recall that  $\eta > 0$  and  $\sigma(y) > 0$ ). Therefore  $\bar{\phi}$  is a test function of the supper solution  $f$  at  $x = 0$ , and this implies:

$$\partial_x \bar{\phi}(0) \leq 0.$$

Using that  $\bar{\phi}(0) = 0$ , we get:

$$0 \geq \partial_x \bar{\phi}(0)\varepsilon = f(\varepsilon) - f(0) - \gamma + \int_\varepsilon^0 \int_0^u \left( \eta + \frac{\lambda f(z)}{\sigma(z)} \right) dz du,$$

and therefore:

$$\int_0^\varepsilon \int_0^u \left( \eta + \frac{\lambda f(z)}{\sigma(z)} \right) dz du \geq f(\varepsilon) - f(0) - \gamma. \quad (11)$$

Analogously for the sub solution  $g$ , we define  $\underline{\phi} = \underline{\phi}(\varepsilon, \eta, \gamma)$  the solution of the following ordinary differential equation:

$$\begin{cases} -\partial_x^2 \underline{\phi}(x) + \frac{\lambda g(x)}{\sigma(x)} = \eta, & x \in (0, \varepsilon), \\ \underline{\phi}(0) = 0, & \underline{\phi} = g(\varepsilon) - g(0) + \gamma, \end{cases} \quad (12)$$

The solution satisfies:

$$\forall x \in [0, \varepsilon], \quad \underline{\phi}(x) = g(\varepsilon) - g(0) + \gamma + \underline{\phi}(x - \varepsilon) + \int_\varepsilon^x \int_0^u \left( -\eta + \frac{\lambda g(z)}{\sigma(z)} \right) dz du,$$

and with the same arguments behind, we can show that  $\underline{\phi}$  is a test function of the sub solution  $g$  at  $x = 0$  (the maximum of  $g - \underline{\phi}$  on the compact set  $[0, \varepsilon]$  is necessary reached at  $x = 0$ ). We get then:

$$\partial_x \underline{\phi}(0) \geq 0,$$

and therefore using that  $\underline{\phi}(0) = 0$ :

$$\int_0^\varepsilon \int_0^u \left( -\eta + \frac{\lambda g(z)}{\sigma(z)} \right) dz du \leq g(\varepsilon) - g(0) + \gamma. \quad (13)$$

Now combining both (11) and (13), we obtain:

$$\int_0^\varepsilon \int_0^u \left( -2\eta + \frac{\lambda(g(z) - f(z))}{\sigma(z)} \right) dz du \leq g(\varepsilon) - g(0) - f(\varepsilon) + f(0) + 2\gamma.$$

Sending  $\gamma \searrow 0$  and  $\eta \searrow 0$ , we have:

$$\limsup_{\eta \searrow 0} \int_0^\varepsilon \int_0^u \left( -2\eta + \frac{\lambda(g(z) - f(z))}{\sigma(z)} \right) dz du \leq \liminf_{\gamma \searrow 0} (g(\varepsilon) - g(0) - f(\varepsilon) + f(0) + 2\gamma),$$

which leads to:

$$\int_0^\varepsilon \int_0^u \frac{\lambda(g(z) - f(z))}{\sigma(z)} dz du \leq g(\varepsilon) - g(0) - f(\varepsilon) + f(0).$$

Therefore as expected at the beginning of the sketch of proof, if we assume (9) that is:

$$\sup_{x \in [0, R]} \{ g(x) - f(x) \} = g(0) - f(0) > 0,$$

we get:

$$\int_0^\varepsilon \int_0^u \frac{\lambda(g(z) - f(z))}{\sigma(z)} dz du \leq 0.$$

Dividing by  $\varepsilon^2$  we obtain:

$$\frac{1}{\varepsilon^2} \int_0^\varepsilon \int_0^u \frac{\lambda(g(z) - f(z))}{\sigma(z)} dz du \leq 0,$$

and finally sending  $\varepsilon \searrow 0$ :

$$\lim_{\varepsilon \searrow 0} \frac{1}{\varepsilon^2} \int_0^\varepsilon \int_0^u \frac{\lambda(g(z) - f(z))}{\sigma(z)} dz du \leq 0,$$

we get:

$$\frac{\lambda}{\sigma(0)} (g(0) - f(0)) \leq 0,$$

and this leads to a contradiction with the assumption (9) using that  $\lambda > 0$  and the ellipticity condition at  $x = 0$ : ( $\sigma(0) > 0$ ). We obtain therefore the result expected at the beginning of this sketch of proof.

#### 4. PROOF OF THEOREM 2.2

We state first the following Proposition, that will be useful for the construction of the test functions at the neighborhood of the vertex  $\mathbf{0}$ .

**Proposition 4.1.** *Let  $r > 0$  and  $(B, H) \in \mathbb{R}^2$ . Fix  $\varepsilon > 0$ ,  $\kappa > 0$  and  $(\eta, \gamma) \in \mathbb{R}^2$ , be four small parameters ( $\varepsilon, \kappa, |\eta|, |\gamma| \ll 1$ ) (expected to be sent to 0.) Assume that  $\varepsilon$  is small enough in order to satisfy:*

$$1 - |B|\varepsilon \exp(|B|\varepsilon) > 0. \tag{14}$$

Let  $w^\kappa$  and  $(z_i^{\varepsilon, \kappa})_{i \in [1, I]}$  be real bounded sequences indexed by  $(\varepsilon, \kappa)$ :

$$\exists w \geq 0, \exists z \geq 0, \sup_{\kappa \geq 0} |w^\kappa| \leq w, \max_{i \in [1, I]} \sup_{\varepsilon \geq 0} \sup_{\kappa \geq 0} |z_i^{\varepsilon, \kappa}| \leq z.$$

For a given  $\ell \in (0, +\infty)$  and  $S \geq 0$ :

(i)-the following parametric ordinary differential equation posed on the domain  $\mathcal{N}_\varepsilon \times [\ell -$

$\kappa, \ell + \kappa]$ :

$$\begin{cases} r\psi_i(x, l) - \partial_x^2 \psi_i(x, l) + B|\partial_x \psi_i(x, l)| + H + \eta = 0, & x \in (0, \varepsilon), \quad l \in (\ell - \kappa, \ell + \kappa), \\ \psi(0, l) = w^\kappa + S(l - \ell), \quad \psi_i(\varepsilon, l) = z_i^{\varepsilon, \kappa} + S(l - \ell) + \gamma, & l \in [\ell - \kappa, \ell + \kappa], \end{cases}, \quad (15)$$

admits a unique solution:

$$\psi = \psi\left(\varepsilon, \kappa, \eta, \gamma, S, w^\kappa, (z_i^{\varepsilon, \kappa})_{i \in \llbracket 1, I \rrbracket}\right)$$

in the class  $\mathcal{C}_{0,1}^{2,0}(\mathcal{N}_\varepsilon \times [\ell - \kappa, \ell + \kappa])$ .

(ii)-Fix  $\beta > 0$ . As soon as we impose  $\kappa = \kappa_\varepsilon$  small enough in order to verify

$$1 - \frac{\beta}{\varepsilon} r\kappa (\exp(|B|\varepsilon) - 1) - \beta \frac{r\kappa|B| \exp(2|B|\varepsilon)}{1 - |B|\varepsilon \exp(|B|\varepsilon)} > 0, \quad (16)$$

then there exists

$$S(\beta) = S\left(\beta, \varepsilon, \kappa, \eta, \gamma, w^\kappa, (z_i^{\varepsilon, \kappa})_{i \in \llbracket 1, I \rrbracket}\right) \geq 0$$

such that:

$$\forall l \in [\ell - \kappa, \ell + \kappa], \quad \partial_l \psi(0, l) = S(\beta) \geq \frac{\beta}{\varepsilon} \left( |B| |\partial_x \psi| + |H| + |\eta| \right). \quad (17)$$

(iii)-Assume moreover that the sequences  $\left(w^\kappa, (z_i^{\varepsilon, \kappa})_{i \in \llbracket 1, I \rrbracket}\right)$  satisfy:

$$\forall i \in \llbracket 1, I \rrbracket, \quad \lim_{\varepsilon \searrow 0} \limsup_{\kappa \searrow 0} |z_i^{\varepsilon, \kappa} - w^\kappa| = 0.$$

Then for all  $\beta > 0$  and the choice of the parameter  $S(\beta) = S\left(\beta, \varepsilon, \kappa, \eta, \gamma, w^\kappa, (z_i^{\varepsilon, \kappa})_{i \in \llbracket 1, I \rrbracket}\right)$  satisfying (17), the solution:

$$\psi = \psi\left(\varepsilon, \kappa, \eta, \gamma, S(\beta), w^\kappa, (z_i^{\varepsilon, \kappa})_{i \in \llbracket 1, I \rrbracket}\right) = \psi_\beta^{\varepsilon, \kappa, \eta, \gamma}$$

of (15) satisfies:

$$\limsup_{\varepsilon \searrow 0} \limsup_{\kappa \searrow 0} \limsup_{\eta \searrow 0} \limsup_{\gamma \searrow 0} \max_{i \in \llbracket 1, I \rrbracket} \sup_{l \in [\ell - \kappa, \ell + \kappa]} \left| \frac{2}{\varepsilon^2} \int_0^\varepsilon \int_0^u \psi_{i,\beta}^{\varepsilon, \kappa, \eta, \gamma}(z, l) dz du - w^\kappa \right| = 0. \quad (18)$$

*Proof.* Fix for a while  $l \in [\ell - \kappa, \ell + \kappa]$ , viewed as an external parameter. We obtain easily the unique solvability on each rays  $\mathcal{R}_{\varepsilon, i} := [0, \varepsilon] \times \{i\}$ ,  $i \in \llbracket 1, I \rrbracket$  of (15) (using for instance

the Corollary 1.9-II given in [10]). Remarking that the sequence  $w^\kappa$  is independent of  $i \in \llbracket 1, I \rrbracket$ , we obtain by extension the unique solvability in the class  $\mathcal{C}^2(\mathcal{N}_\varepsilon)$ .

Set

$$M = M(\kappa, \gamma, \eta, S) = w + z + \kappa S + |\gamma| + \frac{|H| + |\eta|}{r}. \quad (19)$$

It is easy to check that constant map  $x \mapsto M$  (resp.  $x \mapsto -M$ ) is a supper solution (resp. sub solution) of (15), which implies that:

$$\forall i \in \llbracket 1, I \rrbracket, \quad \forall (x, l) \in [0, \varepsilon] \times [\ell - \kappa, \ell + \kappa], \quad |\psi_i(x, l)| \leq M. \quad (20)$$

Since we have for all  $i \in \llbracket 1, I \rrbracket$  and for all  $(x, l) \in (0, \varepsilon) \times (\ell - \kappa, \ell + \kappa)$ :

$$|\partial_x^2 \psi_i(x, l)| \leq |B| |\partial_x \psi_i(x, l)| + rM + |H| + |\eta|,$$

it follows from Grönwall's Lemma that for all  $(x, l) \in (0, \varepsilon) \times (\ell - \kappa, \ell + \kappa)$ :

$$|\partial_x \psi_i(x, l)| \leq |\partial_x \psi_i(0, l)| \exp(|B|\varepsilon) + \frac{rM + |H| + |\eta|}{|B|} (\exp(|B|\varepsilon) - 1), \quad (21)$$

(with the convention  $((\exp(|B|\varepsilon) - 1)/|B|)|_{|B|=0} = \varepsilon$ ).

Observe now that for all  $l \in (\ell - \kappa, \ell + \kappa)$  and for all  $i \in \llbracket 1, I \rrbracket$ :

$$\gamma + z_i^{\varepsilon, \kappa} - w^\kappa = \varepsilon \partial_x \psi_i(0, l) + \int_0^\varepsilon \int_0^u \left( r \psi_i(z, l) + B |\partial_x \psi_i(z, l)| + H + \eta \right) dz du, \quad (22)$$

and therefore from (19), (20) and (22):

$$|\partial_x \psi_i(0, l)| \leq \frac{1}{\varepsilon} |\gamma + z_i^{\varepsilon, \kappa} - w^\kappa| + \varepsilon \exp(|B|\varepsilon) \left( C_1(\kappa, \gamma, \eta, S) + |B| |\partial_x \psi_i(0, l)| \right), \quad (23)$$

$$C_1 = C_1(\kappa, \gamma, \eta, S) = r(w + z + \kappa S + |\gamma|) + 2(|H| + |\eta|) \quad (24)$$

From (23), we see that as soon as we impose  $\varepsilon$  small enough to get:

$$1 - |B|\varepsilon \exp(|B|\varepsilon) > 0,$$

we obtain that for all  $l \in (\ell - \kappa, \ell + \kappa)$  and for all  $i \in \llbracket 1, I \rrbracket$ :

$$|\partial_x \psi_i(0, l)| \leq \frac{1}{1 - |B|\varepsilon \exp(|B|\varepsilon)} \left[ \frac{1}{\varepsilon} |\gamma + z_i^{\varepsilon, \kappa} - w^\kappa| + C_1 \varepsilon \exp(|B|\varepsilon) \right]. \quad (25)$$

Since the sequences  $w^\kappa$  and  $z_i^{\varepsilon, \kappa}$  are uniformly bounded and the parameter  $\kappa \ll 1$  is small enough, we obtain clearly that the sequence  $(\psi(\cdot, l))_{l \in [\ell - \kappa, \ell + \kappa]}$  is equicontinuous

in the class  $\mathcal{C}^{2+\alpha}(\mathcal{N}_\varepsilon)$  (for some  $\alpha \in (0, 1)$ ), uniformly with respect to the parameter  $l \in [\ell - \kappa, \ell + \kappa]$ . In other words:

$$\sup_{l \in [\ell - \kappa, \ell + \kappa]} \|\psi(\cdot, l)\|_{\mathcal{C}^{2+\alpha}(\mathcal{N}_\varepsilon)} \leq C_\varepsilon, \quad (26)$$

where  $C_\varepsilon \geq 0$  is a positive constant independent of  $\kappa$ .

Let us show that  $\psi$  and its first-second derivative are continuous with respect to variable  $l$ . For this purpose, let  $(l_n)$  be a sequence of  $[\ell - \kappa, \ell + \kappa]$  converging to  $l \in [\ell - \kappa, \ell + \kappa]$ . We deduce with the aid of (26), Ascoli's theorem and the boundary conditions, that  $\psi(\cdot, l_n)$  will converge up to sub sequence in  $\mathcal{C}^{2+\alpha}(\mathcal{N}_\varepsilon)$  to a solution of (15), that is indeed  $\psi(\cdot, l)$  by uniqueness. Therefore the map:

$$\begin{cases} [\ell - \kappa, \ell + \kappa] \rightarrow \mathcal{C}^{2+\alpha}(\mathcal{N}_\varepsilon) \\ l \mapsto \psi(\cdot, l), \end{cases},$$

is continuous. We conclude that  $\psi$  is in the class  $\mathcal{C}_{0,1}^{2,0}(\mathcal{N}_\varepsilon \times [\ell - \kappa, \ell + \kappa])$ , since clearly  $l \mapsto \psi(0, l) \in \mathcal{C}^1([\ell - \kappa, \ell + \kappa])$ .

Let us show now (17). Fix  $\beta > 0$ . With the aid of (21) and (25), we have:

$$\begin{aligned} |\partial_x \psi| &= \max_{i \in [1, I]} \left\{ \sup \left\{ |\partial_x \psi_i(x, l)|, \quad (x, l) \in [0, \varepsilon] \times [\ell - \kappa, \ell + \kappa] \right\} \right\} \leq \\ & \frac{\exp(|B|\varepsilon)}{1 - |B|\varepsilon \exp(|B|\varepsilon)} \left[ \max_{i \in [1, I]} \frac{1}{\varepsilon} |\gamma + z_i^{\varepsilon, \kappa} - w^\kappa| + C_1(\kappa, \gamma, \eta, S) \varepsilon \exp(|B|\varepsilon) \right] + \\ & \frac{C_1(\kappa, \gamma, \eta, S)}{|B|} (\exp(|B|\varepsilon) - 1), \end{aligned} \quad (27)$$

where we recall that  $C_1 = C_1(\kappa, \gamma, \eta, S)$  is given in (24) by:

$$C_1(\kappa, \gamma, \eta, S) = r(w + z + \kappa S + |\gamma|) + 2(|H| + |\eta|).$$

Let choose  $\kappa_\varepsilon > 0$  small enough, such that for all  $\kappa \leq \kappa_\varepsilon$ :

$$1 - \frac{\beta}{\varepsilon} r \kappa (\exp(|B|\varepsilon) - 1) - \beta \frac{r \kappa |B| \exp(2|B|\varepsilon)}{1 - |B|\varepsilon \exp(|B|\varepsilon)} > 0.$$

We observe from (27) and the expression of the constant  $C_1(\kappa, \gamma, \eta, S)$ , that if we set the parameter  $S \geq 0$ :

$$S = S(\beta) = S\left(\beta, \varepsilon, \kappa, \eta, \gamma, w^\kappa, (z_i^{\varepsilon, \kappa})_{i \in [1, I]}\right) \geq 0,$$

such that:

$$\begin{aligned}
S(\beta) := & \frac{\beta|B|}{\varepsilon} \frac{\exp(|B|\varepsilon)}{1 - |B|\varepsilon \exp(|B|\varepsilon)} \left[ \max_{i \in [1, I]} \frac{1}{\varepsilon} |\gamma + z_i^{\varepsilon, \kappa} - w^\kappa| + C_2(\gamma, \eta) \varepsilon \exp(|B|\varepsilon) \right], \\
& \frac{1 - \frac{\beta}{\varepsilon} r\kappa (\exp(|B|\varepsilon) - 1) - \beta \frac{r\kappa|B| \exp(2|B|\varepsilon)}{1 - |B|\varepsilon \exp(|B|\varepsilon)}}{1 - \frac{\beta}{\varepsilon} r\kappa (\exp(|B|\varepsilon) - 1) - \beta \frac{r\kappa|B| \exp(2|B|\varepsilon)}{1 - |B|\varepsilon \exp(|B|\varepsilon)}}, \\
& + \frac{\beta|B|}{\varepsilon} \frac{\frac{C_2(\gamma, \eta)}{|B|} (\exp(|B|\varepsilon) - 1)}{1 - \frac{\beta}{\varepsilon} r\kappa (\exp(|B|\varepsilon) - 1) - \beta \frac{r\kappa|B| \exp(2|B|\varepsilon)}{1 - |B|\varepsilon \exp(|B|\varepsilon)}} + \frac{\beta|B|}{\varepsilon} (|H| + |\eta|) \quad (28)
\end{aligned}$$

with:

$$C_2(\gamma, \eta) = r(w + z + |\gamma|) + 2(|H| + |\eta|). \quad (29)$$

we will obtain:

$$S(\beta) \geq \frac{\beta}{\varepsilon} (|B| |\partial_x \psi| + |H| + |\eta|),$$

namely (17) holds true.

To conclude observe first that since both sequences  $w^\kappa$  and  $z_i^{\varepsilon, \kappa}$  are uniformly bounded, we will obtain in (28) that the parameter:

$$S(\beta) = S\left(\beta, \varepsilon, \kappa, \eta, \gamma, w^\kappa, (z_i^{\varepsilon, \kappa})_{i \in [1, I]}\right) = S(\beta, \varepsilon, \kappa, \eta, \gamma)$$

will satisfy:

$$\limsup_{\kappa \searrow 0} \limsup_{\eta \searrow 0} \limsup_{\gamma \searrow 0} \kappa S(\beta, \varepsilon, \kappa, \eta, \gamma) = 0. \quad (30)$$

Let  $l \in [\ell - \kappa, \ell + \kappa]$ . Now for  $\psi = \psi_\beta^{\varepsilon, \kappa, \eta, \gamma}$ , which satisfies:

$$\partial_l \psi(0, l) = S(\beta, \varepsilon, \kappa, \eta, \gamma),$$

we have for all  $i \in [1, I]$  (recall that:  $\psi_\beta^{\varepsilon, \kappa, \eta, \gamma}(0, l) = w^\kappa + S(\beta, \varepsilon, \kappa, \eta, \gamma)(l - \ell)$ ):

$$\begin{aligned}
& \left| \frac{2}{\varepsilon^2} \int_0^\varepsilon \int_0^u \psi_{i, \beta}^{\varepsilon, \kappa, \eta, \gamma}(z, l) dz du - w^\kappa \right| \leq \\
& \left| \frac{2}{\varepsilon^2} \int_0^\varepsilon \int_0^u \left( \psi_{i, \beta}^{\varepsilon, \kappa, \eta, \gamma}(z, l) - (w^\kappa + S(\beta, \varepsilon, \kappa, \eta, \gamma)(l - \ell)) \right) dz du \right| + \kappa S(\beta, \varepsilon, \kappa, \eta, \gamma) \leq \\
& \left| \frac{1}{2\varepsilon^2} \int_0^\varepsilon \int_0^u \int_0^z \partial_x \psi_{i, \beta}^{\varepsilon, \kappa, \eta, \gamma}(t) dt dz du \right| + \kappa S(\beta, \varepsilon, \kappa, \eta, \gamma) \leq \\
& \frac{\varepsilon}{2} |\partial_x \psi_\beta^{\varepsilon, \kappa, \eta, \gamma}| + \kappa S(\beta, \varepsilon, \kappa, \eta, \gamma).
\end{aligned}$$

Hence if the sequences  $\left(w^\kappa, (z_i^{\varepsilon, \kappa})_{i \in \llbracket 1, I \rrbracket}\right)$  satisfy:

$$\forall i \in \llbracket 1, I \rrbracket, \quad \lim_{\varepsilon \searrow 0} \limsup_{\kappa \searrow 0} |z_i^{\varepsilon, \kappa} - w^\kappa| = 0,$$

we see from the expression given (27) and the convergence (30), that we will obtain:

$$\limsup_{\varepsilon \searrow 0} \limsup_{\kappa \searrow 0} \limsup_{\eta \searrow 0} \limsup_{\gamma \searrow 0} \max_{i \in \llbracket 1, I \rrbracket} \sup_{l \in [\ell - \kappa, \ell + \kappa]} \left| \frac{2}{\varepsilon^2} \int_0^\varepsilon \int_0^u \psi_{i, \beta}^{\varepsilon, \kappa, \eta, \gamma}(z, l) dz du - w^\kappa \right| = 0.$$

The proof is complete.  $\square$

We are able now to prove the central result of this work: Theorem 2.2.

*Proof.* Let  $f$  (and resp.  $g$ ) be a supper (resp. sub) continuous viscosity solution of (8).

**Fix in the sequel**  $\underline{\ell} \in [0, K]$ . We argue by contradiction assuming that:

$$\sup \left\{ g_i(x) - f_i(x), \quad ((x, i), l) \in \mathcal{N}_R \times [\underline{\ell}, K] \right\} > 0.$$

Since  $f$  and  $g$  are in the class  $\mathcal{C}(\mathcal{N}_R \times [0, K])$ , using the boundary condition given in the assumptions of the Theorem, the last supremum is necessary reached at a point:

$$(x_\star, i_\star, l_\star) \in [0, R) \times \llbracket 1, I \rrbracket \times [\underline{\ell}, K).$$

**Step 1: Classical arguments at the interior of each rays.** Assume first that:  $x_\star > 0$ .

Let  $\mathcal{V}(x_\star)$  be a neighborhood of  $x_\star$  strictly included in the ray  $\mathcal{R}_{i_\star}$ . When  $l_\star \in [\underline{\ell}, K]$  is fixed, the following Hamiltonian (parameterized by  $l_\star$ ) :

$$H_{i_\star}^{l_\star} := \begin{cases} \overline{\mathcal{V}(x_\star)} \times \mathbb{R}^3 \rightarrow \mathbb{R} \\ (x, u, p, S) \mapsto \lambda u + \sup_{\beta_{i_\star} \in \mathcal{B}_{i_\star}} \left\{ -\sigma_{i_\star}^2(x, l_\star, \beta_{i_\star}) S + \right. \\ \left. b_{i_\star}(x, l_\star, \beta_{i_\star}) p + h_{i_\star}(x, l_\star, \beta_{i_\star}) \right\}, \end{cases}$$

is continuous because:

-linear in  $u$ ,

-convex in  $(p, S)$ ,

-assumption  $(\mathcal{H} - \mathbf{R})$  states that all the coefficients  $(\sigma_i, b_i, h_i)_{i \in \llbracket 1, I \rrbracket}$  are Lipschitz continuous, uniformly in the control variables  $(\beta_i \in \mathcal{B}_i)_{i \in \llbracket 1, I \rrbracket}$ . This imply the Lipschitz continuity of  $H_{i_\star}^{l_\star}$  with respect to the variable  $x$ .

Moreover, we have that the classical assumptions introduced in the seminal work [5] (Theorem 3.3) hold true, which are:

(i)  $\forall(x, u, v, p, S) \in [0, R] \times \mathbb{R}^3$ , if  $u \geq v$ , then

$$H_{i_\star}^{l_\star}(x, u, p, S) - H_{i_\star}^{l_\star}(x, v, p, S) \geq \lambda(u - v), \quad \text{with } \lambda > 0,$$

(ii)  $\exists \omega \in \mathcal{C}(\mathbb{R}_+, \mathbb{R}_+)$ ,  $\omega(0) = 0$ ,  $\forall \alpha > 0$ ,  $\forall(x, y, u, p, X, Y) \in [0, R]^2 \times \mathbb{R}^4$ ,

$$\text{such that: } -3\alpha \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \leq \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq 3\alpha \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \quad \text{then:}$$

$$H_{i_\star}^{l_\star}(y, u, \beta(x - y), Y) - H_{i_\star}^{l_\star}(x, u, \alpha(x - y), X) \leq \omega(\alpha|x - y|^2 + |x - y|).$$

(Recall that the last property (ii) holds true since the coefficients  $(\sigma_i, b_i, h_i)_{i \in [1, I]}$ , are Lipschitz continuous uniformly in the control variables  $(\beta_i \in \mathcal{B}_i)_{i \in [1, I]}$  - see Example 3.6 in [5] for instance - and  $l = l_\star$  is fixed).

Since the Hamiltonian  $H_{i_\star}^{l_\star}$  does not have any dependency with some derivative with respect to the variable  $l$ , we can proceed with the classical arguments of the theory of viscosity solution to obtain a contradiction, that are:

-applying the doubling variable method with the function (here parameterized by  $(i_\star, l_\star)$ ) defined by :

$$\forall \varepsilon > 0, \quad w_\varepsilon^{(i_\star, l_\star)}(x, y) = g_{i_\star}(x, l_\star) - f_{i_\star}(x, l_\star) - \frac{1}{2\varepsilon^2}|x - y|^2, \quad (x, y) \in \mathcal{V}_{i_\star}(x_\star) \times \mathcal{V}_{i_\star}(x_\star),$$

(recall that  $\mathcal{V}_{i_\star}(x_\star)$  is a neighborhood of  $x_\star$  strictly included in  $\mathcal{R}_{i_\star}$ ).

-use the Ishii's matrix lemma (see for example Theorem 3.2 in [5]). (Recall that  $H_{i_\star}^{l_\star}$  is continuous at the neighborhood of  $x_\star$ , which implies that the equivalent definition of a supper (resp. sub solutions) with the closure of the second-order *supper jet* (resp. *sub jet*) of  $f$  (resp.  $g$ ) at  $x_\star$  holds true).

**In the rest of this proof we assume that:**

$$\sup \left\{ g_i(x, l) - f_i(x, l), \quad ((x, i), l) \in \mathcal{N}_R \times [\underline{\ell}, K] \right\} = g(0, l_\star) - f(0, l_\star) > 0, \quad (31)$$

where we recall that:  $l_\star \in [\underline{\ell}, K)$ ,  $\underline{\ell} \in (0, K)$ .

**Step 2: introduction of test functions depending on the speed of the Hamiltonians.** We scale first  $f$  and  $g$  at the vertex  $\mathbf{0}$ . Set:

$$\Theta(f, g) = \frac{1}{2}(f(0, l_\star) + g(0, l_\star)).$$

It is easy to verify that:

$$u = f - \Theta(f, g), \quad v = g - \Theta(f, g), \quad (32)$$

are respectively super solution and sub solution of the following system with *non linear local time's Kirchhoff's boundary transmission*, posed on the domain  $\mathcal{N}_R \times [0, K]$ :

$$\left\{ \begin{array}{l} \lambda w_i(x, l) + \lambda \Theta(f, g) + \sup_{\beta_i \in \mathcal{B}_i} \left\{ -\sigma_i(x, l, \beta_i) \partial_x^2 w_i(x, l) + \right. \\ \left. b_i(x, l, \beta_i) \partial_x w_i(x, l) + h_i(x, l, \beta_i) \right\} = 0, \quad (x, l) \in (0, R) \times (0, K), \\ \partial_l w(0, l) + \inf_{\vartheta \in \mathcal{O}} \left\{ \sum_{i=1}^I \mathbb{S}(\theta_i, l, \vartheta) \partial_x w_i(l, 0) + h_0(l, \vartheta) \right\} = 0, \quad l \in (0, K) \\ w_i(R, l) = \chi_i(l) - \Theta(f, g), \quad l \in [0, K], \\ w_i(x, K) = \mathcal{T}_i(x) - \Theta(f, g), \quad x \in [0, R] \\ \forall (i, j) \in \llbracket 1, I \rrbracket^2, \quad \forall l \in [0, K], \quad w_i(0, l) = w_j(0, l). \end{array} \right. \quad (33)$$

Remark that:

$$\begin{aligned} u(0, l_\star) &= \frac{1}{2}(f(0, l_\star) - g(0, l_\star)) < 0, \quad v(0, l_\star) = \frac{1}{2}(g(0, l_\star) - f(0, l_\star)) > 0, \\ v(0, l_\star) - u(0, l_\star) &= g(0, l_\star) - f(0, l_\star) > 0. \end{aligned} \quad (34)$$

In the sequel, drawing on the method introduced in the last Section 3, we will build in the neighborhood of  $(0, l_\star)$  test functions for the super solution  $u$  (resp. sub solution  $v$ ) of (33) that will solve ordinary differential equations possessing only constant coefficients. This coefficients may be viewed as a kind a supreme envelope of all possible first order errors of the speed of the Hamiltonians  $\left( \mathcal{S}_{\text{peed}}(H_i), i \in \llbracket 1, I \rrbracket \right)$  defined by:

$$\mathcal{S}_{\text{peed}}(H_i) := \left\{ \begin{array}{l} [0, R] \times [0, K] \times \mathbb{R} \rightarrow \mathbb{R}, \\ (x, l, p) \mapsto \sup_{\beta_i \in \mathcal{B}_i} \left\{ \frac{b_i(x, l, \beta_i)p + h_i(x, l, \beta_i)}{\sigma_i(x, l, \beta_i)} \right\} \end{array} \right. ,$$

where the Hamiltonian  $(H_i, i \in \llbracket 1, I \rrbracket)$  are given by:

$$H_i := \begin{cases} [0, R] \times [0, K] \times \mathbb{R}^3 \rightarrow \mathbb{R}, \\ (x, l, u, p, S) \mapsto \lambda u + \sup_{\beta_i \in \mathcal{B}_i} \left\{ -\sigma_i(x, l, \beta_i)S + b_i(x, l, \beta_i)p + h_i(x, l, \beta_i) \right\}. \end{cases}$$

The key point in the construction is to impose a derivative with respect to the 'local-time' variable  $l$  at  $x = 0$  that will absorb all the errors induced by *the Kirchhoff's speed of the Hamiltonians*, given by:

$$\mathcal{K}_f \mathcal{S}_{\text{peed}} := \begin{cases} [0, R] \times [0, K] \times \mathbb{R} \rightarrow \mathbb{R}, \\ (x, l, p) \mapsto \inf_{\vartheta \in \mathcal{O}} \left\{ \sum_{i=1}^I \mathbb{S}(\theta_i, l, \vartheta) \sup_{\beta_i \in \mathcal{B}_i} \left\{ \frac{b_i(x, l, \beta_i)p + h_i(x, l, \beta_i)}{\sigma_i(x, l, \beta_i)} \right\} \right\} \end{cases} \quad (35)$$

Fix  $\varepsilon > 0$  and  $\kappa > 0$  two small parameters expected to be sent to 0. Using (34) and the continuity of  $u$  and  $v$ , there exists a neighborhood of the vertex  $(0, l_\star)$  denoted by  $\mathcal{V}\left((0, l_\star), (\varepsilon, \kappa)\right)$  and defined by:

$$\mathcal{V}\left((0, l_\star), (\varepsilon, \kappa)\right) := \left\{ ((x, i), l) \in \mathcal{N}_R \times [\ell, K], \quad x \leq \varepsilon, \quad 0 < l_\star - \kappa \leq l \leq l_\star + \kappa < K \right\},$$

such that:

$$\forall ((x, i), l) \in \mathcal{V}\left((0, l_\star), (\varepsilon, \kappa)\right), \quad v_i(x, l) \geq 0, \quad u_i(x, l) \leq 0, \quad v_i(x, l) - u_i(x, l) \geq 0. \quad (36)$$

In the sequel we introduce also  $\eta > 0$  and  $\gamma > 0$  two small strictly positive parameters, designed to drive the construction of the test functions at the neighborhood of the vertex  $(0, l_\star)$ .

Set in the sequel:

$$\begin{aligned}\bar{t}(\lambda, \Theta(f, g)) &= \lambda \Theta(f, g) \left( \frac{1}{\underline{\sigma}} \mathbf{1}_{\Theta(f, g) > 0} + \frac{1}{\bar{\sigma}} \mathbf{1}_{\Theta(f, g) \leq 0} \right), \\ \underline{t}(\lambda, \Theta(f, g)) &= \lambda \Theta(f, g) \left( \frac{1}{\underline{\sigma}} \mathbf{1}_{\Theta(f, g) \leq 0} + \frac{1}{\bar{\sigma}} \mathbf{1}_{\Theta(f, g) > 0} \right),\end{aligned}\tag{37}$$

$$\begin{aligned}\bar{u}^\kappa(0) &= \sup \{ u(0, l), \quad l \in [l_\star - \kappa, l_\star + \kappa] \}, \\ \forall i \in \llbracket 1, I \rrbracket, \quad \underline{u}_i^\kappa(\varepsilon) &= \inf \{ u_i(\varepsilon, l), \quad l \in [l_\star - \kappa, l_\star + \kappa] \}, \\ \underline{v}^\kappa(0) &= \inf \{ v(0, l), \quad l \in [l_\star - \kappa, l_\star + \kappa] \}, \\ \forall i \in \llbracket 1, I \rrbracket, \quad \bar{v}_i^\kappa(\varepsilon) &= \sup \{ v_i(\varepsilon, l), \quad l \in [l_\star - \kappa, l_\star + \kappa] \},\end{aligned}\tag{38}$$

Let  $\bar{S} \geq 0$  and  $\underline{S} \geq 0$  be two parameters designed to characterize the 'local-time' derivative of the tests functions at  $\mathbf{0}$ . Using Proposition 4.1, we introduce

$$\bar{\phi} = \bar{\phi}(u, \varepsilon, \eta, \gamma, \bar{S}), \quad \underline{\phi} = \underline{\phi}(v, \varepsilon, \eta, \gamma, \underline{S})$$

(denoted in the next lines lines  $(\bar{\phi}, \underline{\phi})$  for the seek of clarity) the two solutions of the two following ordinary parametric differential equation systems posed on  $\mathcal{N}_\varepsilon \times [0, \kappa]$

$$\begin{cases} \frac{\lambda}{\bar{\sigma}} \bar{\phi}_i - \partial_x^2 \bar{\phi}_i(x, l) + \bar{t}(\lambda, \Theta(f, g)) \\ \quad + \left( |b| |\partial_x \bar{\phi}_i(x, l)| + |h| \right) / \underline{\sigma} = -\eta, \quad (x, l) \in (0, \varepsilon) \times (l_\star - \kappa, l_\star + \kappa), \\ \bar{\phi}(0, l) = \bar{u}^\kappa(0) + \bar{S}(l - l_\star), \\ \bar{\phi}_i(\varepsilon, l) = \underline{u}_i^\kappa(\varepsilon) - \gamma + \bar{S}(l - l_\star), \quad l \in [l_\star - \kappa, l_\star + \kappa], \quad i \in \llbracket 1, I \rrbracket. \end{cases}, \tag{39}$$

and:

$$\begin{cases} \frac{\lambda}{\underline{\sigma}} \underline{\phi}_i - \partial_x^2 \underline{\phi}_i(x, l) + \underline{t}(\lambda, \Theta(f, g)) \\ \quad - \left( |b| |\partial_x \underline{\phi}_i(x, l)| + |h| \right) / \underline{\sigma} = \eta, \quad (x, l) \in (0, \varepsilon) \times (l_\star - \kappa, l_\star + \kappa), \\ \underline{\phi}(0, l) = \underline{v}^\kappa(0) - \underline{S}(l - l_\star), \\ \underline{\phi}_i(\varepsilon, l) = \bar{v}_i^\kappa(\varepsilon) + \gamma - \underline{S}(l - l_\star), \quad l \in [l_\star - \kappa, l_\star + \kappa], \quad i \in \llbracket 1, I \rrbracket. \end{cases}. \tag{40}$$

Assumption  $(\mathcal{H})$  combined with Proposition 4.1 state that both solutions  $\bar{\phi}$  and  $\underline{\phi}$  are unique and in the class of test functions  $\mathcal{C}_{0,1}^{2,0}(\mathcal{N}_\varepsilon \times [l_\star - \kappa, l_\star + \kappa])$ .

**Step 3:**  $\bar{\phi}$  (resp.  $\underline{\phi}$ ) is a test function of the supper solution  $u$  (resp. the sub solution  $v$ ) at the vertex  $\mathbf{0}$  of the Walsh's spider HJB system (33).

We start to show that  $\bar{\phi}$  is a test function of the supper solution  $u$  at  $\mathbf{0}$  of the HJB system (33). The second case involving  $\underline{\phi}$  for the sub solution  $v$  can be treated with the same arguments.

We are then going to show that the minimum of  $u - \bar{\phi}$  on the domain  $\mathcal{N}_\varepsilon \times [l_\star - \kappa, l_\star + \kappa]$ , is necessarily reached at some point  $(0, \bar{l}_\kappa)$  with  $\bar{l}_\kappa \in [l_\star - \kappa, l_\star + \kappa]$ . Using the boundary conditions satisfied by  $\bar{\phi}$  and the expressions given in (37), we obtain that  $\forall i \in \llbracket 1, I \rrbracket$  and  $\forall l \in [l_\star - \kappa, l_\star + \kappa]$ :

$$\begin{aligned} u(0, l) - \bar{\phi}(0, l) &= u(0, l) - \bar{u}^\kappa(0) - \bar{S}(l - l_\star) \leq -\bar{S}(l - l_\star), \\ \forall i \in \llbracket 1, I \rrbracket, \quad u_i(\varepsilon, l) - \bar{\phi}_i(\varepsilon, l) &= u_i(\varepsilon, l) - \underline{u}_i^\kappa(\varepsilon) - \bar{S}(l - l_\star) + \gamma > -\bar{S}(l - l_\star), \end{aligned}$$

which implies that the minimum of  $u - \bar{\phi}$  can not be reached at  $(x = \varepsilon, l)$  with  $l \in [l_\star - \kappa, l_\star + \kappa]$ .

Assume now that minimum of  $u - \bar{\phi}$  is reached at some point

$$(y, j, \ell) \in (0, \varepsilon) \times \llbracket 1, I \rrbracket \times [l_\star - \kappa, l_\star + \kappa].$$

As it is classical in the viscosity formulation, we can without lose of generality assume that:

$$u_j(y, \ell) = \bar{\phi}_j(y, \ell).$$

Since  $u$  is a supper solution of the system (33), we have by definition (recall that  $0 < l_\star - \kappa < l_\star + \kappa < K$ ):

$$\begin{aligned} &\lambda u_j(y, \ell) + \lambda \Theta(f, g) + \\ &\sup_{\beta_j \in \mathcal{B}_j} \left\{ -\sigma_j(y, \ell, \beta_j) \partial_x^2 \bar{\phi}_j(y, \ell) + b_j(y, \ell, \beta_j) \partial_x \bar{\phi}_j(y, \ell) + h_j(y, \ell, \beta_j) \right\} \geq 0. \end{aligned} \quad (41)$$

Hence using the expression of the system of ordinary differential equation (39) satisfied by  $\bar{\phi}$  we get:

$$\begin{aligned} & \lambda u_j(y, \ell) + \lambda \Theta(f, g) + \sup_{\beta_j \in \mathcal{B}_j} \left\{ -\sigma_j(y, \ell, \beta_j) \eta - \sigma_j(y, \ell, \beta_j) \frac{\lambda u_j(y, \ell)}{\bar{\sigma}} \right. \\ & - \sigma_j(y, \ell, \beta_j) \bar{t}(\lambda, \Theta(f, g)) - \sigma_j(y, \ell, \beta_j) (|b| |\partial_x \bar{\phi}_j(y, \ell)| + |h|) / \underline{\sigma} \\ & \left. + b_j(y, \ell, \beta_j) \partial_x \bar{\phi}_j(y, \ell) + h_j(y, \ell, \beta_j) \right\} \geq 0. \end{aligned} \quad (42)$$

Using now that  $\eta > 0$ ,  $u_j(y, \ell) \leq 0$  (see (36)), the expression of  $\bar{t}(\lambda, \Theta(f, g))$  given in (37) and the central ellipticity condition satisfied by the coefficient  $\sigma_j$  (assumption  $\mathcal{H} - \mathbf{E}$ ), we get that the last quantity (42) is smaller than:

$$\begin{aligned} & \left[ \lambda u_j(y, \ell) + \sup_{\beta_j \in \mathcal{B}_j} \left\{ -\sigma_j(y, \ell, \beta_j) \frac{\lambda u_j(y, \ell)}{\bar{\sigma}} \right\} \right] \\ & + \left[ \lambda \Theta(f, g) + \sup_{\beta_j \in \mathcal{B}_j} \left\{ -\sigma_j(y, \ell, \beta_j) \bar{t}(\lambda, \Theta(f, g)) \right\} \right] \\ & + \left[ (|b| |\partial_x \bar{\phi}_j(y, \ell)| + |h|) - (|b| |\partial_x \bar{\phi}_j(y, \ell)| + |h|) \right] \\ & + \sup_{\beta_j \in \mathcal{B}_j} \left\{ -\sigma_j(y, \ell, \beta_j) \eta \right\} \leq -\underline{\sigma} \eta < 0, \end{aligned}$$

and that leads to a contradiction with (41). We conclude that  $\bar{\phi}$  is a test function of the supper solution  $u$  at the vertex  $\mathbf{0}$ , namely there exists  $\bar{l}_\kappa \in [l_\star - \kappa, l_\star + \kappa]$  such that:

$$\partial_l \bar{\phi}(0, \bar{l}_\kappa) + \inf_{\vartheta \in \mathcal{O}} \left\{ \sum_{i=1}^I \mathbb{S}(\theta_i, \bar{l}_\kappa, \vartheta) \partial_x \bar{\phi}_i(0, \bar{l}_\kappa) + h_0(\bar{l}_\kappa, \vartheta) \right\} \leq 0. \quad (43)$$

As announced at the beginning of this Step, using the same arguments, we can show easily that  $\underline{\phi}$  is a test function of the sub solution  $v$  at  $\mathbf{0}$ , namely there exists  $\underline{l}_\kappa \in [l_\star - \kappa, l_\star + \kappa]$  such that:

$$\partial_l \underline{\phi}(0, \underline{l}_\kappa) + \inf_{\vartheta \in \mathcal{O}} \left\{ \sum_{i=1}^I \mathbb{S}(\theta_i, \underline{l}_\kappa, \vartheta) \partial_x \underline{\phi}_i(0, \underline{l}_\kappa) + h_0(\underline{l}_\kappa, \vartheta) \right\} \geq 0. \quad (44)$$

**Step 3: Conclusion.** Fix  $\vartheta \in \mathcal{O}$ . Without loss of generality, in the construction of the tests functions in **Step 1** and the results obtained in **Step 2**, we know from Proposition

4.1, that for  $\varepsilon \ll 1$ ,  $\kappa \ll 1$ , there exists

$$\begin{aligned}\bar{S} &= \bar{S}(\bar{\zeta}, \varepsilon, \kappa, \eta, \gamma) = \bar{S}\left(\bar{\zeta}, \varepsilon, \kappa, |\bar{t}(\lambda, \Theta(f, g))| + \eta, |h|, |\partial_x \bar{\phi}|, \bar{u}^\kappa(0), (\underline{u}_i^\kappa(\varepsilon))_{i \in \llbracket 1, I \rrbracket}\right), \\ \underline{S} &= \underline{S}(\bar{\zeta}, \varepsilon, \kappa, \eta, \gamma) = \underline{S}\left(\bar{\zeta}, \varepsilon, \kappa, |\underline{t}(\lambda, \Theta(f, g))| + \eta, |h|, |\partial_x \underline{\phi}|, \underline{v}^\kappa(0), (\bar{v}_i^\kappa(\varepsilon))_{i \in \llbracket 1, I \rrbracket}\right),\end{aligned}$$

such that:

$$\partial_l \bar{\phi}(0, \bar{l}_\kappa) = \bar{S}(\bar{\zeta}, \varepsilon, \kappa, \eta, \gamma) \geq \frac{I\bar{\zeta}}{\varepsilon} \left( |\bar{t}(\lambda, \Theta(f, g))| + \eta + \frac{|b| |\partial_x \bar{\phi}| + |h|}{\underline{\sigma}} \right), \quad (45)$$

$$\partial_l \phi(0, \underline{l}_\kappa) = \underline{S}(\bar{\zeta}, \varepsilon, \kappa, \eta, \gamma) \geq \frac{I\bar{\zeta}}{\varepsilon} \left( |\underline{t}(\lambda, \Theta(f, g))| + \eta + \frac{|b| |\partial_x \underline{\phi}| + |h|}{\underline{\sigma}} \right). \quad (46)$$

Recall that  $\bar{\zeta}$  is given in assumption  $(\mathcal{H} - \mathbf{R})$ , and we have denoted for  $w \in \mathcal{C}^2(\mathcal{N}_\varepsilon \times [0, \kappa])$ ,  $|\partial_x w| = \max_i \sup_{(x, l)} |\partial_x w_i(x, l)|$ . As an interpretation, roughly speaking (45) and (46) implies that the 'local-time' time derivative of the test function  $(\bar{\phi}, \underline{\phi})$  at  $\mathbf{0}$  absorb the error term induced by the Kirchhoff's speed of the Hamiltonians defined in (35), scaled by  $1/\varepsilon$ .

Remark first that since  $\bar{l}_\kappa \in [l_\star - \kappa, l_\star + \kappa]$  and  $\underline{l}_\kappa \in [l_\star - \kappa, l_\star + \kappa]$ , we have:

$$\lim_{\kappa \searrow 0} \bar{l}_\kappa = l_\star, \quad \lim_{\kappa \searrow 0} \underline{l}_\kappa = l_\star \quad (47)$$

and from the expressions given in (37) and the continuity of  $u$  and  $v$  we get:

$$\begin{aligned}\forall i \in \llbracket 1, I \rrbracket, \quad \lim_{\varepsilon \searrow 0} \limsup_{\kappa \searrow 0} \bar{v}_i^\kappa(\varepsilon) &= v(0, l_\star), \quad \lim_{\varepsilon \searrow 0} \limsup_{\kappa \searrow 0} \underline{u}_i^\kappa(\varepsilon) = u(0, l_\star), \\ \lim_{\kappa \searrow 0} \underline{v}^\kappa(0) &= v(0, l_\star), \quad \lim_{\kappa \searrow 0} \bar{u}^\kappa(0) = u(0, l_\star).\end{aligned} \quad (48)$$

Using (31) and (32), we know that for all  $i \in \llbracket 1, I \rrbracket$ :

$$u_i(\varepsilon, l_\star) - u(0, l_\star) - v_i(\varepsilon, l_\star) + v(0, l_\star) = f_i(\varepsilon, l_\star) - f(0, l_\star) - g_i(\varepsilon, l_\star) + g(0, l_\star) \geq 0.$$

Hence we deduce that for all  $i \in \llbracket 1, I \rrbracket$ :

$$\lim_{\kappa \searrow 0} \underline{u}_i^\kappa(\varepsilon) - \bar{u}^\kappa(0) - \bar{v}_i^\kappa(\varepsilon) + \underline{v}^\kappa(0) \geq 0,$$

and therefore:

$$\exists \kappa_\varepsilon > 0, \quad \forall \kappa \leq \kappa_\varepsilon, \quad \underline{u}_i^\kappa(\varepsilon) - \bar{u}^\kappa(0) - \bar{v}_i^\kappa(\varepsilon) + \underline{v}^\kappa(0) \geq -\varepsilon^3. \quad (49)$$

From the expressions of the ordinary differential equations (39)-(40) satisfied by that  $\bar{\phi}$  and  $\underline{\phi}$ , we obtain:

$$\begin{aligned} \partial_x \bar{\phi}_i(0, \bar{l}_\kappa) &= -\frac{\gamma}{\varepsilon} + \frac{1}{\varepsilon} \left( \underline{u}_i^\kappa(\varepsilon) - \bar{u}^\kappa(0) - \int_0^\varepsilon \int_0^u \frac{\lambda \bar{\phi}_i(z, \bar{l}_\kappa)}{\bar{\sigma}} dz du \right. \\ &\quad \left. - \int_0^\varepsilon \int_0^u \left[ \bar{t}(\lambda, \Theta(f, g)) + \frac{|b| |\partial_x \bar{\phi}_i(z, \bar{l}_\kappa)| + |h|}{\underline{\sigma}} + \eta \right] dz du \right), \\ \partial_x \underline{\phi}_i(0, \underline{l}_\kappa) &= \frac{\gamma}{\varepsilon} + \frac{1}{\varepsilon} \left( \bar{v}_i^\kappa(\varepsilon) - \underline{v}^\kappa(0) - \int_0^\varepsilon \int_0^u \frac{\lambda \underline{\phi}_i(z, \underline{l}_\kappa)}{\bar{\sigma}} dz du \right. \\ &\quad \left. - \int_0^\varepsilon \int_0^u \left[ \underline{t}(\lambda, \Theta(f, g)) - \frac{|b| |\partial_x \underline{\phi}_i(z, \underline{l}_\kappa)| + |h|}{\underline{\sigma}} - \eta \right] dz du \right) \\ \partial_l \bar{\phi}(0, \bar{l}_\kappa) &= \bar{S}(\bar{\zeta}, \varepsilon, \kappa, \eta, \gamma), \quad \partial_l \underline{\phi}(0, \underline{l}_\kappa) = -\underline{S}(\bar{\zeta}, \varepsilon, \kappa, \eta, \gamma). \end{aligned} \quad (50)$$

Now we have the necessary tools to conclude. Recall that  $\vartheta \in \mathcal{O}$  is fixed. Write:

$$\begin{aligned} &\partial_l \bar{\phi}(0, \bar{l}_\kappa) + \sum_{i=1}^I \mathbb{S}(\theta_i, \bar{l}_\kappa, \vartheta) \partial_x \bar{\phi}_i(0, \bar{l}_\kappa) + h_0(\bar{l}_\kappa, \vartheta) - \\ &\left( \partial_l \underline{\phi}(0, \underline{l}_\kappa) + \sum_{i=1}^I \mathbb{S}(\theta_i, \underline{l}_\kappa, \vartheta) \partial_x \underline{\phi}_i(0, \underline{l}_\kappa) + h_0(\underline{l}_\kappa, \vartheta) \right) = \\ &\bar{S}(\bar{\zeta}, \varepsilon, \kappa, \eta, \gamma) - \sum_{i=1}^I \mathbb{S}(\theta_i, \bar{l}_\kappa, \vartheta) \frac{1}{\varepsilon} \left( \int_0^\varepsilon \int_0^u \left[ \bar{t}(\lambda, \Theta(f, g)) + \right. \right. \\ &\quad \left. \left. \frac{|b| |\partial_x \bar{\phi}_i(z, \bar{l}_\kappa)| + |h|}{\underline{\sigma}} + \eta \right] dz du \right) + \underline{S}(\bar{\zeta}, \varepsilon, \kappa, \eta, \gamma) - \\ &\sum_{i=1}^I \mathbb{S}(\theta_i, \bar{l}_\kappa, \vartheta) \frac{1}{\varepsilon} \left( \int_0^\varepsilon \int_0^u \left[ -\underline{t}(\lambda, \Theta(f, g)) + \frac{|b| |\partial_x \bar{\phi}_i(z, \bar{l}_\kappa)| + |h|}{\underline{\sigma}} + \eta \right] dz du \right) - \\ &\frac{\gamma}{\varepsilon} \left( \sum_{i=1}^I \mathbb{S}(\theta_i, \bar{l}_\kappa, \vartheta) + \sum_{i=1}^I \mathbb{S}(\theta_i, \underline{l}_\kappa, \vartheta) \right) + \frac{1}{\varepsilon} \sum_{i=1}^I \mathbb{S}(\theta_i, \bar{l}_\kappa, \vartheta) \left( \underline{u}_i^\kappa(\varepsilon) - \bar{u}^\kappa(0) - \bar{v}_i^\kappa(\varepsilon) + \underline{v}^\kappa(0) \right) + \\ &\frac{1}{\varepsilon} \sum_{i=1}^I \left( \mathbb{S}(\theta_i, \bar{l}_\kappa, \vartheta) - \mathbb{S}(\theta_i, \underline{l}_\kappa, \vartheta) \right) \left( \bar{v}_i^\kappa(\varepsilon) - \underline{v}^\kappa(0) \right) + h_0(\bar{l}_\kappa, \vartheta) - h_0(\underline{l}_\kappa, \vartheta) + \\ &\sum_{i=1}^I \mathbb{S}(\theta_i, \underline{l}_\kappa, \vartheta) \frac{1}{\varepsilon} \int_0^\varepsilon \int_0^u \frac{\lambda \underline{\phi}_i(z, \underline{l}_\kappa)}{\bar{\sigma}} dz du - \sum_{i=1}^I \mathbb{S}(\theta_i, \bar{l}_\kappa, \vartheta) \frac{1}{\varepsilon} \int_0^\varepsilon \int_0^u \frac{\lambda \bar{\phi}_i(z, \bar{l}_\kappa)}{\bar{\sigma}} dz du. \end{aligned}$$

Now using (45)-(46)-(49), together with assumption  $(\mathcal{H} - \mathbf{R})$  we obtain that there exists

$\exists \kappa_\varepsilon > 0, \forall \kappa \leq \kappa_\varepsilon$ :

$$\begin{aligned}
& \partial_l \bar{\phi}(0, \bar{l}_\kappa) + \sum_{i=1}^I \mathbb{S}(\theta_i, \bar{l}_\kappa, \vartheta) \partial_x \partial_x \bar{\phi}_i(0, \bar{l}_\kappa) + h_0(\bar{l}_\kappa, \vartheta) - \\
& \left( \partial_l \underline{\phi}(0, \underline{l}_\kappa) + \sum_{i=1}^I \mathbb{S}(\theta_i, \bar{l}_\kappa, \vartheta) \partial_x \underline{\phi}_i(0, \underline{l}_\kappa) + h_0(\underline{l}_\kappa, \vartheta) \right) \\
& \geq -\frac{\gamma}{\varepsilon} 2I\bar{\zeta} - \varepsilon^2 I\bar{\zeta} - \frac{2|v|I\bar{\zeta}}{\varepsilon} |\bar{l}_\kappa - \underline{l}_\kappa| - |h| |\bar{l}_\kappa - \underline{l}_\kappa| \\
& + \sum_{i=1}^I \mathbb{S}(\theta_i, \underline{l}_\kappa, \vartheta) \frac{1}{\varepsilon} \int_0^\varepsilon \int_0^u \frac{\lambda \underline{\phi}_i(z, \underline{l}_\kappa)}{\bar{\sigma}} dz du - \sum_{i=1}^I \mathbb{S}(\theta_i, \bar{l}_\kappa, \vartheta) \frac{1}{\varepsilon} \int_0^\varepsilon \int_0^u \frac{\lambda \bar{\phi}_i(z, \bar{l}_\kappa)}{\bar{\sigma}} dz du. \quad (51)
\end{aligned}$$

Taking now the infimum over all the  $\vartheta \in \mathcal{O}$  in (51), we obtain with aid of (43) and (44):

$$\begin{aligned}
0 & \geq -\frac{\gamma}{\varepsilon} 2I\bar{\zeta} - \varepsilon^2 I\bar{\zeta} - \frac{2|v|I\bar{\zeta}}{\varepsilon} |\bar{l}_\kappa - \underline{l}_\kappa| - |h| |\bar{l}_\kappa - \underline{l}_\kappa| \\
& + \inf_{\vartheta \in \mathcal{O}} \left[ \sum_{i=1}^I \mathbb{S}(\theta_i, \underline{l}_\kappa, \vartheta) \frac{1}{\varepsilon} \int_0^\varepsilon \int_0^u \frac{\lambda \underline{\phi}_i(z, \underline{l}_\kappa)}{\bar{\sigma}} dz du - \sum_{i=1}^I \mathbb{S}(\theta_i, \bar{l}_\kappa, \vartheta) \frac{1}{\varepsilon} \int_0^\varepsilon \int_0^u \frac{\lambda \bar{\phi}_i(z, \bar{l}_\kappa)}{\bar{\sigma}} dz du \right]. \quad (52)
\end{aligned}$$

For all  $n > 0$ , there exists  $\vartheta_n = \vartheta_n(\varepsilon, \kappa, \eta, \gamma)$  such that:

$$\begin{aligned}
& \inf_{\vartheta \in \mathcal{O}} \left[ \sum_{i=1}^I \mathbb{S}(\theta_i, \underline{l}_\kappa, \vartheta) \frac{1}{\varepsilon} \int_0^\varepsilon \int_0^u \frac{\lambda \underline{\phi}_i(z, \underline{l}_\kappa)}{\bar{\sigma}} dz du - \right. \\
& \left. \sum_{i=1}^I \mathbb{S}(\theta_i, \bar{l}_\kappa, \vartheta) \frac{1}{\varepsilon} \int_0^\varepsilon \int_0^u \frac{\lambda \bar{\phi}_i(z, \bar{l}_\kappa)}{\bar{\sigma}} dz du \right] + 1/n \geq \\
& \left[ \sum_{i=1}^I \mathbb{S}(\theta_i, \underline{l}_\kappa, \vartheta_n) \frac{1}{\varepsilon} \int_0^\varepsilon \int_0^u \frac{\lambda \underline{\phi}_i(z, \underline{l}_\kappa)}{\bar{\sigma}} dz du - \sum_{i=1}^I \mathbb{S}(\theta_i, \bar{l}_\kappa, \vartheta_n) \frac{1}{\varepsilon} \int_0^\varepsilon \int_0^u \frac{\lambda \bar{\phi}_i(z, \bar{l}_\kappa)}{\bar{\sigma}} dz du \right].
\end{aligned}$$

Dividing (52) by  $\varepsilon/2$  we get:

$$0 \geq -\frac{\gamma}{\varepsilon^2}4I\bar{\zeta} - \varepsilon 2I\bar{\zeta} - \frac{|v|4I\bar{\zeta}}{\varepsilon^2}|\bar{l}_\kappa - l_\kappa| - \frac{2}{\varepsilon}|h||\bar{l}_\kappa - l_\kappa| - \frac{2}{n\varepsilon} +$$

$$\left[ \sum_{i=1}^I \mathbb{S}(\theta_i, l_\kappa, \vartheta_n) \frac{2}{\varepsilon^2} \int_0^\varepsilon \int_0^u \frac{\lambda \phi_i(z, l_\kappa)}{\bar{\sigma}} dz du - \sum_{i=1}^I \mathbb{S}(\theta_i, \bar{l}_\kappa, \vartheta_n) \frac{2}{\varepsilon^2} \int_0^\varepsilon \int_0^u \frac{\lambda \bar{\phi}_i(z, \bar{l}_\kappa)}{\bar{\sigma}} dz du \right] \geq$$

$$-\frac{\gamma}{\varepsilon^2}4I\bar{\zeta} - \varepsilon 2I\bar{\zeta} - \frac{|v|4I\bar{\zeta}}{\varepsilon^2}|\bar{l}_\kappa - l_\kappa| - \frac{2}{\varepsilon}|h||\bar{l}_\kappa - l_\kappa| - \frac{2}{n\varepsilon} + \quad (53)$$

$$\frac{\lambda}{\bar{\sigma}} \underline{v}^\kappa(0) \sum_{i=1}^I \mathbb{S}(\theta_i, l_\kappa, \vartheta_n) - \frac{\lambda}{\bar{\sigma}} \bar{u}^\kappa(0) \sum_{i=1}^I \mathbb{S}(\theta_i, \bar{l}_\kappa, \vartheta_n) + \quad (54)$$

$$\sum_{i=1}^I \mathbb{S}(\theta_i, l_\kappa, \vartheta_n) \frac{2}{\varepsilon^2} \int_0^\varepsilon \int_0^u \frac{\lambda(\phi_i(z, l_\kappa) - \underline{v}^\kappa(0))}{\bar{\sigma}} dz du - \quad (55)$$

$$\sum_{i=1}^I \mathbb{S}(\theta_i, \bar{l}_\kappa, \vartheta_n) \frac{2}{\varepsilon^2} \int_0^\varepsilon \int_0^u \frac{\lambda(\bar{\phi}_i(z, \bar{l}_\kappa) - \bar{u}^\kappa(0))}{\bar{\sigma}} dz du \quad (56)$$

To conclude we have first in (53):

$$\lim_{\varepsilon \searrow 0} \limsup_{\kappa \searrow 0} \limsup_{\gamma \searrow 0} \limsup_{n \rightarrow +\infty} \left[ -\frac{\gamma}{\varepsilon^2}4I\bar{\zeta} - \varepsilon 2I\bar{\zeta} - \frac{|v|4I\bar{\zeta}}{\varepsilon^2}|\bar{l}_\kappa - l_\kappa| - \frac{2}{\varepsilon}|h||\bar{l}_\kappa - l_\kappa| - \frac{2}{n\varepsilon} \right] = 0.$$

Secondly, from (34) and (48), we obtain that in (54) (because  $\underline{v}^\kappa(0) \geq 0$ , and  $\bar{u}^\kappa(0) \leq 0$ ):

$$\limsup_{\kappa \searrow 0} \left( \frac{\lambda}{\bar{\sigma}} \underline{v}^\kappa(0) \sum_{i=1}^I \mathbb{S}(\theta_i, l_\kappa, \vartheta_n) - \frac{\lambda}{\bar{\sigma}} \bar{u}^\kappa(0) \sum_{i=1}^I \mathbb{S}(\theta_i, \bar{l}_\kappa, \vartheta_n) \right) \geq I \underline{\zeta} \frac{\lambda}{\bar{\sigma}} [v(0, l_\star) - u(0, l_\star)].$$

Finally, for the term (55), since the boundary condition of  $(\bar{\phi} = (\varepsilon, \kappa, \eta, \gamma), \bar{\phi} = (\varepsilon, \kappa, \eta, \gamma))$  satisfy the point (iii) of Proposition 4.1, namely (see (48)):

$$\forall i \in \llbracket 1, I \rrbracket, \quad \lim_{\varepsilon \searrow 0} \limsup_{\kappa \searrow 0} |\underline{u}_i^\kappa(\varepsilon) - \bar{u}^\kappa(0)| = 0, \quad \lim_{\varepsilon \searrow 0} \limsup_{\kappa \searrow 0} |\bar{v}_i^\kappa(\varepsilon) - \underline{v}^\kappa(0)| = 0,$$

we have:

$$\lim_{\varepsilon \searrow 0} \limsup_{\kappa \searrow 0} \limsup_{\eta \searrow 0} \limsup_{\gamma \searrow 0} \left| \sum_{i=1}^I \mathbb{S}(\theta_i, l_\kappa, \vartheta_n) \frac{2}{\varepsilon^2} \int_0^\varepsilon \int_0^u \frac{\lambda(\phi_i^{\varepsilon, \kappa, \eta, \gamma}(z, l_\kappa) - \underline{v}^\kappa(0))}{\bar{\sigma}} dz du \right| \leq$$

$$\lim_{\varepsilon \searrow 0} \limsup_{\kappa \searrow 0} \limsup_{\eta \searrow 0} \limsup_{\gamma \searrow 0} I \bar{\zeta} \max_i \sup_{l \in [l_\star - \kappa, l_\star + \kappa]} \left| \frac{2}{\varepsilon^2} \int_0^\varepsilon \int_0^u \frac{\lambda(\phi_i^{\varepsilon, \kappa, \eta, \gamma}(z, l) - \underline{v}^\kappa(0))}{\bar{\sigma}} dz du \right|$$

$$= 0,$$

and also:

$$\lim_{\varepsilon \searrow 0} \limsup_{\kappa \searrow 0} \limsup_{\eta \searrow 0} \limsup_{\gamma \searrow 0} \left| \sum_{i=1}^I \mathbb{S}(\theta_i, \bar{l}_\kappa, \vartheta_n) \frac{2}{\varepsilon^2} \int_0^\varepsilon \int_0^u \frac{\lambda(\bar{\phi}_i^{\varepsilon, \kappa, \eta, \gamma}(z, \bar{l}_\kappa) - \bar{u}^\kappa(0))}{\bar{\sigma}} dz du \right| = 0.$$

In the end we get:

$$0 \geq I \zeta \frac{\lambda}{\bar{\sigma}} (v(0, l_\star) - u(0, l_\star)).$$

Therefore (32) imply

$$0 \geq g(0, l_\star) - f(0, l_\star),$$

and that leads to a contradiction with (31). We conclude that for all  $\underline{\ell} \in (0, K)$ , for all  $(x, i) \in \mathcal{N}_R$  and for all  $l \in [\underline{\ell}, K]$ :

$$f_i(x, l) \geq g_i(x, l).$$

We conclude the proof using the continuity of  $f$  and  $g$  with respect to variable  $l$ .  $\square$

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