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► **To cite this version:**

| Jin Tan. Weak solutions of Moffatt's magnetic relaxation equations. 2023. hal-04314903v2

HAL Id: hal-04314903

<https://hal.science/hal-04314903v2>

Preprint submitted on 22 Jan 2024

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WEAK SOLUTIONS OF MOFFATT'S MAGNETIC RELAXATION EQUATIONS

JIN TAN

ABSTRACT. We investigate the global-in-time existence and uniqueness of weak solutions for a family of equations introduced by Moffatt to model magnetic relaxation. These equations are topology-preserving and admit all stationary solutions to the classical incompressible Euler equations as steady states. In the work [7] of Beekie, Friedlander and Vicol, global regularity results have been established for initial magnetic field $B_0 \in H^s(\mathbb{T}^d)$ ($s > d/2 + 1$) when the regularization parameter γ in the equations satisfies $\gamma > \gamma_c := d/2 + 1$. Global regularity for $\gamma \in [0, \gamma_c]$ is left as an open problem, as well as the existence of weak solutions with rough initial data for any $\gamma \geq 0$. In this paper, we show that for any solenoidal magnetic field $B_0 \in L^2(\mathbb{T}^d)$ there exists a unique global weak solution when $\gamma > \gamma_c$. Moreover, the solution can propagate higher-order Sobolev regularity. These results hold true for the borderline case $\gamma = \gamma_c$ only if $B_0 \in L^{2+}(\mathbb{T}^d)$. Finally, we discuss Moffatt's magnetic relaxation procedure.

In order to prove the existence and uniqueness of weak solutions, we establish a key compactness result for an active vector equation and provide some estimates for fractional Laplacian after the Lagrangian change of variables, which may have independent interest.

1. INTRODUCTION

In Arnolds seminal work [2], he developed a new set of geometric ideas for the classical incompressible Euler equations:

$$\partial_t u + u \cdot \nabla u + \nabla P = 0, \quad \operatorname{div} u = 0. \quad (1.1)$$

Since then there are numerous literature devoted to the subject of topological hydrodynamics, for example Ebin and Marsden [16], Holm et al. [18], Arnold and Khesin [4]. In the other hand, Arnold [3] suggested a process which demonstrates the existence of an Euler equilibria (i.e. steady state of system (1.1)) that has the same topological structure as an arbitrary divergence free magnetic field. The idea is to use the evolution dynamics of the magnetic field to reach an Euler equilibria which is topology-preserving (see Brenier [9] for an illustration). This concept was developed later by Moffatt [22]. The magnetic relaxation procedure envisioned by Moffatt formally preserves the streamline topology of an initial divergence free three-dimensional magnetic field, and whose solutions are conjectured to converge in the infinite time limit towards ideal Euler equilibria. There are other magnetic relaxation equations for which the steady states are incompressible Euler equilibria. We mention for instance the models of Vallis-Carnevale-Young[27] and Nishiyama

2010 *Mathematics Subject Classification.* 35Q35; 35D30; 35B40; 78A30.

Key words and phrases. Moffatt's magnetic relaxation equations, Euler/magnetostatic equilibria, weak solutions, global regularity.

[24]. For more details about topological aspects of fluid dynamics, see Moffatt's recent overview [23].

In this paper, we consider the Cauchy problem of the magnetic relaxation equations (MRE) proposed by Moffatt [22] on the flat torus $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ with $d \geq 2$:

$$\begin{cases} \partial_t B + u \cdot \nabla B = B \cdot \nabla u, \\ (-\Delta)^\gamma u = B \cdot \nabla B + \nabla P, \\ \operatorname{div} u = \operatorname{div} B = 0, \end{cases} \quad (1.2)$$

supplemented with the initial conditions

$$B(0, x) = B_0(x), \quad x \in \mathbb{T}^d. \quad (1.3)$$

The unknowns are the incompressible magnetic vector field B , the incompressible velocity vector field u which is taken to have zero mean on \mathbb{T}^d , and the scalar pressure P . The parameter $\gamma \geq 0$ is a regularization parameter of the constitutive law $B \mapsto u$:

$$u = (-\Delta)^{-\gamma} \mathbb{P}(B \cdot \nabla B) = (-\Delta)^{-\gamma} \mathbb{P} \operatorname{div}(B \otimes B), \quad (1.4)$$

where \mathbb{P} is the Leray projector onto divergence free vector fields.

First and foremost, let us observe from (1.2)₃ that for any smooth solution (B, u) of (1.2) we have

$$\frac{d}{dt} \|B(t, \cdot)\|_{L^2}^2 + 2\|u(t, \cdot)\|_{H^\gamma}^2 \leq 0. \quad (1.5)$$

From above energy inequality, it seems proper to define a distribution u via (1.2)₂. Note that in order to define weak solutions for (1.2)₁ a minimal requirement for $u(t, \cdot)$ is the square integrability, when $B(t, \cdot)$ is square integrable. By the Sobolev embedding and (1.2)₂, this would require $\gamma > d/4 + \frac{1}{2}$. The energy equality (1.5) comes to the rescue, it provides for any $\gamma \geq 0$ the required square integrability for both u and B in space. However, the existence of global weak solutions for (1.2) supplemented with rough initial data (e.g. merely bounded data) is challenging even when $\gamma > d/4 + \frac{1}{2}$, since energy inequality (1.5) does not yield robust compactness properties for the magnetic field to pass to the limit in the nonlinear term $B \cdot \nabla B$ in (1.2)₂. To our best knowledge, the only existence result with rough initial data is due to Brenier [9]. In the two dimensional case and with $\gamma = 0$, he obtained global-in-time dissipative weak solutions in the spirit of Lions' dissipative weak solutions to the Euler equations [26], a notion of solution which is weaker than the one of an usual weak solution, but which holds a weak-strong uniqueness property. On the other hand, let us mention Constantin and Pasqualotto [11] recently constructed Euler/magnetostatic equilibria from certain Voigt regularizations of the classical incompressible MHD system without resistivity. The regularization considered in [11] gives additional compactness for the magnetic field, which allowing pass to the limit in the expression $B \cdot \nabla B$, and obtain global-in-time solutions. However, the price for that Voigt type regularization is poor control on the topology of the limiting magnetic field ($t \rightarrow \infty$), because the topology of the magnetic field is not preserving along time evolution. We also mention the work [19] of Kim and Kwon on the global existence of weak solutions of Stokes-Magneto system with fractional diffusions.

At this stage, let us recall the results obtained for (1.2) in Beekie-Friedlander-Vicol [7]. Local-in-time existence and uniqueness of solutions were established for initial data B_0 in Sobolev spaces H^s with $s > d/2 + 1$, for any $\gamma \geq 0$. Global well-posedness result was obtained if further $\gamma > d/2 + 1$. The question of global regularity or finite time blowup for (1.2) when $\gamma \in [0, d/2 + 1]$ was listed as an open problem in [7]. In order to investigate Moffatt's conjecture on magnetic relaxation for (1.2), they proved that ∇u converges to 0 in L^∞ as time diverges, for $\gamma > d/2 + 1$. On the other hand, they showed that the Lipschitz norm of u is not integrable in the whole time interval \mathbb{R}_+ in the case $\gamma = 0$. Thus the relaxation towards Euler equilibria from (1.2) is not clear for general initial data. This fact motivates them established in [7, Theorem 5.1] the asymptotic stability of a special Euler equilibria $e_1 := (1, 0)^T$ under Sobolev smooth perturbations in the two dimensional case with $\gamma = 0$. And in the three dimensional case, they constructed an explicit example of a magnetic field $B(t, \cdot)$ which relaxes in L^2 as $t \rightarrow \infty$ to a steady solution B^∞ , but this limiting solution is not smooth and exhibits magnetic current sheets: the current $j(t, \cdot) = \nabla \times B$ grows as $t^{1/4}$ in L^2 , in other words the magnetic field $B(t, \cdot)$ is not uniformly (in time) bounded in H^1 . Moreover, motivated by the work of Elgindi and Masmoudi [17] on the Euler equations, they exhibited an example of current that grows exponentially in time. These examples suggest that for general initial data one cannot expect magnetic relaxation of (1.2) with respect to strong norms, at least for $\gamma = 0$.

The main goal of this paper is to establish existence, uniqueness and regularity property of global weak solutions for (1.2) subject to rough initial data when the regularization parameter $\gamma \geq d/2 + 1$. Before stating our main results, let us first give the following definition of weak solutions for Moffatt's MRE (1.2) with general $\gamma \geq 0$.

Definition 1.1. *Let $\gamma \geq 0$. Let initial data $B_0 \in L^2(\mathbb{T}^d)$ and $\operatorname{div} B_0 = 0$. A pair of vector field $(B, u)(t, x)$ is called a global weak solution to the Cauchy problem (1.2)-(1.3), if $(B, u) \in L^\infty(0, T; L^2(\mathbb{T}^d)) \times L^2(0, T; \dot{H}^\gamma(\mathbb{T}^d))$ for all $0 < T < \infty$, and it satisfies for almost every $0 < t < T$,*

$$\begin{aligned} \int_0^t \int_{\mathbb{T}^d} [B \cdot \partial_t v + (B \otimes u) : \nabla v - (u \otimes B) : \nabla v] dx d\tau \\ = \int_{\mathbb{T}^d} B(t, \cdot) \cdot v(t, \cdot) dx - \int_{\mathbb{T}^d} B_0 \cdot v(0, \cdot) dx, \end{aligned}$$

for any vector test function $v \in C^\infty([0, T] \times \mathbb{T}^d)$, and

$$\int_{\mathbb{T}^d} [u(t, \cdot) \cdot (-\Delta)^\gamma w + (B(t, \cdot) \otimes B(t, \cdot)) : \nabla w] dx = 0,$$

for any vector test function $w \in C^\infty(\mathbb{T}^d)$ with $\operatorname{div} w = 0$, and

$$\int_{\mathbb{T}^d} B(t, \cdot) \cdot \nabla \phi dx = \int_{\mathbb{T}^d} u(t, \cdot) \cdot \nabla \psi dx = 0,$$

for any scalar test functions $\phi, \psi \in C^\infty(\mathbb{T}^d)$.

Moreover, the energy inequality holds, namely,

$$\|B(t, \cdot)\|_{L^2}^2 + 2 \int_0^t \|u(\tau, \cdot)\|_{\dot{H}^\gamma}^2 d\tau \leq \|B_0\|_{L^2}^2. \quad (1.6)$$

1.1. Main results. Now, we are ready to state our main results. The first one address the global regularity for Moffatt's MRE (1.2) with borderline regularization parameter $\gamma = d/2 + 1$. It reads

Theorem 1.2. *Let $s > d/2 + 1$ and $\gamma = \gamma_c := d/2 + 1$. Consider any initial data $B_0 \in H^s$ such that $\operatorname{div} B_0 = 0$. Then, there exists a unique solution $B \in \mathcal{C}([0, \infty); H^s)$ with associated velocity $u \in \mathcal{C}([0, \infty); H^{s-1+2\gamma_c}) \cap L^2(0, \infty; H^{s+\gamma_c})$. Moreover, the pair (B, u) satisfies the following energy equality:*

$$\|B(t, \cdot)\|_{L^2}^2 + 2 \int_{t_0}^t \|u(\tau, \cdot)\|_{H^{\gamma_c}}^2 d\tau = \|B(t_0, \cdot)\|_{L^2}^2, \quad (1.7)$$

for all $0 \leq t_0 \leq t < \infty$, and also the bound:

$$\begin{aligned} \|B(t, \cdot)\|_{H^s}^2 + \int_0^t \|u(\tau, \cdot)\|_{H^{s+\gamma_c}}^2 d\tau \\ \leq \|B_0\|_{H^s}^2 e^{e^{C(1+t\|B_0\|_{H^s}^2)} e^{C\{(1+t\|B_0\|_{L^\infty}^2)\}^3} e^{C\sqrt{t}\|B_0\|_{L^2}}}, \end{aligned} \quad (1.8)$$

where C is a positive constant depends only on s and d .

The next theorem shows the existence and uniqueness of weak solutions in the sense of Definition 1.1 whenever the regularization parameter satisfying $\gamma \geq \gamma_c$.

Theorem 1.3. *Let $\gamma \geq \gamma_c$. Consider any initial data $B_0 \in L^2(\mathbb{T}^d)$ such that $\operatorname{div} B_0 = 0$. It holds that*

- (i) *In the case $\gamma > \gamma_c$, the Cauchy problem (1.2)–(1.3) admits a unique global weak solution $(B, u) \in \mathcal{C}([0, \infty); L^2) \times L^\infty(\mathbb{R}_+; H^\gamma)$, and such that energy inequality (1.6) becomes an equality.*
- (ii) *In the case $\gamma = \gamma_c$, if in addition $B_0 \in L^{2+\varepsilon}$ for any $\varepsilon > 0$, then the Cauchy problem (1.2)–(1.3) admits a unique global weak solution $(B, u) \in (\mathcal{C}([0, \infty); L^2) \cap L^\infty(0, \infty; L^{2+\varepsilon})) \times L^\infty(0, \infty; W^{1, \infty})$, which satisfies energy equality (1.7) and the bound:*

$$\|B(t, \cdot)\|_{L^{2+\varepsilon}}^2 \leq e^{\{C(1+t\|B_0\|_{L^{2+\varepsilon}}^2)\} e^{C\sqrt{t}\|B_0\|_{L^2}}}, \quad (1.9)$$

for almost every $t \in [0, \infty)$, where C is a positive constant depends only on d and $1/(2 + \varepsilon)$.

In the last, we investigate Moffatt's magnetic relaxation procedure. This is a subtle matter at least for the case $\gamma = 0$, as we can see from Beekie-Friedlander-Vicol's examples. One may interested in the case of large γ and the possible asymptotic behavior of the solutions obtained in Theorem 1.3. The result below shows the relaxation of the velocity field of these solutions as $t \rightarrow \infty$.

Corollary 1.4. *Let (B, u) be the weak solutions obtained in Theorem 1.3 (i), then*

$$\lim_{t \rightarrow \infty} \|u(t, \cdot)\|_{H^\alpha} = 0, \quad \text{for all } 0 \leq \alpha < 2\gamma - \gamma_c. \quad (1.10)$$

Also, in Theorem 1.3 (ii)

$$\lim_{t \rightarrow \infty} \|u(t, \cdot)\|_{H^\beta} = 0, \quad \text{for all } 0 \leq \beta < \gamma_c. \quad (1.11)$$

However, we are not sure that the above magnetic field $B(t, \cdot)$ will relax to a steady state B^∞ of the Euler equations (1.1) as $t \rightarrow \infty$. We note that the constitutive law (1.4) endows a cubic nature of the nonlinear term in the magnetic equation (1.2)₁, which may yields potential growth of the magnetic field (see (1.8) and (1.9)). As such, Moffatt's magnetic relaxation procedure seems mysterious even for large γ .

In the next, we manage to identify the limiting profiles of Moffatt's magnetic relaxation procedure for a class of initial magnetic fields. For that purpose, we recall the following theorem from [7, Theorem 5.1].

Theorem 1.5 ([7]). *Let $d = 2$. Let $k \geq 4$ and $m \geq k + 9$. Choose $\delta \in (0, 1)$. There exists small enough η_0 such that for any solenoidal vector field B_0 if $\|B_0 - e_1\|_{H^m} = \eta \leq \eta_0$, and $\mathbb{P}_0(B_0 - e_1) = 0$, then we have that (1.2)-(1.3) has a unique global in time solution (B, u) , which satisfies $\|B - e_1\|_{L^2} \leq \eta$, $\mathbb{P}_0 B^2 \equiv 0$ and*

$$\|\mathbb{P}_\perp(B - e_1)\|_{\dot{H}^k} \leq 4\eta e^{-(1-\delta)t} \quad (1.12)$$

$$\|\mathbb{P}_0(B^1 - 1)\|_{H^{k+2}} \leq 4\eta \quad (1.13)$$

$$\|B - e_1\|_{\dot{H}^m}^2 \leq 4\eta e^{\eta t} \quad (1.14)$$

for $t \in [0, \infty)$. In above, for any function $\psi : \mathbb{T}^2 \rightarrow \mathbb{R}$, we used the notations

$$\begin{aligned} (\mathbb{P}_0\psi)(x_2) &:= \frac{1}{|\mathbb{T}|} \int_{\mathbb{T}} \psi(x_1, x_2) dx_1, \\ (\mathbb{P}_\perp)\psi(x_1, x_2) &:= \psi(x_1, x_2) - (\mathbb{P}_0\psi)(x_2). \end{aligned}$$

As a consequence, the velocity field satisfies $u(t, \cdot) \rightarrow 0$ as $t \rightarrow \infty$, whereas the magnetic field $B(t, \cdot)$ relaxes to a steady state B^∞ with $\|B^\infty - e_1\|_{H^{k+2}} \leq 4\eta$, both convergences taking place with respect to strong topologies.

Then we have

Corollary 1.6. *The steady state B^∞ obtained in Theorem 1.5 is a vector-valued function of the single variable $x_2 \in \mathbb{T} = [0, 1)$ (that corresponds to a shear flow of the Euler equations) given by¹*

$$B^\infty(x_2) = (-\dot{\phi}^\infty(x_2), 0)$$

with

$$\phi^\infty(x_2) := \int_0^\infty 1_{\{|\phi_0| > \lambda\}}|_{\geq x_2}(\lambda) d\lambda \quad \text{for } x_2 \in \mathbb{T},$$

(i.e. $\phi^\infty(x_2) = \inf\{\lambda \in [0, \infty) : |\{|\phi_0| > \lambda\}| \geq x_2\}$) where $\phi_0 : \mathbb{T}^2 \rightarrow \mathbb{R}$ is defined by $\phi_0 := \Delta^{-1}\nabla^\perp B_0$ such that $(\mathbb{P}_0\phi_0)(x_2) = -x_2$, with $\nabla^\perp := (-\partial_2, \partial_1)$.

Proof. The proof is essentially motivated by a recent work [12] of Dalibard-Guillod-Leblond. In the two-dimensional case, notice that for initial magnetic field B_0 satisfied $\mathbb{P}_0(B_0 - e_1) = 0$ we may identify a unique scalar magnetic stream function $\phi = \Delta^{-1}\nabla^\perp \cdot B$, which satisfies the active scalar equation

$$\begin{cases} \partial_t \phi + u \cdot \nabla \phi = 0 \\ \phi(t, x)|_{t=0} = \phi_0. \end{cases} \quad (1.15)$$

¹We use dots for differentiation with spatial variable x_2 .

Since $B = \nabla^\perp \phi$, we see from (1.12) and $\mathbb{P}_0 B^2 \equiv 0$ that

$$\|\partial_1 \phi(t, \cdot)\|_{\dot{H}^k} = \|\mathbb{P}_\perp B^2\|_{\dot{H}^k} \leq 4\eta e^{-(1-\delta)t} \rightarrow 0, \quad \text{as } t \rightarrow \infty.$$

Then thanks to Theorem 1.5 we know that $\phi(t, \cdot) \rightarrow \phi^*$ in strong topologies as $t \rightarrow \infty$ for some $\phi^* = \phi^*(x_2)$ satisfying

$$\|\dot{\phi}^* + 1\|_{H^{k+2}} \leq 4\eta < 1.$$

In other words, ϕ^* is a vertical decreasing rearrangement² of ϕ_0 . The classical rearrangement theory e.g. in [21, Chapter 3] tells us ϕ^∞ is the unique vertical decreasing rearrangement of ϕ_0 , therefore $\phi^* = \phi^\infty$. \square

1.2. A few remarks on the main results. First, Theorem 1.2 answered a borderline case of the open problem \mathbf{Q}_1 that proposed in [7, Section 7]. Notice that the global regularity problem of the borderline case $\gamma = \gamma_c$ is not accessible by the methods of [7]. Indeed, in this case, the energy inequality (1.5) does not ensure the velocity field is Lipschitz continuous but only log-Lipschitz continuous, globally-in-time. To deal with this critical situation, we take use of some known logarithmic interpolation inequalities and observe that the $L^{2+\varepsilon}$ norm of the magnetic field can be controlled by its initial norm through a double exponential inequality, see (1.9). As a consequence, we obtain global-in-time bound of Lipschitz norm of the velocity field in (2.13). Still, with these new bounds do not ensure previous known regularity criteria apply. In order to obtain global regularity, we provide a refined Beal-Kato-Majda type regularity criteria i.e. Proposition 2.1.

Second, Theorem 1.3 partially answered the open problem \mathbf{Q}_4 that proposed in [7, Section 7], which concerns the existence of weak solutions for $\gamma \geq 0$ and the validity of energy inequality (1.6) for $\gamma \in (d/4 + 1/2, d/2 + 1]$. As a consequence of Theorem 1.3, the existence of weak solutions for $\gamma \in [0, d/2 + 1)$ remains open. We note that the essential difficulty for proving existence of a weak solution for MRE (1.2) is the lack of robust compactness property of the magnetic field, which can not obtained directly from energy inequality (1.5) for any $\gamma \geq 0$. To obtain strong convergence property of the magnetic field, we establish a key compact result for the following linear active vector equations

$$\begin{cases} \partial_t A_n + \operatorname{div}(A_n \otimes v_n) = \operatorname{div}(v_n \otimes A_n), \\ \operatorname{div} A_n = \operatorname{div} v_n = 0. \end{cases} \quad (1.16)$$

The result stated in Lemma 3.1 maybe understood as a generalization of Lions' fundamental compactness result [26] for scalar transport equation.

A great deal of our analysis is based on the Littlewood-Paley decomposition and Besov spaces (see Appendix A for definitions), because we found this framework is convenient for studying the well-posedness issue of rough solutions for Moffatt's MRE (1.2). Indeed, noticing that the nonlinear term $B \otimes B$ may only lies in L^1 and it is well-known that the Calderón-Zygmund type operator is unbounded in L^1 (nor L^∞), so one cannot apply standard L^1 estimate to the Stokes type equation which is satisfied by the velocity field. About the proof of uniqueness, notice that the magnetic field satisfies an active vector equation, it naturally causes losing

²We say that two maps $\phi_1, \phi_2 : \mathbb{T}^d \rightarrow \mathbb{R}$ are rearrangement of each other if $|\{\phi_1 > \lambda\}| = |\{\phi_2 > \lambda\}|$ for all $\lambda \geq 0$.

of one derivative when performing estimates for the difference of two solutions. Thus we handle the uniqueness issue by the Lagrangian approach, and it relies on some estimates in Besov spaces (instead of L^1) for related Stokes system. Let us mention that after transforming the problem in Lagrangian coordinates, the difference analysis is not standard for the case $\gamma \notin \mathbb{N}$. Indeed, since the fractional Laplacian operator is *nonlocal* we have to formulate an associated operator in the Lagrangian coordinates, and provide some important estimates for such operator.

From a mathematical perspective, the analysis of the MRE (1.2) is quite interesting and challenging. On the one hand, the constitutive law (1.4) together with energy inequality (1.6) connotes a yet to be discovered (nonlinear) dissipative mechanism of the magnetic field, see the linearization near e_1 in [7]. Regarding this topic, we consider the propagation of higher-order Sobolev regularity of weak solutions along time evolution. In fact, it can be easily seen from estimate (2.11) that, in Theorem 1.3 if in addition $B_0 \in H^s$ ($s > d/2$) then $B \in L^\infty(0, \infty; H^s)$ and $u \in L^\infty(0, \infty; H^{s-1+2\gamma})$. Concerning regularity of the pressure and more regularity properties of the weak solutions, see Remark 1. On the other hand, MRE (1.2) shares some similarities with the incompressible Euler equations (1.1) in vorticity formulation and its solutions are conjectured asymptotically converge to the steady states of (1.1) as time diverges. In this aspect, Corollary 1.6 seems the first result that provide non-trivial explicit limiting profiles in the context of magnetic relaxation problem, which may helps us understand better about Moffatt's magnetic relaxation procedure.

After the completion of the current manuscript (arXiv:2311.18407), we were informed that the global regularity result for the borderline case $\gamma = \gamma_c$ was obtained recently in Bae-Kwon-Shin [5] (arXiv:2310.03255). Comparing to their result, the bounds (1.8), (2.12), (2.13) and Proposition 2.1 in the present manuscript are new. More importantly, the main contribution of our results lies on the existence and uniqueness of weak solutions for any (almost) finite energy initial data.

Notation. We end this introductory part by presenting a few notations. We denote by C harmless positive ‘constants’ that may change from one line to the other, and we sometimes write $A \lesssim B$ instead of $A \leq CB$. Likewise, $A \sim B$ means that $C_1B \leq A \leq C_2B$ with absolute constants C_1, C_2 . Throughout the paper, i -th coordinate of a vector v will be denoted by v^i . For a real-valued matrix M , M^T represents its transpose, while for two multidimensional real-valued matrices M_1, M_2 , $M_1 : M_2$ denotes their standard inner product. We use the notation $\Lambda := (-\Delta)^{1/2}$, and $[\mathbb{A}, \mathbb{B}] := \mathbb{A}\mathbb{B} - \mathbb{B}\mathbb{A}$ for the commutator of two operators. For \mathcal{X} a Banach space, $p \in [1, \infty]$ and $T \in (0, \infty]$, the notation $L^p(0, T; \mathcal{X})$ designates the set of measurable functions $f : [0, T] \rightarrow \mathcal{X}$ with $t \mapsto \|f(t)\|_{\mathcal{X}}$ in $L^p(0, T)$, endowed with the norm $\|\cdot\|_{L^p(0, T; \mathcal{X})}$. For any interval I of \mathbb{R} , we agree that $\mathcal{C}(I; \mathcal{X})$ denotes the set of continuous functions from I to \mathcal{X} . We keep the same notation for functions with several components.

The rest of the paper unfolds as follows. In the next section, we focus on the proof of Theorem 1.2. Section 3 is devoted to the proof of Theorem 1.3 and Corollary 1.4. For the reader's convenience, results concerning Besov spaces, LittlewoodPaley decomposition on \mathbb{T}^d are recalled in Appendix A. Finally, in Appendix B we prove Lemma 3.1.

2. GLOBAL REGULARITY FOR THE BORDERLINE CASE $\gamma = d/2 + 1$

This section is devoted to the proof of Theorem 1.2. To do that, at first we need the following Beale-Kato-Majda type regularity criteria.

2.1. A refined regularity criteria for $\gamma > d/2$.

Proposition 2.1. *Let $\gamma > d/2, s > d/2 + 1$. Assume that the initial magnetic field $B_0 \in H^s$ and $\operatorname{div} B_0 = 0$. Then, there exists a time $T^* = T^*(\|B_0\|_{H^s})$, such that the problem (1.2)–(1.3) has a unique solution $B \in \mathcal{C}([0, T^*]; H^s)$ with associated velocity $u \in \mathcal{C}([0, T^*]; H^{s-1+2\gamma}) \cap L^2(0, T^*; H^{s+\gamma})$. Moreover, it holds that*

$$\|B(t, \cdot)\|_{H^s}^2 \leq \|B_0\|_{H^s}^2 e^{C(1+t\|B_0\|_{H^s}^2)e^{CV(t)}}, \quad (2.1)$$

for $t \in [0, T^*]$, where C is a positive constant depending only on γ, s, d , and

$$V(t) := \int_0^t (\|\nabla u(\tau, \cdot)\|_{L^\infty} + \|\Lambda u(\tau, \cdot)\|_{L^\infty}) d\tau.$$

Proof. Thanks to [7, Theorem 2.2], we only need to prove the bound (2.1). Applying the operator Λ^r ($r > 0$) to both sides of MRE (1.2) and testing the resulting equations with $\Lambda^r B$, $\Lambda^r u$, respectively, we then obtain that

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|B(t, \cdot)\|_{\dot{H}^r}^2 + \|u(t, \cdot)\|_{\dot{H}^{r+\gamma}}^2 \\ &= \langle [\Lambda^r, B \cdot \nabla] u, \Lambda^r B \rangle - \langle [\Lambda^r, u \cdot \nabla] B, \Lambda^r B \rangle + \langle [\Lambda^r, B \cdot \nabla] B, \Lambda^r u \rangle. \end{aligned} \quad (2.2)$$

Recalling the following Kato-Ponce type commutator estimates from Lemma A.5: for all $r > 0$,

$$\|[\Lambda^r f, g]\|_{L^2} \lesssim \|\Lambda^r f\|_{L^2} \|g\|_{L^\infty} + \|\nabla f\|_{L^2} \|\Lambda^{r-1} g\|_{L^\infty} \quad (2.3)$$

and

$$\|[\Lambda^r f, g]\|_{L^2} \lesssim \|\Lambda^r f\|_{L^\infty} \|g\|_{L^2} + \|\nabla f\|_{L^\infty} \|\Lambda^{r-1} g\|_{L^2}. \quad (2.4)$$

Now, use (2.3) and the Sobolev embedding $H^\gamma \hookrightarrow L^\infty$ (since $\gamma > d/2$), we have

$$\begin{aligned} |\langle [\Lambda^r, B \cdot \nabla] u, \Lambda^r B \rangle| &\lesssim \|[\Lambda^r, B \cdot \nabla] u\|_{L^2} \|\Lambda^r B\|_{L^2} \\ &\lesssim (\|\Lambda^r B\|_{L^2} \|\nabla u\|_{L^\infty} + \|\nabla B\|_{L^2} \|\Lambda^{r-1} \nabla u\|_{L^\infty}) \|B\|_{\dot{H}^r} \\ &\lesssim (\|B\|_{\dot{H}^r} \|\nabla u\|_{L^\infty} + \|\nabla B\|_{L^2} \|u\|_{H^{r+\gamma}}) \|B\|_{\dot{H}^r}. \end{aligned} \quad (2.5)$$

Similarly, (2.4) yields that

$$\begin{aligned} |\langle [\Lambda^r, u \cdot \nabla] B, \Lambda^r B \rangle| &\lesssim \|[\Lambda^r, u \cdot \nabla] B\|_{L^2} \|\Lambda^r B\|_{L^2} \\ &\lesssim (\|\Lambda^r u\|_{L^\infty} \|\nabla B\|_{L^2} + \|\nabla u\|_{L^\infty} \|\Lambda^{r-1} \nabla B\|_{L^2}) \|B\|_{\dot{H}^r} \\ &\lesssim (\|u\|_{H^{r+\gamma}} \|\nabla B\|_{L^2} + \|\nabla u\|_{L^\infty} \|B\|_{\dot{H}^r}) \|B\|_{\dot{H}^r}. \end{aligned} \quad (2.6)$$

We decompose the third term in (2.2), by the fact that $\operatorname{div} B = 0$, into

$$\langle [\Lambda^r, B \cdot \nabla] B, \Lambda^r u \rangle = -\langle \Lambda^r (B \otimes B), \nabla \Lambda^r u \rangle + \langle B \cdot \nabla \Lambda^r u, \Lambda^r B \rangle.$$

Then, Lemma A.6 implies that

$$\begin{aligned} |\langle \Lambda^r (B \otimes B), \nabla \Lambda^r u \rangle| &\lesssim \|B \otimes B\|_{\dot{H}^r} \|u\|_{\dot{H}^{r+1}} \\ &\lesssim \|B\|_{H^r} \|B\|_{L^\infty} \|u\|_{\dot{H}^{r+1}}, \end{aligned}$$

and Hölder's inequality implies that

$$|\langle B \cdot \nabla \Lambda^r u, \Lambda^r B \rangle| \lesssim \|B\|_{L^\infty} \|\nabla \Lambda^r u\|_{L^2} \|\Lambda^r B\|_{L^2}$$

$$\lesssim \|B\|_{L^\infty} \|u\|_{\dot{H}^{r+1}} \|B\|_{\dot{H}^r}.$$

One finds from above two inequalities and $\gamma > d/2 \geq 1$ that

$$|\langle [\Lambda^r, B \cdot \nabla] B, \Lambda^r u \rangle| \lesssim \|B\|_{L^\infty} \|B\|_{H^r} \|u\|_{H^{r+\gamma}}. \quad (2.7)$$

Putting inequalities (2.5)-(2.7) and energy inequality (1.5) into inequality (2.2), and using Young's inequality and the fact that u has zero mean, it gives that

$$\frac{d}{dt} \|B\|_{H^r}^2 + \|u\|_{H^{r+\gamma}}^2 \leq C \|B\|_{H^r}^2 (\|\nabla u\|_{L^\infty} + \|\nabla B\|_{L^2}^2 + \|B\|_{L^\infty}^2). \quad (2.8)$$

At this stage, we have to estimate $\|\nabla B\|_{L^2}$ and $\|B\|_{L^\infty}$. In fact, in the case $r = 1$, from (2.7) and the last two inequality in (2.5) and (2.6), we can improve (2.8) as

$$\frac{d}{dt} \|B\|_{H^1}^2 + \|u\|_{H^{1+\gamma}}^2 \lesssim \|B\|_{H^1}^2 (\|\nabla u\|_{L^\infty} + \|\Lambda u\|_{L^\infty} + \|B\|_{L^\infty}^2). \quad (2.9)$$

Applying the maximal principal for magnetic equation (1.2)₁ yields that

$$\begin{aligned} \|B(t, \cdot)\|_{L^\infty} &\leq \|B_0\|_{L^\infty} \exp\left(C \int_0^t \|\nabla u(\tau, \cdot)\|_{L^\infty} d\tau\right) \\ &\leq \|B_0\|_{L^\infty} \exp(CV(t)), \end{aligned} \quad (2.10)$$

which together with (2.9) imply that

$$\begin{aligned} \|B(t, \cdot)\|_{H^1}^2 + \|B(t, \cdot)\|_{L^\infty}^2 \\ \leq (\|B_0\|_{H^1}^2 + \|B_0\|_{L^\infty}^2) \exp\{C(1+t\|B_0\|_{L^\infty}^2) \exp(CV(t))\}. \end{aligned}$$

Finally, taking use of the above inequality and (2.8) we get for any $r > 0$,

$$\|B(t, \cdot)\|_{H^r}^2 \leq \|B_0\|_{H^r}^2 \exp\{\exp\{C(1+tC_0) \exp(CV(t))\}\}, \quad (2.11)$$

with $C_0 = \|B_0\|_{H^1}^2 + \|B_0\|_{L^\infty}^2$.

Since $s > d/2$, one has $C_0 \lesssim \|B_0\|_{H^s}^2$. Thus by taking $r = s$ in (2.11) we complete the proof of Proposition 2.1. \square

2.2. The global regularity. Now, we are ready to prove Theorem 1.2. Thanks to Proposition 2.1, we only need to obtain the boundness of $V(t)$ for any $t > 0$. But, in the case $\gamma = \gamma_c$ energy inequality (1.5) does not give the bound directly, since the failure of the Sobolev embedding $H^{\gamma_c-1} \hookrightarrow L^\infty$. To deal with this difficulty, we show global-in-time bound of the magnetic field in L^p for all $p > 2$, which in turn gives the desired estimate of u thanks to the constitutive law (1.4).

Proposition 2.2. *Let $\gamma = \gamma_c$. Let (B, u) be a smooth solution for the MRE (1.2), then for all $p \in (2, \infty]$, it holds that*

$$\|B(t, \cdot)\|_{L^p}^2 \leq \exp\{C(1+t + \|B_0\|_{L^p}^2) \exp(C\sqrt{t}\|B_0\|_{L^2})\}, \quad (2.12)$$

and

$$\int_0^t \|(\nabla u, \Lambda u)(\tau, \cdot)\|_{L^\infty} d\tau \leq C \left((1+t + \|B_0\|_{L^p}^2)^3 \exp(C\sqrt{t}\|B_0\|_{L^2}) \right), \quad (2.13)$$

for all $0 \leq t < \infty$, where C is a positive constant depends only on d and $1/p$.

Proof. Let us first discuss for the case of general γ and p . Taking advantage of the Besov spaces (for definitions, see Appendix A), we see from the constitute law (1.4) that for all $\gamma \geq 0$ and $p \in [2, \infty]$,

$$\begin{aligned} \|u\|_{\dot{B}_{\infty,\infty}^{2\gamma-1-\frac{2d}{p}}} &\lesssim \|u\|_{\dot{B}_{p/2,\infty}^{2\gamma-1}} \\ &\lesssim \|\Lambda^{-1}\mathbb{P}\text{div}(B \otimes B)\|_{\dot{B}_{p/2,\infty}^0} \\ &\lesssim \|B \otimes B\|_{L^{p/2}} \lesssim \|B\|_{L^p}^2, \end{aligned} \quad (2.14)$$

where we used the properties that $L^{p/2} \hookrightarrow \dot{B}_{p/2,\infty}^0 \hookrightarrow \dot{B}_{\infty,\infty}^{-2d/p}$ and the operator $\Lambda^{-1}\mathbb{P}\text{div}$ is bounded on $\dot{B}_{p/2,\infty}^0$ from Proposition A.3.

Whenever $2\gamma - 1 - 2d/p > 1$, the logarithmic interpolation inequality from Lemma A.4 together with inequality (2.14) imply that

$$\begin{aligned} \|u\|_{\dot{B}_{\infty,1}^1} &\lesssim 1 + \|u\|_{\dot{B}_{\infty,\infty}^1} \log \left(e + \|u\|_{\dot{B}_{\infty,\infty}^1} + \|u\|_{\dot{B}_{\infty,\infty}^{2\gamma-1-\frac{2d}{p}}} \right) \\ &\lesssim 1 + \|u\|_{\dot{B}_{\infty,\infty}^1} \log \left(e + \|u\|_{\dot{B}_{\infty,\infty}^1} + \|B\|_{L^p}^2 \right). \end{aligned} \quad (2.15)$$

Thus in our setting, i.e. $\gamma = \gamma_c = d/2 + 1$ with $p > 2$, we know that inequality (2.15) is satisfied. And thanks to the embedding $\dot{H}^{\gamma_c} \hookrightarrow \dot{B}_{\infty,\infty}^1$, (2.15) can be further rewritten as

$$\|u\|_{\dot{B}_{\infty,1}^1} \lesssim 1 + \|u\|_{\dot{H}^{\gamma_c}}^2 + \|u\|_{\dot{H}^{\gamma_c}} \log(e + \|B\|_{L^p}^2). \quad (2.16)$$

Taking advantage of energy inequality (1.5), that is

$$\int_0^\infty \|u(t, \cdot)\|_{\dot{H}^{\gamma_c}}^2 dt \leq \|B_0\|_{L^2}^2, \quad (2.17)$$

it remains to estimate $\|B\|_{L^p}$. In fact, similar to (2.10), by the standard L^p -estimate of transport equation, one has

$$\begin{aligned} \|B(t, \cdot)\|_{L^p}^2 &\leq \|B_0\|_{L^p}^2 \exp \left(C \int_0^t \|\nabla u(\tau, \cdot)\|_{L^\infty} d\tau \right) \\ &\leq \|B_0\|_{L^p}^2 \exp \left(C \int_0^t \|u(\tau, \cdot)\|_{\dot{B}_{\infty,1}^1} d\tau \right). \end{aligned}$$

This combined with (2.16) gives that

$$\begin{aligned} &\|B(t, \cdot)\|_{L^p}^2 \\ &\leq \|B_0\|_{L^p}^2 \exp \left(C \int_0^t (1 + \|u(\tau, \cdot)\|_{\dot{H}^{\gamma_c}}^2 + \|u(\tau, \cdot)\|_{\dot{H}^{\gamma_c}} \log(e + \|B(\tau, \cdot)\|_{L^p}^2)) d\tau \right). \end{aligned}$$

Define $X(t) := \log(e + \|B(t, \cdot)\|_{L^p}^2)$, the above inequality recasts to

$$X(t) \leq X(0) + C \int_0^t (1 + \|u(\tau, \cdot)\|_{\dot{H}^{\gamma_c}}^2 + \|u(\tau, \cdot)\|_{\dot{H}^{\gamma_c}} X(\tau)) d\tau.$$

Gronwall's lemma then implies that

$$X(t) \leq \left(X(0) + C \int_0^t (1 + \|u(\tau, \cdot)\|_{\dot{H}^{\gamma_c}}^2) d\tau \right) \exp \left(C \int_0^t \|u(\tau, \cdot)\|_{\dot{H}^{\gamma_c}} d\tau \right). \quad (2.18)$$

In virtue of (2.17), we obtain from (2.18) that

$$X(t) \leq C(X(0) + t + \|B_0\|_{L^2}^2) \exp(C\sqrt{t}\|B_0\|_{L^2}),$$

which gives (2.12). Finally, use Young's inequality, (2.18) and (2.17) again, we infer from (2.16) that

$$\int_0^t \|u(\tau, \cdot)\|_{\dot{B}_{\infty,1}^1} d\tau \leq C \left((1+t + \|B_0\|_{L^p}^2)^3 \exp(C\sqrt{t}\|B_0\|_{L^2}) \right).$$

It completes the proof of Proposition 2.2. \square

Proof of Theorem 1.2. Estimate (2.1) shows that the local-in-time H^s solution obtained in Proposition 2.1 may be uniquely continued past to time T , if $\|(\nabla u, \Lambda u)(t, \cdot)\|_{L^\infty}$ is integrable on $[0, T]$. Moreover, the a priori estimates in Proposition 2.2 give that $\|(\nabla u, \Lambda u)\|_{L^1(0,T;L^\infty)}$ is bounded only in terms of T and norm $\|B_0\|_{L^p}$. Thus the global existence of H^s solutions is obtained by contradiction argument. Taking into account the bounds (2.1) and (2.13), we obtain the bound (1.8). The proof of energy equality (1.7) is classic, we omit its details here. \square

3. WEAK SOLUTIONS OF MOFFATT'S MRE FOR $\gamma \geq d/2 + 1$

This section mainly concerns the proof of existence and uniqueness of weak solutions of MRE (1.2) for $\gamma \geq \gamma_c$. Because the initial data is rough, the energy inequality (1.5) or inequality (2.12) do not provide robust compactness properties of the magnetic field by classic compactness argument. To show strong convergence property of the magnetic field, we develop a compactness result for the corresponding linearized equation, which is motivated by Lions' well-known renormalization argument [26] for transport equation.

Lemma 3.1. *Let $d = 2, 3$ and $T > 0$. Given a pair of sequence $\{(A_n, v_n)\}_{n \in \mathbb{N}}$ satisfying*

$$A_n \in \mathcal{C}([0, T]; L^2(\mathbb{T}^d)), v_n \in L^2(0, T; H^1(\mathbb{T}^d)) \cap L^1(0, T; W^{1,\infty}(\mathbb{T}^d)),$$

and

$$\|A_n\|_{L^\infty(0,T;L^2(\mathbb{T}^d))} + \|v_n\|_{L^2(0,T;H^1(\mathbb{T}^d))} + \|v_n\|_{L^1(0,T;W^{1,\infty}(\mathbb{T}^d))} \leq M, \quad (3.1)$$

for some constant M independent of n . Assume that

$$\begin{cases} \partial_t A_n + \operatorname{div}(A_n \otimes v_n) = \operatorname{div}(v_n \otimes A_n) & \text{in } \mathcal{D}'((0, T) \times \mathbb{T}^d), \\ \operatorname{div} A_n = \operatorname{div} v_n = 0 & \text{in } \mathcal{D}'((0, T) \times \mathbb{T}^d), \\ A_n|_{t=0} = A_{0,n}, \end{cases}$$

and

$$\begin{aligned} A_{0,n} &\rightarrow A_0 \quad \text{strongly in } L^2(\mathbb{T}^d), \\ v_n &\rightarrow v \quad \text{strongly in } L^1(0, T; W^{1,\infty}(\mathbb{T}^d)). \end{aligned}$$

Then, up to extraction, A_n converges strongly in $\mathcal{C}([0, T]; L^2(\mathbb{T}^d))$ to a function $A \in \mathcal{C}([0, T]; L^2(\mathbb{T}^d))$, which is the unique solution of the problem

$$\begin{cases} \partial_t A + \operatorname{div}(A \otimes v) = \operatorname{div}(v \otimes A) & \text{in } \mathcal{D}'((0, T) \times \mathbb{T}^d), \\ \operatorname{div} A = \operatorname{div} v = 0 & \text{in } \mathcal{D}'((0, T) \times \mathbb{T}^d), \\ A|_{t=0} = A_0 & \text{a.e. in } \mathbb{T}^d. \end{cases} \quad (3.2)$$

The proof of Lemma 3.1 is given in Appendix B.

3.1. Proof of existence. In order to prove the existence part of Theorem 1.3, we shall resort to the following procedure:

- (1) smooth out the initial data and get a sequence $\{(B_n, u_n)\}_{n \in \mathbb{N}}$ of global-in-time smooth solutions to MRE (1.2);
- (2) use compactness results to prove that $\{(B_n, u_n)\}_{n \in \mathbb{N}}$ converges, up to extraction, to a solution of MRE (1.2) supplemented with initial data B_0 ;
- (3) prove energy inequality.

For expository purposes, we focus on the case $\gamma = \gamma_c$, i.e. statement (ii), just explaining at the end of the subsection what has to be modified to handle the case $\gamma > \gamma_c$. To proceed, let us smooth out the initial data $B_0 \in L^{2+\varepsilon}$ such that

$$B_{0,n} \in C^\infty(\mathbb{T}^d), \quad B_{0,n} \rightarrow B_0 \quad \text{strongly in } L^{2+\varepsilon}(\mathbb{T}^d). \quad (3.3)$$

Then, in light of the global regularity result stated in Theorem 1.2 there exists a unique global smooth solution (B_n, u_n) satisfying

$$\begin{cases} \partial_t B_n + \operatorname{div}(B_n \otimes u_n) = \operatorname{div}(u_n \otimes B_n), \\ (-\Delta)^{\gamma_c} u_n = \operatorname{div}(B_n \otimes B_n) + \nabla P_n, \\ \operatorname{div} u_n = \operatorname{div} B_n = 0, \end{cases} \quad (3.4)$$

corresponding to initial data $B_{0,n}$. Being smooth, the pair (B_n, u_n) satisfies all the a priori estimates in Proposition 2.2, and thus in particular for all $0 < T < \infty$,

$$\sup_{t \in [0, T]} \|B_n(t, \cdot)\|_{L^{2+\varepsilon}}^2 \leq \exp\{C(1 + T + \|B_0\|_{L^{2+\varepsilon}}^2) \exp(C\sqrt{T}\|B_0\|_{L^2})\}, \quad (3.5)$$

$$\begin{aligned} \int_0^T \|(\nabla u_n, \Lambda u_n)(\tau, \cdot)\|_{L^\infty} d\tau \\ \leq C \left((1 + T + \|B_0\|_{L^{2+\varepsilon}}^2)^3 \exp(C\sqrt{T}\|B_0\|_{L^2}) \right). \end{aligned} \quad (3.6)$$

And also,

$$\|B_n\|_{L^\infty(\mathbb{R}_+; L^2)}^2 + \|u_n\|_{L^2(\mathbb{R}_+; \dot{H}^{\gamma_c})}^2 \leq \|B_0\|_{L^2}^2. \quad (3.7)$$

In the next, we are going to derive strong convergence property of $\{u_n\}_{n \in \mathbb{N}}$. To do that, we first need to derive the evolution equation of the velocity. Noticing that the matrix $B_n \otimes B_n$ satisfies (see [7, Page 1319])

$$\begin{aligned} \partial_t(B_n \otimes B_n) + \operatorname{div}(u_n \odot (B_n \otimes B_n)) \\ = \nabla u_n(B_n \otimes B_n) + (B_n \otimes B_n)(\nabla u_n)^T, \end{aligned} \quad (3.8)$$

where $\{\operatorname{div}(v \odot (w \otimes z))\}^{i,j} := \sum_{k=1}^d \partial_k(v^k w^i z^j)$ for vectors $v, w, z \in \mathbb{R}^d$.

Therefore

$$\begin{aligned} \partial_t u_n &= (-\Delta)^{-\gamma} \mathbb{P} \operatorname{div} \partial_t(B_n \otimes B_n) \\ &= (-\Delta)^{-\gamma} \mathbb{P} \operatorname{div} (\nabla u_n(B_n \otimes B_n) + (B_n \otimes B_n)(\nabla u_n)^T - \operatorname{div}(u_n \odot (B_n \otimes B_n))). \end{aligned}$$

Since $\gamma_c \geq 1$, one can deduce from the above equality that

$$\begin{aligned} \|\partial_t u_n\|_{L^2} &\lesssim \|\nabla u_n(B_n \otimes B_n)\|_{L^1} + \|(B_n \otimes B_n)(\nabla u_n)^T\|_{L^1} + \|u_n \odot (B_n \otimes B_n)\|_{L^1} \\ &\lesssim \|\nabla u_n\|_{L^\infty} \|B_n\|_{L^2}^2. \end{aligned} \quad (3.9)$$

In particular, $\{\partial_t u_n\}_{n \in \mathbb{N}}$ is uniformly bounded in $L^1(0, T; L^2)$, thanks to estimates (3.6), (3.7). At this stage, classical compactness arguments imply that there exists a subsequence, still denoted by $\{u_n\}_{n \in \mathbb{N}}$, and a function

$$u \in L^2(0, T; H^{\gamma_c}) \cap L^1(0, T; W^{1, \infty}),$$

such that

$$u_n \rightarrow u \quad \text{strongly in } L^2(0, T; L^2). \quad (3.10)$$

Moreover, since $2\gamma_c - 1 - 2d/(2 + \varepsilon) > 1$, then inequalities (2.14), (3.5) and interpolation inequality in Proposition A.3 together with convergence property (3.10) give that

$$u_n \rightarrow u \quad \text{strongly in } L^1(0, T; W^{1, \infty}). \quad (3.11)$$

In virtue of convergence properties (3.11), (3.3) and estimates (3.6), (3.7), Lemma 3.1 implies that, up to extraction, B_n converges strongly in $\mathcal{C}([0, \infty); L^2)$ to a function

$$B \in \mathcal{C}([0, \infty); L^2) \cap L^\infty(0, \infty; L^{2+\varepsilon}),$$

such that the pair (B, u) solves the problem (1.2)-(1.3) on $(0, \infty) \times \mathbb{T}^d$. The bound (1.9) is essentially due to estimate (3.5).

In order to prove energy equality (1.7), notice that the smooth approximate solution (B_n, u_n) fulfills

$$\|B_n(t, \cdot)\|_{L^2}^2 + 2 \int_0^t \|u_n(\tau, \cdot)\|_{H^{\gamma_c}}^2 d\tau = \|B_{0,n}\|_{L^2}^2. \quad (3.12)$$

Thanks to the strong convergence property of the magnetic field $\{B_n\}_{n \in \mathbb{N}}$ in the space $\mathcal{C}([0, \infty); L^2(\mathbb{T}^d))$, we have for all $0 \leq t < \infty$

$$\lim_{n \rightarrow \infty} \|B_n(t, \cdot)\|_{L^2}^2 = \|B(t, \cdot)\|_{L^2}^2.$$

As for the velocity field, using the Besov embedding and (2.14) we see that for any $p \in (2, 4]$

$$\|u_n\|_{\dot{B}_{2, \infty}^{\gamma_c + d - 2d/p}} \lesssim \|u_n\|_{\dot{B}_{p/2, \infty}^{2\gamma_c - 1}} \lesssim \|B_n\|_{L^p}^2.$$

This, together with interpolation inequality in Proposition A.3 and convergence property (3.10) and uniform bound (3.5) then give that $u_n \rightarrow u$ strongly in $L^2(0, T; H^{\gamma_c})$. By passing to the limit in (3.12), one can finally concludes that energy equality (1.7) is true.

Let us now briefly explain how to adapt the proof to the case when $B_0 \in L^2$ and $\gamma > \gamma_c$. First, we smooth out the data and use the global well-posedness result in [7, Theorem 3.1], which gives the existence of global smooth solutions that satisfying the energy equality

$$\|B_n(t, \cdot)\|_{L^2}^2 + 2 \int_0^t \|u_n(\tau, \cdot)\|_{H^\gamma}^2 d\tau = \|B_{0,n}\|_{L^2}^2. \quad (3.13)$$

Thanks to (3.13), the strong convergence property (3.11) of the velocity can be proved similarly as previous case. Indeed, use $\gamma > \gamma_c$ we have the Sobolev embedding $H^\gamma \hookrightarrow W^{1, \infty}$. From this point, the strong convergence in L^2 of the magnetic field is obtained via applying Lemma 3.1 again. Energy equality thus follows from the fact that $u_n \rightarrow u$ strongly in $L^2(0, T; H^\gamma)$ and by passing to the limit in (3.13).

We complete the proof of existence part in Theorem 1.3.

Remark 1 (regularity property of weak solutions). Here, we list some additional regularity properties of the weak solutions that constructed in this subsection. In the case that $\gamma > \gamma_c$, by testing the equation (1.2)₂ with u we get

$$\begin{aligned} \|u\|_{\dot{H}^\gamma}^2 &= |\langle B \otimes B, \nabla u \rangle| \\ &\leq \|B\|_{L^2}^2 \|\nabla u\|_{L^\infty} \\ &\leq \|B\|_{L^2}^2 \|u\|_{\dot{H}^\gamma}^2. \end{aligned}$$

Then by Poincaré's inequality we have $u \in L^\infty(\mathbb{R}_+; H^\gamma)$. Meanwhile, we know from estimate (2.14) that $u \in L^\infty(\mathbb{R}_+; \dot{B}_{1,\infty}^{2\gamma-1})$ with the associated mean-free pressure $P \in L^\infty(\mathbb{R}_+; \dot{B}_{1,\infty}^0)$.

When $\gamma = \gamma_c$, thanks to (1.9) and (2.14) we see that

$$u \in L^\infty(0, \infty; \dot{B}_{(2+\varepsilon)/2, \infty}^{2\gamma_c-1}) \cap L^\infty(\mathbb{R}_+; \dot{B}_{1,\infty}^{2\gamma_c-1})$$

with the associated mean-free pressure

$$P \in L^\infty(0, \infty; \dot{B}_{(2+\varepsilon)/2, \infty}^0) \cap L^\infty(\mathbb{R}_+; \dot{B}_{1,\infty}^0).$$

3.2. Proof of uniqueness. Because the magnetic field is rough, we shall first prove uniqueness of weak solutions written in the Lagrangian coordinates and then obtain uniqueness in the Eulerian coordinates. To this end, we first present here the definition and properties of the Lagrangian change of variables. More details and proofs can be found in e.g. [13, 14].

3.2.1. The Lagrangian coordinates. Let X be the flow associated to the divergence free velocity field v , that is, the solution to

$$\frac{d}{dt} X_v(t, y) = v(t, X_v(t, y)), \quad X_v(0, y) = y, \quad \forall y \in \mathbb{T}^d. \quad (3.14)$$

Equation (3.14) describes the relation between the Eulerian coordinates $x = X_v(t, y)$ and the Lagrangian coordinates y . In the sequel, we shall use the notation $\underline{v}(t, y) := v(t, X_v(t, y))$. We also define by $\Delta_{\underline{v}}$, $\nabla_{\underline{v}}$ and $\text{div}_{\underline{v}}$ the operators corresponding to the original operators Δ , ∇ and div , respectively, after performing the change to the Lagrangian coordinates. Index \underline{v} underlines the dependency on \underline{v} . Moreover, the chain rule yields that

$$\nabla_{\underline{v}} = M_{\underline{v}}^T \nabla_y, \quad \text{div}_{\underline{v}} = M_{\underline{v}} : \nabla_y = \text{div}_y(M_{\underline{v}} \cdot), \quad \Delta_{\underline{v}} \underline{w} = \text{div}_y(M M^T \nabla_y \cdot). \quad (3.15)$$

In particular, for any scalar function, one has

$$\nabla_{\underline{v}} = \text{div}_y(M_{\underline{v}} \cdot). \quad (3.16)$$

These algebraic relations will be of fundamental importance in our analysis.

Let us now list a few basic properties for the Lagrangian change of variables.

Proposition 3.2. *Assume that $v \in L^1(0, T; W^{1,\infty})$. Then the solution to the system (3.14) exists on the time interval $[0, T]$ and*

$$\|\nabla_y X_v(t)\|_{L^\infty} \leq \exp\left(\int_0^t \|\nabla_x v(\tau, \cdot)\|_{L^\infty} d\tau\right),$$

and

$$X_v(t, y) = y + \int_0^t \underline{v}(\tau, y) d\tau, \quad \nabla_y X_v(t, y) = \text{Id} + \int_0^t \nabla_y \underline{v}(\tau, y) d\tau.$$

Let $Y_v(t, \cdot)$ be the inverse diffeomorphism of $X_v(t, \cdot)$, then

$$\nabla_x Y_v(t, x) = (\nabla_y X_v(t, y))^{-1}.$$

Furthermore, if

$$\int_0^t \|\nabla_y \underline{v}(\tau, \cdot)\|_{\dot{B}_{q,1}^{d/q}} d\tau \leq \frac{1}{2} \quad \text{for some } q \in [1, \infty), \quad (3.17)$$

then $M_{\underline{v}}(t, y) := (\nabla_y X_v(t, y))^{-1}$ satisfies

$$M_{\underline{v}}(t, y) = (\text{Id} + (\nabla_y X_v - \text{Id}))^{-1} = \sum_{j=0}^{\infty} (-1)^j \left(\int_0^t \nabla_y \underline{v} d\tau \right)^j, \quad (3.18)$$

and the following inequalities:

$$\begin{aligned} \|M_{\underline{v}} - \text{Id}\|_{L^\infty(0,t;\dot{B}_{q,1}^{d/q})} &\lesssim \|\nabla_y \underline{v}\|_{L^1(0,t;\dot{B}_{q,1}^{d/q})}, \\ \|\nabla_y M_{\underline{v}}\|_{L^\infty(0,t;\dot{B}_{q,1}^{d/q})} &\lesssim \|\nabla_y^2 \underline{v}\|_{L^1(0,t;\dot{B}_{q,1}^{d/q})}. \end{aligned}$$

In Lagrangian coordinates, we will use repeatedly the fact that $\delta M_{\underline{v}}(t, y) := M_{\underline{v}_1}(t, y) - M_{\underline{v}_2}(t, y)$ satisfies

$$\delta M_{\underline{v}}(t, y) = (D_1(t, y) - D_2(t, y)) \cdot \left(\sum_{k \geq 1} \sum_{0 \leq j < k} D_1^j(t, y) D_2^{k-j-1}(t, y) \right), \quad (3.19)$$

with $D_i(t, y) := \int_0^t \nabla_y \underline{v}_i(\tau, y) d\tau$ for $i = 1, 2$.

Then, we have the proposition below.

Proposition 3.3. *Let \underline{v}_1 and \underline{v}_2 be two vector fields satisfying (3.17) (with same q) and define $\delta \underline{v} := \underline{v}_1 - \underline{v}_2$. Then*

$$\begin{aligned} \|\delta M_{\underline{v}}\|_{L^\infty(0,t;\dot{B}_{q,1}^{d/q})} &\lesssim \|\nabla_y \delta \underline{v}\|_{L^1(0,t;\dot{B}_{q,1}^{d/q})}, \\ \|\nabla \delta M_{\underline{v}}\|_{L^\infty(0,t;\dot{B}_{q,1}^{d/q})} &\lesssim \|\nabla_y^2 \delta \underline{v}\|_{L^1(0,t;\dot{B}_{q,1}^{d/q})} \\ &\quad + \|\nabla_y \delta \underline{v}\|_{L^1(0,t;\dot{B}_{q,1}^{d/q})} \|(\nabla_y^2 \underline{v}_1, \nabla_y^2 \underline{v}_2)\|_{L^1(0,t;\dot{B}_{q,1}^{d/q})}. \end{aligned}$$

Let us now derive the MRE (1.2) in the Lagrangian coordinates. We set

$$\begin{aligned} \underline{B}(t, y) &:= B(t, X_u(t, y)), \quad \underline{u}(t, y) := u(t, X_u(t, y)) \\ \underline{P}(t, y) &:= P(t, X_u(t, y)). \end{aligned}$$

Using the fact that the solution of the magnetic equation (1.2)₁ is given by

$$\underline{B}(t, y) = \nabla_y X_u(t, y) B_0(y),$$

and relations in (3.15), we find that MRE (1.2)-(1.3) recasts to

$$\left\{ \begin{array}{l} \underline{B}(t, y) = B_0(y) + \int_0^t B_0(y) \cdot \nabla_y \underline{u}(\tau, y) d\tau, \\ (-\Delta_y)^\gamma \underline{u} - M_{\underline{u}}^T \nabla_y \underline{P} = \text{div}_y (M_{\underline{u}}(\underline{B} \otimes \underline{B})) + ((-\Delta_y)^\gamma - (-\Delta_{\underline{u}})^\gamma) \underline{u}, \\ \text{div}_y \underline{u} = (\text{Id} - M_{\underline{u}}^T) : \nabla_y \underline{u}. \end{array} \right. \quad (3.20)$$

We are ready to prove the uniqueness part of Theorem 1.3. Consider two solutions (B_1, u_1, P_1) and (B_2, u_2, P_2) to MRE (1.2), emanating from the same initial

data B_0 , and fulfilling the properties in Theorem 1.3, and denote by $(\underline{B}_1, \underline{u}_1, \underline{P}_1)$ and $(\underline{B}_2, \underline{u}_2, \underline{P}_2)$ the corresponding triplets in Lagrangian coordinates. That is

$$\begin{aligned}\underline{B}_i(t, y) &:= B_i(t, X_{u_i}(t, y)), & \underline{u}_i(t, y) &:= u_i(t, X_{u_i}(t, y)) \\ \underline{P}_i(t, y) &:= P_i(t, X_{u_i}(t, y)),\end{aligned}$$

where X_{u_i} is the solution of (3.14) with velocity u_i , for $i = 1, 2$. Recall from Remark 1 that we have

$$u_i \in L^\infty(0, \infty; \dot{B}_{p, \infty}^{2\gamma-1}) \quad \text{for } i = 1, 2,$$

with $\gamma > \gamma_c$, $p = 1$ or $\gamma = \gamma_c$, $p = (2 + \varepsilon)/2$. Since in our framework the Lagrangian and Eulerian formulations are equivalent (see, e.g. [13, 14]), hence taking T_0 small enough, we may assume, with no loss of generality, that

$$C(T_0) := \int_0^{T_0} \|\underline{u}_1(t, \cdot)\|_{\dot{B}_{p, \infty}^{2\gamma-1}} dt + \int_0^{T_0} \|\underline{u}_2(t, \cdot)\|_{\dot{B}_{p, \infty}^{2\gamma-1}} dt \leq \frac{1}{2}. \quad (3.21)$$

Particularly, this ensures inequality (3.17) is satisfied and thus Proposition 3.2 and 3.3 apply. Indeed, since $2\gamma - 1 > d/p + 1$, the interpolation inequality from Proposition A.3 implies that for any function $w \in \dot{B}_{p, \infty}^{2\gamma-1}$ which has zero mean on \mathbb{T}^d , it satisfies

$$\|w\|_{\dot{B}_{p, 1}^{d/p+1}} \lesssim \|w\|_{\dot{B}_{p, \infty}^{2\gamma-1}}. \quad (3.22)$$

Before proceeding, we have to emphasize that the proof will be divided into two parts $\gamma \in \mathbb{N}$ and $\gamma \notin \mathbb{N}$. This seems unavoidable, since in the case $\gamma \notin \mathbb{N}$ the operator $(-\Delta)^\gamma$ is *nonlocal* so does the associated operator $(-\Delta_{\underline{u}})^\gamma$, in other words the ‘magic formulas’ in (3.15) is not available anymore. This fact gives us difficulties when performing difference estimates. For expository purposes, we focus our attention on statement (i) i.e. $\gamma > \gamma_c$, $p = 1$. In the end, we will explain how to adapt the argument to prove uniqueness in the case $\gamma = \gamma_c$, $p = (2 + \varepsilon)/2$.

3.2.2. The case $\gamma \in \mathbb{N}$. Define $(\delta \underline{B}, \delta \underline{u}, \delta \underline{P}) := (\underline{B}_1 - \underline{B}_2, \underline{u}_1 - \underline{u}_2, \underline{P}_1 - \underline{P}_2)$ and $\delta \underline{M} := \underline{M}_{\underline{u}_1} - \underline{M}_{\underline{u}_2}$. In virtue of formulas (3.15)-(3.16), the system for $(\delta \underline{B}, \delta \underline{u}, \delta \underline{P})$ reads

$$\begin{cases} \delta \underline{B}(t, y) = \int_0^t B_0(y) \cdot \nabla_y \delta \underline{u}(\tau, y) d\tau, \\ (-\Delta_y)^\gamma \delta \underline{u} - \nabla_y \delta \underline{P} = R_1 + R_2 + R_3, \\ \operatorname{div}_y \delta \underline{u} = R_4, \end{cases} \quad (3.23)$$

where

$$\begin{aligned}R_1 &:= \operatorname{div}_y(\delta \underline{M} \underline{P}_1) + \operatorname{div}_y((\underline{M}_{\underline{u}_2} - \operatorname{Id}) \delta \underline{P}), \\ R_2 &:= \operatorname{div}_y(\delta \underline{M}(\underline{B}_1 \otimes \underline{B}_1)) + \operatorname{div}_y(\underline{M}_{\underline{u}_2}(\delta \underline{B} \otimes \underline{B}_1) + \underline{M}_{\underline{u}_2}(\underline{B}_2 \otimes \delta \underline{B})), \\ R_3 &:= ((-\Delta_{\underline{u}_2})^\gamma - (-\Delta_{\underline{u}_1})^\gamma) \underline{u}_2 + ((-\Delta_y)^\gamma - (-\Delta_{\underline{u}_1})^\gamma) \delta \underline{u}, \\ R_4 &:= (\operatorname{Id} - \underline{M}_{\underline{u}_2}) : \nabla_y \delta \underline{u} - \delta \underline{M} : \nabla \underline{u}_1.\end{aligned}$$

Thanks to Hölder’s inequality, the embedding $\dot{B}_{p, 1}^{d/p} \hookrightarrow L^\infty$ and (3.22), one has

$$\begin{aligned}\|\delta \underline{B}(t, \cdot)\|_{L^2} &\leq \|B_0\|_{L^2} \|\nabla_y \delta \underline{u}\|_{L^1(0, t; L^\infty)} \\ &\leq \|B_0\|_{L^2} \|\delta \underline{u}\|_{L^1(0, t; \dot{B}_{1, 1}^{d+1})} \\ &\lesssim \|B_0\|_{L^2} \|\delta \underline{u}\|_{L^1(0, t; \dot{B}_{1, \infty}^{2\gamma-1})}\end{aligned} \quad (3.24)$$

on $t \in [0, T_0]$. Meanwhile, applying Lemma A.8 to the equation of $\delta \underline{u}$ gives

$$\|\delta \underline{u}(t, \cdot)\|_{\dot{B}_{1,\infty}^{2\gamma-1}} + \|\delta \underline{P}\|_{\dot{B}_{1,\infty}^0} \lesssim \|(R_1, R_2, R_3)(t, \cdot)\|_{\dot{B}_{1,\infty}^{-1}} + \|R_4(t, \cdot)\|_{\dot{B}_{1,\infty}^{2\gamma-2}}. \quad (3.25)$$

In what follows, we estimate the right-hand side of (3.25) term by term. For that purpose, we recall the following product law from Lemma A.7:

$$\dot{B}_{1,1}^d \times \dot{B}_{1,\infty}^0 \hookrightarrow \dot{B}_{1,1}^0. \quad (3.26)$$

Then, by (3.26) and Proposition 3.2 and 3.3, for the term R_1 we have

$$\begin{aligned} \|R_1(t, \cdot)\|_{\dot{B}_{1,\infty}^{-1}} &\lesssim \|\delta M\|_{\dot{B}_{1,1}^d} \|\underline{P}_1\|_{\dot{B}_{1,\infty}^0} + \|M_{\underline{u}_2} - \text{Id}\|_{\dot{B}_{1,1}^d} \|\delta \underline{P}\|_{\dot{B}_{1,\infty}^0} \\ &\lesssim \|\underline{P}_1\|_{\dot{B}_{1,\infty}^0} \|\delta \underline{u}\|_{L^1(0,t;\dot{B}_{1,1}^{d+1})} + \|\underline{u}_2\|_{L^1(0,t;\dot{B}_{1,1}^{d+1})} \|\delta \underline{P}\|_{\dot{B}_{1,\infty}^0} \\ &\lesssim \|\underline{P}_1\|_{\dot{B}_{1,\infty}^0} \|\delta \underline{u}\|_{L^1(0,t;\dot{B}_{1,\infty}^{2\gamma-1})} + C(T_0) \|\delta \underline{P}\|_{\dot{B}_{1,\infty}^0}, \end{aligned} \quad (3.27)$$

where in the last line inequality (3.22) was used again. Let us now bound R_2 . It is clear that

$$\begin{aligned} \|\text{div}_y(\delta M(\underline{B}_1 \otimes \underline{B}_1))\|_{\dot{B}_{1,\infty}^{-1}} &\lesssim \|\delta M\|_{\dot{B}_{1,1}^d} \|\underline{B}_1\|_{\dot{B}_{1,\infty}^0}^2 \\ &\lesssim \|\underline{B}_1\|_{L^2}^2 \|\delta \underline{u}\|_{L^1(0,t;\dot{B}_{1,\infty}^{2\gamma-1})}. \end{aligned}$$

Then, we write the decomposition

$$\text{div}_y(M_{\underline{u}_2}(\delta \underline{B} \otimes \underline{B}_1)) = \text{div}_y((M_{\underline{u}_2} - \text{Id})(\delta \underline{B} \otimes \underline{B}_1)) + \text{div}_y(\delta \underline{B} \otimes \underline{B}_1)$$

and find that

$$\begin{aligned} \|\text{div}_y((M_{\underline{u}_2} - \text{Id})(\delta \underline{B} \otimes \underline{B}_1))\|_{\dot{B}_{1,\infty}^{-1}} &\lesssim \|M_{\underline{u}_2} - \text{Id}\|_{\dot{B}_{1,1}^d} \|\delta \underline{B} \otimes \underline{B}_1\|_{\dot{B}_{1,\infty}^0} \\ &\lesssim C(T_0) \|\underline{B}_1\|_{L^2} \|\delta \underline{B}\|_{L^2} \\ \|\text{div}_y(\delta \underline{B} \otimes \underline{B}_1)\|_{\dot{B}_{1,\infty}^{-1}} &\lesssim \|\delta \underline{B} \otimes \underline{B}_1\|_{L^1} \\ &\lesssim \|\underline{B}_1\|_{L^2} \|\delta \underline{B}\|_{L^2}. \end{aligned}$$

This yields that

$$\|\text{div}_y(M_{\underline{u}_2}(\delta \underline{B} \otimes \underline{B}_1))\|_{\dot{B}_{1,\infty}^{-1}} \lesssim (1 + C(T_0)) \|\underline{B}_1\|_{L^2} \|\delta \underline{B}\|_{L^2},$$

so does for the last term in R_2 , that is

$$\|\text{div}_y(M_{\underline{u}_2}(\underline{B}_2 \otimes \delta \underline{B}))\|_{\dot{B}_{1,\infty}^{-1}} \lesssim (1 + C(T_0)) \|\underline{B}_2\|_{L^2} \|\delta \underline{B}\|_{L^2},$$

from which one has

$$\|R_2\|_{\dot{B}_{1,\infty}^{-1}} \lesssim \|\underline{B}_1\|_{L^2}^2 \|\delta \underline{u}\|_{L^1(0,t;\dot{B}_{1,\infty}^{2\gamma-1})} + (1 + C(T_0)) \|\underline{B}_1, \underline{B}_2\|_{L^2} \|\delta \underline{B}\|_{L^2}. \quad (3.28)$$

Next, we estimate R_4 . Obviously, we have

$$\|w\|_{L^\infty} \lesssim \|w\|_{\dot{B}_{1,1}^d} \lesssim \|w\|_{\dot{B}_{1,\infty}^{2\gamma-2}}.$$

Then Lemma A.6 implies that $\dot{B}_{1,\infty}^{2\gamma-2}$ is an algebra for mean-free functions on \mathbb{T}^d . Thus by formulas (3.18), (3.19) and inequality (3.21), we have

$$\begin{aligned} \|R_4\|_{\dot{B}_{1,\infty}^{2\gamma-2}} &\lesssim \|\text{Id} - M_{\underline{u}_2} : \nabla_y \delta \underline{u}\|_{\dot{B}_{1,\infty}^{2\gamma-2}} + \|\delta M : \nabla_y \underline{u}_1\|_{\dot{B}_{1,\infty}^{2\gamma-2}} \\ &\lesssim \|\underline{u}_2\|_{L^1(0,t;\dot{B}_{1,\infty}^{2\gamma-1})} \|\delta \underline{u}\|_{\dot{B}_{1,\infty}^{2\gamma-1}} + \|\delta \underline{u}\|_{L^1(0,t;\dot{B}_{1,\infty}^{2\gamma-1})} \|\underline{u}_1\|_{\dot{B}_{1,\infty}^{2\gamma-1}} \\ &\lesssim T_0 \|(\underline{u}_1, \underline{u}_2)\|_{L^\infty(0,T_0;\dot{B}_{1,\infty}^{2\gamma-1})} \|\delta \underline{u}\|_{L^\infty(0,T_0;\dot{B}_{1,\infty}^{2\gamma-1})}, \end{aligned} \quad (3.29)$$

where in the last line we used Hölder's inequality in time.

Finally, the term R_3 maybe handled along the same lines as previous and by induction on $\gamma \in \mathbb{N}$ and $\gamma > \gamma_c$. More precisely, define

$$\mathcal{R}_1^\gamma(w) := ((-\Delta_{\underline{u}_2})^\gamma - (-\Delta_{\underline{u}_1})^\gamma)w, \quad \mathcal{R}_2^\gamma(w) := ((-\Delta_y)^\gamma - (-\Delta_{\underline{u}_1})^\gamma)w,$$

we claim that we have

$$\begin{aligned} & \|\mathcal{R}_1^\gamma(w)\|_{\dot{B}_{1,\infty}^{-1}} \\ & \lesssim T_0(1 + \|(\underline{u}_1, \underline{u}_2)\|_{L^\infty(0,T_0;\dot{B}_{1,\infty}^{2\gamma-1})})\|\delta\underline{u}\|_{L^\infty(0,T_0;\dot{B}_{1,\infty}^{2\gamma-1})}\|w\|_{L^\infty(0,T_0;\dot{B}_{1,\infty}^{2\gamma-1})} \end{aligned} \quad (3.30)$$

and

$$\|\mathcal{R}_2^\gamma(w)\|_{\dot{B}_{1,\infty}^{-1}} \lesssim T_0\|\underline{u}_1\|_{L^\infty(0,T_0;\dot{B}_{1,\infty}^{2\gamma-1})}\|w\|_{L^\infty(0,T_0;\dot{B}_{1,\infty}^{2\gamma-1})}. \quad (3.31)$$

Notice that we have

$$\begin{aligned} \mathcal{R}_1^\gamma(w) &= \left((\operatorname{div}_y(M_{\underline{u}_2}M_{\underline{u}_2}^T \nabla_y))^\gamma - (\operatorname{div}_y(M_{\underline{u}_1}M_{\underline{u}_1}^T \nabla_y))^\gamma \right) w, \\ \mathcal{R}_2^\gamma(w) &= \left((\operatorname{div}_y \nabla_y)^\gamma - (\operatorname{div}_y(M_{\underline{u}_1}M_{\underline{u}_1}^T \nabla_y))^\gamma \right) w, \end{aligned}$$

and

$$M_{\underline{u}_2}M_{\underline{u}_2}^T - M_{\underline{u}_1}M_{\underline{u}_1}^T = \delta M(M_{\underline{u}_1} - \operatorname{Id}) + (M_{\underline{u}_2} - \operatorname{Id})\delta M^T + \delta M + \delta M^T, \quad (3.32)$$

so it is easy to check that (3.30) and (3.31) are satisfied when $\gamma = 3$. Now, assuming that they are satisfied for $\gamma - 1$, we write

$$\begin{aligned} & ((-\Delta_{\underline{u}_2})^\gamma - (-\Delta_{\underline{u}_1})^\gamma)w \\ &= ((-\Delta_{\underline{u}_2})^{\gamma-1} - (-\Delta_{\underline{u}_1})^{\gamma-1})\Delta_{\underline{u}_2}w + (-\Delta_{\underline{u}_1})^{\gamma-1}(\Delta_{\underline{u}_2} - \Delta_{\underline{u}_1})w, \end{aligned}$$

and find that

$$\begin{aligned} & \|((-\Delta_{\underline{u}_2})^\gamma - (-\Delta_{\underline{u}_1})^\gamma)w\|_{\dot{B}_{1,\infty}^{-1}} \\ & \leq \|\mathcal{R}_1^{\gamma-1}(\Delta_{\underline{u}_2}w)\|_{\dot{B}_{1,\infty}^{-1}} + \|(-\Delta_{\underline{u}_1})^{\gamma-1}(\Delta_{\underline{u}_2} - \Delta_{\underline{u}_1})w\|_{\dot{B}_{1,\infty}^{-1}} \\ & \lesssim T_0(1 + \|(\underline{u}_1, \underline{u}_2)\|_{L^\infty(0,T_0;\dot{B}_{1,\infty}^{2\gamma-3})})\|\delta\underline{u}\|_{L^\infty(0,T_0;\dot{B}_{1,\infty}^{2\gamma-3})}\|\Delta_{\underline{u}_2}w\|_{L^\infty(0,T_0;\dot{B}_{1,\infty}^{2\gamma-3})} \\ & \quad + \|(\Delta_{\underline{u}_2} - \Delta_{\underline{u}_1})w\|_{\dot{B}_{1,\infty}^{2\gamma-3}} \\ & \lesssim T_0(1 + \|(\underline{u}_1, \underline{u}_2)\|_{L^\infty(0,T_0;\dot{B}_{1,\infty}^{2\gamma-1})})\|\delta\underline{u}\|_{L^\infty(0,T_0;\dot{B}_{1,\infty}^{2\gamma-1})}\|w\|_{L^\infty(0,T_0;\dot{B}_{1,\infty}^{2\gamma-1})}, \end{aligned}$$

where in the last inequality we used the facts that the transform X_{u_1}, X_{u_2} are volume preserving, and $\dot{B}_{1,\infty}^{2\gamma-2}$ is an algebra together with formula (3.32). As such, we conclude by induction that (3.30) is satisfied for all $\gamma \in \mathbb{N}$ and $\gamma > \gamma_c$. The proof of (3.31) can be handled along the same lines.

With (3.30) and (3.31) in hand, we see that

$$\begin{aligned} \|R_3\|_{\dot{B}_{1,\infty}^{-1}} & \leq \|\mathcal{R}_1^\gamma(\underline{u}_2)\|_{\dot{B}_{1,\infty}^{-1}} + \|\mathcal{R}_2^\gamma(\delta\underline{u})\|_{\dot{B}_{1,\infty}^{-1}} \\ & \lesssim T_0(1 + \|(\underline{u}_1, \underline{u}_2)\|_{L^\infty(0,T_0;\dot{B}_{1,\infty}^{2\gamma-1})})^2\|\delta\underline{u}\|_{L^\infty(0,T_0;\dot{B}_{1,\infty}^{2\gamma-1})}. \end{aligned} \quad (3.33)$$

Note that, owing to our assumptions on $(\underline{B}_1, \underline{u}_1, \underline{P}_1)$ and $(\underline{B}_2, \underline{u}_2, \underline{P}_2)$, all the factors involving the functions $(\delta\underline{B}, \delta\underline{u}, \delta\underline{P})$ in the estimate of R_1, \dots, R_4 tend to 0 when T_0 goes to 0. Putting together all the previous inequalities (3.27), (3.28), (3.29) and (3.33) into inequality (3.25) and using (3.24), we get $(\delta\underline{B}, \delta\underline{u}) \equiv 0$ on $[0, T_0] \times \mathbb{T}^d$ if T_0 is small enough. Finally, thanks to a standard connectivity argument uniqueness is proved on the whole \mathbb{R}_+ .

3.2.3. *The case $\gamma \notin \mathbb{N}$.* Let us recall that, in \mathbb{R}^d , for all $\sigma \in (0, 1)$ the fractional Laplacian $(-\Delta_x)^\sigma = \Lambda^{2\sigma}$ enjoys a singular integral definition:

$$\Lambda_x^{2\sigma} f = \text{p.v.} \int_{\mathbb{R}^d} (f(x) - f(\underline{z})) K^\sigma(x - \underline{z}) d\underline{z} \quad (3.34)$$

with the kernel $K^\sigma(\underline{z}) = c_{d,\sigma}/|\underline{z}|^{d+2\sigma}$, where $c_{d,\sigma}$ is a constant depending on dimension d and σ . For periodic functions, the representation is still valid due to a sufficient decay of K^σ at infinity. Thus in Lagrangian coordinates $\Lambda_x^{2\sigma} u_i$ ($i = 1, 2$) recasts to

$$\begin{aligned} \Lambda_{\underline{u}_i}^{2\sigma} u_i &= \text{p.v.} \int_{\mathbb{R}^d} (u_i(X_{\underline{u}_i}(t, y)) - u_i(\underline{z})) K^\sigma(X_{\underline{u}_i}(t, y) - \underline{z}) d\underline{z} \\ &= \text{p.v.} \int_{\mathbb{R}^d} (\underline{u}_i(y) - \underline{u}_i(z)) K^\sigma(X_{\underline{u}_i}(t, y) - X_{\underline{u}_i}(t, z)) dz, \end{aligned}$$

where we used the fact that the transform X_{u_1}, X_{u_2} are volume preserving. The above formula has its limitation when considering difference estimates in the Lagrangian coordinates, so we are motivated to naturally extend its definition: Given any divergence free vector-valued function $v \in L^1(0, T; W^{1,\infty})$ and X_v defined by (3.14), $\forall \underline{f} : y \mapsto \underline{f}(y)$ we define

$$\Lambda_{\underline{v}}^{2\sigma} \underline{f} := \text{p.v.} \int_{\mathbb{R}^d} (\underline{f}(y) - \underline{f}(z)) K^\sigma(X_v(t, y) - X_v(t, z)) dz. \quad (3.35)$$

Now, since for any given $\gamma \notin \mathbb{N}$ and $\gamma > \gamma_c$, $\gamma = 1 + \theta + [\gamma] - 1$ with $\theta = \gamma - [\gamma] \in (0, 1)$, and in Eulerian coordinates $(-\Delta)^\gamma = -\text{div} \Lambda^{2\theta} \nabla (-\Delta)^{[\gamma]-1}$, thus one finds by definition (3.35) that

$$(-\Delta_{\underline{u}_i})^\gamma \underline{u}_i = -\text{div}_{\underline{u}_i} \Lambda_{\underline{u}_i}^{2\theta} \nabla_{\underline{u}_i} (-\Delta_{\underline{u}_i})^{[\gamma]-1} \underline{u}_i. \quad (3.36)$$

Based on above formula, we are able to rewrite (3.23) as

$$\begin{cases} \delta \underline{B}(t, y) = \int_0^t B_0(y) \cdot \nabla_y \delta \underline{u}(\tau, y) d\tau, \\ (-\Delta_y)^\gamma \delta \underline{u} - \nabla_y \delta \underline{P} = R_1 + R_2 + R'_3, \\ \text{div}_y \delta \underline{u} = R_4, \end{cases} \quad (3.37)$$

with

$$\begin{aligned} R'_3 &:= (\text{div}_{\underline{u}_1} - \text{div}_{\underline{u}_2}) \Lambda_{\underline{u}_2}^{2\theta} \nabla_{\underline{u}_2} (-\Delta_{\underline{u}_2})^{[\gamma]-1} \underline{u}_2 \\ &\quad + \text{div}_{\underline{u}_1} (\Lambda_{\underline{u}_1}^{2\theta} - \Lambda_{\underline{u}_2}^{2\theta}) \nabla_{\underline{u}_2} (-\Delta_{\underline{u}_2})^{[\gamma]-1} \underline{u}_2 \\ &\quad + \text{div}_{\underline{u}_1} \Lambda_{\underline{u}_1}^{2\theta} (\nabla_{\underline{u}_1} - \nabla_{\underline{u}_2}) (-\Delta_{\underline{u}_2})^{[\gamma]-1} \underline{u}_2 \\ &\quad + \text{div}_{\underline{u}_1} \Lambda_{\underline{u}_1}^{2\theta} \nabla_{\underline{u}_1} ((-\Delta_{\underline{u}_1})^{[\gamma]-1} - (-\Delta_{\underline{u}_2})^{[\gamma]-1}) \underline{u}_2 \\ &\quad + ((-\Delta_y)^\gamma + \text{div}_{\underline{u}_1} \Lambda_{\underline{u}_1}^{2\theta} \nabla_{\underline{u}_1} (-\Delta_{\underline{u}_1})^{[\gamma]-1}) \delta \underline{u} \\ &=: R'_{31} + R'_{32} + R'_{33} + R'_{34} + R'_{35}. \end{aligned}$$

In order to give an unified proof for $\theta \in (0, \infty)$, we have to perform estimates for $\delta \underline{u}$ in slightly larger Besov space compared to the case $\gamma \in \mathbb{N}$. More precisely, define $p^* = d/(d - \theta)$ i.e. $\theta = d - d/p^*$, and apply Lemma A.8 gives that

$$\begin{aligned} \|\delta \underline{u}(t, \cdot)\|_{\dot{B}_{p^*, \infty}^{2\gamma-1-d+d/p^*}} + \|\delta \underline{P}\|_{\dot{B}_{p^*, \infty}^{-d+d/p^*}} \\ \lesssim \|(R_1, R_2, R'_3)(t, \cdot)\|_{\dot{B}_{p^*, \infty}^{-1-d+d/p^*}} + \|R_4(t, \cdot)\|_{\dot{B}_{p^*, \infty}^{2\gamma-2-d+d/p^*}}. \end{aligned} \quad (3.38)$$

Moreover, using $2\gamma - 1 - d > 1$ and thanks to the embedding property in Proposition A.3, it is clear that

$$\begin{aligned} \|\delta \underline{B}(t, \cdot)\|_{L^2} &\lesssim \|B_0\|_{L^2} \|\nabla_y \delta \underline{u}\|_{L^1(0,t;L^\infty)} \\ &\lesssim \|B_0\|_{L^2} \|\delta \underline{u}\|_{L^1(0,t;\dot{B}_{p^*,\infty}^{2\gamma-1-d+d/p^*})}. \end{aligned} \quad (3.39)$$

At this stage, let us remark that the estimates of the terms R_1, R_2, R_4 in the right-hand side of (3.38) are quite similar to what we did in the previous case. Indeed, since $d/p^* - \theta > 0$, one has the product law (from Lemma A.7)

$$\dot{B}_{p^*,1}^{d/p^*} \times \dot{B}_{p^*,\infty}^{-\theta} \hookrightarrow \dot{B}_{p^*,\infty}^{-\theta}, \quad (3.40)$$

and the fact that the function space $\dot{B}_{p^*,\infty}^{2\gamma-2-d+d/p^*}$ is an algebra in our framework due to $2\gamma > 2 + d$ again.

As such, we will only focus on the estimate of R'_3 in the next. To do that, we claim that we have the following estimates, for which proof is presented in Appendix A.

Lemma 3.4. *Let $0 < s_1 < 1$, $s_1 < s_2 < 1 + s_1$ and $p_1, r_1 \in [1, \infty]$. Assume that $\underline{v}, \underline{v}_1, \underline{v}_2$ satisfied (3.17). Let $w \in \dot{B}_{p_1, r_1}^{s_2 - s_1}$, then it holds that*

$$\|\Lambda_{\underline{v}}^{s_2} w\|_{\dot{B}_{p_1, r_1}^{-s_1}} \lesssim \|w\|_{\dot{B}_{p_1, r_1}^{s_2 - s_1}} \quad (3.41)$$

and

$$\|(\Lambda_{\underline{v}_1}^{s_2} - \Lambda_{\underline{v}_2}^{s_2})w\|_{\dot{B}_{p_1, r_1}^{-s_1}} \lesssim \|w\|_{\dot{B}_{p_1, r_1}^{s_2 - s_1}} \|\delta \underline{v}\|_{L^1(0,T;W^{1,\infty})}. \quad (3.42)$$

Then, thanks to product law (3.40) and Lemma 3.4, and keeping in mind that $\theta = d - d/p^*$, for $C_0 := 1 + \|(\underline{u}_1, \underline{u}_2)\|_{L^\infty(0,T_0;\dot{B}_{1,1}^{d+1})}$ the terms in R'_3 can be estimated as follows,

$$\begin{aligned} \|R'_{31}\|_{\dot{B}_{1,\infty}^{-1-d+d/p^*}} &\lesssim \|\delta M \Lambda_{\underline{u}_2}^{2\theta} \nabla_{\underline{u}_2} (-\Delta_{\underline{u}_2})^{[\gamma]-1} \underline{u}_2\|_{\dot{B}_{1,\infty}^{-\theta}} \\ &\lesssim \|\delta \underline{u}\|_{L^1(0,T_0;\dot{B}_{p^*,1}^{d/p^*+1})} \|\Lambda_{\underline{u}_2}^{2\theta} \nabla_{\underline{u}_2} (-\Delta_{\underline{u}_2})^{[\gamma]-1} \underline{u}_2\|_{L^\infty(0,T_0;\dot{B}_{p^*,\infty}^{-\theta})} \\ &\lesssim C_0 T_0 \|\delta \underline{u}\|_{L^1(0,T_0;\dot{B}_{p^*,1}^{d/p^*+1})} \|\nabla_{\underline{u}_2} (-\Delta_{\underline{u}_2})^{[\gamma]-1} \underline{u}_2\|_{L^\infty(0,T_0;\dot{B}_{q^*,\infty}^\theta)} \\ &\lesssim C_0 T_0 \|\delta \underline{u}\|_{L^\infty(0,T_0;\dot{B}_{p^*,1}^{d/p^*+1})} \|\underline{u}_2\|_{L^\infty(0,T_0;\dot{B}_{p^*,\infty}^{2[\gamma]-1+\theta})} \end{aligned}$$

and

$$\begin{aligned} \|R'_{32}\|_{\dot{B}_{p^*,\infty}^{-1-d+d/p^*}} &\lesssim C_0 T_0 \|(\Lambda_{\underline{u}_1}^{2\theta} - \Lambda_{\underline{u}_2}^{2\theta}) \nabla_{\underline{u}_2} (-\Delta_{\underline{u}_2})^{[\gamma]-1} \underline{u}_2\|_{L^\infty(0,T_0;\dot{B}_{p^*,\infty}^{-\theta})} \\ &\lesssim C_0 T_0 \|\delta \underline{u}\|_{L^1(0,T_0;W^{1,\infty})} \|\nabla_{\underline{u}_2} (-\Delta_{\underline{u}_2})^{[\gamma]-1} \underline{u}_2\|_{L^\infty(0,T_0;\dot{B}_{p^*,\infty}^\theta)} \\ &\lesssim C_0 T_0 \|\delta \underline{u}\|_{L^\infty(0,T_0;W^{1,\infty})} \|\underline{u}_2\|_{L^\infty(0,T_0;\dot{B}_{p^*,\infty}^{2[\gamma]-1+\theta})}. \end{aligned}$$

Furthermore, noticing that we have

$$\dot{B}_{p^*,1}^{d/p^*} \times \dot{B}_{p^*,\infty}^\theta \hookrightarrow \dot{B}_{p^*,\infty}^\theta \quad (\text{since } \theta - d/p^* < 0),$$

thus

$$\begin{aligned} \|R'_{33}\|_{\dot{B}_{p^*,\infty}^{-1-d+d/p^*}} &\lesssim C_0 T_0 \|\Lambda_{\underline{u}_1}^{2\theta} (\nabla_{\underline{u}_1} - \nabla_{\underline{u}_2}) (-\Delta_{\underline{u}_2})^{[\gamma]-1} \underline{u}_2\|_{\dot{B}_{p^*,\infty}^{-\theta}} \\ &\lesssim C_0 T_0 \|(\nabla_{\underline{u}_1} - \nabla_{\underline{u}_2}) (-\Delta_{\underline{u}_2})^{[\gamma]-1} \underline{u}_2\|_{\dot{B}_{p^*,\infty}^\theta} \\ &\lesssim C_0 T_0 \|\delta \underline{u}\|_{L^1(0,T_0;\dot{B}_{p^*,1}^{d/p^*+1})} \|\nabla_y (-\Delta_{\underline{u}_2})^{[\gamma]-1} \underline{u}_2\|_{\dot{B}_{p^*,\infty}^\theta} \end{aligned}$$

$$\lesssim C_0 T_0 \|\delta \underline{u}\|_{L^\infty(0, T_0; \dot{B}_{p^*, 1}^{d/p^*+1})} \|\underline{u}_2\|_{\dot{B}_{p^*, \infty}^{2[\gamma]-1+\theta}}.$$

And, similar to (3.30) (using $2[\gamma] - 1 + \theta = 2\gamma - 1 - d + d/p^* > d/p^* + 1$)

$$\begin{aligned} \|R'_{34}\|_{\dot{B}_{p^*, \infty}^{-1-d+d/p^*}} &\lesssim C_0 T_0 \|\Lambda_{\underline{u}_1}^{2\theta} \nabla_{\underline{u}_1} ((-\Delta_{\underline{u}_1})^{[\gamma]-1} - (-\Delta_{\underline{u}_2})^{[\gamma]-1}) \underline{u}_2\|_{\dot{B}_{p^*, \infty}^{-\theta}} \\ &\lesssim C_0 T_0 \|((-\Delta_{\underline{u}_1})^{[\gamma]-1} - (-\Delta_{\underline{u}_2})^{[\gamma]-1}) \underline{u}_2\|_{\dot{B}_{p^*, \infty}^{1+\theta}} \\ &\lesssim C_0^2 T_0 \|\delta \underline{u}\|_{L^\infty(0, T_0; \dot{B}_{p^*, \infty}^{2[\gamma]-1+\theta})} \|\underline{u}_2\|_{\dot{B}_{p^*, \infty}^{2[\gamma]-1+\theta}}. \end{aligned}$$

The last term R'_{35} may be handled along the same lines. Indeed, we have

$$R'_{35} = -\operatorname{div}_y \Lambda_y^{2\theta} \nabla(-\Delta_y)^{[\gamma]-1} + \operatorname{div}_{\underline{u}_1} \Lambda_{\underline{u}_1}^{2\theta} \nabla_{\underline{u}_1} (-\Delta_{\underline{u}_1})^{[\gamma]-1} \delta \underline{u}.$$

Finally, putting together all the previous inequalities into (3.38) and using (3.39), we conclude uniqueness on a sufficiently small time interval, and then on the whole interval \mathbb{R}_+ .

Let us now explain how the arguments have to be modified in the case $\gamma = \gamma_c$, $p = (2 + \varepsilon)/2$. In fact, one only needs to consider the case that ε is small enough (say $\varepsilon < 1/3$), and then follow the above proof with small modification. In particular, in the case $\gamma \in \mathbb{N}$, one may perform estimates for the equation of $\delta \underline{u}$ in $\dot{B}_{p, \infty}^{-1}$, while for the case $\gamma \in \mathbb{N}$ one may choose the corresponding space to $\dot{B}_{p^*, \infty}^{-1-d/p+d/p^*}$ with $p^* = 2dp/(2d - p\theta)$ i.e. $\theta/2 = d/p - d/p^*$. The details are left to the interested reader. Theorem 1.3 is thus proved. \square

Proof of Corollary 1.4. We remark that the argument is the same for (1.10) and (1.11). Applying the time derivative to the velocity equation and using the equality of (3.8), we get

$$(-\Delta)^\gamma \partial_t u = \mathbb{P} \operatorname{div} (\nabla u (B \otimes B) + (B \otimes B) (\nabla u)^\top - \operatorname{div} (u \odot (B \otimes B))). \quad (3.43)$$

Observing from Lemma A.7 the following product law:

$$\dot{B}_{1,1}^{d-1} \times \dot{B}_{1,\infty}^0 \hookrightarrow \dot{B}_{1,1}^{-1}$$

and recalling the product law (3.26), it yields

$$\begin{aligned} \|\mathbb{P} \operatorname{div} (\nabla u (B \otimes B) + (B \otimes B) (\nabla u)^\top)\|_{\dot{B}_{1,1}^{-2}} &\lesssim \|\nabla u (B \otimes B)\|_{\dot{B}_{1,1}^{-1}} \\ &\lesssim \|\nabla u\|_{\dot{B}_{1,1}^{d-1}} \|B \otimes B\|_{\dot{B}_{1,\infty}^0} \\ &\lesssim \|u\|_{\dot{B}_{1,1}^d} \|B\|_{L^2}^2, \end{aligned}$$

and

$$\begin{aligned} \|\mathbb{P} \operatorname{div} \operatorname{div} (u \odot (B \otimes B))\|_{\dot{B}_{1,1}^{-2}} &\lesssim \|u \odot (B \otimes B)\|_{\dot{B}_{1,1}^0} \\ &\lesssim \|u\|_{\dot{B}_{1,1}^d} \|B \otimes B\|_{\dot{B}_{1,\infty}^0} \\ &\lesssim \|u\|_{\dot{B}_{1,1}^d} \|B\|_{L^2}^2. \end{aligned}$$

Taking account of above two inequalities and $\gamma \geq \gamma_c$, we see from (3.43) that

$$\begin{aligned} \|\partial_t u\|_{L^\infty(\mathbb{R}_+; \dot{B}_{1,1}^{2\gamma-2})} &\lesssim \|u\|_{L^\infty(\mathbb{R}_+; \dot{B}_{1,1}^d)} \|B\|_{L^\infty(\mathbb{R}_+; L^2)}^2 \\ &\lesssim \|u\|_{L^\infty(\mathbb{R}_+; \dot{B}_{1,\infty}^{2\gamma-1})} \|B_0\|_{L^2}^2. \end{aligned}$$

This, together with the embedding $\dot{B}_{1,1}^{2\gamma-2} \hookrightarrow \dot{H}^{2\gamma-2-d/2}$ and $\gamma \geq \gamma_c$ again imply that $\partial_t u \in L^\infty(\mathbb{R}_+; H^{d/2})$. Now, noticing that $u \in L^2(\mathbb{R}_+; H^{d/2})$ and

$$\begin{aligned} \|u(t_1, \cdot) - u(t_2, \cdot)\|_{H^{d/2}} &\leq \left| \int_{t_1}^{t_2} \|\partial_t u(t, \cdot)\|_{H^{d/2}} dt \right| \\ &\leq \|\partial_t u\|_{L^\infty(\mathbb{R}_+; H^{d/2})} |t_1 - t_2|. \end{aligned}$$

we get $\|u(t, \cdot)\|_{H^{d/2}} \rightarrow 0$ as $t \rightarrow \infty$ by [15, Lemma 3.1], which assesses that a uniformly continuous and integrable function must vanish at infinity. Finally, (1.10) and (1.11) are proved by interpolating with the $\dot{B}_{1,\infty}^{2\gamma-1}$ norm of u , respectively, which is uniformly bounded in time thanks to Remark 1. \square

APPENDIX A. FUNCTION SPACES AND USEFUL INEQUALITIES

Here, we recall the homogeneous Littlewood-Paley decomposition for periodic functions, the definition of Besov space on torus \mathbb{T}^d and some useful properties. For reference, we refer to [10, Appendix A]. In fact the analysis for periodic case is essentially the same with the whole-space case, and the details for the latter can be found in e.g. [6].

Let $\varphi \in \mathcal{D}(\mathcal{A})$ be a smooth function supported in the annulus $\mathcal{A} = \{\xi \in \mathbb{R}^d, \frac{3}{4} \leq |\xi| \leq \frac{8}{3}\}$ and such that

$$\sum_{j \in \mathbb{Z}} \varphi(2^{-j}\xi) = 1, \quad \forall \xi \in \mathbb{R}^d \setminus \{0\}.$$

For any $u \in \mathcal{S}'(\mathbb{T}^d)$, we have the Fourier series representation

$$u(x) = \sum_{k \in \mathbb{Z}^d} \widehat{u}_k e^{ik \cdot x} \quad \text{with} \quad \widehat{u}_k := \frac{1}{|\mathbb{T}^d|} \int_{\mathbb{T}^d} u(x) e^{-ik \cdot x} dx.$$

Note that in the sequel of this section our functions will always have mean zero. The periodic dyadic block $\dot{\Delta}_j$ and the low frequency cut-off \dot{S}_j are defined by

$$\forall j \in \mathbb{Z}, \quad \dot{\Delta}_j u := \sum_{k \in \mathbb{Z}^d} \varphi(2^{-j}k) \widehat{u}_k e^{ik \cdot x} \quad \text{and} \quad \dot{S}_j u := \sum_{\ell \leq j-1} \dot{\Delta}_\ell u.$$

Then we have the decomposition

$$u = \sum_{j \in \mathbb{Z}} \dot{\Delta}_j u, \quad \forall u \in \mathcal{S}'(\mathbb{T}^d).$$

Moreover, the homogeneous Littlewood-Paley decomposition satisfies the property of almost orthogonality:

$$\dot{\Delta}_j \dot{\Delta}_{j'} u = 0, \quad \text{if } |j - j'| \geq 2, \quad \dot{\Delta}_j (S_{j'-1} u \dot{\Delta}_{j'} u) = 0, \quad \text{if } |j - j'| \geq 5,$$

and it is obvious that $\dot{\Delta}_j u \equiv 0$ for negative enough j , i.e. low frequency vanishes.

We now recall the definition of homogeneous Besov spaces.

Definition A.1. Let s be a real number and (p, r) be in $[1, \infty]^2$, we set

$$\|u\|_{\dot{B}_{p,r}^s(\mathbb{T}^d)} := \begin{cases} \|2^{js} \|\dot{\Delta}_j u\|_{L^p(\mathbb{T}^d)}\|_{\ell^r(\mathbb{Z})} & \text{for } 1 \leq r < \infty, \\ \sup_{j \in \mathbb{Z}} 2^{js} \|\dot{\Delta}_j u\|_{L^p(\mathbb{T}^d)} & \text{for } r = \infty. \end{cases}$$

The homogeneous Besov space $\dot{B}_{p,r}^s(\mathbb{T}^d) := \{u \in \mathcal{S}' : \|u\|_{\dot{B}_{p,r}^s(\mathbb{T}^d)} < \infty\}$.

It is clear that $\|\cdot\|_{\dot{H}^s(\mathbb{T}^d)} = \|\cdot\|_{\dot{B}_{2,2}^s(\mathbb{T}^d)}$, where $\dot{H}^s(\mathbb{T}^d)$ is the standard homogeneous Sobolev spaces.

We will now give a characterization of Besov spaces in terms of finite differences.

Proposition A.2 ([6, Theorem 2.36]). *Let $s \in (0, 1)$ and $p, r \in [1, \infty]$. Then for any $u \in \dot{B}_{p,r}^s(\mathbb{T}^d)$,*

$$\left\| \frac{\|u(\cdot + y) - u(\cdot)\|_{L^p(\mathbb{T}^d)}}{|y|^s} \right\|_{L^r(\mathbb{T}^d; \frac{dy}{|y|^d})} \sim \|u\|_{\dot{B}_{p,r}^s(\mathbb{T}^d)},$$

and also

$$\left\| \frac{\|u(\cdot + z) - u(\cdot)\|_{L^p(\mathbb{T}^d)}}{|z|^s} \right\|_{L^r(\mathbb{R}^d; \frac{dz}{|z|^d})} \sim \|u\|_{\dot{B}_{p,r}^s(\mathbb{T}^d)}. \quad (\text{A.1})$$

Next, we recall some basic facts on the Besov spaces.

Proposition A.3. *Let $1 \leq p \leq \infty$. Then there hold:*

- for all $s \in \mathbb{R}$ and $1 \leq p, r \leq \infty$, we have

$$\|D^k u\|_{\dot{B}_{p,r}^s(\mathbb{T}^d)} \sim \|u\|_{\dot{B}_{p,r}^{s+k}(\mathbb{T}^d)}.$$

- for any $\theta \in (0, 1)$ and $\underline{s} < \bar{s}$, we have

$$\|u\|_{\dot{B}_{p,1}^{\theta \underline{s} + (1-\theta)\bar{s}}(\mathbb{T}^d)} \lesssim \|u\|_{\dot{B}_{p,\infty}^{\underline{s}}(\mathbb{T}^d)} \|u\|_{\dot{B}_{p,\infty}^{\bar{s}}(\mathbb{T}^d)}^{1-\theta}.$$

- *Embedding: we have the following continuous embedding*

$$\dot{B}_{p,r}^s(\mathbb{T}^d) \hookrightarrow \dot{B}_{\infty,\infty}^{s-\frac{d}{p}}(\mathbb{T}^d) \quad \text{whenever } 1 \leq p, r \leq \infty,$$

and

$$\dot{B}_{p,1}^0(\mathbb{T}^d) \hookrightarrow L^p(\mathbb{T}^d) \hookrightarrow \dot{B}_{p,\infty}^0(\mathbb{T}^d),$$

$$\dot{B}_{p,\infty}^{\bar{s}}(\mathbb{T}^d) \hookrightarrow \dot{B}_{p,1}^{\underline{s}}(\mathbb{T}^d) \quad \text{whenever } \underline{s} < \bar{s}.$$

- Let f be a bounded function on $\mathbb{Z}^d \setminus \{0\}$ which is homogeneous of degree 0. Define $f(D)$ on $\mathcal{S}(\mathbb{T}^d)$ by

$$(f(D)u)(x) := \sum_{k \in \mathbb{Z}^d} f(k) \hat{u}_k e^{ik \cdot x}.$$

Then, for all exponents (s, p, r) , we have the estimate

$$\|f(D)u\|_{\dot{B}_{p,r}^s(\mathbb{T}^d)} \lesssim \|u\|_{\dot{B}_{p,r}^s(\mathbb{T}^d)}.$$

The following logarithmic interpolation inequality is available.

Lemma A.4. [6, Remark 2.105] *There exists a constant C such that for all $s \in \mathbb{R}$, $\theta > 0$ and $1 \leq p \leq \infty$, we have*

$$\|f\|_{\dot{B}_{p,1}^s(\mathbb{T}^d)} \leq C \|f\|_{\dot{B}_{p,\infty}^s(\mathbb{T}^d)} \left(1 + \log \left(\frac{\|f\|_{\dot{B}_{p,\infty}^{s-\theta}(\mathbb{T}^d)} + \|f\|_{\dot{B}_{p,\infty}^{s+\theta}(\mathbb{T}^d)}}{\|f\|_{\dot{B}_{p,\infty}^s(\mathbb{T}^d)}} \right) \right). \quad (\text{A.2})$$

The following Kato-Ponce type inequalities can be found in [20, Corollary 5.2].

Lemma A.5. *Let $s > 0$ and $1 < p < \infty$. Then for any $f, g \in \mathcal{S}(\mathbb{T}^d)$,*

$$\|[\Lambda^s, f]g\|_{L^p(\mathbb{T}^d)} \leq C(\|\Lambda^s f\|_{L^{p_1}(\mathbb{T}^d)}\|g\|_{L^{p_2}(\mathbb{T}^d)} + \|\nabla f\|_{L^{p_3}(\mathbb{T}^d)}\|\Lambda^{s-1}g\|_{L^{p_4}(\mathbb{T}^d)}),$$

where $1 < p_1, p_2, p_3, p_4 \leq \infty$, and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}$.

Lemma A.6. [6, Corollary 2.54] *Let $s > 0$ and $p, r \in [1, \infty]$. Then $\dot{B}_{p,r}^s \cap L^\infty$ is an algebra and we have*

$$\|fg\|_{\dot{B}_{p,r}^s(\mathbb{T}^d)} \leq C(\|f\|_{\dot{B}_{p,r}^s(\mathbb{T}^d)}\|g\|_{L^\infty(\mathbb{T}^d)} + \|f\|_{L^\infty(\mathbb{T}^d)}\|g\|_{\dot{B}_{p,r}^s(\mathbb{T}^d)}).$$

We frequently used the following product law (see e.g. [1, Proposition 2.1]).

Lemma A.7. *Let $1 \leq p < \infty$ and $s_1 + s_2 > 0$ with $s_1 \leq d/p$, $s_2 < d/p$, then*

$$\dot{B}_{p,1}^{s_1}(\mathbb{T}^d) \times \dot{B}_{p,\infty}^{s_2}(\mathbb{T}^d) \hookrightarrow \dot{B}_{p,1}^{s_1+s_2-d/p}(\mathbb{T}^d). \quad (\text{A.3})$$

Next, we need the regularity estimates in Besov spaces for the fractional Stokes equation:

$$\begin{cases} (-\Delta)^\gamma u - \nabla P = \operatorname{div} F & \text{on } \mathbb{T}^d, \\ \operatorname{div} u = g. \end{cases} \quad (\text{A.4})$$

Lemma A.8. *Let $\gamma \geq 0$, $1 \leq p, r \leq \infty$ and $s \in \mathbb{R}$. Assume that $F \in \dot{B}_{p,r}^s(\mathbb{T}^d)$. Then, (A.4) has a unique solution (u, P) in $\dot{B}_{p,r}^{2\gamma+s-1}(\mathbb{T}^d) \times \dot{B}_{p,r}^s(\mathbb{T}^d)$ and there exists a constant C depending only on d such that*

$$\|u\|_{\dot{B}_{p,r}^{2\gamma+s-1}(\mathbb{T}^d)} + \|P\|_{\dot{B}_{p,r}^s(\mathbb{T}^d)} \leq C(\|F\|_{\dot{B}_{p,r}^s(\mathbb{T}^d)} + \|g\|_{\dot{B}_{p,r}^{2\gamma+s-2}(\mathbb{T}^d)}).$$

Proof. The proof of the case $g \equiv 0$ is classical. The general case follows from this particular case by considering $v := u + \nabla(-\Delta)^{-1}g$, which satisfies

$$\begin{cases} (-\Delta)^\gamma v - \nabla(P + (-\Delta)^{\gamma-1}g) = \operatorname{div} F & \text{on } \mathbb{T}^d, \\ \operatorname{div} v = 0. \end{cases}$$

□

In the last of this section, we give the proof of Lemma 3.4.

Proof of Lemma 3.4. The proof is via duality argument. We first prove (3.41). For any $f \in \mathcal{S}(\mathbb{T}^d) \cap \dot{B}_{p_1',r_1'}^{s_1}(\mathbb{T}^d)$, we use the periodic representation of the kernel K^{s_2} to write

$$\begin{aligned} & \langle \Lambda_{\underline{v}}^{s_2} w, f \rangle \\ &= \text{p.v.} \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} f(y)(w(y) - w(z)) K^{s_2}(X_{\underline{v}}(t, y) - X_{\underline{v}}(t, z)) dz dy \\ &= \text{p.v.} \sum_{j \in \mathbb{Z}^d} \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} f(y)(w(y) - w(z)) K^{s_2}(X_{\underline{v}}(t, y) - X_{\underline{v}}(t, z) + j) dz dy \\ &= \frac{1}{2} \text{p.v.} \sum_{j \in \mathbb{Z}^d} \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} (f(y) - f(z))(w(y) - w(z)) K^{s_2}(X_{\underline{v}}(t, y) - X_{\underline{v}}(t, z) + j) dz dy \\ &= \frac{1}{2} \text{p.v.} \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} (f(y) - f(z))(w(y) - w(z)) K^{s_2}(X_{\underline{v}}(t, y) - X_{\underline{v}}(t, z)) dz dy \\ &= \frac{1}{2} \text{p.v.} \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} (f(y) - f(z+y))(w(y) - w(z+y)) \\ & \quad \cdot K^{s_2}(X_{\underline{v}}(t, y) - X_{\underline{v}}(t, z+y)) dz dy. \end{aligned}$$

Note that in our framework

$$|X_{\underline{v}}(t, y) - X_{\underline{v}}(t, z + y)| \sim |z|.$$

Indeed, we have

$$X_{\underline{v}}(t, y) - X_{\underline{v}}(t, z + y) = -z - z \cdot \int_0^1 \int_0^t (\nabla_y \underline{v})(\tau, y + az) d\tau da. \quad (\text{A.5})$$

Thus by Hölder's inequality and (A.1) in Proposition A.2 (note that our assumptions ensure $0 < s_2$, $s_2 - s_1 < 1$), we get

$$\begin{aligned} & |\langle \Lambda_{\underline{v}}^{s_2} w, f \rangle| \\ & \lesssim \left\| \frac{\|w(\cdot + z) - w(\cdot)\|_{L^{p_1}(\mathbb{T}^d)}}{|z|^{s_2 - s_1}} \right\|_{L^{r_1}(\mathbb{R}^d; \frac{dz}{|z|^d})} \left\| \frac{\|f(\cdot + z) - f(\cdot)\|_{L^{p'_1}(\mathbb{T}^d)}}{|z|^{s_1}} \right\|_{L^{r'_1}(\mathbb{R}^d; \frac{dz}{|z|^d})} \\ & \lesssim \|w\|_{\dot{B}_{p_1, r_1}^{s_2 - s_1}} \|f\|_{\dot{B}_{p'_1, r'_1}^{s_1}} \quad (\text{with } 1/p'_1 + 1/p_1 = 1/r'_1 + 1/r_1 = 1), \end{aligned}$$

which finally implies (3.41) by duality. Similarly, we can write

$$\begin{aligned} & \langle (\Lambda_{\underline{v}_1}^{s_2} - \Lambda_{\underline{v}_2}^{s_2})w, f \rangle \\ & = \text{p.v.} \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} f(y)(w(y) - w(z)) \\ & \quad \cdot \left(K^{s_2}(X_{\underline{v}_1}(t, y) - X_{\underline{v}_1}(t, z)) - K^{s_2}(X_{\underline{v}_2}(t, y) - X_{\underline{v}_2}(t, z)) \right) dz dy \\ & = \frac{1}{2} \text{p.v.} \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} (f(y) - f(z))(w(y) - w(z)) \\ & \quad \cdot \left(K^{s_2}(X_{\underline{v}_1}(t, y) - X_{\underline{v}_1}(t, z)) - K^{s_2}(X_{\underline{v}_2}(t, y) - X_{\underline{v}_2}(t, z)) \right) dz dy \\ & = \frac{1}{2} \text{p.v.} \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} (f(y) - f(z + y))(w(y) - w(z + y)) \\ & \quad \cdot \left(K^{s_2}(X_{\underline{v}_1}(t, y) - X_{\underline{v}_1}(t, z + y)) - K^{s_2}(X_{\underline{v}_2}(t, y) - X_{\underline{v}_2}(t, z + y)) \right) dz dy. \end{aligned}$$

And, thanks to (A.5) we have

$$\begin{aligned} & |K^{s_2}(X_{\underline{v}_1}(t, y) - X_{\underline{v}_1}(t, z + y)) - K^{s_2}(X_{\underline{v}_2}(t, y) - X_{\underline{v}_2}(t, z + y))| \\ & \lesssim \frac{\|\nabla_y \delta \underline{v}\|_{L^1(0, t; L^\infty(\mathbb{T}^d))}}{|z|^{d + s_2}}. \end{aligned}$$

Thus, applying Hölder's inequality and (A.1) again yields (3.42). \square

APPENDIX B. A COMPACTNESS RESULT FOR AN ACTIVE VECTOR EQUATION

Here, we provide the proof of Lemma 3.1. Before proceed it, let us recall a classical lemma of commutator estimates for mollifiers. Let function $\eta(x)$ be the standard mollifier on \mathbb{T}^d , for any function $g \in L^1(\mathbb{T}^d)$ we denote $(g)_\varepsilon := g * \eta_\varepsilon$ with $\eta_\varepsilon(x) := \eta(x/\varepsilon)/\varepsilon^d$.

Lemma B.1. [26, Lemma 2.3] *Let $w \in W^{1, \infty}(\mathbb{T}^d; \mathbb{R}^d)$, $g \in L^2(\mathbb{T}^d; \mathbb{R})$. Then, $(\text{div}(wg))_\varepsilon - \text{div}(w(g)_\varepsilon)$ converges to 0 in $L^2(\mathbb{T}^d)$ as ε goes to 0.*

Proof of Lemma 3.1. The proof is divided into several steps.

Step 1. Existence. At first, from the bound (3.1) and the equality of $\partial_t A_n$, we have

$$\begin{aligned} \|\partial_t A_n\|_{L^2(0,T;\dot{W}^{-1,3/2}(\mathbb{T}^d))} &\lesssim \|A_n \otimes v_n\|_{L^2(0,T;L^{3/2}(\mathbb{T}^d))} + \|v_n \otimes A_n\|_{L^2(0,T;L^{3/2}(\mathbb{T}^d))} \\ &\lesssim \|A_n\|_{L^\infty(0,T;L^2(\mathbb{T}^d))} \|v_n\|_{L^2(0,T;L^6(\mathbb{T}^d))} \\ &\lesssim \|A_n\|_{L^\infty(0,T;L^2(\mathbb{T}^d))} \|v_n\|_{L^2(0,T;H^1(\mathbb{T}^d))} \\ &\lesssim M^2. \end{aligned}$$

At this stage, classical functional analysis arguments (including the well-known Aubin-Lions-Simon theorem [8, Theorem II. 5.16]) imply that there exists a subsequence, still denoted by $\{(A_n, v_n)\}_{n \in \mathbb{N}}$, and

$$A \in C_w(0, T; L^2(\mathbb{T}^d)) \cap \mathcal{C}([0, T]; \dot{H}^{-1}(\mathbb{T}^d))$$

such that

$$\begin{aligned} A_n &\rightharpoonup A \quad \text{weak } * \text{ in } L^\infty(0, T; L^2(\mathbb{T}^d)), \\ A_n &\rightarrow A \quad \text{strongly in } L^\infty(0, T; \dot{H}^{-1}(\mathbb{T}^d)), \\ v_n &\rightharpoonup v \quad \text{weakly in } L^2(0, T; H^1(\mathbb{T}^d)). \end{aligned}$$

These convergence properties are enough to enable us pass to the limit in the weak formulation of (3.2)₁:

$$\begin{aligned} \int_{\mathbb{T}^d} A_n(t, x) \cdot w(t, x) dx - \int_0^t \int_{\mathbb{T}^d} A_n \cdot \partial_t w dx d\tau + \int_0^t \int_{\mathbb{T}^d} (v_n \otimes A_n) : \nabla w dx d\tau \\ = \int_0^t \int_{\mathbb{T}^d} (A_n \otimes v_n) : \nabla w dx d\tau + \int_{\mathbb{T}^d} A_{0,n}(x) \cdot w(0, x) dx, \end{aligned}$$

for any vector function $w \in \mathcal{C}^\infty([0, T] \times \mathbb{T}^d)$, and conclude that A is a solution of problem (3.2).

Step 2. Renormalization. In this step, we show that

$$\partial_t |A|^2 + v \cdot \nabla |A|^2 = 2(A \cdot \nabla v) \cdot A \quad \text{in } \mathcal{D}'((0, T) \times \mathbb{T}^d). \quad (\text{B.1})$$

In fact, since

$$\partial_t A + \text{div}(A \otimes v) = \text{div}(v \otimes A) \quad \text{in } \mathcal{D}'((0, T) \times \mathbb{T}^d),$$

testing the above equation by $(A)_\varepsilon \phi$ with ϕ any test function on $[0, T] \times \mathbb{T}^d$, we have

$$\begin{aligned} \int_{\mathbb{T}^d} A(t, x) \cdot (A)_\varepsilon(t, x) \phi(t, x) dx - \int_{\mathbb{T}^d} A_0(x) \cdot (A_0)_\varepsilon(x) \phi(0, x) dx \\ - \int_0^t \int_{\mathbb{T}^d} (A \cdot (A)_\varepsilon \partial_t \phi + (A \cdot \nabla v)_\varepsilon \cdot A \phi + (A \cdot \nabla v) \cdot (A)_\varepsilon \phi) dx d\tau \\ = - \int_0^t \int_{\mathbb{T}^d} ((v \cdot \nabla A)_\varepsilon \cdot A \phi + (v \cdot \nabla A) \cdot (A)_\varepsilon \phi) dx d\tau. \end{aligned} \quad (\text{B.2})$$

In above, we used equality $\partial_t (A)_\varepsilon = -(v \cdot \nabla A)_\varepsilon + (A \cdot \nabla v)_\varepsilon$.

Now, using that $\text{div } v = 0$ one has

$$\begin{aligned} \int_0^t \int_{\mathbb{T}^d} ((v \cdot \nabla A)_\varepsilon \cdot A \phi - (v \cdot \nabla (A)_\varepsilon) \cdot A \phi) dx d\tau \\ = \int_0^t \int_{\mathbb{T}^d} (\text{div}(A \otimes v)_\varepsilon - \text{div}((A)_\varepsilon \otimes v)) \cdot A \phi dx d\tau \end{aligned}$$

$$= \sum_{i=1}^d \int_0^t \int_{\mathbb{T}^d} (\operatorname{div}(A^i v)_\varepsilon - \operatorname{div}((A^i)_\varepsilon v)) A^i \phi \, dx d\tau.$$

Hölder's inequality and Lemma B.1 with the assumptions that $v \in L^1(0, T; W^{1, \infty}(\mathbb{T}^d))$, $A \in L^\infty(0, T; L^2(\mathbb{T}^d))$ then yield that

$$\begin{aligned} & \left| \int_0^t \int_{\mathbb{T}^d} ((v \cdot \nabla A)_\varepsilon \cdot A \phi + (v \cdot \nabla A) \cdot (A)_\varepsilon \phi) \, dx d\tau \right| \\ & \leq \sum_{i=1}^d \left| \int_0^t \int_{\mathbb{T}^d} ((\operatorname{div}(A^i v))_\varepsilon - \operatorname{div}((A^i)_\varepsilon v)) A^i \phi \, dx d\tau \right| \\ & \lesssim \sum_{i=1}^d \|(\operatorname{div}(A^i v))_\varepsilon - \operatorname{div}((A^i)_\varepsilon v)\|_{L^1(0, T; L^2(\mathbb{T}^d))} \|A^i\|_{L^\infty(0, T; L^2(\mathbb{T}^d))} \|\phi\|_{L^\infty([0, T] \times \mathbb{T}^d)} \\ & \rightarrow 0, \quad \text{as } \varepsilon \rightarrow 0. \end{aligned}$$

Thus for the last line in equality (B.2),

$$\begin{aligned} & - \int_0^t \int_{\mathbb{T}^d} ((v \cdot \nabla A)_\varepsilon \cdot A \phi + (v \cdot \nabla A) \cdot (A)_\varepsilon \phi) \, dx d\tau \\ & = - \int_0^t \int_{\mathbb{T}^d} ((v \cdot \nabla A)_\varepsilon \cdot A \phi - (v \cdot \nabla(A)_\varepsilon) \cdot A \phi) \, dx d\tau + \int_0^t \int_{\mathbb{T}^d} (v \cdot \nabla \phi) A \cdot (A)_\varepsilon \, dx d\tau \\ & \rightarrow \int_0^t \int_{\mathbb{T}^d} (v \cdot \nabla \phi) |A|^2 \, dx d\tau, \quad \text{as } \varepsilon \rightarrow 0, \end{aligned}$$

while the remain terms in equality (B.2) converge obviously. Precisely, one has

$$\begin{aligned} & \int_{\mathbb{T}^d} |A(t, x)|^2 \phi(t, x) \, dx - \int_{\mathbb{T}^d} |A_0(x)|^2 \phi(0, x) \, dx \\ & = \int_0^t \int_{\mathbb{T}^d} (|A|^2 \partial_t \phi + 2(A \cdot \nabla v) \cdot A \phi + (v \cdot \nabla \phi) |A|^2) \, dx d\tau, \quad (\text{B.3}) \end{aligned}$$

i.e. (B.1) is satisfied.

Step 3. Continuity in time and uniqueness. To show that $A \in \mathcal{C}([0, T]; L^2)$, we first prove $\lim_{t \rightarrow 0^+} \|A(t, \cdot) - A_0(\cdot)\|_{L^2(\mathbb{T}^d)} = 0$. In fact, by taking $\phi \equiv 1$ in (B.3), we see that

$$\begin{aligned} & \left| \int_{\mathbb{T}^d} |A(t, x)|^2 \, dx - \int_{\mathbb{T}^d} |A_0(x)|^2 \, dx \right| \\ & = 2 \left| \int_0^t \int_{\mathbb{T}^d} (A \cdot \nabla v) \cdot A \, dx d\tau \right| \\ & \lesssim \|A\|_{L^\infty(0, T; L^2(\mathbb{T}^d))}^2 \|\nabla v\|_{L^1(0, t; L^\infty(\mathbb{T}^d))} \rightarrow 0, \quad \text{as } t \rightarrow 0^+. \end{aligned}$$

With above estimate and the fact that $A \in \mathcal{C}_w([0, T]; L^2(\mathbb{T}^d))$, we get strong right continuity of A at $t = 0$. A standard connectivity argument enables us to conclude that $A \in \mathcal{C}([0, T]; L^2(\mathbb{T}^d))$.

Next, we explain why A is the unique solution of problem (3.2) in the class $L^\infty(0, T; L^2(\mathbb{T}^d))$. By contradiction, assume that there exists another solution $\tilde{A} \in L^\infty(0, T; L^2(\mathbb{T}^d))$ with same initial data A_0 . Then

$$\partial_t(A - \tilde{A}) + \operatorname{div}((A - \tilde{A}) \otimes v) = \operatorname{div}(v \otimes (A - \tilde{A})) \quad \text{in } \mathcal{D}'((0, T) \times \mathbb{T}^d).$$

Similar to (B.1), this implies that

$$\partial_t |A - \tilde{A}|^2 + v \cdot \nabla |A - \tilde{A}|^2 = 2((A - \tilde{A}) \cdot \nabla v) \cdot (A - \tilde{A}) \quad \text{in } \mathcal{D}'((0, T) \times \mathbb{T}^d),$$

and integrating in space-time further gives that

$$\begin{aligned} \int_{\mathbb{T}^d} |A(t, x) - \tilde{A}(t, x)|^2 dx &= 2 \int_0^t \int_{\mathbb{T}^d} ((A - \tilde{A}) \cdot \nabla v) \cdot (A - \tilde{A}) dx d\tau \\ &\lesssim \int_0^t \| (A - \tilde{A})(\tau, \cdot) \|_{L^2(\mathbb{T}^d)}^2 \| \nabla v(\tau, \cdot) \|_{L^\infty(\mathbb{T}^d)} d\tau. \end{aligned}$$

By Gronwall's lemma, one gets $A \equiv \tilde{A}$ on $[0, T] \times \mathbb{T}^d$.

Step 4. Strong convergence. Let us use the notation \bar{f} to denote the weak limit of $\{f_n\}_{n \in \mathbb{N}}$ in $L^1(\mathbb{T}^d)$. Due to the fact that convex lower semicontinuous function give rise to L^1 -sequentially weakly lower semicontinuous functionals (see e.g. [25, Theorem 3]), we have

$$|f|^2 \leq \overline{|f|^2} \quad \text{a.e. on } \mathbb{T}^d, \quad (\text{B.4})$$

for any sequence $\{f_n\}_{n \in \mathbb{N}}$ such that

$$f_n \rightharpoonup f \quad \text{and} \quad |f_n|^2 \rightharpoonup \overline{|f|^2} \quad \text{weakly in } L^1(\mathbb{T}^d).$$

In what follows, we show

$$|\overline{A \otimes A} - A \otimes A| \leq C(|\overline{A|^2} - |A|^2) \quad \text{a.e. on } [0, T] \times \mathbb{T}^d. \quad (\text{B.5})$$

For fixed $i, j \in \{1, \dots, d\}$, using identity

$$\begin{aligned} 2(A_n^i A_n^j - A^i A^j) &= (A_n^i + A_n^j)^2 - (A^i + A^j)^2 \\ &\quad - ((A_n^i)^2 - (A^i)^2 + (A_n^j)^2 - (A^j)^2), \end{aligned}$$

we see that

$$\begin{aligned} 2(\overline{A^i A^j} - A^i A^j) &= \overline{(A^i + A^j)^2} - (A^i + A^j)^2 \\ &\quad - \left(\overline{(A^i)^2} - (A^i)^2 + \overline{(A^j)^2} - (A^j)^2 \right). \end{aligned}$$

Thanks to (B.4), we get

$$\begin{aligned} 2(\overline{A^i A^j} - A^i A^j) &\geq - \left(\overline{(A^i)^2} - (A^i)^2 + \overline{(A^j)^2} - (A^j)^2 \right) \\ &\geq - 2(|\overline{A|^2} - |A|^2). \end{aligned} \quad (\text{B.6})$$

Similarly, using identity

$$\begin{aligned} 2(A_n^i A_n^j - A^i A^j) &= -(A_n^i - A_n^j)^2 + (A^i - A^j)^2 \\ &\quad + (A_n^i)^2 - (A^i)^2 + (A_n^j)^2 - (A^j)^2, \end{aligned}$$

we see that

$$\begin{aligned} 2(\overline{A^i A^j} - A^i A^j) &= -\overline{(A^i - A^j)^2} + (A^i - A^j)^2 \\ &\quad + \overline{(A^i)^2} - (A^i)^2 + \overline{(A^j)^2} - (A^j)^2. \end{aligned}$$

Thanks to (B.4) again, we get

$$\begin{aligned} 2(\overline{A^i A^j} - A^i A^j) &\leq \overline{(A^i)^2} - (A^i)^2 + \overline{(A^j)^2} - (A^j)^2 \\ &\leq 2(|\overline{A|^2} - |A|^2). \end{aligned} \quad (\text{B.7})$$

Combining (B.6) and (B.7) gives rise to (B.5).

Now, testing the equation (3.2)₁ by A_n and using that $\operatorname{div} v_n = \operatorname{div} A_n = 0$, one gets

$$\int_{\mathbb{T}^d} |A_n(t, x)|^2 dx = \int_{\mathbb{T}^d} |A_{0,n}(x)|^2 dx - \int_0^t \int_{\mathbb{T}^d} (A_n \otimes A_n) : \nabla v_n dx d\tau.$$

Recall from (B.3) that

$$\int_{\mathbb{T}^d} |A(t, x)|^2 dx = \int_{\mathbb{T}^d} |A_0(x)|^2 dx - \int_0^t \int_{\mathbb{T}^d} (A \otimes A) : \nabla v dx d\tau.$$

From above two equalities, one finds that

$$\begin{aligned} \|A_n(t, \cdot)\|_{L^2(\mathbb{T}^d)}^2 - \|A(t, \cdot)\|_{L^2(\mathbb{T}^d)}^2 &= \|A_{0,n}\|_{L^2(\mathbb{T}^d)}^2 - \|A_0\|_{L^2(\mathbb{T}^d)}^2 \\ &+ \int_0^t \int_{\mathbb{T}^d} ((A \otimes A) : \nabla(v - v_n) + (A \otimes A - A_n \otimes A_n) : \nabla v_n) dx d\tau. \end{aligned} \quad (\text{B.8})$$

Thanks to our assumptions in Lemma 3.1, we know that the first and second term on the right-hand side of (B.8) converges to 0 as $n \rightarrow \infty$, respectively, while the third term converges to

$$\int_0^t \int_{\mathbb{T}^d} (A \otimes A - \overline{A \otimes A}) : \nabla v dx d\tau.$$

Thus pass to the limit in (B.8) yields

$$\begin{aligned} \int_{\mathbb{T}^d} \overline{|A(t, x)|^2} dx &= \lim_{n \rightarrow \infty} \|A_n(t, \cdot)\|_{L^2(\mathbb{T}^d)}^2 \\ &= \|A(t, \cdot)\|_{L^2(\mathbb{T}^d)}^2 + \int_0^t \int_{\mathbb{T}^d} (A \otimes A - \overline{A \otimes A}) : \nabla v dx d\tau. \end{aligned}$$

Inequality (B.5) and Hölder's inequality then imply that

$$\begin{aligned} \int_{\mathbb{T}^d} \left(\overline{|A(t, x)|^2} - |A(t, x)|^2 \right) dx \\ \lesssim \int_0^t \left(\int_{\mathbb{T}^d} \left(\overline{|A(\tau, x)|^2} - |A(\tau, x)|^2 \right) dx \right) \|\nabla v(\tau, \cdot)\|_{L^\infty} d\tau, \end{aligned}$$

from which Gronwall's lemma finally yields

$$\lim_{n \rightarrow \infty} \|A_n(t, \cdot)\|_{L^2(\mathbb{T}^d)}^2 = \|A(t, \cdot)\|_{L^2(\mathbb{T}^d)}^2 \quad \text{a.e. on } [0, T].$$

This convergence property combined with the bound (3.1) and previously established $A \in \mathcal{C}([0, T]; L^2)$ yield the strong convergence $A_n \rightarrow A$ in $\mathcal{C}([0, T]; L^2)$. We complete the proof of the lemma. \square

ACKNOWLEDGE

The author is partially supported by the CY Initiative of Excellence, project CYNA (CY Nonlinear Analysis) and PEPS JCJC 2023 of the CNRS. The author thank Christophe Prange for introducing him the work [12].

REFERENCES

- [1] ABIDI, H., GUI, G., ZHANG, P.: Well-posedness of 3-D inhomogeneous Navier-Stokes equations with highly oscillating initial velocity field. *J. Math. Pures Appl.* **100**(2), 166–203 (2013)
- [2] ARNOLD, V.I.: Sur la gomtrie diffrentielle des groupes de Lie de dimension infinie et ses applications lhydrodynamique des fluides parfaits. *Ann. Inst. Fourier (Grenoble)* **16**(fasc. 1), 319–361 (1966)
- [3] ARNOLD, V.I.: The asymptotic Hopf invariant and it applications. In: Proceedings of Summer School in Differential Equations. Armenian Academy of Sciences, English translation, vol. **561** (1974)

- [4] ARNOLD, V.I., KHESIN, B.A.: Topological methods in hydrodynamics. In: Applied Mathematical Sciences, vol. **125**. Springer, New York (1998)
- [5] BAE, H., KWON, H., SHIN, J.: Global solutions to Stokes-Magneto equations with fractional dissipations. *arXiv:2310.03255* (2023)
- [6] BAHOURI, H., CHEMIN, J. Y., DANCHIN, R.: Fourier Analysis and Nonlinear Partial Differential Equations. Grundlehren der Mathematischen Wissenschaften, Vol. **343**. Springer, Berlin (2011)
- [7] BEEKIE, R., FRIEDLANDER, S., VICOL, V.: On Moffatts magnetic relaxation equations. *Commun. Math. Phys.* **390**(3), 1311–1339 (2022)
- [8] BOYER F. AND FABRIE P.: Mathematical tools for the study of the incompressible Navier-Stokes equations and related models. Vol.**183**. Applied Mathematical Sciences. Springer, New York (2013)
- [9] BRENIER, Y.: Topology-preserving diffusion of divergence-free vector fields and magnetic relaxation. *Commun. Math. Phys.* **330**(2), 757–770 (2014)
- [10] CAMERON, S., STRAIN, R.M.: Critical local well-posedness for the fully nonlinear Peskin problem. *Comm. Pure Appl. Math.* (2023)
- [11] CONSTANTIN, P., PASQUALOTTO, F.: Magnetic Relaxation of a VoigtMHD System. *Commun. Math. Phys.* **402**(2), 1931–1952 (2023)
- [12] DALIBARD, A.-L., GUILLOD, J., LEBLOND, A.: Long-time behavior of the Stokes-transport system in a channel. *arXiv:2306.00780v1*
- [13] DANCHIN R., MUCHA P.B.: A Lagrangian approach for the incompressible Navier-Stokes equations with variable density. *Comm. Pure Appl. Math.* **65**, 1458–1480 (2012)
- [14] DANCHIN R., MUCHA P.B.: Incompressible flows with piecewise constant density. *Arch. Ration. Mech. Anal.* **207**, 991–1023 (2013)
- [15] DOERING, C.R., WU, J., ZHAO, K., ZHENG, X.: Long time behavior of the two-dimensional Boussinesq equations without buoyancy diffusion. *Physica D* **376**(377), 144–159 (2018)
- [16] EBIN, D.G., MARSDEN, J.E.: Groups of diffeomorphisms and the motion of an incompressible fluid. *Ann. Math.* **2**(92), 102–163 (1970)
- [17] ELGINDI, T.M., MASMOUDI, N.: L^∞ ill-posedness for a class of equations arising in hydrodynamics. *Arch. Ration. Mech. Anal.* **235**(3), 1979–2025 (2020)
- [18] HOLM, D.D., MARSDEN, J.E., RATIU, T., WEINSTEIN, A.: Nonlinear stability of fluid and plasma equilibria. *Phys. Rep.* **123**(12), 116 (1985)
- [19] KIM, H., KWON, H.: Global existence and uniqueness of weak solutions of a Stokes-Magneto system with fractional diffusions. *J. Diff. Equat.* **374**, 497–547 (2023)
- [20] LI, D.: On Kato-Ponce and fractional Leibniz. *Rev. Mat. Iberoam.* **35**, 23–100 (2019)
- [21] LIEB, H E., LOSS, M.: Analysis. volume **14**. American Mathematical Society (2001)
- [22] MOFFATT, H.K.: Magnetostatic equilibria and analogous Euler flows of arbitrarily complex topology. I. Fundamentals. *J. Fluid Mech.* **159**, 359–378 (1985)
- [23] MOFFATT, H.K.: Some topological aspects of fluid dynamics. *J. Fluid Mech.* **914**:Paper No. P1 (2021)
- [24] NISHIYAMA, T.: Magneto-hydrodynamic approach to solvability of the three-dimensional stationary Euler equations. *Glasgow Math. J.* **44**, 411–418 (2002)
- [25] NOVOTNÝ, A., PETZELTOVÁ, H.: Weak Solutions for the Compressible Navier-Stokes Equations: Existence, Stability, and Longtime Behavior. Handbook of mathematical analysis in mechanics of viscous fluids, 1381–1546, Springer, Cham, (2018)
- [26] LIONS, P.-L.: Mathematical topics in fluid mechanics. Incompressible models, Oxford Lecture Series in Mathematics and its Applications, **3** (1996)
- [27] VALLIS, G.K., CARNEVALE, G.F., YOUNG, W.R.: Extremal energy properties and construction of stable solutions of the Euler equations. *J. Fluid Mech.* **207**, 133–152 (1989)

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