A μ-analysis based approach to probabilistic delay margin analysis of uncertain linear systems*

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***Abstract*— Monte Carlo and deterministic μ-analysis are well known tools in aerospace industry for Validation and Verification (V&V) purposes. On the one hand, Monte Carlo analysis can provide an accurate measure of probability for sufficiently frequent phenomena. Nevertheless, it is often time-consuming and may fail to compute worst-case stability margins and performance levels. Deterministic μ-analysis, on the other hand, cannot miss worst-case scenarios. But in the absence of a measure of probability, a control system may be invalidated on the basis of extremely rare events. About twenty years ago, probabilistic μ-analysis was therefore introduced to bridge the analysis gap between both techniques. It combines efficient μ-based algorithms with a branch-and-bound scheme to explore the whole uncertainty domain, also considering probability distributions on the uncertain parameters. In this context, this paper describes a novel algorithm for probabilistic delay margin analysis, which provides upper and lower bounds on the probability of an uncertain system violating a desired delay margin.**

I. Introduction

Time delays occur in almost every control engineering problem, and generally have a significant impact on system behavior. Most studies, including this work, focus on constant time delays. A wide variety of analysis techniques are available in both the time and frequency domains.

Time domain techniques mainly use Lyapunov–Krasovskii functionals or Lyapunov–Razumikhin functions [1]. These approaches are based on the classical Lyapunov theory, and aim at constructing nontraditional energy-storage functions/functionals for time-delay closed-loop systems. Much literature deals with deterministic linear time-invariant (LTI) problems in presence as well as absence of uncertainty, see e.g. [2], [3]. The use of Lyapunov function(al)s usually leads to criteria that can be expressed in terms of Linear Matrix Inequalities (LMI) and solved with dedicated solvers.

Frequency domain approaches are generally based on the small-gain theorem and use μ-based techniques [4] or Integral Quadratic Constraints (IQC) [5]. First, μ-analysis makes use of the structured singular value μ to study robust stability and performance problems accounting for the structure of the uncertainties in linear models. Examples of application to uncertain time-delay systems can be found in [6] and [7]. Then, IQC analysis can be interpreted as the combination of (scaled) small gain techniques, mainly μ-analysis, and positivity/passivity techniques which study the interconnection of a linear operator with a non-linearity ("Lur'e problem"). It is used to study robust stability and performance properties of uncertain, time-varying and nonlinear systems. Some applications of IQC to robustness analysis of time-delay systems can be found in [5] and [8].

Most frequency domain approaches require to replace the constant delay element $e^{-s\tau}$, where s is the Laplace variable and τ the considered time delay in seconds, with a rational expression. This step has a major influence on the conservatism of the analysis. In some studies, time-delay elements are eliminated by covering their value sets with unit disks, which leads to overly conservative results [9]. Other approaches replace the exponential term by Padé approximations [10], Taylor series expansions [11] or the Laguerre formula [12]... The accuracy of these approximations can often be improved by introducing more complexity. In contrast, an exact representation of the time delay effects is proposed in [7]. The replacement function has the same properties (unit gain and phase varying linearly with frequency) as the actual time delay. The price to pay is the introduction of a parameter with frequency dependent bounds.

With this literature review in mind, let us now describe the context and the motivations of this work. Due to their simplicity, Monte Carlo (MC) simulations [13] have long been the preferred Verification and Validation (V&V) approach in the industry. They can provide an accurate measure of probability for sufficiently frequent phenomena. Nevertheless, they are often time-consuming and may fail to compute worst-case stability margins and performance levels. In contrast, deterministic simulation-free techniques such as μ and IQC analysis have reached a good level of maturity. Unlike MC simulations, rare scenarios are no longer missed, but their probability of occurrence is not measured. Analysis results can therefore be overly conservative [14] and a control system may be invalidated on the basis of very rare and therefore extremely unlikely events [15]. So there is a real need to develop new methods to fill the gap between MC simulations and worst-case analysis. The work presented in this paper seeks precisely to make a step in this direction.

However efficient and accurate they may be, LMI-based time-domain approaches seem difficult to implement at the present time when it comes to be fast while analyzing realistic systems characterized by high-dimensional models with multiple uncertainties. On the other hand, frequency-domain approaches such as μ and IQC analysis have shown their ability to handle some real-world systems. IQC analysis has the advantage of being able to deal with uncertainties, time-varying parameters and non-linearities at the same time. But it can be conservative and usually requires to use an

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LMI solver, which significantly impacts the computational time as already highlighted above. On the opposite, μ-analysis is restricted to the analysis of uncertain systems, but guaranteed and very accurate bounds on stability margins and performance levels can usually be obtained very quickly [16]. And probabilistic μ-analysis has even been identified for many years as a serious candidate to the analysis of rare events [17], [14]. A significant effort has been made in recent years to propose new probabilistic μ tools, which are both versatile and computationally efficient [18], [19]. The objective of this paper is now to focus on delay margin. Considering the aforementioned literature review on time-delay systems and V&V needs, it seems relevant to propose a μ-analysis based approach to probabilistic delay margin analysis, able to compute tight bounds on the probability of delay margin violation for linear systems in the presence of parametric uncertainties with given probability distributions.

The paper is organized as follows. The considered problem is first stated in Section II. Section III gives a brief overview to the proposed solution. The main theoretical results are then detailed in Section IV, as well as a practical algorithm. The potential of this algorithm is finally evaluated in Section V.

II. Problem statement

Let us consider the following continuous-time uncertain LTI system:

$$\left\{ \begin{array}{rcl} \dot{x} & = & A(\delta)x + B(\delta)u \\ y & = & C(\delta)x + D(\delta)u \end{array} \right. \quad (1)$$

The real uncertain parameters $\delta = (\delta_1, \ldots, \delta_N)$ are bounded and without loss of generality normalized, so that the whole set of admissible uncertainties is covered when $\delta \in \mathcal{B}_\delta = [-1\ 1]^N$. They are independent random variables, whose probability density functions f are supported on the bounded interval $[-1\ 1]$. It is assumed that $A(\delta)$, $B(\delta)$, $C(\delta)$, $D(\delta)$ are polynomial or rational functions of the δ_i and that system (1) can be transformed into a Linear Fractional Representation (LFR) as shown in Fig. 1: the uncertainties are separated from the nominal LTI system $M(s)$ and isolated in a block-diagonal operator $\Delta = \text{diag}(\delta_1 I_{n_1}, \ldots, \delta_N I_{n_N}) \in \mathbb{R}^{p\times p}$, where I_{n_i} is the $n_i \times n_i$ identity matrix. The set of matrices with the same block-diagonal structure as Δ is denoted $\mathbf{\Delta}$. Let $\mathcal{B}_{\mathbf{\Delta}} = \{\Delta \in \mathbf{\Delta} : \delta_i \in \mathcal{B}_\delta\}$ and $D_{\mathbf{\Delta}} = \{\Delta \in \mathbf{\Delta} : \delta_i \in D\}$ be the subsets of $\mathbf{\Delta}$ corresponding to $\mathcal{B}_\delta$ and to a given box $D \in \mathcal{B}_\delta$ respectively.

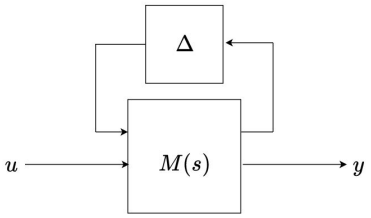

Fig. 1: Linear Fractional Representation (LFR)

In this work, system (1) and Fig. 1 describe the control loop opened at the place where the delay margin should be computed. The closed-loop interconnection is therefore recovered by applying a unit negative feedback between y and u, *i.e.* by setting $G(s) = 1$ in Fig. 2.

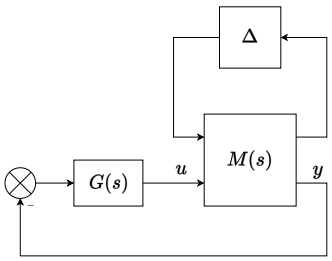

Fig. 2: Feedback loop for delay margin analysis

Using these notations, probabilistic delay margin analysis can be formalized as follows:

Problem 2.1: Compute the probability $\overline{P}^{\phi}_{\mathbf{\Delta},f}(M(s))$ that the delay margin is smaller than a given threshold ϕ when $\Delta \in \mathcal{B}_{\mathbf{\Delta}}$ for the negative feedback loop obtained by connecting y to u in Fig. 1.

So, $\overline{P}^{\phi}_{\mathbf{\Delta},f}(M(s))$ is the probability that the interconnection of Fig. 2 is not stable for all $G(s) = e^{-\tau s}$, $\tau \in [0\ \phi]$. A control system can then be rejected or validated depending on whether $\overline{P}^{\phi}_{\mathbf{\Delta},f}(M(s))$ does or does not exceed a given threshold ϵ. A practical approach is presented in Section IV to compute tight bounds on $\overline{P}^{\phi}_{\mathbf{\Delta},f}(M(s))$ for Single-Input Single-Output (SISO) systems, *i.e.* when $u \in \mathbb{R}$ and $y \in \mathbb{R}$. The global underlying idea is now summarized in Section III.

III. Overview of the proposed solution

During the past 5 years, a significant effort has been put in the development of probabilistic μ theory and its implementation in the Stochastic Worst Case Analysis Toolbox (STOWAT) [18]. Stability and H_∞ performance were studied first [20], [21], followed by gain/phase/disk margins [19]. The novelty of this paper lies in the introduction of a delay margin analysis algorithm to solve Problem 2.1, and to enrich this set of probabilistic μ-based tools.

Probabilistic μ-analysis builds upon classical μ. A preliminary step is therefore to transform the interconnection of Fig. 2 into an LFR, which requires to replace the exponential delay $G(s) = e^{-\tau s}$ with a rational function $\Phi(\alpha)$. A simple but exact representation is proposed in Section IV-A, in the sense that it has the same properties (unit gain and phase varying linearly with frequency) as the actual time delay. This is the first main contribution of the paper.

From an algorithmic point of view, probabilistic μ-analysis combines classical μ-based tools with a branch-and-bound (B&B) algorithm to explore the whole uncertainty domain $\mathcal{B}_\delta$. The delay margin is first computed at the center of $\mathcal{B}_\delta$, *i.e.* for $\Delta = 0$. If it is larger (resp. smaller) than the desired threshold ϕ, it is then checked whether the delay margin requirement is satisfied (resp. violated) on the entire domain $\mathcal{B}_\delta$ using sufficient conditions involving μ upper bound computations (see Propositions 4.1 and 4.3 in Sections IV-B and IV-C). If this cannot be guaranteed, $\mathcal{B}_\delta$ is finally partitioned into smaller boxes and this process is repeated until

each box has guaranteed sufficient/insufficient margin, or is small enough to be neglected (see Section IV-D). Guaranteed upper and lower bounds on the exact probability of delay margin violation $\overline{P}^{\phi}_{\mathbf{\Delta},f}(M(s))$ are finally obtained, based on the probability distributions of the uncertain parameters δ, thus solving Problem 2.1.

A major difficulty with respect to the gain/phase/disk margins considered in [19] is that the aforementioned exact rational representation of the delay function introduces a parameter with frequency-dependent bounds that adds to the uncertainties already present in Δ. An algorithm is proposed in Section IV-B to take this dependence on frequency into account, which is not possible with state-of-the-art μ-based tools. This is the second main contribution of the paper. As explained in Section IV-C, this exact representation can for the moment only be used for the delay margin satisfaction test, and its applicability to the violation test remains an open issue currently under investigation. A second-order Padé approximation is therefore introduced in Section IV-A and used in a more classical way to perform the delay margin violation test in Section IV-C.

IV. Probabilistic delay margin

A. Interconnection for delay margin analysis

The negative feedback loop of Fig. 2 is first built from the LFR of Fig. 1. $G(s)$ represents a time-delay $\tau \in [0\ \phi]$:

$$G(s) = e^{-\tau s} \tag{2}$$

and the nominal closed-loop uncertain system is obtained for $\tau = 0$, *i.e.* $G(s) = 1$. $G(s)$ does not have a rational dependence on τ. Therefore, it should first be replaced with a rational expression to be able to build an LFR and apply robustness tools such as μ-analysis. Two expressions are used: one to check whether the delay margin is larger than ϕ over an entire box $D \subset \mathcal{B}_\delta$ (satisfaction test, see Section IV-B) and the other to check whether it is lower than ϕ (violation test, see Section IV-C).

The transformation used for the delay margin satisfaction test is based on the same idea as in [7], but it is much simpler. $G(s)$ has a unitary gain and introduces a frequency-dependent phase shift equal to $-\tau\omega$ at the frequency ω. The delay margin is thus guaranteed to be larger than ϕ if the system remains stable for any phase shift in the interval $\mathcal{I} = [-\min(\phi\omega, 2\pi)\ 0]$ at the frequency ω. In this context, $G(s)$ can be replaced with any rational expression $\Phi(\alpha)$ such that $|\Phi(\alpha)| = 1$, and whose phase $\angle\Phi(\alpha)$ covers the interval $\mathcal{I}$ when α covers $[0\ \alpha_{max}(\omega)]$. The following choice is made:

$$\Phi(\alpha) = \frac{2j\alpha^2 - 2(1+j)\alpha + 1}{-2j\alpha^2 - 2(1-j)\alpha + 1} \tag{3}$$

for which:

$$\alpha_{max} = \begin{cases} \dfrac{1-\beta-\sqrt{1+\beta^2}}{2} & \text{if} \quad \omega < \pi/\phi \\ 0.5 & \text{if} \quad \omega = \pi/\phi \\ \dfrac{1-\beta+\sqrt{1+\beta^2}}{2} & \text{if} \quad \omega > \pi/\phi \end{cases} \tag{4}$$

and $\beta = \tan\frac{-\min(\phi\omega,2\pi)}{2}$. To get equation (4), just note that:

$$\angle\Phi(\alpha) = 2\arctan\frac{2\alpha(\alpha-1)}{1-2\alpha} \tag{5}$$

which gives for $\alpha = \alpha_{max}$:

$$\tan\frac{\angle\Phi(\alpha_{max})}{2} = \beta = \frac{2\alpha_{max}(\alpha_{max}-1)}{1-2\alpha_{max}} \tag{6}$$

and finally $2\alpha_{max}^2 + 2(\beta-1)\alpha_{max} - \beta = 0$. The motivation for this choice of $\Phi(\alpha)$ is that $\angle\Phi(\alpha)$ and $\alpha_{max}(\omega)$ are simple functions with a smooth behavior, as shown in Fig. 3.

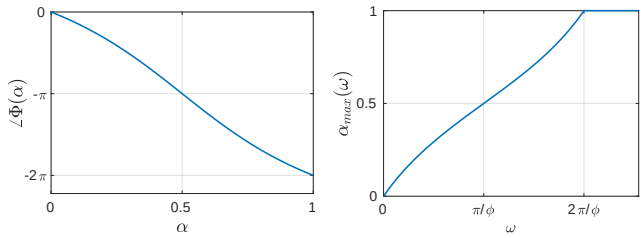

Fig. 3: $\Phi(\alpha)$ and $\alpha_{max}(\omega)$ used for the satisfaction test

Standard matrix manipulations based on the Redheffer star product allow to equivalently transform the interconnection of Fig. 2, where $G(s)$ is replaced with $\Phi(\alpha)$, into that of Fig. 4, where $\Delta_m = \alpha I_2$, $\alpha \in [0\ \alpha_{max}(\omega)]$, contains a single parameter with frequency dependent bounds. Note that a rational function $\Phi(\alpha)$ of degree 1 in α would lead to a simpler LFR, where α is not repeated in Δ_m. This is however not possible here. α would indeed tend to $\pm\infty$ to allow $\angle\Phi(\alpha)$ to cover $[-2\pi\ 0]$, but the μ-based tools used in the sequel require the Δ block of the LFR to be bounded.

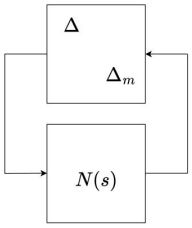

Fig. 4: Transformation of the interconnection of Fig. 2

For the delay margin violation test, the exponential term (equation 2) is replaced by a Padé approximation, whose order is chosen based on a balance between accuracy, complexity and validity. Indeed, the higher the order, the better the accuracy, but the larger the number of repetitions of τ in the resulting LFR. Moreover, the negative phase shift caused by the approximation should be less in absolute value than the one induced by the true time delay for all frequencies, as clarified in Section III.C, and this property may not be satisfied for all orders. In view of these elements, replacing $G(s)$ by a second-order Padé approximation:

$$\Psi(s) = \frac{(\tau s)^2 - 6\tau s + 12}{(\tau s)^2 + 6\tau s + 12} \tag{7}$$

appears to be a good compromise. It has indeed been observed in practice that the first-order approximation is not accurate enough, whereas the second-order one is usually

sufficient, as shown in Section V. Moreover, the second-order Padé approximation satisfies the condition on the phase. The use of a higher-order approximation does not raise any technical difficulty, provided that the condition on the phase is satisfied. To equivalently transform the interconnection of Fig. 2, where $G(s)$ is replaced with $\Psi(s)$, into that of Fig. 4, where $\Delta_m = \tau I_2$, $\tau \in [0\ \phi]$, an approach similar to the one used for first order Padé approximations in [10] is applied.

B. Checking stability margin satisfaction on a box

Let us consider a box $D \in \mathcal{B}_\delta$ and the exact characterization (3) of the delay function. The interconnection $N(s) - \text{diag}\,(\Delta, \alpha I_2)\,, \Delta \in D_{\boldsymbol{\Delta}}$, of Fig. 4 is normalized and equivalently replaced by $\tilde{N}(s) - \text{diag}\,(\tilde{\Delta}, \alpha I_2)$, $\tilde{\Delta} \in \mathcal{B}_{\boldsymbol{\Delta}}$. Checking whether the delay margin requirement is satisfied on the entire box D can then be done using the necessary and sufficient condition of Proposition 4.1.

Proposition 4.1: The delay margin is larger than ϕ on a given box $D \subset \mathcal{B}_\delta$ if and only if the interconnection $\tilde{N}(s) - \text{diag}\,(\tilde{\Delta}, \alpha I_2)$ is stable $\forall \tilde{\Delta} \in \mathcal{B}_{\boldsymbol{\Delta}}$ and $\forall \alpha \in [0\ \alpha_{max}(\omega)]$.

Proposition 4.1 requires solving a non-standard μ-analysis problem, where the bound on α is frequency-dependent. A classical approach is to reduce the whole frequency range to a finite grid $(\omega_k)_k$. α is then normalized at each ω_k, *i.e.* $\alpha = \frac{\alpha_{max}(\omega_k)}{2}(\delta_m + 1), \delta_m \in [-1\ 1]$, resulting in a fully normalized interconnection $\tilde{N}(s) - \text{diag}\,(\tilde{\Delta}, \delta_m I_2)$. It is finally checked if $\mu(\tilde{N}(j\omega_k)) \leq 1$ for all k (due to NP-hardness, an upper bound $\overline{\mu}(\tilde{N}(j\omega_k))$ is computed in practice, see [22]). But this strategy is doomed to fail. When the box D gets closer to the limit between the domains of delay margin satisfaction and violation, the peak value of $\mu(\tilde{N}(j\omega))$ tends to 1 and the risk of missing critical frequencies where $\mu > 1$ increases. Adding more frequencies to the grid can be tempting, but the computational time quickly becomes prohibitive. Moreover, several tests have shown that all frequencies where $\mu > 1$ always end up being missed, thus leading to the erroneous claim that the delay margin requirement is met. A better approach uses the Hamiltonian-based algorithm of [23], which computes a guaranteed μ upper bound over an entire frequency interval. A three-step strategy is proposed here, as detailed in Algorithm 1. The method of [23] is first applied at a given frequency ω_k assuming that all uncertainties have constant bounds, which leads to an initial interval $[\omega_{min}\ \omega_{max}]$ including ω_k on which $\overline{\mu}(\tilde{N}(j\omega)) \leq 1$ (step 1). But α has an increasing frequency-dependent upper bound, as shown in Fig. 3 (right). Therefore, ω_{min} is an admissible lower bound on the validity interval for the considered delay margin problem, but it can be further decreased (step 2). On the opposite, ω_{max} is too large and is reduced by means of a dichotomic search (step 3). Algorithm 1 is repeated until the union of all validity intervals covers the whole frequency range, which guarantees that the delay margin is larger than ϕ, or until an ω_k is found such that $\overline{\mu}(\tilde{N}(j\omega_k)) > 1$. In the latter case, either the delay margin is not larger than ϕ, or the μ upper bound is too conservative to prove the converse.

Algorithm 1 Validity interval $[\omega_{min}\ \omega_{max}]$ around ω_k

Step 1 - Initialization:

1) select an unevaluated frequency ω_k and normalize α at ω_k to get the fully normalized interconnection $\tilde{N}(s) - \text{diag}\,(\tilde{\Delta}, \delta_m I_2)$, $\tilde{\Delta} \in \mathcal{B}_{\boldsymbol{\Delta}}$, $\delta_m \in [-1\ 1]$
2) check whether $\overline{\mu}(\tilde{N}(j\omega_k)) \leq 1$
3) **if** not **then** STOP **else** use [23] to compute an initial interval $[\omega_{min}\ \omega_{max}]$ around ω_k on which $\overline{\mu}(\tilde{N}(j\omega)) \leq 1$

Step 2 - Lower bound improvement:

initialization: set $\omega_{test} = \omega_k$
while $\omega_{test} - \omega_{min} > \epsilon$ **do**

1) set $\omega_{test} = \omega_{min}$
2) normalize α at ω_{test} and compute $\tilde{N}(s)$
3) use [23] to compute an interval $[\omega_1\ \omega_2]$ around ω_{test} on which $\overline{\mu}(\tilde{N}(j\omega)) \leq 1$
4) set $\omega_{min} = \omega_1$

end while

Step 3 - Upper bound improvement:

initialization: set $\omega_{low} = \omega_k$ and $\omega_{high} = \omega_{max}$
while $\omega_{high} - \omega_{low} > \epsilon$ **do**

1) set $\omega_{test} = \frac{(\omega_{high} + \omega_{low})}{2}$
2) normalize α at ω_{test} and compute $\tilde{N}(s)$
3) check if the D and G scaling matrices associated to the μ upper bound of step 1 are still valid at ω_{low}
4) **if** valid **then**
   - use [23] to determine the validity interval $[\omega_1\ \omega_2]$
   - **if** $\omega_2 > \omega_{test}$ **then** set $\omega_{max} = \omega_{test}$ and $\omega_{low} = \omega_{test}$ **else** set $\omega_{high} = \omega_{test}$

end while

C. Checking stability margin violation on a box

The Padé approximation (7) of the delay function is now used. The interconnection $N(s) - \text{diag}\,(\Delta, \tau I_2)\,, \Delta \in D_{\boldsymbol{\Delta}}, \tau \in [0\ \phi]$, of Fig. 4 is normalized and equivalently replaced by $\tilde{N}(s) - \text{diag}\,(\tilde{\Delta}, \delta_m I_2)$, $\tilde{\Delta} \in \mathcal{B}_{\boldsymbol{\Delta}}, \delta_m \in [-1\ 1]$, *i.e.* $\tau = \frac{\phi}{2}(\delta_m + 1)$. Proposition 4.2 then provides a sufficient condition to check whether the delay margin requirement is violated on the entire box $D \subset \mathcal{B}_\delta$.

Proposition 4.2: The delay margin is lower than ϕ on a given box $D \subset \mathcal{B}_\delta$ if $\forall \tilde{\Delta} \in \mathcal{B}_{\boldsymbol{\Delta}}$, $\exists\, \hat{\delta}_m \in [-1\ 1]$ such that the interconnection $\tilde{N}(s) - \text{diag}\,(\tilde{\Delta}, \hat{\delta}_m I_2)$ is unstable.

Proposition 4.2 cannot be directly evaluated using standard μ-based tools and is therefore replaced by Proposition 4.3.

Proposition 4.3: The delay margin is lower than ϕ on a given box $D \subset \mathcal{B}_\delta$ if $\exists\, \hat{\delta}_m \in [-1\ 1]$ such that the interconnection $\tilde{N}(s) - \text{diag}\,(\tilde{\Delta}, \hat{\delta}_m I_2)$ is unstable $\forall \tilde{\Delta} \in \mathcal{B}_{\boldsymbol{\Delta}}$.

As suggested in [19], $\hat{\delta}_m$ is determined by studying the stability of the nominal interconnection $\tilde{N}(s) - \text{diag}\,(0_{p\times p}, \delta_m I_2)$. A finite number of values $(\delta_m^k)_k$ which grid the interval $[-1\ 1]$ are considered, and the one which moves a pole of the interconnection the farthest in the right

half-plane is selected to be $\hat{\delta}_m$. As $\hat{\delta}_m$ remains constant in Proposition 4.3, it can be integrated into $\tilde{N}(s)$ to form a reduced normalized interconnection $\tilde{N}_r(s)-\tilde{\Delta}$, where $\tilde{N}_r(s)$ is unstable. It is then sufficient to check whether this reduced interconnection remains unstable $\forall\tilde{\Delta} \in \mathcal{B}_{\mathbf{\Delta}}$. This can be done easily by determining with standard μ-based tools if $\overline{\mu}(\tilde{N}_r(j\omega)) \leq 1$ on the whole frequency range.

Unlike the delay margin satisfaction test of Section IV-B, some conservatism is introduced here. First, the Padé approximation (7) is used instead of the exact delay (2). This is admissible, since its phase is smaller in absolute value than the one of $G(s)$ for all frequencies. But unlike Proposition 4.1, this makes the condition of Proposition 4.2 non-necessary. Higher-order Padé approximations can be used to mitigate this conservatism, at the price of a slight increase in the computational time. Second, $\hat{\delta}_m$ is fixed to a constant value in Proposition 4.3, which makes the condition even more non-necessary. But as the size of D decreases along the iterations of the B&B algorithm (see Section IV-D), it is reasonable to think that there is more and more chance that a single value of $\hat{\delta}_m$ will fit.

Remark 4.1: The exact characterization (3) cannot be used as for the satisfaction test. Indeed, the upper bound $\alpha_{max}(\omega)$ on α is frequency-dependent, and tends to 0 when $\omega \to 0$. So $\hat{\delta}_m$ in Proposition 4.3 could not be constant and would necessarily depend on ω. As a result, the state-state matrices of $\tilde{N}_r(s)$ would also be frequency-dependent, thus preventing to use the Hamiltonian-based approach of [23] in Algorithm 1.

D. Algorithmic issues

The conditions for determining whether a given delay margin requirement is satisfied (Proposition 4.1) or violated (Proposition 4.3) on an entire box are embedded into the same B&B algorithm as in [19], which is not reproduced here for the sake of brevity. $\mathcal{B}_\delta$ is divided into smaller and smaller boxes until each box has guaranteed sufficient/insufficient delay margin, or has a probability lower than a user-defined threshold p_{min}. Before that, a preliminary stability analysis should be performed with Algorithm 1 of [20], leading to:

$$\mathcal{B}_\delta = D_s \cup D_{\overline{s}} \cup D_{s_u} \tag{8}$$

where D_s, $D_{\overline{s}}$ and D_{s_u} are the domains of guaranteed stability, guaranteed instability and undetermined stability respectively. The delay margin can indeed only be evaluated for stable systems, so the application of the delay margin algorithm is restricted to D_s, leading to:

$$D_s = D_m \cup D_{\overline{m}} \cup D_{m_u} \tag{9}$$

where D_m, $D_{\overline{m}}$ and D_{m_u} are the domains of guaranteed delay margin satisfaction, guaranteed delay margin violation and undetermined delay margin, with corresponding probabilities $p(D_m)$, $p(D_{\overline{m}})$ and $p(D_{m_u})$. Combing (8) and (9), the final partitioning of $\mathcal{B}_\delta$ can be obtained as:

$$\mathcal{B}_\delta = D_m \cup D_{\overline{m}} \cup D_{m_u} \cup D_{\overline{s}} \cup D_{s_u} \tag{10}$$

Guaranteed bounds on the exact probability $\overline{P}^{\phi}_{\mathbf{\Delta},f}(M(s))$ of delay margin violation are finally obtained, which solves Problem 2.1:

$$p(D_{\overline{m}}) \leq \overline{P}^{\phi}_{\mathbf{\Delta},f}(M(s)) \leq p(D_{\overline{m}}) + p(D_{m_u}) = p(D_s) - p(D_m)$$

V. Numerical results

The proposed algorithm has been implemented in the STOWAT and is validated here on a simple satellite model with two uncertainties, adapted from [4]. The low number of uncertainties indeed allows the results to be represented graphically, which enhances clarity. The satellite is represented as a symmetric cylinder spinning around the symmetry axis z with a constant angular rate Ω. The angular rates ω_x and ω_y around the x and y axes are controlled using torques T_x and T_y. Let I_x, $I_y = I_x$ and I_z be the inertia of the satellite with respect to the x, y and z axes respectively. The system rotational motion can be described by:

$$\begin{cases} T_x = I_x\dot{\omega}_x - \omega_y\Omega(I_x - I_z) \\ T_y = I_x\dot{\omega}_y - \omega_x\Omega(I_z - I_x) \end{cases} \tag{11}$$

Uniformly distributed uncertain parameters $\delta_1 \in [-0.5, 2.5]$ and $\delta_2 \in [0, 2]$ are introduced, leading to:

$$\begin{bmatrix} \dot{\omega}_x \\ \dot{\omega}_y \end{bmatrix} = \begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \end{bmatrix} + \begin{bmatrix} \delta_1 & 0 \\ 0 & \delta_2 \end{bmatrix} \begin{bmatrix} u_x \\ u_y \end{bmatrix} \tag{12}$$

where $u_x = \frac{T_x}{I_x}, u_y = \frac{T_y}{I_x}$. Two measures ν_x, ν_y are available:

$$\begin{bmatrix} \nu_x \\ \nu_y \end{bmatrix} = \begin{bmatrix} 1 & a \\ -a & 1 \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \end{bmatrix} \tag{13}$$

and a static controller K is applied:

$$\begin{bmatrix} u_x \\ u_y \end{bmatrix} = -K \begin{bmatrix} \nu_x \\ \nu_y \end{bmatrix} = - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \nu_x \\ \nu_y \end{bmatrix} \tag{14}$$

It is assumed in the sequel that $a = 10$. A preliminary probabilistic stability analysis is performed on the uncertain closed-loop system (12)-(14) using Algorithm 1 of [20]. The first channel of the control loop is then opened, the second one remaining closed. An LFR is obtained as in Fig. 1, where $u = u_x$ and $y = \nu_x$. A probabilistic delay margin analysis is finally performed with $\phi = 0.2$ using the proposed algorithm. The B&B algorithm stops when $p(D) \leq p_{min} = 1/5000\%$ for all $D \in \mathcal{L}$. It can be determined in a few seconds that:

1) 48.1% of the uncertainty domain $[-0.5\ 2.5] \times [0\ 2]$ is guaranteed to be stable in closed loop,
2) the probability of delay margin violation satisfies $\overline{P}^{\phi}_{\mathbf{\Delta},f}(M(s)) \in [26.4\%\ 28.2\%]$.

Results are presented in Fig. 5 with the following color code:

- green: the delay margin is guaranteed to be $\geq \phi$,
- red: the delay margin is guaranteed to be $< \phi$,
- blue: the delay margin is undetermined,
- orange: instability is guaranteed,
- gray: stability is undetermined.

A comparison is made with a classical grid-based approach:

- magenta: the delay margin is guaranteed to be $\geq \phi$,
- yellow: the delay margin is guaranteed to be $< \phi$,
- black: the system is unstable.

Both analyses are in good agreement, but the STOWAT provides guaranteed results on the whole uncertainty domain, and not only for a finite number of points.

However, the second-order Padé approximation can sometimes lead to some conservatism. If the same analysis is performed with $\phi = 0.4$, a substantial part of the uncertainty domain remains undetermined, as seen in Fig. 6 (blue region in the area $\delta_1 \in [0\ 1]$ and $\delta_2 \in [0\ 0.75]$). A comparison with a grid-based approach indeed shows that in this region, the system is unstable when perturbed by a true time delay (upper plot), but remains stable when perturbed by a second-order Padé approximation (lower plot). The conservatism in this region cannot be reduced by changing the stopping criterion of the B&B algorithm, but probably by considering higher-order Padé approximations, at the price of a slight increase in the computational time.

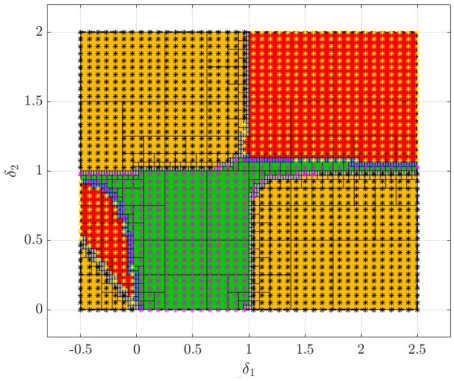

Fig. 5: Delay margin analysis with $\phi = 0.2s$

VI. Conclusion

An algorithm for probabilistic delay margin analysis of SISO systems has been outlined and integrated into the STOWAT. Its capabilities and limitations have been presented through the application to a demonstrative benchmark. Follow-up research will focus on replacing the Padé approximation-based violation test with an exact computational method, as it is already the case for the margin satisfaction test, so as to reduce conservatism.

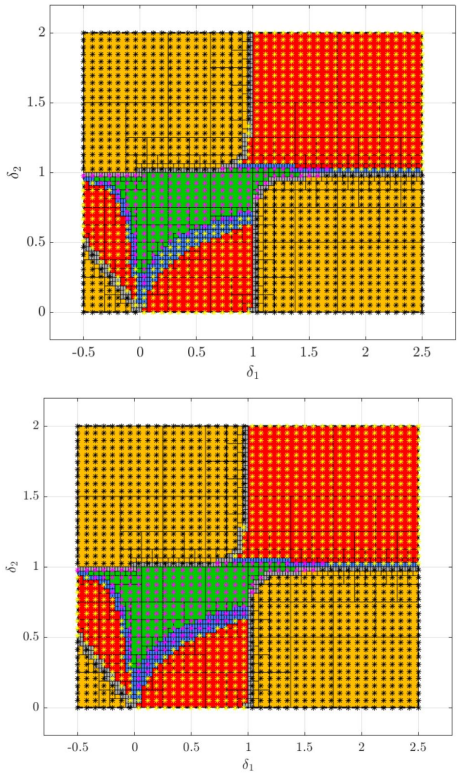

Fig. 6: Delay margin analysis with $\phi = 0.4s$ and comparison with a delay-based / Padé-based grid (upper / lower plot)

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