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STABILIZATION OF THE KAWAHARA EQUATION WITH SATURATED INTERNAL OR BOUNDARY FEEDBACK CONTROLS

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ABSTRACT. In this paper, we study the stabilizability of the Kawahara equation in a bounded interval $(0, L)$ by a saturated internal control and boundary feedback control. We prove the exponential stabilization of the corresponding closed-loop system. We first show the well-posedness of the closed-loop system using linear, nonlinear semigroup theory and fixed-point arguments. Next, we prove two stabilization results by means of interior-saturating feedback and boundary-saturated feedback as well, using observability inequalities and certain unique continuation principle.

1. Introduction and main results.

1.1. **System under study.** Let us consider the following nonlinear Kawahara equation

$$\begin{cases} \partial_t u + \partial_x u + \partial_x^3 u - \partial_x^5 u + u^p \partial_x u + \mathbf{f}_1(t, x) = 0, & (t, x) \in (0, \infty) \times (0, L), \\ u(t, 0) = u(t, L) = \partial_x u(t, 0) = \partial_x u(t, L) = 0, & t \in (0, \infty), \\ \partial_x^2 u(t, L) = \mathbf{f}_2(t), & t \in (0, \infty), \\ u(0, x) = u_0(x), & x \in (0, L), \end{cases} \quad (1)$$

with $p \in [1, 2)$, and the functions \mathbf{f}_1 and \mathbf{f}_2 are the controls acting internally and through the boundary conditions, respectively. The objective of this work is to show the exponential stabilization of (1) by acting with either internal or boundary controls, in the case where these controls are subject to saturation.

The Kawahara equation $\partial_t u + \partial_x u + \partial_x^3 u - \partial_x^5 u + u^p \partial_x u = 0$ is a fifth-order, nonlinear one-dimensional equation, and is typically used to model the propagation

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of small-amplitude long waves in fluid dynamics, water waves, plasma dynamics, among other dispersive phenomena (see [19, 18, 14] and the references therein). This equation can also be considered as an extension of the Korteweg-de Vries equation (KdV), including high order dispersion effects. It has been studied as a fifth-degree KdV equation and singularly perturbed KdV equation in [2, 31] respectively. Discovered in 1972 by Kawahara [19], it was also obtained by Hasimoto for shallow water systems in 1970 [17]. Since then, the Kawahara equation has been a subject of study from different points of view. Recently, the control and stabilization properties have attracted the attention of the control community. Let us provide a brief overview of this subject.

As far as we know, the first works dealing with these kinds of control problems for Kawahara equation were [39, 40, 41]. In these works, the stabilization of global solutions of the linear Kawahara equation using localized damping was addressed following the ideas presented in the works [33, 30], where similar problems were analyzed for the KdV equation. These results were then extended to the nonlinear Kawahara equation, with the nonlinearity $u\partial_x u$. In [1], a similar problem was studied for the generalized Kawahara equation with the nonlinearity $u^p\partial_x u$ for $p \in [1, 4)$. It is important to note that in [1], the energy of solutions decay exponentially for arbitrary initial data in $L^2(0, L)$ for $p \in [1, 2]$. However, this decay was only proved for small initial data if $p \in (2, 4)$. Internal and boundary delayed stabilization problems for the Kawahara equation were considered in [10, 12, 5]. The exponential stability was shown using multipliers, Lyapunov functionals, and compactness ideas. In the work [4], the authors investigate the exponential stability issue of the Kawahara equation with feedback consisting of damping and finite memory terms acting in the boundary. Exponential stabilization of the Kawahara equation with distributed infinite memory has been addressed in [3]. We also mention [6], where Massera's theorems were obtained, [11], where the internal null controllability was shown via Carleman estimates, [9], where controllability in weighted Sobolev spaces was proved, and [7, 8], where the control with overdetermination condition was considered.

As we already mentioned, the focus of this paper is to analyze the case of saturated controllers. Saturation in control systems is related to the fact that in most real-life applications, the actuators are subject to constraints, limitations due to physical, economical, operational reasons, etc. It is well-known that even in the finite-dimensional case the introduction of saturation in the controller could destabilize the system [37, Example 1.1], making the saturated control problem a challenging task. In the finite-dimensional framework, the stability analysis could be done by studying an appropriate Lyapunov function and a sector condition related to the saturation map [36, 37, 38].

With respect to saturation in infinite-dimensional control systems, there is not a general method to tackle these problems. Noteworthy works include [32] where the stabilization of a one dimensional wave equation was considered with saturated internal and boundary controls. In this work, stability results were proved by using Lyapunov functions and LaSalle's invariance principle. In the case of parabolic equations in [21], several results are collected in the one-dimensional case. The principal idea is to decompose the system into its stable and stable parts via a spectral approach. The unstable finite dimensional part is tackled as in [37], and for the remaining part, an appropriated Lyapunov function is considered. In the case of KdV-type equations, the approach is to follow the ideas of the works [33, 30]

and to prove the exponential stability via compactness arguments. This requires, in particular, the use of unique continuation results for nonlinear PDEs. In this direction, we mention the works [22, 23, 28, 25]. This work can be considered a continuation of the works [23, 25], where saturated stabilization problems were analyzed for the KdV equation. In this sense, we extend these ideas to a high order dispersive equation. Moreover, we address a slightly more general nonlinearity.

1.2. Setting of the problem. This paper deals with the feedback stabilization of the system (1) by means of saturated control acting in-domain or through a boundary action. In this work, we consider two types of saturation function \mathbf{sat} which can be given by the following. Let $M > 0$ be the saturation level. We define

$$\mathbf{sat}_{loc}(s) = \begin{cases} s, & \text{if } |s| \leq M, \\ M \operatorname{sgn}(s), & \text{if } |s| \geq M, \end{cases} \quad (2)$$

and the second one is

$$\mathbf{sat}_2(s) = \begin{cases} s, & \text{if } \|s\|_{L^2(0,L)} \leq M, \\ \frac{sM}{\|s\|_{L^2(0,L)}}, & \text{if } \|s\|_{L^2(0,L)} \geq M. \end{cases} \quad (3)$$

1.2.1. Internal control. We start by describing the case where the control action is in-domain.

$$\begin{cases} \partial_t u + \partial_x u + \partial_x^3 u - \partial_x^5 u + u^p \partial_x u + \mathbf{sat}(a(x)u(t, x)) = 0, & (t, x) \in (0, \infty) \times (0, L), \\ u(t, 0) = u(t, L) = \partial_x u(t, 0) = \partial_x u(t, L) = 0, & t \in (0, \infty), \\ \partial_x^2 u(t, L) = 0, & t \in (0, \infty), \\ u(0, x) = u_0(x), & x \in (0, L), \end{cases} \quad (4)$$

where $a \in L^\infty(0, L)$ is a nonnegative localized function given by¹

$$a_1 \geq a(x) \geq a_0 > 0 \text{ on } \omega \subset [0, L], \omega \text{ is a nonempty open subset of } [0, L]. \quad (5)$$

1.2.2. Boundary control. In this work, we also explore the case where the saturated actuator is at the boundary, consider the following equation:

$$\begin{cases} \partial_t u + \partial_x u + \partial_x^3 u - \partial_x^5 u + u^p \partial_x u = 0, & (t, x) \in (0, \infty) \times (0, L), \\ u(t, 0) = u(t, L) = \partial_x u(t, 0) = \partial_x u(t, L) = 0, & t \in (0, \infty), \\ \partial_x^2 u(t, L) = \mathbf{sat}_{loc}(\alpha \partial_x^2 u(t, 0)), & t \in (0, \infty), \\ u(0, x) = u_0(x), & x \in (0, L), \end{cases} \quad (6)$$

where $|\alpha| < 1$.

Remark 1.1. It is important to remark that in the case of internal control, we can deal with both types of saturation \mathbf{sat}_{loc} and \mathbf{sat}_2 while in the boundary case, we only analyze \mathbf{sat}_{loc} .

Remark 1.2. In the boundary case, an alternative formulation could be adding the saturation term as follows

$$\partial_x^2 u(t, L) = \alpha \mathbf{sat}_{loc}(\partial_x^2 u(t, 0)), \quad t \in (0, \infty).$$

¹For instance we can take $a_1 = \|a\|_\infty$.

The results obtained in this work can be easily adapted to this case with no major differences. Observe that these cases are not equivalent; for instance, for s such that $|s| > M > |\alpha s|$ we have

$$\alpha s = \mathbf{sat}_{loc}(\alpha s) \neq \alpha \mathbf{sat}_{loc}(s) = M\alpha \mathbf{sgn}(s).$$

The main objective of this paper is to show the exponential stability of (4) and (6) in the presence of saturated control. More precisely, we investigate the semiglobal exponential stabilization result (see [Theorem 3.5](#)) for the Kawahara equation (4) with internal saturation. That means, for any $r > 0$, we can find a positive number $\mu(r) > 0$ such that the energy

$$E(t) = \frac{1}{2} \int_0^L u(t, x)^2 dx$$

of the solution of (4) decays exponentially with a rate $\mu(r)$, provided the initial data satisfying the following smallness assumption $\|u_0\|_{L^2(0,L)} \leq r$. For the saturated boundary feedback case, we study local exponential stabilization result (see [Theorem 2.11](#)) for the system (6) with $|\alpha| < 1$. That is, there exist positive constants $r, \mu, C > 0$ such that whenever $\|u_0\|_{L^2(0,L)} \leq r$, the solution of (6) satisfies

$$\|u(t, \cdot)\|_{L^2(0,L)} \leq Ce^{-\mu t} \|u_0\|_{L^2(0,L)}, \quad \forall t \geq 0. \quad (7)$$

Our study is based on deriving appropriate observability inequalities for the saturated systems. These inequalities are obtained using compactness ideas and Carleman estimates. We also put special emphasis on the global well-posedness of the associated systems. Indeed, as we can observe from the definition of the saturation maps (2) and (3), if the argument of the map is small enough (in the appropriated norms), then the saturation is not really active. For instance, this was one of the main points in [\[28\]](#).

The structure of this paper is the following: In [Section 2](#), we study the well-posedness result and exponential stabilizability associated with (6). [Section 3](#) discusses the existence, uniqueness and stability analysis for the Kawahara equation (4) with internal saturation. At the end, we conclude the paper with some additional remarks and open questions.

2. Boundary saturation. This section is devoted to the well-posedness and stability analysis of the Kawahara equation with saturated boundary feedback. At first, we show the existence and uniqueness of the system (6) with the saturated feedback at the boundary. We will follow the work [\[25\]](#) (see also [\[26\]](#)).

2.1. Well-posedness. First, we consider the system (6) without the internal non-linearity, and we prove the well-posedness based on nonlinear semigroup theory. Then, we will add an internal source term in order to take in account the nonlinear term $u^p \partial_x u$. Consider the system

$$\begin{cases} \partial_t u + \partial_x u + \partial_x^3 u - \partial_x^5 u = 0, & (t, x) \in (0, \infty) \times (0, L), \\ u(t, 0) = u(t, L) = \partial_x u(t, 0) = \partial_x u(t, L) = 0, & t \in (0, \infty), \\ \partial_x^2 u(t, L) = \mathbf{sat}_{loc}(\alpha \partial_x^2 u(t, 0)), & t \in (0, \infty), \\ u(0, x) = u_0(x), & x \in (0, L). \end{cases} \quad (8)$$

Let $\mathcal{A}_b : D(\mathcal{A}_b) \rightarrow L^2(0, L)$ the nonlinear operator defined by

$$\mathcal{A}_b w := -w' - w''' + w'''' \quad (9)$$

$$D(\mathcal{A}_b) = \{w \in H^5(0, L) \cap H_0^2(0, L) : w''(L) = \mathbf{sat}_{loc}(\alpha w''(0))\}.$$

Note that the operator \mathcal{A}_b is nonlinear by the action of the boundary saturation. Following [25, 22], we show that \mathcal{A}_b is a closed, dissipative operator and that for some $\lambda > 0$, $D(\mathcal{A}_b) \subset R(\mathcal{I} - \lambda \mathcal{A}_b)$, where $\mathcal{I} : L^2(0, L) \rightarrow L^2(0, L)$, is the identity operator and R denotes the range of an operator. The following result ensures the existence of mild and strong solutions for (8).

Proposition 2.1. *Let $u_0 \in L^2(0, L)$ and $|\alpha| < 1$, then there exists a unique mild solution $u \in C([0, \infty), L^2(0, L))$ of (8). If $u_0 \in D(\mathcal{A}_b)$, this solution is strong and $u \in C([0, \infty), D(\mathcal{A}_b)) \cap C^1((0, \infty), L^2(0, L))$. Moreover, for $T > 0$ and $u_0 \in L^2(0, L)$, the following estimate follows*

$$\|u_0\|_{L^2(0, L)}^2 \leq \frac{1}{T} \|u\|_{L^2((0, T), L^2(0, L))}^2 + \|\partial_x^2 u(\cdot, 0)\|_{L^2(0, T)}^2. \quad (10)$$

Proof. **\mathcal{A}_b is a closed operator.** Consider the operator $\mathcal{A}_b^0 : D(\mathcal{A}_b^0) \rightarrow L^2(0, L)$ defined by $\mathcal{A}_b^0 w := \mathcal{A}_b w$, where $D(\mathcal{A}_b^0) = \{w \in H^5(0, L) \cap H_0^2(0, L) : w''(L) = 0\}$. Then, clearly \mathcal{A}_b^0 is a closed operator. Similarly, by [20, Page 91] the operator \mathbf{sat}_{loc} satisfies

$$\forall (s, \tilde{s}) \in \mathbb{R}^2, \quad |\mathbf{sat}_{loc}(s) - \mathbf{sat}_{loc}(\tilde{s})| \leq |s - \tilde{s}|. \quad (11)$$

Thus, \mathbf{sat}_{loc} is a Lipschitz continuous function and hence the operator \mathcal{A}_b is closed. **\mathcal{A}_b is dissipative.** Let $w, v \in D(\mathcal{A}_b)$, then after some integration by parts and (11) we have

$$\begin{aligned} \langle \mathcal{A}_b w - \mathcal{A}_b v, w - v \rangle_{L^2(0, L)} &= \int_0^L (-w' - u''' + w'''' + v' + v''' - v'''')(w - v) dx \\ &= \frac{1}{2} [(\mathbf{sat}_{loc}(\alpha w''(0)) - \mathbf{sat}_{loc}(\alpha v''(0)))^2 \\ &\quad - (w''(0) - v''(0))^2] \\ &\leq \frac{\alpha^2 - 1}{2} (w''(0) - v''(0))^2 \leq 0, \end{aligned}$$

from where we obtain that \mathcal{A}_b is dissipative.

For $\lambda > 0$ small enough, $D(\mathcal{A}_b) \subset R(\mathcal{I} - \lambda \mathcal{A}_b)$. Take $\lambda > 0$, then $D(\mathcal{A}_b) \subset R(\mathcal{I} - \lambda \mathcal{A}_b)$, is equivalent to given $w \in D(\mathcal{A}_b)$ find $v \in D(\mathcal{A}_b)$ such that $(\mathcal{I} - \lambda \mathcal{A}_b)v = w$. Define $\tilde{\lambda} = \frac{1}{\lambda}$, then we have to find solutions $v \in D(\mathcal{A}_b)$ of

$$\begin{cases} \tilde{\lambda} v + v' + v''' - v'''' = \tilde{\lambda} w, \\ v(0) = v(L) = v'(0) = v'(L) = 0, \quad v''(L) = \mathbf{sat}_{loc}(\alpha v''(0)). \end{cases} \quad (12)$$

Consider the operator $\mathcal{T} : H^3(0, L) \rightarrow L^2(0, L)$ defined by $\mathcal{T}(y) = z$, where z is the solution of

$$\begin{cases} \tilde{\lambda} z + z' + z''' - z'''' = \tilde{\lambda} w, \\ z(0) = z(L) = z'(0) = z'(L) = 0, \quad z''(L) = \mathbf{sat}_{loc}(\alpha y''(0)). \end{cases} \quad (13)$$

Lemma 2.2. *For all $w \in D(\mathcal{A}_b)$, the operator \mathcal{T} is well-defined, i.e., for all $y \in H^3(0, L)$, there exists a unique solution to (13).*

The proof of [Lemma 2.2](#) is given in the [Appendix A](#). Let us show that \mathcal{T} has a fixed point. Take $C > 0$ to be chosen later and define the convex set

$$K_C = \{z \in H_0^2(0, L) : \|z\|_{H_0^2(0, L)} \leq C\}. \quad (14)$$

As $H_0^2(0, L)$ is compactly embedded in $L^2(0, L)$, the set K_C is a bounded set in $H_0^2(0, L)$ and relatively compact in $L^2(0, L)$. Furthermore, as K_C is closed in $L^2(0, L)$, we can deduce that K_C is compact in $L^2(0, L)$. Our idea now is to apply the Schauder fixed point theorem [[13](#), Theorem B.19]. Thus, we have to prove the following lemma

Lemma 2.3. *There exists $C > 0$ such that $\mathcal{T}(H^3(0, L)) \subset K_C$.*

Proof. Multiplying [\(13\)](#) by z and integrating on $(0, L)$ we get after some integration by parts

$$\tilde{\lambda} \int_0^L z^2 dx + \frac{1}{2} z''(0)^2 = \frac{1}{2} \mathbf{sat}_{loc}(\alpha y''(0))^2 + \tilde{\lambda} \int_0^L wz dx.$$

Then, using the definition of \mathbf{sat}_{loc} given in [\(2\)](#) and Cauchy-Schwartz, we obtain

$$\frac{\tilde{\lambda}}{2} \|z\|_{L^2(0, L)}^2 \leq \frac{M^2}{2} + \frac{\tilde{\lambda}}{2} \|w\|_{L^2(0, L)}^2. \quad (15)$$

Multiplying [\(13\)](#) by xz and integrating on $(0, L)$ we get after some integration by parts

$$\begin{aligned} \tilde{\lambda} \int_0^L xz^2 dx + \frac{3}{2} \int_0^L (z')^2 dx + \frac{5}{2} \int_0^L (z'')^2 dx &= \frac{1}{2} \int_0^L z^2 dx + \frac{L}{2} \mathbf{sat}(\alpha y''(0))^2 \\ &\quad + \tilde{\lambda} \int_0^L xwz dx, \end{aligned}$$

$$\frac{3}{2} \|z'\|_{L^2(0, L)}^2 + \frac{5}{2} \|z''\|_{L^2(0, L)}^2 \leq \left(\frac{L\tilde{\lambda}}{2} + \frac{1}{2} \right) \|z\|_{L^2(0, L)}^2 + \frac{LM^2}{2} + \frac{L\tilde{\lambda}}{2} \|w\|_{L^2(0, L)}^2, \quad (16)$$

and hence using [\(15\)](#) and [\(16\)](#) we conclude, $\|z''\|_{L^2(0, L)}^2 \leq C$, where $C > 0$ only depends on the level of saturation M , L , $\tilde{\lambda}$ and w which ends the proof of [Lemma 2.3](#). \square

Using the Schauder fixed point theorem, we manage to find $v \in H^3(0, L)$ solution of [\(12\)](#), we get $v \in D(\mathcal{A}_b)$ using that the boundary conditions and $w \in D(\mathcal{A}_b)$. We conclude that for all $\lambda > 0$, $D(\mathcal{A}_b) \subset R(\mathcal{I} - \lambda \mathcal{A}_b)$. Using [[24](#), Theorem 4.2] \mathcal{A}_b is the generator of a semigroup of contractions $S(t)$, $t > 0$. By [[24](#), Theorem 4.5] for all $T > 0$, $u_0 \in D(\mathcal{A}_b)$, $u = S(t)u_0 \in C([0, T]; D(\mathcal{A}_b)) \cap C^1((0, T); L^2(0, L))$ is a strong solution to [\(8\)](#). Moreover, by [[24](#), Theorem 4.10] this solution is unique. Also, we can deduce that for all $T > 0$, for all $u_0 \in L^2(0, L)$, there exists a unique mild solution of [\(8\)](#) $u \in C([0, T]; L^2(0, L))$.

To deduce [\(10\)](#), and multiply [\(8\)](#) by $(T - t)u$ and integrate $(0, T) \times (0, L)$, after some integration by parts we get

$$T \int_0^L u_0^2 dx + \int_0^T (T - t) [\mathbf{sat}_{loc}(\partial_x^2 u(t, 0))^2 - \partial_x^2 u(t, 0)^2] dt = \int_0^T \int_0^L u^2 dx dt,$$

using [\(11\)](#) we obtain [\(10\)](#). The result can be extended by density to mild solutions. Therefore, the proof of [Proposition 2.1](#) is finished. \square

Next, we study the well-posedness of the system (8) with an extra source term f .

$$\begin{cases} \partial_t u + \partial_x u + \partial_x^3 u - \partial_x^5 u = f, & (t, x) \in (0, \infty) \times (0, L), \\ u(t, 0) = u(t, L) = \partial_x u(t, 0) = \partial_x u(t, L) = 0, & t \in (0, \infty), \\ \partial_x^2 u(t, L) = \mathbf{sat}_{loc}(\alpha \partial_x^2 u(t, 0)), & t \in (0, \infty), \\ u(0, x) = u_0(x), & x \in (0, L). \end{cases} \quad (17)$$

Formally, this system can be seen as $\partial_t u = \mathcal{A}_b u + f$. The following result can be obtained using [29, Theorem 10.1, Theorem 10.2].

Proposition 2.4. *Let $u_0 \in L^2(0, L)$, $|\alpha| < 1$, $f \in L^1((0, \infty), L^2(0, L))$, then there exists a unique mild solution $u \in C([0, \infty), L^2(0, L))$ of (17). Moreover, if $u_0 \in D(\mathcal{A}_b)$ and $f \in W^{1,1}((0, \infty), L^2(0, L))$ this solution is strong and $u \in C([0, \infty), L^2(0, L))$, $u(t, \cdot) \in D(\mathcal{A}_b)$, for all $t \geq 0$ and $u \in C^1((0, \infty), L^2(0, L))$.*

In order to deal with the nonlinear term $u^p \partial_x u$, we show that mild solutions of (17) have an extra regularity. Consider the space $B_T = C([0, T], L^2(0, L)) \cap L^2((0, T), H^2(0, L))$, endowed with the norm

$$\|u\|_{B_T}^2 = \|u\|_{C([0, T], L^2(0, L))}^2 + \|u\|_{L^2((0, T), H^2(0, L))}^2.$$

Proposition 2.5. *Let $u_0 \in L^2(0, L)$, $|\alpha| < 1$, $f \in L^1((0, T), L^2(0, L))$ and $T > 0$, consider $u \in C([0, T], L^2(0, L))$, the restriction of the solution of (17) on $[0, T]$, then $u \in B_T$. Moreover, the following estimate holds,*

$$\|u\|_{B_T} \leq C(1 + \sqrt{T}) (\|u_0\|_{L^2(0, L)} + \|f\|_{L^1((0, T), L^2(0, L))}). \quad (18)$$

Proof. Suppose u is a strong solution of (17), let $T > 0$ and $0 \leq s \leq T$. Multiplying the first equation of (17) by u and integrating on $(0, s) \times (0, L)$, we get after some integration by parts

$$\begin{aligned} \frac{1}{2} \int_0^L u(s, x)^2 dx + \frac{1}{2} \int_0^s (\partial_x^2 u(t, 0))^2 - \mathbf{sat}(\alpha \partial_x^2 u(t, 0))^2 dt &= \frac{1}{2} \int_0^L u_0^2 dx \\ &+ \int_0^s \int_0^L f u dx dt. \end{aligned} \quad (19)$$

Note that for all $s \in [0, T]$

$$\begin{aligned} \int_0^s \int_0^L f u dx dt &\leq \int_0^T \|u(t, \cdot)\|_{L^2(0, L)} \|f(t, \cdot)\|_{L^2(0, L)} dt \\ &\leq \|u\|_{C([0, T], L^2(0, L))} \|f\|_{L^1((0, T), L^2(0, L))}. \end{aligned}$$

Using the above equation, (11), (19), we get for all $s \in [0, T]$,

$$\begin{aligned} \frac{1}{2} \int_0^L u(s, x)^2 dx + C_\alpha \int_0^s \partial_x^2 u(t, 0)^2 dt &\leq \frac{1}{2} \int_0^L u_0^2 dx \\ &+ \|u\|_{C([0, T], L^2(0, L))} \|f\|_{L^1((0, T), L^2(0, L))}, \end{aligned} \quad (20)$$

where $C_\alpha = \frac{1 - |\alpha|^2}{2} > 0$. Taking the supremum for $s \in [0, T]$, and with Young's inequality, we get

$$\|u\|_{C([0,T],L^2(0,L))}^2 + \int_0^T \partial_x^2 u(t,0)^2 dt \leq C \left(\|u_0\|_{L^2(0,L)}^2 + \|f\|_{L^1((0,T),L^2(0,L))}^2 \right). \quad (21)$$

Let us multiply the first equation of (17) by xu and integrate on $(0, T) \times (0, L)$. Performing integration by parts, we observe

$$\begin{aligned} \frac{1}{2} \int_0^L xu(T, \cdot)^2 dx + \frac{3}{2} \int_0^T \int_0^L \partial_x u^2 dx dt + \frac{5}{2} \int_0^T \int_0^L \partial_x^2 u^2 dx dt &= \frac{1}{2} \int_0^L xu_0^2 dx \\ &+ \frac{1}{2} \int_0^T \int_0^L u^2 dx dt + \frac{L}{2} \int_0^T \mathbf{sat}_{loc}(\alpha \partial_x^2 u(t,0))^2 dt + \int_0^T \int_0^L xuf dx dt. \end{aligned} \quad (22)$$

Note that

$$\int_0^T \int_0^L u^2 dx dt \leq T \|u\|_{C([0,T],L^2(0,L))}^2$$

and

$$\int_0^T \int_0^L xuf dx dt \leq L \|u\|_{C([0,T],L^2(0,L))} \|f\|_{L^1((0,T),L^2(0,L))}.$$

Using the above two inequalities (21) and (22), we derive

$$\|\partial_x^2 u\|_{L^2((0,T),L^2(0,L))}^2 \leq C(1+T) \left(\|u_0\|_{L^2(0,L)}^2 + \|f\|_{L^1((0,T),L^2(0,L))}^2 \right), \quad (23)$$

where $C > 0$ does not depend on T . Thus $u \in L^2((0, T), H^2(0, L))$. Joining (21) and (23) we deduce (18). \square

Before going to prove the global well-posedness result for the nonlinear system (6), we first state some important estimates related to the nonlinear term and whose proof can be founded in [34, Lemma 2.3, Lemma 2.4]:

Proposition 2.6. *Let $u \in B_T$. Then for $1 \leq p \leq 2$, $u^p \partial_x u \in L^1((0, T), L^2(0, L))$ and the map $u \in B_T \mapsto u^p \partial_x u \in L^1((0, T), L^2(0, L))$ is continuous. Moreover, there exists $C > 0$ such that, for any $u, v \in B_T$*

$$\|u^p \partial_x v\|_{L^1((0,T),L^2(0,L))} \leq C(T^{(2-p)/4} + \sqrt{T}) \|u\|_{B_T}^p \|v\|_{B_T}. \quad (24)$$

$$\begin{aligned} \|u^p \partial_x u - v^p \partial_x v\|_{L^1((0,T),L^2(0,L))} &\leq C(T^{(2-p)/4} + \sqrt{T}) \left(\|u\|_{B_T} \|v\|_{B_T}^{p-1} \right. \\ &\quad \left. + \|v\|_{B_T}^p + \|u\|_{B_T}^p \right) \|u - v\|_{B_T}. \end{aligned} \quad (25)$$

We conclude this subsection by proving the global well-posedness result for nonlinear equation (6).

Theorem 2.7. *Let $u_0 \in L^2(0, L)$ and $|\alpha| < 1$, then there exists a unique mild solution $u \in C([0, \infty), L^2(0, L)) \cap L^2(0, \infty, H^2(0, L))$ of (6).*

Proof. Let $u_0 \in D(\mathcal{A}_b)$ and $R, \theta > 0$ to be chosen later. Consider the ball $\mathcal{B}_{B_\theta}(0, R) := \{v \in B_\theta, \|v\|_{B_\theta} \leq R\}$. Then $\mathcal{B}_{B_\theta}(0, R)$ is a complete metric space. Let us introduce the map $\Phi : B_\theta \mapsto B_\theta$, defined by $\Phi(v) = u$, where u solves

$$\begin{cases} \partial_t u + \partial_x u + \partial_x^3 u - \partial_x^5 u = -v^p \partial_x v, & (t, x) \in (0, \theta) \times (0, L), \\ u(t, 0) = u(t, L) = \partial_x u(t, 0) = \partial_x u(t, L) = 0, & t \in (0, \theta), \\ \partial_x^2 u(t, L) = \mathbf{sat}_{loc}(\alpha \partial_x^2 u(t, 0)), & t \in (0, \theta), \\ u(0, x) = u_0(x), & x \in (0, L). \end{cases}$$

We observe first that Φ is well-defined, since by [Proposition 2.6](#) $v^p \partial_x v \in L^1((0, \theta), L^2(0, L))$. By [Proposition 2.5](#) and [Proposition 2.6](#) we get

$$\begin{aligned} \|\Phi(v)\|_{B_\theta} &\leq C(1 + \sqrt{\theta}) (\|u_0\|_{L^2(0, L)} + \|v^p \partial_x v\|_{L^1((0, \theta), L^2(0, L))}) \\ &\leq C_1(1 + \sqrt{\theta}) \|u_0\|_{L^2(0, L)} + C_2(\theta^{(2-p)/4} + \sqrt{\theta}) \|v\|_{B_T}^{p+1}, \end{aligned}$$

for some $C_1, C_2 > 0$. Then taking $R = 3C_1 \|u_0\|_{L^2(0, L)}$ and $\theta > 0$ small enough such that

$$\begin{cases} C_2(\theta^{\frac{2-p}{4}} + \sqrt{\theta}) R^p < \frac{1}{3}, \\ (1 + \sqrt{\theta}) < 2, \end{cases}$$

we have that $\|\Phi(v)\|_{B_\theta} \leq R$ and thus Φ maps $\mathcal{B}_{B_\theta}(0, R)$ into itself. Let us take $v_1, v_2 \in \mathcal{B}_{B_\theta}(0, R)$, and consider $w = u_1 - u_2 = \Phi(v_1) - \Phi(v_2)$. We observe that w solves

$$\begin{cases} \partial_t w + \partial_x w + \partial_x^3 w - \partial_x^5 w = f, & (t, x) \in (0, \theta) \times (0, L), \\ w(t, 0) = w(t, L) = \partial_x w(t, 0) = \partial_x w(t, L) = 0, & t \in (0, \theta), \\ \partial_x^2 w(t, L) = h, & t \in (0, \theta), \\ w(0, x) = 0, & x \in (0, L), \end{cases}$$

where $f = -v_1^p \partial_x v_1 + v_2^p \partial_x v_2$ and $h = \mathbf{sat}_{loc}(\alpha \partial_x^2 u_1(t, 0)) - \mathbf{sat}_{loc}(\alpha \partial_x^2 u_2(t, 0))$. Similar integration by parts as in [Proposition 2.5](#) allows to write

$$\|w\|_{B_\theta} \leq C (\|h\|_{L^2(0, \theta)} + \|f\|_{L^1((0, \theta), L^2(0, L))}). \quad (26)$$

Our idea is to obtain an estimate of $\|h\|_{L^2(0, \theta)}$. Proceeding as [\(19\)](#) we obtain

$$\frac{1}{2} \int_0^L w(\theta, \cdot)^2 dx + \frac{1}{2} \int_0^\theta (\partial_x^2 w(t, 0))^2 - h(t)^2 dt = \int_0^\theta \int_0^L f w dx dt.$$

Thanks to [\(11\)](#), we get

$$\begin{aligned} \|h\|_{L^2(0, \theta)} &= \|\mathbf{sat}_{loc}(\alpha \partial_x^2 u_1(\cdot, 0)) - \mathbf{sat}_{loc}(\alpha \partial_x^2 u_2(\cdot, 0))\|_{L^2(0, \theta)} \\ &\leq \|\alpha \partial_x^2 u_1(\cdot, 0) - \alpha \partial_x^2 u_2(\cdot, 0)\|_{L^2(0, \theta)} \\ &\leq |\alpha| \|\partial_x^2 w(\cdot, 0)\|_{L^2(0, \theta)}, \end{aligned}$$

which implies

$$\frac{1}{2} \int_0^L w(\theta, \cdot)^2 dx + C_\alpha \int_0^\theta \partial_x^2 w(t, 0)^2 dt \leq \int_0^\theta \int_0^L f w dx dt,$$

where C_α is same as mentioned in [\(20\)](#). Thus by Young's inequality we have $\|w\|_{C((0, \theta), L^2(0, L))} \leq C \|f\|_{L^1((0, \theta), L^2(0, L))}$, then by the above inequality

$$\|h\|_{L^2(0, \theta)} \leq |\alpha| \|\partial_x^2 w(\cdot, 0)\|_{L^2(0, \theta)} \leq C \|f\|_{L^1((0, \theta), L^2(0, L))}.$$

Using this estimate in [\(26\)](#) we obtain

$$\begin{aligned} \|\Phi(v_1) - \Phi(v_2)\|_{B_\theta} &= \|w\|_{B_\theta} \\ &\leq C \|f\|_{L^1((0, \theta), L^2(0, L))} \\ &= C \|v_1^p \partial_x v_1 - v_2^p \partial_x v_2\|_{L^1((0, \theta), L^2(0, L))}. \end{aligned}$$

Finally, by [Proposition 2.6](#)

$$\begin{aligned} \|\Phi(v_1) - \Phi(v_2)\|_{B_\theta} &\leq C_3(\theta^{(2-p)/4} + \sqrt{\theta}) \left(\|v_1\|_{B_\theta} \|v_2\|_{B_\theta}^{p-1} + \|v_1\|_{B_\theta}^p \right. \\ &\quad \left. + \|v_2\|_{B_\theta}^p \right) \|v_1 - v_2\|_{B_\theta}, \end{aligned}$$

for some $C_3 > 0$. Then, taking $\theta > 0$, small enough such that $3C_3(\theta^{(2-p)/4} + \sqrt{\theta})R^p < 1$, we get that Φ is a contraction, and by the Banach fixed point theorem and a density argument, we deduce the local in time well-posedness.

To conclude the global well-posedness result since the energy of the system is decreasing, we only need some global a priori estimates. Let $T > 0$ and u be a strong solution of (6). Multiplying (6) by u and integrating over $(0, L) \times (0, s)$, we get after some integration by parts

$$\frac{1}{2} \int_0^L u(s, \cdot)^2 dx + \frac{1}{2} \int_0^s (\partial_x^2 u(t, 0))^2 - \mathbf{sat}(\alpha \partial_x^2 u(t, 0))^2 dt = \frac{1}{2} \int_0^L u_0^2 dx.$$

Using (2) we derive

$$\frac{1}{2} \int_0^L u^2(s, \cdot) dx + C_\alpha \int_0^s \partial_x^2 u(t, 0)^2 dt \leq \frac{1}{2} \int_0^L u_0^2 dx, \quad (27)$$

where C_α is the same as mentioned in (20). Next, taking the supremum for $s \in [0, T]$ in (27),

$$\|u\|_{C([0, T], L^2(0, L))}^2 \leq \|u_0\|_{L^2(0, L)}^2. \quad (28)$$

On the other hand, taking $s = T$ in (27),

$$\int_0^T \partial_x^2 u(t, 0)^2 dt \leq \frac{1}{2} C_\alpha^{-1} \|u_0\|_{L^2(0, L)}^2. \quad (29)$$

Let us multiply the first equation of (6) by xu and integrate on $(0, L) \times (0, T)$,

$$\begin{aligned} \frac{1}{2} \int_0^L xu^2(T, \cdot) dx + \frac{3}{2} \int_0^T \int_0^L \partial_x u^2 dx dt + \frac{5}{2} \int_0^T \int_0^L \partial_x^2 u^2 dx dt &= \frac{1}{2} \int_0^L xu_0^2(x) dx \\ &+ \frac{1}{p+2} \int_0^T \int_0^L u^{p+2} dx dt + \frac{1}{2} \int_0^T \int_0^L u^2 dx dt + \frac{L}{2} \int_0^T \partial_x^2 u(t, 0)^2 dt. \end{aligned}$$

For the term involving $\int_0^T \int_0^L u^2 dx dt$

$$\int_0^T \int_0^L u^2(t, x) dx \leq T \|u\|_{C([0, T], L^2(0, L))}^2.$$

For the term involving $\int_0^T \int_0^L u^{p+2}(t, x) dx dt$, observe that

$$\begin{aligned} \frac{1}{p+2} \int_0^T \int_0^L u^{p+2} dx dt &\leq \frac{1}{p+2} \int_0^T \|u(t, \cdot)\|_{L^\infty(0, L)}^p \|u(t, \cdot)\|_{L^2(0, L)}^2 dt \\ &\leq \frac{2^{p/2}}{p+2} \int_0^T \|\partial_x u(t, \cdot)\|_{L^2(0, L)}^{p/2} \|u(t, \cdot)\|_{L^2(0, L)}^{2+p/2} dt. \end{aligned}$$

Therefore using (28) we have

$$\begin{aligned} \frac{1}{p+2} \int_0^T \int_0^L u^{p+2} dx dt &\leq C \frac{2^{p/2}}{p+2} \|u_0\|_{L^2(0, L)}^{2+p/2} \int_0^T \|\partial_x u(t, \cdot)\|_{L^2(0, L)}^{p/2} dt \\ &\leq C \frac{2^{p/2} T^{1-p/4}}{p+2} \|u_0\|_{L^2(0, L)}^{2+p/2} \left(\int_0^T \|\partial_x u(t, \cdot)\|_{L^2(0, L)}^2 dt \right)^{p/4} \\ &\leq C_p T \|u_0\|_{L^2(0, L)}^{\frac{8+2p}{4-p}} + \frac{1}{2} \left(\int_0^T \|\partial_x u(t, \cdot)\|_{L^2(0, L)}^2 dt \right), \quad (30) \end{aligned}$$

for some positive constant C_p depending on p and C . Thus, from (28), (29) and Young's inequality we derive

$$\int_0^T \int_0^1 \partial_x u^2 dx dt + \int_0^T \int_0^L \partial_x^2 u^2 dx dt \leq C_p \left[(1+T) \|u_0\|_{L^2(0,L)}^2 + T \|u_0\|_{L^2(0,L)}^{\frac{8+2p}{4-p}} \right]. \quad (31)$$

Finally, with (28) and (31) and arguing by density, the proof of [Theorem 2.7](#) is finished. \square

2.2. Exponential stabilization. In this part we prove our main result with respect to the exponential stability of system (6). The proof is based on showing an appropriate observability inequality, using a contradiction argument and compactness ideas. The main difference in the proof of internal and boundary saturation is that for the boundary case we first prove such an observability inequality by analyzing system (8) and then we pass to the fully nonlinear system (6) by a perturbation approach. Due to this argument, our exponential stability result for (6) is of local nature. On the other side, in the proof of the observability inequality for (4), we work directly with the nonlinear system, which allows us to obtain a semiglobal exponential stability result. In what follows, we consider here the classical L^2 -energy,

$$E(t) = \frac{1}{2} \int_0^L u(t,x)^2 dx. \quad (32)$$

We are now going to prove the exponential stabilization result for the case of boundary saturation, that is the system (6). In this case, the following unique continuation property is crucial in the proof

Lemma 2.8. ([5, Lemma 4.3]) *Let $L > 0$, if $\lambda, \psi \in \mathbb{C} \times H^5(0, L)$ satisfies*

$$\begin{cases} \lambda \psi(x) + \psi'(x) + \psi'''(x) - \psi''''(x) = 0, & x \in (0, L) \\ \psi(0) = \psi(L) = \psi'(0) = \psi'(L) = 0, \\ \psi''(0) = \psi''(L) = 0, \end{cases} \quad (33)$$

then $\psi \equiv 0$.

Remark 2.9. This lemma was recently proved in [5] using complex analysis tools. For a sake of completeness, we explain here some ideas. By extending ψ to the whole space \mathbb{R} we obtain that u satisfies

$$\lambda \psi + \psi' + \psi''' - \psi'''' = -\psi''''(0)\delta'_0 + \psi''''(L)\delta'_L - \psi'''(0)\delta_0 + \psi'''(L)\delta_L, \quad \text{in } \mathcal{S}'(\mathbb{R}).$$

It can be shown that $\lambda = ir$, $r \in \mathbb{R}$, thus taking the Fourier transform of the above expression we get

$$i\hat{\psi}(\xi) = f(\xi, L) = \frac{\alpha_1 i\xi - \alpha_2 i\xi e^{-i\xi L} + \alpha_3 - \alpha_4 e^{-i\xi L}}{\xi^5 + \xi^3 - \xi + r}.$$

Therefore, the existence of a non-trivial ψ satisfying (33) is related to the analyticity of $f(\xi, L)$. This question was studied in [5] by analyzing the possibility to build a Möbius transformation (see also [15] where several other cases have been studied).

For a strong solution of (6) (or (8)) performing integration by parts, we get using $|\alpha| < 1$

$$\dot{E}(t) = \frac{1}{2} (\text{sat}_{loc}(\alpha \partial_x^2 u(t, 0))^2 - \partial_x^2 u(t, 0)^2) \leq -C_\alpha \partial_x^2 u(t, 0)^2 \leq 0. \quad (34)$$

Thus, the energy is a non-increasing function of time. In fact, we will prove that the energy decays exponentially to 0. Our approach will be based on an observability inequality for the system (8).

Proposition 2.10. *Let $L > 0$, $|\alpha| < 1$ and $u_0 \in L^2(0, L)$. For any solution u of (8), the following inequality holds:*

$$\int_0^L u_0^2 dx \leq C_{obs} \int_0^T \partial_x^2 u(t, 0)^2 dt, \quad (35)$$

for some $C_{obs} > 0$, that does not depend on u_0 .

Proof. In order to prove the observability inequality, we follow [33] and use a contradiction argument. Let us suppose that (35) is false, then there exists $(u_0^n)_{n \in \mathbb{N}} \subset L^2(0, L)$ such that

$$\int_0^L (u_0^n)^2 dx = 1, \quad \text{and} \quad \int_0^T \partial_x^{2,n} u(t, 0)^2 dt \longrightarrow 0$$

when $n \rightarrow \infty$ and where u^n is the associate solution of (8) with initial data u_0^n . Using (18) with $f = 0$ we get that $(u^n)_{n \in \mathbb{N}}$ is bounded in $L^2((0, T), H^2(0, L))$. Then as $\partial_t u^n = -\partial_x u^n - \partial_x^3 u^n + \partial_x^5 u^n$, we can see that $(\partial_t u^n)_{n \in \mathbb{N}}$ is bounded in $L^2((0, T), H^{-3}(0, L))$, thus by Aubin-Lions lemma $(u^n)_{n \in \mathbb{N}}$ is relatively compact in $L^2((0, T), L^2(0, L))$, therefore we can assume that $u^n \rightarrow u$ in $L^2((0, T), L^2(0, L))$. Using (10) we derive that $(u_0^n)_{n \in \mathbb{N}}$ is a Cauchy sequence in $L^2(0, L)$. Let $u_0 = \lim_{n \rightarrow \infty} u_0^n$, then

$$\int_0^L u_0^2 dx = 1, \quad \text{and} \quad \int_0^T \partial_x^2 u(t, 0)^2 dt = 0.$$

Thus, as $\text{sat}_{loc}(0) = 0$, we conclude that u solves the following linear equation:

$$\begin{cases} \partial_t u + \partial_x u + \partial_x^3 u - \partial_x^5 u = 0, & (t, x) \in (0, \infty) \times (0, L), \\ u(t, 0) = u(t, L) = 0, & t \in (0, \infty), \\ \partial_x u(t, 0) = \partial_x u(t, L) = 0, & t \in (0, \infty), \\ \partial_x^2 u(t, 0) = \partial_x^2 u(t, L) = 0, & t \in (0, \infty). \end{cases} \quad (36)$$

But by Lemma 2.8 we can show that there is no function satisfying this system and $\|u_0\|_{L^2(0, L)} = 1$, which gives us the contradiction and finishes the proof of Proposition 2.10. \square

Next, we prove our main result related to the exponential stability in the boundary saturated case.

Theorem 2.11. *Let $L > 0$, $|\alpha| < 1$ and $u_0 \in L^2(0, L)$. For any solution u of (8), the energy of the system decays exponentially to zero i.e, there exists $C, \gamma > 0$ such that $E(t) \leq CE(0)e^{-\gamma t}$, for all $t \geq 0$. Moreover, if $\|u_0\|_{L^2(0, L)} \leq \varepsilon$ for $\varepsilon > 0$ small enough, the same property holds for (6).*

Proof. Let $u_0 \in D(\mathcal{A}_b)$, using (34) and (11) we get for a strong solution of (8)

$$\dot{E}(t) \leq -C_\alpha \partial_x^2 u(t, 0)^2,$$

thus we can see

$$E(T) - E(0) \leq -C_\alpha \int_0^T \partial_x^2 u(t, 0)^2 dt.$$

Using the last expression, (34) and (35) we get

$$E(T) \leq E(0) \leq C_{obs} \int_0^T \partial_x^2 u(t, 0)^2 dt \leq \frac{C_{obs}}{C_\alpha} (E(0) - E(T)),$$

which implies that

$$E(T) \leq \gamma E(0), \quad \text{with } \gamma = \frac{C_{obs}}{C_\alpha + C_{obs}} < 1. \quad (37)$$

As (8) is invariant by translation in time and repeating this argument on $[(m-1)T, mT]$ for $m = 1, 2, \dots$ to obtain,

$$E(mT) \leq \gamma E((m-1)T) \leq \dots \leq \gamma^m E(0).$$

Hence, we have $E(mT) \leq e^{-\mu mT} E(0)$ where $\mu = \frac{1}{T} \ln\left(\frac{1}{\gamma}\right) > 0$. Let $t > 0$ then there exists $m \in \mathbb{N}^*$ such that $(m-1)T < t \leq mT$, and then using again the non-increasing property of the energy, we get

$$E(t) \leq E((m-1)T) \leq e^{-\mu(m-1)T} E(0) \leq \frac{1}{\gamma} e^{-\mu t} E(0).$$

Finally, we conclude the exponential stability of (8) by a density argument.

Now we study the nonlinear case, note that (6) is also invariant by translation in time; thus, to prove the stability, the objective is to show that (37) holds for u solution of (6) if the initial data is small enough. This idea was used in several works, for instance, [25, 27]. Let $\|u_0\|_{L^2(0,L)} \leq \epsilon$ for $\epsilon > 0$ to fix later. Take u the solution of (6), then we can split u as $u = \bar{u} + \tilde{u}$, where

$$\begin{cases} \partial_t \bar{u} + \partial_x \bar{u} + \partial_x^3 \bar{u} - \partial_x^5 \bar{u} = 0, & (t, x) \in (0, \infty) \times (0, L), \\ \bar{u}(t, 0) = \bar{u}(t, L) = \partial_x \bar{u}(t, 0) = \partial_x \bar{u}(t, L) = 0, & t \in (0, \infty), \\ \partial_x^2 \bar{u}(t, L) = \mathbf{sat}_{loc}(\alpha \partial_x^2 \bar{u}(t, 0)), & t \in (0, \infty), \\ \bar{u}(0, x) = u_0(x), & x \in (0, L), \end{cases}$$

and

$$\begin{cases} \partial_t \tilde{u} + \partial_x \tilde{u} + \partial_x^3 \tilde{u} - \partial_x^5 \tilde{u} = u^p \partial_x u, & (t, x) \in (0, \infty) \times (0, L), \\ \tilde{u}(t, 0) = \tilde{u}(t, L) = \partial_x \tilde{u}(t, 0) = \partial_x \tilde{u}(t, L) = 0, & t \in (0, \infty), \\ \partial_x^2 \tilde{u}(t, L) = h(t), & t \in (0, \infty), \\ \tilde{u}(0, x) = 0, & x \in (0, L). \end{cases}$$

where $h(t) = \mathbf{sat}_{loc}(\alpha \partial_x^2 u(t, 0)) - \mathbf{sat}_{loc}(\alpha \partial_x^2 \bar{u}(t, 0))$. Next, we observe that

$$\|u(T, \cdot)\|_{L^2(0,L)} \leq \|\bar{u}(T, \cdot)\|_{L^2(0,L)} + \|\tilde{u}(T, \cdot)\|_{L^2(0,L)}. \quad (38)$$

From the other side as \bar{u} satisfies (8), we get using (37) and $\bar{u}(0, x) = u_0(x)$ that there exists $\gamma < 1$ such that

$$\|\bar{u}(T)\|_{L^2(0,L)} \leq \gamma \|u_0\|_{L^2(0,L)}. \quad (39)$$

For \tilde{u} , by (18) and Proposition 2.6,

$$\|\tilde{u}(T, \cdot)\|_{L^2(0,L)} \leq \|\tilde{u}\|_{C([0,T], L^2(0,L))} \leq C \|u^p \partial_x u\|_{L^1((0,T), L^2(0,L))} \leq C \|u\|_{B_T}^{p+1},$$

using (28)-(31), we derive

$$\|u\|_{B_T}^{p+1} \leq C \left(\|u_0\|_{L^2(0,L)}^{p+1} + \|u_0\|_{L^2(0,L)}^{\frac{(4+p)(p+1)}{4-p}} \right).$$

Thus, we get for \tilde{u}

$$\|\tilde{u}(T, \cdot)\|_{L^2(0,L)} \leq \left(C\|u_0\|_{L^2(0,L)}^p + C\|u_0\|_{L^2(0,L)}^{\frac{p(6+p)}{4-p}} \right) \|u_0\|_{L^2(0,L)}.$$

Finally, as $\|u_0\|_{L^2(0,L)} \leq \epsilon$ we deduce from (38)

$$\begin{aligned} \|u(T, \cdot)\|_{L^2(0,L)} &\leq \|\bar{u}(T, \cdot)\|_{L^2(0,L)} + \|\tilde{u}(T, \cdot)\|_{L^2(0,L)} \\ &\leq \gamma\|u_0\|_{L^2(0,L)} + (C\epsilon^p + C\epsilon^{\frac{p(6+p)}{4-p}})\|u_0\|_{L^2(0,L)} \\ &= (\gamma + C\epsilon^p + C\epsilon^{\frac{p(6+p)}{4-p}})\|u_0\|_{L^2(0,L)}. \end{aligned}$$

We conclude by choosing $\epsilon > 0$ small enough such that $\tilde{\gamma} = \gamma + C\epsilon^p + C\epsilon^{\frac{p(6+p)}{4-p}} < 1$. \square

3. Internal Saturation. In this section, we discuss the stability analysis of the Kawahara equation (4) with internal saturated feedback. We will follow the works [22, 23, 28] to study the stabilization result. As before, we first show the well-posedness of the internal system. Then, using the multiplier approach, we prove the exponential stability of the system.

3.1. Well-posedness with internal feedback. In this section, we study the global well-posedness of (4), some of the computations are quite similar to those exposed in Section 2.1 for that we focus on the main difference.

3.1.1. Well-posedness of linear system. Let us first investigate the well-posedness of the linear Kawahara system, i.e., when we drop the nonlinear internal terms $u^p \partial_x u$ and $\text{sat}(au)$:

$$\begin{cases} \partial_t u + \partial_x u + \partial_x^3 u - \partial_x^5 u = 0, & (t, x) \in (0, \infty) \times (0, L), \\ u(t, 0) = u(t, L) = \partial_x u(t, 0) = \partial_x u(t, L) = \partial_x^2 u(t, L) = 0, & t \in (0, \infty), \\ u(0, x) = u_0(x), & x \in (0, L). \end{cases} \quad (40)$$

Thus we write the above system as:

$$\begin{cases} \dot{u}(t) = \mathcal{A}u(t), & t > 0, \\ u(0) = u_0, \end{cases} \quad (41)$$

where the operator \mathcal{A} can be written as

$$\mathcal{A}u = -u' - u''' + u''''$$

with the domain

$$D(\mathcal{A}) = \{u \in H^5(0, L) \cap H_0^2(0, L) : u_{xx}(L) = 0\}.$$

Remark 3.1. Observe that contrary to the case of boundary saturation, we first analyze a system which is linear so that we use linear semigroup theory for the study of (40). In what follows, we add the nonlinear terms as source terms. In the case of boundary saturation, this strategy is not too clear, and for that reason we just dropped the internal nonlinear term, and we used nonlinear semigroup arguments. That made the well-posedness of (40) easier than the boundary case.

It can be checked that the operator \mathcal{A} and its adjoint operator

$$\mathcal{A}^* u = u' + u''' - u''''$$

with the domain

$$D(\mathcal{A}^*) = \{u \in H^5(0, L) \cap H_0^2(0, L) : u_{xx}(0) = 0\},$$

are both dissipative. Therefore, by the Lumer-Philips theorem, it follows that \mathcal{A} generates a strongly continuous semigroup of contraction $\{S(t)\}_{t \geq 0}$, (see [39, 41] for details). Hereinafter, we have the following well-posedness result for the linear Kawahara system (see [39, Theorem 2.1]):

Theorem 3.2. *Let us take $u_0 \in D(\mathcal{A})$. Then the system (40) possesses a unique strong solution $u \in C([0, T]; D(\mathcal{A}) \cap C^1(0, T; L^2(0, L)))$. If the initial data $u_0 \in L^2(0, L)$, the unique mild solution of (40) lies in $C([0, T]; L^2(0, L)) \cap L^2(0, T; H^2(0, L))$. Moreover, we have a generic positive constant $C > 0$ such that the solution satisfies the following estimates*

- $\|u\|_{C([0, T]; L^2(0, L))} + \|u\|_{L^2(0, T; H^2(0, L))} \leq C \|u_0\|_{L^2(0, L)},$
- $\|\partial_x^2 u(\cdot, 0)\|_{L^2(0, T)} \leq C \|u_0\|_{L^2(0, L)}.$

Next, we study the well-posedness result for the linear Kawahara equation with a source term:

$$\begin{cases} \partial_t u + \partial_x u + \partial_x^3 u - \partial_x^5 u = f, & (t, x) \in (0, \infty) \times (0, L), \\ u(t, 0) = u(t, L) = \partial_x u(t, 0) = \partial_x u(t, L) = \partial_x^2 u(t, L) = 0, & t \in (0, \infty), \\ u(0, x) = u_0(x), & x \in (0, L). \end{cases} \quad (42)$$

Thus, we write the above system in the form of an abstract nonhomogeneous evolution equation:

$$\begin{cases} \dot{u} = \mathcal{A}u + f, & t > 0, \\ u(0) = u_0. \end{cases} \quad (43)$$

By classical semigroup theory result, it follows that for any $f \in L_{loc}^1(0, \infty; L^2(0, L))$, the solution of the system (43)

$$u(t) = S(t)u_0 + \int_0^t S(t-s)f(s)ds$$

belongs to $C([0, \infty); L^2(0, L))$. Moreover, for any $T > 0$ we have the following continuity estimate:

$$\|u\|_{C([0, T]; L^2(0, L))} \leq \left(\|u_0\|_{L^2(0, L)} + \|f\|_{L^1(0, T; L^2(0, L))} \right). \quad (44)$$

We also have the following estimates:

Theorem 3.3. *Let us take $f \in L^1(0, T; L^2(0, L))$. Then, the solution u satisfies the following estimates*

$$\|u\|_{B_T} + \|\partial_x^2 u(\cdot, 0)\|_{L^2(0, T)} \leq C(1 + \sqrt{T}) \left(\|u_0\|_{L^2(0, L)} + \|f\|_{L^1(0, T; L^2(0, L))} \right). \quad (45)$$

Proof. First, we multiply the equation (42) by u and integrate it over $(0, t) \times (0, L)$ and next we perform integration by parts to get:

$$\int_0^1 u^2(t, x)dx + \int_0^t \partial_x^2 u(s, 0)ds \leq 2 \int_0^t \|f(t, \cdot)\|_{L^2(0, L)} \|u(t, \cdot)\|_{L^2(0, L)} + \|u_0\|_{L^2(0, L)}^2$$

$$\leq 2\|u\|_{C([0,T];L^2(0,L))}\|f\|_{L^1((0,T);L^2(0,L))} + \|u_0\|_{L^2(0,L)}^2$$

Taking supremum over $[0, T]$ and using (44) we have

$$\|u\|_{C([0,T];L^2(0,L))}^2 + \|\partial_x^2 u(\cdot, 0)\|_{L^2(0,T)}^2 \leq C \left(\|u_0\|_{L^2(0,L)}^2 + \|f\|_{L^1((0,T);L^2(0,L))}^2 \right). \quad (46)$$

Similarly, multiplying xu in the equation (42) and doing a similar type of energy estimate as [39, Lemma 2.1], we can include the norm $\|u\|_{L^2(0,T;H^2(0,L))}$ in the above estimate. Thus, the proof is complete. \square

Now we are in a position to prove the global well-posedness result for the system (4).

Theorem 3.4. *Let $u_0 \in L^2(0, L)$. Then there exists a unique mild solution $u \in C([0, \infty); L^2(0, L)) \cap L^2(0, \infty); H^2(0, L)$ of (4). Moreover, for any $T > 0$, there exists $C > 0$, such that the solution of the system (4) satisfies the following:*

$$\|u\|_{L^2(0,T;H^2(0,L))}^2 \leq C \left(\|u_0\|_{L^2(0,L)}^2 + T \|u_0\|_{L^2(0,L)}^{\frac{8+2p}{4-p}} \right), \quad (47)$$

$$\|\partial_x^2 u(\cdot, 0)\|_{L^2(0,L)} \leq C \|u_0\|_{L^2(0,L)}, \quad (48)$$

$$\begin{aligned} \|u_0\|_{L^2(0,L)}^2 &\leq \frac{1}{T} \int_0^T \int_0^L u^2(t, x) dx dt + \int_0^T |\partial_x^2 u(t, 0)|^2 dt \\ &\quad + 2 \int_0^T \int_0^L \mathbf{sat}(a(x)u(t, x))u(t, x) dx dt. \end{aligned} \quad (49)$$

Proof. Let $u_0 \in D(\mathcal{A})$ and $R, \theta > 0$ to be chosen later. Consider again the ball $\mathcal{B}_{B_\theta}(0, R) := \{v \in B_\theta, \|v\|_{B_\theta} \leq R\}$. Let Ψ , the map defined by $\Psi : B_\theta \mapsto B_\theta$, $\Psi(v) = u$, where u solves

$$\begin{cases} \partial_t u + \partial_x u + \partial_x^3 u - \partial_x^5 u = -v^p \partial_x v - \mathbf{sat}(a(x)v(t, x)), & (t, x) \in (0, \theta) \times (0, L), \\ u(t, 0) = u(t, L) = \partial_x u(t, 0) = \partial_x u(t, L) = 0, \partial_x^2 u(t, L) = 0, & t \in (0, \theta), \\ u(0, x) = u_0(x), & x \in (0, L). \end{cases}$$

Observe that Φ is well-defined, since by Proposition 2.6 $v^p \partial_x v \in L^1((0, \theta); L^2(0, L))$ and by [22, Proposition 3.4]

$$\|\mathbf{sat}(av)\|_{L^1((0,\theta);L^2(0,L))} \leq 3\sqrt{L}a_1\sqrt{\theta}\|v\|_{B_\theta}.$$

By Theorem 3.3, Proposition 2.6 and the above inequality we get

$$\|\Phi(v)\|_{B_\theta} \leq C_1(1 + \sqrt{\theta})\|u_0\|_{L^2(0,L)} + C_2(\theta^{(2-p)/4} + \sqrt{\theta})\|v\|_{B_T}^{p+1} + C_2\sqrt{\theta}\|v\|_{B_T},$$

for some $C_1, C_2 > 0$. Then taking $R = 3C_1\|u_0\|_{L^2(0,L)}$ and $\theta > 0$ small enough such that

$$\begin{cases} C_2(\theta^{\frac{2-p}{4}} R^p + \sqrt{\theta} R^p + \sqrt{\theta}) < \frac{1}{3}, \\ (1 + \sqrt{\theta}) < 2, \end{cases}$$

we have that $\|\Psi(v)\|_{B_\theta} \leq R$ and thus Φ maps $\mathcal{B}_{B_\theta}(0, R)$ into itself. Let us take $v_1, v_2 \in \mathcal{B}_{B_\theta}(0, R)$, and consider $w = u_1 - u_2 = \Psi(v_1) - \Psi(v_2)$. Following the same idea as in the boundary saturation case, we obtain

$$\begin{aligned} \|\Psi(v_1) - \Psi(v_2)\|_{B_\theta} &= \|w\|_{B_\theta} \\ &\leq C \left(\|f\|_{L^1((0,\theta),L^2(0,L))} + 3\sqrt{L}a_1\sqrt{\theta}\|v_1 - v_2\|_{B_\theta} \right) \end{aligned}$$

$$\begin{aligned} &\leq C \left((\theta^{(2-p)/4} + \sqrt{\theta}) \left(\|v_1\|_{B_\theta} \|v_2\|_{B_\theta}^{p-1} + \|v_1\|_{B_\theta}^p + \|v_2\|_{B_\theta}^p \right) \right. \\ &\quad \left. + 3\sqrt{L}a_1\sqrt{\theta} \right) \|u - v\|_{B_\theta}. \end{aligned}$$

Then, taking $\theta > 0$, small enough such that $C \left((\theta^{(2-p)/4} + \sqrt{\theta})R^p + 3\sqrt{L}a_1\sqrt{\theta} \right) < 1$, we get that Ψ is a contraction, and by the Banach fixed point theorem and a density argument, we deduce the local in time well-posedness. To conclude the global well-posedness result, since the energy of the system is decreasing, we need some a priori estimates. Let $T > 0$ and u be a strong solution of (4). Multiplying the equation (4) by u and integrating on $[0, L]$ we have after integration by parts and boundary data

$$\frac{1}{2} \frac{d}{dt} \int_0^L u^2 dx + \frac{1}{2} \partial_x^2 u(t, 0)^2 + \int_0^L \mathbf{sat}(a(x)u(t, x))u(t, x) dx = 0. \quad (50)$$

Integrating over $[0, T]$ and using the fact that \mathbf{sat} is an odd function, one can prove the estimate (48). Similarly to (28), integrating (50) over $[0, s]$ we can show that

$$\|u\|_{C([0, T]; L^2(0, L))}^2 \leq \|u_0\|_{L^2(0, L)}^2. \quad (51)$$

Multiplying the first equation of (4) by xu and taking integration on both sides over $[0, T] \times (0, L)$, applying successive integration by parts and using boundary conditions, we obtain

$$\begin{aligned} &\frac{1}{2} \int_0^L xu^2(T, \cdot) dx + \frac{3}{2} \int_0^T \int_0^L u_x^2 dx + \frac{5}{2} \int_0^T \int_0^L u_{xx}^2 dx dt + \int_0^T \int_0^L xus\mathbf{sat}(au) dx dt \\ &= \frac{1}{2} \int_0^L xu_0^2(\cdot) dx + \frac{1}{2} \int_0^T \int_0^L u^2 dx dt + \frac{1}{p+2} \int_0^T \int_0^L u^{p+2} dx dt. \end{aligned}$$

Noting the fact \mathbf{sat} is an odd function, using (51) and following the computations of (30) we finally have (47).

Next multiplying first equation of (4) by $(T-t)u$ and taking integration over $[0, L]$ on both sides, we write

$$\begin{aligned} &\int_0^L (T-t)u\partial_t u dx + \int_0^L (T-t)u\partial_x u dx + \int_0^L (T-t)u\partial_x^3 u dx - \int_0^L (T-t)u\partial_x^5 u dx \\ &\quad + \int_0^L (T-t)u\partial_x u dx + \int_0^L (T-t)\mathbf{sat}(a(x)u(t, x))u(t, x) dx = 0. \end{aligned} \quad (52)$$

Integrating over $[0, T]$ we have

$$\begin{aligned} &\frac{1}{2} \int_0^T \int_0^L u^2 dx dt - \frac{T}{2} \int_0^L u_0^2 dx + \frac{1}{2} \int_0^T (T-t)\partial_x^2 u(t, 0)^2 dt \\ &\quad + \int_0^T \int_0^L (T-t)\mathbf{sat}(a(x)u(t, x))u(t, x) dx dt = 0. \end{aligned}$$

Thus we get the estimate (49). \square

3.2. Stability result. We pass now to study the case of internal saturation. We are interested to prove the following exponential stabilization result:

Theorem 3.5 (Semi global stabilization). *Let us consider the system (4). Then for any $r > 0$ there exists a constant $\mu(r) > 0$ such that for any $u_0 \in L^2(0, L)$ satisfying $\|u_0\|_{L^2(0, L)} \leq r$, the solution of (4) satisfy the following:*

$$E(t) \leq e^{-\mu t} E(0), \forall t \geq 0,$$

where $E(\cdot)$ is defined in (32).

Before going to prove **Theorem 3.5**, first we state some useful results borrowed from the article [23].

Lemma 3.6 ([23]). *Let $r > 0$ and $a : [0, L] \mapsto \mathbb{R}$ be a function which satisfies (5). Let us define*

$$k(r) = \min \left\{ \frac{M}{a_1 r}, 1 \right\},$$

where M and a_1 are defined in (2) and (5) respectively. The following two hold:

- For $s \in L^2(0, L)$ with $\|s\|_{L^2(0, L)} \leq r$, we have

$$(\mathbf{sat}_2(a(x)s(x)) - k(r)a(x)s(x)) s(x) \geq 0, \forall x \in [0, L]. \quad (53)$$

- For $s \in L^\infty(0, L)$ with $|s(x)| \leq r$, we have

$$(\mathbf{sat}_{loc}(a(x)s(x)) - k(r)a(x)s(x)) s(x) \geq 0, \forall x \in [0, L]. \quad (54)$$

3.2.1. **Proof of Theorem 3.5 with $\omega = [0, L]$ and $\mathbf{sat} = \mathbf{sat}_2$.** In this case, the proof is direct. Indeed, thanks to the estimate (50), we have

$$\frac{1}{2} \frac{d}{dt} \int_0^L u^2 dx \leq - \int_0^L \mathbf{sat}_2(a(x)u(t, x)) u(t, x) dx. \quad (55)$$

Using (51), we have $\|u(t, \cdot)\|_{L^2(0, L)} \leq r$, where $r > 0$ be any number such that $\|u_0\|_{L^2} \leq r$. Applying first estimate (53) of **Lemma 3.6** we deduce

$$\frac{1}{2} \frac{d}{dt} \int_0^L u^2 dx \leq - \int_0^L k(r)a_0 u^2(t, x) dx. \quad (56)$$

Thus the proof is complete with $\mu = \min\{a_0, k(r)\}$.

3.2.2. **Proof of Theorem 3.5 with $\omega \subset [0, L]$ and $\mathbf{sat} = \mathbf{sat}_2$.** In this case, we will use a contradiction argument as [34]. First, let us state a unique continuation lemma for the Kawahara equation, which plays an integral part in the proof of stabilization result:

Lemma 3.7 (Unique continuation principle, [16]). *Let $u_0 \in L^2(0, L)$ and $u \in B_T$ be the solution of the equation*

$$\begin{cases} \partial_t u + \partial_x u + \partial_x^3 u - \partial_x^5 u + u^p \partial_x u = 0, & (t, x) \in (0, T) \times (0, L), \\ u(t, 0) = u(t, L) = \partial_x u(t, 0) = \partial_x u(t, L) = \partial_x^2 u(t, L) = 0, & t \in (0, T), \\ u(0, x) = u_0(x), & x \in (0, L), \end{cases} \quad (57)$$

such that $u(t, x) = 0, \forall x \in \omega, \forall t \in (0, T)$, with $\omega \subset (0, L)$. Then, $u(t, x) = 0, \forall x \in (0, L), \forall t \in (0, T)$.

For the sake of completeness, we give a sketch of the proof of [Lemma 3.7](#) in the [Appendix B](#).

Let us consider $\|u_0\|_{L^2(0,L)} \leq r$. Using [\(51\)](#), we deduce

$$\|u(t, \cdot)\|_{L^2(0,L)} \leq r, \forall t \geq 0. \quad (58)$$

Integrating [\(50\)](#) over $[0, T]$ we have

$$\int_0^L u^2(T, \cdot) dx = \int_0^L u_0^2 dx - \int_0^T \partial_x^2 u(t, 0)^2 dt - 2 \int_0^T \int_0^L \mathbf{sat}_2(a(x)u(t, x))u(t, x) dx dt \quad (59)$$

As [\(58\)](#) holds, we apply the first part of [Lemma 3.6](#), and deduce

$$\int_0^L u^2(T, \cdot) dx \leq \int_0^L u_0^2 dx - \int_0^T \partial_x^2 u(t, 0)^2 dt - 2 \int_0^T \int_0^L k(r)a(x)|u(t, x)|^2 dx dt. \quad (60)$$

Next, for the time being we assume the following result:

Lemma 3.8. *Let us take $T > 0$ and let r be any positive number. Then for any $u_0 \in L^2(0, L)$, with $\|u_0\|_{L^2(0,L)} \leq r$, the solution of the equation [\(4\)](#) satisfies the following*

$$\|u_0\|_{L^2(0,L)}^2 \leq C_2 \left(\int_0^T |\partial_x^2 u(t, 0)|^2 dt + 2 \int_0^T \int_0^L k(r)a(x)|u(t, x)|^2 dx dt \right). \quad (61)$$

Therefore combining [\(60\)](#) and [\(61\)](#), we finally have the existence of $\gamma \in (0, 1)$ such that

$$\|u(kT, \cdot)\|_{L^2(0,L)}^2 \leq \gamma^k \|u_0\|_{L^2(0,L)}^2, \forall k \geq 0. \quad (62)$$

Using [\(51\)](#), we have $\|u(t, \cdot)\|_{L^2(0,L)}^2 \leq \|u(kT, \cdot)\|_{L^2(0,L)}^2$, where $kT \leq t \leq (k+1)T$. Finally, we obtain that

$$\|u(t, \cdot)\|_{L^2(0,L)} \leq \frac{1}{\gamma} e^{-\mu t} \|u_0\|_{L^2(0,L)}, \forall t \geq 0, \quad (63)$$

where $\mu = -\frac{\log \gamma}{T}$.

3.2.3. Proof of Lemma 3.8. Thanks to [\(49\)](#), we can claim that in order to prove [Lemma 3.8](#), it is enough to show that the following

$$\|u\|_{L^2(0,T;L^2(0,L))}^2 \leq C_2 \left(\int_0^T |\partial_x^2 u(t, 0)|^2 dt + 2 \int_0^T \int_0^L k(r)a(x)|u(t, x)|^2 dx dt \right), \quad (64)$$

for some $C > 0$, provided $\|u_0\|_{L^2(0,L)} \leq r$. We prove this inequality by a contradiction approach. Suppose that [\(64\)](#) is not true. Hence, there exists a sequence of functions $u_0^n \in L^2(0, L)$ with $\|u_0^n\|_{L^2(0,L)} \leq r$ and $\{u^n\}_{n \in \mathbb{N}} \in \mathcal{B}_T$ be the corresponding solution of the equation

$$\begin{cases} (\partial_t + \partial_x + \partial_x^3 - \partial_x^5)u^n + u^{np}\partial_x u^n + \mathbf{sat}_2(a(x)u^n(t, x)) = 0, & (t, x) \in (0, \infty) \times (0, L), \\ u^n(t, 0) = u^n(t, L) = \partial_x u^n(t, 0) = \partial_x u^n(t, L) = \partial_x^2 u^n(t, L) = 0, & t \in (0, \infty), \\ u^n(0, x) = u_0^n(x), & x \in (0, L), \end{cases} \quad (65)$$

where $p \in [1, 2)$, and

$$\lim_{n \rightarrow \infty} \frac{\|u^n\|_{L^2(0,T;L^2(0,L))}^2}{\left(\int_0^T |\partial_x^2 u^n(t,0)|^2 dt + 2 \int_0^T \int_0^L k(r)a|u^n(t,x)|^2 dx dt \right)} = +\infty. \quad (66)$$

Also (51) gives that

$$\|u^n(t, \cdot)\|_{L^2(0,L)} \leq r, \quad \forall t \geq 0. \quad (67)$$

Let us denote $\lambda_n = \|u^n\|_{L^2(0,T;L^2(0,L))}$ and $v_n(t,x) = \frac{u^n(t,x)}{\lambda_n}$. $\{\lambda_n\}_{n \in \mathbb{N}}$ is a bounded sequence by (67). Therefore it has a subsequence, denoted by $\{\lambda_n\}_{n \in \mathbb{N}}$ such that there exists $\lambda \geq 0$ with

$$\lambda_n \rightarrow \lambda \text{ as } n \rightarrow \infty. \quad (68)$$

Note that v_n satisfies the following equation:

$$\begin{cases} (\partial_t + \partial_x + \partial_x^3 - \partial_x^5)v^n + \lambda_n v^{np} \partial_x v^n + \frac{\mathbf{sat}_2(a(x)\lambda_n v^n(t,x))}{\lambda_n} = 0, & (t,x) \in (0,\infty) \times (0,L), \\ v^n(t,0) = v^n(t,L) = \partial_x v^n(t,0) = \partial_x v^n(t,L) = \partial_x^2 v^n(t,L) = 0, & t \in (0,\infty), \\ \|v_n\|_{L^2(0,T;L^2(0,L))} = 1, \end{cases} \quad (69)$$

and also by (66) we further have

$$\left(\int_0^T |\partial_x^2 v^n(t,0)|^2 dt + 2 \int_0^T \int_0^L k(r)a|v^n(t,x)|^2 dx dt \right) \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (70)$$

In addition, using (49), we have that $\{\|v_n(0, \cdot)\|_{L^2(0,L)}\}_{n \in \mathbb{N}}$ is a bounded sequence and due to the following

$$\begin{cases} \|v^n\|_{L^2(0,T;H^2(0,L))}^2 \leq C \left(\|v^n(0, \cdot)\|_{L^2(0,L)}^2 + T \|v^n(0, \cdot)\|_{L^2(0,L)}^{\frac{8+2p}{4-p}} \right), \\ \|v^n(t, \cdot)\|_{L^2(0,L)} \leq \|v^n(0, \cdot)\|_{L^2(0,L)}, \end{cases} \quad (71)$$

the sequence $\{v_n\}_{n \in \mathbb{N}}$ is bounded in the space $L^2(0,T;H^2(0,L))$ and $L^\infty([0,T];L^2(0,L))$. Therefore there exists a subsequence still denoted by $\{v_n\}_{n \in \mathbb{N}}$ and a function $v \in L^\infty(0,T;L^2(0,L)) \cap L^2(0,T;H^2(0,L))$ such that

$$\begin{cases} v_n \xrightarrow{*} v & \text{in } L^\infty(0,T;L^2(0,L)) \\ v_n \rightharpoonup v & \text{in } L^2(0,T;H^2(0,L)) \end{cases} \quad (72)$$

To derive the equation satisfied by v , we need to pass the limit in (69) and thus, in the following steps, we need to take care of the convergence of the terms related to the nonlinearity.

First of all, we have

$$\|v^{np} \partial_x v^n\|_{L^1((0,T),L^2(0,L))} \leq C(T^{(2-p)/4} + \sqrt{T}) \|v^n\|_{B_T}^p \|v^n\|_{B_T} \leq C. \quad (73)$$

In the next step, we prove that $\{(v^n)^{p+1}\}_{n \in \mathbb{N}}$ is bounded in $L^2(0,T;L^2(0,L))$.

$$\int_0^T \int_0^L (v^n)^{2p+2} dx dt \leq \|v^n\|_{L^2(0,T;H^1(0,L))}^2 \|v^n\|_{L^\infty(0,T;L^2(0,L))}^{3+\epsilon} \leq C, \quad (74)$$

where $p = 1 + \epsilon, 0 < \epsilon < 1$. This shows that $\{v^{np} \partial_x v^n\}_{n \in \mathbb{N}}$ is bounded in the space $L^2(0,T;H^{-1}(0,L)) \subset L^2(0,T;H^{-3}(0,L))$. Also, we know that $\{\partial_x^5 v^n\}_{n \in \mathbb{N}}$

is bounded in $L^2(0, T; H^{-3}(0, L))$. Furthermore, it is easy to check the following estimate (see [23, Lemma 3.2] for more details)

$$\left\| \frac{\text{sat}_2(a(x)\lambda_n v^n(t, x))}{\lambda_n} \right\|_{L^2(0, T; L^2(0, L))} \leq C \|v^n\|_{L^2(0, T; H^1(0, L))}. \quad (75)$$

Thus combining all the above arguments ((71)-(75)), we can deduce that that

$$\partial_t v^n = - \left(\partial_x v^n + \partial_x^3 v^n - \partial_x^5 v^n + \lambda_n v^{np} \partial_x v^n + \frac{\text{sat}_2(a(x)\lambda_n v^n(t, x))}{\lambda_n} \right)$$

is bounded in $L^2(0, T; H^{-3}(0, L))$ (as each term on the right side of the above identity lies in the same).

Lemma 3.9. *Lions-Aubin lemma, [35, Corollary 4] Let us consider three Banach spaces $X \subset B \subset Y$, with X, Y are reflexive, X is compactly embedded in B and B is continuously embedded in Y . Then the following two embedding*

$$\begin{aligned} \left\{ u \in L^2(0, T; X) \mid u_t \in L^2(0, T; Y) \right\} &\hookrightarrow L^2(0, T; B) \\ \left\{ u \in L^\infty(0, T; X) \mid u_t \in L^2(0, T; Y) \right\} &\hookrightarrow C([0, T]; B) \end{aligned}$$

are compact.

Thanks to the Lions-Aubin lemma with the embedding $H^2(0, L) \subset L^2(0, L) \subset H^{-3}(0, L)$, $\{v_n\}_{n \in \mathbb{N}}$ is relatively compact in $L^2(0, T; L^2(0, L))$. It follows that there exists a subsequence, again denoted by $\{v_n\}_{n \in \mathbb{N}}$ such that $v_n \rightharpoonup v$ in $L^2(0, T; L^2(0, L))$. This implies that $v^n \rightarrow v$ a.e. $(t, x) \in (0, T) \times (0, L)$. Further, it is immediate to have the following $(v^n)^{p+1} \rightarrow v^{p+1}$ a.e. $(t, x) \in (0, T) \times (0, L)$.

Next, since $\{(v^n)^{p+1}\}_{n \in \mathbb{N}}$ is bounded in $L^2(0, T; L^2(0, L))$ (by the virtue of (74)), there exists some $h \in L^2(0, T; L^2(0, L))$ such that $(v^n)^{p+1} \rightharpoonup h$ in $L^2(0, T; L^2(0, L))$. Uniqueness of limit gives $h = v^{p+1}$ a.e. (see [34, Lemma 2.15]). As $(v^n)^{p+1} \rightharpoonup v^{p+1}$ in $L^2(0, T; L^2(0, 1))$, it is straightforward to note from the definition of distribution that $(v^n)^{p+1} \rightarrow v^{p+1}$ in $\mathcal{D}'((0, T) \times (0, L))$. Taking the derivative with respect to space, we further have $\lambda_n v^{np} \partial_x v^n \rightarrow \lambda v^p \partial_x v$ in $\mathcal{D}'((0, T) \times (0, L))$.

Furthermore, as $\{v^n\}_{n \in \mathbb{N}}$ and $\{\partial_t v^n\}_{n \in \mathbb{N}}$ are bounded in the spaces $L^\infty(0, T; L^2(0, L))$ and $L^2(0, T; H^{-3}(0, L))$ respectively, using the embedding $L^2(0, L) \subset H^{-1}(0, L) \subset H^{-3}(0, L)$, we get by [35, Corollary 4], for a subsequence denoted by $\{v^n\}_{n \in \mathbb{N}}$ that $v^n \rightarrow v$ in $C([0, T]; H^{-1}(0, L))$.

Thus in a nutshell, interpolation theory gives us the following

$$\begin{cases} v_n \xrightarrow{*} v & \text{in } L^\infty(0, T; L^2(0, L)) \\ v_n \rightharpoonup v & \text{in } L^2(0, T; H^2(0, L)) \\ v_n \rightarrow v & \text{in } L^2(0, T; L^2(0, L)) \\ v_n \rightarrow v & \text{in } C([0, T]; H^{-1}(0, L)) \\ \lambda_n v^{np} \partial_x v^n \rightarrow \lambda v^p \partial_x v & \text{in } \mathcal{D}'((0, T) \times (0, L)). \end{cases} \quad (76)$$

Moreover, from (70) we deduce:

$$\left(\int_0^T |\partial_x^2 v(t, 0)|^2 dt + 2 \int_0^T \int_0^L k(r) a(x) |v(t, x)|^2 dx dt \right)$$

$$\leq \liminf_{n \rightarrow \infty} \left(\int_0^T |\partial_x^2 v^n(t, 0)|^2 dt + 2 \int_0^T \int_0^L k(r)a(x)|v^n(t, x)|^2 dx dt \right) = 0.$$

Therefore $a(x)k(x)v = 0, x \in (0, L)$ and $\partial_x^2 v(\cdot, 0) = 0, t \in (0, T)$. This further implies $v = 0, x \in \omega, t \in (0, T)$. We also obtain the limit v solves the following equation in the sense of distribution

$$\begin{cases} \partial_t v + \partial_x v + \partial_x^3 v - \partial_x^5 v + \lambda v^p \partial_x v = 0, & (t, x) \in (0, \infty) \times (0, L), \\ v(t, 0) = v(t, L) = \partial_x v(t, 0) = \partial_x v(t, L) = \partial_x^2 v(t, L) = 0, & t \in (0, \infty), \\ \|v\|_{L^2(0, T; L^2(0, L))} = 1. \end{cases} \quad (77)$$

Thanks to the unique continuation principle [Lemma 3.7](#), we have $v = 0, x \in (0, L), t \in (0, T)$. This contradicts the fact, $\|v\|_{L^2(0, T; L^2(0, L))} = 1$. Therefore [Lemma 3.8](#) holds.

3.2.4. Proof of [Theorem 3.5](#) with $\omega \subset [0, L]$ and $\mathbf{sat} = \mathbf{sat}_{loc}$. We will prove the exponential stabilization result [Theorem 3.5](#) with $\mathbf{sat} = \mathbf{sat}_{loc}$ as the same strategy of above. It is clear that the stabilizability follows if we can prove the following lemma:

Lemma 3.10. *Let us take $T > 0$ and let r be any positive number. Then for any $u_0 \in L^2(0, L)$, with $\|u_0\|_{L^2(0, L)} \leq r$, the solution of the equation [\(4\)](#) satisfies the following*

$$\|u_0\|_{L^2(0, L)}^2 \leq C_2 \left(\int_0^T |\partial_x^2 u(t, 0)|^2 dt + 2 \int_0^T \int_0^L \mathbf{sat}_{loc}(au(t, x))u(t, x) dx dt \right). \quad (78)$$

Proof. Using [\(49\)](#), we can say that to prove [Lemma 3.8](#), it is enough to show that the following

$$\|u\|_{L^2(0, T; L^2(0, L))}^2 \leq C_2 \left(\int_0^T |\partial_x^2 u(t, 0)|^2 dt + 2 \int_0^T \int_0^L \mathbf{sat}_{loc}(au(t, x))u(t, x) dx dt \right), \quad (79)$$

for some $C > 0$, provided $\|u_0\|_{L^2(0, L)} \leq r$. Using contradiction, we will prove the estimate [\(79\)](#). If possible, let us assume that there exists a sequence of functions $u_0^n \in L^2(0, L)$ with $\|u_0^n\|_{L^2(0, L)} \leq r$ and $\{u^n\}_{n \in \mathbb{N}} \in \mathcal{B}_T$ be the corresponding solution of the equation

$$\begin{cases} (\partial_t + \partial_x + \partial_x^3 - \partial_x^5)u^n + u^{np} \partial_x u^n + \mathbf{sat}_{loc}(a(x)u^n(t, x)) = 0, & (t, x) \in (0, \infty) \times (0, L), \\ u^n(t, 0) = u^n(t, L) = \partial_x u^n(t, 0) = \partial_x u^n(t, L) = \partial_x^2 u^n(t, L) = 0, & t \in (0, \infty), \\ u^n(0, x) = u_0^n(x), & x \in (0, L), \end{cases} \quad (80)$$

where $p \in [1, 2)$, and

$$\lim_{n \rightarrow \infty} \frac{\|u^n\|_{L^2(0, T; L^2(0, L))}^2}{\left(\int_0^T |\partial_x^2 u^n(t, 0)|^2 dt + 2 \int_0^T \int_0^L \mathbf{sat}_{loc}(au(t, x))u(t, x) dx dt \right)} = +\infty. \quad (81)$$

Thanks to [\(51\)](#), we obtain

$$\|u^n(t, \cdot)\|_{L^2(0, L)} \leq r, \forall t \geq 0. \quad (82)$$

Also, using (49), we have that $\{v_n(0, \cdot)\}_{n \in \mathbb{N}}$ is a bounded sequence in $L^2(0, L)$ and we further deduce

$$\|v^n\|_{L^2(0, T; H^2(0, L))}^2 \leq C \left(\|v^n(0, \cdot)\|_{L^2(0, L)}^2 + T \|v^n(0, \cdot)\|_{L^2(0, L)}^{\frac{8+2p}{4-p}} \right) \leq Cr^2 + CTr^{\frac{8+2p}{4-p}}. \quad (83)$$

Using Poincare inequality, we have

$$\|u^n(t, \cdot)\|_{L^\infty(0, L)} \leq C \|u^n\|_{H^2(0, L)} \quad \forall t \in [0, T].$$

This further gives

$$\|u^n(\cdot, x)\|_{L^2(0, T)}^2 \leq C_1, \quad (84)$$

where $C_1 = C^2(Cr^2 + CTr^{\frac{8+2p}{4-p}})$. Let us consider the subsets $\Omega_i \subset [0, T]$ defined as

$$\Omega_i = \{t \in [0, T] : \|u^n(t, \cdot)\|_{L^\infty(0, L)} > i\}. \quad (85)$$

Its complement is then given by

$$\Omega_i^c = \{t \in [0, T] : \|u^n(t, \cdot)\|_{L^\infty(0, L)} \leq i\}. \quad (86)$$

Therefore we have

$$\|u^n\|_{L^2(0, T; L^\infty(0, L))}^2 \geq \|u^n\|_{L^2(\Omega_i; L^\infty(0, L))}^2 \geq i^2 \mu(\Omega_i), \quad (87)$$

where $\mu(\Omega_i)$ is the Lebesgue measure of the set Ω_i . Next, using (84) we have $\mu(\Omega_i) \leq \frac{C_1}{i^2}$. This implies that

$$\max \left\{ T - \frac{C_1}{i^2}, 0 \right\} \leq \mu(\Omega_i^c) \leq T. \quad (88)$$

Thanks to the estimate (54) of Lemma 3.6, we have

$$\begin{aligned} \int_0^T \int_0^L \mathbf{sat}_{loc}(au^n) u^n dt dx &= \int_{\Omega_i} \int_0^L \mathbf{sat}_{loc}(au^n) u^n dt dx + \int_{\Omega_i^c} \int_0^L \mathbf{sat}_{loc}(au^n) u^n dt dx \\ &\geq \int_{\Omega_i^c} \int_0^L \mathbf{sat}_{loc}(au^n) u^n dt dx \geq \int_{\Omega_i^c} \int_0^L ak(i)(u^n)^2 dt dx. \end{aligned} \quad (89)$$

Let us denote $\lambda_n = \|u^n\|_{L^2(0, T; L^2(0, L))}$ and $v_n(t, x) = \frac{u^n(t, x)}{\lambda_n}$. $\{\lambda_n\}_{n \in \mathbb{N}}$ is a bounded sequence by (67). Therefore it has a subsequence, denoted by $\{\lambda_n\}_{n \in \mathbb{N}}$ such that there exists $\lambda \geq 0$ with

$$\lambda_n \rightarrow \lambda \text{ as } n \rightarrow \infty. \quad (90)$$

Note that v_n satisfies the following equation:

$$\begin{cases} (\partial_t + \partial_x + \partial_x^3 - \partial_x^5)v^n + \lambda_n v^{np} \partial_x v^n + \frac{\mathbf{sat}_{loc}(a\lambda_n v^n)}{\lambda_n} = 0, & (t, x) \in (0, \infty) \times (0, L), \\ v^n(t, 0) = v^n(t, L) = \partial_x v^n(t, 0) = \partial_x v^n(t, L) = \partial_x^2 v^n(t, L) = 0, & t \in (0, \infty), \\ \|v_n\|_{L^2(0, T; L^2(0, L))} = 1, \end{cases} \quad (91)$$

and also by (81) and (89) and we further have

$$\left(\int_0^T |\partial_x^2 v^n(t, 0)|^2 dt + 2 \int_0^T \int_0^L k(i)a(x)|v^n(t, x)|^2 dx dt \right) \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (92)$$

Moreover, it is easy to check the following estimate (see [23, Lemma 3.2])

$$\left\| \frac{\mathbf{sat}_{loc}(a(x)\lambda_n v^n(t, x))}{\lambda_n} \right\|_{L^2(0, T; L^2(0, L))} \leq C \|v^n\|_{L^2(0, T; H^1(0, L))}. \quad (93)$$

Like previous section, we have that

$$\partial_t v^n = - \left(\partial_x v^n + \partial_x^3 v^n - \partial_x^5 v^n + \lambda_n v^{np} \partial_x v^n + \frac{\mathbf{sat}_{loc}(a(x)\lambda_n v^n(t, x))}{\lambda_n} \right)$$

is bounded in $L^2(0, T; H^{-3}(0, L))$. Thanks to the Lions-Aubin lemma with the embedding $H^2(0, L) \subset L^2(0, L) \subset H^{-3}(0, L)$, $\{v_n\}_{n \in \mathbb{N}}$ is relatively compact in $L^2(0, T; L^2(0, L))$. It follows that there exists a subsequence, again denoted by $\{v_n\}_{n \in \mathbb{N}}$ such that $v_n \rightarrow v$ in $L^2(0, T; L^2(0, L))$. Moreover with (92) we have

$$ak(i)v(t, x) = 0, \quad \forall x \in (0, L), \forall t \in \Omega_i^c \text{ and } \partial_x^2 v(t, 0) = 0, \forall t \in (0, T).$$

As $k(i)$ is strictly positive, we further have $v(t, x) = 0, \forall x \in \omega, \forall t \in \cup_{i \in \mathbb{N}} \Omega_i^c$ and $\partial_x^2 v(t, 0) = 0, \forall t \in (0, T)$. As $\mu(\cup_{i \in \mathbb{N}} \Omega_i^c) = T$, we have $v(t, x) = 0, \forall x \in \omega, \forall t \in \cup_{i \in \mathbb{N}} \Omega_i^c$ and $\partial_x^2 v(t, 0) = 0, \forall t \in (0, T)$. This implies that $v(t, x) = 0, \forall x \in \omega$ with almost every $t \in (0, T)$. We also obtain that the limit v solves the following equation in the sense of distribution

$$\begin{cases} \partial_t v + \partial_x v + \partial_x^3 v - \partial_x^5 v + \lambda v^p \partial_x v = 0, & (t, x) \in (0, \infty) \times (0, L), \\ v(t, 0) = v(t, L) = \partial_x v(t, 0) = \partial_x v(t, L) = \partial_x^2 v(t, L) = 0, & t \in (0, \infty), \\ \|v\|_{L^2(0, T; L^2(0, L))} = 1. \end{cases} \quad (94)$$

Using the unique continuation principle Lemma 3.7 for the Kawahara equation we have $v(t, x) = 0, \forall x \in (0, L)$. Which is a contradiction. Hence the proof follows. \square

4. Conclusion and final comments. In this work, the exponential stabilization of the nonlinear Kawahara equation was addressed. The control was considered acting internally or in the boundary. In both cases, a global well-posedness result was obtained. In the case of boundary saturation, a local exponential stability result was derived while for the internal saturated actuator a semiglobal exponential stability result was shown. The main strategies in the proof of those results were observability inequalities, contradiction arguments, and compactness ideas. We finish this work with several comments and remarks, in order:

4.1. Sign of the first-order derivative. One can wonder if our results are still valid if we slightly modify our systems. This extension is, in fact not true in all the cases. To fix ideas, consider the following equation

$$\begin{cases} \partial_t u - \partial_x u + \partial_x^3 u - \partial_x^5 u + u^p \partial_x u = 0, & (t, x) \in (0, \infty) \times (0, L), \\ u(t, 0) = u(t, L) = \partial_x u(t, 0) = \partial_x u(t, L) = 0, & t \in (0, \infty), \\ \partial_x^2 u(t, L) = \mathbf{sat}_{loc}(\alpha \partial_x^2 u(t, 0)), & t \in (0, \infty), \\ u(0, x) = u_0(x), & x \in (0, L). \end{cases} \quad (95)$$

Observe that (95) it is almost the same equation as (6); the unique difference is the sign of the first order derivative. If we reply the arguments in Theorem 2.11 to

show exponential stability, we are asked to analyze the solutions of

$$\begin{cases} \partial_t u - \partial_x u + \partial_x^3 u - \partial_x^5 u = 0, & (t, x) \in (0, \infty) \times (0, L), \\ u(t, 0) = u(t, L) = 0, & t \in (0, \infty), \\ \partial_x u(t, 0) = \partial_x u(t, L) = 0, & t \in (0, \infty), \\ \partial_x u(t, 0) = \partial_x^2 u(t, 0) = \partial_x^2 u(t, L) = 0, & t \in (0, \infty). \end{cases} \quad (96)$$

As we already mentioned, when we study whether the only solution to (96) is the trivial one, a critical lengths phenomenon could appear. For instance, in [1] the existence of a critical set \mathcal{N} for a similar equation was shown. More specifically, by considering the following constants

$$a = \sqrt{\frac{\sqrt{5} + 1}{2}}, \quad b = \sqrt{\frac{\sqrt{5} - 1}{2}}, \quad A = C_2 + C_3, \quad B = C_2 - C_3,$$

$$C_2 = 1 - e^{-aL}, \quad C_3 = e^{aL} - 1, \quad C_1 = -\left(1 + \frac{a^2}{b^2}\right)A, \quad C_4 = \frac{a^2}{b^2}A, \quad C_5 = -\frac{a}{b}B,$$

and the set

$$\mathcal{N} = \left\{ L > 0 : e^{ibL} = \left(\frac{C_4 + iC_5}{|C_4 + iC_5|} \right)^2 \right\} \subset \mathbb{R}^+.$$

The authors of [1] showed that if $L \in \mathcal{N}$, then

$$\psi(x) = C_1 + C_2 e^{ax} + C_3 e^{-ax} + C_4 \cos(bx) + C_5 \sin(bx) \not\equiv 0, \quad x \in (0, L),$$

solves $\psi' - \psi''' - \psi'''' = 0$, with boundary conditions $\psi(0) = \psi(L) = \psi'(0) = \psi'(L) = \psi''(0) = \psi''(L) = 0$.

It happens that for system (96), there is no critical lengths phenomenon. In particular, the following result holds

Lemma 4.1. *If $\lambda, \psi \in \mathbb{C} \times H^5(0, L)$ satisfies*

$$\begin{cases} \lambda \psi(x) - \psi'(x) + \psi'''(x) - \psi''''(x) = 0, & x \in (0, L) \\ \psi(0) = \psi(L) = \psi'(0) = \psi'(L) = 0, \\ \psi''(0) = \psi''(L) = 0, \end{cases} \quad (97)$$

then $\psi \equiv 0$.

The proof of this lemma is given in [Appendix C](#).

Now we consider the same equation, but we drop the first order term; i.e.,

$$\begin{cases} \partial_t u + \partial_x^3 u - \partial_x^5 u = 0, & (t, x) \in (0, \infty) \times (0, L), \\ u(t, 0) = u(t, L) = 0, & t \in (0, \infty), \\ \partial_x u(t, 0) = \partial_x u(t, L) = 0, & t \in (0, \infty), \\ \partial_x u(t, 0) = \partial_x^2 u(t, 0) = \partial_x^2 u(t, L) = 0, & t \in (0, \infty). \end{cases} \quad (98)$$

In [39, 40], it was shown that the set of critical lengths related to this equation is empty. Therefore, we expect that our stability result by saturated boundary feedback is still valid in this both context.

In the case of internal saturation, the situation is better. Consider the following system

$$\begin{cases} \partial_t u + \eta \partial_x u + \partial_x^3 u - \partial_x^5 u + u^p \partial_x u + \mathbf{sat}(a(x)u(t, x)) = 0, & (t, x) \in (0, \infty) \times (0, L), \\ u(t, 0) = u(t, L) = \partial_x u(t, 0) = \partial_x u(t, L) = \partial_x^2 u(t, L) = 0, & t \in (0, \infty), \\ u(0, x) = u_0(x), & x \in (0, L), \end{cases} \quad (99)$$

where $\eta = 0, -1$ and $a \in L^\infty(0, L)$ satisfies (5). Straightforward modifications of [Theorem 3.4](#) allow us to show that there exists a unique mild solution $u \in C([0, \infty), L^2(0, L)) \cap L^2(0, \infty), H^2(0, L)$ of (99). With respect to the stabilization, following the proof of [Theorem 3.5](#) we are asking to prove similar unique continuation principle as [Lemma 3.7](#), for a solution of

$$\begin{cases} \partial_t u + \eta \partial_x u + \partial_x^3 u - \partial_x^5 u + u^p \partial_x u = 0, & (t, x) \in (0, \infty) \times (0, L), \\ u(t, 0) = u(t, L) = \partial_x u(t, 0) = \partial_x u(t, L) = \partial_x^2 u(t, L) = 0, & t \in (0, \infty), \\ u(t, x) = 0, & (t, x) \in (0, \infty) \times \omega, \\ u(0, x) = u_0(x), & x \in (0, L). \end{cases}$$

In particular, we can write

$$\partial_t u + \partial_x u + \partial_x^3 u - \partial_x^5 u + u^p \partial_x u = (1 - \eta) \partial_x u.$$

Then, we can apply the Carleman estimate (100) and absorb the remaining right hand-side terms by the left hand side terms. From where we can conclude the semiglobal exponential stability of (99).

4.2. More general nonlinearities. In the introduction of this work, we mention that in [1] the internal stabilization was studied in the case $p \in [1, 4)$, but a smallness condition in the initial data was required for $p \in (2, 4)$. This is one of the main reasons why we only considered the case $p \in [1, 2)$. In fact, if the initial data is too small, then by continuity, the solution remains small, which could imply that $\mathbf{sat}(au) = au$, then the system is not really saturated.

In the same direction, a natural question arises regarding the types of saturation we can handle using this method. Here, we provide some examples in the case of \mathbf{sat}_{loc} . In the case of boundary saturation concerning the proof of well-posedness and exponential stability, we can replace the boundary feedback $\partial_x^2 u(t, L) = \mathbf{sat}_{loc}(\alpha \partial_x^2 u(t, 0))$ by $\partial_x^2 u(t, L) = F(\alpha \partial_x^2 u(t, 0))$, where $F : \mathbb{R} \mapsto \mathbb{R}$ satisfies:

- $|F(x_1) - F(x_2)| \leq |x_1 - x_2|^2$, for all $x_1, x_2 \in \mathbb{R}$.
- There exists $M > 0$, such that for all $x \in \mathbb{R}$, $|F(x)| \leq M$.
- $F(0) = 0$.

For the internal saturation problem, instead of $\mathbf{sat}_{loc}(a(x)u(t, x))$ we can consider feedback $F(a(x)u(t, x))$, where $F : \mathbb{R} \mapsto \mathbb{R}$ is such that:

- $\|F(h) - F(g)\|_{L^2(0, L)} \leq C \|h - g\|_{L^2(0, L)}$, for all $h, g \in L^2(0, L)$.
- F is an odd function.
- $F(0) = 0$.
- F satisfies the second point of [Lemma 3.6](#).

²Actually if $|F(x_1) - F(x_2)| \leq C|x_1 - x_2|$, the results are still valid if $|\alpha| \leq \frac{1}{C}$

For instance, given a saturation level, $M > 0$ we can take $F(x) = M \arctan(\frac{x}{M})$ or $F(x) = M \tanh(\frac{x}{M})$. Most of the aforementioned properties are clearly satisfied. To conclude, we prove this function satisfies the second point of **Lemma 3.6**. To simplify the computations, we take $a(x) = 1$ for all $x \in (0, L)$ and $M = 1$. Therefore, for $s \in L^\infty$ given $r > 0$ and $|s(x)| < r$, we need to find $k(r)$ such that

$$(F(s(x)) - k(r)s(x))s(x) \geq 0, \text{ for all } x \in (0, L).$$

We have just to check that if $|x| < r$, then $F(x)x - k(r)x^2 \geq 0$. Note that the function $\theta(x) = F(x)x - k(r)x^2$ is even. Therefore, we just consider $x \geq 0$. Moreover $\theta(0) = 0$, for $F(x) = \arctan(x)$, we have

$$\begin{aligned} \theta'(x) &= \frac{x}{1+x^2} + \arctan(x) - 2k(r)x \\ &\geq x \left(\frac{1}{1+r^2} - 2k(r) \right) + \arctan(x) \geq 0 \end{aligned}$$

by choosing $k(r) = \frac{1}{2(1+r^2)}$. A similar analysis can be developed for $\tanh(x)$.

Appendix A. Proof of Lemma 2.2. In this part, we prove that the operator \mathcal{T} is well-defined. Indeed, take ψ defined by

$$\psi(x) = \frac{L^2}{8\pi^2} \sin^2\left(\frac{2\pi}{L}x\right) \mathbf{sat}_{loc}(\alpha y''(0)).$$

We can easily observe that $\psi(0) = \psi(L) = \psi'(0) = \psi'(L) = 0$, and $\psi''(L) = \mathbf{sat}_{loc}(\alpha y''(0))$. Then z solves (13) if and only if $\varphi = z - \psi$ solves

$$\begin{cases} \tilde{\lambda}\varphi + \varphi' + \varphi''' - \varphi'''' = f = \tilde{\lambda}w - \tilde{\lambda}\psi - \psi' - \psi''' + \psi'''' , \\ \varphi(0) = \varphi(L) = \varphi'(0) = \varphi'(L) = \varphi''(L) = 0. \end{cases}$$

which is equivalent to $(\mathcal{A}_b^0 - \tilde{\lambda}I)\varphi = -f$. Now, let (λ, y) a pair eigenvalue, eigenfunction of \mathcal{A}_b^0 , i.e $\mathcal{A}_b^0 y = \lambda y$, in particular we have

$$\begin{cases} -y' - y''' + y'''' = \lambda y, \\ -\bar{y}' - \bar{y}''' + \bar{y}'''' = \bar{\lambda} \bar{y}. \end{cases}$$

We multiply the first equation by \bar{y} , the second one by y and we integrate over $(0, L)$, we get after some integration by parts

$$\int_0^L y \bar{y}' dx + \int_0^L y \bar{y}''' dx - \int_0^L y \bar{y}'''' dx - |y'(0)|^2 = \bar{\lambda} |y|^2 dx$$

and

$$-\int_0^L \bar{y} y' dx - \int_0^L \bar{y} y''' dx + \int_0^L \bar{y} y'''' dx = \bar{\lambda} \int_0^L |y|^2 dx.$$

By adding these two equations, we deduce

$$-\frac{1}{2}|y'(0)|^2 = \operatorname{Re}(\lambda) \int_0^L |y|^2 dx \implies \operatorname{Re}(\lambda) \leq 0.$$

Then, as $\tilde{\lambda} > 0$, we get $\tilde{\lambda} \notin \sigma(\mathcal{A}_b^0)$, thus $\mathcal{A}_b^0 - \tilde{\lambda}I$ is invertible, and finally, $\varphi = -(\mathcal{A}_b^0 - \tilde{\lambda}I)^{-1}f$, is well defined and also z .

Appendix B. Proof of Lemma 3.7. This section is devoted to the proof of the unique continuation principle for the Kawahara equation with interior observation. Let us first recall the following existence result of the Kawahara equation in a regular set-up.

Proposition B.1. *Let us assume that $u_0 \in H^5(0, L)$. Then the equation (57) with $u(0, \cdot) = u_0$ possesses a unique solution $u \in C([0, T]; H^5(0, L)) \cap L^2(0, T; H^7(0, L))$. Also by interpolation, if $u_0 \in H^s(0, L)$, then the equation (77) possesses a unique solution $u \in C([0, T]; H^s(0, L)) \cap L^2(0, T; H^{s+2}(0, L))$, $0 \leq s \leq 5$.*

One can prove the above result in the same spirit of [34].

Let us consider a function $\psi \in C^\infty(0, 1)$ with $\psi > 0$ in $(0, 1)$, $\psi(0) = \psi(1) = 0$, $\|\psi\|_{L^\infty(0,1)} = 1$, $|\psi_x| > 0$ in $[0, 1] \setminus \omega$, $\psi_x(0) > 0$, $\psi_x(1) < 0$. For any $\epsilon > 0$, we consider

$$a(t, x) = \frac{e^{\mu(\psi(x)+3)} - e^{5\mu}}{(t - \epsilon)(T - t)}, \quad \xi = e^{\lambda a}, \quad \varphi = \frac{e^{\mu(\psi(x)+3)}}{(t - \epsilon)(T - t)}, \quad t \in (\epsilon, T).$$

Next, we will use a global Carleman estimate [16, Proposition 1] for the Kawahara equation. Let us introduce the following space

$$\mathcal{U}_\epsilon = \{u \in L^2(\epsilon, T; H^5(0, L)) \mid u(t, 0) = u(t, L) = \partial_x u(t, 0) = \partial_x u(t, L) = 0, \\ \partial_x^2 u(t, L) = 0, \quad Pu = \partial_t u + \partial_x u + \partial_x^3 u - \partial_x^5 u \in L^2(\epsilon, T; L^2(0, L))\}.$$

Proposition B.2. *There exists positive constants $\mu_0 > 1$, $C_0 > 0$ and $C_1 > 0$ such that for $\mu = \mu_0$ and for every $s \geq C_0(T + T^2)$, for all $u \in \mathcal{U}_\epsilon$, we have*

$$\int_\epsilon^T \int_0^L (s\varphi\xi^2\partial_x^4 u^2 + s^3\varphi^3\xi^2\partial_x^3 u^2 + s^5\varphi^5\xi^2\partial_x^2 u^2 + s^7\varphi^7\xi^2\partial_x u^2 + s^9\varphi^9\xi^2 u^2) \\ \leq C \int_\epsilon^T \int_\omega (\xi^2|Pu|^2 + s\varphi\xi^2\partial_x^4 u^2 + s^9\varphi^9\xi^2 u^2). \quad (100)$$

B.1. Proof of Lemma 3.7. As, $u \in L^2(0, T; H^2(0, L))$, for any $\epsilon \in (0, T)$, there exists a candidate $t_1 \in (0, \epsilon)$ such that $u(t_1, \cdot) \in H^2(0, L)$. Using Proposition B.1, we get that $u \in C([t_1, T]; H^2(0, L)) \cap L^2(t_1, T; H^4(0, L))$. A similar argument gives a time $t_2 \in (t_1, \epsilon)$ such that $u(t_2, \cdot) \in H^4(0, L)$. Again from Proposition B.1, it follows that $u \in C([t_2, T]; H^4(0, L)) \cap L^2(t_2, T; H^6(0, L))$. Proceeding with similar argument we finally have $t_3 \in (t_2, \epsilon)$ such that $u \in C([t_3, T]; H^5(0, L)) \cap L^2(t_3, T; H^7(0, L))$. Therefore, we eventually have $u \in C([\epsilon, T]; H^5(0, L)) \cap L^2(\epsilon, T; H^7(0, L))$. Thus we can use (100) for the solution u as $u \in \mathcal{U}_\epsilon$.

Plugging $Pu = u^p \partial_x u$ in the Global Carleman estimate (100), we have

$$\int_\epsilon^T \int_0^L (s\varphi\xi^2\partial_x^4 u^2 + s^3\varphi^3\xi^2\partial_x^3 u^2 + s^5\varphi^5\xi^2\partial_x^2 u^2 + s^7\varphi^7\xi^2\partial_x u^2 + s^9\varphi^9\xi^2 u^2) \\ \leq C \int_\epsilon^T \int_\omega (\xi^2|\partial_x u u^p|^2 + s\varphi\xi^2\partial_x^4 u^2 + s^9\varphi^9\xi^2 u^2). \quad (101)$$

Note that $u \in L^\infty((\epsilon, T) \times (0, L))$. Therefore one can find a large constant $C_1 = \|u\|_{L^\infty((\epsilon, T) \times (0, L))}^p$ and consider s large enough so we can absorb the last term in the

right hand sides by the term $\frac{1}{2} \int_\epsilon^T \int_0^L s^7 \varphi^7 \xi^2 \partial_x u^2$. Indeed taking $s \geq C_1(T + T^2)$,

we have

$$C \int_\epsilon^T \int_\omega \xi^2 |u^p \partial_x u|^2 \leq \frac{1}{2} \int_\epsilon^T \int_0^L s^7 \varphi^7 \xi^2 \partial_x u^2.$$

Finally we have the following:

$$\begin{aligned} & \int_{\epsilon}^T \int_0^L (s\varphi\xi^2\partial_x^4u^2 + s^3\varphi^3\xi^2\partial_x^3u^2 + s^5\varphi^5\xi^2\partial_x^2u^2 + s^7\varphi^7\xi^2\partial_xu^2 + s^9\varphi^9\xi^2u^2) \\ & \leq C \int_{\epsilon}^T \int_{\omega} (s\varphi\xi^2\partial_x^4u^2 + s^9\varphi^9\xi^2u^2). \end{aligned}$$

Therefore $u = 0$ in $(0, T) \times \omega$ implies that $u = 0$ in $(\epsilon, T) \times (0, L)$ for all $\epsilon > 0$. As a result, we have $u = 0$ for all $(t, x) \in (0, T) \times (0, L)$.

Appendix C. Proof of Lemma 4.1. In this section, we provide a proof of Lemma 4.1. The proof is similar to [5, Lemma 4.3], therefore we just focus on the differences. Multiplying (97) by $\bar{\psi}$ and integrating, we easily get $\lambda = -ir$, $r \in \mathbb{R}$. Extending ψ by 0 for $x \notin [0, L]$ and taking Fourier transform, we derive

$$-i\hat{\psi}(\xi) = f(\xi, L) = \frac{\alpha_1 i\xi - \alpha_2 i\xi e^{-i\xi L} + \alpha_3 - \alpha_4 e^{-i\xi L}}{\xi^5 + \xi^3 + \xi + r},$$

where α_i , for $i = 1, 2, 3, 4$ are the traces of ψ''' and ψ'''' . Thus, we aim to study when the function $f(\cdot, L)$ defines an entire function, which is equivalent to ensuring that all the zeros (with multiplicity) of

$$q(\xi) = \xi^5 + \xi^3 + \xi + r$$

are also zeros of

$$N(\xi, L) = \alpha_1 i\xi - \alpha_2 i\xi e^{-i\xi L} + \alpha_3 - \alpha_4 e^{-i\xi L}.$$

Let ξ_i , $i = 1, 2, 3, 4, 5$ the zeros of $q(\cdot)$. Then $f(\cdot, L)$ is entire if

$$\frac{\alpha_1 i\xi_i + \alpha_3}{\alpha_2 i\xi_i + \alpha_4} = e^{-i\xi_i L}, \quad i = 1, 2, 3, 4, 5.$$

For $\vec{\alpha} = (\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in \mathbb{C}^4 \setminus \{0\}$, define the discriminant

$$d(\vec{\alpha}) = \alpha_1\alpha_3 - \alpha_2\alpha_4.$$

Thus, for $\vec{\alpha}$ such that $d(\vec{\alpha}) \neq 0$ we can introduce the Möbius transformation

$$M(\xi_i) = e^{-iL\xi_i}, \quad i = 1, 2, 3, 4, 5.$$

The following results are borrowed from [15, Lemma 2.1 and Lemma 2.2]

Lemma C.1. *Let $\vec{\alpha} \in \mathbb{C}^4 \setminus \{0\}$ with $d(\vec{\alpha}) = 0$ and $L > 0$. Then, the set of the imaginary parts of the zeros of $N(\cdot, L)$ has at most two elements.*

Lemma C.2. *For $L > 0$, there is no Möbius transformation M , such that*

$$M(\xi) = e^{-iL\xi}, \xi \in \{\xi_1, \xi_2, \bar{\xi}_1, \bar{\xi}_2\},$$

with $\xi_1, \xi_2, \bar{\xi}_1, \bar{\xi}_2$ all distinct in \mathbb{C} .

Using these lemmas, it suffices to prove that the set of imaginary parts of zeros of $q(\cdot)$ has three elements.

- If $r = 0$, $q(\xi) = \xi^5 + \xi^3 + \xi = \xi(\xi^4 + \xi^2 + 1)$, and its roots are given by:

$$x_0 = 0, \quad x_1 = -\frac{1}{2} + i\frac{\sqrt{3}}{2}, \quad x_2 = \frac{1}{2} + i\frac{\sqrt{3}}{2}, \quad x_3 = \bar{x}_1, \quad x_4 = \bar{x}_2.$$

- Let $r \neq 0$. We claim that $q(\cdot)$ has only one real root. First, observe that

$$\lim_{x \rightarrow \pm\infty} q(x) = \pm\infty$$

That ensures the existence of such a real root. Now, note that $q'(x) = 5x^4 + 3x^2 + 1 > 0$ for $x \in \mathbb{R}$, thus, $q(\cdot)$ has only one real and simple root. Finally, the roots of q are given by

$$x_0 \in \mathbb{R}, \quad x_1, x_2, \overline{x_1}, \overline{x_2} \in \mathbb{C}.$$

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