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# STABILIZATION OF KAWAHARA EQUATION WITH INPUT SATURATION AND SATURATED BOUNDARY FEEDBACK

SUBRATA MAJUMDAR<sup>†</sup>, AND HUGO PARADA<sup>‡\*</sup>

ABSTRACT. In this paper, we study the stabilizability of the Kawahara equation in a bounded interval  $(0, L)$  by a saturated internal control and boundary feedback. We prove the exponential stabilization of the corresponding closed-loop system. We first show the well-posedness of the nonlinear system using linear, nonlinear semigroup theory and fixed-point arguments. Next, we prove two stabilization results by means of interior-saturating feedback and boundary-saturated feedback as well, using observability inequalities and certain unique continuation principle.

## 1. INTRODUCTION AND MAIN RESULTS

1.1. **System under study.** Let us consider the following nonlinear Kawahara equation

$$\begin{cases} \partial_t u + \partial_x u + \partial_x^3 u - \partial_x^5 u + u^p \partial_x u + \mathbf{f}_1(t, x) = 0, & (t, x) \in (0, \infty) \times (0, L), \\ u(t, 0) = u(t, L) = \partial_x u(t, 0) = \partial_x u(t, L) = 0, & t \in (0, \infty), \\ \partial_x^2 u(t, L) = \mathbf{f}_2(t), & t \in (0, \infty), \\ u(0, x) = u_0(x), & x \in (0, L), \end{cases} \quad (1.1)$$

with  $p \in [1, 2)$ , and the functions  $\mathbf{f}_1$  and  $\mathbf{f}_2$  are the controls acting internally and through the boundary conditions, respectively. The objective of this work is to show the exponential stabilization of (1.1) by acting with either internal or boundary controls, in the case where these controls are subject to saturation. The Kawahara equation  $\partial_t u + \partial_x u + \partial_x^3 u - \partial_x^5 u + u^p \partial_x u = 0$  is fifth-order, nonlinear one dimensional equation and is typically used to model the propagation of small-amplitude long waves in fluid dynamics, water waves, plasma dynamics among other dispersive phenomena (see [12, 15, 16] and the references therein). This equation can be also considered as an extension of the Korteweg-de Vries equation (KdV), including high order dispersion effects. It has been studied as a fifth degree KdV equation and singularly perturbed KdV equation in [2, 28] respectively. Discovered in 1972 by Kawahara [16], it was also obtained by Hasimoto for shallow water system in 1970 [14]. Since then, the Kawahara equation has been subject of study for different points of views. Recently, the control and stabilization properties have attracted the attention of the control community. Let us provide a brief overview about this subject.

As far as we know the first works dealing with this kind of control problems for Kawahara equation were [36, 37]. In these works, the stabilization of global solutions of the linear Kawahara equation using localized damping was addressed following the ideas presented in the works [27, 30], where similar problems was analyzed for the KdV equation. These results were then extended to the nonlinear Kawahara equation, with nonlinearity  $u \partial_x u$ . In [1], a similar problem was studied for the generalized Kawahara equation with nonlinearity  $u^p \partial_x u$  for  $p \in [1, 4)$ . It is important to remark that in [1], the energy of solutions decay exponentially for arbitrary initial data in  $L^2(0, L)$  for  $p \in [1, 2]$ . However, this decay was only proved for small initial data if  $p \in (2, 4)$ . Internal and boundary delayed stabilization problems for Kawahara equation were considered in [3, 8, 10]. The exponential stability was shown using multipliers, Lyapunov functionals and compactness ideas. We also mention [4], where Massera's theorems were obtained, [9], where the internal null controllability was shown via Carleman estimates, [7], where controllability in weighted Sobolev spaces was proved, and [5, 6], where the control with overdetermination condition was considered.

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<sup>†</sup>Department of Mathematics, Indian Institute of Technology Bombay, Powai, Mumbai, 400076, India. e-mail: subratamajumdar634@gmail.com.

<sup>‡</sup>Université Paul Sabatier, Institut de Mathématiques de Toulouse, 118 Route de Narbonne, 31062 Toulouse, France. e-mail: hugo.parada@math.univ-toulouse.fr.

\*Corresponding author: Hugo Parada.

As we already mentioned, the focus of this paper is to analyze the case of saturated controllers. Saturation in control systems is related to the fact that in most real-life applications, the actuators are subject to constraints, limitations due to physical, economical, operational reasons, etc. It is well-known that even in the finite-dimensional case the introduction of saturation in the controller could destabilize the system [34, Example 1.1], making the saturated control problem a challenging task. In the finite-dimensional framework, the stability analysis could be done by studying an appropriate Lyapunov function and a sector condition related with the saturation map [33–35].

With respect to saturation in infinite-dimensional control systems there is not a general method to tackle these problems. Noteworthy works include [29] where the stabilization of a one dimensional wave equation was considered with saturated internal and boundary controls. In this work, stability results were proved by using Lyapunov functions and LaSalle’s invariance principle. In the case of parabolic equations in [18], several results are collected in the one-dimensional case, the principal idea is to decompose the system into its stable and stable parts via a spectral approach. The unstable finite dimensional part is tackled as in [34], and for the remaining part, an appropriated Lyapunov function is considered. In the case of KdV-type equations, the approach is to follow the ideas of the works [27, 30] and to prove the exponential stability via compactness arguments. This requires, in particular, the use of unique continuation results for nonlinear PDEs. In this direction we mention the works [19, 20, 22, 25].

**1.2. Setting of the problem.** This paper deals with the feedback stabilization of the system (1.1) by means of saturated control acting in-domain or through a boundary action. In this work, we consider two types of saturation function  $\mathbf{sat}$  which can be given by the following. Let  $M > 0$  be the saturation level. We define

$$\mathbf{sat}_{loc}(s) = \begin{cases} s, & \text{if } |s| \leq M, \\ M\mathbf{sgn}(s), & \text{if } |s| \geq M, \end{cases} \quad (1.2)$$

and the second one is

$$\mathbf{sat}_2(s) = \begin{cases} s, & \text{if } \|s\|_{L^2(0,L)} \leq M, \\ \frac{sM}{\|s\|_{L^2(0,L)}}, & \text{if } \|s\|_{L^2(0,L)} \geq M. \end{cases} \quad (1.3)$$

**1.2.1. Internal control.** We start by describing the case where the control action is in-domain.

$$\begin{cases} \partial_t u + \partial_x u + \partial_x^3 u - \partial_x^5 u + u^p \partial_x u + \mathbf{sat}(a(x)u(t, x)) = 0, & (t, x) \in (0, \infty) \times (0, L), \\ u(t, 0) = u(t, L) = \partial_x u(t, 0) = \partial_x u(t, L) = 0, & t \in (0, \infty), \\ \partial_x^2 u(t, L) = 0, & t \in (0, \infty), \\ u(0, x) = u_0(x), & x \in (0, L), \end{cases} \quad (1.4)$$

where  $a \in L^\infty(0, L)$  is a localized function given by

$$a_1 \geq a(x) \geq a_0 > 0 \text{ on } \omega \subset [0, L], \omega \text{ is a nonempty open subset of } [0, L]. \quad (1.5)$$

**1.2.2. Boundary control.** In this work, we also explore the case where the saturated actuator is at the boundary, consider the following equation:

$$\begin{cases} \partial_t u + \partial_x u + \partial_x^3 u - \partial_x^5 u + u^p \partial_x u = 0, & (t, x) \in (0, \infty) \times (0, L), \\ u(t, 0) = u(t, L) = \partial_x u(t, 0) = \partial_x u(t, L) = 0, & t \in (0, \infty), \\ \partial_x^2 u(t, L) = \mathbf{sat}_{loc}(\alpha \partial_x^2 u(t, 0)), & t \in (0, \infty), \\ u(0, x) = u_0(x), & x \in (0, L), \end{cases} \quad (1.6)$$

where  $|\alpha| < 1$ .

The main objective of this paper is to show the exponential stability of (1.4) and (1.6) in the presence of saturated control. More precisely, we investigate semiglobal exponential stabilization result (see **Theorem 3.4**) for the Kawahara equation (1.4) with internal saturation. That means, for any  $r > 0$ , we can find a positive number  $\mu(r) > 0$  such that the energy

$$E(t) = \frac{1}{2} \int_0^L u(t, x)^2 dx$$

of the solution of (1.4) decays exponentially with a rate  $\mu(r)$ , provided the initial data satisfying the following smallness assumption  $\|u_0\|_{L^2(0,L)} \leq r$ . For the saturated boundary feedback case, we study

local exponential stabilization result (see [Theorem 3.3](#)) for the system [\(1.6\)](#) with  $|\alpha| < 1$ . That is, there exist positive constants  $r, \mu, C > 0$  such that whenever  $\|u_0\|_{L^2(0,L)} \leq r$ , the solution of [\(1.6\)](#) satisfies

$$\|u(t, \cdot)\|_{L^2(0,L)} \leq Ce^{-\mu t} \|u_0\|_{L^2(0,L)}, \quad \forall t \geq 0. \quad (1.7)$$

Our study is based on deriving appropriate observability inequalities for the saturated systems. These inequalities are obtained using compactness ideas and Carleman estimates. We also put special emphasis on the global well-posedness of the associated systems. Indeed, as we can observe from the definition of the saturation maps [\(1.2\)](#) and [\(1.3\)](#) if the argument of the map, it is small enough (in the appropriated norms) then the saturation is not really active. This was one of the main points for instance in [\[25\]](#).

The structure of this paper is the following. In [Section 2](#), we study the well-posedness results associated to [\(1.4\)](#) and [\(1.6\)](#). Then, in [Section 3](#), we study the two exponential stabilization results for the case of boundary saturation and saturated interior feedback, respectively. At the end, we conclude the paper with some additional remarks and open questions.

## 2. WELL-POSEDNESS

This section is devoted to the existence and uniqueness of the system [\(1.4\)](#) and [\(1.6\)](#) with the saturated feedback. We will follow the works [\[19, 20, 22, 25\]](#) see also [\[23\]](#).

**2.1. Well-posedness with boundary feedback.** We start by studying the well-posedness results for [\(1.6\)](#). First, we consider the system [\(1.6\)](#) without the internal nonlinearity, and we prove the well-posedness based on nonlinear semigroup theory. Then, we will add an internal source term in order to take in account the nonlinear term  $u^p \partial_x u$ . Consider the system

$$\begin{cases} \partial_t u + \partial_x u + \partial_x^3 u - \partial_x^5 u = 0, & (t, x) \in (0, \infty) \times (0, L), \\ u(t, 0) = u(t, L) = \partial_x u(t, 0) = \partial_x u(t, L) = 0, & t \in (0, \infty), \\ \partial_x^2 u(t, L) = \mathbf{sat}_{loc}(\alpha \partial_x^2 u(t, 0)), & t \in (0, \infty), \\ u(0, x) = u_0(x), & x \in (0, L). \end{cases} \quad (2.1)$$

Let  $\mathcal{A}_b : D(\mathcal{A}_b) \rightarrow L^2(0, L)$  the nonlinear operator defined by

$$\mathcal{A}_b w := -w' - w''' + w'''' \quad (2.2)$$

$$D(\mathcal{A}_b) = \{w \in H^5(0, L) \cap H_0^2(0, L) : w''(L) = \mathbf{sat}_{loc}(\alpha w''(0))\}.$$

Note that the operator  $\mathcal{A}_b$  is nonlinear by the action of the boundary saturation. Following [\[19, 22\]](#), we show that  $\mathcal{A}_b$  is a closed, dissipative operator and that for some  $\lambda > 0$ ,  $D(\mathcal{A}_b) \subset R(\mathcal{I} - \lambda \mathcal{A}_b)$ , where  $\mathcal{I} : L^2(0, L) \rightarrow L^2(0, L)$ , is the identity operator and  $R$  denotes the range of an operator. The following result, ensure the existence of mild and strong solutions for [\(2.1\)](#).

**Proposition 2.1.** *Let  $u_0 \in L^2(0, L)$  and  $|\alpha| < 1$ , then there exists a unique mild solution  $u \in C([0, \infty), L^2(0, L))$  of [\(2.1\)](#). Moreover, if  $u_0 \in D(\mathcal{A}_b)$ , this solution is strong and  $u \in C([0, \infty), D(\mathcal{A}_b)) \cap C^1((0, \infty), L^2(0, L))$ .*

*Proof.*  **$\mathcal{A}_b$  is a closed operator.** Consider the operator  $\mathcal{A}_b^0 : D(\mathcal{A}_b^0) \rightarrow L^2(0, L)$  defined by  $\mathcal{A}_b^0 w := \mathcal{A}_b w$ , where  $D(\mathcal{A}_b^0) = \{w \in H^5(0, L) \cap H_0^2(0, L) : w''(L) = 0\}$ . Then, clearly  $\mathcal{A}_b^0$  is a closed operator. Similarly, by [\[17, Page 91\]](#) the operator  $\mathbf{sat}_{loc}$  satisfies

$$\forall (s, \tilde{s}) \in \mathbb{R}^2, \quad |\mathbf{sat}_{loc}(s) - \mathbf{sat}_{loc}(\tilde{s})| \leq |s - \tilde{s}|. \quad (2.3)$$

Thus,  $\mathbf{sat}_{loc}$  is a Lipschitz continuous function and hence the operator  $\mathcal{A}_b$  is closed.

**$\mathcal{A}_b$  is dissipative.** Let  $w, v \in D(\mathcal{A}_b)$ , then after some integration by parts and [\(2.3\)](#) we have

$$\begin{aligned} \langle \mathcal{A}_b w - \mathcal{A}_b v, w - v \rangle_{L^2(0,L)} &= \int_0^L (-w' - u''' + w'''' + v' + v''' - v''''')(w - v) dx \\ &= \frac{1}{2} [(w''(L) - v''(L))^2 - (w''(0) - v''(0))^2] \\ &= \frac{1}{2} [(\mathbf{sat}_{loc}(\alpha w''(0)) - \mathbf{sat}_{loc}(\alpha v''(0)))^2 - (w''(0) - v''(0))^2] \\ &\leq \frac{\alpha^2 - 1}{2} (w''(0) - v''(0))^2 \leq 0, \end{aligned}$$

from where we obtain that  $\mathcal{A}_b$  is dissipative.

**For  $\lambda > 0$  small enough,  $D(\mathcal{A}_b) \subset R(\mathcal{I} - \lambda\mathcal{A}_b)$ .** Take  $\lambda > 0$ , then  $D(\mathcal{A}_b) \subset R(\mathcal{I} - \lambda\mathcal{A}_b)$ , is equivalent to given  $w \in D(\mathcal{A}_b)$  find  $v \in D(\mathcal{A}_b)$  such that  $(\mathcal{I} - \lambda\mathcal{A}_b)v = w$ . Define  $\tilde{\lambda} = \frac{1}{\lambda}$ , then we have to find solutions  $v \in D(\mathcal{A}_b)$  of

$$\begin{cases} \tilde{\lambda}v + v' + v''' - v'''' = \tilde{\lambda}w, \\ v(0) = v(L) = v'(0) = v'(L) = 0, \quad v''(L) = \mathbf{sat}_{loc}(\alpha v''(0)). \end{cases} \quad (2.4)$$

Consider the operator  $\mathcal{T} : H^3(0, L) \rightarrow L^2(0, L)$  defined by  $\mathcal{T}(y) = z$ , where  $z$  is the solution of

$$\begin{cases} \tilde{\lambda}z + z' + z''' - z'''' = \tilde{\lambda}w, \\ z(0) = z(L) = z'(0) = z'(L) = 0, \quad z''(L) = \mathbf{sat}_{loc}(\alpha y''(0)). \end{cases} \quad (2.5)$$

**Lemma 2.2.** *For all  $w \in D(\mathcal{A}_b)$ , the operator  $\mathcal{T}$  is well-defined, i.e. for all  $y \in H^3(0, L)$  there exists a unique solution to (2.5).*

The proof of Lemma 2.2 is given in the Appendix A. Let us show now that  $\mathcal{T}$  has a fixed point. Take  $C > 0$  to be chosen later and define

$$K_C = \{z \in H_0^2(0, L) : \|z\|_{H_0^2(0, L)} \leq C\}. \quad (2.6)$$

As  $H_0^2(0, L)$  is compactly embedded in  $L^2(0, L)$ , the set  $K_C$  is a bounded set in  $H_0^2(0, L)$  and relatively compact in  $L^2(0, L)$ . Furthermore, as  $K_C$  is closed in  $L^2(0, L)$  we can deduce that  $K_C$  is compact in  $L^2(0, L)$ . Our idea now is to apply the Schauder fixed point theorem [11, Theorem B.19]. Thus, we have to prove the following lemma

**Lemma 2.3.** *There exists  $C > 0$  such that  $\mathcal{T}(H^3(0, L)) \subset K_C$ .*

*Proof.* Multiplying (2.5) by  $z$  and integrating on  $(0, L)$  we get after some integration by parts

$$\tilde{\lambda} \int_0^L z^2 dx + \frac{1}{2} z''(0)^2 = \frac{1}{2} \mathbf{sat}_{loc}(\alpha y''(0))^2 + \tilde{\lambda} \int_0^L wz dx.$$

Then, using the definition of  $\mathbf{sat}_{loc}$  given in (1.2) and Cauchy-Schwartz, we obtain

$$\frac{\tilde{\lambda}}{2} \|z\|_{L^2(0, L)}^2 \leq \frac{M^2}{2} + \frac{\tilde{\lambda}}{2} \|w\|_{L^2(0, L)}^2. \quad (2.7)$$

Now, multiplying (2.5) by  $xz$  and integrating on  $(0, L)$  we get after some integration by parts

$$\begin{aligned} \tilde{\lambda} \int_0^L xz^2 dx + \frac{3}{2} \int_0^L (z')^2 dx + \frac{5}{2} \int_0^L (z'')^2 dx &= \frac{1}{2} \int_0^L z^2 dx + \frac{L}{2} \mathbf{sat}(\alpha y''(0))^2 + \tilde{\lambda} \int_0^L xwz dx, \\ \frac{3}{2} \|z'\|_{L^2(0, L)}^2 + \frac{5}{2} \|z''\|_{L^2(0, L)}^2 &\leq \left( \frac{L\tilde{\lambda}}{2} + \frac{1}{2} \right) \|z\|_{L^2(0, L)}^2 + \frac{LM^2}{2} + \frac{L\tilde{\lambda}}{2} \|w\|_{L^2(0, L)}^2, \end{aligned} \quad (2.8)$$

and hence using (2.7) and (2.8) we conclude,  $\|z''\|_{L^2(0, L)}^2 \leq C$ , where  $C > 0$  only depends on the level of saturation  $M$ ,  $L$ ,  $\tilde{\lambda}$  and  $w$  which ends the proof of Lemma 2.3.  $\square$

Using the Schauder fixed point theorem, we manage to find  $v \in H^3(0, L)$  solution of (2.4), we get  $v \in D(\mathcal{A}_b)$  using that the boundary conditions and  $w \in D(\mathcal{A}_b)$ . We conclude that for all  $\lambda > 0$ ,  $D(\mathcal{A}_b) \subset R(\mathcal{I} - \lambda\mathcal{A}_b)$ . Using [21, Theorem 4.2]  $\mathcal{A}_b$  is the generator of a semigroup of contractions  $S(t)$ ,  $t > 0$ . By [21, Theorem 4.5] for all  $T > 0$ ,  $u_0 \in D(\mathcal{A}_b)$ ,  $u = S(t)u_0 \in C([0, T]; D(\mathcal{A}_b)) \cap C^1((0, T); L^2(0, L))$  is a strong solution to (2.1). Moreover, by [21, Theorem 4.10] this solution is unique. Also, we can deduce that for all  $T > 0$ , for all  $u_0 \in L^2(0, L)$ , there exists a unique mild solution of (2.1)  $u \in C([0, T]; L^2(0, L))$  and then, the proof of Proposition 2.1 is finished.  $\square$

Now we study the well-posedness of the system (2.1) with an extra source term  $f$ .

$$\begin{cases} \partial_t u + \partial_x u + \partial_x^3 u - \partial_x^5 u = f, & (t, x) \in (0, \infty) \times (0, L), \\ u(t, 0) = u(t, L) = \partial_x u(t, 0) = \partial_x u(t, L) = 0, & t \in (0, \infty), \\ \partial_x^2 u(t, L) = \mathbf{sat}_{loc}(\alpha \partial_x^2 u(t, 0)), & t \in (0, \infty), \\ u(0, x) = u_0(x), & x \in (0, L). \end{cases} \quad (2.9)$$

Formally, this system can be seen as  $\partial_t u = \mathcal{A}_b u + f$ . The following result can be obtained using [26, Theorem 10.1, Theorem 10.2].

**Proposition 2.4.** *Let  $u_0 \in L^2(0, L)$ ,  $|\alpha| < 1$ ,  $f \in L^1((0, \infty), L^2(0, L))$ , then there exists a unique mild solution  $u \in C([0, \infty), L^2(0, L))$  of (2.9). Moreover, if  $u_0 \in D(\mathcal{A}_b)$  and  $f \in W^{1,1}((0, \infty), L^2(0, L))$  this solution is strong and  $u \in C([0, \infty), L^2(0, L))$ ,  $u(t, \cdot) \in D(\mathcal{A}_b)$ , for all  $t \geq 0$  and  $u \in C^1((0, \infty), L^2(0, L))$ .*

Now, in order to deal with the nonlinear term  $u^p \partial_x u$ , we show that mild solutions of (2.9) have an extra regularity. Consider the space  $B_T = C([0, T], L^2(0, L)) \cap L^2((0, T), H^2(0, L))$ , endowed with the norm

$$\|u\|_{B_T}^2 = \|u\|_{C([0, T], L^2(0, L))}^2 + \|u\|_{L^2((0, T), H^2(0, L))}^2.$$

**Proposition 2.5.** *Let  $u_0 \in L^2(0, L)$ ,  $|\alpha| < 1$ ,  $f \in L^1((0, T), L^2(0, L))$  and  $T > 0$ , consider  $u \in C([0, T], L^2(0, L))$ , the restriction of the solution of (2.9) on  $[0, T]$ , then  $u \in B_T$ . Moreover, the following estimate holds,*

$$\|u\|_{B_T} \leq C(1 + \sqrt{T}) (\|u_0\|_{L^2(0, L)} + \|f\|_{L^1((0, T), L^2(0, L))}). \quad (2.10)$$

For  $f = 0$ , the following estimate follows

$$\|u_0\|_{L^2(0, L)}^2 \leq \frac{1}{T} \|u\|_{L^2((0, T), L^2(0, L))}^2 + \|\partial_x^2 u(\cdot, 0)\|_{L^2(0, T)}^2. \quad (2.11)$$

*Proof.* Suppose  $u$  is a strong solution of (2.9), let  $T > 0$  and  $0 \leq s \leq T$ . Multiplying the first line of (2.9) by  $u$  and integrating on  $(0, L) \times (0, s)$ , we get

$$\int_0^s \int_0^L u \partial_t u dx dt + \int_0^s \int_0^L u \partial_x u dx dt + \int_0^s \int_0^L u \partial_x^3 u dx dt - \int_0^s \int_0^L u \partial_x^5 u dx dt = \int_0^s \int_0^L u f dx dt.$$

After some integration by parts, we observe

$$\frac{1}{2} \int_0^L u(s, x)^2 dx + \frac{1}{2} \int_0^s (\partial_x^2 u(t, 0)^2 - \mathbf{sat}(\alpha \partial_x^2 u(t, 0))^2) dt = \frac{1}{2} \int_0^L u_0^2 dx + \int_0^s \int_0^L f u dx dt. \quad (2.12)$$

Note that for all  $s \in [0, T]$

$$\int_0^s \int_0^L f u dx dt \leq \int_0^T \|u(t, \cdot)\|_{L^2(0, L)} \|f(t, \cdot)\|_{L^2(0, L)} dt \leq \|u\|_{C([0, T], L^2(0, L))} \|f\|_{L^1((0, T), L^2(0, L))}$$

Now, using the above equation, (2.3), (2.12), we get for all  $s \in [0, T]$ ,

$$\frac{1}{2} \int_0^L u(s, x)^2 dx + C_\alpha \int_0^s \partial_x^2 u(t, 0)^2 dt \leq \frac{1}{2} \int_0^L u_0^2 dx + \|u\|_{C([0, T], L^2(0, L))} \|f\|_{L^1((0, T), L^2(0, L))}, \quad (2.13)$$

where  $C_\alpha = \frac{1 - |\alpha|^2}{2} > 0$ . Now, taking the supremum for  $s \in [0, T]$ , and with Young's inequality, we get

$$\|u\|_{C([0, T], L^2(0, L))}^2 + \int_0^T \partial_x^2 u(t, 0)^2 dt \leq C \left( \|u_0\|_{L^2(0, L)}^2 + \|f\|_{L^1((0, T), L^2(0, L))}^2 \right). \quad (2.14)$$

Now, we multiply the first equation of (2.9) by  $xu$  and integrate on  $(0, L) \times (0, T)$  and we get

$$\int_0^T \int_0^L xu \partial_t u dx dt + \int_0^T \int_0^L xu \partial_x u dx dt + \int_0^T \int_0^L xu \partial_x^3 u dx dt - \int_0^T \int_0^L xu \partial_x^5 u dx dt = \int_0^T \int_0^L xu f dx dt. \quad (2.15)$$

Thus, performing integration by parts

$$\begin{aligned} \frac{1}{2} \int_0^L xu(T, \cdot)^2 dx + \frac{3}{2} \int_0^T \int_0^L \partial_x u^2 dx dt + \frac{5}{2} \int_0^T \int_0^L \partial_x^2 u^2 dx dt &= \frac{1}{2} \int_0^L xu_0^2 dx + \frac{1}{2} \int_0^T \int_0^L u^2 dx dt \\ &+ \frac{L}{2} \int_0^T \mathbf{sat}(\alpha \partial_x^2 u(t, 0))^2 dt + \int_0^T \int_0^L xu f dx dt. \end{aligned} \quad (2.16)$$

Note that

$$\int_0^T \int_0^L u^2 dx dt \leq T \|u\|_{C([0, T], L^2(0, L))}^2$$

and

$$\int_0^T \int_0^L x u f dx dt \leq L \|u\|_{C([0,T],L^2(0,L))} \|f\|_{L^1((0,T),L^2(0,L))}.$$

Using these two inequalities, (2.14) we derive

$$\|\partial_x^2 u\|_{L^2((0,T),L^2(0,L))}^2 \leq C(1+T) \left( \|u_0\|_{L^2(0,L)}^2 + \|f\|_{L^1((0,T),L^2(0,L))}^2 \right), \quad (2.17)$$

where  $C > 0$  does not depend on  $T$ . Thus  $u \in L^2((0,T),H^2(0,L))$ . Joining (2.14) and (2.17) we deduce (2.10). To deduce (2.11), consider  $f = 0$  and multiply (2.1) by  $(T-t)u$  and integrate  $(0,T) \times (0,L)$ , after some integration by parts we get

$$T \int_0^L u_0^2 dx + \int_0^T (T-t) [\mathbf{sat}_{loc}(\partial_x^2 u(t,0))^2 - \partial_x^2 u(t,0)^2] dt = \int_0^T \int_0^L u^2 dx dt,$$

using (2.3) we obtain (2.11). The result can be extended by density to mild solutions.  $\square$

Before going to prove the global well-posedness result for the nonlinear system (1.6), we first show some important estimates related to the nonlinear term:

**Proposition 2.6.** *Let  $u \in B_T$ . Then for  $1 \leq p \leq 2$ ,  $u^p \partial_x u \in L^1((0,T),L^2(0,L))$  and the map  $u \in B_T \mapsto u^p \partial_x u \in L^1((0,T),L^2(0,L))$  is continuous. Moreover, there exists  $C > 0$  such that, for any  $u, v \in B_T$*

$$\|u^p \partial_x v\|_{L^1((0,T),L^2(0,L))} \leq C(T^{(2-p)/4} + \sqrt{T}) \|u\|_{B_T}^p \|v\|_{B_T}. \quad (2.18)$$

$$\|u^p \partial_x u - v^p \partial_x v\|_{L^1((0,T),L^2(0,L))} \leq C(T^{(2-p)/4} + \sqrt{T}) \left( \|u\|_{B_T} \|v\|_{B_T}^{p-1} + \|v\|_{B_T}^p + \|u\|_{B_T}^p \right) \|u - v\|_{B_T}. \quad (2.19)$$

*Proof.* Estimate (2.18) trivially follows from [31, Lemma 2.3]. Next, we give a sketch of the proof of the estimate (2.19).

$$\begin{aligned} \int_0^T \|u^p \partial_x u - v^p \partial_x v\|_{L^2(0,L)} &= \int_0^T \|u^p \partial_x (u-v) + (u^p - v^p) \partial_x v\|_{L^2(0,L)} \\ &\leq C(T^{(2-p)/4} + \sqrt{T}) \|u\|_{B_T}^p \|u-v\|_{B_T} + \int_0^T \left( |u|^{p-1} + |v|^{p-1} \right) (u-v) \partial_x v \|_{L^2(0,L)} \end{aligned}$$

Using [31, Lemma 2.4], we estimate the last term of the above inequality as:

$$\int_0^T \left( |u|^{p-1} + |v|^{p-1} \right) (u-v) \partial_x v \|_{L^2(0,L)} \leq C T^{(2-p)/4} \|u-v\|_{B_T} \left( \|u\|_{B_T} \|v\|_{B_T}^{p-1} + \|v\|_{B_T}^p \right).$$

$\square$

We conclude this subsection by proving the global well-posedness result for nonlinear equation (1.6).

**Theorem 2.7.** *Let  $u_0 \in L^2(0,L)$  and  $|\alpha| < 1$ , then there exists a unique mild solutions  $u \in C([0,\infty),L^2(0,L)) \cap L^2(0,\infty,H^2(0,L))$  of (1.6).*

*Proof.* Let  $u_0 \in D(\mathcal{A}_b)$  and  $R, \theta > 0$  to be chosen later. Consider the ball  $\mathcal{B}_{B_\theta}(0,R) := \{v \in B_\theta, \|v\|_{B_\theta} \leq R\}$ . Then  $\mathcal{B}_{B_\theta}(0,R)$  it is a complete metric space. Let us introduce the map  $\Phi : B_\theta \mapsto B_\theta$ , defined by  $\Phi(v) = u$ , where  $u$  solves

$$\begin{cases} \partial_t u + \partial_x u + \partial_x^3 u - \partial_x^5 u = -v^p \partial_x v, & (t,x) \in (0,\theta) \times (0,L), \\ u(t,0) = u(t,L) = \partial_x u(t,0) = \partial_x u(t,L) = 0, & t \in (0,\theta), \\ \partial_x^2 u(t,L) = \mathbf{sat}_{loc}(\alpha \partial_x^2 u(t,0)), & t \in (0,\theta), \\ u(0,x) = u_0(x), & x \in (0,L). \end{cases}$$

We observe first that  $\Phi$  is well-defined, since by Proposition 2.6  $v^p \partial_x v \in L^1((0,\theta),L^2(0,L))$ . By Proposition 2.5 and Proposition 2.6 we get

$$\begin{aligned} \|\Phi(v)\|_{B_\theta} &\leq C(1 + \sqrt{\theta}) \left( \|u_0\|_{L^2(0,L)} + \|v^p \partial_x v\|_{L^1((0,\theta),L^2(0,L))} \right) \\ &\leq C_1(1 + \sqrt{\theta}) \|u_0\|_{L^2(0,L)} + C_2(\theta^{(2-p)/4} + \sqrt{\theta}) \|v\|_{B_T}^{p+1}, \end{aligned}$$

for some  $C_1, C_2 > 0$ . Then taking  $R = 3C_1 \|u_0\|_{L^2(0,L)}$  and  $\theta > 0$  small enough such that

$$\begin{cases} C_2(\theta^{2-p/4} + \sqrt{\theta}) R^p < \frac{1}{3}, \\ (1 + \sqrt{\theta}) < 2, \end{cases}$$



we have that  $\|\Phi(v)\|_{B_\theta} \leq R$  and thus  $\Phi$  maps  $\mathcal{B}_{B_\theta}(0, R)$  into itself. Now take  $v_1, v_2 \in \mathcal{B}_{B_\theta}(0, R)$ , and consider  $w = u_1 - u_2 = \Phi(v_1) - \Phi(v_2)$ . We observe that  $w$  solves

$$\begin{cases} \partial_t w + \partial_x w + \partial_x^3 w - \partial_x^5 w = f, & (t, x) \in (0, \theta) \times (0, L), \\ w(t, 0) = w(t, L) = \partial_x w(t, 0) = \partial_x w(t, L) = 0, & t \in (0, \theta), \\ \partial_x^2 w(t, L) = h, & t \in (0, \theta), \\ w(0, x) = 0, & x \in (0, L), \end{cases}$$

where  $f = -v_1^p \partial_x v_1 + v_2^p \partial_x v_2$  and  $h = \mathbf{sat}_{loc}(\alpha \partial_x^2 u_1(t, 0)) - \mathbf{sat}_{loc}(\alpha \partial_x^2 u_2(t, 0))$ . Similar integration by parts as in [Proposition 2.5](#) allows to write

$$\|w\|_{B_\theta} \leq C (\|h\|_{L^2(0, \theta)} + \|f\|_{L^1((0, \theta), L^2(0, L))}). \quad (2.20)$$

Our idea now is to obtain an estimate of  $\|h\|_{L^2(0, \theta)}$ . Proceeding as [\(2.12\)](#) we obtain

$$\frac{1}{2} \int_0^L w(\theta, \cdot)^2 dx + \frac{1}{2} \int_0^\theta (\partial_x^2 w(t, 0))^2 - h(t)^2 dt = \int_0^\theta \int_0^L f w dx dt.$$

Using [\(2.3\)](#) we get

$$\begin{aligned} \|h\|_{L^2(0, \theta)} &= \|\mathbf{sat}_{loc}(\alpha \partial_x^2 u_1(\cdot, 0)) - \mathbf{sat}_{loc}(\alpha \partial_x^2 u_2(\cdot, 0))\|_{L^2(0, \theta)} \\ &\leq \|\alpha \partial_x^2 u_1(\cdot, 0) - \alpha \partial_x^2 u_2(\cdot, 0)\|_{L^2(0, \theta)} \\ &\leq |\alpha| \|\partial_x^2 w(\cdot, 0)\|_{L^2(0, \theta)}, \end{aligned}$$

which implies

$$\frac{1}{2} \int_0^L w(\theta, \cdot)^2 dx + C_\alpha \int_0^\theta \partial_x^2 w(t, 0)^2 dt \leq \int_0^\theta \int_0^L f w dx dt.$$

Thus by Young's inequality  $\|w\|_{C([0, \theta], L^2(0, L))} \leq C \|f\|_{L^1((0, \theta), L^2(0, L))}$ , then by the above equation

$$\|h\|_{L^2(0, \theta)} \leq |\alpha| \|\partial_x^2 w(\cdot, 0)\|_{L^2(0, \theta)} \leq C \|f\|_{L^1((0, \theta), L^2(0, L))}.$$

Using this estimate in [\(2.20\)](#) we obtain

$$\|\Phi(v_1) - \Phi(v_2)\|_{B_\theta} = \|w\|_{B_\theta} \leq C \|f\|_{L^1((0, \theta), L^2(0, L))} = C \|v_1^p \partial_x v_1 - v_2^p \partial_x v_2\|_{L^1((0, \theta), L^2(0, L))}.$$

Finally by [Proposition 2.6](#)

$$\|\Phi(v_1) - \Phi(v_2)\|_{B_\theta} \leq C_3 (\theta^{(2-p)/4} + \sqrt{\theta}) \left( \|v_1\|_{B_\theta} \|v_2\|_{B_\theta}^{p-1} + \|v_1\|_{B_\theta}^p + \|v_2\|_{B_\theta}^p \right) \|v_1 - v_2\|_{B_\theta},$$

for some  $C_3 > 0$ . Then, taking  $\theta > 0$ , small enough such that  $3C_3(\theta^{(2-p)/4} + \sqrt{\theta})R^p < 1$ , we get that  $\Phi$  is a contraction and by the Banach fixed point theorem and a density argument, we deduce the local in time well-posedness.

To conclude the global well-posedness result, we need some a priori estimates. Let  $T > 0$  and  $u$  be a strong solution of [\(1.6\)](#). Multiplying [\(1.6\)](#) by  $u$  and integrating over  $(0, L) \times (0, s)$ , we get after some integration by parts

$$\frac{1}{2} \int_0^L u(s, \cdot)^2 dx + \frac{1}{2} \int_0^s (\partial_x^2 u(t, 0))^2 - \mathbf{sat}(\alpha \partial_x^2 u(t, 0))^2 dt = \frac{1}{2} \int_0^L u_0^2 dx.$$

Using [\(1.2\)](#) we derive

$$\frac{1}{2} \int_0^L u^2(s, \cdot) dx + C_\alpha \int_0^s \partial_x^2 u(t, 0)^2 dt \leq \frac{1}{2} \int_0^L u_0^2 dx. \quad (2.21)$$

Then, taking the supremum for  $s \in [0, T]$  in [\(2.21\)](#),

$$\|u\|_{C([0, T], L^2(0, L))}^2 \leq \|u_0\|_{L^2(0, L)}^2. \quad (2.22)$$

On the other hand, taking  $s = T$  in [\(2.21\)](#),

$$\int_0^T \partial_x^2 u(t, 0)^2 dt \leq \frac{1}{2} C_\alpha^{-1} \|u_0\|_{L^2(0, L)}^2. \quad (2.23)$$

Now, we multiply the first line of [\(1.6\)](#) by  $xu$  and integrate on  $(0, L) \times (0, T)$ ,

$$\begin{aligned} \frac{1}{2} \int_0^L x u^2(T, \cdot) dx + \frac{3}{2} \int_0^T \int_0^L \partial_x u^2 dx dt + \frac{5}{2} \int_0^T \int_0^L \partial_x^2 u^2 dx dt &= \frac{1}{2} \int_0^L x u_0^2(x) + \frac{1}{p+2} \int_0^T \int_0^L u^{p+2} dx dt \\ &\quad + \frac{1}{2} \int_0^T \int_0^L u^2 dx dt + \frac{L}{2} \int_0^T \partial_x^2 u(t, 0)^2 dt. \end{aligned}$$



For the term involving  $\int_0^T \int_0^L u^2 dx dt$

$$\int_0^T \int_0^L u^2(t, x) dx \leq T \|u\|_{C([0, T], L^2(0, L))}^2.$$

For the term involving  $\int_0^T \int_0^L u^{p+2}(t, x) dx dt$ , observe that

$$\begin{aligned} \frac{1}{p+2} \int_0^T \int_0^L u^{p+2} dx dt &\leq \frac{1}{p+2} \int_0^T \|u(t, \cdot)\|_{L^\infty(0, L)}^p \|u(t, \cdot)\|_{L^2(0, L)}^2 dt \\ &\leq \frac{2^{p/2}}{p+2} \int_0^T \|\partial_x u(t, \cdot)\|_{L^2(0, L)}^{p/2} \|u(t, \cdot)\|_{L^2(0, L)}^{2+p/2} dt. \end{aligned}$$

Therefore using (2.22) we have

$$\begin{aligned} \frac{1}{p+2} \int_0^T \int_0^L u^{p+2} dx dt &\leq C \frac{2^{p/2}}{p+2} \|u_0\|_{L^2(0, L)}^{2+p/2} \int_0^T \|\partial_x u(t, \cdot)\|_{L^2(0, L)}^{p/2} dt \\ &\leq C \frac{2^{p/2} T^{1-p/4}}{p+2} \|u_0\|_{L^2(0, L)}^{2+p/2} \left( \int_0^T \|\partial_x u(t, \cdot)\|_{L^2(0, L)}^2 dt \right)^{p/4} \\ &\leq C_p T \|u_0\|_{L^2(0, L)}^{\frac{8+2p}{4-p}} + \frac{1}{2} \left( \int_0^T \|\partial_x u(t, \cdot)\|_{L^2(0, L)}^2 dt \right), \end{aligned} \quad (2.24)$$

for some positive constant  $C_p$  depending on  $p$  and  $C$ . Thus, from (2.22), (2.23) and Young's inequality we derive

$$\int_0^T \int_0^1 \partial_x u^2 dx dt + \int_0^T \int_0^L \partial_x^2 u^2 dx dt \leq C_p \left[ (1+T) \|u_0\|_{L^2(0, L)}^2 + T \|u_0\|_{L^2(0, L)}^{\frac{8+2p}{4-p}} \right]. \quad (2.25)$$

Finally, with (2.22) and (2.25) and arguing by density, the proof of [Theorem 2.7](#) is finished.  $\square$

**2.2. Well-posedness with internal feedback.** In this section, we study the global well-posedness of (1.4), some of the computations are quite similar to those exposed in Section 2.1 for that we focus in the main difference.

*2.2.1. Well-posedness of linear system.* Let us first investigate the well-posedness of the linear Kawahara system, i.e when we drop the nonlinear internal terms  $u^p \partial_x u$  and  $\text{sat}(au)$ :

$$\begin{cases} \partial_t u + \partial_x u + \partial_x^3 u - \partial_x^5 u = 0, & (t, x) \in (0, \infty) \times (0, L), \\ u(t, 0) = u(t, L) = \partial_x u(t, 0) = \partial_x u(t, L) = \partial_x^2 u(t, L) = 0, & t \in (0, \infty), \\ u(0, x) = u_0(x), & x \in (0, L). \end{cases} \quad (2.26)$$

Thus we write the above system as:

$$\begin{cases} \dot{u}(t) = \mathcal{A}u(t), & t > 0, \\ u(0) = u_0, \end{cases} \quad (2.27)$$

where the operator  $\mathcal{A}$  can be written as

$$\mathcal{A}u = -u' - u''' + u''''$$

with the domain

$$D(\mathcal{A}) = \{u \in H^5(0, L) \cap H_0^2(0, L) : u_{xx}(L) = 0\}.$$

*Remark 2.8.* Observe that contrary to the case of boundary saturation, we first analyse a system which is linear so that we use linear semigroup theory for the study of (2.26). In what follows, we add the nonlinear terms as source terms. In the case of boundary saturation this strategy it is not clear, and for that reason we just drop the internal nonlinear term and we used nonlinear semigroup arguments. That made the well-posedness of (2.26) easier than the boundary case.

It can be checked that the operator  $\mathcal{A}$  and its adjoint operator

$$\mathcal{A}^*u = u' + u''' - u''''$$

with the domain

$$D(\mathcal{A}^*) = \{u \in H^5(0, L) \cap H_0^2(0, L) : u_{xx}(0) = 0\},$$

are both are dissipative. Therefore by Lumer-Philips theorem, it follows that  $\mathcal{A}$  generates a strongly continuous semigroup of contraction  $\{S(t)\}_{t \geq 0}$ . Hereinafter, we have the following well-posedness result for the linear Kawahara system:

**Theorem 2.9.** *Let us take  $u_0 \in D(\mathcal{A})$ . Then the equation (2.26) possesses a unique strong solution  $u \in C([0, T]; D(\mathcal{A}) \cap C^1(0, T; L^2(0, L)))$ . If the initial data  $u_0 \in L^2(0, L)$ , the unique mild solution of (2.26) lies in  $C([0, T]; L^2(0, L)) \cap L^2(0, T; H^2(0, L))$ . Moreover, we have a generic positive constant  $C > 0$  such that the solution satisfies the following estimates*

- $\|u\|_{C([0, T]; L^2(0, L))} + \|u\|_{L^2(0, T; H^2(0, L))} \leq C \|u_0\|_{L^2(0, L)},$
- $\|u_{xx}(\cdot, 0)\|_{L^2(0, T)} \leq C \|u_0\|_{L^2(0, L)}.$

Next, we study the well-posedness result for the linear Kawahara equation with a source term:

$$\begin{cases} \partial_t u + \partial_x u + \partial_x^3 u - \partial_x^5 u = f, & (t, x) \in (0, \infty) \times (0, L), \\ u(t, 0) = u(t, L) = \partial_x u(t, 0) = \partial_x u(t, L) = \partial_x^2 u(t, L) = 0, & t \in (0, \infty), \\ u(0, x) = u_0(x), & x \in (0, L). \end{cases} \quad (2.28)$$

Thus, we write the above system in the form of an abstract nonhomogeneous evolution equation:

$$\begin{cases} \dot{u} = \mathcal{A}u + f, & t > 0, \\ u(0) = u_0. \end{cases} \quad (2.29)$$

By classical semigroup theory result, it follows that for any  $f \in L^1_{loc}(0, \infty; L^2(0, L))$ , the solution of the system (2.29)

$$u(t) = S(t)u_0 + \int_0^t S(t-s)f(s)ds$$

belongs to  $C([0, \infty); L^2(0, L))$ . We also have the following estimates:

**Theorem 2.10.** *Let us take  $f \in L^1(0, T; L^2(0, L))$ . Then, the solution  $u$  satisfies the following estimates*

$$\|u\|_{B_T} + \|\partial_x^2 u(\cdot, 0)\|_{L^2(0, T)} \leq C(1 + \sqrt{T}) (\|y_0\|_{L^2(0, L)} + \|f\|_{L^1((0, T), L^2(0, L))}). \quad (2.30)$$

Now we are in a position to prove the global well-posedness result for the system (1.4).

**Theorem 2.11.** *Let  $u_0 \in L^2(0, L)$ . Then there exists a unique mild solution  $u \in C([0, \infty), L^2(0, L)) \cap L^2(0, \infty), H^2(0, L)$  of (1.4). Moreover, for any  $T > 0$ , there exists  $C > 0$ , such that the solution of the system (1.4) with (1.5), satisfies the following:*

$$\|u\|_{L^2(0, T; H^2(0, L))}^2 \leq C \left( \|u_0\|_{L^2(0, L)}^2 + T \|u_0\|_{L^2(0, L)}^{\frac{8+2p}{4-p}} \right), \quad (2.31)$$

$$\|\partial_x^2 u(\cdot, 0)\|_{L^2(0, L)} \leq C \|u_0\|_{L^2(0, L)}, \quad (2.32)$$

$$\begin{aligned} \|u_0\|_{L^2(0, L)}^2 &\leq \frac{1}{T} \int_0^T \int_0^L u^2(t, x) dx dt + \int_0^T |\partial_x^2 u(t, 0)|^2 dt \\ &\quad + 2 \int_0^T \int_0^L \mathbf{sat}(a(x)u(t, x))u(t, x) dx dt. \end{aligned} \quad (2.33)$$

*Proof.* Let  $u_0 \in D(\mathcal{A})$  and  $R, \theta > 0$  to be chosen later. Consider again the ball  $\mathcal{B}_{B_\theta}(0, R) := \{v \in B_\theta, \|v\|_{B_\theta} \leq R\}$ . Let  $\Psi$ , the map defined by  $\Psi : B_\theta \mapsto B_\theta$ ,  $\Psi(v) = u$ , where  $u$  solves

$$\begin{cases} \partial_t u + \partial_x u + \partial_x^3 u - \partial_x^5 u = -v^p \partial_x v - \mathbf{sat}(a(x)v(t, x)), & (t, x) \in (0, \theta) \times (0, L), \\ u(t, 0) = u(t, L) = \partial_x u(t, 0) = \partial_x u(t, L) = 0, \partial_x^2 u(t, L) = 0, & t \in (0, \theta), \\ u(0, x) = u_0(x), & x \in (0, L). \end{cases}$$

Observe that  $\Phi$  is well-defined, since by Proposition 2.6  $v^p \partial_x v \in L^1((0, \theta), L^2(0, L))$  and by [19, Proposition 3.4]

$$\|\mathbf{sat}(av)\|_{L^1((0, \theta); L^2(0, L))} \leq 3\sqrt{L}a_1\sqrt{\theta}\|v\|_{B_\theta}.$$

By Theorem 2.10, Proposition 2.6 and the above inequality we get

$$\begin{aligned} \|\Phi(v)\|_{B_\theta} &\leq C(1 + \sqrt{\theta}) (\|u_0\|_{L^2(0, L)} + \|v^p \partial_x v\|_{L^1((0, \theta), L^2(0, L))} + \|\mathbf{sat}(a(x)v(t, x))\|_{L^1((0, \theta), L^2(0, L))}) \\ &\leq C_1(1 + \sqrt{\theta})\|u_0\|_{L^2(0, L)} + C_2(\theta^{(2-p)/4} + \sqrt{\theta})\|v\|_{B_T}^{p+1} + C_2\sqrt{\theta}\|v\|_{B_T}, \end{aligned}$$

for some  $C_1, C_2 > 0$ . Then taking  $R = 3C_1\|u_0\|_{L^2(0,L)}$  and  $\theta > 0$  small enough such that

$$\begin{cases} C_2(\theta^{\frac{2-p}{4}}R^p + \sqrt{\theta}R^p + \sqrt{\theta}) < \frac{1}{3}, \\ (1 + \sqrt{\theta}) < 2, \end{cases}$$

we have that  $\|\Psi(v)\|_{B_\theta} \leq R$  and thus  $\Phi$  maps  $\mathcal{B}_{B_\theta}(0, R)$  into itself. Now take  $v_1, v_2 \in \mathcal{B}_{B_\theta}(0, R)$ , and consider  $w = u_1 - u_2 = \Psi(v_1) - \Psi(v_2)$ . We observe that  $w$  solves

$$\begin{cases} \partial_t w + \partial_x w + \partial_x^3 w - \partial_x^5 w = f + h, & (t, x) \in (0, \theta) \times (0, L), \\ w(t, 0) = w(t, L) = \partial_x w(t, 0) = \partial_x w(t, L) = 0, \partial_x^2 w(t, L) = 0, & t \in (0, \theta), \\ w(0, x) = 0, & x \in (0, L), \end{cases}$$

where  $f = -v_1^p \partial_x v_1 + v_2^p \partial_x v_2$  and  $h = -\mathbf{sat}(a(x)v_1(t, x)) + \mathbf{sat}(a(x)v_2(t, x))$ . Similar integration by parts as in [Theorem 2.10](#) allows to write

$$\|w\|_{B_\theta} \leq C \left( \|h\|_{L^1((0,\theta),L^2(0,L))} + \|f\|_{L^1((0,\theta);L^2(0,L))} \right). \quad (2.34)$$

Further we have (see [\[19, Proposition 3.4\]](#))

$$\|h\|_{L^1((0,\theta);L^2(0,L))} = \|-\mathbf{sat}(av_1) + \mathbf{sat}(av_2)\|_{L^1((0,\theta);L^2(0,L))} \leq 3\sqrt{L}a_1\sqrt{\theta}\|v_1 - v_2\|_{B_\theta}.$$

Using this estimate along with [\(2.19\)](#) in [\(2.34\)](#) we obtain

$$\begin{aligned} \|\Psi(v_1) - \Psi(v_2)\|_{B_\theta} &= \|w\|_{B_\theta} \leq C \left( \|f\|_{L^1((0,\theta),L^2(0,L))} + 3\sqrt{L}a_1\sqrt{\theta}\|v_1 - v_2\|_{B_\theta} \right) \\ &\leq C \left( (\theta^{(2-p)/4} + \sqrt{\theta}) \left( \|v_1\|_{B_\theta} \|v_2\|_{B_\theta}^{p-1} + \|v_1\|_{B_\theta}^p + \|v_2\|_{B_\theta}^p \right) + 3\sqrt{L}a_1\sqrt{\theta} \right) \|u - v\|_{B_\theta}. \end{aligned}$$

Then, taking  $\theta > 0$ , small enough such that  $C \left( (\theta^{(2-p)/4} + \sqrt{\theta})R^p + 3\sqrt{L}a_1\sqrt{\theta} \right) < 1$ , we get that  $\Psi$  is a contraction and by the Banach fixed point theorem and a density argument, we deduce the local in time well-posedness. To conclude the global well-posedness result, we need some a priori estimates. Let  $T > 0$  and  $u$  be a strong solution of [\(1.4\)](#). Multiplying the equation [\(1.4\)](#) by  $u$  and integrating on  $[0, L]$  we have after integration by parts and boundary data

$$\frac{1}{2} \frac{d}{dt} \int_0^L u^2 dx + \frac{1}{2} \partial_x^2 u(t, 0)^2 + \int_0^L \mathbf{sat}(a(x)u(t, x))u(t, x) dx = 0. \quad (2.35)$$

Integrating over  $[0, T]$  and using the fact that  $\mathbf{sat}$  is an odd function, one can prove the estimate [\(2.32\)](#). Similarly to [\(2.22\)](#), integrating [\(2.35\)](#) over  $[0, s]$  we can show that

$$\|u\|_{C([0,T],L^2(0,L))}^2 \leq \|u_0\|_{L^2(0,L)}^2. \quad (2.36)$$

Multiplying first equation of [\(1.4\)](#) by  $xu$  and taking integration on both sides over  $[0, T] \times (0, L)$ , applying successive integration by parts and using boundary conditions, we obtain

$$\begin{aligned} \frac{1}{2} \int_0^L xu^2(T, \cdot) dx + \frac{3}{2} \int_0^T \int_0^L u_x^2 dx + \frac{5}{2} \int_0^T \int_0^L u_{xx}^2 dx dt + \int_0^T \int_0^L xus\mathbf{sat}(au) dx dt &= \frac{1}{2} \int_0^L xu_0^2(\cdot) dx \\ &+ \frac{1}{2} \int_0^T \int_0^L u^2 dx dt \\ &+ \frac{1}{p+2} \int_0^T \int_0^L u^{p+2} dx dt. \end{aligned}$$

Noting the fact  $\mathbf{sat}$  is an odd function, using [\(2.36\)](#) and following the computations of [\(2.24\)](#) we finally have [\(2.31\)](#).

Next multiplying first equation of [\(1.4\)](#) by  $(T-t)u$  and taking integration over  $[0, L]$  on both sides, we write

$$\begin{aligned} \int_0^L (T-t)u\partial_t u dx + \int_0^L (T-t)u\partial_x u dx + \int_0^L (T-t)u\partial_x^3 u dx \\ - \int_0^L (T-t)u\partial_x^5 u dx + \int_0^L (T-t)u\partial_x u dx + \int_0^L (T-t)\mathbf{sat}(a(x)u(t, x))u(t, x) dx = 0. \end{aligned} \quad (2.37)$$

Integrating over  $[0, T]$  we have

$$\frac{1}{2} \int_0^T \int_0^L u^2 dx dt - \frac{T}{2} \int_0^L u_0^2 dx + \frac{1}{2} \int_0^T (T-t)\partial_x^2 u(t, 0)^2 dt + \int_0^T \int_0^L (T-t)\mathbf{sat}(a(x)u(t, x))u(t, x) dx dt = 0.$$

Thus we get the estimate (2.33).  $\square$

### 3. EXPONENTIAL STABILIZATION

In this part we prove our main results with respect to the exponential stability of systems (1.4) and (1.6). Both proofs are based on showing an appropriated observability inequality, using a contradiction argument and compactness ideas. The main difference in the proof of internal and boundary saturation is that for the boundary case we first prove such a observability inequality by analyzing system (2.1) and then we pass to the fully nonlinear system (1.6) by a perturbation approach, due to this argument our exponential stability result for (1.6) it is of local nature. From the other side, in the proof of the observability inequality for (1.4) we work directly with the nonlinear system which allow us to obtain a semiglobal exponential stability result. In what follows, we consider here the classical  $L^2$ -energy,

$$E(t) = \frac{1}{2} \int_0^L u(t, x)^2 dx. \quad (3.1)$$

**3.1. Boundary saturation.** In this part we are going to prove our main result in the case of boundary saturation related to the exponential stability of system (1.6). In the case of boundary saturation, the following unique continuation property is crucial in the proof

**Lemma 3.1.** ([3, Lemma 4.3]) *Let  $L > 0$ , if  $\lambda, \psi \in \mathbb{C} \times H^5(0, L)$  satisfies*

$$\begin{cases} \lambda\psi(x) + \psi'(x) + \psi'''(x) - \psi''''(x) = 0, & x \in (0, L) \\ \psi(0) = \psi(L) = \psi'(0) = \psi'(L) = 0, \\ \psi''(0) = \psi''(L) = 0, \end{cases} \quad (3.2)$$

then  $\psi \equiv 0$ .

For a strong solution of (1.6) (or (2.1)) performing integration by parts, we get using  $|\alpha| < 1$

$$\dot{E}(t) = \frac{1}{2} (\text{sat}(\alpha \partial_x^2 u(t, 0))^2 - \partial_x^2 u(t, 0)^2) \leq -C_\alpha \partial_x^2 u(t, 0)^2 \leq 0. \quad (3.3)$$

Thus, the energy is a non-increasing function of time. In fact, we will prove that the energy decays exponentially to 0. Our approach will be based on an observability inequality for the system (2.1).

**Proposition 3.2.** *Let  $L > 0$ ,  $|\alpha| < 1$  and  $u_0 \in L^2(0, L)$ . For any solution  $u$  of (2.1), the following inequality holds:*

$$\int_0^L u_0^2 dx \leq C_{obs} \int_0^T \partial_x^2 u(t, 0)^2 dt, \quad (3.4)$$

for some  $C_{obs} > 0$ , that does not depend on  $u_0$ .

*Proof.* In order to prove the observability inequality, we follow [30] and use a contradiction argument. Let us suppose that (3.4) is false, then there exists  $(u_0^n)_{n \in \mathbb{N}} \subset L^2(0, L)$  such that

$$\int_0^L (u_0^n)^2 dx = 1, \text{ and } \int_0^T \partial_x^{2,n} u(t, 0)^2 dt \longrightarrow 0$$

when  $n \rightarrow \infty$  and where  $u^n$  is the associate solution of (2.1) with initial data  $u_0^n$ . Now, using (2.10) with  $f = 0$  we get that  $(u^n)_{n \in \mathbb{N}}$  is bounded in  $L^2((0, T), H^2(0, L))$ . Then as  $\partial_t u^n = -\partial_x u^n - \partial_x^3 u^n + \partial_x^5 u^n$ , we can see that  $(\partial_t u^n)_{n \in \mathbb{N}}$  is bounded in  $L^2((0, T), H^{-3}(0, L))$ , thus by Aubin-Lions lemma  $(u^n)_{n \in \mathbb{N}}$  is relatively compact in  $L^2((0, T), L^2(0, L))$ , therefore we can assume that  $u^n \rightarrow u$  in  $L^2((0, T), L^2(0, L))$ . Now, using (2.11) we derive that  $(u_0^n)_{n \in \mathbb{N}}$  is a Cauchy sequence in  $L^2(0, L)$ . Let  $u_0 = \lim_{n \rightarrow \infty} u_0^n$ , then

$$\int_0^L u_0^2 dx = 1, \text{ and } \int_0^T \partial_x^2 u(t, 0)^2 dt = 0$$

Thus, as  $\text{sat}_{loc}(0) = 0$ , we conclude that  $y$  solves the following linear equation:

$$\begin{cases} \partial_t u + \partial_x u + \partial_x^3 u - \partial_x^5 u = 0, & (t, x) \in (0, \infty) \times (0, L), \\ u(t, 0) = u(t, L) = 0, & t \in (0, \infty), \\ \partial_x u(t, 0) = \partial_x u(t, L) = 0, & t \in (0, \infty), \\ \partial_x u(t, 0) = \partial_x^2 u(t, 0) = \partial_x^2 u(t, L) = 0, & t \in (0, \infty). \end{cases} \quad (3.5)$$

But by Lemma 3.1 we can show that there is no function satisfying this system and  $\|u_0\|_{L^2(0, L)} = 1$ , which gives us the contradiction and finishes the proof of Proposition 3.2.  $\square$

Now, we prove our main result related with the exponential stability in the boundary saturated case.

**Theorem 3.3.** *Let  $L > 0$ ,  $|\alpha| < 1$  and  $u_0 \in L^2(0, L)$ . For any solution  $u$  of (2.1), the energy of the system decays exponentially to zero i.e, there exists  $C, \gamma > 0$  such that  $E(t) \leq CE(0)e^{-\gamma t}$ , for all  $t \geq 0$ . Moreover, if  $\|u_0\|_{L^2(0, L)} \leq \varepsilon$  for  $\varepsilon > 0$  small enough, the same property holds for (1.6).*

*Proof.* Let  $u_0 \in D(\mathcal{A}_b)$ , using (3.3) and (2.3) we get for a strong solution of (2.1)

$$\dot{E}(t) \leq -C_\alpha \partial_x^2 u(t, 0)^2,$$

thus we can see

$$E(T) - E(0) \leq -C_\alpha \int_0^T \partial_x^2 u(t, 0)^2 dt.$$

Using the last expression, (3.3) and (3.4) we get

$$E(T) \leq E(0) \leq C_{obs} \int_0^T \partial_x^2 u(t, 0)^2 dt \leq \frac{C_{obs}}{C_\alpha} (E(0) - E(T)),$$

which implies that

$$E(T) \leq \gamma E(0), \quad \text{with } \gamma = \frac{C_{obs}}{C_\alpha + C_{obs}} < 1. \quad (3.6)$$

As (2.1) is invariant by translation in time and repeating this argument on  $[(m-1)T, mT]$  for  $m = 1, 2, \dots$  to obtain,

$$E(mT) \leq \gamma E((m-1)T) \leq \dots \leq \gamma^m E(0).$$

Hence, we have  $E(mT) \leq e^{-\mu mT} E(0)$  where  $\mu = \frac{1}{T} \ln\left(\frac{1}{\gamma}\right) > 0$ . Let  $t > 0$  then there exists  $m \in \mathbb{N}^*$  such that  $(m-1)T < t \leq mT$ , and then using again the non-increasing property of the energy we get

$$E(t) \leq E((m-1)T) \leq e^{-\mu(m-1)T} E(0) \leq \frac{1}{\gamma} e^{-\mu t} E(0).$$

Finally, we conclude the exponential stability of (2.1) by a density argument.

We study now the nonlinear case, note that as (1.6) is also invariant by translation in time, thus to prove the stability, the objective is to show that (3.6) holds for  $u$  solution of (1.6) if the initial data it is small enough. This idea was used in several works, for instance [22, 24]. Let  $\|u_0\|_{L^2(0, L)} \leq \varepsilon$  for  $\varepsilon > 0$  to fix later. Take  $u$  the solution of (1.6), then we can split  $u$  as  $u = \bar{u} + \tilde{u}$ , where

$$\begin{cases} \partial_t \bar{u} + \partial_x \bar{u} + \partial_x^3 \bar{u} - \partial_x^5 \bar{u} = 0, & (t, x) \in (0, \infty) \times (0, L), \\ \bar{u}(t, 0) = \bar{u}(t, L) = \partial_x \bar{u}(t, 0) = \partial_x \bar{u}(t, L) = 0, & t \in (0, \infty), \\ \partial_x^2 \bar{u}(t, L) = \mathbf{sat}_{loc}(\alpha \partial_x^2 \bar{u}(t, 0)), & t \in (0, \infty), \\ \bar{u}(0, x) = u_0(x), & x \in (0, L), \end{cases}$$

and

$$\begin{cases} \partial_t \tilde{u} + \partial_x \tilde{u} + \partial_x^3 \tilde{u} - \partial_x^5 \tilde{u} = u^p \partial_x u, & (t, x) \in (0, \infty) \times (0, L), \\ \tilde{u}(t, 0) = \tilde{u}(t, L) = \partial_x \tilde{u}(t, 0) = \partial_x \tilde{u}(t, L) = 0, & t \in (0, \infty), \\ \partial_x^2 \tilde{u}(t, L) = h(t), & t \in (0, \infty), \\ \tilde{u}(0, x) = 0, & x \in (0, L). \end{cases}$$

where  $h(t) = \mathbf{sat}(\alpha \partial_x^2 u(t, 0)) - \mathbf{sat}(\alpha \partial_x^2 \bar{u}(t, 0))$ . Now, we observe that

$$\|u(T, \cdot)\|_{L^2(0, L)} \leq \|\bar{u}(T, \cdot)\|_{L^2(0, L)} + \|\tilde{u}(T, \cdot)\|_{L^2(0, L)}. \quad (3.7)$$

From the other side as  $\bar{u}$  satisfies (2.1), we get using (3.6) and  $\bar{u}(0, x) = u_0(x)$  that there exists  $\gamma < 1$  such that

$$\|\bar{u}(T)\|_{L^2(0, L)} \leq \gamma \|u_0\|_{L^2(0, L)}. \quad (3.8)$$

For  $\tilde{u}$ , by (2.10) and Proposition 2.6,

$$\|\tilde{u}(T, \cdot)\|_{L^2(0, L)} \leq \|\tilde{u}\|_{C([0, T], L^2(0, L))} \leq C \|u^p \partial_x u\|_{L^1((0, T), L^2(0, L))} \leq C \|u\|_{B_T}^{p+1},$$

using (2.22)-(2.25), we derive

$$\|u\|_{B_T}^{p+1} \leq C \left( \|u_0\|_{L^2(0, L)}^{p+1} + \|u_0\|_{L^2(0, L)}^{\frac{(4+p)(p+1)}{4-p}} \right).$$

Thus, we get for  $\tilde{u}$

$$\|\tilde{u}(T, \cdot)\|_{L^2(0,L)} \leq \left( C\|u_0\|_{L^2(0,L)}^p + C\|u_0\|_{L^2(0,L)}^{\frac{p(6+p)}{4-p}} \right) \|u_0\|_{L^2(0,L)}.$$

Finally, as  $\|u_0\|_{L^2(0,L)} \leq \epsilon$  we deduce from (3.7)

$$\begin{aligned} \|u(T, \cdot)\|_{L^2(0,L)} &\leq \|\bar{u}(T, \cdot)\|_{L^2(0,L)} + \|\tilde{u}(T, \cdot)\|_{L^2(0,L)} \\ &\leq \gamma\|u_0\|_{L^2(0,L)} + (C\epsilon^p + C\epsilon^{\frac{p(6+p)}{4-p}})\|u_0\|_{L^2(0,L)} \\ &= (\gamma + C\epsilon^p + C\epsilon^{\frac{p(6+p)}{4-p}})\|u_0\|_{L^2(0,L)}. \end{aligned}$$

We conclude by choosing  $\epsilon > 0$  small enough such that  $\tilde{\gamma} = \gamma + C\epsilon^p + C\epsilon^{\frac{p(6+p)}{4-p}} < 1$ .  $\square$

**3.2. Internal saturation.** We pass now to study the case of internal saturation. We are interested to prove the following exponential stabilization result:

**Theorem 3.4** (Semi global stabilization). *Let us consider the system (1.4). Then for any  $r > 0$  there exists a constant  $\mu(r) > 0$  such that for any  $u_0 \in L^2(0, L)$  satisfying  $\|u_0\|_{L^2(0,L)} \leq r$ , the solution of (1.4) satisfy the following:*

$$E(t) \leq e^{-\mu t} E(0), \forall t \geq 0,$$

where  $E(\cdot)$  is defined in (3.1).

Before going to prove **Theorem 3.4**, first we state some useful results borrowed from the article [20].

**Lemma 3.5** ([20]). *Let  $r > 0$  and  $a : [0, L] \mapsto \mathbb{R}$  be a function which satisfies (1.5). Let us define*

$$k(r) = \min \left\{ \frac{M}{a_1 r}, 1 \right\},$$

where  $M$  and  $a_1$  are defined in (1.2) and (1.5) respectively. The following two hold:

- For  $s \in L^2(0, L)$  with  $\|s\|_{L^2(0,L)} \leq r$ , we have

$$(\mathbf{sat}_2(a(x)s(x)) - k(r)a(x)s(x))s(x) \geq 0, \forall x \in [0, L]. \quad (3.9)$$

- For  $s \in L^\infty(0, L)$  with  $|s(x)| \leq r$ , we have

$$(\mathbf{sat}_{loc}(a(x)s(x)) - k(r)a(x)s(x))s(x) \geq 0, \forall x \in [0, L]. \quad (3.10)$$

**3.2.1. Proof of Theorem 3.4 with  $\omega = [0, L]$  and  $\mathbf{sat} = \mathbf{sat}_2$ .** In this case, the proof is direct. Indeed, thanks to the estimate (2.35), we have

$$\frac{1}{2} \frac{d}{dt} \int_0^L u^2 dx \leq - \int_0^L \mathbf{sat}(a(x)u(t, x))u(t, x) dx. \quad (3.11)$$

Using (2.36), we have  $\|u(t, \cdot)\|_{L^2(0,L)} \leq r$ , where  $r > 0$  be any number such that  $\|u_0\|_{L^2} \leq r$ . Applying first estimate (3.9) of **Lemma 3.5** we deduce

$$\frac{1}{2} \frac{d}{dt} \int_0^L u^2 dx \leq - \int_0^L k(r)a_0 u^2(t, x) dx. \quad (3.12)$$

Thus the proof is complete with  $\mu = \min\{a_0, k(r)\}$ .

**3.2.2. Proof of Theorem 3.4 with  $\omega \subset [0, L]$  and  $\mathbf{sat} = \mathbf{sat}_2$ .** In this case, we will use a contradiction argument as [31]. First, let us state a unique continuation lemma for the Kawahara equation, which plays an integral part in the proof of stabilization result:

**Lemma 3.6** (Unique continuation principle, [13]). *Let  $u_0 \in L^2(0, L)$  and  $u \in B_T$  be the solution of the equation*

$$\begin{cases} \partial_t u + \partial_x u + \partial_x^3 u - \partial_x^5 u + u^p \partial_x u = 0, & (t, x) \in (0, \infty) \times (0, L), \\ u(t, 0) = u(t, L) = \partial_x u(t, 0) = \partial_x u(t, L) = \partial_x^2 u(t, L) = 0, & t \in (0, \infty), \\ u(0, x) = u_0(x), & x \in (0, L), \end{cases} \quad (3.13)$$

such that  $u(t, x) = 0, \forall x \in \omega, \forall t \in (0, T)$ , with  $\omega \subset (0, L)$ . Then  $u(t, x) = 0, \forall x \in (0, L), \forall t \in (0, T)$ .

For the sake of completeness, we give a sketch of the proof of [Lemma 3.6](#) in the [Appendix B](#).

Let us consider  $\|u_0\|_{L^2(0,L)} \leq r$ . Using [\(2.36\)](#), we deduce

$$\|u(t, \cdot)\|_{L^2(0,L)} \leq r, \forall t \geq 0. \quad (3.14)$$

Integrating [\(2.35\)](#) over  $[0, T]$  we have

$$\int_0^L u^2(T, \cdot) dx = \int_0^L u_0^2 dx - \int_0^T \partial_x^2 u(t, 0)^2 dt - 2 \int_0^T \int_0^L \mathbf{sat}(a(x)u(t, x))u(t, x) dx dt \quad (3.15)$$

As [\(3.14\)](#) holds, we apply the first part of [Lemma 3.5](#), and deduce

$$\int_0^L u^2(T, \cdot) dx \leq \int_0^L u_0^2 dx - \int_0^T \partial_x^2 u(t, 0)^2 dt - 2 \int_0^T \int_0^L k(r)a(x)|u(t, x)|^2 dx dt. \quad (3.16)$$

Next, for the time being we assume the following result:

**Lemma 3.7.** *Let us take  $T > 0$  and let  $r$  be any positive number. Then for any  $u_0 \in L^2(0, L)$ , with  $\|u_0\|_{L^2(0,L)} \leq r$ , the solution of the equation [\(1.4\)](#) satisfies the following*

$$\|u_0\|_{L^2(0,L)}^2 \leq C_2 \left( \int_0^T |\partial_x^2 u(t, 0)|^2 dt + 2 \int_0^T \int_0^L k(r)a(x)|u(t, x)|^2 dx dt \right). \quad (3.17)$$

Therefore combining [\(3.16\)](#) and [\(3.17\)](#), we finally have the existence of  $\gamma \in (0, 1)$  such that

$$\|u(kT, \cdot)\|_{L^2(0,L)}^2 \leq \gamma^k \|u_0\|_{L^2(0,L)}^2, \forall k \geq 0. \quad (3.18)$$

Using [\(2.36\)](#), we have  $\|u(t, \cdot)\|_{L^2(0,L)}^2 \leq \|u(kT, \cdot)\|_{L^2(0,L)}^2$ , where  $kT \leq t \leq (k+1)T$ . Finally, we obtain that

$$\|u(t, \cdot)\|_{L^2(0,L)} \leq \frac{1}{\gamma} e^{-\mu t} \|u_0\|_{L^2(0,L)}, \forall t \geq 0, \quad (3.19)$$

where  $\mu = -\frac{\log \gamma}{T}$ .

**3.2.3. Proof of Lemma 3.7.** Thanks to [\(2.33\)](#), we can say that to prove [Lemma 3.7](#), it is enough to show that the following

$$\|u\|_{L^2(0,T;L^2(0,L))}^2 \leq C_2 \left( \int_0^T |\partial_x^2 u(t, 0)|^2 dt + 2 \int_0^T \int_0^L k(r)a(x)|u(t, x)|^2 dx dt \right), \quad (3.20)$$

for some  $C > 0$ , provided  $\|u_0\|_{L^2(0,L)} \leq r$ . We prove this inequality by contradiction approach. Suppose that [\(3.20\)](#) is not true. Hence there exists a sequence of functions  $u_0^n \in L^2(0, L)$  with  $\|u_0^n\|_{L^2(0,L)} \leq r$  and  $\{u^n\}_{n \in \mathbb{N}} \in \mathcal{B}_T$  be the corresponding solution of the equation

$$\begin{cases} \partial_t u^n + \partial_x u^n + \partial_x^3 u^n - \partial_x^5 u^n + u^{np} \partial_x u^n + \mathbf{sat}(a(x)u^n(t, x)) = 0, & (t, x) \in (0, \infty) \times (0, L), \\ u^n(t, 0) = u^n(t, L) = \partial_x u^n(t, 0) = \partial_x u^n(t, L) = \partial_x^2 u^n(t, L) = 0, & t \in (0, \infty), \\ u^n(0, x) = u_0^n(x), & x \in (0, L), \end{cases} \quad (3.21)$$

where  $p \in [1, 2)$ , and

$$\lim_{n \rightarrow \infty} \frac{\|u^n\|_{L^2(0,T;L^2(0,L))}^2}{\left( \int_0^T |\partial_x^2 u^n(t, 0)|^2 dt + 2 \int_0^T \int_0^L k(r)a|u^n(t, x)|^2 dx dt \right)} = +\infty. \quad (3.22)$$

Also [\(2.36\)](#) gives that

$$\|u^n(t, \cdot)\|_{L^2(0,L)} \leq r, \forall t \geq 0. \quad (3.23)$$

Let us denote  $\lambda_n = \|u^n\|_{L^2(0,T;L^2(0,L))}$  and  $v_n(t, x) = \frac{u^n(t, x)}{\lambda_n}$ .  $\{\lambda_n\}_{n \in \mathbb{N}}$  is a bounded sequence by [\(3.23\)](#).

Therefore it has a subsequence, denoted by  $\{\lambda_n\}_{n \in \mathbb{N}}$  such that there exists  $\lambda \geq 0$  with

$$\lambda_n \rightarrow \lambda \text{ as } n \rightarrow \infty. \quad (3.24)$$

Note that  $v_n$  satisfies the following equation:

$$\begin{cases} \partial_t v^n + \partial_x v^n + \partial_x^3 v^n - \partial_x^5 v^n + \lambda_n v^{np} \partial_x v^n + \frac{\mathbf{sat}(a(x)\lambda_n v^n(t, x))}{\lambda_n} = 0, & (t, x) \in (0, \infty) \times (0, L), \\ v^n(t, 0) = v^n(t, L) = \partial_x v^n(t, 0) = \partial_x v^n(t, L) = \partial_x^2 v^n(t, L) = 0, & t \in (0, \infty), \\ \|v_n\|_{L^2(0,T;L^2(0,L))} = 1. \end{cases} \quad (3.25)$$



and also by (3.22) we further have

$$\left( \int_0^T |\partial_x^2 v^n(t, 0)|^2 dt + 2 \int_0^T \int_0^L k(r) a |v^n(t, x)|^2 dx dt \right) \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (3.26)$$

In addition, using (2.33), we have that  $\{\|v_n(0, \cdot)\|_{L^2(0, L)}\}_{n \in \mathbb{N}}$  is a bounded sequence and due to the following

$$\begin{cases} \|v^n\|_{L^2(0, T; H^2(0, L))}^2 \leq C \left( \|v^n(0, \cdot)\|_{L^2(0, L)}^2 + T \|v^n(0, \cdot)\|_{L^2(0, L)}^{\frac{8+2p}{4-p}} \right) \\ \|v^n(t, \cdot)\|_{L^2(0, L)} \leq \|v^n(0, \cdot)\|_{L^2(0, L)}, \end{cases} \quad (3.27)$$

the sequence  $\{v_n\}_{n \in \mathbb{N}}$  is bounded in the space  $L^2(0, T; H^2(0, L))$  and  $L^\infty([0, T]; L^2(0, L))$ . Therefore there exists a subsequence still denoted by  $\{v_n\}_{n \in \mathbb{N}}$  and a function  $v \in L^\infty(0, T; L^2(0, L)) \cap L^2(0, T; H^2(0, L))$  such that

$$\begin{cases} v_n \xrightarrow{*} v & \text{in } L^\infty(0, T; L^2(0, L)) \\ v_n \rightharpoonup v & \text{in } L^2(0, T; H^2(0, L)) \end{cases} \quad (3.28)$$

To derive the equation satisfied by  $v$ , we need to pass the limit in (3.25) and thus, in the following steps, we need to take care of the convergence of the terms related to the nonlinearity.

First of all, we have

$$\|v^{np} \partial_x v^n\|_{L^1((0, T), L^2(0, L))} \leq C(T^{(2-p)/4} + \sqrt{T}) \|v^n\|_{B_T}^p \|v^n\|_{B_T} \leq C. \quad (3.29)$$

In the next step, we prove that  $\{(v^n)^{p+1}\}_{n \in \mathbb{N}}$  is bounded in  $L^2(0, T; L^2(0, L))$ .

$$\int_0^T \int_0^L (v^n)^{2p+2} dx dt \leq \|v^n\|_{L^2(0, T; H^1(0, L))}^2 \|v^n\|_{L^\infty(0, T; L^2(0, L))}^{3+\epsilon} \leq C, \quad (3.30)$$

where  $p = 1 + \epsilon$ ,  $0 < \epsilon < 1$ . This shows that  $\{v^{np} \partial_x v^n\}_{n \in \mathbb{N}}$  is bounded in the space  $L^2(0, T; H^{-1}(0, L)) \subset L^2(0, T; H^{-3}(0, L))$ . Also, we know that  $\{\partial_x^5 v^n\}_{n \in \mathbb{N}}$  is bounded in  $L^2(0, T; H^{-3}(0, L))$ . Furthermore, it is easy to check the following estimate (see [20, Lemma 3.2] for more details)

$$\left\| \frac{\text{sat}(a(x) \lambda_n v^n(t, x))}{\lambda_n} \right\|_{L^2(0, T; L^2(0, L))} \leq C \|v^n\|_{L^2(0, T; H^1(0, L))}. \quad (3.31)$$

Thus combining we have that  $\partial_t v^n = - \left( \partial_x v^n + \partial_x^3 v^n - \partial_x^5 v^n + \lambda_n v^{np} \partial_x v^n + \frac{\text{sat}(a(x) \lambda_n v^n(t, x))}{\lambda_n} \right)$  is bounded in  $L^2(0, T; H^{-3}(0, L))$ . Thanks to the Lions-Aubin lemma with the embedding  $H^2(0, L) \subset L^2(0, L) \subset H^{-3}(0, L)$ ,  $\{v_n\}_{n \in \mathbb{N}}$  is relatively compact in  $L^2(0, T; L^2(0, L))$ . It follows that there exists a subsequence, again denoted by  $\{v_n\}_{n \in \mathbb{N}}$  such that  $v_n \rightarrow v$  in  $L^2(0, T; L^2(0, L))$ . Further, it implies  $(v^n)^{p+1} \rightarrow v^{p+1}$  a.e.  $(t, x) \in (0, T) \times (0, L)$ . Since  $\{(v^n)^{p+1}\}_{n \in \mathbb{N}}$  is bounded in  $L^2(0, T; L^2(0, L))$ , therefore there exists some  $h \in L^2(0, T; L^2(0, L))$  such that  $(v^n)^{p+1} \rightharpoonup h$  in  $L^2(0, T; L^2(0, L))$ . Uniqueness of limit gives  $h = v^{p+1}$  a.e. and we have  $(v^n)^{p+1} \rightarrow v^{p+1}$  in  $\mathcal{D}'((0, T) \times (0, L))$ . Taking the derivative with respect to space, we further have  $\lambda_n v^{np} \partial_x v^n \rightarrow \lambda v^p \partial_x v$  in  $\mathcal{D}'((0, T) \times (0, L))$ . As  $\{v^n\}_{n \in \mathbb{N}}$  and  $\{\partial_t v^n\}_{n \in \mathbb{N}}$  are bounded in the spaces  $L^\infty(0, T; L^2(0, L))$  and  $L^2(0, T; H^{-3}(0, L))$  respectively. Using the embedding  $L^2(0, L) \subset H^{-1}(0, L) \subset H^{-3}(0, L)$  (see [32, Corollary 4]), we get by for a subsequence denoted by  $\{v^n\}_{n \in \mathbb{N}}$  that  $v^n \rightarrow v$  in  $C([0, T]; H^{-1}(0, L))$ .

Thus in a nutshell, interpolation theory gives us the following

$$\begin{cases} v_n \xrightarrow{*} v & \text{in } L^\infty(0, T; L^2(0, L)) \\ v_n \rightharpoonup v & \text{in } L^2(0, T; H^2(0, L)) \\ v_n \rightarrow v & \text{in } L^2(0, T; L^2(0, L)) \\ v_n \rightarrow v & \text{in } C([0, T]; H^{-1}(0, L)) \\ \lambda_n v^{np} \partial_x v^n \rightarrow \lambda v^p \partial_x v & \text{in } \mathcal{D}'((0, T) \times (0, L)). \end{cases} \quad (3.32)$$

Moreover, from (3.26) we deduce:

$$\begin{aligned} & \left( \int_0^T |\partial_x^2 v(t, 0)|^2 dt + 2 \int_0^T \int_0^L k(r) a(x) |v(t, x)|^2 dx dt \right) \\ & \leq \liminf_{n \rightarrow \infty} \left( \int_0^T |\partial_x^2 v^n(t, 0)|^2 dt + 2 \int_0^T \int_0^L k(r) a(x) |v^n(t, x)|^2 dx dt \right) = 0. \end{aligned}$$

Therefore  $a(x)k(x)v = 0, x \in (0, L)$  and  $\partial_x^2 v(\cdot, 0) = 0, t \in (0, T)$ . This further implies  $v = 0, x \in \omega, t \in (0, T)$ . We also obtain the limit  $v$  solves the following equation in the sense of distribution

$$\begin{cases} \partial_t v + \partial_x v + \partial_x^3 v - \partial_x^5 v + \lambda v^p \partial_x v = 0, & (t, x) \in (0, \infty) \times (0, L), \\ v(t, 0) = v(t, L) = \partial_x v(t, 0) = \partial_x v(t, L) = \partial_x^2 v(t, L) = 0, & t \in (0, \infty), \\ \|v\|_{L^2(0, T; L^2(0, L))} = 1. \end{cases} \quad (3.33)$$

by unique continuation principle [Lemma 3.6](#), we have  $v = 0, x \in (0, L), t \in (0, T)$ . This contradicts the fact  $\|v\|_{L^2(0, T; L^2(0, L))} = 1$ . Therefore [Lemma 3.7](#) holds.

**3.2.4. Proof of [Theorem 3.4](#) with  $\omega \subset [0, L]$  and  $\mathbf{sat} = \mathbf{sat}_{loc}$ .** We will prove the exponential stabilization result [Theorem 3.4](#) with  $\mathbf{sat} = \mathbf{sat}_{loc}$  as the same strategy of above. It is clear that the stabilizability follows if we can prove the following lemma:

**Lemma 3.8.** *Let us take  $T > 0$  and let  $r$  be any positive number. Then for any  $u_0 \in L^2(0, L)$ , with  $\|u_0\|_{L^2(0, L)} \leq r$ , the solution of the equation [\(1.4\)](#) satisfies the following*

$$\|u_0\|_{L^2(0, L)}^2 \leq C_2 \left( \int_0^T |\partial_x^2 u(t, 0)|^2 dt + 2 \int_0^T \int_0^L \mathbf{sat}_{loc}(au(t, x))u(t, x) dx dt \right). \quad (3.34)$$

*Proof.* Using [\(2.33\)](#), we can say that to prove [Lemma 3.7](#), it is enough to show that the following

$$\|u\|_{L^2(0, T; L^2(0, L))}^2 \leq C_2 \left( \int_0^T |\partial_x^2 u(t, 0)|^2 dt + 2 \int_0^T \int_0^L \mathbf{sat}_{loc}(au(t, x))u(t, x) dx dt \right), \quad (3.35)$$

for some  $C > 0$ , provided  $\|u_0\|_{L^2(0, L)} \leq r$ . Using contradiction we will prove the estimate [\(3.35\)](#). If possible, let us assume that there exists a sequence of functions  $u_0^n \in L^2(0, L)$  with  $\|u_0^n\|_{L^2(0, L)} \leq r$  and  $\{u^n\}_{n \in \mathbb{N}} \in \mathcal{B}_T$  be the corresponding solution of the equation

$$\begin{cases} \partial_t u^n + \partial_x u^n + \partial_x^3 u^n - \partial_x^5 u^n + u^{n,p} \partial_x u^n + \mathbf{sat}_{loc}(a(x)u^n(t, x)) = 0, & (t, x) \in (0, \infty) \times (0, L), \\ u^n(t, 0) = u^n(t, L) = \partial_x u^n(t, 0) = \partial_x u^n(t, L) = \partial_x^2 u^n(t, L) = 0, & t \in (0, \infty), \\ u^n(0, x) = u_0^n(x), & x \in (0, L), \end{cases} \quad (3.36)$$

where  $p \in [1, 2)$ , and

$$\lim_{n \rightarrow \infty} \frac{\|u^n\|_{L^2(0, T; L^2(0, L))}^2}{\left( \int_0^T |\partial_x^2 u^n(t, 0)|^2 dt + 2 \int_0^T \int_0^L \mathbf{sat}_{loc}(au(t, x))u(t, x) dx dt \right)} = +\infty. \quad (3.37)$$

Thanks to [\(2.36\)](#), we obtain

$$\|u^n(t, \cdot)\|_{L^2(0, L)} \leq r, \forall t \geq 0. \quad (3.38)$$

Also, using [\(2.33\)](#), we have that  $\{v_n(0, \cdot)\}_{n \in \mathbb{N}}$  is a bounded sequence in  $L^2(0, L)$  and we further deduce

$$\|v^n\|_{L^2(0, T; H^2(0, L))}^2 \leq C \left( \|v^n(0, \cdot)\|_{L^2(0, L)}^2 + T \|v^n(0, \cdot)\|_{L^2(0, L)}^{\frac{8+2p}{4-p}} \right) \leq Cr^2 + CTr^{\frac{8+2p}{4-p}}. \quad (3.39)$$

Using Poincare inequality, we have

$$\|u^n(t, \cdot)\|_{L^\infty(0, L)} \leq C \|u^n\|_{H^2(0, L)} \forall t \in [0, T].$$

This further gives

$$\|u^n(\cdot, x)\|_{L^2(0, T)}^2 \leq C_1, \quad (3.40)$$

where  $C_1 = C^2(Cr^2 + CTr^{\frac{8+2p}{4-p}})$ . Let us consider the subsets  $\Omega_i \subset [0, T]$  defined as

$$\Omega_i = \{t \in [0, T] : \|u^n(t, \cdot)\|_{L^\infty(0, L)} > i\}. \quad (3.41)$$

Its complement is then given by

$$\Omega_i^c = \{t \in [0, T] : \|u^n(t, \cdot)\|_{L^\infty(0, L)} \leq i\}. \quad (3.42)$$

Therefore we have

$$\|u^n\|_{L^2(0, T; L^\infty(0, L))}^2 \geq \|u^n\|_{L^2(\Omega_i; L^\infty(0, L))}^2 \geq i^2 \mu(\Omega_i), \quad (3.43)$$

where  $\mu(\Omega_i)$  is the Lebesgue measure of the set  $\Omega_i$ . Next, using (3.40) we have  $\mu(\Omega_i) \leq \frac{C_1}{i^2}$ . This implies that

$$\max \left\{ T - \frac{C_1}{i^2}, 0 \right\} \leq \mu(\Omega_i^c) \leq T. \quad (3.44)$$

Thanks to the estimate (3.10) of Lemma 3.5, we have

$$\begin{aligned} \int_0^T \int_0^L \mathbf{sat}_{loc}(au^n)u^n dt dx &= \int_{\Omega_i} \int_0^L \mathbf{sat}_{loc}(au^n)u^n dt dx + \int_{\Omega_i^c} \int_0^L \mathbf{sat}_{loc}(au^n)u^n dt dx \\ &\geq \int_{\Omega_i^c} \int_0^L \mathbf{sat}_{loc}(au^n)u^n dt dx \geq \int_{\Omega_i^c} \int_0^L ak(i)(u^n)^2 dt dx. \end{aligned} \quad (3.45)$$

Let us denote  $\lambda_n = \|u^n\|_{L^2(0,T;L^2(0,L))}$  and  $v_n(t, x) = \frac{u^n(t, x)}{\lambda_n}$ .  $\{\lambda_n\}_{n \in \mathbb{N}}$  is a bounded sequence by (3.23). Therefore it has a subsequence, denoted by  $\{\lambda_n\}_{n \in \mathbb{N}}$  such that there exists  $\lambda \geq 0$  with

$$\lambda_n \rightarrow \lambda \text{ as } n \rightarrow \infty. \quad (3.46)$$

Note that  $v_n$  satisfies the following equation:

$$\begin{cases} \partial_t v^n + \partial_x v^n + \partial_x^3 v^n - \partial_x^5 v^n + \lambda_n v^{np} \partial_x v^n + \frac{\mathbf{sat}_{loc}(a(x)\lambda_n v^n(t, x))}{\lambda_n} = 0, & (t, x) \in (0, \infty) \times (0, L), \\ v^n(t, 0) = v^n(t, L) = \partial_x v^n(t, 0) = \partial_x v^n(t, L) = \partial_x^2 v^n(t, L) = 0, & t \in (0, \infty), \\ \|v_n\|_{L^2(0,T;L^2(0,L))} = 1, \end{cases} \quad (3.47)$$

and also by (3.37) and (3.45) and we further have

$$\left( \int_0^T |\partial_x^2 v^n(t, 0)|^2 dt + 2 \int_0^T \int_0^L k(i)a(x)|v^n(t, x)|^2 dx dt \right) \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (3.48)$$

Moreover, it is easy to check the following estimate (see [20, Lemma 3.2])

$$\left\| \frac{\mathbf{sat}_{loc}(a(x)\lambda_n v^n(t, x))}{\lambda_n} \right\|_{L^2(0,T;L^2(0,L))} \leq C \|v^n\|_{L^2(0,T;H^1(0,L))}. \quad (3.49)$$

Like previous section, we have that  $\partial_t v^n = -\left(\partial_x v^n + \partial_x^3 v^n - \partial_x^5 v^n + \lambda_n v^{np} \partial_x v^n + \frac{\mathbf{sat}(a(x)\lambda_n v^n(t, x))}{\lambda_n}\right)$  is bounded in  $L^2(0, T; H^{-3}(0, L))$ . Thanks to the Lions-Aubin lemma with the embedding  $H^2(0, L) \subset L^2(0, L) \subset H^{-3}(0, L)$ ,  $\{v_n\}_{n \in \mathbb{N}}$  is relatively compact in  $L^2(0, T; L^2(0, L))$ . It follows that there exists a subsequence, again denoted by  $\{v_n\}_{n \in \mathbb{N}}$  such that  $v_n \rightarrow v$  in  $L^2(0, T; L^2(0, L))$ . Moreover with (3.48) we have  $ak(i)v(t, x) = 0, \forall x \in (0, L), \forall t \in \Omega_i^c$  and  $\partial_x^2 v(t, 0) = 0, \forall t \in (0, T)$ . As  $k(i)$  is strictly positive, we further have  $v(t, x) = 0, \forall x \in \omega, \forall t \in \cup_{i \in \mathbb{N}} \Omega_i^c$  and  $\partial_x^2 v(t, 0) = 0, \forall t \in (0, T)$ . As  $\mu(\cup_{i \in \mathbb{N}} \Omega_i^c) = T$ , we have  $v(t, x) = 0, \forall x \in \omega, \forall t \in \cup_{i \in \mathbb{N}} \Omega_i^c$  and  $\partial_x^2 v(t, 0) = 0, \forall t \in (0, T)$ . This implies that  $v(t, x) = 0, \forall x \in \omega$  with almost every  $t \in (0, T)$ . We also obtain that the limit  $v$  solves the following equation in the sense of distribution

$$\begin{cases} \partial_t v + \partial_x v + \partial_x^3 v - \partial_x^5 v + \lambda v^p \partial_x v = 0, & (t, x) \in (0, \infty) \times (0, L), \\ v(t, 0) = v(t, L) = \partial_x v(t, 0) = \partial_x v(t, L) = \partial_x^2 v(t, L) = 0, & t \in (0, \infty), \\ \|v\|_{L^2(0,T;L^2(0,L))} = 1. \end{cases} \quad (3.50)$$

Using the unique continuation principle Lemma 3.6 for Kawahara equation we have  $v(t, x) = 0, \forall x \in (0, L)$ . Which is a contradiction. Hence the proof follows.  $\square$

#### 4. CONCLUSION AND FINAL COMMENTS

In this work the exponential stabilization of the nonlinear Kawahara equation was addressed. The control was considered acting internally or in the boundary. In both cases, a global well-posedness result was obtained. In the case of boundary saturation a local exponential stability result was derived while for internal saturated actuator a semiglobal exponential stability result was shown. The main strategies in the proof of those results were observability inequalities, contradiction arguments and compactness ideas. We finish this work with several comments and remarks, in order:

**4.1. Sign of the first-order derivative.** One can wonder if our results are still valid if we modify slightly our systems. This, extension is in fact not true in all the cases. To fix ideas, consider the following equation

$$\begin{cases} \partial_t u - \partial_x u + \partial_x^3 u - \partial_x^5 u + u^p \partial_x u = 0, & (t, x) \in (0, \infty) \times (0, L), \\ u(t, 0) = u(t, L) = \partial_x u(t, 0) = \partial_x u(t, L) = 0, \partial_x^2 u(t, L) = \mathbf{sat}_{loc}(\alpha \partial_x^2 u(t, 0)), & t \in (0, \infty), \\ u(0, x) = u_0(x), & x \in (0, L). \end{cases} \quad (4.1)$$

Observe that (4.1) it is almost the same equation than (1.6), the unique difference is the sign of the first order derivative. If we reply the arguments [Theorem 3.3](#) to show exponential stability, we are ask to analyze the solutions of

$$\begin{cases} \partial_t u - \partial_x u + \partial_x^3 u - \partial_x^5 u = 0, & (t, x) \in (0, \infty) \times (0, L), \\ u(t, 0) = u(t, L) = 0, & t \in (0, \infty), \\ \partial_x u(t, 0) = \partial_x u(t, L) = 0, & t \in (0, \infty), \\ \partial_x u(t, 0) = \partial_x^2 u(t, 0) = \partial_x^2 u(t, L) = 0, & t \in (0, \infty). \end{cases} \quad (4.2)$$

For this system, it was shown in [1], the existence of a critical set  $\mathcal{N}$ , such that if  $L \in \mathcal{N}$ , there exists a steady-state solution of (4.2). Therefore, the argument does not follow. In the case of the KdV equation similar phenomena appears, but for the KdV equation studied in [30] the set of critical lengths is totally know, that allows in [22] to show the exponential stability with saturated boundary control if the length is not critical. By contrast, for (4.2) it is not known if the set  $\mathcal{N}$  represents all the critical lengths or just some of them. With this in mind, the exponential stability with boundary saturation holds as long the following unique continuation property is true for give  $L > 0$

**Assumption 4.1.** *If  $\lambda, \psi \in \mathbb{C} \times H^5(0, L)$  satisfies*

$$\begin{cases} \lambda \psi(x) - \psi'(x) + \psi'''(x) - \psi''''(x) = 0, & x \in (0, L) \\ \psi(0) = \psi(L) = \psi'(0) = \psi'(L) = 0, \\ \psi''(0) = \psi''(L) = 0, \end{cases}$$

then  $\psi \equiv 0$ .

Now we consider the same equation, but we drop the first order term; i.e.

$$\begin{cases} \partial_t u + \partial_x^3 u - \partial_x^5 u = 0, & (t, x) \in (0, \infty) \times (0, L), \\ u(t, 0) = u(t, L) = 0, & t \in (0, \infty), \\ \partial_x u(t, 0) = \partial_x u(t, L) = 0, & t \in (0, \infty), \\ \partial_x u(t, 0) = \partial_x^2 u(t, 0) = \partial_x^2 u(t, L) = 0, & t \in (0, \infty). \end{cases} \quad (4.3)$$

In [36, 37], it was shown that the set of critical lengths related to this equation it is empty. Therefore, we expect that our stability result by saturated boundary feedback is still valid in this context.

In the case of internal saturation the situation is better. Consider the following system

$$\begin{cases} \partial_t u + \eta \partial_x u + \partial_x^3 u - \partial_x^5 u + u^p \partial_x u + \mathbf{sat}(a(x)u(t, x)) = 0, & (t, x) \in (0, \infty) \times (0, L), \\ u(t, 0) = u(t, L) = \partial_x u(t, 0) = \partial_x u(t, L) = 0, \partial_x^2 u(t, L) = 0, & t \in (0, \infty), \\ u(0, x) = u_0(x), & x \in (0, L), \end{cases} \quad (4.4)$$

where  $\eta = 0, -1$  and  $a \in L^\infty(0, L)$  satisfies (1.5). Straightforward modifications of [Theorem 2.11](#) allow us to show that there exists a unique mild solution  $u \in C([0, \infty), L^2(0, L)) \cap L^2(0, \infty), H^2(0, L)$  of (4.4). With respect to the stabilization, following the proof of [Theorem 3.4](#) we are asking to prove similar unique continuation principle as [Lemma 3.6](#), for a solution of

$$\begin{cases} \partial_t u + \eta \partial_x u + \partial_x^3 u - \partial_x^5 u + u^p \partial_x u = 0, & (t, x) \in (0, \infty) \times (0, L), \\ u(t, 0) = u(t, L) = \partial_x u(t, 0) = \partial_x u(t, L) = 0, \partial_x^2 u(t, L) = 0, & t \in (0, \infty), \\ u(t, x) = 0, & (t, x) \in (0, \infty) \times \omega, \\ u(0, x) = u_0(x), & x \in (0, L). \end{cases}$$

In particular, we can write

$$\partial_t u + \partial_x u + \partial_x^3 u - \partial_x^5 u + u^p \partial_x u = (1 - \eta) \partial_x u.$$

Then, we can apply the Carleman estimate (B.1) and absorb the remaining right hand-side terms by the left hand side terms. From where we can conclude the semiglobal exponential stability of (4.4).

**4.2. More general nonlinearities.** In the introduction of this work, we mention that in [1] the internal stabilization was studied in the case  $p \in [1, 4)$ , but a smallness condition in the initial data was required for  $p \in (2, 4)$ . This is one of the main reasons why we only considered the case  $p \in [1, 2)$ . In fact, if the initial data is too small, then by continuity, the solution remains small, which could imply that  $\text{sat}(au) = au$ , then the system is not really saturated.

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### APPENDIX A. PROOF OF LEMMA 2.2

In this part, we prove that the operator  $\mathcal{T}$  is well-defined. Indeed, take  $\psi$  defined by

$$\psi(x) = \frac{L^2}{8\pi^2} \sin^2\left(\frac{2\pi}{L}x\right) \text{sat}_{loc}(\alpha y''(0)).$$

We can easily observe that  $\psi(0) = \psi(L) = \psi'(0) = \psi'(L) = 0$ , and  $\psi''(L) = \text{sat}_{loc}(\alpha y''(0))$ . Then  $z$  solves (2.5) if and only if  $\varphi = z - \psi$  solves

$$\begin{cases} \tilde{\lambda}\varphi + \varphi' + \varphi''' - \varphi'''' = f = \tilde{\lambda}w - \tilde{\lambda}\psi - \psi' - \psi''' + \psi'''' , \\ \varphi(0) = \varphi(L) = \varphi'(0) = \varphi'(L) = \varphi''(L) = 0. \end{cases}$$

which is equivalent to  $(\mathcal{A}_b^0 - \tilde{\lambda}I)\varphi = -f$ . Now, let  $(\lambda, y)$  a pair eigenvalue, eigenfunction of  $\mathcal{A}_b^0$ , i.e  $\mathcal{A}_b^0 y = \lambda y$ , in particular we have

$$\begin{cases} -y' - y''' + y'''' = \lambda y, \\ -\bar{y}' - \bar{y}''' + \bar{y}'''' = \bar{\lambda}\bar{y}. \end{cases}$$

We multiply the first equation by  $\bar{y}$ , the second one by  $y$  and we integrate over  $(0, L)$ , we get after some integration by parts

$$\int_0^L y\bar{y}' dx + \int_0^L y\bar{y}''' dx - \int_0^L y\bar{y}'''' dx - |y'(0)|^2 = \bar{\lambda}|y|^2 dx$$

and

$$-\int_0^L y\bar{y}' dx - \int_0^L y\bar{y}''' dx + \int_0^L y\bar{y}'''' dx = \bar{\lambda} \int_0^L |y|^2 dx.$$

By adding these two equations, we deduce

$$-\frac{1}{2}|y'(0)|^2 = \text{Re}(\lambda) \int_0^L |y|^2 dx \implies \text{Re}(\lambda) \leq 0.$$

Then as  $\tilde{\lambda} > 0$ , we get  $\tilde{\lambda} \notin \sigma(\mathcal{A}_b^0)$ , thus  $\mathcal{A}_b^0 - \tilde{\lambda}I$  is invertible and finally,  $\varphi = -(\mathcal{A}_b^0 - \tilde{\lambda}I)^{-1}f$ , is well defined and also  $z$ .

### APPENDIX B. PROOF OF LEMMA 3.6

This section is devoted to the proof of the unique continuation principle for the Kawahara equation with interior observation. Let us first recall the following existence result of the Kawahara equation in a regular set-up.

**Proposition B.1.** *Let us assume that  $u_0 \in H^5(0, L)$ . Then the equation (3.13) with  $u(0, \cdot) = u_0$  possesses a unique solution  $u \in C([0, T], H^5(0, L)) \cap L^2(0, T; H^7(0, L))$ . Also by interpolation, if  $u_0 \in H^s(0, L)$ , then the equation (3.33) possesses a unique solution  $u \in C([0, T], H^s(0, L)) \cap L^2(0, T; H^{s+2}(0, L))$ ,  $0 \leq s \leq 5$ .*

One can prove the above result in the same spirit of [31].

Let us consider a function  $\psi \in C^\infty(0, 1)$  with  $\psi > 0$  in  $(0, 1)$ ,  $\psi(0) = \psi(1) = 0$ ,  $\|\psi\|_{L^\infty(0,1)} = 1$ ,  $|\psi_x| > 0$  in  $[0, 1] \setminus \omega$ ,  $\psi_x(0) > 0$ ,  $\psi_x(1) < 0$ . For any  $\epsilon > 0$ , we consider

$$a(t, x) = \frac{e^{\mu(\psi(x)+3)} - e^{5\mu}}{(t-\epsilon)(T-t)}, \quad \xi = e^{\lambda a}, \quad \varphi = \frac{e^{\mu(\psi(x)+3)}}{(t-\epsilon)(T-t)}, \quad t \in (\epsilon, T).$$

Now we will use a global Carleman estimate [13, Proposition 1] for Kawahara equation. Let us denote the following space

$$\mathcal{U}_\epsilon = \left\{ u \in L^2(\epsilon, T; H^5(0, L)) \mid u(t, 0) = u(t, L) = \partial_x u(t, 0) = \partial_x u(t, L) = \partial_x^2 u(t, L) = 0, \right. \\ \left. Pu = \partial_t u + \partial_x u + \partial_x^3 u - \partial_x^5 u \in L^2(\epsilon, T; L^2(0, L)) \right\}$$

**Proposition B.2.** *There exists positive constants  $\mu_0 > 1$ ,  $C_0 > 0$  and  $C_1 > 0$  such that for  $\mu = \mu_0$  and for every  $\lambda \geq C_0(T + T^2)$ , for all  $u \in \mathcal{U}_\epsilon$ , we have*

$$\int_\epsilon^T \int_0^L (s\varphi\xi^2\partial_x^4 u^2 + s^3\varphi^3\xi^2\partial_x^3 u^2 + s^5\varphi^5\xi^2\partial_x^2 u^2 + s^7\varphi^7\xi^2\partial_x u^2 + s^9\varphi^9\xi^2 u^2) \\ \leq C \int_\epsilon^T \int_\omega (\xi^2 |Pu|^2 + s\varphi\xi^2\partial_x^4 u^2 + s^9\varphi^9\xi^2 u^2). \quad (\text{B.1})$$

**B.1. Proof of Lemma 3.6.** As,  $u \in L^2(0, T; H^2(0, L))$ , for any  $\epsilon \in (0, T)$ , there exists a candidate  $t_1 \in (0, \epsilon)$  such that  $u(t_1, \cdot) \in H^2(0, L)$ . Using Proposition B.1, we get that  $u \in C([t_1, T]; H^2(0, L)) \cap L^2(t_1, T; H^4(0, L))$ . Similar argument gives a time  $t_2 \in (t_1, \epsilon)$  such that  $u(t_2, \cdot) \in H^4(0, L)$ . Again from Proposition B.1, it follows that  $u \in C([t_2, T]; H^4(0, L)) \cap L^2(t_2, T; H^6(0, L))$ . Proceeding with similar argument we finally have  $t_3 \in (t_2, \epsilon)$  such that  $u \in C([t_3, T]; H^5(0, L)) \cap L^2(t_3, T; H^7(0, L))$ . Therefore, we eventually have  $u \in C([\epsilon, T]; H^5(0, L)) \cap L^2(\epsilon, T; H^7(0, L))$ . Thus we can use (B.1) for the solution  $u$  as  $u \in \mathcal{U}_\epsilon$ .

Plugging  $Pu = u^p \partial_x u$  in the Global Carleman estimate (B.1), we have

$$\int_\epsilon^T \int_0^L (s\varphi\xi^2\partial_x^4 u^2 + s^3\varphi^3\xi^2\partial_x^3 u^2 + s^5\varphi^5\xi^2\partial_x^2 u^2 + s^7\varphi^7\xi^2\partial_x u^2 + s^9\varphi^9\xi^2 u^2) \\ \leq C \int_\epsilon^T \int_\omega (\xi^2 |\partial_x u u^p|^2 + s\varphi\xi^2\partial_x^4 u^2 + s^9\varphi^9\xi^2 u^2). \quad (\text{B.2})$$

Note that  $u \in L^\infty((\epsilon, T) \times (0, L))$ . Therefore one can find a large constant  $C_1 = \|u\|_{L^\infty((\epsilon, T) \times (0, L))}^p$  and consider  $s$  large enough so we can absorb the last term in the right hand sides by the term  $\frac{1}{2} \int_\epsilon^T \int_0^L s^7\varphi^7\xi^2\partial_x u^2$ . Indeed choose  $s \geq C_1(T + T^2)$  we have

$$C \int_\epsilon^T \int_\omega \xi^2 |u^p \partial_x u|^2 \leq \frac{1}{2} \int_\epsilon^T \int_0^L s^7\varphi^7\xi^2\partial_x u^2.$$

Finally we have the following:

$$\int_\epsilon^T \int_0^L (s\varphi\xi^2\partial_x^4 u^2 + s^3\varphi^3\xi^2\partial_x^3 u^2 + s^5\varphi^5\xi^2\partial_x^2 u^2 + s^7\varphi^7\xi^2\partial_x u^2 + s^9\varphi^9\xi^2 u^2) \\ \leq C \int_\epsilon^T \int_\omega (s\varphi\xi^2\partial_x^4 u^2 + s^9\varphi^9\xi^2 u^2).$$

Therefore  $u = 0$  in  $(0, T) \times \omega$  implies that  $u = 0$  in  $(\epsilon, T) \times (0, L)$  for all  $\epsilon > 0$ . As a result we get have  $u = 0$ ,  $(t, x) \in (0, T) \times (0, L)$ .

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DEPARTMENT OF MATHEMATICS, INDIAN INSTITUTE OF TECHNOLOGY BOMBAY, POWAI, MUMBAI, 400076, INDIA  
Email address: subratamajumdar634@gmail.com

UNIVERSITÉ PAUL SABATIER, INSTITUT DE MATHÉMATIQUES DE TOULOUSE, 118 ROUTE DE NARBONNE, 31062 TOULOUSE, FRANCE.  
Email address: hugo.parada@math.univ-toulouse.fr