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A few words about maps[★]

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Abstract

In this paper, we survey some properties, encoding, and bijections involving combinatorial maps, double occurrence words, and chord diagrams. We particularly study quasi-trees from a purely combinatorial point of view and derive a topological representation of maps with a given spanning quasi-tree using two fundamental polygons, which extends the representation of planar maps based on the equivalence with bipartite circle graphs. Then, we focus on Depth-First Search trees and their connection with a poset we define on the spanning quasi-trees of a map. We apply the bijections obtained in the first section to the problem of enumerating loopless rooted maps. Finally, we return to the planar case and discuss a decomposition of planar rooted loopless maps and its consequences on planar rooted loopless map enumeration.

Keywords: combinatorial map, depth-first search, permutation, circle graph, chord diagram, double occurrence word, Delta-matroid, quasi-tree, map enumeration, map representation.

[†]This project started at the occasion of the talk presented in 2009 by Pierre Rosenstiehl in Bordeaux. Pierre left before this paper was eventually written, but it is fair to include him as a coauthor.

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1. Introduction

Combinatorial maps is a natural bridge linking combinatorics to topological graph theory. At the heart of this connection is the property that a cellular embedding of a graph on a surface can (basically) be described by means of the cyclic order of incident edges at vertex. This property, already noticed by Heffter [13] at the end of the XIXth century, led to a combinatorial study of graph embeddings, initiated by Edmonds [9] and Youngs [26], who independently gave a precise description of this correspondence, as well as a derived computation of the faces and genus of the embedding. This framework was then popularized by White's book "Graphs, Group, and Surfaces" [25]. Further generalization have been proposed, with the notions of hypermaps [5, 14], of dessins d'enfants [22], and of ribbon graphs [20]. Also, instead of a description of combinatorial maps by means of a permutation and an involution, Tutte proposed a description based on three involutions.

From these combinatorial descriptions of graphs on surfaces, some coding schemes of maps by words were derived, which allowed to shed a new light in [5] on Tutte's enumeration formulas of maps [23], or a bijection between (rooted) hypermaps and indecomposable permutations (also called connected or irreducible) [18] (see also [6]). The connection between maps and words also impacted several other fields, from graph drawing [12] to matroid theory [10, 11], and was central to the characterization of Gauss codes [19, 8].

In this paper, we survey some aspects of combinatorial maps, and how they are intrinsically linked with chord diagrams, double occurrence words, Euler tours, Δ -matroids, polynomials, etc., with an emphasis on words. In particular, we explicit few bijections involving rooted maps and words. As an application, we derive some enumerative properties of loopless rooted maps. Our approach will be purely combinatorial, though it relates to topological properties of graph embeddings on orientable surfaces. The reader interested in the topological aspects is referred to the monograph [16] (See also [17]).

2. Combinatorial maps and hypermaps

Recall that the group of all permutations of a set B is the *symmetric group* $\text{Sym}(B)$ of B . The composition of permutations is denoted multiplicatively: $\sigma\tau = \sigma \circ \tau$. A permutation $\tau \in \text{Sym}(B)$ is an *involution* if τ^2 is the identity permutation; it is *fixed-point free* if $\tau(x) \neq x$ for every $x \in B$. We denote by $\langle \tau_1, \dots, \tau_k \rangle$ the subgroup of $\text{Sym}(B)$ generated by the permutations τ_1, \dots, τ_k , which is the smallest subgroup of $\text{Sym}(B)$ which contains τ_1, \dots, τ_k . The *orbit* of an element $b \in B$ in a subgroup Γ of $\text{Sym}(B)$ is the set $\Gamma \cdot b = \{\gamma(b) : \gamma \in \Gamma\}$. A subgroup Γ of $\text{Sym}(B)$ *acts transitively* on B if, for every $b, b' \in B$ there exists some $\tau \in \Gamma$ such that $\tau(b) = b'$. In other words, Γ acts transitively on B if Γ has a single orbit. The *cycles* of a permutation $\gamma \in \text{Sym}(B)$ are the orbits of $\langle \gamma \rangle$, that is, the sets of the form $\{\gamma^i(b) : i \in \mathbb{Z}\}$. The set of all the cycles of γ is denoted by $Z(\gamma)$.

For $\mu \in \text{Sym}(B)$ and $B' \subseteq B$ we define the *restriction* of σ to B' as the permutation $\sigma|_{B'} \in \text{Sym}(B')$ defined as follows: for every $b \in B'$, $\mu|_{B'}(b) = \mu^k(b)$, where k is the minimum positive integer such that $\mu^k(b) \in B'$. We further define the *cutting-out* of μ on B' as the permutation $\mu_{B'} \in \text{Sym}(B)$, where $\mu_{B'}(b)$ is equal to $\mu|_{B'}(b)$ if $b \in B'$, and to b if $b \notin B'$. (Intuitively, $\mu_{B'}$ is $\mu|_{B'}$ on B' and the identity mapping on $B \setminus B'$.)

The next easy lemma will be useful.

Lemma 1. Let $\sigma, \mu \in \text{Sym}(B)$ and let $B' \subseteq B$. Then,

$$\sigma_{B'}\mu_{B'} = (\sigma\mu_{B'})_{B'} \quad \text{and thus} \quad \sigma|_{B'}\mu|_{B'} = (\sigma\mu_{B'})|_{B'}.$$

Proof. If $b \notin B'$, then b is a fixed point of both $\sigma_{B'}\mu_{B'}$ and $(\sigma\mu_{B'})_{B'}$. So, assume $b \in B'$. Let k be the minimum positive integer with $(\sigma\mu_{B'})^k(b) \in B'$. We prove by induction on i that for every $1 \leq i \leq k$ we have $(\sigma\mu_{B'})^i(b) = \sigma^i\mu_{B'}(b)$. The base case, $i = 1$, is straightforward. Assume that the property holds for all $1 \leq j < i$ and $1 < i \leq k$. Then,

$$\begin{aligned} (\sigma\mu_{B'})^i(b) &= \sigma\mu_{B'}(\sigma\mu_{B'})^{i-1}(b) \\ &= \sigma(\sigma\mu_{B'})^{i-1}(b) && \text{(as } (\sigma\mu_{B'})^{i-1}(b) \notin B') \\ &= \sigma(\sigma^{i-1}\mu_{B'})(b) && \text{(by induction hypothesis)} \\ &= \sigma^i\mu_{B'}(b) \end{aligned}$$

Thus, it follows that k is also the minimum positive integer with $\sigma^k\mu_{B'}(b) \in B'$. Hence, $(\sigma\mu_{B'})_{B'}^k(b) = (\sigma\mu_{B'})^k(b) = \sigma^k\mu_{B'}(b) = \sigma_{B'}\mu_{B'}(b)$. \square

A (*combinatorial*) *map* is a triple $\mathcal{M} = (B, \sigma, \alpha)$, where B is the set of the *flags* of \mathcal{M} , $\sigma, \alpha \in \text{Sym}(B)$, α is a fixed-point free involution, and $\langle \sigma, \alpha \rangle$ acts transitively on B . When we relax the condition that $\langle \sigma, \alpha \rangle$ acts transitively on B , we say that $\mathcal{M} = (B, \sigma, \alpha)$ is a *general map*, and we refer to the orbits of $\langle \sigma, \alpha \rangle$ as the *components* of \mathcal{M} . The *edges* of a map \mathcal{M} are the cycles of α , and its *vertices* are the cycles of σ . For a flag $b \in B$ it will be convenient to define

$$\underline{b} = \langle \alpha \rangle \cdot b = \{b, \alpha(b)\},$$

which is the edge of \mathcal{M} that contains b .

Two maps $\mathcal{M} = (B, \sigma, \alpha)$ and $\mathcal{M}' = (B', \sigma', \alpha')$ are *isomorphic* if there exists a bijection $f : B \rightarrow B'$ (referred to as a (*map*) *isomorphism* of \mathcal{M} and \mathcal{M}') with $\sigma' = f \circ \sigma \circ f^{-1}$ and $\alpha' = f \circ \alpha \circ f^{-1}$.

The *underlying graph* of a map \mathcal{M} is a graph $G(\mathcal{M})$ with vertex set $Z(\sigma)$ and edge set $Z(\alpha)$. In this graph, an edge $e \in Z(\alpha)$ is incident to a vertex $v \in Z(\sigma)$ if $e \cap v \neq \emptyset$. If $e \subseteq v$ (i.e. if e and v intersect on two flags), then e is a *loop* attached to v . It will be convenient to identify \mathcal{M} with an embedding of $G(\mathcal{M})$. When we speak about cycles, trees, and cuts of \mathcal{M} , we mean the cycles, trees, and cuts of $G(\mathcal{M})$, as subsets of edges.

It is well known that a map \mathcal{M} defines a cellular embedding of $G(\mathcal{M})$ on an orientable surface, whose genus g is given by Euler's formula

$$2 - 2g = |Z(\sigma\alpha)| - |Z(\alpha)| + |Z(\sigma)|. \quad (1)$$

In this paper, by the *dual* of a general map $\mathcal{M} = (B, \sigma, \alpha)$ we mean the general map

$$\mathcal{M}^* = (B, \sigma\alpha, \alpha).$$

Note that \mathcal{M}^* has the same components as \mathcal{M} ; in particular, the dual of a map is a map. Remark that this duality differs from the geometric dual by the orientation of the dual surface (see Figure 1).

A *rooted map* is a pair $\mathcal{M}_\bullet = (\mathcal{M}, b_\bullet)$, where b_\bullet is a flag of \mathcal{M} . Two rooted combinatorial maps (\mathcal{M}, b_\bullet) and $(\mathcal{M}', b'_\bullet)$ are *isomorphic* if there exists a map isomorphism f of \mathcal{M} and \mathcal{M}' with $b'_\bullet = f(b_\bullet)$.

For an introduction to combinatorial maps (and hypermaps) we refer the interested reader to [7].

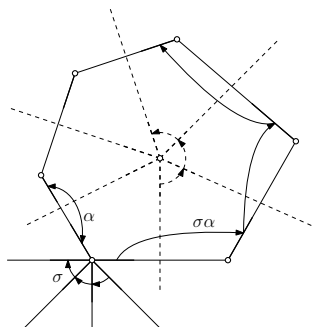


Figure 1: Map duality. From a geometric point of view, the orientation of the surface is reversed when we consider the dual map.

3. Planar maps

Before we consider the general setting of maps, we take time to comment on the planar case. It follows immediately from Euler's formula that a map \mathcal{M} is planar if and only if the complement of a spanning tree of \mathcal{M} is a spanning tree of the dual map \mathcal{M}^* . Fixing a spanning tree T of a planar map \mathcal{M} allows defining several bijections. A first one is obtained by considering a tubular neighborhood of T . Traversing this tubular neighborhood, either we follow some edge of T , or we cross some edge of the complement of T . This way, the traversal defines a circular sequence, in which every edge appears twice. This circular sequence, in turn, naturally defines a bipartite chord diagram (hence a bipartite circle graph). Conversely, every proper coloration of a bipartite circle graph uniquely determines a planar map and a spanning tree of it (see Figure 2). This property is at the heart of Fraysseix's characterization of circle graphs [10, 11] and Rosenstiehl's characterization of planarity by the algebraic diagonal [21].

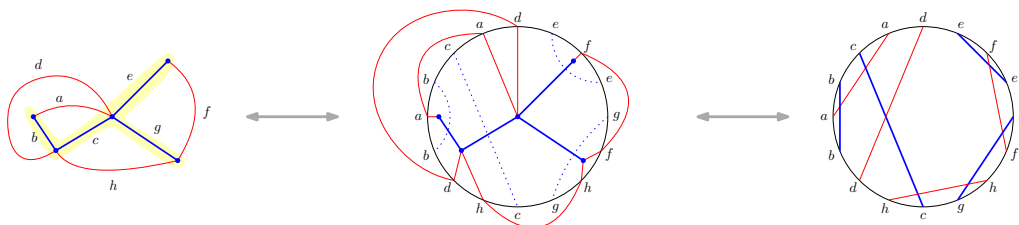


Figure 2: From a spanning tree T of a planar graph G to the bipartite circle graph $\Lambda_T(G)$, and back.

For an edge $f \notin T$ we denote by $\gamma_T(f)$ the *fundamental cycle* of f , that is the edge set of the unique cycle in $T \cup \{f\}$; For an edge $e \in T$ we denote by $\omega_T(e)$ the *fundamental cocycle* of e , that is the unique inclusion minimal cut of G intersecting T only at e . It is easily proved that for an edge $e \in T$ and an edge $f \notin T$ we have $e \in \gamma_T(f) \iff f \in \omega_T(e)$. The *fundamental interlacement graph* of the tree T in the graph G is the bipartite graph $\Lambda_T(G)$ with vertex set $T \cup (E(G) \setminus T)$, where $e \in T$ is adjacent to $f \in E(G) \setminus T$ if $e \in \gamma_T(f)$. The above described construction emphasizes that the fundamental interlacement graph $\Lambda_T(G)$ of a spanning tree T in a planar graph G is a bipartite circle graph.

The *local complementation* of a graph G at a vertex v is the graph $G * v$ obtained by replacing the subgraph induced by G on the neighbors of v by its complement (see Figure 3).

This transformation was introduced by Kotzig [15] in relations with κ -transformations. The *pivoting* (or *switching*) of an edge uv in a graph G results in the graph $G \wedge uv = G * u * v * u = G * v * u * v$.

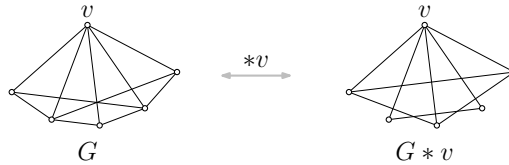


Figure 3: Local complementation of a graph G at a vertex v

The chord diagrams and circle graphs obtained when one considers a different spanning tree are nicely related. Indeed, if a spanning tree T' is obtained from a spanning tree T by removing a tree edge e and replacing it with a non-tree edge f , then $f \in \omega(e)$ (that is: ef is an edge of $\Lambda_T(G)$), and we have $\Lambda_{T'}(G) = \Lambda_T(G) \wedge ef$ (that is: the circle graph obtained from $\Lambda_T(G)$ by pivoting the edge ef ; see Figure 4).

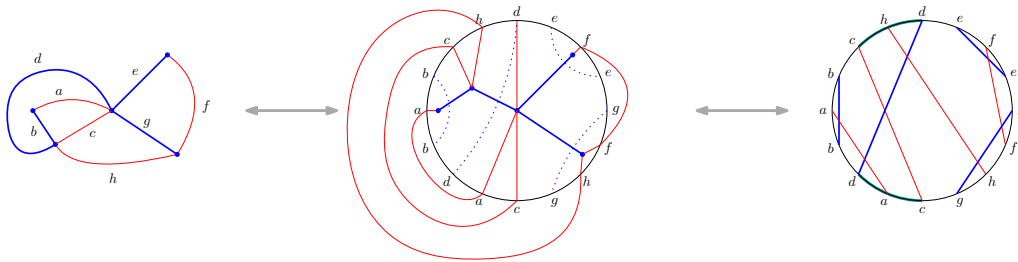


Figure 4: From a planar graph to a bipartite circle graph, and back... with another tree obtained by replacing $c \in T$ by $d \notin T$. The corresponding chord diagram is obtained by pivoting cd .

While pivoting is defined on circle graphs, its effect on a chord diagram representation is easy to describe (see Figure 5).

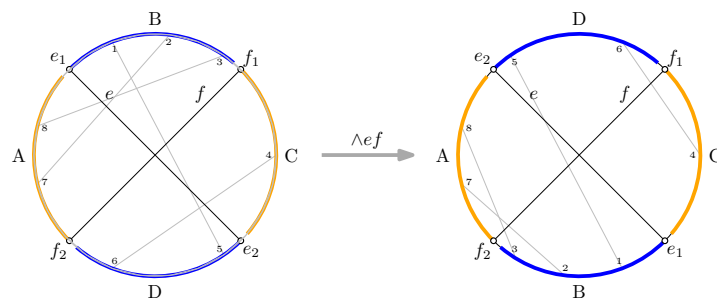


Figure 5: Pivoting ef consists in rotating the parts between the endpoints of the chords e and f as in the picture.

On the other hand, contracting an edge in the tree T (or deleting an edge in its complement)

results in the deletion of the corresponding chord from the chord diagram. This connection extends to general (loopless simple) graphs, and this nice interplay partly explains the development of a study of vertex minors and pivot minors, which was built as an analog to Robertson and Seymour's graph minor project.

However, considering graphs instead of maps, and circle graphs instead of maps, we might lose some geometrical and topological aspects. Therefore, we shall keep chord diagrams and maps as primitive objects.

4. Tours, Quasi-trees, and Map Minors

Definition 2 (tour). Let $\mathcal{M} = (B, \sigma, \alpha)$ be a general map and let F be a subset of edges of \mathcal{M} . The *tour* of F in \mathcal{M} is the permutation τ defined by

$$\tau(b) = \sigma\alpha_F(b) = \begin{cases} \sigma\alpha(b) & \text{if } \underline{b} \in F \\ \sigma(b) & \text{otherwise} \end{cases} \quad (2)$$

where, by a slight abuse of notation, we denote by α_F the cutting out of α on the union of all the edges in F , that is on the set $\{b \in B : \underline{b} \in F\}$.

Note that the tour of F in \mathcal{M} is the same as the tour of the complement of F in the dual general map \mathcal{M}^* .

When \mathcal{M} has a single component (i.e. \mathcal{M} is a map), and F is (the edge set of) a spanning tree of $G(\mathcal{M})$, it is well known that the tour of F in \mathcal{M} is a cycle (see Figure 6).

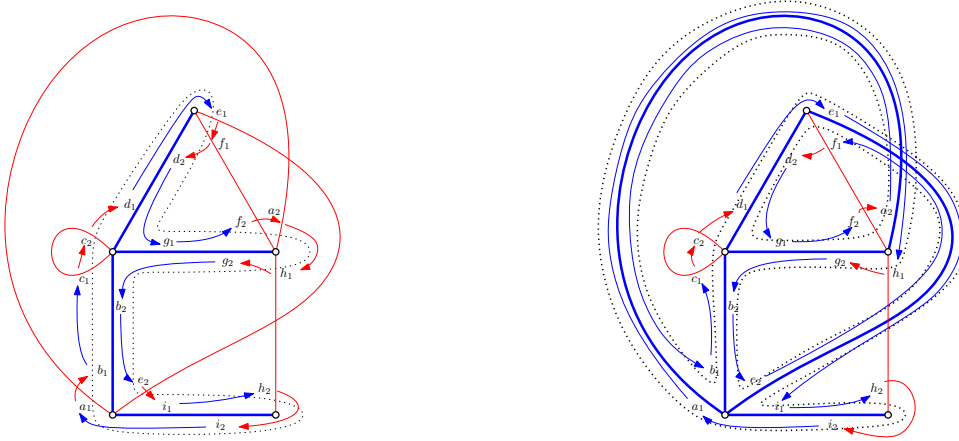


Figure 6: Tour of a spanning tree (on the left) and of a quasi-tree (on the right). (Quasi-)tree edges are depicted as thick blue edges and the other edges are depicted as thin red edges. The blue arrows correspond to $\sigma\alpha$ and the red ones to σ .

We use this property to generalize the notion of spanning tree.

Definition 3 (quasi-tree). A (spanning) *quasi-tree* of a general map \mathcal{M} is a subset F of edges of \mathcal{M} , whose tour in \mathcal{M} is a cycle.

Note that the standard definition of a quasi-tree is slightly different, as it is (topologically) defined as a map (or, more generally, a ribbon graph) with one boundary component [3]. However, it is not difficult to check that our definition coincides with the notion of a (spanning) quasi-tree in its original sense. The *genus* of a quasi-tree S of a map \mathcal{M} is the value $(|S| - (n - 1))/2$, where n is the number of vertices of \mathcal{M} .

The existence of a quasi-tree characterizes maps, the same way that the existence of a spanning tree characterizes connected graphs.

Lemma 4. A general map $\mathcal{M} = (B, \sigma, \alpha)$ has a single component (i.e. is a map) if and only if \mathcal{M} has a quasi-tree.

Proof. If \mathcal{M} is a map, then any spanning tree of $G(\mathcal{M})$ is a quasi-tree of \mathcal{M} . Conversely, assume that \mathcal{M} has a quasi-tree S . As $\langle \sigma, \alpha \rangle$ acts transitively on each cycle of the tour of S in \mathcal{M} , it acts transitively on B . Hence, \mathcal{M} is a map. \square

Quasi-trees are compatible with map duality, in a way which is similar to trees with matroid duality.

Lemma 5. A subset S of edges is a quasi-tree of a map \mathcal{M} if and only if its complement is a quasi-tree of the dual map \mathcal{M}^* .

Proof. This directly follows from the fact that the tour of S in \mathcal{M} is the same as the tour of the complement of S in \mathcal{M}^* . \square

A *bridge* (or *isthmus*) of a map \mathcal{M} is an edge e , such that $G(\mathcal{M}) - e$ is disconnected, while a *separating loop* of \mathcal{M} is an edge e , such that $G(\mathcal{M}^*) - e$ is disconnected. Hence, a separating loop of a map is a bridge of the dual map. Remark that bridges (resp. separating loops) are exactly the edges of \mathcal{M} that belong to all quasi-trees (resp. to no quasi-tree) of \mathcal{M} . Note that, from the map point of view, a bridge does not always disconnect the map as it can be an edge incident to a degree 1 vertex. Similarly, a separating loop can be a loop that enclosing a single face (see Figure 7).

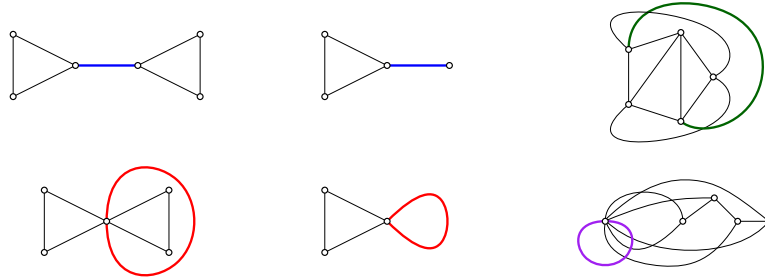


Figure 7: Bridges (in fat blue) and separating loops (in fat red). Bridges belong to all quasi-trees, while separating loops belong to no quasi-tree. On the right, two dual maps (you can check it...), where the purple edge is dual to the green edge. On the right top, the green edge is an example of a non-bridge with two incidences with a same face; on the right bottom, the purple edge is an example of a non-separating loop.

Definition 6 (edge deletion). Let \mathcal{M} be a map, and let e be an edge of \mathcal{M} that is not a bridge of \mathcal{M} . The map obtained by deleting e in \mathcal{M} is the map $\mathcal{M} \setminus e = (B \setminus e, \sigma|_{B \setminus e}, \alpha|_{B \setminus e})$.

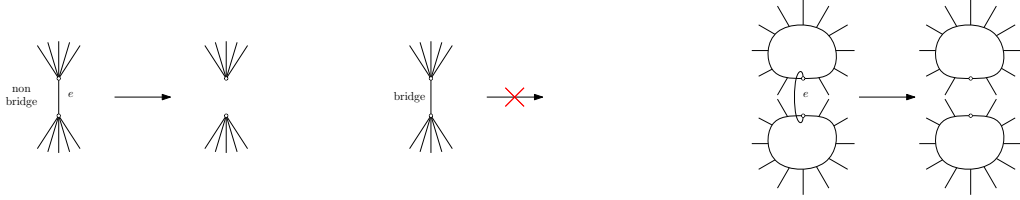


Figure 8: Deleting an edge e of a map \mathcal{M} . The number of edges decreases by 1, while the number of vertices remains unchanged. Note that the number of faces can increase.

We now prove that this definition is valid, meaning that $\mathcal{M} \setminus e$ is indeed a map. This will follow from the next lemma applied to any spanning tree F of $G(\mathcal{M})$ that avoids e , the transitive action of $\langle \sigma', \alpha' \rangle$ on $B \setminus e$ being witnessed by the existence of a quasi-tree of $\mathcal{M} \setminus e$. Remark that $G(\mathcal{M} \setminus e) = G(\mathcal{M}) \setminus e$.

Lemma 7. Let $\mathcal{M} = (B, \sigma, \alpha)$ be a general map, let $S \subset B$, let $e \in B \setminus S$, let τ be the tour of S in \mathcal{M} , and let τ' be the tour of S in $\mathcal{M} \setminus e$. Then, $\tau' = \tau|_{B \setminus e}$.

In particular, S is a quasi-tree of \mathcal{M} if and only if S is a quasi-tree of $\mathcal{M} \setminus e$.

Proof. By definition, $\tau = \sigma\alpha_S$. As $e \notin S$ we have $(\alpha_S)_{B \setminus e} = \alpha_S$. According to Theorem 1, we have $\tau|_{B \setminus e} = (\sigma\alpha_S)|_{B \setminus e} = \sigma|_{B \setminus e}(\alpha_S)|_{B \setminus e} = \sigma|_{B \setminus e}(\alpha|_{B \setminus e})_S = \tau'$. \square

Lemma 8. Let \mathcal{M} be a map and let e be a bridge of \mathcal{M} . Then e belongs to all the quasi-trees of \mathcal{M} .

Proof. Assume for contradiction that e is a bridge of \mathcal{M} and \mathcal{M} has a quasi-tree S that does not contain e . Then $S \setminus e$ is a quasi-tree of $\mathcal{M} \setminus e$, thus $\mathcal{M} \setminus e$ is a map. However, it is easily checked that $G(\mathcal{M} \setminus e) = G(\mathcal{M}) \setminus e$, thus is not connected. \square

Note that if we delete from \mathcal{M} all the edges that are not in a quasi-tree S of \mathcal{M} , we get a map with a single face (which is the tour of S on this map). Also note that deleting an edge keeps the number of vertices constant.

Definition 9 (edge contraction). Let \mathcal{M} be a map, and let e be an edge of \mathcal{M} that is not a separating loop of \mathcal{M} . The map obtained by contracting e in \mathcal{M} is the map $\mathcal{M}/e = (B \setminus e, (\sigma\alpha)|_{B \setminus e} \alpha|_{B \setminus e}, \alpha|_{B \setminus e})$.

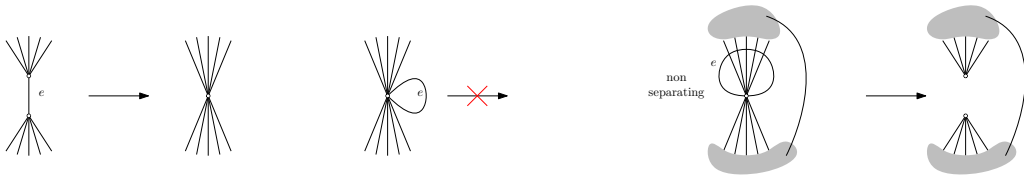


Figure 9: Contracting an edge e of a map \mathcal{M} . Note the special case of a non-separating loop. The number of edges decreases by 1, while the number of faces remains unchanged.

It is immediate from the definition that we have

$$\mathcal{M}/e = (\mathcal{M}^* \setminus e)^*.$$

As a consequence, if S is a quasi-tree of \mathcal{M} and $e \in S$ then $S \setminus e$ is a quasi-tree of \mathcal{M}/e .

Using the properties we know about quasi-trees and minors, we can easily prove that deletion and contraction commute:

Lemma 10. Let $e \neq f$ be edges of \mathcal{M} . If $\mathcal{M} \setminus e / f$ is defined then so is $\mathcal{M} / f \setminus e$ and we have $\mathcal{M} \setminus e / f = \mathcal{M} / f \setminus e$.

Proof. Let T be spanning tree of $\mathcal{M} \setminus e / f$. Then T is a quasi-tree of $\mathcal{M} \setminus e$ and $T \cup \{e\}$ is a quasi-tree of \mathcal{M} . As $f \notin T \cup \{e\}$ the edge f is contractible in \mathcal{M} and $T \cup \{e\}$ is a quasi-tree of \mathcal{M} / f . As $e \in T \cup \{e\}$, the edge e is erasable in \mathcal{M} / f and T is a quasi-tree of $\mathcal{M} / f \setminus e$. As the tours of T in $\mathcal{M} \setminus e / f$ and $\mathcal{M} / f \setminus e$ are the same, the two maps are the same. \square

Note that a similar proof can be used to prove that deletions commute and that contractions commute. From the properties of quasi-trees with respect to deletion and contraction, we deduce the following.

Lemma 11. Let $\zeta(\mathcal{M})$ be the number of quasi-trees of \mathcal{M} . Then ζ satisfies the following contraction/deletion formula

$$\zeta(\mathcal{M}) = \begin{cases} \zeta(\mathcal{M} / e) & \text{if } e \text{ is a bridge} \\ \zeta(\mathcal{M} \setminus e) & \text{if } e \text{ is a separating loop} \\ \zeta(\mathcal{M} / e) + \zeta(\mathcal{M} \setminus e) & \text{otherwise} \end{cases}$$

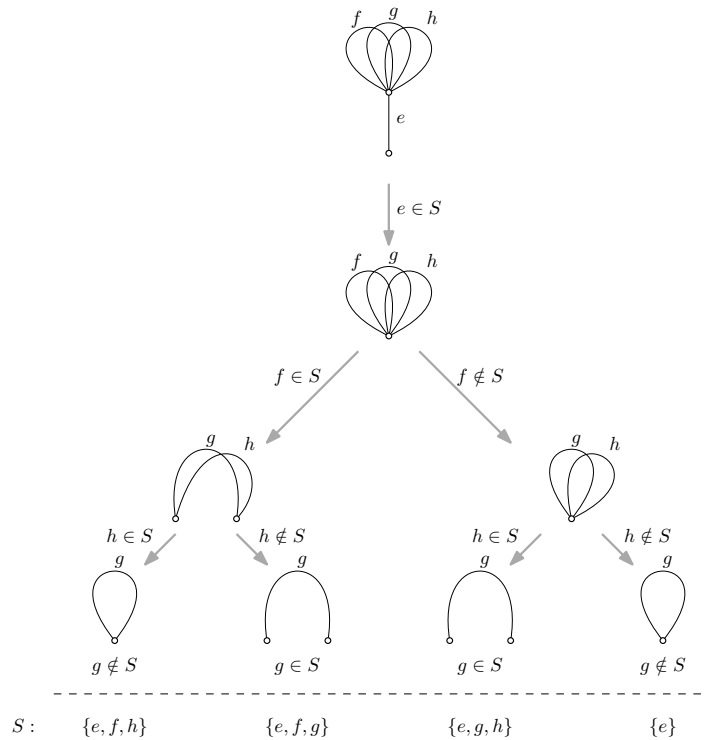


Figure 10: A map with 4 quasi-trees

Notice that a map \mathcal{M} defines two graphs, namely $G(\mathcal{M})$ and $G(\mathcal{M}^*)$. We noticed that edge deletion corresponds to a deletion in $G(\mathcal{M})$, thus edge contraction corresponds to a deletion in $G(\mathcal{M}^*)$.

For more on this subject and on the extension to maps (and, more generally, to ribbon graphs) of the Tutte polynomial, we refer the reader to [3] and to the survey [4].

5. Bicolored chord diagrams and the Δ -matroid of quasi-trees

Given a cyclic permutation τ , we define the *interlace chord diagram* $\Lambda(\tau)$ of τ as the chord diagram obtained by putting on a circle the flags in B in τ -order, the chords linking the pairs of flags in a same edge. Given a quasi-tree S of a map \mathcal{M} with tour τ , we define $\tilde{\Lambda}(\mathcal{M}, S)$ as the chord diagram $\Lambda(\tau)$ with chords in S colored 1 and those not in S colored 2 (see Figure 12). Every chord diagram is a representation of a *circle graph*, whose vertices are the chords and whose edges are the pairs of intersecting chords. It will be convenient to denote $\tilde{\Lambda}(\mathcal{M}, S)$ both the chord diagram and its associated circle graph.

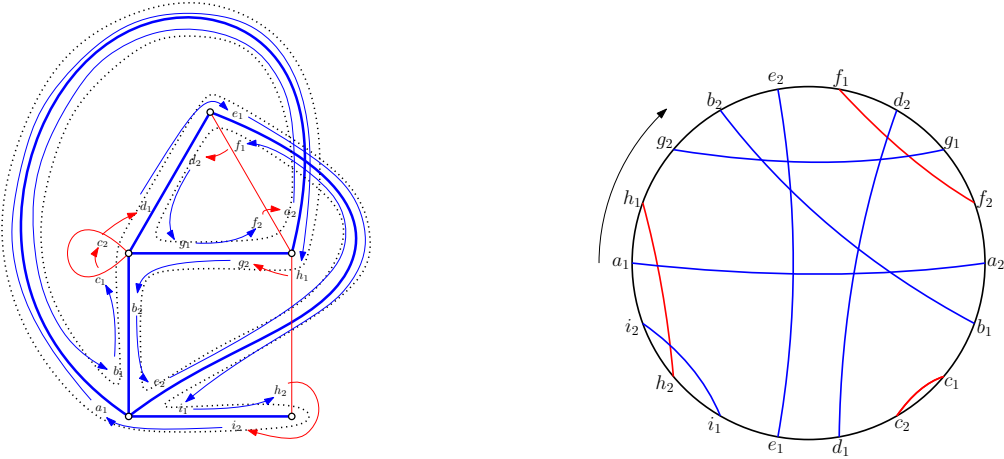


Figure 11: Tour of a quasi-tree and the corresponding bicolored chord diagram (color 1 is blue, color 2 is red).

We further define as $I(\mathcal{M}, S)$ the bipartite subgraph of $\tilde{\Lambda}(\mathcal{M}, S)$ induced by S and its complement. In other words, two edges are adjacent in $I(\mathcal{M}, S)$ if they are adjacent in $\tilde{\Lambda}(\mathcal{M}, S)$, one is in S and not the other. The property mentioned in the planar case extends in the general case:

Fact 12. Let \mathcal{M} be a map and let S be a spanning tree of \mathcal{M} . Then

- for each $e \in S$, the neighbors of e in $I(\mathcal{M}, S)$ are the elements of $\omega(e) \setminus \{e\}$;
- for each $f \notin S$, the neighbors of f in $I(\mathcal{M}, S)$ are the elements of $\gamma(f) \setminus \{f\}$.

Note that a direct consequence of Theorem 12 is that $I(\mathcal{M}, S)$ (hence $\tilde{\Lambda}(\mathcal{M}, S)$) is connected if \mathcal{M} is 2-connected.

It is well known that two spanning trees S and S' differ by two elements, i.e. $S' = S \Delta \{e, f\}$ if and only if e and f are adjacent in $I(\mathcal{M}, S)$, and that we have $I(\mathcal{M}, S') = I(\mathcal{M}, S) \wedge ef$. This property extends to the chord diagrams of quasi-trees (considering $\tilde{\Lambda}(\mathcal{M}, S)$ instead of $I(\mathcal{M}, S)$).

Lemma 13. Let S be a quasi-tree of a map \mathcal{M} and let e, f be distinct edges of \mathcal{M} that are adjacent in $\widetilde{\Lambda}(\mathcal{M}, S)$. Then $S' = S \triangle \{e, f\}$ is a quasi-tree of \mathcal{M} and $\widetilde{\Lambda}(\mathcal{M}, S')$ is obtained from $\widetilde{\Lambda}(\mathcal{M}, S) \wedge ef$ by flipping the colors of e and f .

Conversely, if $S' = S \triangle \{e, f\}$ is a quasi-tree of \mathcal{M} , then e and f are adjacent in $\widetilde{\Lambda}(\mathcal{M}, S)$.

Proof. Let τ be the tour of S in \mathcal{M} , and let e_1, f_1, e_2, f_2 be the flags in e and f (following the τ -order). The cycle τ then rewrites as $(w_1, e_1, w_2, f_1, w_3, e_2, w_4, f_2)$, where w_1, w_2, w_3 , and w_4 are sequences of flags. It is easily checked that the tour τ' of S' is $(w_1, e_2, w_4, f_1, w_3, e_1, w_2)$. In particular, τ' is a cycle thus S' is a quasi-tree of \mathcal{M} .

Assume that e and f are not adjacent in $\widetilde{\Lambda}(\mathcal{M}, S)$. Let e_1, e_2, f_1, f_2 be the flags in e and f (following the τ -order). The cycle τ then rewrites as $(w_1, e_1, w_2, e_2, w_3, f_1, w_4, f_2)$, where w_1, w_2, w_3 , and w_4 are sequences of flags. Then, it is easily checked that the tour τ' of S' has 3 cycles, namely (w_1, e_1, w_3, f_1) , (w_2, e_2) and (w_4, f_2) . Thus, S' is not a quasi-tree of \mathcal{M} . \square

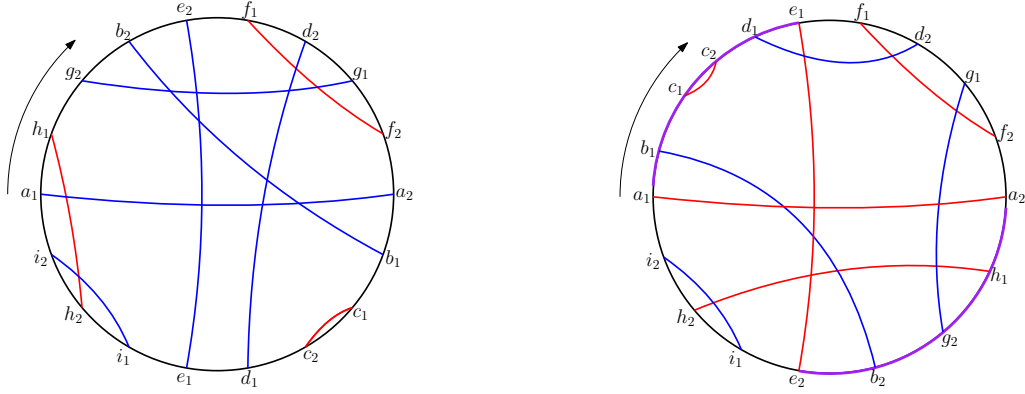


Figure 12: Pivoting ae transforms the chord diagram of the quasi-tree $\{a, b, d, e, g, i\}$ into the one of the spanning-tree $\{b, d, g, i\}$.

Lemma 14. For every quasi-tree S of \mathcal{M} , there exist quasi-trees S_0 and S_1 of \mathcal{M} with $S_0 \subseteq S \subseteq S_1$, S_0 is a spanning tree of \mathcal{M} , and S_1 is the complement of a spanning tree of \mathcal{M}^* .

Proof. By duality, it is sufficient to prove the existence of S_0 . Assume that S contains two edges e, f that are adjacent in $\widetilde{\Lambda}(S)$. Then $S' = S \triangle \{e, f\} \subseteq S$ is a quasi-tree of \mathcal{M} . So, assume that S does not contain two edges e, f that are adjacent in $\widetilde{\Lambda}(\mathcal{M}, S)$. By deleting all the edges not in S , we get a map \mathcal{M}' , where $G(\mathcal{M}')$ is a spanning subgraph of $G(\mathcal{M})$ induced by the edges in S . As this map has a single face and no interlaced edges, we deduce that \mathcal{M}' is a tree, hence S is a spanning tree of \mathcal{M} . \square

The next lemma is an easy consequence of our study of minors.

Lemma 15. Let S be a quasi-tree of a map \mathcal{M} .

- If $e \in S$, then $\widetilde{\Lambda}(\mathcal{M} / e, S \setminus e) = \widetilde{\Lambda}(\mathcal{M}, S) - e$;
- If $e \notin S$, then $\widetilde{\Lambda}(\mathcal{M} \setminus e, S) = \widetilde{\Lambda}(\mathcal{M}, S) - e$.

We now state an analog of the edge exchange property of spanning trees, which implies that the set of all quasi-trees of a map forms an *even Δ -matroid* [2].

Lemma 16. Let $S_1 \neq S_2$ be two quasi-trees of a map \mathcal{M} . Then, for every $e \in S_1 \Delta S_2$ there exists $f \in S_1 \Delta S_2$ with $f \neq e$, such that $S_1 \Delta \{e, f\}$ is a quasi-tree of \mathcal{M} .

Proof. By contracting all the edges in $S_1 \cap S_2$ and deleting those in the complement of $S_1 \cup S_2$, we reduce to the case where (S_1, S_2) is a partition of the edge set. Note that \mathcal{M} has no bridges and no separating loops. Let τ_1 be the tour of S_1 in \mathcal{M} . If there exists $f \neq e$ such that ef is an edge of $\widetilde{\Lambda}(\mathcal{M}, S_1)$ we are done. We prove by contradiction that no other case can occur.

Assume $e \in S_1$. Let $\tau_1 = (w_1, e_1, w_2, e_2)$. Note that w_1 and w_2 are not empty. Let $\mathcal{M} \setminus e = (B \setminus e, \sigma', \alpha')$. Let $b \in w_2$. By assumption, $\alpha(b) \in w_2$. Assume $b \notin S_1$. If b is not the last flag of w_2 then $\sigma'(b) = \sigma(b) \in w_2$. Otherwise, $\sigma'(b) = \sigma^2(b) = \sigma(e_2) = \sigma\alpha(e_1) \in w_2$. Assume $b \in S_1$ (hence $\alpha(b) \in S_1$). If $\alpha(b)$ is not the last flag of w_2 (i.e. $\sigma(b) \neq e_2$) then $\sigma'(b) = \sigma\alpha(\alpha(b)) \in w_2$. Otherwise, $\sigma'(b) = \sigma^2(b) = \sigma(e_2) = \sigma\alpha(e_1) \in w_2$. In all cases, $\sigma'(b) \in w_2$. Thus, we get that w_2 is closed under the action of $\langle \sigma', \alpha' \rangle$, contradicting the hypothesis that e is not a bridge (i.e. that $\mathcal{M} \setminus e$ is a map).

Assume $e \notin S_1$. Let $\tau_1 = (w_1, e_1, w_2, e_2)$. Note that w_1 and w_2 are not empty. Let $\mathcal{M} / e = (B \setminus e, \sigma', \alpha')$. Let $b \in w_2$. By assumption, $\alpha(b) \in w_2$. Assume $b \in S_1$. If b is not the last flag of w_2 then $\sigma'\alpha'(b) = \sigma\alpha(b) \in w_2$. Otherwise, $\sigma'\alpha'(b) = \sigma\alpha\sigma\alpha(b) = \sigma\alpha(e_2) = \sigma(e_1) \in w_2$. Assume $b \notin S_1$. If $\alpha(b)$ is not the last flag of w_2 (i.e. $\sigma(b) \neq e_2$) then $\sigma'\alpha'(b) = \sigma\alpha(b) \in w_2$. Otherwise, $\sigma'\alpha'(b) = \sigma\alpha\sigma\alpha(b) = \sigma\alpha(e_2) = \sigma(e_1) \in w_2$. In all cases, $\sigma'\alpha'(b) \in w_2$. Thus, we get that w_2 is closed under the action of $\langle \sigma'\alpha', \alpha' \rangle$, contradicting the hypothesis that e is not a separating loop (i.e. that \mathcal{M} / e is a map). \square

Corollary 17. Let $S_1 \neq S_2$ be two quasi-trees of a map \mathcal{M} . Then there exists a sequence $(\{e_1, f_1\}, \dots, \{e_k, f_k\})$ of pairs of edges such that $S_2 = S_1 \Delta \{e_1, f_1\} \Delta \dots \Delta \{e_k, f_k\}$ and, for all $1 \leq i < k$, $S_1 \Delta \{e_1, f_1\} \Delta \dots \Delta \{e_i, f_i\}$ is a quasi-tree of \mathcal{M} .

Remark 18. In this setting, the bridges (resp. the separating loops) of \mathcal{M} are the *coloops* (resp. the *loops*) of the Δ -matroid of the quasi-trees of \mathcal{M} , meaning that bridges are exactly those edges that belong to all quasi-trees (Theorem 8) and, dually, separating loops are exactly those edges that belong to no quasi-tree.

Also, as a consequence of Theorems 13 and 17, the isolated vertices of $\widetilde{\Lambda}(\mathcal{M}, S)$ are exactly the bridges and the separating loops of \mathcal{M} .

6. Representation of maps using quasi-trees

We extend the bijection between bipartite circle graphs and planar graphs shown Figure 2 to a bijection between bicolored chord diagrams and pairs (\mathcal{M}, S) , where \mathcal{M} is a map and S is a quasi-tree of \mathcal{M} .

We now present how to derive a representation of a map \mathcal{M} from the choice of a quasi-tree S of \mathcal{M} . Let g be the genus of \mathcal{M} and let g_S be the genus of S .

Recall that an orientable surface of genus $g > 1$ can be represented using a polygon with $4g$ sides (the *Fricke canonical polygon* or *fundamental polygon* of the surface), where the sides are matched in distinct pairs with opposite orientation (see Figure 13 and, e.g. [17]).

In our representation of maps, we use up to two polygons. Precisely, we fix a circle Γ , a polygonal Π_i inside Γ with $4g_S$ sides and a polygon Π_e enclosing Γ with $4(g - g_S)$ sides. (If

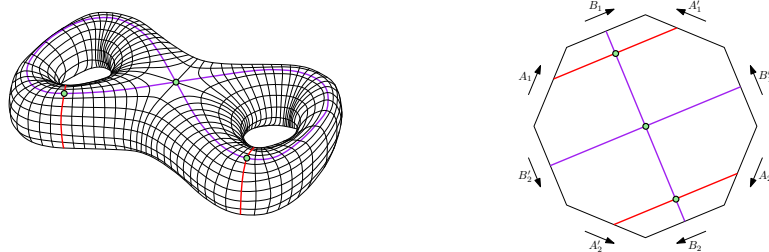


Figure 13: Representation of a double torus by an octagon.

$g_s = 0$ we don't need Π_i and if $g = g_s$ we don't need Π_e .) Each of these two polygons has his sides matched by distinct pairs, as in Fricke representation. We put on Γ the points of the chord diagram $\widetilde{\Lambda}(\mathcal{M}, S)$ and draw (without crossings, using the fundamental polygons) the chords of S inside Γ and the other chords outside Γ . These chords are drawn in such a way that they do not cross Γ (see Figure 14).

If one contracts Γ into a single point, our drawing defines two maps: the exterior map with edges "outside" Γ , which has genus g_s , and the interior map with edges "inside" Γ , which has genus $g - g_s$. Note that each of these maps has a single vertex (corresponding to Γ). We now draw the dual of the interior map in our representation, inside Γ . Note that the obtained map has a number of vertices equal to $|S| - 2g_s + 1$. Then, we connect each point of Γ corresponding an endpoint of a chord not in S to the vertex that can be reached without crossing any other edge.

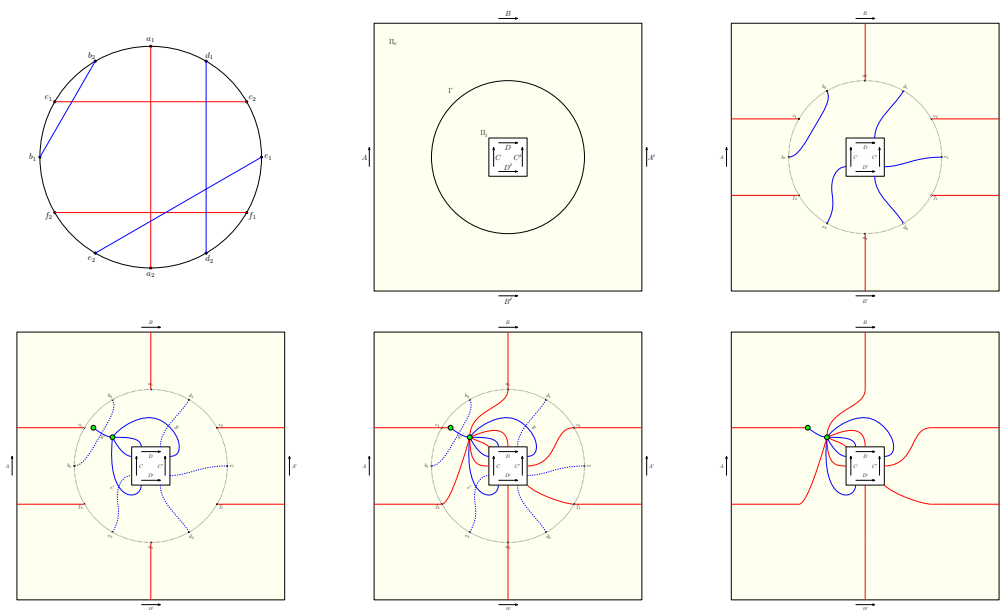


Figure 14: A bicolored chord diagram (top left), a representation of the double torus using two squares (center), and a drawing of the chord diagram without crossings (top right). Dualization of the interior map (bottom left), connection to the endpoints of the chords not in S (bottom center), and final drawing of the map (and quasi-tree) associated to the bicolored chord diagram (bottom right).

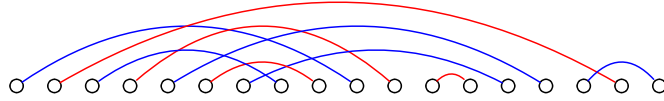
7. Rooted maps, Ordered matchings, and Double occurrence words

Recall that rooting a map $\mathcal{M} = (B, \sigma, \alpha)$ consists in selecting a root flag $b_\bullet \in B$. Note that rooting a map kills all the automorphisms of the map, as every flag can be written as $\mu(b_\bullet)$, for some $\mu \in \langle \sigma, \alpha \rangle$. It will be convenient to denote \mathcal{M}_\bullet the map \mathcal{M} rooted at b_\bullet .



Figure 15: A rooted map \mathcal{M}_\bullet . The root flag is either indicated using a cross placed just (anti-clockwise) before the root flag (here a_1) at the incident vertex (on the left), or by orienting the corresponding edge from the vertex incident to the root flag (on the right).

Similarly, rooting a chord diagram consists in selecting a root chord endpoint. Rooting $\tilde{\Lambda}(\mathcal{M}, S)$ and reading all the flags from the root, while traversing the circle clockwise, we obtain a linear order on B or, equivalently, a *single occurrence word*, that is a word whose letters are the flags, which contains every flag exactly once. This way, we can identify rooted bicolored chord diagrams and bicolored ordered matchings. For example, the bicolored chord diagram depicted on the left of Figure 12 and rooted at a_1 corresponds to the bicolored ordered matching



Lemma 19. The following objects are in bijection:

- rooted bicolored chord diagrams with m chords,
- bicolored ordered matchings of size $2m$,
- pairs (\mathcal{M}_\bullet, S) , where \mathcal{M}_\bullet is a rooted map with m edges and S is a quasi-tree of \mathcal{M}_\bullet .

Proof. The chord diagram is defined by τ , which can be computed from \mathcal{M} and S . The bicolouration is given by S . Conversely, from the bicolored chord diagram, we can define α (by the chords) and σ (from τ , α and the bicolouration). \square

Corollary 20. The sum of $\zeta(\mathcal{M}_\bullet)$ over all rooted maps \mathcal{M}_\bullet with m edges is $\frac{(2m)!}{m!}$.

Proof. This sum is 2^m times the number of ordered matchings of size $2m$. \square

The *pivoting class* of a chord diagram Λ is the class of all chord diagrams that can be reached from Λ by a sequence of pivoting operations.

Lemma 21. The number of distinct pivoting classes of rooted chord diagrams with m chords equals the number of rooted maps with m edges, and the number $\zeta(\mathcal{M})$ of quasi-trees of a rooted map \mathcal{M}_\bullet is the size of the pivoting class associated to \mathcal{M}_\bullet .

Applying the transformation $b \mapsto \underline{b}$, which maps each flag to the edge it belongs to, we transform the single occurrence word w defined by the tour of a quasi-tree into a word, whose letters are edges, which uses every edge exactly twice. We call such a word a *double occurrence word*. When using the above labeling scheme, a tour of a rooted map is fully determined by the

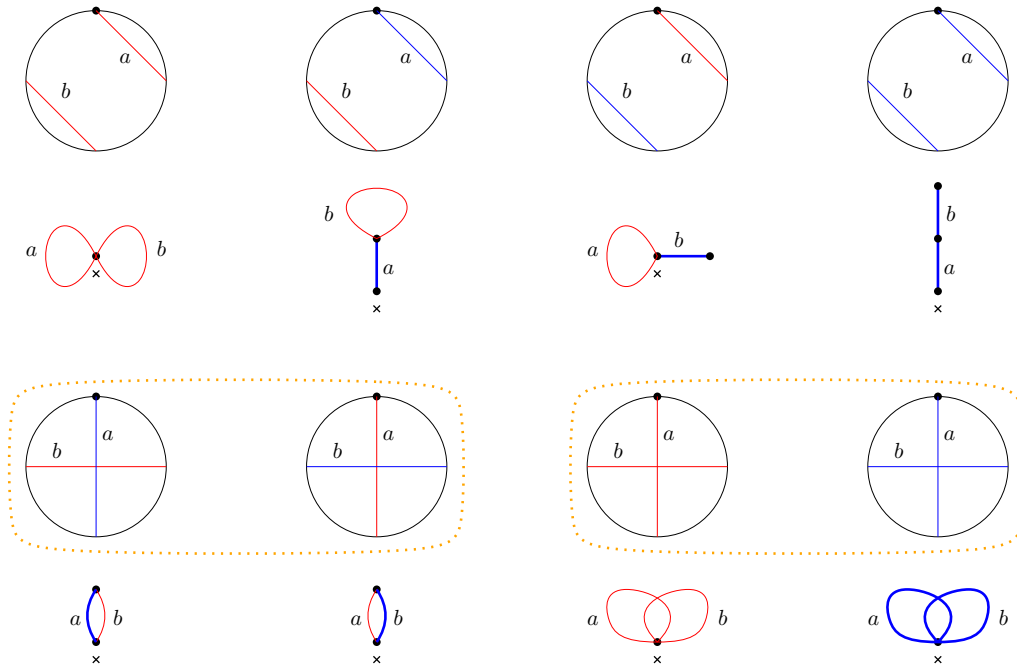


Figure 16: Bijections in the case $m = 2$. There are 6 rooted maps and 8 pairs (M_\bullet, S) , where M_\bullet is a rooted map and S is a quasi-tree of M . The set S is in blue and dotted curves delineate the pivoting classes.

associated double occurrence word. A natural question is whether we can use a standard quasi-tree, which could also be recovered from the mere double occurrence word. In full generality, it is obvious that the answer is negative, as the double occurrence word e, e corresponds to two distinct maps, one with a single edge linking two vertices (with quasi-tree $\{e\}$) and one with a single loop attached to a vertex (with empty quasi-tree). However, if we restrict ourselves to loopless maps, we shall prove that some particular quasi-tree (actually, some specific spanning tree) can be recovered from its associated double occurrence word and that, moreover, the admissible double occurrence words can be characterized.

Note that the circle graph defined by a double occurrence word w gets a natural orientation: the edge ef of the circle graph is oriented from e to f if $e f e f$ is a subword of w . We denote by $\vec{\Lambda}(w)$ the oriented circle graph defined by the single occurrence word w (see Figure 17).

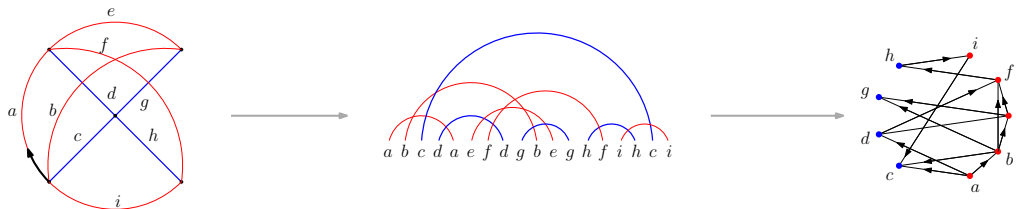


Figure 17: From a spanning tree of a rooted map (on the left) to a directed circle graph (on the right), via a double occurrence word (in the middle).

8. Depth-First Search trees

The *depth-first search* (DFS) is traversal procedure of a connected graph G that produces a rooted spanning tree T of G with the property that the two incidences of every edge of G lie in a leaf-to-root path of T (*Trémaux property*). It will be convenient to describe this procedure on a rooted combinatorial map (\mathcal{M}, b_0) . The DFS starts at the vertex incident to b_0 , which is the *root* of T , and explores the graph as far as possible before backtracking. In order to formalize the DFS, we introduce some terminology and notations. It is convenient to introduce a dummy flag b^0 , inserted just before b_0 (i.e. $\sigma(b^0) = b_0$). The current position of the search is a flag b , initially set to b_0 ; We denote by M_V (resp M_B) the set of *visited vertices* (resp. of *visited flags*). Initially, $M_B = \{b^0\}$ and $M_V = \{r\}$, where r is the vertex incident to b_0 . To each non-root visited vertex $v \in M_V \setminus \{r\}$ will be associated a *discovery flag* b_v , which will be the flag incident to v in the edge linking v to its parent in the constructed spanning tree (or b^0 if v is the root). The DFS can be described as the repetition of the following two steps:

- we look for a non-visited flag incident to the current vertex. If such a flag exists, this becomes the current flag b (which is now visited and consequently added to M_B); otherwise, if the current vertex is not the root, we backtrack, meaning that the current flag becomes the discovery flag of v (which is now visited and consequently added to M_B); otherwise (i.e. if all the flags incident to the current vertex have been visited and the current vertex is the root), the procedure ends.
- if the opposite flag $b' = \alpha(b)$ of the current flag b is not incident to a visited vertex, then we traverse the edge, meaning that the current vertex v is now the vertex incident to b' (which is now visited and consequently added to M_V), the discovery flag of this vertex is set to b' , and the current flag is b' (which is now visited and consequently added to M_B).

Notice that in this procedure we do not precise how an non-visited flag is chosen when multiple choices are possible. In this sense, the DFS is not (formally speaking) a fully deterministic procedure.

When dealing with combinatorial map, there are two obvious ways of making the above algorithm fully deterministic, by always selecting the first possible choice or by always selecting the last possible choice. This way, we construct two special DFS-tree, the *Early DFS-tree* and the *Late DFS-tree* of the rooted map. Note that by reversing the orientation of the map, that is by considering the map (B, σ^{-1}, α) instead of (B, σ, α) , the Early DFS-tree (resp. the Late DFS-tree) becomes the Late DFS-tree (resp. the Early DFS-tree).

Given a rooted map \mathcal{M}_\bullet and a spanning tree S of \mathcal{M}_\bullet , it is easily checked whether S has the Trémaux property, i.e. whether S is a DFS-tree.

Lemma 22. Let S be a spanning tree of a rooted map \mathcal{M}_\bullet , and let \underline{w} be the associated double occurrence word.

Then S has the Trémaux property if and only if no f has both (in $\vec{\Lambda}(w)$) an in-neighbor and an out-neighbor in S . In other words S has the Trémaux property if and only if w does not contain the pattern $e_1 f e_1 e_2 f e_2$, with $e_1, e_2 \in S$.

Proof. First note that if an edge f has a neighbor in S , then $f \notin S$. Assume S is a DFS-tree. Contracting tree edges and deleting cotree edges preserves the property of being a DFS-tree. This allows us to reduce to the case where M is a cycle. Then either $w = f e_1 \dots e_k f e_k \dots e_1$ or $w = e_1 \dots e_k f e_k \dots e_1 f$.

Conversely, assume that for every $f \notin S$ either all the S -neighbors of f in $\vec{\Lambda}(w)$ are in-neighbors of f or all the S -neighbors of f in $\vec{\Lambda}(w)$ are out-neighbors of f and assume for contradiction that $E \setminus S$ contains a transversal edge $f = uv$. Let x be the least common ancestor of u and v in S , and let e_1, \dots, e_k (resp. e'_1, \dots, e'_ℓ) be the path linking x and u (resp. x and v) in S . Again, we can reduce to case where M is the cycle $\gamma(f)$. Then we get $w = e_1 \dots e_k f e_k \dots e_1 e'_1 \dots e'_\ell f e'_\ell \dots e_1$ (up to exchange between the paths $e_1 \dots e_k$ and $e'_1 \dots e'_\ell$). Hence, f has both in-neighbors and out-neighbors in S , contradicting our hypothesis. \square

Lemma 23. Let S be a spanning tree of a rooted map \mathcal{M}_\bullet , and let \underline{w} be the associated double occurrence word. Then,

- S is the Early DFS of \mathcal{M}_\bullet if every $e \in S$ is a source of $\vec{\Lambda}(w)$ (or, equivalently, if w does not contain the pattern $f e f e$ with $e \in S$ and $f \notin S$);
- S is the Late DFS of \mathcal{M}_\bullet if every $e \in S$ is a sink of $\vec{\Lambda}(w)$ (or, equivalently, if w does not contain the pattern $e f e f$ with $e \in S$ and $f \notin S$).

We now deduce a duality results about DFS-trees of planar maps (see Figure 18).

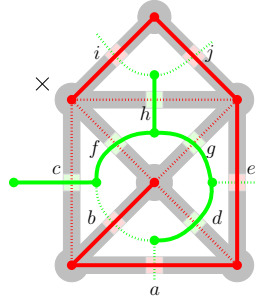


Figure 18: Illustration of the duality between the Early DFS of a planar map and the Late DFS of its dual. Note that because of our definition of the duality of maps, the Early DFS of the primal map \mathcal{M} corresponds to a DFS giving precedence to the leftmost admissible edge. Re-rooting the dual DFS-tree at $\sigma^{-1}(b_\bullet)$ we get also a DFS giving precedence to the leftmost admissible edge.

Corollary 24. Let S be a DFS-tree of a rooted 2-connected planar map \mathcal{M}_\bullet . Then the complement \bar{S} of S is a DFS-tree of \mathcal{M}_\bullet^* (the dual map rooted at the same flag b_\bullet) if and only if

- either S is the Early DFS of \mathcal{M}_\bullet and \bar{S} is the Late DFS of \mathcal{M}_\bullet^* ,
- or S is the Late DFS of \mathcal{M}_\bullet and \bar{S} is the Early DFS of \mathcal{M}_\bullet^* .

Proof. As \mathcal{M} is planar, \bar{S} is a spanning tree of \mathcal{M}^* .

Assume S is the Early (resp. the Late) DFS of \mathcal{M}_\bullet . As the double occurrence word associated to the spanning tree S of \mathcal{M}_\bullet is the same as the double occurrence word associated to the quasi-tree \bar{S} of \mathcal{M}_\bullet^* , we deduce from Theorem 23 that \bar{S} is the Late (resp. the Early) DFS of \mathcal{M}_\bullet^* .

Assume \bar{S} is a DFS-tree of \mathcal{M}_\bullet^* . According to Theorem 22, for every $f \notin S$, either all the S -neighbors of f in $\vec{\Lambda}(w)$ are in-neighbors of f (we say that f has type 1) or all the S -neighbors of f in $\vec{\Lambda}(w)$ are out-neighbors of f (we say that f has type 2). Dually, for every $e \in S$, either all the \bar{S} -neighbors of e in $\vec{\Lambda}(w)$ are in-neighbors of e (we say that e has type 2) or all the \bar{S} -neighbors of

f in $\vec{\Lambda}(w)$ are out-neighbors of f (we say that e has type 1). Let $\vec{\Gamma}$ be the bipartite sub-digraph of $\vec{\Lambda}(w)$ induced by the parts S and \bar{S} . It is immediate that the neighbors of an edge of type i in $\vec{\Gamma}$ have also the type i . As \mathcal{M} is 2-connected, the graph $\vec{\Gamma}$ is connected (as easily follows from Theorem 12). Hence, all the edges have the same type. According to Theorem 23, we deduce that either S is the Early DFS of \mathcal{M}_\bullet (and \bar{S} is the Late DFS of \mathcal{M}_\bullet^*) or S is the Late DFS of \mathcal{M}_\bullet (and \bar{S} is the Early DFS of \mathcal{M}_\bullet^*). \square

9. The Quasi-tree poset

One can define a poset on the set of all the quasi-trees of a rooted map as follows: to each pair (w, I) , where w is a double occurrence word and I is a subset of letters of w we associate the binary word $b(w, I)$ obtained by extracting from w the subword formed by all the first occurrences of the letters, replacing each letter not in I by 0 and all letters in I by 1. Denote by $<_{\text{bin}}$ the lexicographic order on the binary strings. Then, a quasi-tree S of \mathcal{M} is smaller than another quasi-tree S' of \mathcal{M} if there exists a sequence $S_0 = S, S_1, \dots, S_k = S'$ of quasi-trees of \mathcal{M} , where $|S_i \Delta S_{i-1}| = 2$ and $b(w(S_{i-1}), S_{i-1}) <_{\text{bin}} b(w(S_i), S_i)$ (for $1 \leq i \leq k$), where $w(S)$ denotes the first-occurrence word defined by the quasi-tree S . Note that one interpret this poset as a partial order on a pivoting class of (rooted) circle graphs.

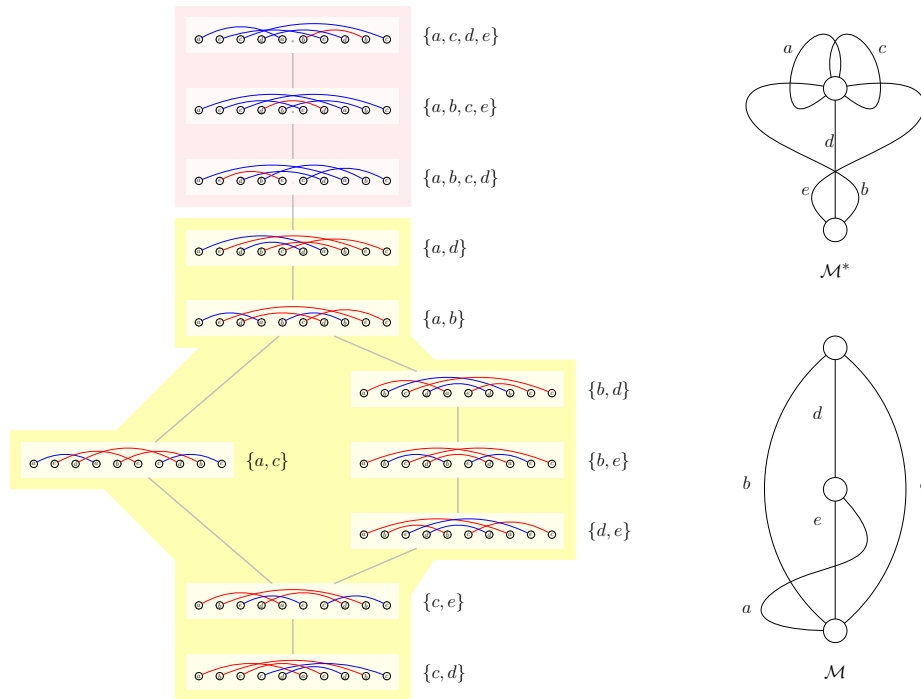


Figure 19: The poset of the quasi-trees of a rooted map. The yellow zone corresponds to spanning trees, while the pink zone corresponds to quasi-trees with genus 1. The minimum element of the poset is the Late DFS-tree, while the maximum one is the complement of the Late DFS-tree of the dual rooted map.

As the Late DFS-tree is the minimum of this poset, it is a good candidate to serve as a standard quasi-tree. An immediate property of this tree T is that no tree edge is interlaced on the right. In

other words, tree edges are sinks of $\vec{\Lambda}(w)$. On the other hand, if an edge is not in the tree, it is interlaced with at least one tree edge, as we assumed that the map is loopless. Thus, the set of sinks of $\vec{\Lambda}(w)$ dominates $\vec{\Lambda}(w)$.

Lemma 25. Double occurrence words w with $|w| = 2m$ such that the set of sinks of $\vec{\Lambda}(w)$ dominates $\vec{\Lambda}(w)$ are in bijection with rooted loopless maps with m edges.

Example 26. There are 14 orientable rooted loopless maps with 3 edges, which correspond to the double occurrence sequences distinct from $abacbc$ (see Figure 20).

$w \backslash I$	\emptyset	$\{a\}$	$\{b\}$	$\{c\}$	$\{a, b\}$	$\{a, c\}$	$\{b, c\}$	$\{a, b, c\}$
aabbcc								
aabcbe								
aabccb								
ababcc								
abacbc								
abaacb								
abbacc								
abbcaac								
abbceca								
abcabc								
abcacb								
abcbac								
abcbca								
abccab								
abccba								

Figure 20: Maps defined by double occurrence words on 3 symbols. In $\vec{\Lambda}(abacbc)$, the only sink is c and it does not dominate a .

Theorem 25 can easily be specialized to planar maps.

Lemma 27. Double occurrence words w with $|w| = 2m$ every vertex of $\vec{\Lambda}(w)$ is a source or a sink (i.e. such that no symbol of w is interlaced both on the left and on the right) are in bijection with rooted loopless planar maps with m edges.

Proof. Consider the bijection between double occurrence words w with $|w| = 2m$ such that the set of sinks of $\vec{\Lambda}(w)$ dominates $\vec{\Lambda}(w)$ and rooted loopless maps with m edges (see Theorem 25). In this bijection, the set of all vertices $T(w)$ of $\vec{\Lambda}(w)$ that have no out-neighbors forms a spanning tree of the map. Thus, the map associated to a word w is planar if and only if the partition $(T(w), V(\vec{\Lambda}(w)) \setminus T(w))$ is a partition into two independent sets, that is if and only if no two symbols of w with an out-neighbor in $\vec{\Lambda}(w)$ are adjacent in $\vec{\Lambda}(w)$. This is clearly equivalent to the property that no vertex has both an in-neighbor and an out-neighbor. \square

10. Counting Loopless Rooted Maps

A 1-2 *occurrence word* is a word in which each letter appears either once or twice. Let w be an 1-2 occurrence word. The symbols that appear only once in w are the *unmatched* symbols of w , and they form the set $U(w)$. The symbols that appear twice are the *matched* symbols of w . Removing the unmatched symbols from w , we get the double occurrence word w° . An unmatched symbol a is *covered* by a matched symbol b in w if $b a b$ is a pattern of w .

We define two properties for words:

- Property P: a word w has property P if it is a double occurrence word such that the sinks of $\vec{\Lambda}(w)$ form a dominating set; in other words, every symbol interlaced on the right (non-sink) is interlaced on the right by a symbol that is not interlaced on the right.
- Property Q: a word w has property Q if it is a 1-2 occurrence word, w° has property P, and no unmatched symbol of w is covered in w by a sink of $\vec{\Lambda}(w)$ (i.e. by a matched symbol of w not interlaced on the right).

Note that property P obviously implies property Q and that theorem 25 expresses that rooted loopless maps with n edges are in bijection with the double occurrence words with n symbols that satisfy the property P.

Lemma 28. Let w be a 1-2 occurrence word.

Then w has property Q if and only if

- either w is empty,
- or $w = w' a$, where a is an unmatched symbol of w and w' has the property Q,
- or $w = w_1 a w_2 a$, where no unmatched symbol of w is in w_2 , w_2° has property P, and w_1 has property Q.

Moreover, such a decomposition, if it exists, is unique.

Proof. First, notice that the uniqueness of the decomposition (when it exists) is straightforward. We prove the statement by induction on the number of symbols in w . If w is empty, the statement is obviously satisfied. If the last symbol a of w is unmatched in w (so $w = w' a$) it is immediate that w has property Q if and only if w' has property Q. So we can assume that w decomposes as $w = w_1 a w_2 a$.

Assume w has property Q. As a is not interlaced on the right (i.e. is a sink of $\vec{\Lambda}(w)$) we get that no symbol in w_2 is unmatched in w . That w_1° and w_2° have property P follows directly from the hypothesis that w° has property Q. Finally, consider an unmatched symbol f of w_1 and assume for contradiction that f is covered by a matched symbol e of w_1 that is not interlaced on the right in w_1 (i.e. is a sink of $\vec{\Lambda}(w_1)$). Assume first that e is interlaced on the right in w by some matched symbol g of w (i.e. that e is not a sink of $\vec{\Lambda}(w)$). Then g is interlaced on the right by a in w . As this holds for all possible choices of g it follows that e is not dominated by a sink in $\vec{\Lambda}(w)$, contradicting the assumption that w has property Q. Hence, e is a sink of $\vec{\Lambda}(w)$. As w has property Q we get that f is matched in w . But then e is interlaced on the right by f in w , contradicting the fact that e is a sink of $\vec{\Lambda}(w)$. It follows that w_1 has property Q.

Conversely, assume that $w = w_1 a w_2 a$, no unmatched symbol of w is in w_2 , w_2° has property P, and w_1 has property Q. We first prove the next claim, which determines the unmatched symbols of w .

▷ *Claim 1.* A symbol e is a sink of $\vec{\Lambda}(w)$ if and only if it is a sink of $\vec{\Lambda}(w_1)$ or $\vec{\Lambda}(w_2)$.

Proof of the claim. The left to right implication is straightforward, so we are left with the right to left implication.

The symbol a is not interlaced on the right, thus is a sink of $\vec{\Lambda}(w)$. Every source of $\vec{\Lambda}(w_2)$ is not interlaced on the right in w_2 so cannot be interlaced on the right in w (as the first occurrences of matched symbols of w that are unmatched in w_2 are on the left of w_2) hence are sinks of $\vec{\Lambda}(w)$. Consider a sink e of $\vec{\Lambda}(w_1)$. If e is not a sink of $\vec{\Lambda}(w)$, then it covers the first occurrence of a matched symbol f of w , whose second occurrence is in w_2 . But then e covers f in w_1 and f is unmatched in w_1 , contradicting the assumption that w_1 has property Q. ◁

Consider a matched symbol f that is not a sink of w . If both occurrences of f belong to w_1 (resp. to w_2) then f is interlaced on the left by a sink of $\vec{\Lambda}(w_1)$ (resp. a source of $\vec{\Lambda}(w_2)$), which is (by above claim) a source of $\vec{\Lambda}(w)$. Otherwise, if the first occurrence of f belongs to w_1 and the second to w_2 , then f is interlaced on the right by the sink a of w . Altogether, we get that w° has property P.

Assume for contradiction that some unmatched symbol of w is covered in w by some sink e of $\vec{\Lambda}(w)$. According to the assumptions, as f is unmatched in w , it belongs to w_1 . The sink e of $\vec{\Lambda}(w)$, having its first occurrence in w_1 cannot be a or a sink of $\vec{\Lambda}(w_2)$. Hence, by the above claim, it is a sink of $\vec{\Lambda}(w_1)$. It follows that f is covered in w_1 by a sink of $\vec{\Lambda}(w_1)$ contradicting the assumption that w_1 has property Q. ◻

Let $G_{n,m}$ denote the number of 1-2 occurrence words with n matched symbols and m unmatched symbols that satisfy the property Q if n, m are both non-negative integers, and let $G_{n,m} = 0$ otherwise.

Lemma 29. For every $n \geq 1$ and $m \geq 0$ we have $G_{0,m} = 1$ and

$$G_{n,m} = G_{n,m-1} + \sum_{i=0}^{n-1} \sum_{j=0}^{n-1-i} \frac{(2i+j)!}{(2i)!} \binom{m+j}{j} G_{i,0} G_{n-1-i-j,m+j} \quad (3)$$

Proof. This is a direct consequence of theorem 28: every 1-2 occurrence word w with n matched symbol and m unmatched symbols decomposes either as $w' a$, where w' as n matched symbols and $m-1$ unmatched symbols, or is obtained (by $w = w_1 a w_2 a$), for some pair (i, j) of non-negative integers with $i+j \leq n-1$, from a double occurrence word with property P with i symbols (w_2°) and an 1-2 occurrence word w_1 with $n-1-(i+j)$ matched symbols and $m+j$ unmatched symbols with property Q (w_1) by inserting j place-holders in w_2° to match j unmatched symbols of w_1 (this gives $\binom{2i+j}{j}$ choices), selecting in w_1 the j symbols that will be matched in w_2 ($\binom{m+j}{j}$ choices) and choosing a matching between these symbols and the place-holders in w_2 ($j!$ choices). ◻

To compute values of $G_{n,0}$ it may be helpful to introduce $M_{n,m} = m! G_{n,m}$. (Note that $G_{n,0} = M_{n,0}$.) Then $M_{0,m} = m!$ and we have the following recurrence:

$$M_{n,m} = m M_{n,m-1} + \sum_{i=0}^{n-1} \sum_{j=0}^{n-1-i} \binom{2i+j}{j} M_{i,0} M_{n-1-i-j,m+j}.$$

Using this recurrence, we computed the values of $G_{n,0}$ for $1 \leq n \leq 20$, namely:

1, 3, 14, 87, 672, 6204, 66719, 820395, 11370212, 175583880, 2992513416, 55838871492, 1132934744671, 24846387327825, 585953052416226, 14791975514747882, 398109420366969728, 11382340640393570304, 344600158836813725696, 11015256001205535506432.

Let $s = 2n + m, t = n + m$, and $N_{s,t} = M_{s-t,2t-s}$. Then we have

$$N_{s,t} = (2t - s)N_{s-1,t-1} + \sum_{k=0}^{s-t-1} \sum_{\ell=k}^{2k} \binom{\ell}{k} N_{2(\ell-k),\ell-k} N_{s-2-\ell,t-1}.$$

11. Back to the Planar Case

Interestingly, a variant of the problem of enumerating rooted maps in which only separating loops are forbidden was solved by Walsh and Lehman [24] for any given genus. As every loop of a planar map is separating, it results from their work that the number of loopless rooted planar maps with n edges is given by the closed expression $\frac{2(4n+1)!}{(n+1)!(3n+2)!} = \binom{4n+1}{n+1} - 9\binom{4n+1}{n-1}$ (sequence A000260 of the On-line encyclopedia of integer sequences). We refer the interested reader to [1] for more results on enumerations of loopless planar maps.

In this section, we consider how this problem is connected to the bijections we defined in the general case. We define two new properties for words:

- Property N: a word w has property N if it is a double occurrence word such that every vertex of $\vec{\Lambda}(w)$ is a source or a sink or, equivalently, if no symbol of w is interlaced both on the left and on the right;
- Property N': a word w has property N' if it is a 1-2 occurrence word, w° has property N, and every symbol covering an unmatched symbol of w is a sink of $\vec{\Lambda}(w)$ (i.e. is not interlaced on the right).

Note that property N obviously implies property N' and that theorem 27 expresses that rooted loopless planar maps with n edges are in bijection with the double occurrence words with n symbols that satisfy the property N.

For a word w , we denote by \bar{w} the *reversal* of w , that is the word obtained from w by reversing the order of the letters.

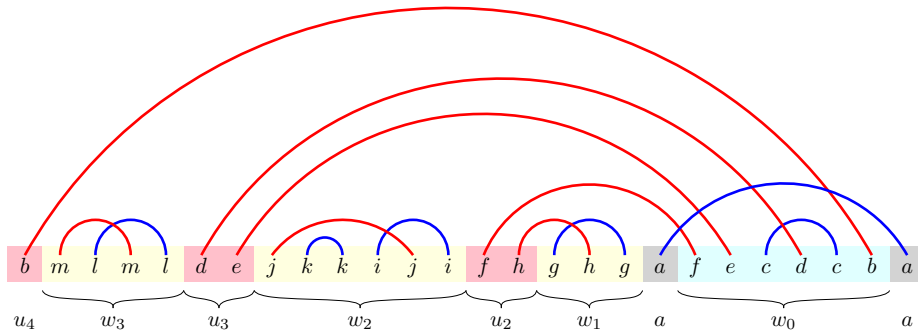


Figure 21: Example of a decomposition of a double occurrence word with property N. The words u_1 and w_4 are empty.

Lemma 30. Let w be a non-empty 1-2 occurrence word. Then w has property N' if and only if

- either w is empty,
- or $w = w' a$ where a is an unmatched symbol of w and w' has property N' ,
- or $w = w_k u_k \dots w_1 u_1 a w_0 a$, where ‘
 - a is a symbol,
 - the words u_1, \dots, u_k contain (together) exactly the first occurrences of the matched symbols of w that belong to w_0 but are unmatched in w_0 ,
 - the words u_2, \dots, u_k and w_1, \dots, w_{k-1} are not empty,
 - the pattern z of w_0 with all symbols in $U(w_0) \setminus U(w)$ satisfy $u_k \dots u_1 = \tilde{z}$,
 - every symbol in z occurs before all the elements of $U(w_0) \cap U(w)$,
 - w_0 and w_k have property N' ,
 - every w_i with $1 \leq i < k$ has property N .

Moreover, the decomposition, if it exists, is unique.

Proof. Let w be a non-empty 1-2 occurrence word whose last symbol is matched in w and let $w = w_k u_k \dots w_1 u_1 a w_0 a$, where a is a symbol, u_1, \dots, u_k contain the first occurrences of the matched symbols of w that belong to w_0 but are unmatched in w_0 , u_2, \dots, u_k are not empty, and w_i is not empty if $1 \leq i < k$. Note that this decomposition is uniquely defined. Let z be the pattern of w_0 with all symbols in $U(w_0) \setminus U(w)$.

Assume w has property N' . As every symbol in z is interlaced on the right by a , no symbol in z is interlaced on the left, thus no symbol in u_1, \dots, u_k is covered in $w_k u_k \dots w_1 u_1$. As u_k is not empty, there exists a symbol f with second occurrence in w_0 and first occurrence in u_k . The symbol f is interlaced on the right by a in w . By property N' , f does not cover any unmatched symbol. In particular, w_{k-1}, \dots, w_1 do not contain any unmatched symbol of w . Moreover no symbol f can have a first occurrence in w_j and a second occurrence in w_i for $1 \leq i < j \leq k$ as otherwise it would interlace on the left a symbol e with second occurrence in w_0 and first occurrence in u_j (such a symbol exists as u_j is not empty), which is itself interlaced on the right by a , contradicting the hypothesis that w° has property N . It follows that all the w_i with $1 \leq i < k$ are double occurrence words. It is straightforward that if w' is a pattern of w with $U(w') \subseteq U(w)$ then w' has property N' . Hence we get $u_k \dots u_1 = \tilde{z}$ (by considering the pattern including the two occurrences of a of the symbols in z), that w_i has property N for $1 \leq i < k$, and that w_k has property N' . Also, assume that some $e \in U(w) \cap U(w_0)$ appears before some $f \in U(w_0) \setminus U(w)$ in w_0 . Then f covers e in w and f is interlaced on the right by a in w , contradicting the hypothesis that w has property N' . We now prove that w_0 has property N' . First note that every symbol in $U(w) \cap U(w_0)$ is not covered in w_0 by a symbol that is interlaced on the right in w_0 , for otherwise the same would hold in w , contradicting property N' . Let $f \in U(w_0) \setminus U(w)$ and assume for contradiction that f is covered by e in w_0 that is interlaced on the right by g . Then e is interlaced (in w) on the right by g and on the left by f , contradicting the hypothesis that w has property N' .

Conversely, assume that $u_k \dots u_1 = \tilde{z}$ and that w_0, w_k have property N' , that all w_i with $1 \leq i < k$ have property N , and that all the elements in $U(w_0) \cap U(w)$ are on the left of all the elements in $U(w_0) \setminus U(w)$. Assume for contradiction that w does not have the property N' . Then either there exists a matched symbol f interlaced both on the left and on the right, or there exists an unmatched symbol f covered by a symbol e that is interlaced on the right. Assume that there exists a matched symbol f interlaced both on the left and on the right. Obviously $f \neq a$. Also, f is not a symbol in

z as these symbols are interlaced only on the right. If f belongs to w_i with $1 \leq i \leq k$ then the two occurrences of f as well as the two occurrences of every symbol interlaced with f in w belong to w_i , contradicting the assumption that w_i has property N' . Hence the two occurrences of f belong to w_0 . As w_0 has property N' the symbol f is not interlaced both on the left and on the right in w_0 . It follows that f is interlaced on the right in w_0 and is interlaced on the left by some symbol g in z , contradicting property N' . Now assume that there exists an unmatched symbol f covered by a symbol e that is interlaced on the right. Then either f belongs to w_0 or f belongs to w_k . If f belongs to w_k so do the two occurrences of e as well as the two occurrences of any symbol interlaced with e , contradicting the hypothesis that w_k has property N' . If f belongs to w_0 then either the two occurrences of e (as well as the two occurrences of the symbol interlaced with e on the left) belong to w_0 , contradicting the assumption that w_0 has property N' , or $e \in U(w_0) \setminus U(w)$, contradicting the hypothesis that all the elements of $U(w_0) \cap U(w)$ (like f) are on the right of all the elements in $U(w_0) \setminus U(w)$ (like the second occurrence of e). \square

Remark that the condition $u_k \dots u_1 = \widetilde{z}$ and the condition that all the elements in $U(w_0) \cap U(w)$ are on the right of all the elements in $U(w_0) \setminus U(w)$ simply mean that we do not have any choice when we will match unmatched symbols of w_0 , what will simplify the counting.

Let $T_{n,m}$ be the number of 1-2 occurrence words with n matched symbols and m unmatched symbols that satisfy property N' .

Lemma 31. Let

$$F(x, y) = \sum_{n \geq 0} \sum_{m \geq 0} T_{n,m} x^n y^m.$$

Then $F(x, y)$ is a solution of the equation

$$F(xy, y) = 1 + yF(xy, y) + xy \frac{F(xy, y)^2}{1 - yF(xy, 0)} \quad (4)$$

Proof. The first summand is 1, for the empty word; the second is $yF(xy, y)$ for w' followed by a ; the third summand corresponds to a matched $a(xy)$, the word w_k with property N' ($F(xy, y)$), a sequence $((1 - (F(xy, 0) - 1)y(1 - y)^{-1}))^{-1}$ of words formed a non-empty sequence of unmatched symbols $(y(1 - y)^{-1})$ followed by a non-empty words with property N ($F(xy, 0) - 1$), a sequence of unmatched symbols $((1 - y)^{-1})$, and finally the word w_0 with property N' ($F(xy, y)$). \square

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