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A comparison of smooth basis constructions for isogeometric analysis

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Abstract

In order to perform isogeometric analysis with increased smoothness on complex domains, trimming, variational coupling or unstructured spline methods can be used. The latter two classes of methods require a multi-patch segmentation of the domain, and provide continuous bases along patch interfaces. In the context of shell modeling, variational methods are widely used, whereas the application of unstructured spline methods on shell problems is rather scarce. In this paper, we therefore provide a qualitative and a quantitative comparison of a selection of unstructured spline constructions, in particular the D-Patch, Almost- C^1 , Analysis-Suitable G^1 and the Approximate C^1 constructions. Using this comparison, we aim to provide insight into the selection of methods for practical problems, as well as directions for future research. In the qualitative comparison, the properties of each method are evaluated and compared. In the quantitative comparison, a selection of numerical examples is used to highlight different advantages and disadvantages of each method. In the latter, comparison with weak coupling methods such as Nitsche's method or penalty methods is made as well. In brief, it is concluded that the Approximate C^1 and Analysis-Suitable G^1 converge optimally in the analysis of a bi-harmonic problem, without the need of special refinement procedures. Furthermore, these methods provide accurate stress fields. On the other hand, the Almost- C^1 and D-Patch provide relatively easy construction on complex geometries. The Almost- C^1 method does not have limitations on the valence of boundary vertices, unlike the D-Patch, but is only applicable to biquadratic local bases. Following from these conclusions, future research directions are proposed, for example towards making the Approximate C^1 and Analysis-Suitable G^1 applicable to more complex geometries.

Keywords: Isogeometric Analysis, Unstructured Splines, Kirchhoff–Love shell, Biharmonic Equation

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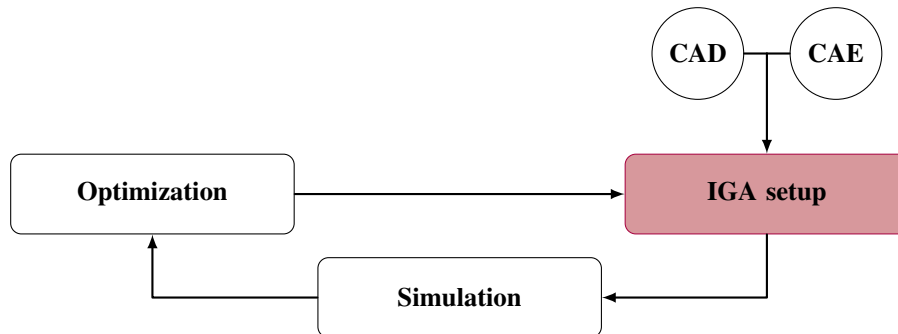


Figure 1: General workflow for solving a physics problem and optimizing a geometry or topology coming from CAD and CAE processes. Starting from *CAD* and *CAE*, the *IGA Setup* is performed. In this block, a computational basis is extracted from the geometry, to be used for simulation. Then, the *Simulation* block involves assembly of the operators of the physics problem on the computational basis coming from the *IGA Setup*. In case of shape or topology optimization problems, the simulation results are evaluated and the shape/topology is modified. From this changed shape/topology, a new computational basis can be obtained and the process can be repeated. The *IGA Setup* block is marked to be elaborated further on in Fig. 2.

1. Introduction

Present day engineering disciplines depend on Computer-aided design (CAD) and numerical simulation models for physics for design and analysis. Typically, geometries designed in CAD are converted to meshes for an analysis with numerical techniques like Finite Element Methods (FEMs). Since the geometry description in CAD is based on splines whereas meshes for simulation are based on linear geometry approximations, geometric data is lost during this conversion. Isogeometric analysis [1] is the bridge between CAD and Computer-Aided Engineering (CAE), since it is employing splines as a basis for geometric design and numerical analysis. In practice, an isogeometric analysis and optimization workflow can be seen as depicted in Fig. 1. Starting with a geometry from CAD as well as from material parameters, boundary conditions et cetera from CAE, isogeometric simulations and eventually geometry or topology optimization can be performed. The step connecting the inputs from CAD and CAE is referred to as *IGA Setup* in Fig. 1. This step takes care of the preparation for the simulation step, including the pre-processing of the geometry, if needed, and the construction of the isogeometric discretization space.

Due to the arbitrary smoothness of spline basis functions, isogeometric analysis has several advantages over conventional finite element methods. For example: (i) the introduction of k -refinements, which are proven to provide high accuracy per degree of freedom [2, 3]; (ii) high accuracy in eigenvalue problems, e.g. for structural vibrations [4–6]; or (iii) geometric exactness in parametric design and interface problems, e.g. applied to the parametric design of prosthetic heart valves [7]. Furthermore, the C^1 -smooth discretization spaces allow to solve equations such as the biharmonic equation, the Cahn–Hilliard equations or the Kirchhoff–Love shell equations without introducing auxiliary variables. However, due to the tensor-product structure of the spline basis, higher-order smoothness can be enforced easily only on domains that allow simple patch partitions (e.g. an L-shape or an annulus), whereas on geometrically and topologically more complicated domains alternative approaches are required to solve equations that require basis functions of higher-order continuity.

29 For more complicated domains, the *IGA setup* block in Fig. 1 involves a pre-processing step
30 of either the geometry, the system of equations or the solution space to solve the original sys-
31 tem of equations. In Fig. 2, this pre-processing step is subdivided into three options: trimmed
32 domain approaches, unstructured splines and variational coupling methods. Given an initial ge-
33 ometry (cf. Fig. 3a), the trimmed domain approaches alter the tensor-product domain by defining
34 parts of the domain that are physical or non-physical (cf. Fig. 3b). In case of unstructured splines
35 or variational coupling methods, the geometry is decomposed into multiple different patches (cf.
36 Fig. 3c) on which continuity conditions are enforced by constructing a smooth basis (unstructured
37 splines) or by adding extra terms to the system of equations (variational coupling approaches). In
38 Section 2 of this paper, a review of trimmed domain approaches, unstructured splines and varia-
39 tional coupling methods is provided. Examples include immersed methods, degenerate patches
40 and Nitsche’s method, respectively. In case of simple geometries (and given the right inputs) the
41 methods are identical.

42
43 As shown in Fig. 2, each class of methods has its own characteristics and previous work
44 has provided several comparisons of methods among each other, which are elaborated more in
45 Section 2. In the context of the workflow sketched in Fig. 1, unstructured splines provide a
46 valuable alternative to the other methods, since they are constructed for a fixed topology and
47 hence the computational costs of their construction are not related to changing shapes or moving
48 domains. However, recent developments mainly focused on different unstructured spline meth-
49 ods separately, rather than providing a valuable comparison. In this paper, we therefore provide
50 a qualitative and a quantitative comparison of a selection of unstructured spline constructions.
51 We consider finite, piece-wise polynomial spline constructions, hence we do not include ratio-
52 nal constructions or infinite representations, such as subdivision surfaces. More precisely, we
53 compare examples of (globally) G^1 -smooth multi-patch constructions (the Analysis-Suitable G^1
54 construction of [8] and the Approximate C^1 construction of [9]), the D-Patch method of [10] and
55 the Almost- C^1 construction of [11], motivated in Section 2.3. The selected methods are qualita-
56 tively compared based on their properties, and quantitatively based on several different examples
57 with biharmonic and Kirchhoff–Love shell equations. The aim of this paper is to provide a fair
58 comparison¹ of these methods, providing a good overview of the strengths and weaknesses of
59 each method in different cases.

60
61 The paper is outlined as follows: in Section 2, a detailed overview of the methods appearing
62 in Fig. 2 is provided. In Section 3 we provide a qualitative analysis of the four constructions that
63 are discussed in this paper, while in Section 4 we provide a quantitative analysis of all methods.
64 There, we present five benchmark problems solving either a biharmonic or a Kirchhoff–Love
65 equation. These benchmark problems serve different purposes and we compare which method,
66 in which setting, performs best. In Section 5, we conclude this paper based on the findings from
67 the previous sections and we provide directions for future research.

68

¹We believe that a comparison like the one presented in this paper is never fully unbiased, since the authors have contributed to different methods in previous publications and do not represent the entire research community.

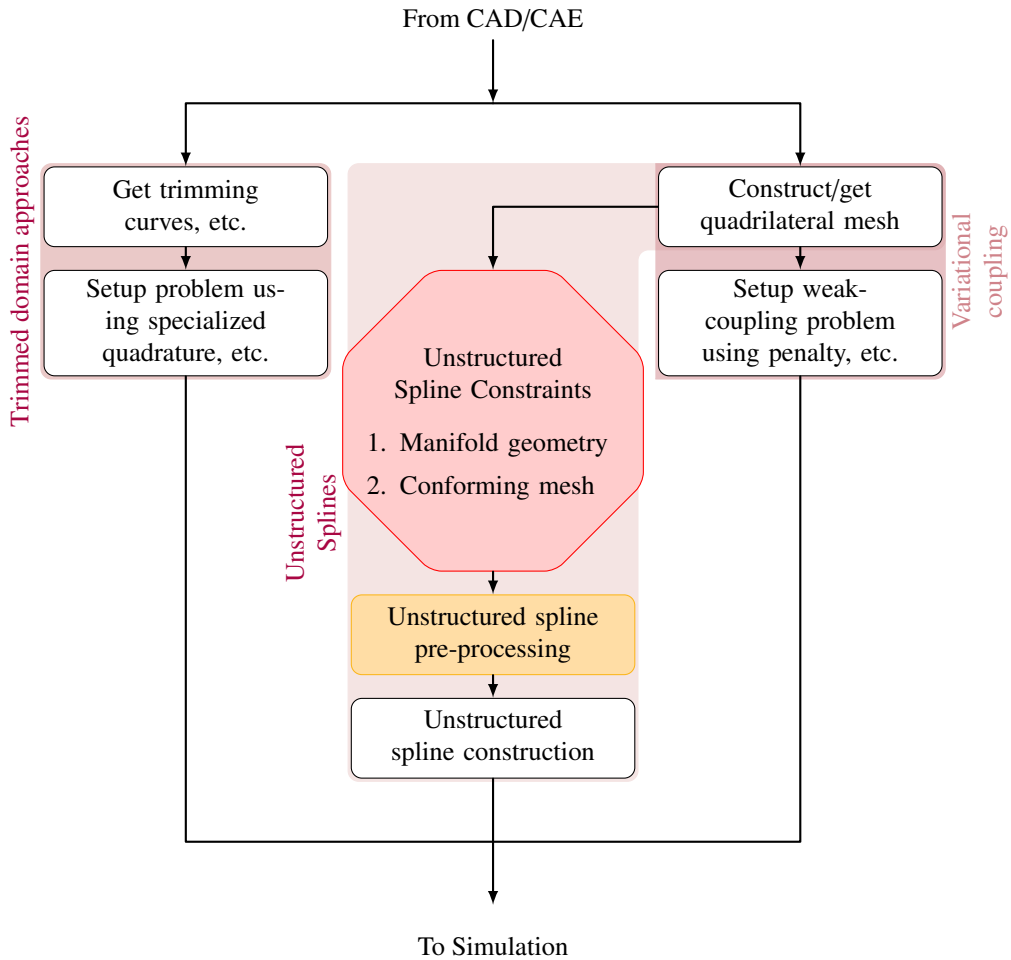


Figure 2: Inside the *IGA Setup* block from Fig. 1, three methods are distinguished. Firstly, *trimmed domain approaches* use trimming curves or surfaces to identify parts of a tensor-product domain as the actual domain. However, since elements can be trimmed poorly, *specialized quadrature* rules and solver preconditioners are typically needed. Alternatives to trimming are *weak coupling* or *unstructured spline* methods. For both classes of methods, a geometry with a given topology needs to be decomposed into multiple sub-domains (i.e. patches) via *quadrilateral meshing*. Given a quadrilateral mesh, *weak methods* assemble extra penalty terms into the equation to be solved, or add extra equations to be solved to satisfy continuity constraints. Lastly, *unstructured spline* constructions can be used to couple multiple domains by constructing a continuous basis. These methods, however can only be used on *manifold geometries* and *conforming meshes*. When these requirements are satisfied, *unstructured spline pre-processing* is required before the *unstructured spline construction* can take place. The *pre-processing* is highlighted and will be elaborated on more in Fig. 7 in Section 3.

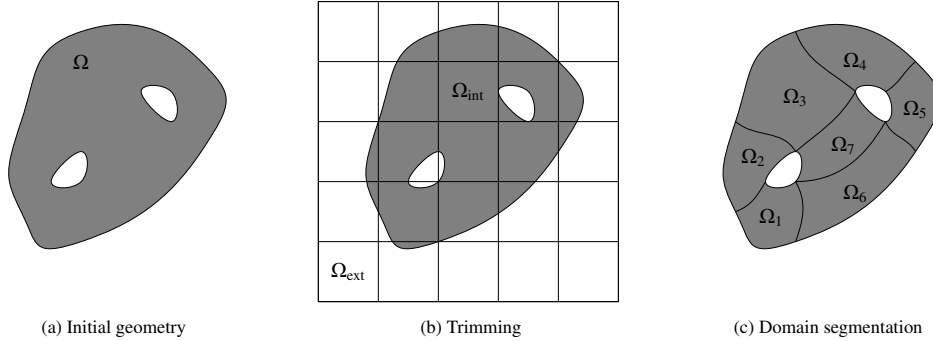


Figure 3: Given an initial geometry Ω (a), trimming (b) uses the curves of the boundary of the original geometry to define the interior domain Ω_{int} and the exterior domain Ω_{ext} . An alternative approach for modelling the domain is to use domain segmentation (c). Here, the domain is decomposed into several patches Ω_i which together define the full domain Ω .

69 2. Multi-patch isogeometric analysis: literature review

70 As discussed in the introduction of this paper, in particular in Fig. 2, three classes of methods
 71 for the modelling of complicated domains can be characterised: trimmed domain approaches,
 72 variational coupling methods and unstructured splines. The goal of all methods is to achieve a
 73 certain level of continuity across the whole analysis domain such that multi-patch isogeometric
 74 analysis can be performed for example for the Kirchhoff–Love shell model [12], the biharmonic
 75 equation or the Cahn–Hilliard equation [13]. As shown in Fig. 3, trimmed domain approaches
 76 use the fact that parts of tensor-product geometries are trimmed away, using trimming curves to
 77 separate regions of interest and regions that should be omitted, see Fig. 3b. Variational coupling
 78 approaches and unstructured splines are defined on multi-patch domains, typically following
 79 from a segmentation of the original domain, see Fig. 3c. In case of variational coupling methods,
 80 the system of equations is enriched with terms that will enforce continuity (typically in a weak
 81 sense) between the patches. In case of unstructured spline constructions, a basis is constructed
 82 on the multi-patch object, where certain smoothness is enforced strongly. When starting from a
 83 trimmed geometry, the step of creating a multi-patch domain decomposition (i.e. *untrimming*)
 84 from an arbitrary geometry with an arbitrary topology is a very important step in the application
 85 of weak coupling methods and unstructured spline constructions, as can be seen in the flowchart
 86 in Fig. 2. In this paper, however, the topic of untrimming will not be discussed as it is out of
 87 scope of our study. Hence, the reader is referred to [14, 15] for an overview of these methods.

88
 89 In this section, an overview of the trimmed domain approaches (Section 2.1), variational cou-
 90 pling methods (Section 2.2) and unstructured splines (Section 2.3) is provided. A fourth method,
 91 which will not be discussed in this section, is to introduce auxiliary variables for derivatives of
 92 the solution, so that C^1 continuity requirements are reduced to C^0 and standard interface coupling
 93 can be used. These so-called mixed formulations are common in conventional FEM, although
 94 recent advances have also been made for Kirchhoff–Love plates and shells and the biharmonic
 95 eigenvalue problem [16–18].

96 2.1. Trimming approaches

97 Trimming is a technique where so-called trimming curves or surfaces separate parts of tensor-
 98 product spline domains to define a geometry. Trimming is a common technique to represent

99 complex geometries in CAD, and typically geometries consist of multiple trimmed patches with
100 boundary and interface curves trimming the actual patches. We refer to the work [19] for an
101 overview of trimming methods in isogeometric analysis. Generalizing the idea of trimming to
102 techniques where curves or surfaces are used to define the domain of interest as *trimmed domain*
103 *approaches*, several approaches have been proposed to perform simulations on complex geome-
104 tries, including the finite cell method [20–22], Cut-FEM [23] or immersed methods [24, 25].
105 The advantage of these methods is that the trimmed CAD geometries could directly be used for
106 analysis. However, when only small parts of the physical domain are cut, leading to small cut
107 elements, numerical difficulties can occur in the conditioning of the system, leading to solver
108 instabilities or accuracy problems [26]. Therefore, the analysis of complex trimmed geome-
109 tries via methods like the FCM typically require special quadrature schemes to take into account
110 small cut cells [27] or preconditioners to stabilize the numerical analysis [28]. In the context
111 of Kirchhoff–Love shell modelling, isogeometric analysis on trimmed geometries has been per-
112 formed in several studies [29–31] including some with focus on multi-patch coupling [31–35].

113

114 2.2. Variational coupling methods

115 We define variational coupling methods as methods that modify the system of equations to
116 enforce certain continuity across patch interfaces. Examples of these methods are penalty meth-
117 ods, Nitsche’s methods, mortar methods or Lagrangian penalized methods. In the context of
118 Kirchhoff–Love shell analysis, these weak coupling methods have received a lot of attention in
119 previous studies and an overview is provided by [36]. Firstly, an in-plane coupling was proposed
120 in [37] together with a method for coupling non-manifold patches using the so-called bending
121 strip method [38]. Later, weak coupling approaches have been developed for multi-patch do-
122 mains. Here, coupling terms can be added inside the existing variational formulation (referred
123 to as Nitsche’s or penalty methods) or imposed by Lagrange multipliers (referred to as mortar
124 methods).

125

126 Several works on Nitsche techniques (cf. [39]) for isogeometric analysis have been published
127 starting from the imposition of boundary conditions [40], towards multi-patch coupling and the
128 coupling of patches [41], later using a non-symmetric parameter-free Nitsche’s method [42].
129 Nitsche’s methods have been applied to Kirchhoff plates [43], Kirchhoff–Love shells [29, 44, 45],
130 hyperelastic 2D elasticity [46] and the biharmonic equation [9, 47] and for modelling local sub-
131 domains [48] for elasticity simulations. The advantages of Nitsche’s methods are that the for-
132 mulation is variationally consistent and requires only mild stabilization, which can be performed
133 automatically, by estimating the stability parameter. However, the involved integral terms are
134 complicated expressions that impose high implementation and assembly efforts. Therefore, cou-
135 pling approaches using only penalization have been developed [49–53]. Although several im-
136 provements have been made in these works, the main disadvantage of penalty methods is that
137 a suitable penalty parameter has to be chosen. Using the super penalty approach [33, 54], the
138 computation of the penalty parameter can be automated. However, this method has not yet been
139 tested for non-linear shell problems or on ‘dirty’ geometries. Both Nitsche’s and penalty methods
140 can be used to couple geometries that are non-manifold, i.e. geometries that have out-of-plane
141 connections like stiffened structures by penalizing changes in the angle of patches on an inter-
142 face. Furthermore, the methods can handle interfaces with non-matching parameterizations.

143

144 Instead of adding coupling terms in the variational form, as is done in Nitsche’s and penalty
 145 methods, mortar methods [55] add extra degrees of freedom by introducing Lagrange multipli-
 146 ers which are required to resolve additional coupling conditions. The use of mortar methods to
 147 couple non-conforming isogeometric sub-domains was first done by [56]. In [57] the FEA-based
 148 approach of [55] was extended for NURBS-based IGA, but the aim was to develop a method
 149 for C^0 -coupling for Reissner-Mindlin shells, hence insufficient for isogeometric Kirchhoff–Love
 150 shells. A mortar method aiming to establish C^1 coupling is given in [58] and a method that
 151 provides C^n continuity was given by [59, 60]. Furthermore, G^1 mortar coupling, referred to as
 152 extended mortar coupling, was presented in [61] for Kirchhoff–Love shells, based on a coupling
 153 in least square sense. On the other hand, in [62] a mortar method to enforce C^1 coupling for
 154 the biharmonic equation was developed, where the Lagrange multiplier spaces are constructed
 155 similarly to [63] for C^0 -coupling. An approach to reduce computational costs involved in finding
 156 Lagrange multipliers is called dual mortaring [64], where Lagrange multipliers are eliminated
 157 using a compact dual basis. This approach has been developed for Bezier elements [65] and it
 158 has been applied for Kirchhoff–Love shells [66] and a bi-orthogonal spline space has been pre-
 159 sented for weak dual mortaring for patch coupling [67]. In [68], a hybrid method was provided
 160 and applied to Kirchhoff plates, which combines mortar methods and penalty methods. Lastly,
 161 a comparison of Nitsche, penalty and mortar methods is given by [69]. For a more complete
 162 overview of mortar methods for isogeometric analysis, the reader is referred to [70]. In gen-
 163 eral, mortar methods have the advantage over Nitsche’s methods that there are no parameters
 164 involved and that the implementation efforts are lower. However, the disadvantage is that a suit-
 165 able spline space needs to be found for the Lagrange multipliers [62, 63, 71]. Like Nitsche’s and
 166 penalty methods, mortar methods can handle non-matching parameterizations and non-manifold
 167 interfaces, the latter by similar penalization of interfacing patches.

168 2.3. Unstructured splines

169 Compared to weak coupling methods, unstructured spline constructions do not alter the sys-
 170 tem of equations to be solved. Instead, the computational basis is modified such that it satisfies
 171 continuity conditions across patch interfaces. Unstructured splines are typically constructed for
 172 in-plane (i.e. manifold) interfaces and not on out-of-plane (i.e. non-manifold) interfaces, since
 173 the notion of smoothness is uniquely defined only in the former setting. However, unstructured
 174 spline constructions for non-manifold interfaces are possible, e.g. as in [72–74] in the con-
 175 text of subdivision. Furthermore, unstructured spline constructions are typically constructed on
 176 conforming interfaces, i.e. interfaces with matching meshes, but, as long as the patch param-
 177 eterizations are matching, this can be overcome by taking the knot vector union of the interface
 178 patches. However, the advantage of unstructured spline constructions is that as soon as the basis
 179 is constructed for a certain untrimmed geometry, there are no additional costs involved other than
 180 evaluation costs for changing shapes, which make unstructured spline bases suitable for shape
 181 optimization problems. In case of topology changes or large changes of the shape, however, the
 182 mesh topology of the unstructured spline space has to be changed as well. Unlike weak methods,
 183 which are typically based on the introduction of penalties (e.g. in terms of energy), unstructured
 184 spline constructions are typically provided as generic geometric methods that are applicable to
 185 any equation that requires C^1 coupling across multi-patch interfaces. With the advance of isoge-
 186 ometric analysis, the interest in parametrically C^1 and geometrically G^1 splines has grown. An
 187 overview of smooth multi-patch discretizations for isogeometric analysis can be found in [75],
 188 and a small overview is provided below. We distinguish between enforcing parametric conti-
 189 nuity, i.e., the type of continuity between mesh elements within a regular tensor-product spline

190 patch, and general geometric continuity, cf. [76]. In the following, three types of constructions
191 are classified, depending on their continuity on patch interfaces, around vertices and in the patch
192 interior:

- 193 • Patch coupling with geometric continuity on patch interfaces and parametric continuity
194 inside patches.
- 195 • Patch coupling with parametric continuity everywhere.
- 196 • Patch coupling with parametric continuity almost everywhere.

197 Although other constructions outside of these categories exist, e.g., [77, 78], our review is
198 restricted to the aforementioned categories since the methods considered in Sections 3 and 4 fall
199 into these categories.

200 *Geometric continuity on patch interfaces and parametric continuity inside patches*

201 This first category of unstructured spline constructions assumes that a fixed C^0 -matching
202 multi-patch parametrization is given. On this multi-patch domain, a C^1 -smooth isogeometric
203 space is constructed. As shown in [76], for any isogeometric function the C^1 condition over each
204 interface is equivalent to a G^1 geometric continuity condition of the graph surface correspond-
205 ing to the function. If the domain is planar and the patches are bilinear, then the C^1 constraints
206 can be resolved and a C^1 spline space was constructed by [79] and applied to the isogeometric
207 analysis of the biharmonic equation in [80]. It could be shown in [81] and [82] that C^1 splines
208 over bilinear quadrilaterals and mixed (bi)linear quadrilateral/triangle meshes possess optimal
209 approximation properties. Furthermore, the work [83] studied the arbitrary C^n -smooth spline
210 space for bi-linear multi-patch parameterizations, based on their previously published findings.

211
212 Considering general C^0 -matching multi-patch domains, the work of [84] introduces the class
213 of analysis-suitable G^1 (AS- G^1) multi-patch parameterizations which includes bi-linear patches.
214 This AS- G^1 condition is in general required to obtain optimal approximation properties. The
215 condition implies that the gluing data for G^1 continuity is linear, which is explained in more
216 detail in Section 3.1. While it could be shown in [85] that all planar multi-patch domains possess
217 AS- G^1 reparameterizations, creating AS- G^1 surface domains is more difficult. Several strategies
218 to achieve this were introduced in [8], thus making C^1 -smooth multi-patch parameterizations
219 applicable to biharmonic equations and isogeometric Kirchhoff–Love shell models [86]. In the
220 work of [87] the construction of [84] is used to develop a scaled-boundary model for smooth
221 Kirchhoff–Love shells, similar to the approach of [88] for Kirchhoff plates.

222
223 Alternatively to constructing an AS- G^1 parameterization, one can relax the smoothness con-
224 ditions. This was done in [9], where the construction of an approximate C^1 (Approx. C^1) space
225 is presented. The basis construction is explicit, possesses the same degree-of-freedom structure
226 as an AS- G^1 space, but the C^1 condition is not satisfied exactly but only approximately. It de-
227 faults to the AS- G^1 construction when the AS- G^1 requirements are met. In [47] a comparison of
228 the presented space with Nitsche’s method was performed, yielding optimal convergence results
229 without the need of a coupling terms. More details on the Approx. C^1 method are provided in
230 Section 3.2.

231

232 *Parametric continuity everywhere*

233 The starting point for this class of constructions is different from the previous. Here we cre-
234 ate smooth splines in a parametric sense between neighboring mesh elements. Such parametric
235 C^1 conditions are easy to resolve, but they lead to singularities at vertices of valencies other than
236 four, so-called extraordinary vertices. This is due to the conflicting coupling conditions on partial
237 derivatives around the EVs, which lead to all partial derivatives to vanish there. Inspired by the
238 Degenerate Patch (D-Patch) approach from [89], the works of [10, 90] provide C^1 smooth spline
239 spaces for multi-patch geometries with parametric smoothness everywhere. On extraordinary
240 vertices (EVs), which is a junction between 3 or 5 or more patches (i.e. valence $\nu > 2, \nu \neq 4$), the
241 original D-Patch method shows a singularity of the basis in EVs combined with a reduction of
242 degrees of freedom in this point. An improvement of the D-Patch method was presented in [90],
243 by splitting elements around the EVs such that every element is associated to four degrees of
244 freedom. However, this construction does not have non-negativity and is based on PHT splines,
245 which have limited smoothness. A new design and analysis framework for multi-patch geome-
246 tries was presented in [10], based on D-Patches with T-splines for refinement and non-negative
247 splines yielding optimal convergence properties. This was also demonstrated in [91] for isoge-
248 ometric Kirchhoff–Love shells. In the work [92], it is motivated that this construction can also
249 be used if only one element around the EV is isolated. More details on the D-Patch method are
250 provided in Section 3.3.

251

252 Alternatively, subdivision surface based constructions lead to unstructured splines that are
253 parametrically continuous everywhere, cf. [93–97]. However, such approaches require an infi-
254 nite number of polynomial pieces around each EV. Thus, we discard them for our comparison.
255 Moreover, in general their approximation properties are severely reduced near EVs [98].

256

257 *Parametric continuity almost everywhere*

258 As mentioned previously, imposing parametric continuity everywhere leads to singularities
259 at all EVs. Thus, instead of constructing a space with full parametric continuity, spaces with
260 parametric continuity almost everywhere except around the EVs can also be considered. This
261 way, one ends up with regular, smooth rings around EVs which then need to be filled in some
262 way. Such so-called *hole-filling* techniques are commonplace in geometric modeling and can
263 also be used to construct smooth spaces for isogeometric analysis, cf. [99–104]. We focus here
264 on the simplest possible way of resolving this issue, which is to enforce only C^0 -smoothness
265 near the EVs and G^1 at the EV, namely the Almost- C^1 construction proposed in [11]. Simi-
266 lar constructions, which enforce no additional smoothness near EVs were proposed for mixed
267 quadrilateral/triangle meshes in [105] and for arbitrary degree multi-patch B-splines with en-
268 hanced smoothness (MPBES) in [106].

269

270 The Almost- C^1 construction we consider here yields piece-wise biquadratic splines which
271 are C^1 in regular regions and which have reduced smoothness around extraordinary vertices, in-
272 dependent of the valence or the location (i.e. interior or boundary EVs). In contrast to that, most
273 commonly used hole-filling approaches yield exactly C^1 -smooth spaces but introduce locally
274 polynomials of higher degree, or require a higher degree to start with, such as the construction
275 presented in [107], which converts Catmull–Clark subdivision surfaces to G^1 -smooth piece-wise
276 biquintic elements. While exact smoothness is of relevance for geometric modeling, it is not

277 necessary from an analysis point of view.

278

279 3. Qualitative comparison

280 In the qualitative comparison of this paper, we focus on the properties of different unstruc-
281 tured spline constructions and their implication on the application of these constructions in a
282 workflow as in Fig. 1. More precisely, we comment on the continuity of each construction and
283 their nestedness properties and we aim to provide a set of requirements for the *unstructured*
284 *spline pre-processing* block in Fig. 2. Since the qualitative comparison of the considered meth-
285 ods in this paper mostly covers properties of the methods and their implications, mathematical
286 details about the construction or convergence properties are not provided. For more details, the
287 reader is referred to [8] for the Analysis-Suitable G^1 (AS- G^1) method, which extends the 2D
288 construction from [108], to [9] for the Approximate C^1 (Approx. C^1) method, to [10] for the De-
289 generate Patches (D-Patch) and to [11] for the Almost- C^1 method. However, for the qualitative
290 comparison, some key terms are introduced as preliminaries.

291

292 Firstly, a *quadrilateral mesh* (*quad mesh*) is a mesh of quadrilateral elements, representing
293 a (planar) surface geometry. The quadrilaterals can be represented by tensor B-splines of any
294 degree which can be mapped onto a parametric unit-square. Typically, when the tensor B-spline
295 quadrilaterals have different sizes in different directions or even different refinement levels, as-
296 semblies of these *patches* are typically referred to as *multi-patches*. An example of a multi-patch
297 is given in Fig. 3c. The conversion of a quad-mesh with many elements to a multi-patch with
298 a smaller number of patches derived from groups of elements can be done using the procedure
299 described in Fig. 4. Here, a half-edge mesh is traversed and elements are collected into groups
300 corresponding to final patches. The vertices of the elements in one group (i.e. patch) form the
301 control net of the bi-linear patch.

302

303 Secondly, for parametrically smooth constructions, different classes of vertices are consid-
304 ered. For so-called *extraordinary vertices* (EVs), these constructions typically are different. An
305 *interior extraordinary vertex* (*interior EV*) is a vertex on a quad mesh on which three or more
306 than four patches meet. The number of patches coming together at a vertex is referred to as the
307 *valence*, denoted by ν . Furthermore a *boundary extraordinary vertex* (*boundary EV*) is a vertex
308 on the boundary of the quad mesh with valence $\nu \geq 3$. For geometrically smooth constructions,
309 the construction depends on the geometry around the vertex rather than the valence of the vertex.
310 Hence, for these constructions the notion of EVs is irrelevant.

311

312 Lastly, a refinement of a spline space is called *nested* if the refined spline space is fully con-
313 tained in the unrefined space. As a consequence, the geometry is exact under element refinement,
314 which is beneficial from an analysis point of view.

315 3.1. Analysis-suitable G^1

316 The analysis-suitable G^1 (AS- G^1) construction is a novel approach in isogeometric analysis
317 that was introduced for planar geometries and surfaces in [84], but a construction which extends
318 [108] for planar domains to surfaces is detailed in [8]. This construction ensures that basis func-
319 tions at interfaces have C^1 continuity, while basis functions at vertices have C^2 continuity. The

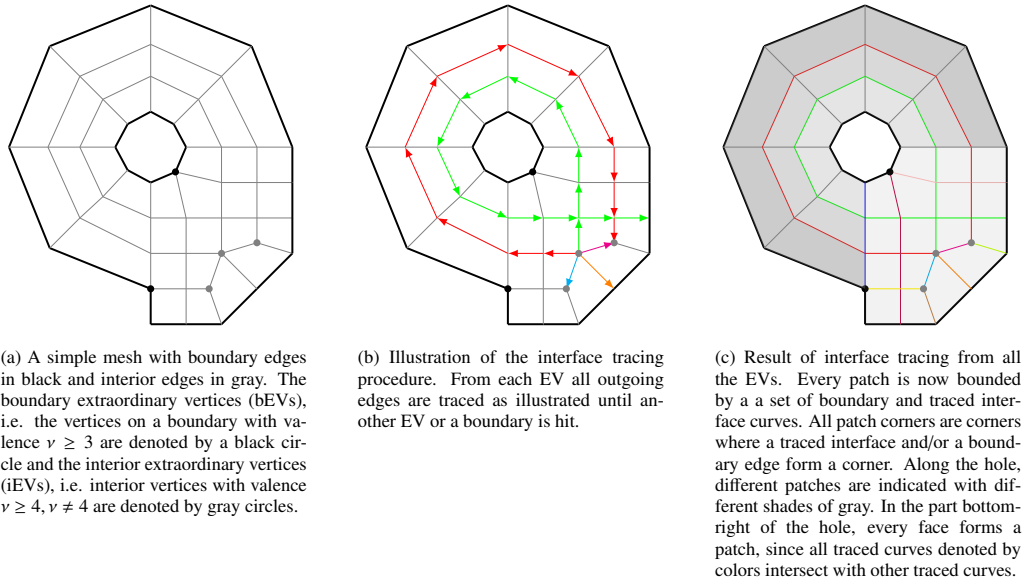


Figure 4: Procedure to find a multi-patch segmentation from a given mesh. The original mesh in (a) has 46 vertices, 81 edges and 45 faces and the final multi-patch (c) has 20 patches.

320 approach is based on the concept that G^k -smooth surfaces can produce C^k -smooth isogeometric
 321 functions [76]. When dealing with general C^0 -matching multi-patch domains, the so-called AS-
 322 G^1 conditions must be satisfied to ensure optimal approximation. If these conditions are met,
 323 a C^1 -smooth subspace of the isogeometric space can be constructed, which is sufficiently large.
 324 Such geometries are referred to as analysis-suitable geometries. However, it should be noted that
 325 the C^1 -smooth multi-patch isogeometric space generally depends on the geometry, as discussed
 326 in [109]. To overcome this issue, an Argyris-like space was proposed in [108], which has a dimen-
 327 sion that is independent of the geometry.

328
 329 Given an interface between two patches, the C^1 continuity condition at the interface is de-
 330 fined by a linear combination of tangent vectors and transversal derivatives, which is referred to
 331 as gluing data [84]. The C^1 smooth basis functions at the interface, or more generally at the edge,
 332 can be described by the first order Taylor expansion of the trace and the transversal derivative.
 333 It is shown in [84], that the ideal choice for the space-representation of the trace and transver-
 334 sal derivative is $\mathcal{S}(\mathbf{p}, \mathbf{r} - 1, \mathbf{h})^2$ and $\mathcal{S}(\mathbf{p} - 1, \mathbf{r} - 2, \mathbf{h})$, respectively. These basis functions have
 335 local support and are linearly independent, but they depend on the gluing data and, therefore,
 336 on the geometry reparameterization itself. To ensure that the basis functions form a C^1 -smooth
 337 subspace of the isogeometric space, and to maintain the nestedness of the spline spaces, it is
 338 necessary to have gluing data as a linear function which fulfils all analysis-suitable geometries.
 339 For instance, all bi-linear patches meet this requirement. However, if a geometry is not analysis-
 340 suitable, it can be reparameterized using the technique presented in [85].

²The notation $\mathcal{S}(\mathbf{p} = (p, p), \mathbf{r} = (r, r), \mathbf{h} = (h, h))$ indicates a two-dimensional spline space with p as the polynomial degree, r as the regularity and h as the mesh size in both directions.

341

342 For any vertices in the quad mesh, to describe the C^1 condition is not that straight-forward. In
343 order to keep it general, the vertex basis functions is constructed by the C^2 interpolation using the
344 C^1 basis functions from the corresponding edges. As a consequence, the vertex basis functions
345 also have local support and are linearly independent.

346

347 Summarizing, the AS- G^1 construction can be constructed by three different, linearly inde-
348 pendent sub-spaces: interior, edge and vertex space. They can be described as follows:

- 349 • Interior space: basis functions that have zero values and derivatives on the patch edges and
350 vertices.
- 351 • Interface space: basis functions that have vanishing function values up to the second
352 derivatives at the vertices.
- 353 • Vertex space: C^2 interpolating functions at the vertex, i.e., basis functions that have non-
354 vanishing C^2 data at the vertex.

355 The AS- G^1 construction with the interface and vertex constructions as described above are fully
356 C^1 over the whole domain. In addition, the AS- G^1 construction can only be constructed when
357 the degree of the basis is $p \geq 3$ and the regularity is reduced as $r \leq p - 2$.

358

359 Figure 5a presents a local region around and EV with valence five with line styles indicating
360 different continuity levels on patch or element boundaries (see the caption of Fig. 5). For the
361 AS- G^1 construction, the continuity at the vertex is C^2 by construction. Furthermore, the conti-
362 nuity at the interior element interfaces is C^{p-2} due to the restriction on keeping the isoparametric
363 concept. Lastly, since the AS- G^1 construction provides a G^1 surface, the patch interfaces are C^1
364 by construction [76].

365

366 In sum, the core ideas behind the AS- G^1 construction are as follows:

367 • **Degree, regularity, continuity**

368 The spline space is fully C^1 , hence suitable to solve fourth-order problems. However, the
369 computation of the space requires analysis-suitability of the parameterization as well as
370 degree $p \geq 3$ and regularity $r \leq p - 2$ for the basis functions.

371 • **Limitations on construction**

372 The space can be constructed on fully unstructured quadrilateral meshes with both interior
373 and boundary extraordinary vertices. The construction of the basis functions is indepen-
374 dent of the location or valence of the EVs. However, the analysis-suitability condition
375 imposes a requirement on the geometries on which the construction can be constructed.
376 Furthermore, the geometry parameterization is not changed.

377 • **Nestedness**

378 The spline spaces are nested.

379 • **Refinement procedure**

380 Refinement procedure is standard (by knot insertion) since the parameterization does not
381 change.

382 3.2. Approximate C^1

383 The Approximate C^1 construction [9] provides, as the name suggests, approximately C^1 con-
384 tinuity on interfaces and vertices, more precisely the construction provides C^1 continuity in the
385 refinement limit. The Approx. C^1 construction shares similarities with the AS- G^1 construction,
386 but the main difference between the construction of the Approx. C^1 and the AS- G^1 spaces is that
387 it relaxes the AS- G^1 condition on the geometry, i.e., it allows geometries with non-linear gluing
388 data. In fact, the exact gluing data are splines with higher polynomial degree and lower regular-
389 ity or even piece-wise rational. As a consequence, trying to extend the construction for AS- G^1
390 parameterizations directly to non-AS- G^1 geometries yields complicated basis functions that are
391 challenging to evaluate and integrate accurately. To overcome this issue and obtain a construc-
392 tion with more easily definable basis functions, the gluing data are approximated. However, this
393 approximation means that the C^1 condition is no longer satisfied exactly but only approximately.
394

395 By utilizing the approximation of the gluing data, the Approximate C^1 construction incor-
396 porates the concept of different spline spaces found in the AS- G^1 construction. In this case, the
397 interior, vertex, and interface basis functions fulfill the same conditions as in the AS- G^1 con-
398 struction, but the degree and regularity differ between these spaces. Specifically, the sub-spaces
399 for the AS- G^1 construction have $p \geq 3$ and $r \leq p - 2$, while the Approximate C^1 construction
400 employs an interior space with $p \geq 3$ and $r \leq p - 1$, along with vertex and interface spaces that
401 have locally reduced smoothness based on the approximation of the gluing data. Consequently,
402 on the one hand the Approximate C^1 construction restores the potential for maximal smoothness
403 of isogeometric functions in the refinement limit, but the nestedness of the basis is lost. On the
404 other hand, the approximation of the gluing data in the Approximate C^1 construction does not
405 require analysis-suitability for the optimal convergence rate, unlike the AS- G^1 construction. This
406 feature makes the method applicable to more complex geometries. When the Approximate C^1
407 construction is applied to an analysis-suitable geometry with $p \geq 3$ and $r \leq p - 2$, and the gluing
408 data approximation is exact, the construction becomes equivalent to the AS- G^1 construction.
409

410 Figure 5b presents a local region around and EV with valence five with line styles indicat-
411 ing different continuity levels on patch or element boundaries (see the caption of Fig. 5). For
412 the Approx. C^1 construction on a fully smooth basis ($p \geq 3$ and $r = p - 1$), the interior basis
413 recovers full smoothness on element boundaries, hence C^{p-1} continuity. In the shaded region
414 around the interfaces and the EV, the continuity is locally reduced by construction of the locally
415 reduced continuous subspace and the approximation of the gluing data. Similar to the AS- G^1
416 construction, the continuity on the EV is C^2 by construction and the element boundaries are C^1
417 approximately.
418

419 In sum, the core ideas behind the Approx. C^1 construction are as follows:

- 420 • **Degree, regularity, continuity**
421 The spline space is approximately C^1 and fully C^1 in the limit of refinement. This makes
422 the spline space suitable to solve fourth-order problems. Contrary to the AS- G^1 construc-
423 tion, the spline space approximates the gluing data, allowing maximal smoothness in the
424 interior space ($r = p - 1$) for degrees $p \geq 3$.
- 425 • **Limitations on construction**
426 The space can be constructed on fully unstructured quadrilateral meshes with both interior

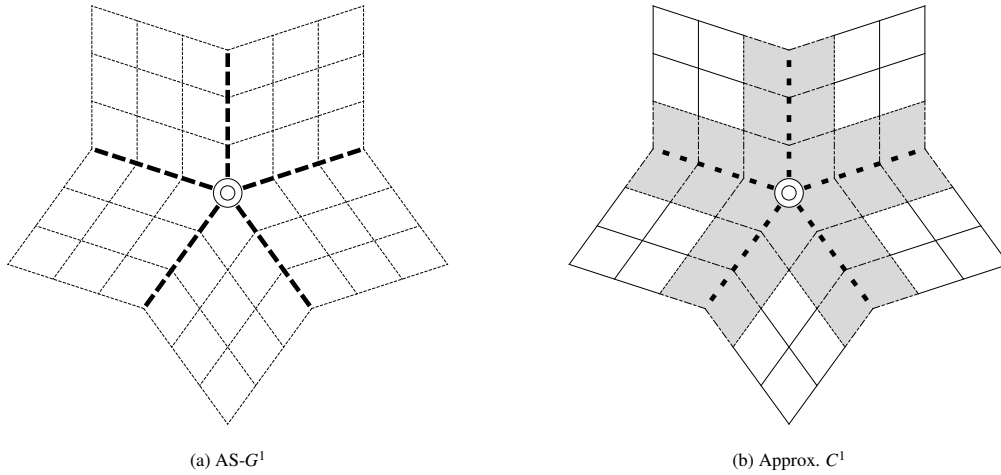


Figure 5: Schematic representation of the continuity across element boundaries and patch interfaces for the (a) AS- G^1 construction, (b) Approx. C^1 constructions. Thin lines indicate element boundaries and thick lines indicate patch interfaces. Solid lines represent C^{p-1} continuity, dashed lines represent C^{p-2} continuity, thick dashed lines represent C^1 interfaces and loosely dashed lines represent approximate C^1 interfaces. A double lined circle represent a C^2 continuous vertex, a filled circle represent a singular vertex and a white filled circled with a single line represents a C^1 continuous vertex. The gray shaded area for the Approx. C^1 represents local reduced continuity.

427 and boundary extraordinary vertices. The construction of the basis functions is independent of the location or valence of the vertices. Contrary to AS- G^1 the analysis-suitability condition is not needed. However, the construction requires a G^1 condition at the interfaces of surfaces.

431 • **Nestedness**

432 The spline spaces are not nested.

433 • **Refinement procedure**

434 Refinement procedure is standard since the parameterization does not change.

435 **3.3. D-patch**

436 The relative ease of imposing parametric smoothness for splines has led to the development of degenerate Bezier patches, or D-patches [89], which can be used to build C^1 smooth splines on unstructured quadrilateral meshes with no boundary extraordinary vertices. The constructions can be formulated for splines of any bi-degree [75], and there are no restrictions on their smoothness in the locally-structured regions of the mesh. In the locally-unstructured regions of the mesh (i.e., in a neighbourhood of an extraordinary vertex), the splines are C^1 smooth and first-order degenerate. Note that this degeneracy means that the spline spaces are not necessarily H^2 -conforming, but numerical evidence shows that they can still be used to solve fourth-order problems.

446 Specifically, imposition of strong C^1 smoothness around an extraordinary vertex requires that the splines vanish up to first order at the extraordinary vertex. This degeneracy trivially implies

448 matching first derivatives at the extraordinary vertex (since all of them vanish) but does not im-
 449 ply C^1 smoothness of the resulting spline functions and the geometries built using them. As
 450 shown in [89], additional conditions can be imposed upon certain higher-order mixed derivatives
 451 to ensure this desired C^1 smoothness. Furthermore, the effect of these additional constraints can
 452 be localised to a neighbourhood of the extraordinary vertex by imposing them on a subdivided
 453 representation of the splines [90]. This means that a patch-based representation of C^1 D-patch
 454 splines takes functions that are in $\mathcal{S}(\mathbf{p}, \mathbf{r}, \mathbf{h}/2)$ on each patch, where almost all basis functions are
 455 in $\mathcal{S}(\mathbf{p}, \mathbf{r}, \mathbf{h})$, except a few basis functions supported in a neighbourhood of extraordinary points
 456 (the number of basis functions depends on the valence).

457
 458 The D-patch construction allows for nested refinements of the spline spaces [89]. If different
 459 orders of smoothness are being imposed in locally-structured and locally-unstructured regions of
 460 the mesh, then nested refinements produce spline spaces with a higher number of basis functions
 461 supported in the vicinity of extraordinary points (the number depends on the refinement-level),
 462 see [10] for instance. On the other hand, a patch-based approach allows for a simpler implemen-
 463 tation by limiting the smoothness across patch interfaces to C^1 ; the smoothness in patch-interiors
 464 can still be arbitrarily chosen. However, special care should be taken when using D-patches with
 465 nested refinements – the degeneracy of the splines near extraordinary vertices means that, with
 466 mesh refinements, the shape regularity of the mesh starts to worsen with refinements and the
 467 finite element matrices become very ill-conditioned.

468
 469 In sum, the core ideas behind the D-patch spline construction are the following:

- 470 • **Degree, regularity, continuity**
 471 The spline space is fully C^1 . In general, the degeneracy of derivatives means that the
 472 spaces are H^2 -nonconforming, however numerical evidence supports their use in solving
 473 fourth-order problems. The construction can be formulated for splines of any degree and
 474 the smoothness away from extraordinary vertices can be chosen arbitrarily.
- 475 • **Limitations on construction**
 476 The space can be constructed on unstructured quadrilateral meshes with no boundary ex-
 477 traordinary vertices.
- 478 • **Nestedness**
 479 The spline spaces can be refined in a nested manner, however the resulting mesh have poor
 480 shape regularity and the corresponding finite element matrices may be very ill-conditioned.
- 481 • **Refinement procedure**
 482 Refinement procedures can be derived from standard B-spline knot insertion.

483 3.4. Almost C^1

484 Almost- C^1 splines are defined on a general, conforming quadrilateral mesh. They are piece-
 485 wise biquadratic and possess mixed smoothness, i.e., they are C^1 in regular regions, while the
 486 smoothness near extraordinary vertices, i.e., vertices with valence different from four, is reduced.
 487 To be precise, they are C^1 smooth at all vertices (including extraordinary vertices) and across all
 488 edges except for the ones emanating from an extraordinary vertex. Moreover, while they are de-
 489 fined to be biquadratic on all regular elements, they are piece-wise biquadratic splines (with one
 490 inner knot in each direction) on all elements that are neighboring an extraordinary vertex. Details

491 can be found in [11]. As a consequence, a patch based representation of Almost- C^1 splines takes
 492 functions that are in $\mathcal{S}(\mathbf{2}, \mathbf{1}, \mathbf{h}/2)$ on each patch, where almost all basis functions are in $\mathcal{S}(\mathbf{2}, \mathbf{1}, \mathbf{h})$,
 493 except a few basis functions supported in a 1-ring neighbourhood of extraordinary points (the
 494 number depends on the valence).

495
 496 A central feature of Almost- C^1 splines is the mixed smoothness imposition described above.
 497 In particular, this choice of mixed smoothness only depends on the current refinement level of
 498 the mesh. That is, standard C^1 -smoothness is enforced across all edges at the current refinement
 499 level except the ones that are incident upon extraordinary vertices, where only C^0 smoothness is
 500 enforced. Additionally, these smoothness conditions are combined with G^1 smoothness imposi-
 501 tion at each extraordinary vertex. This means that almost- C^1 splines do not yield nested spaces
 502 when refining. As a result, the refinement process essentially amounts to a projection of coarse
 503 Almost- C^1 splines onto the refined Almost- C^1 spline space. This projection can be chosen in
 504 many different ways and can have a significant impact on the limit surface description as well as
 505 isogeometric simulations using these spaces. In [11] a smoothing and refinement procedure is
 506 proposed that results in a C^1 -smooth limit surface for sufficiently regular input data.

507
 508 Let us briefly summarize the refinement procedure here. We assume that we are given a quad
 509 mesh and associate a control point with each face of the mesh. The initial smoothing step guaran-
 510 tees that all control points associated to the one ring around an extraordinary vertex are coplanar.
 511 Having given such an initial control point grid, we then refine the geometry using explicit sub-
 512 division rules as specified in [11, 105]. The rules are the same as for quadratic tensor-product
 513 B-splines in regular regions and maintain the coplanarity near extraordinary vertices.

514
 515 In sum, the core ideas behind the Almost- C^1 spline construction are the following:

- 516 • **Degree, regularity, continuity**
 517 The spline space locally reproduces biquadratic polynomials and it is sufficiently smooth
 518 to be able to solve fourth order problems.
- 519 • **Limitations on construction**
 520 The splines can be constructed on fully unstructured quadrilateral meshes, in particular,
 521 those that contain both interior and boundary extraordinary vertices.
- 522 • **Nestedness**
 523 Since the spaces are not nested, the convergence behavior of Almost- C^1 splines depends
 524 on how the geometry parameterization is refined.
- 525 • **Refinement procedure**
 526 An initial geometry and a refinement procedure can be constructed in such a way, that the
 527 limit geometry parameterization is normal continuous everywhere.

528 Thus, the concept introduced in [11] is quite flexible, since the initial smoothing procedure
 529 and the refinement procedure are not unique and can be tailored to the needs coming from geo-
 530 metric modeling, e.g., one may want to reproduce Doo-Sabin subdivision surfaces, thus having
 531 to modify the subdivision rule for refinement accordingly. The spline space that is introduced
 532 on each refinement level can be seen as a simple hole-filling construction, which is sufficient for
 533 numerical analysis.

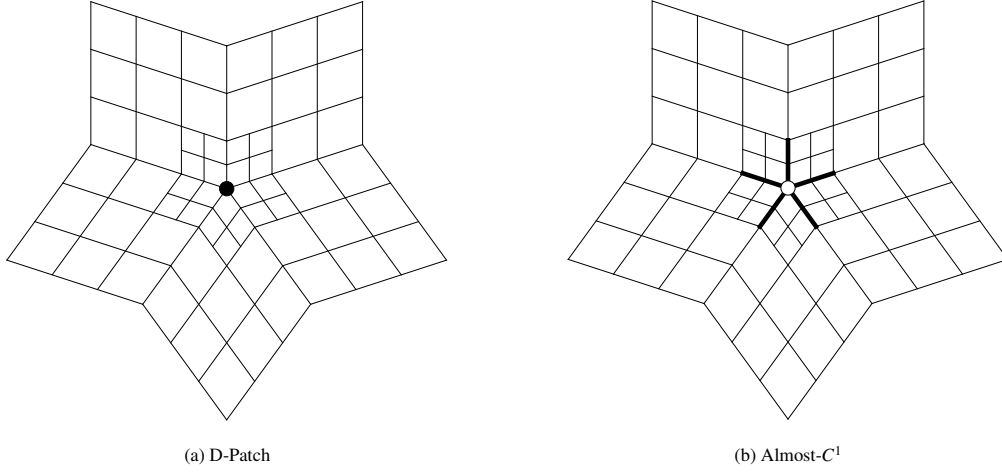


Figure 6: Schematic representation of the continuity across element boundaries and patch interfaces for the (a) D-Patch and (b) Almost- C^1 . Line styles are as in Fig. 5.

534 3.5. Conclusions

535 In this section, a summary of the construction and the properties of the analysis-suitable G^1
 536 (AS- G^1), the approximate C^1 (Approx. C^1), the degenerate patches (D-patch) and the Almost- C^1
 537 methods have been provided, referring to the relevant publications for the mathematical details.
 538 For each method, comments have been provided on the degree, regularity and continuity of the
 539 space, on the limitations of the construction in terms of the quadrilateral mesh, on nestedness for
 540 refinement and on the refinement procedure itself. In addition, Figs. 5 and 6 provides detailed
 541 information on the local continuity of the constructions around an extraordinary vertex.

542
 543 The aim of the qualitative analysis of the methods in this paper is to provide a comparison
 544 of a set of properties and requirements of each method and their implications on their applica-
 545 bility. While the subsections presented before provide a brief description of the properties of the
 546 methods and the reason behind these properties and requirements, Table 1 provides a side-by-
 547 side comparison of each method based on the subsections before. In particular, the table lists the
 548 (i,ii) requirements on degree and regularity for the constructions, iii geometrical or topological
 549 limitations if applicable, (iv,v) the continuity of the constructed bases in the interior and on the
 550 interfaces and element boundaries and vi nestedness of the constructed basis.

551
 552 Following from Table 1, the requirements for construction of the unstructured spline bases
 553 are summarized in Fig. 7 as pre-processing conditions that have to be satisfied for each unstruc-
 554 tured spline construction in the process depicted in Fig. 2. The degree and regularity conditions
 555 (cf. i,ii in Table 1) must be satisfied for each construction, e.g. by performing projections on
 556 suitable spline spaces or by knot insertion routines. Furthermore, the geometric or topological
 557 limitations (cf. iii in Table 1) impose additional constraints that the geometry must satisfy.

558

Table 1: Summary of the requirements for the construction and the properties for each of the considered bases. The construction requirements include the degree and regularity of the basis used for construction as well as geometrical or topological properties of the input geometry. The properties include the continuity on interfaces, vertices and in the interior of the unstructured spline construction, as well as the nestedness property.

Requirements	AS- G^1	Approx. C^1	D-Patch	Almost- C^1
(i) Degree	$p \geq 3$	$p \geq 3$	$p \geq 3$	$p = 2$
(ii) Regularity	$r \leq p - 2$	$r \leq p - 1$	$r \leq p - 1$	$r = 1$
(v) Geometrical / topological limitations	Analysis-suitability	G^2 continuity	BEVs: $v \leq 3$, C^1 continuity	C^1 continuity
Properties	AS- G^1	Approx. C^1	D-Patch	Almost- C^1
(iii) Interface & Vertex Continuity	C^1	C^1 in the limit	C^1	C^1 in the limit
(iv) Interior continuity	C^{p-2}	C^{p-1}	C^{p-1}	C^1
(vi) Nestedness	Yes	No	Yes	No

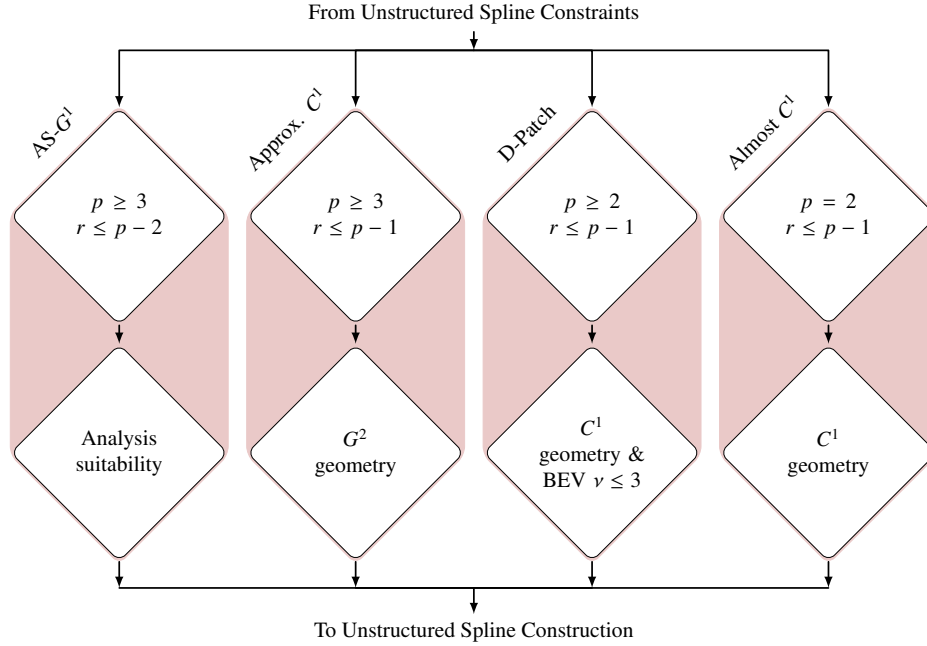


Figure 7: Inside the *unstructured spline pre-processing* block from Fig. 2. The unstructured spline requirements are depicted in diamond-shaped blocks for methods AS- G^1 , Approx. C^1 , D-Patch and Almost C^1 . The first row represents requirements on degree p and regularity r . If not satisfied, the geometry can be projected onto a space that satisfies the requirement, or degree elevation or reduction steps can be performed together with refinement operations. The second row depicts the requirements on the geometry parameterization; these blocks can be satisfied by changing the geometry.

559 **4. Quantitative comparison**

560 In this section a quantitative comparison between the methods provided in Section 3 is pro-
561 vided. In addition, variational coupling methods are compared if applicable. The quantitative
562 comparison is composed of various benchmark problems, each providing a different conclusion
563 with respect to the methods considered:

564 **Biharmonic problem on a planar domain (Section 4.1)** The first example entails solving the
565 biharmonic problem on a planar domain. The goal of this example is to assess the con-
566 vergence properties of all considered unstructured spline constructions, hence the problem
567 will be solved on a simple analysis-suitable geometry without EVs on the boundary, such
568 that every method from Section 3 can be applied and compared to the manufactured solu-
569 tion.

570 **Linear Kirchhoff–Love shell analysis on a surface (Section 4.2)** The second example entails
571 solving the Kirchhoff–Love shell equation on curved domains. The goal of this example
572 is to demonstrate the performance of the unstructured spline construction for simple shell
573 problems. Therefore, comparison will be made to single-patch results and penalty coupling
574 from [51].

575 **Spectral analysis on a planar domain (Section 4.3)** In the third example, spectral analysis of
576 a plate equation is performed. The goal of this example is to assess the spectral properties
577 of the unstructured spline methods compared to a variational approach and a single patch,
578 since the spectral properties of highly continuous bases have been demonstrated to be
579 superior over non-smooth bases [110].

580 **Modal analysis of a complex geometry (Section 4.4)** In the fourth example, a modal analysis
581 is performed on a complex geometry extracted from a quad-mesh. The goal of this example
582 is to demonstrate the applicability and performance of the unstructured spline methods on
583 a large-scale, more complicated geometry.

584 **Stress analysis in a curved shell (Section 4.5)** Lastly, the fifth example involves the analysis of
585 stress fields in shells. The goal of this example is to assess the performance of unstructured
586 spline constructions and a penalty method when it comes to stress reconstruction in shells.
587 For the Kirchhoff–Love shell, the stresses are obtained by taking gradients of the deformed
588 geometry, hence of the solution. This means that for C^1 bases, stresses are C^0 . This might
589 be unfavourable in engineering applications where local stress fields are of importance,
590 e.g. fatigue analysis.

591 In all examples except the complex geometry in Section 4.4, the domain decomposition from
592 Fig. 8 is used to decompose a simple domain into a domain with extraordinary vertices in the
593 interior. Domains with EVs on the boundary are left out of scope, since the D-patch construction
594 would change the outer boundaries of the domain, hence the comparison would involve a signif-
595 icantly different geometry. Since different methods have different constraints on the degree and
596 regularity of the basis, different combinations of the degree p and regularity r are tested through-
597 out the benchmark problems. In Table 2 the combinations of p and r and the methods that are
598 compared for these bases are provided. For the biharmonic problem and the spectral analysis
599 (Sections 4.1 and 4.3) Nitsche’s method is used for comparison, see [47] for more details. When
600 solving the Kirchhoff–Love shell equations, the penalty method is used for comparison, see [51]



Figure 8: Multi-patch decomposition of a simple domain into six patches. The domain has two EVs in the interior (valence 3 and 5) and no boundary EVs.

Table 2: Degree p and regularity r constraints for each considered method from Section 3, see Table 1.

	$p = 2, r = 1$	$p = 3, r = 1$	$p = 3, r = 2$
D-patch	★	★	★
Almost- C^1	★		
Approx. C^1		★	★
AS- G^1		★	
Nitsche/Penalty	★	★	★

601 for more details. In all examples, Dirichlet boundary conditions are applied at the control points
 602 and clamped boundary conditions are applied weakly as in [51]. All results are obtained using
 603 the Geometry + Simulation modules [111, 112] and will be published in a separate publication.

604

605 As discussed in Section 3, the D-patch and Almost- C^1 constructions involve a pre-smoothing
 606 of the geometry. In case of mesh convergence results, refinements can be performed in different
 607 ways. On the one hand, the original geometry can be refined and a new construction with a new
 608 geometry approximation can be performed. On the other hand, the geometry resulting from the
 609 construction in the first refinement level can be refined in a nested way, such that the geometry
 610 does not change after the first mesh. In the quantitative comparison, all refinements are performed
 611 in a nested way, unless specified otherwise.

612 4.1. Biharmonic equation on a planar domain

613 The first benchmark entails the biharmonic equation on a planar domain. The purpose of this
 614 example is to assess the convergence properties of the unstructured spline methods described in
 615 Section 3. We mainly follow the structure of [47]. The biharmonic equation is solved on a unit
 616 square $\Omega = [0, 1]^2$ with the patch segmentation from Fig. 8. The biharmonic equation is defined
 617 by

$$\Delta^2 \varphi = f. \tag{1}$$

618 In the present example, convergence is analysed with respect to a manufactured solution

$$\tilde{\varphi}(x_1, x_2) = (\cos(4\pi x_1) - 1)(\cos(4\pi x_2) - 1), \quad (2)$$

619 such that the right-hand-side function becomes:

$$f(x_1, x_2) = 256\pi^4(4 \cos(4\pi x) \cos(4\pi y) - \cos(4\pi x) - \cos(4\pi y)) \quad (3)$$

620 Furthermore, on all boundaries of the domain, the manufactured solution and its derivatives are
621 imposed as Dirichlet and Neumann boundary conditions, respectively:

$$\left. \begin{aligned} \varphi &= \tilde{\varphi}(x_1, x_2) \\ \partial_{\mathbf{n}}\varphi &= \partial_{\mathbf{n}}\tilde{\varphi} \end{aligned} \right\} \text{on } \Gamma, \quad (4)$$

622 where $\Gamma = \partial\Omega$, \mathbf{n} is the unit outward normal vector on Γ . The biharmonic equation from Eq. (1)
623 with boundary conditions Eq. (4) can be discretized by obtaining the weak formulation, see [47],
624 inserting Eq. (4) and by defining an approximation of the solution φ as φ_h . Furthermore, a weak
625 coupling can be established through Nitsche's method. For the mathematical details behind the
626 discretization of the biharmonic equation and optionally adding Nitsche interface coupling terms,
627 we refer to [9, 47]. For the D-Patch and Almost C^1 constructions, the geometry is smoothed upon
628 construction. The geometry used for evaluation of the weak formulation is constructed by using
629 an L_2 -projection of the geometry from the coarsest space which is projected onto the smooth
630 basis of each refinement level. For the D-Patch, the non-negative smoothness matrix for vertex
631 smoothing is used. Although this matrix produces non-nested meshes, it provides the highest
632 rates of convergence. Furthermore, the factor β (cf. [10, sec. 5.1]) is chosen as $\beta = 0.4$ as used
633 by [10], or $\beta = 1.2$, and halved in each refinement level.

634
635 To evaluate the unstructured spline constructions from Section 3, the numerical approxima-
636 tion φ_h is compared to the manufactured solution $\tilde{\varphi}$ in the L^2 -, H^1 - and H^2 -norms on the multi-
637 patch segmentation from Fig. 8. The bi-linear segmentation is refined and degree elevated until
638 the desired degree p and regularity r from Table 2 are obtained. In addition, a Nitsche coupling
639 of the patches is employed for comparison.

640
641 The results for the comparison are presented in Fig. 9. For degree $p = 2$ and regularity
642 $r = 1$ the Almost- C^1 , D-patch and Nitsche coupling methods are compared. As expected, the
643 results show consistency between the Almost- C^1 , D-patch and Nitsche's method with expected
644 convergence. The results also show a slight dependency on the factor β for the D-Patch. For
645 degree $p = 3$ and regularity $r = 1$, the Approx. C^1 , AS- G^1 , D-patch and Nitsche's method can be
646 compared. The results of the Approx. C^1 and AS- G^1 are exactly the same, since the original ge-
647 ometry is analysis-suitable and contains only bi-linear patches. Then applying the Approx. C^1 to
648 an analysis-suitable geometry with regularity $p - 2$, the approximate gluing data becomes exact,
649 hence the same as in the AS- G^1 construction. The D-patch in this case shows better convergence
650 of the L_2 , H_1 and H_2 errors for $\beta = 1.2$ than for $\beta = 0.4$. Though, for both choices of β , the
651 convergence is sub-optimal, as was observed in the work by [91]. Furthermore, the L_2 -norm in-
652 creases at the last point of the D-Patch results, due to ill-conditioning of the system of equations.
653 Lastly, for degree $p = 3$ and regularity $r = 2$ the Approx. C^1 , D-patch and Nitsche's method are
654 compared. The observations are as for the $p = 3$, $r = 1$ case.

655

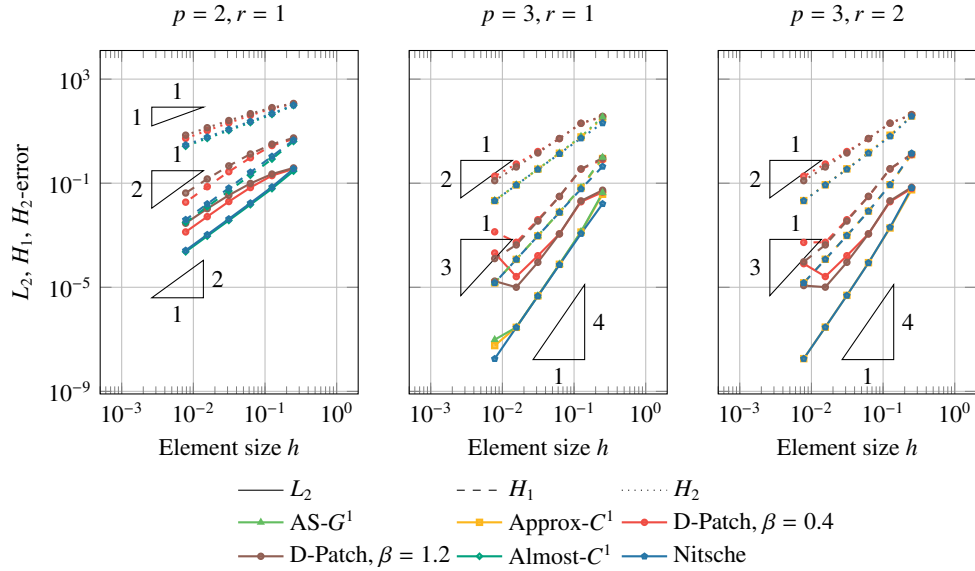


Figure 9: Errors for the AS- G^1 , Approx- C^1 , D-Patch and Almost- C^1 construction for the biharmonic problem on the domain in Fig. 8. The L_2 , H_1 and H_2 errors with respect to the analytical solution are plotted with different line styles in the top row. Furthermore, all results are plotted against the element size h and the expected convergence rates are given by the triangles.

656 Overall, the results show expected convergence behaviour for all considered spline construc-
 657 tions compared to theoretical results and compared to a Nitsche coupling method. However, the
 658 D-Patch method does not converge for very fine meshes, due to ill-conditioning of the system
 659 matrix.

660 4.2. Linear Kirchhoff–Love shell analysis on a surface

661 We solve the linear Kirchhoff–Love shell equations on two geometries to demonstrate the
 662 convergence behaviour of the methods on curved surfaces. To this end, two benchmark examples
 663 are considered. Firstly, a hyperbolic paraboloid surface is constructed with shape, inspired by
 664 [8]:

$$\mathbf{r}(\xi_1, \xi_2) = \begin{bmatrix} \xi_1 & \xi_2 & \xi_1^2 - \xi_2^2 \end{bmatrix} \quad (5)$$

665 The left-side of the hyperbolic paraboloid is clamped ($\mathbf{u} = \mathbf{0}$) and the other sides are free. Fur-
 666 thermore, a distributed load with magnitude $8000t$ is applied with t the thickness, see Fig. 10.
 667 Secondly, an elliptic paraboloid shaped-domain is modelled, with equation

$$\mathbf{r}(\xi_1, \xi_2) = \begin{bmatrix} \xi_1 & \xi_2 & 1 - 2(\xi_1^2 + \xi_2^2) \end{bmatrix} \quad (6)$$

668 For this shape, a point load with magnitude $10^8 t$ is applied in the middle of the domain. The
 669 corners of the domain are only fixed in vertical z direction to allow sliding in the xy -plane. One
 670 corner is fixed in all directions to create a well-posed problem. For both hyperbolic paraboloid
 671 (Fig. 10) and elliptic paraboloid (Fig. 11) the multi-patch segmentation from Fig. 8 is used. In
 672 both cases, the shells are modelled with a thickness of $t = 0.01$ [mm] and with a Saint-Venant

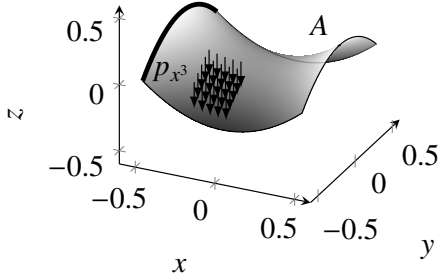


Figure 10: Hyperbolic paraboloid shell geometry with coordinates $\mathbf{r}(\xi_1, \xi_2) = [\xi_1 \quad \xi_2 \quad \xi_1^2 - \xi_2^2]$, $\xi_1, \xi_2 \in [-1/2, 1/2]$. The left-edge of the hyperbolic paraboloid is clamped, i.e. the displacements and rotations are zero ($\mathbf{u} = \mathbf{0}$ and $\frac{\partial u_z}{\partial x} = 0$).

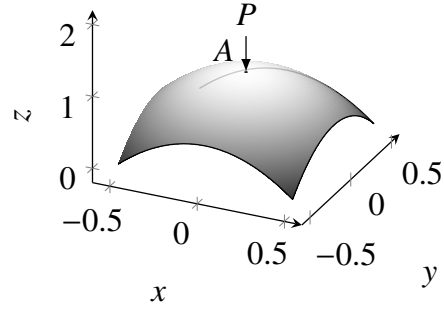


Figure 11: Elliptic paraboloid shell geometry with coordinates $\mathbf{r}(\xi_1, \xi_2) = [\xi_1 \quad \xi_2 \quad 1 - 2(\xi_1^2 + \xi_2^2)]$, $\xi_1, \xi_2 \in [-1/2, 1/2]$. On the corners of the domain, the vertical displacements are set to zero $u_z = 0$ and one corner is fixed in-plane as well. Furthermore, a point load with magnitude $P = 10^8 t$ is applied in the middle of the geometry.

673 Kirchhoff material with Young's modulus $E = 200$ [GPa] and Poisson's ratio $\nu = 0.3$ [-]. The
 674 refinement procedure as described in Section 4.1 is used for the D-Patch and Almost- C^1
 675 constructions.

676
 677 The results of both analyses are given in Figs. 12 and 13. Here, different unstructured spline
 678 constructions are tested on patch-bases with different degrees and regularities, as reported in Ta-
 679 ble 2. For each combination of degree p and regularity r , the energy norm $W_{\text{int}}^h = \frac{1}{2} \mathbf{u}_h^\top K_h \mathbf{u}_h$
 680 is plotted against the number of degrees of freedom, with \mathbf{u}_h the discrete displacement vector and
 681 K_h the discrete linear stiffness matrix. From the results in Figs. 12 and 13, a few observations can
 682 be made. Firstly, the Approx. C^1 and AS- G^1 methods show slow convergence on the hyperbolic
 683 paraboloid geometry, while the convergence on the elliptic paraboloid geometry is similar to the
 684 single-patch convergence. The slow convergence for the hyperbolic paraboloid shell is also ob-
 685 served in [8]. Since the results of the same constructions on the elliptic paraboloid geometries
 686 do not show slower convergence, the slow convergence is hypothetically a result of the double
 687 curvature with different signs of the shell. Secondly, the D-Patch and Approx. C^1 show com-
 688 parable convergence to the penalty method on both geometries, which is slightly slower than the
 689 convergence of the single-patch results. This is explained by the fact that the degrees of freedom
 690 are more optimally allocated for the single-patch parameterization. Lastly, the results obtained
 691 by the penalty method for different penalty parameters α show convergence with a rate similar to
 692 the D-Patch and Almost- C^1 methods for penalty parameters $\alpha \in \{1, 10\}$. For $\alpha = 100$ the penalty
 693 method is still converging to the same solution, but convergence starts after a few refinement
 694 steps.

695 4.3. Spectral analysis on a planar domain

696 In this example, the spectral properties of the unstructured spline constructions on a multi-
 697 patch domain are considered. From [4] it is known that isogeometric analysis has the advan-

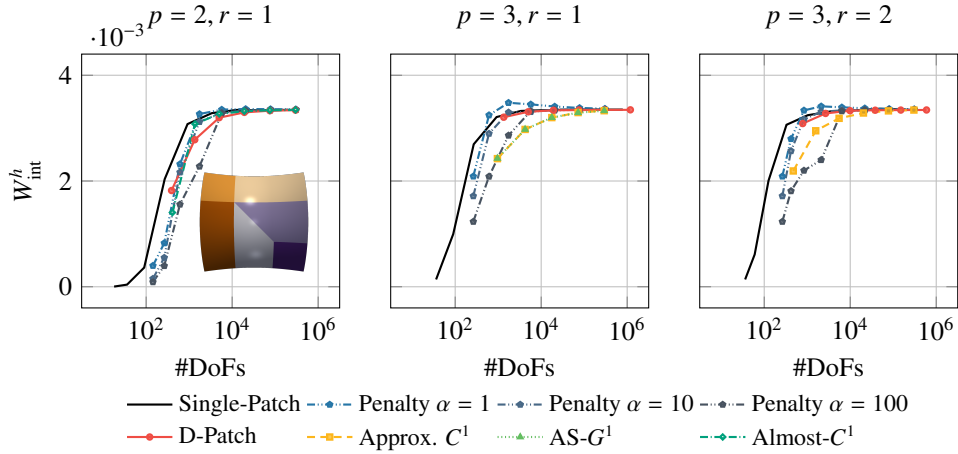


Figure 12: Bending energy norm $W_{\text{int}}^h = \frac{1}{2} \mathbf{u}_h^\top K_h \mathbf{u}_h$ for the hyperboloid geometry from Fig. 10 with a patch segmentation as in Fig. 8. The results are presented for different combinations of the degree p and regularity r for all unstructured spline constructions. In addition, the results for a penalty method with parameter $\alpha \in \{1, 10, 100\}$ are provided for comparison.

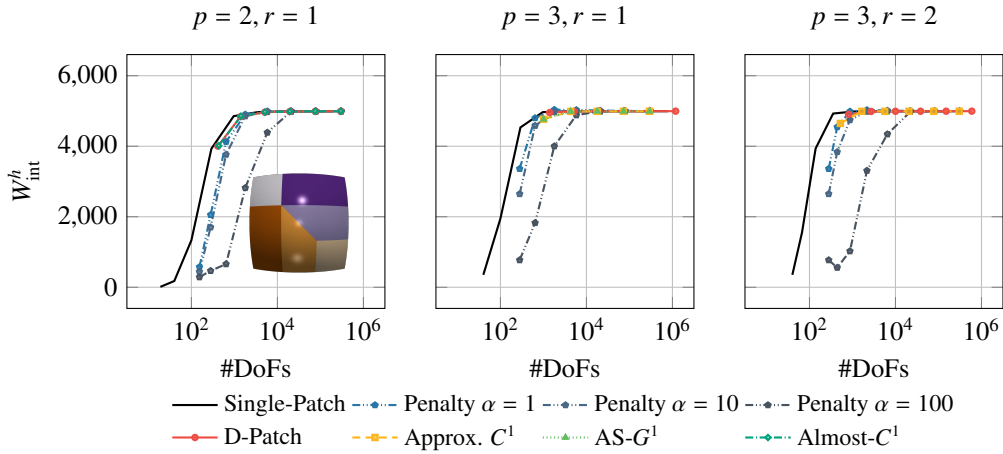


Figure 13: Bending energy norm $W_{\text{int}}^h = \frac{1}{2} \mathbf{u}_h^\top K_h \mathbf{u}_h$ for the paraboloid geometry from Fig. 11 with a patch segmentation as in Fig. 8. The results are presented for different combinations of the degree p and regularity r for all unstructured spline constructions. In addition, the results for a penalty method with parameter $\alpha \in \{1, 10, 100\}$ are provided for comparison.

698 tage over C^0 Finite Element Analysis with respect to spectra for eigenvalue problems. Smooth
699 isogeometric discretization provide converging spectra with spline degree p , whereas the spec-
700 tra obtained by C^0 FEA diverge with p and typically have optical branches. Similarly, when
701 patches with C^0 continuity are considered, optical branches are introduced and the accuracy of
702 the spectral approximation decreases [113]. In this benchmark problem, we compare the basis
703 constructions from Table 2 on Fig. 8 on their spectral properties. For Nitsche's method, we use
704 different values for the coupling parameter to assess its influence on the spectrum.

705
706 For the problem at hand, we consider a unit-square domain with parametric lay-out from
707 Fig. 8 for simplicity. We consider modal analysis using the plate equation. The stiffness operator
708 of the free vibration plate equation is similar to the biharmonic equation from Eq. (1), and the
709 inertia is included on the right-hand-side:

$$D\Delta^2 w = -\rho t \frac{\partial^2 w}{\partial \tau^2} \quad (7)$$

710 Assuming that $w(x, y, \tau)$ is harmonic, i.e. $w(x, y, \tau) = \hat{w}(x, y) \exp\{i\omega\tau\}$ with ω a frequency, the
711 equation simplifies to

$$D\Delta^2 \hat{w} = \omega^2 \frac{\partial^2 \hat{w}}{\partial \tau^2}. \quad (8)$$

712 Here, $D = Et^3/(12(1 - \nu^2))$ is the flexural rigidity of the plate with $E = 10^5$ [Pa] the Young's
713 modulus of the plate, $t = 10^{-2}$ [m] the thickness and $\nu = 0.2$ [-] the Poisson's ratio. Furthermore,
714 $\rho = 10^5$ [kg] is the material density. Equation (8) is a generalized eigenvalue problem with
715 eigenpairs (ω_i, v_i) where ω_i is the i^{th} eigenfrequency and v_i the i^{th} mode shape. The mode shape
716 for a simply supported unit plate with $n \times m$ half-waves is given by

$$v_{nm}(x, y) = \sin(n\pi x) \sin(m\pi y) \quad (9)$$

717 with corresponding eigenfrequency

$$\omega_{nm} = (n^2 + m^2)\pi^2 \sqrt{\frac{D}{\rho t}}. \quad (10)$$

718 In addition, the numerical solution to Eq. (8) is obtained by solving the following generalized
719 eigenvalue problem

$$D \int_{\Omega} \Delta w \Delta \varphi \, d\Omega = \omega^2 \rho t \int_{\Omega} u \varphi \, d\Omega \quad (11)$$

720 With φ a test function, see Section 4.1. In further representation of the solutions, we employ the
721 index i such that $\omega_i < \omega_{i+1}$ and we use the subscript h for numerically obtained solutions.

722
723 Figure 14 presents the spectra for different degrees, regularities and for different methods.
724 Here, the vertical axis represents the ratio of the numerically obtained eigenfrequency over the
725 analytical eigenfrequency with index i , thus $\omega_{h,i}/\omega_i$. The horizontal axis represents the fraction
726 of the eigenfrequency index i over the total number of eigenmodes. The total number of eigen-
727 modes is equal to the number of degrees of freedom in the system. The results are presented for
728 the degrees and regularities as in Table 2.

729

730 Firstly, the $p = 2, r = 1$ plot shows that Nitsche's method oscillates for all considered values
731 of the penalty parameter. Furthermore, in the part where it is not oscillating, the ratio $\omega_{i,h}/\omega_i$ is
732 higher than for the D-patch and Almost- C^1 method. Additionally, the D-patch and Almost- C^1
733 methods show a significant difference with respect to the single patch result, which is due to the
734 non-Cartesian multi-patch segmentation of Fig. 8 and the fact that the analytical mode shapes are
735 Cartesian. For the $p = 3, r = 1$ and $p = 3, r = 2$ bases similar conclusions can be drawn. Al-
736 though for the $p = 3, r = 1$ case the Approx. C^1 method seems worse than the D-patch method,
737 the opposite is true for $p = 3, r = 2$. Hence, it can be concluded that no method outperforms
738 another, but that all unstructured spline constructions perform better than Nitsche's method.
739

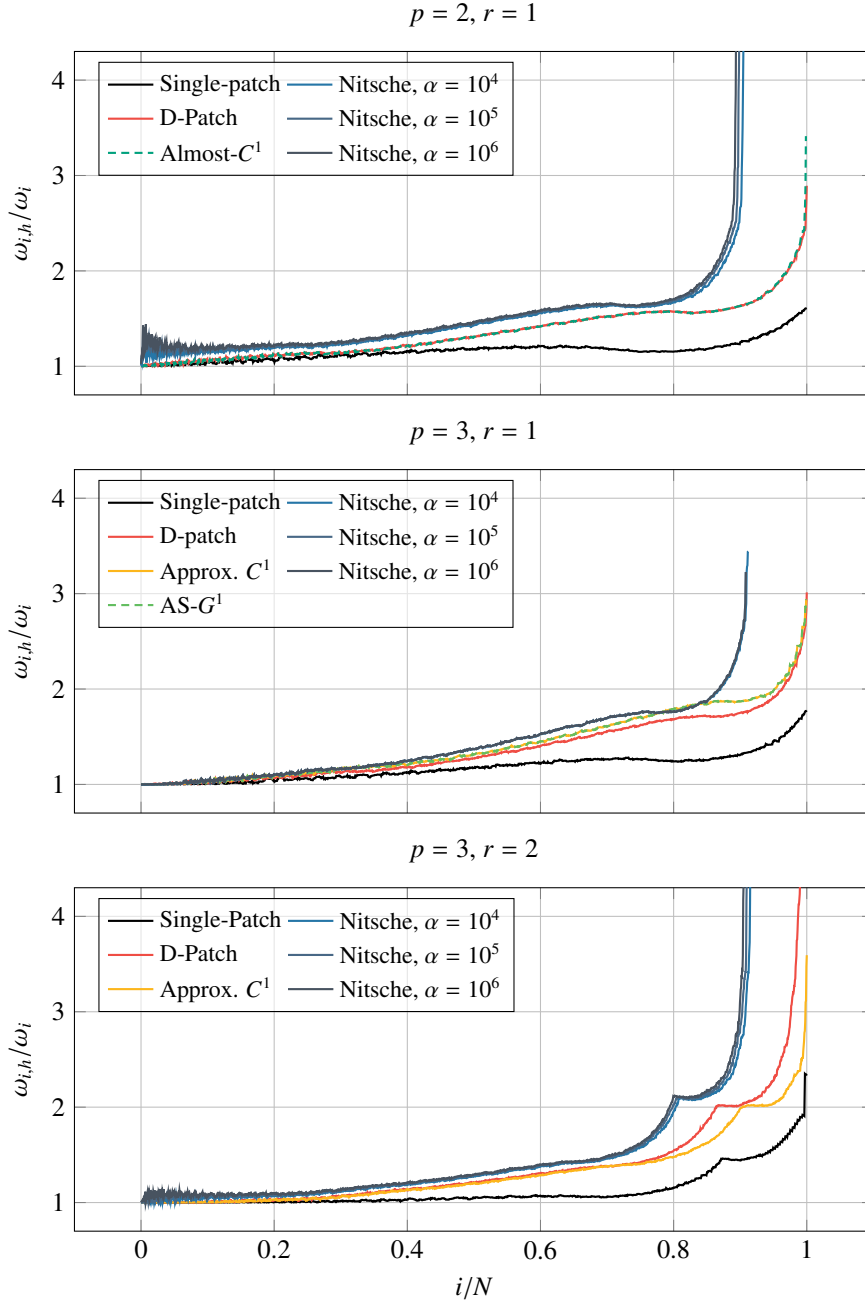


Figure 14: Eigenvalue spectra for the biharmonic eigenvalue problem on the domain from Fig. 8. The horizontal axes depict the eigenvalue index i over the total number of eigenvalues N . The vertical axes represent the numerical eigenvalue $\omega_{i,h}$ over the analytical eigenvalue ω_i , both with index i . The results are plotted for different combinations of the degree p and regularity r of the basis. The results for a Nitsche method are given for different penalty parameters α .

740 4.4. Modal analysis of a complex geometry

741 The next example for the quantitative analysis in this paper involves the modal analysis on
742 a larger-scale complex geometry, depicted in Fig. 15a. The goal of this example is to show the
743 usability of the considered constructions on an off-the-shelf industrial geometry. The geometry
744 is represented as a mesh consisting of 15895 vertices, 31086 edges and 62172 faces. This ge-
745 ometry is converted to bi-linear patches using the procedure discussed in Fig. 4 in Section 3.
746 The interface and boundary curves of the patches are given in Fig. 15b and the final multi-patch
747 object is given in Fig. 15c. The latter has 3 EVs of valence 3, 10 EVs of valence 5 and 16 bEVs.
748 Moreover, the material parameters specified for a steel material. That is, the density of the mate-
749 rial is $\rho = 7850 \cdot 10^{-6}$ [tonnes/mm³], the shell thickness is $t = 10$ [mm], the Young's modulus is
750 $E = 210 \cdot 10^3$ [MPa] and the Poisson's ratio is $\nu = 0.3$ [-]. All the sides of the geometry are kept
751 free, meaning that the modal analysis results will consist of six modes with zero eigenfrequen-
752 cies: the rigid body modes. In the sequel, we list the results for deformation modes only.

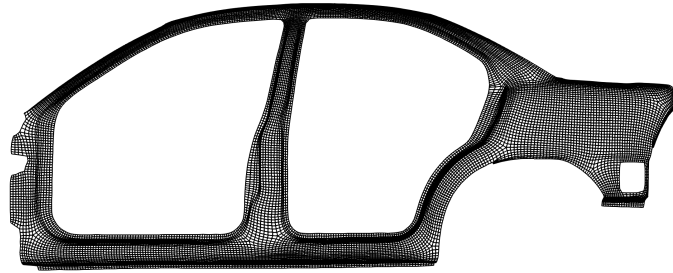
753
754 After the creation of the linear multi-patch object, h -, p - and k -refinement steps can be per-
755 formed to construct a multi-basis corresponding to the patch lay-out on which unstructured
756 splines can be constructed. For the Almost- C^1 and D-Patch constructions, the bases are con-
757 structed by refining and elevating the initial linear basis up to the desired degree and regularity,
758 after which the the Almost- C^1 and D-Patch basis and geometry are computed. An Almost- C^1
759 geometry is provided in Fig. 15c.

760
761 The AS- G^1 construction requires an analysis-suitable geometry, which can be constructed
762 following [8], which is based on the planar construction developed in [85]. However, the ge-
763 ometry from Fig. 15b is only C^0 -smooth due to the original linear mesh it is constructed from.
764 An algorithm to automatically pre-process the geometry to obtain an analysis-suitable G^1 sur-
765 face is not yet developed. The algorithm from [8] requires AS- G^1 gluing data, which cannot be
766 prescribed directly on a C^0 surface. If the surface is not pre-processed to be AS- G^1 , no suitable
767 gluing data can be found and the basis construction is not applicable. Although the Approx. C^1
768 construction does not require an analysis-suitable re-parameterization, it does require G^1 smooth-
769 ness at the interfaces. If this condition is not satisfied there exists no C^1 construction that can
770 be approximated by this method. For both methods, the required pre-processing efforts are non-
771 trivial or not demonstrated on industrial geometries, and therefore left out of the scope of this
772 paper³.

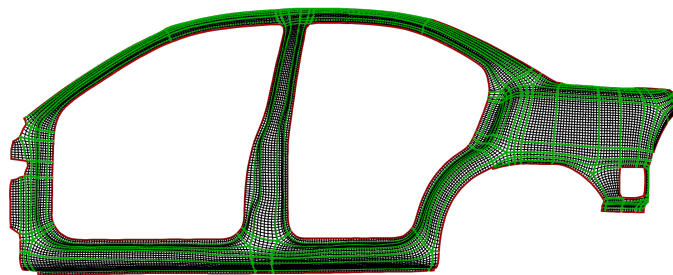
773
774 Furthermore, penalty methods have been used in the context of modal analysis on a 27 patch
775 composite wind-turbine blade in [51], where the variation of the element size of interface ele-
776 ments seems rather small. In the present paper, an attempt was made to apply the penalty method
777 on the geometry in Fig. 15c, but unidentifiable vibration modes were obtained, possibly because
778 of the large variation of element lengths across the interfaces of the domain, challenging the de-
779 termination of a suitable penalty parameter α .

780
781 Table 3 presents the eigenfrequencies for the first four deformation modes of the car side
782 panel for the D-Patch and the Almost- C^1 constructions with degree $p = 2$ and regularity $r = 1$

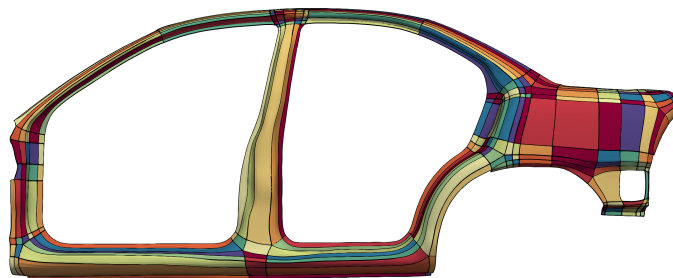
³In the case of a different starting point for this benchmark, such as a smooth mesh composed of higher-order quadri-
lateral elements, e.g. derived from a subdivision surface, instead of a bi-linear mesh, the pre-processing efforts required
for the AS- G^1 and Approx. C^1 will be different.



(a) Original quad mesh with 15895 vertices, 31086 edges and 62172 faces.



(b) Interface (green) and boundary (red) curves.



(c) Final multi-patch segmentation with 307 patches.

Figure 15: Geometry of the side panel of a car. The original mesh (a) is traced with the procedure from Fig. 4, yielding a set of boundary and interface curves (b). From these curves, the multi-patch segmentation (c) for isogeometric analysis is constructed following Fig. 4c.

Table 3: Eigenfrequencies of the Almost- C^1 and D-Patch constructions for the car geometry in Fig. 15. The results of an ABAQUS FEA simulation using the S4R element are provided as a reference. The mode-shapes are plotted in Fig. 16.

Method	# DoFs	Mode 1	Mode 2	Mode 3	Mode 4	
Almost- C^1 , $p = 2$, $r = 1$	13,731	15.740	25.567	43.829	56.654	
	49,758	15.762	25.564	43.429	56.778	
	189,654	15.776	25.552	43.269	56.785	
	740,814	15.774	25.531	43.177	56.746	
D-Patch, $p = 2$, $r = 1$	49,437	15.785	25.607	43.641	56.902	
	189,333	15.780	25.561	43.323	56.807	
	740,493	15.775	25.533	43.191	56.748	
D-Patch, $p = 3$, $r = 1$	136,839	15.749	25.593	43.348	56.786	
	630,459	15.760	25.581	43.231	56.801	
D-Patch, $p = 3$, $r = 2$	71,760	15.771	25.539	43.224	56.744	
	226,524	15.755	25.582	43.235	56.807	
ABAQUS S4R	10mm	126,966	15.303	24.881	42.629	54.887
	5mm	440,076	15.224	24.780	42.516	54.627
	2.5mm	1,653,030	15.119	24.640	42.338	54.277

783 for the Almost- C^1 construction and with $(p, r) = (2, 1)$, $(p, r) = (3, 1)$ and $(p, r) = (3, 2)$ for
784 the D-Patch. Figure 16 provides the corresponding mode shapes on the D-Patch geometry with
785 $p = 3$, $r = 2$ and the mode shapes have been qualitatively matched to construct Table 3. From
786 these results, it can be observed that the Almost- C^1 and D-Patch methods provide eigenfrequen-
787 cies in the same range and that the eigenfrequencies are mostly converging in the second digit.
788 Moreover, the eigenfrequencies of the D-Patch and Almost- C^1 methods for coarse meshes and
789 $p = 2$, $r = 1$ already provide reasonable estimates compared to higher degrees and refinements.
790 On the other hand, the results obtained using an ABAQUS S4R element show convergence in the
791 second digit, and slightly lower frequencies than the IGA results, possibly because the FEM uses
792 a different geometry approximation. Overall, it can be concluded from this benchmark problem
793 that the Almost- C^1 and D-Patch are more robust for industrial and large scale geometries, that
794 are represented by at least C^0 -conforming quadrilateral meshes, compared to the Approx. C^1
795 and AS- G^1 methods due to the pre-processing efforts required by the latter. Furthermore, these
796 methods are parameter-free, making them robust also with respect to penalty methods.

797 4.5. Stress analysis in a curved shell

798 An interesting application for smooth unstructured spline construction is for the use of thin
799 shell analysis for engineering applications. Not only displacements (Section 4.2) or vibrations
800 (Section 4.4) are of interest, but also stress evaluations, for example for fatigue analysis. In
801 the last example, we demonstrate the performance of all methods in Table 2 on the evaluation
802 of stresses in a curved Kirchhoff–Love shell. Since the Kirchhoff–Love shell formulation is
803 displacement-based, the displacements are C^1 continuous across patch interfaces for C^1 con-
804 structions. The stresses, however, are based on the gradients of the displacements, hence their
805 continuity theoretically is C^0 for a perfect C^1 coupling. In this example, we elaborate on the Von
806 Mises membrane stress field resulting from the 6-patch elliptic paraboloid from Fig. 13. The

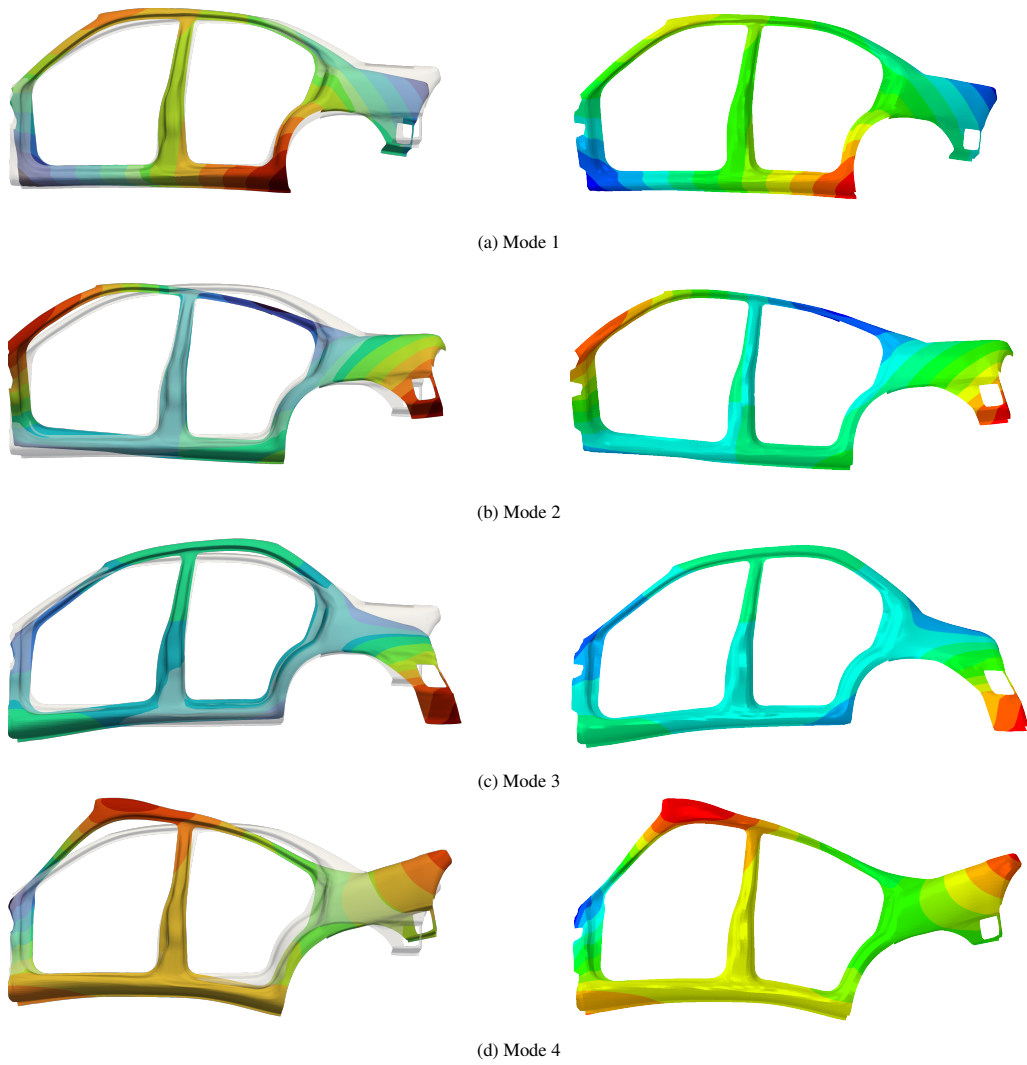


Figure 16: Out-of-plane deformations of the first four vibration modes of the side of the car from Fig. 15. The results on the left represent the results obtained by the D-Patch construction and the results on the right represent results obtained using ABAQUS (10mm). The mode shapes are all deformation modes warped by the deformation vector and plotted over the undeformed (transparent) geometry.

807 stress fields are plotted for bases with degree and regularity from Table 2 and additionally for a
808 basis with $p = 4$, $r = 2$. Note that the regularity r of these bases is the regularity in the patch
809 interior.

810
811 In Fig. 16, the stress fields for the elliptic paraboloid example from Fig. 13 are provided.
812 From these results, it can immediately be seen that the stress field for a single patch parameter-
813 ization with basis $p = 2$, $r = 1$ exposes the elements of the basis because of the C^0 continuity
814 across elements. Similar effects are seen for the D-patch, Almost- C^1 and the penalty method.
815 Increasing the degree of the basis while keeping the regularity the same results in a $p = 3$, $r = 1$
816 basis. The element continuity is still C^0 for the stresses, but the higher continuity of the basis
817 within the element results in a slightly improved stress field, as can be seen from the single patch,
818 the D-patch and penalty methods. The Approx. C^1 and AS- G^1 methods in addition show a better
819 stress field around the EVs compared to the D-patch with only small wiggles in the inner contour.
820 Increasing the smoothness by going to $p = 3$, $r = 2$ shows that the Approx. C^1 method predicts
821 the stress field very well over the whole domain but with the wiggles in the inner contour, and
822 that the D-patch suffers from the singularity at the EVs. Lastly, the $p = 4$, $r = 2$ plots show that
823 the wiggles in the inner contour are eliminated for the Approx. C^1 and the AS- G^1 methods and
824 that the artifacts of the D-patch around the EV are still there but to a lesser extent. Finally, the
825 results of the penalty method in Fig. 16 show it is able to provide an accurate representation of
826 the stress fields. As seen from Fig. 13, penalty factors $\alpha = 1$ and $\alpha = 10$ provide good conver-
827 gence in the bending energy norm. Indeed, the stress fields for the fixed 64×64 element meshes
828 in Fig. 16 confirm that for these penalty factors the stress fields accurately represent the single
829 patch stress fields, despite small artifacts around the EVs for $\alpha = 1$. For a higher penalty factor
830 of $\alpha = 100$, the stress fields following from the penalty method are not guaranteed to be accurate,
831 showing the downside of this method.

832
833 Overall, the stress analysis for multiple combinations and regularities shows that the Almost-
834 C^1 method is generally unfavourable since it is only applicable for $p = 2$, $r = 1$ hence C^0 stress
835 fields, suffering from a lack of continuity over the whole domain. This also makes the D-patch as
836 applicable as the Approx. C^1 method in terms of degree and regularity combinations. Comparing
837 the D-patch with the Approx. C^1 and the AS- G^1 methods, it is shown that the D-patch suffers
838 from the singularity in the EVs when reconstructing stresses, whereas the other two methods are
839 able to recover the stress fields without problems. Moreover, this example has also shown the
840 advantage of smooth unstructured spline constructions for stress analyses, since their continuity
841 across (almost) all of the domain is ensured, contrary to the penalty method. Lastly, this example
842 shows the advantage of IGA in general over lower-order methods like FEA, since the higher-
843 degree bases (e.g. $p = 4$, $r = 2$) provides smooth stress fields compared to lower-degree bases
844 ($p = 2$, $r = 1$).

846 4.6. Conclusions

847 In this section a quantitative comparison of the AS- G^1 , the Approx. C^1 , the D-Patch and the
848 Almost- C^1 constructions is provided. The methods have been assessed on different aspects: i)
849 convergence of the biharmonic equation (Section 4.1); ii) convergence of the linear Kirchhoff-
850 Love shell (Section 4.2); iii) eigenvalue spectrum approximation (Section 4.3); iv) application
851 to a large-scale complex geometry (Section 4.4) and; v) the reconstruction of stress fields (Sec-
852 tion 4.5). From these analyses, the following conclusions can be drawn:

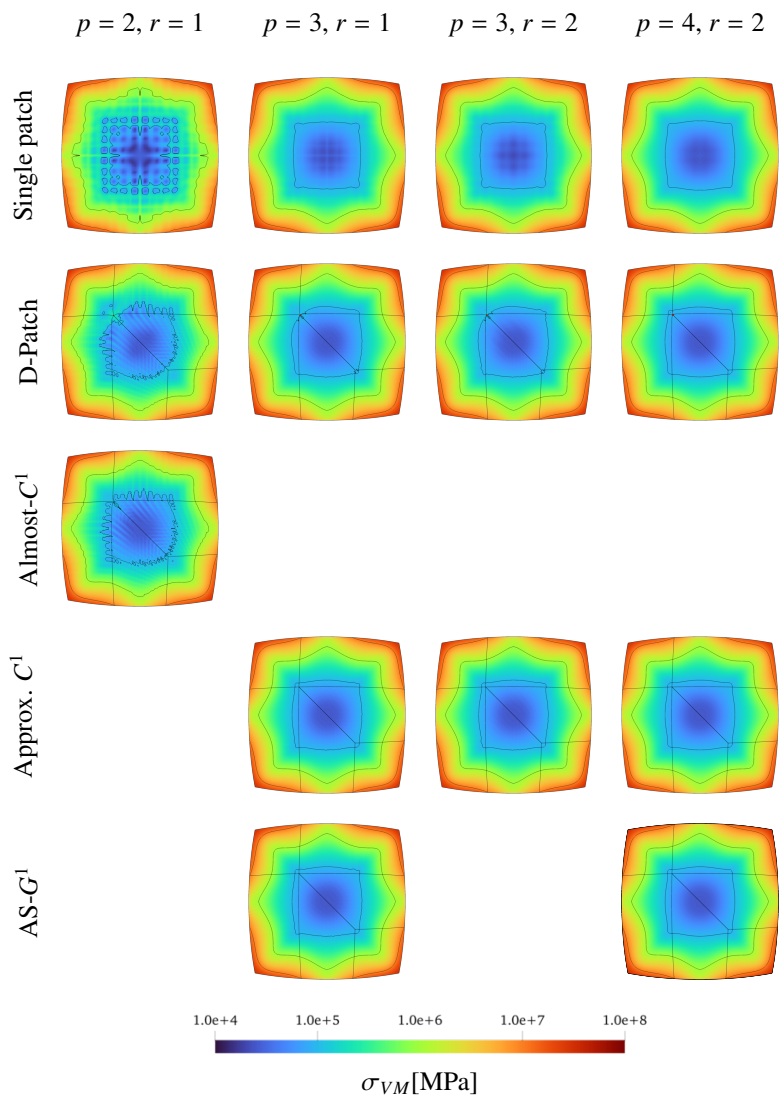


Figure 16: (Caption on next page).

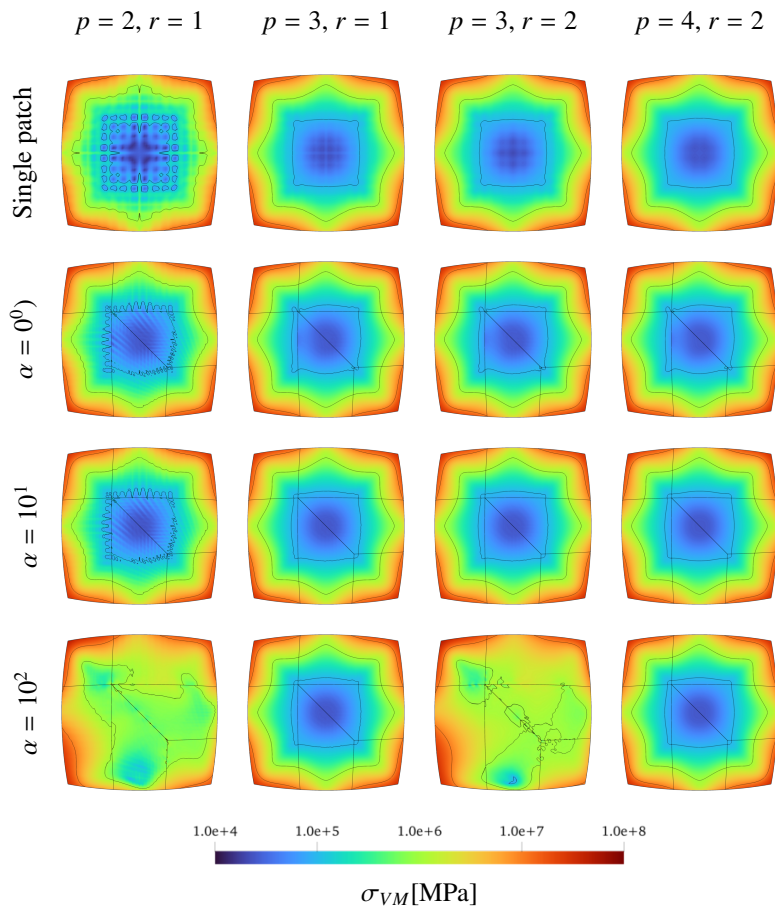


Figure 16: Von Mises membrane stress fields for the single patch, unstructured splines and penalty-coupled multi-patch paraboloid from Section 4.2 and Fig. 13 with 64×64 elements per patch. The results are provided for different combinations of degree p and regularity r . The color bar represents the stress and the contours are plotted for stress levels $\sigma_{VM} \in \{10^5, 10^6, 10^7\}$ [MPa].

- 853 • All methods converge in a theoretical setting to the same solution for the biharmonic equa-
854 tion (Sections 4.1 and 4.2). However, the convergence behaviour of the D-Patch method
855 is sub-optimal and affected by conditioning issues for large meshes. Furthermore, the Ap-
856 prox. C^1 and AS- G^1 methods give worse convergence compared to other methods for the
857 hyperbolic paraboloid shell but good convergence rates for the elliptic paraboloid shell
858 example.
 - 859 • From a spectral analysis on the biharmonic equation Section 4.3 it can be concluded that
860 there is no best unstructured spline construction. Depending on the degree and regularity,
861 small difference in the eigenvalue spectra are observed between the methods. Comparing
862 with Nitsche’s method, however, it is concluded that the unstructured spline constructions
863 considered in this paper perform consistently better. This is also confirmed by the applied
864 modal analysis on the car geometry Section 4.4, where penalty method fails to find accurate
865 eigenfrequencies, possibly because of an unsuitable penalty parameter.
 - 866 • From the applied modal analysis on a complex geometry, it can also be concluded that
867 the Almost- C^1 and D-Patch constructions are more straight-forward to apply to a complex
868 geometry extracted from a mesh. This is due to the fact that the Approx. C^1 and AS-
869 G^1 constructions require, respectively, a G^2 geometry and an analysis-suitable geometry,
870 which are both not trivial to construct from an originally C^0 -continuous mesh. Instead, the
871 D-Patch and Almost- C^1 constructions require a C^1 geometry, which is easier to construct
872 in general.
 - 873 • From the stress fields presented in Section 4.5 following from the analysis in Section 4.2, it
874 can be concluded that the AS- G^1 and Approx. C^1 methods provide excellent stress fields.
875 The D-Patch also provides good stress fields, but inaccuracies are found around the EVs,
876 possibly because of the singularity close to the EV. The Almost- C^1 method is considered
877 inaccurate for stress analysis because of a lack of higher-degree generalizations. Lastly,
878 comparison with penalty methods shows that the unstructured spline constructions gener-
879 ally provide a robust parameter-free approach for coupling, whereas the penalty method
880 requires careful selection of the penalty parameter.
- 881 Overall, our finding suggest that the Almost- C^1 and D-Patch are generally easier to construct,
882 but for certain problems they have limited accuracy. On the other hand, the AS- G^1 or Approx. C^1
883 discretisations require more pre-processing efforts, but provide optimal convergence, hence ac-
884 curacy. This, however, depends on the input geometry: generic quad-meshes might require more
885 pre-processing efforts than C^1 -matching parameterisations. Lastly, the results provided in this
886 section have shown that strong coupling methods have certain advantages over weak methods,
887 and therefore provide an interesting alternative.

888 5. Conclusions and future work

889 In this paper, we provide a qualitative and quantitative comparison of unstructured spline
890 constructions for smooth multi-patches in isogeometric analysis. The general advantage of un-
891 structured spline constructions over trimming or variational coupling methods is that they are
892 parameter-free, do not require specialized solvers and are typically once constructed in a shape
893 optimization workflow. The goal of this paper is to compare the analysis-suitable G^1 (AS- G^1)

894 the approximate C^1 (Approx. C^1), the degenerate patches (D-Patch) and the Almost- C^1 construc-
895 tions with respect to qualitative aspects (i.e. constraints for application) and quantitative aspects
896 (i.e. numerical performance).

897

898 From the qualitative analysis, it followed that each method required a different set of con-
899 straints to be satisfied before the constructions could be applied, see Fig. 7 and Table 1. Degree
900 and regularity constraints can be satisfied by knot insertion routines or re-fitting, which are rel-
901 atively straight-forward. The constraint on analysis-suitability for the AS- G^1 and the constraint
902 on G^2 continuity for the Approx. C^1 method require dedicated reparametrisation routines, such
903 as the one presented by [85]. The fact that D-Patches are restricted to geometries without bound-
904 ary extraordinary vertices requires redefinition of the quadrilateral mesh. Lastly, the fact that
905 the Almost- C^1 method is only defined for bi-quadratic bases ($p = 2$) restricts the inter-element
906 continuity to C^1 through the whole domain. Depending on the application and the availability of
907 existing routines in software, different unstructured spline constructions are favourable, depend-
908 ing on the geometric flexibility or desired degree and regularity .

909

910 From the quantitative analysis, some conclusions can be drawn on the considered unstruc-
911 tured spline constructions and between unstructured spline constructions compared to variational
912 methods such as Nitsche’s method or a penalty method. From the analysis, it was in general ob-
913 served that depending on the problem type, the different methods have their advantages and
914 disadvantages. Firstly, simple biharmonic equations (see Section 4.1) and linear shells (see Sec-
915 tion 4.2) provided good results for all methods. However, the AS- G^1 and Approx. C^1 meth-
916 ods showed slow convergence for the double-curved shell and the D-patch suffered from ill-
917 conditioning for fine meshes. The Almost- C^1 provided good results in general, however it is
918 only applicable on bi-quadratic splines. Secondly, all methods showed superiority over Nitsche’s
919 method for the computation of an eigenvalue spectrum for plate vibrations (see Section 4.3)
920 and no significant differences between the unstructured spline constructions have been observed.
921 Thirdly, the D-Patch and Almost- C^1 showed straight-forward applicability on the problem of a
922 complex geometry (see Section 4.4), whereas the analysis-suitability requirement of the AS- G^1
923 method and the smoothness requirement of the Approx. C^1 method are non-trivial to satisfy on
924 off-the-shelf industrial geometries.

925

926 For the penalty method, no suitable penalty parameter was found, and probably optimal
927 penalty parameters should be chosen per interface rather than globally. Lastly, the AS- G^1 and
928 Approx. C^1 methods provided superior results for stress reconstruction, where the D-Patch suf-
929 fered around the EVs due to its singular parameterization and the Almost- C^1 method provided
930 bad results due to a lack of higher degrees.

931

932 In conclusion, both comparisons provide an overview of the applicability of the methods
933 with respect to the requirements needed to construct them, on the notions of nestedness and in
934 general on the performance of the methods. Overall, it can be concluded from both analyses
935 that among the compared methods, there is no general best construction. More precisely, the
936 quantitative analysis shows that different methods perform differently in different applications,
937 given that they can be constructed. Furthermore, with the backgrounds and properties provided
938 in the qualitative analysis section, we hope that the present paper provides valuable insights for
939 application of the considered methods to multi-patch problems.

940

941 In addition, the comparisons in the present paper provide directions for the improvements of
942 the considered methods. For the AS- G^1 and Approx. C^1 methods, restrictions on geometry and
943 parameterization are a bottleneck in the industrial applications. Therefore, it is recommended
944 to expand the applicability of these methods by developing dedicated geometric pre-processing
945 routines. For the D-Patch construction, the limitation of the construction of the basis near $\nu > 3$
946 boundary EVs calls for the development of routines to eliminate these EVs in quadrilateral multi-
947 patches, as discussed in the qualitative comparison. Furthermore, the example of the bi-harmonic
948 equation has shown that the D-Patch can suffer from ill-conditioned system, hence development
949 of pre-conditioners for D-Patch constructions is advised. Lastly, although the Almost- C^1 resolves
950 the downsides of the D-Patch construction, its restriction on the degree of the spline-space is a
951 major disadvantage when plotting stress fields in shell analysis. Therefore, for the Almost- C^1
952 construction it is recommended to explore expansion to higher degrees.

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