



HAL
open science

Hodge conjecture for projective hypersurfaces

Johann Bouali

► **To cite this version:**

| Johann Bouali. Hodge conjecture for projective hypersurfaces. 2024. hal-04312948v8

HAL Id: hal-04312948

<https://hal.science/hal-04312948v8>

Preprint submitted on 14 Jan 2024 (v8), last revised 29 Jan 2024 (v10)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Hodge conjecture for projective hypersurfaces

Johann Bouali

January 14, 2024

Abstract

We show that a Hodge class of a complex smooth projective hypersurface is an analytic logarithmic De Rham class. On the other hand we show that for a complex smooth projective variety an analytic logarithmic De Rham class of type (d, d) is the class of a codimension d algebraic cycle. We deduce the Hodge conjecture for smooth projective hypersurfaces.

Contents

1	Introduction	1
1.1	Complex analytic De Rham logarithmic classes	1
1.2	Hodge conjecture for smooth projective hypersurfaces	2
2	Preliminaries and Notations	2
2.1	Notations	2
2.2	Complex integral periods	4
2.3	Algebraic cycles and motives	5
2.4	The logarithmic De Rham complexes	7
3	Hypersurface De Rham cohomology and its complex periods	13
4	Complex analytic logarithmic de Rham classes	16
5	Hodge Conjecture for complex smooth projective hypersurfaces	21

1 Introduction

1.1 Complex analytic De Rham logarithmic classes

In [2], we introduced, for X an algebraic variety over a field, the notion of logarithmic De Rham cohomology classes. For X a complex algebraic variety, we also introduce in this work the notion of logarithmic analytic De Rham cohomology classes. In proposition 4 (section 4) we give an analytic analogue of [2] theorem 2 (ii), that is if X is a smooth complex projective variety,

- a logarithmic analytic De Rham cohomology class of type (d, d) is the class of a codimension d algebraic cycle.
- we have $H^j OL_{X^{an},0}(H_{usu}^{j-l}(X^{an}, \Omega_{X^{an},\log,0}^l)) = 0$ for $j, l \in \mathbb{Z}$ such that $2l > j$.

Here $OL_{X^{an},0} : \Omega_{X^{an},\log,0}^l[-l] \rightarrow \Omega_{X^{an}}^\bullet$ is the morphism of the abstract abelian group generated by logarithmic analytic l -forms (see definition 2) to the analytic de Rham complex and usu denote the usual complex topology. The proof of proposition 4 is motivic and similar to the proof of [2] theorem 2 : a logarithmic analytic class of bidegree (p, q) , $p, q \neq 0$, $p \geq q$, $q \neq d_U$ is acyclic for the Zariski topology on

each open subset $U \subset X$ such that there exists an etale map $e : U \rightarrow \mathbb{A}_{\mathbb{C}}^{d_U}$ such that $e : U \rightarrow e(U)$ is finite etale and $\mathbb{A}_{\mathbb{C}}^{d_U} \setminus e(U) \subset \mathbb{A}_{\mathbb{C}}^{d_U}$ is a divisor (proposition 2), this allows us to proceed by a finite induction using the crucial fact that the purity isomorphism for De Rham cohomology preserve logarithmic analytic classes since the purity isomorphism is motivic (c.f. [3], see proposition 1).

1.2 Hodge conjecture for smooth projective hypersurfaces

Let $X = V(f) \subset \mathbb{P}_{\mathbb{C}}^N$ be a smooth projective hypersurface. Denote $j : U := \mathbb{P}_{\mathbb{C}}^N \setminus X \hookrightarrow \mathbb{P}_{\mathbb{C}}^N$ the open complementary subset. Let $(\Omega_{\mathbb{P}^N}^{\bullet} \otimes_{\mathcal{O}_{\mathbb{P}^N}} F^{\bullet-p} j_* \mathcal{O}_U)^{an}$ be the filtered De Rham complex of U . The cone of ι_{∂} :

$$\iota_{\partial} : \Omega_{\mathbb{P}^N, an}^{N-p}(\log X)^{\partial=0}[-p] \hookrightarrow (\Omega_{\mathbb{P}^N}^{\bullet} \otimes_{\mathcal{O}_{\mathbb{P}^N}} F^{\bullet-p} j_* \mathcal{O}_U)^{an}$$

is locally acyclic for the complex topology, more precisely is acyclic on open balls (see e.g.[6]). As a consequence, we show in proposition 3, using an open cover $\mathbb{P}_{\mathbb{C}}^N = \cup_{i=1}^n B_i$ by open balls $\phi_i : B_i \simeq D(0, 1)^N$ such that $\phi_i(B_i \cap X) = 0 \times D(0, 1)^{N-1}$, that a Hodge class $\alpha \in F^{N-p} H^N(U^{an}, \mathbb{Q})$ is logarithmic, that is

$$\alpha \in H^N OL_{U^{an}, 0}(H_{usu}^p(U^{an}, \Omega_{U^{an}, \log, 0}^{N-p})).$$

We get that, if $N = 2p + 1$ and $\lambda \in F^p H^{2p}(X^{an}, \mathbb{Q})$ is a Hodge class,

$$\lambda = Res_{X, \mathbb{P}^N}(\alpha) + m[H] \in H^{2p} OL_{X^{an}}(H_{usu}^p(X^{an}, \Omega_{X^{an}, \log, 0}^p)),$$

where $Res_{X, \mathbb{P}^N} : H^N(U^{an}, \mathbb{C}) \xrightarrow{\sim} H^{2p}(X^{an}, \mathbb{C})$ is the residu map, using the fact that since the residu map is motivic (see definition 1), it preserves the logarithmic De Rham classes. Proposition 4 (ii) then implies that $\lambda = [Z]$, $Z \in \mathcal{Z}^p(X)$, is the class of an algebraic cycle.

To our knowledge, Hodge's conjecture for hypersurfaces was known for only a few cases : hyperplanes, quadrics and Fermat's hypersurfaces of degree ≤ 21 (Shioda). Our new contribution which allowed us to tackle the general case of hypersurfaces is the introduction of the notion of analytical logarithmic de Rham classes on the one hand and the motivic purity isomorphism on the other hand.

I am grateful to Professor F. Mokrane for his help and support during the preparation of this work.

2 Preliminaries and Notations

2.1 Notations

- Denote by Top the category of topological spaces and RTop the category of ringed spaces.
- Denote by Cat the category of small categories and RCat the category of ringed topoi.
- For $\mathcal{S} \in \text{Cat}$ and $X \in \mathcal{S}$, we denote $\mathcal{S}/X \in \text{Cat}$ the category whose objects are $Y/X := (Y, f)$ with $Y \in \mathcal{S}$ and $f : Y \rightarrow X$ is a morphism in \mathcal{S} , and whose morphisms $\text{Hom}((Y', f'), (Y, f))$ consists of $g : Y' \rightarrow Y$ in \mathcal{S} such that $f \circ g = f'$.
- For $(\mathcal{S}, \mathcal{O}_{\mathcal{S}}) \in \text{RCat}$ a ringed topos, we denote by
 - $\text{PSh}(\mathcal{S})$ the category of presheaves of $\mathcal{O}_{\mathcal{S}}$ modules on \mathcal{S} and $\text{PSh}_{\mathcal{O}_{\mathcal{S}}}(\mathcal{S})$ the category of presheaves of $\mathcal{O}_{\mathcal{S}}$ modules on \mathcal{S} , whose objects are $\text{PSh}_{\mathcal{O}_{\mathcal{S}}}(\mathcal{S})^0 := \{(M, m), M \in \text{PSh}(\mathcal{S}), m : M \otimes \mathcal{O}_{\mathcal{S}} \rightarrow M\}$, together with the forgetful functor $o : \text{PSh}(\mathcal{S}) \rightarrow \text{PSh}_{\mathcal{O}_{\mathcal{S}}}(\mathcal{S})$,
 - $C(\mathcal{S}) = C(\text{PSh}(\mathcal{S}))$ and $C_{\mathcal{O}_{\mathcal{S}}}(\mathcal{S}) = C(\text{PSh}_{\mathcal{O}_{\mathcal{S}}}(\mathcal{S}))$ the big abelian category of complexes of presheaves of $\mathcal{O}_{\mathcal{S}}$ modules on \mathcal{S} ,
 - $C_{\mathcal{O}_{\mathcal{S}}(2)fil}(\mathcal{S}) := C_{(2)fil}(\text{PSh}_{\mathcal{O}_{\mathcal{S}}}(\mathcal{S})) \subset C(\text{PSh}_{\mathcal{O}_{\mathcal{S}}}(\mathcal{S}), F, W)$, the big abelian category of (bi)filtered complexes of presheaves of $\mathcal{O}_{\mathcal{S}}$ modules on \mathcal{S} such that the filtration is biregular and $\text{PSh}_{\mathcal{O}_{\mathcal{S}}(2)fil}(\mathcal{S}) := (\text{PSh}_{\mathcal{O}_{\mathcal{S}}}(\mathcal{S}), F, W)$.

- Let $(\mathcal{S}, O_{\mathcal{S}}) \in \text{RCat}$ a ringed topos with topology τ . For $F \in C_{O_{\mathcal{S}}}(\mathcal{S})$, we denote by $k : F \rightarrow E_{\tau}(F)$ the canonical flasque resolution in $C_{O_{\mathcal{S}}}(\mathcal{S})$ (see [1]). In particular for $X \in \mathcal{S}$, $H^*(X, E_{\tau}(F)) \xrightarrow{\sim} \mathbb{H}_{\tau}^*(X, F)$.
- For $f : \mathcal{S}' \rightarrow \mathcal{S}$ a morphism with $\mathcal{S}, \mathcal{S}' \in \text{RCat}$, endowed with topology τ and τ' respectively, we denote for $F \in C_{O_{\mathcal{S}}}(\mathcal{S})$ and each $j \in \mathbb{Z}$,
 - $f^* := H^j \Gamma(\mathcal{S}, k \circ \text{ad}(f^*, f_*)(F)) : \mathbb{H}^j(\mathcal{S}, F) \rightarrow \mathbb{H}^j(\mathcal{S}', f^*F)$,
 - $f^* := H^j \Gamma(\mathcal{S}, k \circ \text{ad}(f^{*mod}, f_*)(F)) : \mathbb{H}^j(\mathcal{S}, F) \rightarrow \mathbb{H}^j(\mathcal{S}', f^{*mod}F)$,

the canonical maps.

- For $m : A \rightarrow B$, $A, B \in C(\mathcal{A})$, \mathcal{A} an additive category, we denote $c(A) : \text{Cone}(m : A \rightarrow B) \rightarrow A[1]$ and $c(B) : B \rightarrow \text{Cone}(m : A \rightarrow B)$ the canonical maps.
- For $X \in \text{Top}$ and $Z \subset X$ a closed subset, denoting $j : X \setminus Z \hookrightarrow X$ the open complementary, we will consider

$$\Gamma_Z^{\vee} \mathbb{Z}_X := \text{Cone}(\text{ad}(j!, j^*)(\mathbb{Z}_X) : j!j^* \mathbb{Z}_X \hookrightarrow \mathbb{Z}_X) \in C(X)$$

and denote for short $\gamma_Z^{\vee} := \gamma_X^{\vee}(\mathbb{Z}_X) := c(\mathbb{Z}_X) : \mathbb{Z}_X \rightarrow \Gamma_Z^{\vee} \mathbb{Z}_X$ the canonical map in $C(X)$.

- Denote by $\text{Sch} \subset \text{RTop}$ the subcategory of schemes (the morphisms are the morphisms of locally ringed spaces). We denote by $\text{PSch} \subset \text{Sch}$ the full subcategory of proper schemes. For a field k , we consider $\text{Sch}/k := \text{Sch}/\text{Spec } k$ the category of schemes over $\text{Spec } k$. The objects are $X := (X, a_X)$ with $X \in \text{Sch}$ and $a_X : X \rightarrow \text{Spec } k$ a morphism and the morphisms are the morphisms of schemes $f : X' \rightarrow X$ such that $f \circ a_{X'} = a_X$. We then denote by
 - $\text{Var}(k) = \text{Sch}^{ft}/k \subset \text{Sch}/k$ the full subcategory consisting of algebraic varieties over k , i.e. schemes of finite type over k ,
 - $\text{PVar}(k) \subset \text{QPVar}(k) \subset \text{Var}(k)$ the full subcategories consisting of quasi-projective varieties and projective varieties respectively,
 - $\text{PSmVar}(k) \subset \text{SmVar}(k) \subset \text{Var}(k)$, $\text{PSmVar}(k) := \text{PVar}(k) \cap \text{SmVar}(k)$, the full subcategories consisting of smooth varieties and smooth projective varieties respectively.
- Denote by $\text{AnSp}(\mathbb{C}) \subset \text{RTop}$ the subcategory of analytic spaces over \mathbb{C} , and by $\text{AnSm}(\mathbb{C}) \subset \text{AnSp}(\mathbb{C})$ the full subcategory of smooth analytic spaces (i.e. complex analytic manifold).
- For $X \in \text{Top}$ and $X = \cup_{i \in I} X_i$ an open cover, we denote $X_{\bullet} \in \text{Fun}(\Delta, \text{Top})$ the associated simplicial space, with for $J \subset I$, $j_{IJ} : X_I := \cap_{i \in I} X_i \hookrightarrow X_J := \cap_{i \in J} X_i$ the open embedding.
- For $X \in \text{AnSm}(\mathbb{C})$ and $X = \cup_{i \in I} X_i$ an open cover, we denote $X_{\bullet} \in \text{Fun}(\Delta, \text{AnSm}(\mathbb{C}))$ the associated simplicial complex manifold, with for $J \subset I$, $j_{IJ} : X_I := \cap_{i \in I} X_i \hookrightarrow X_J := \cap_{i \in J} X_i$ the open embedding.
- Let $(X, O_X) \in \text{RTop}$. We consider its De Rham complex $\Omega_X^{\bullet} := DR(X)(O_X)$.
 - Let $X \in \text{Sch}$. Considering its De Rham complex $\Omega_X^{\bullet} := DR(X)(O_X)$, we have for $j \in \mathbb{Z}$ its De Rham cohomology $H_{DR}^j(X) := \mathbb{H}^j(X, \Omega_X^{\bullet})$.
 - Let $X \in \text{Var}(k)$. Considering its De Rham complex $\Omega_X^{\bullet} := \Omega_{X/k}^{\bullet} := DR(X/k)(O_X)$, we have for $j \in \mathbb{Z}$ its De Rham cohomology $H_{DR}^j(X) := \mathbb{H}^j(X, \Omega_X^{\bullet})$. The differentials of $\Omega_X^{\bullet} := \Omega_{X/k}^{\bullet}$ are by definition k -linear, thus $H_{DR}^j(X) := \mathbb{H}^j(X, \Omega_X^{\bullet})$ has a structure of a k vector space.
 - Let $X \in \text{AnSp}(\mathbb{C})$. Considering its De Rham complex $\Omega_X^{\bullet} := DR(X)(O_X)$, we have for $j \in \mathbb{Z}$ its De Rham cohomology $H_{DR}^j(X) := \mathbb{H}^j(X, \Omega_X^{\bullet})$.

- For $X \in \text{AnSp}(\mathbb{C})$, we denote $\alpha(X) : \mathbb{C}_X \hookrightarrow \Omega_X^\bullet$ the embedding in $C(X)$. For $X \in \text{AnSm}(\mathbb{C})$, $\alpha(X) : \mathbb{C}_X \hookrightarrow \Omega_X^\bullet$ is an equivalence usu local by Poincare lemma.
- We denote $\mathbb{I}^n := [0, 1]^n \in \text{Diff}(\mathbb{R})$ (with boundary). For $X \in \text{Top}$ and R a ring, we consider its singular cochain complex

$$C_{\text{sing}}^*(X, R) := (\mathbb{Z} \text{Hom}_{\text{Top}}(\mathbb{I}^*, X)^\vee) \otimes R$$

and for $l \in \mathbb{Z}$ its singular cohomology $H_{\text{sing}}^l(X, R) := H^n C_{\text{sing}}^*(X, R)$. For $f : X' \rightarrow X$ a continuous map with $X, X' \in \text{Top}$, we have the canonical map of complexes

$$f^* : C_{\text{sing}}^*(X, R) \rightarrow C_{\text{sing}}^*(X', R), \sigma \mapsto f^* \sigma := (\gamma \mapsto \sigma(f \circ \gamma)).$$

In particular, we get by functoriality the complex

$$C_{X, R \text{ sing}}^* \in C_R(X), (U \subset X) \mapsto C_{\text{sing}}^*(U, R)$$

We recall that

- For $X \in \text{Top}$ locally contractible, e.g. $X \in \text{CW}$, and R a ring, the inclusion in $C_R(X)$ $c_X : R_X \rightarrow C_{X, R \text{ sing}}^*$ is by definition an equivalence top local and that we get by the small chain theorem, for all $l \in \mathbb{Z}$, an isomorphism $H^l c_X : H^l(X, R_X) \xrightarrow{\sim} H_{\text{sing}}^l(X, R)$.
- For $X \in \text{Diff}(\mathbb{R})$, the restriction map

$$r_X : \mathbb{Z} \text{Hom}_{\text{Diff}(\mathbb{R})}(\mathbb{I}^*, X)^\vee \rightarrow C_{\text{sing}}^*(X, R), w \mapsto w : (\phi \mapsto w(\phi))$$

is a quasi-isomorphism by Whitney approximation theorem.

- We will consider the morphism of site given by the analytification functor

$$\text{An} : \text{AnSp}(\mathbb{C}) \rightarrow \text{Var}(\mathbb{C}), X \mapsto \text{An}(X) := X^{an}, (f : X' \rightarrow X) \mapsto \text{An}(f) := f^{an}.$$

For $X \in \text{Var}(\mathbb{C})$, we denote by

$$\text{an}_X := \text{An}|_{X^{et}} := X^{an, et} \rightarrow X^{et}, \text{an}_X(U \rightarrow X) := U^{an},$$

its restriction to the small etale site $o_X(X^{et}) \subset \text{Var}(\mathbb{C})$, where $o_X(U \rightarrow X) = U$, in particular we have the following commutative diagram

$$\begin{array}{ccc} \text{AnSp}(\mathbb{C}) & \xrightarrow{\text{An}} & \text{Var}(\mathbb{C}), \\ \downarrow o_X^{an} & & \downarrow o_X \\ X^{an, et} & \xrightarrow{\text{an}_X} & X^{et} \end{array}$$

where $o_X^{an}(U \hookrightarrow X^{an}) = U$. We get the morphism in RTop $\text{an}_X := \text{an}_X|_{o(X)} : X^{an} \rightarrow X$, $o : o(X) \subset X^{et}$ being the embedding of the Zariski open subsets of X .

2.2 Complex integral periods

Let k be a field of characteristic zero and $X \in \text{SmVar}(k)$ be a smooth variety. Let $X = \cup_{i=1}^s X_i$ be an open affine cover. For each embedding $\sigma : k \hookrightarrow \mathbb{C}$, we have the evaluation period embedding map which is the morphism of bi-complexes

$$\begin{aligned} \text{ev}(X)_{\bullet}^\bullet : \Gamma(X_{\bullet}, \Omega_{X_{\bullet}}^\bullet) &\rightarrow \mathbb{Z} \text{Hom}_{\text{Diff}}(\mathbb{I}^\bullet, X_{\mathbb{C}, \bullet}^{an})^\vee \otimes \mathbb{C}, \\ w_I^l \in \Gamma(X_I, \Omega_{X_I}^l) &\mapsto (\text{ev}(X)_I^l(w_I^l) : \phi_I^l \in \mathbb{Z} \text{Hom}_{\text{Diff}}(\mathbb{I}^l, X_{\mathbb{C}, I}^{an})^\vee \otimes \mathbb{C} \mapsto \text{ev}_I^l(w_I^l)(\phi_I^l) := \int_{\mathbb{I}^l} \phi_I^{l*} w_I^l) \end{aligned}$$

given by integration. By taking all the affine open cover $(j_i : X_i \hookrightarrow X)$ of X , we get for $\sigma : k \hookrightarrow \mathbb{C}$, the evaluation period embedding map

$$ev(X) := \varinjlim_{(j_i : X_i \hookrightarrow X)} ev(X)_{\bullet} : \varinjlim_{(j_i : X_i \hookrightarrow X)} \Gamma(X_{\bullet}, \Omega_{X_{\bullet}}^{\bullet}) \rightarrow \varinjlim_{(j_i : X_i \hookrightarrow X)} \mathbb{Z} \text{Hom}_{\text{Diff}(\mathbb{R})}(\mathbb{I}^{\bullet}, X_{\mathbb{C}, \bullet}^{an})^{\vee} \otimes \mathbb{C}$$

It induces in cohomology, for $j \in \mathbb{Z}$, the evaluation period map

$$H^j ev(X) = H^j ev(X)_{\bullet} : H_{DR}^j(X) = H^j \Gamma(X_{\bullet}, \Omega_{X_{\bullet}}^{\bullet}) \rightarrow H_{\text{sing}}^j(X_{\mathbb{C}}^{an}, \mathbb{C}) = H^j(\text{Hom}_{\text{Diff}(\mathbb{R})}(\mathbb{I}^{\bullet}, X_{\mathbb{C}, \bullet}^{an})^{\vee} \otimes \mathbb{C}).$$

which does NOT depend on the choice of the affine open cover by acyclicity of quasi-coherent sheaves on affine noetherian schemes for the left hand side and from Mayer-Vietoris quasi-isomorphism for singular cohomology of topological spaces and Whitney approximation theorem for differential manifolds for the right hand side.

Let $X \in \text{SmVar}(k)$. Then,

$$H^* ev(X_{\mathbb{C}}) = H^* R\Gamma(X_{\mathbb{C}}^{an}, \alpha(X))^{-1} \circ \Gamma(X_{\mathbb{C}}^{an}, E_{zar}(\Omega(\text{an}_X))) : H_{DR}^*(X_{\mathbb{C}}) \xrightarrow{\sim} H_{\text{sing}}^*(X_{\mathbb{C}}^{an}, \mathbb{C})$$

is the canonical isomorphism induced by the analytical functor and the quasi-isomorphism $\alpha(X) : \mathbb{C}_{X_{\mathbb{C}}^{an}} \hookrightarrow \Omega_{X_{\mathbb{C}}^{an}}^{\bullet}$, which gives the periods elements $H^* ev(X)(H_{DR}^*(X)) \subset H_{\text{sing}}^*(X_{\mathbb{C}}^{an}, \mathbb{C})$.

2.3 Algebraic cycles and motives

For $X \in \text{Sch}$ noetherian irreducible and $d \in \mathbb{N}$, we denote by $\mathcal{Z}^d(X)$ the group of algebraic cycles of codimension d , which is the free abelian group generated by irreducible closed subsets of codimension d .

For $X, X' \in \text{Sch}$ noetherian, with X' irreducible, we denote by $\mathcal{Z}^{fs/X'}(X' \times X) \subset \mathcal{Z}_{d_{X'}}(X' \times X)$ which consist of algebraic cycles $\alpha = \sum_i n_i \alpha_i \in \mathcal{Z}_{d_{X'}}(X' \times X)$ such that, denoting $\text{supp}(\alpha) = \cup_i \alpha_i \subset X' \times X$ its support and $p' : X' \times X \rightarrow X'$ the projection, $p'_{|\text{supp}(\alpha)} : \text{supp}(\alpha) \rightarrow X'$ is finite surjective.

Let k be a field. We denote by $\text{Cor SmVar}(k)$ the category whose objects are $\{X \in \text{SmVar}(k)\}$ and

$$\text{Hom}_{\text{Cor SmVar}(k)}(X', X) := \mathcal{Z}^{fs/X'}(X' \times X)$$

(see [3] for the composition law). We denote by $\text{Tr} : \text{Cor SmVar}(k) \rightarrow \text{SmVar}(k)$ the morphism of site given by the embedding $\text{Tr} : \text{SmVar}(k) \hookrightarrow \text{Cor SmVar}(k)$. Let $F \in \text{PSh}(\text{SmVar}(k))$, we say that F admits transfers if $F = \text{Tr}_* F$ with $F \in \text{PSh}(\text{Cor SmVar}(k))$.

Let k be a field. We recall that $\text{DA}(k) := \text{Ho}_{\mathbb{A}^1, et}(C(\text{SmVar}(k)))$ is the derived category of mixed motives.

We have (see e.g. [1]) the Hodge realization functor

$$\mathcal{F}^{Hdg} : \text{DA}(\mathbb{C}) \rightarrow D(MHS), M \mapsto \mathcal{F}^{Hdg}(M) := \iota^{-1}(\mathcal{F}^{FDR}(M), \text{Bti}^* M, \alpha(M)),$$

where $D(MHS) \hookrightarrow D_{2fil}(\mathbb{C}) \times_I D_{fil}(\mathbb{Q})$ is the derived category of mixed Hodge structures.

For $X \in \text{Var}(k)$ and $Z \subset X$ a closed subset, denoting $j : X \setminus Z \hookrightarrow X$ the open complementary, we will consider

$$\Gamma_Z^{\vee} \mathbb{Z}_X := \text{Cone}(\text{ad}(j_{\#}, j^*)(\mathbb{Z}_X) : j_{\#} j^* \mathbb{Z}_X \hookrightarrow \mathbb{Z}_X) \in C(\text{Var}(k)^{sm}/X)$$

and denote for short

$$\gamma_Z^{\vee} := \gamma_Z^{\vee}(\mathbb{Z}_X) := c(\mathbb{Z}_X) : \mathbb{Z}_X \rightarrow \Gamma_Z^{\vee} \mathbb{Z}_X$$

the canonical map in $C(\text{Var}(k)^{sm}/X)$. For $X \in \text{Var}(k)$ and $Z \subset X$ a closed subset, denoting $a_X : X \rightarrow \text{Spec } k$ the structural map, we have then the motive of X with support in Z defined as

$$M_Z(X) := a_{X!} \Gamma_Z^{\vee} a_X^! \mathbb{Z} \in \text{DA}(k).$$

If $X \in \text{SmVar}(k)$, we will also consider

$$a_{X\#} \Gamma_Z^{\vee} \mathbb{Z}_X := \text{Cone}(a_{X\#} \circ \text{ad}(j_{\#}, j^*)(\mathbb{Z}_X) : \mathbb{Z}(U) \hookrightarrow \mathbb{Z}(X)) =: \mathbb{Z}(X, X \setminus Z) \in C(\text{SmVar}(k)),$$

and denote for short

$$\gamma_Z^\vee := a_{X\sharp}\gamma_Z^\vee(\mathbb{Z}_X) := c(\mathbb{Z}(X)) : \mathbb{Z}(X) \rightarrow \mathbb{Z}(X, X \setminus Z)$$

the map in $C(\text{SmVar}(k))$. Then for $X \in \text{SmVar}(k)$ and $Z \subset X$ a closed subset

$$M_Z(X) := a_{X!}\Gamma_Z^\vee a_X^1 \mathbb{Z} = a_{X\sharp}\Gamma_Z^\vee \mathbb{Z}_X =: \mathbb{Z}(X, X \setminus Z) \in \text{DA}(k).$$

- Let $(X, Z) \in \text{Sch}^2$ with $X \in \text{Sch}$ a noetherian scheme and $Z \subset X$ a closed subset. We have the deformation $(D_Z X, \mathbb{A}_Z^1) \rightarrow \mathbb{A}^1$, $(D_Z X, \mathbb{A}_Z^1) \in \text{Sch}^2$ of (X, Z) by the normal cone $C_{Z/X} \rightarrow Z$, i.e. such that

$$(D_Z X, \mathbb{A}_Z^1)_s = (X, Z), s \in \mathbb{A}^1 \setminus 0, (D_Z X, \mathbb{A}_Z^1)_0 = (C_{Z/X}, Z).$$

We denote by $i_1 : (X, Z) \hookrightarrow (D_Z X, \mathbb{A}_Z^1)$ and $i_0 : (C_{Z/X}, Z) \hookrightarrow (D_Z X, \mathbb{A}_Z^1)$ the closed embeddings in Sch^2 .

- Let k be a field of characteristic zero. Let $X \in \text{SmVar}(k)$. For $Z \subset X$ a closed subset of pure codimension c , consider a desingularisation $\epsilon : \tilde{Z} \rightarrow Z$ of Z and denote $n : \tilde{Z} \xrightarrow{\epsilon} Z \subset X$. We have then the morphism in $\text{DA}(k)$

$$G_{Z,X} : M(X) \xrightarrow{D(\mathbb{Z}(n))} M(\tilde{Z})(c)[2c] \xrightarrow{\mathbb{Z}(\epsilon)} M(Z)(c)[2c]$$

where $D : \text{Hom}_{\text{DA}(k)}(M_c(\tilde{Z}), M_c(X)) \xrightarrow{\sim} \text{Hom}_{\text{DA}(k)}(M(X), M(\tilde{Z})(c)[2c])$ is the duality isomorphism from the six functors formalism (moving lemma of Suzlin and Voevodsky) and $\mathbb{Z}(n) := \text{ad}(n_!, n^!)(a_X^1 \mathbb{Z})$, noting that $n_! = n_*$ since n is proper and that $a_X^1 = a_X^*[d_X]$ and $a_Z^1 = a_Z^*[d_Z]$ since X , resp. \tilde{Z} , are smooth (considering the connected components, we may assume X and \tilde{Z} of pure dimension).

We recall the following facts (see [3] and [1]):

Proposition 1. *Let k be a field of characteristic zero. Let $X \in \text{SmVar}(k)$ and $i : Z \subset X$ a smooth subvariety of pure codimension d . Then $C_{Z/X} = N_{Z/X} \rightarrow Z$ is a vector bundle of rank d . The closed embeddings $i_1 : (X, Z) \hookrightarrow (D_Z X, \mathbb{A}_Z^1)$ and $i_0 : (C_{Z/X}, Z) \hookrightarrow (D_Z X, \mathbb{A}_Z^1)$ in $\text{SmVar}^2(k)$ induces isomorphisms of motives $\mathbb{Z}(i_1) : M_Z(X) \xrightarrow{\sim} M_{\mathbb{A}_Z^1}(D_Z X)$ and $\mathbb{Z}(i_0) : M_Z(N_{Z/X}) \xrightarrow{\sim} M_{\mathbb{A}_Z^1}(D_Z X)$ in $\text{DA}(k)$. We get the excision isomorphism in $\text{DA}(k)$*

$$P_{Z,X} := \mathbb{Z}(i_0)^{-1} \circ \mathbb{Z}(i_1) : M_Z(X) \xrightarrow{\sim} M_Z(N_{Z/X}).$$

We have

$$\text{Th}(N_{Z/X}) \circ P_{Z,X} \circ \gamma_Z^\vee(\mathbb{Z}_X) = G_{Z,X} := D(\mathbb{Z}(i)) : M(X) \rightarrow M(Z)(d)[2d].$$

Proof. See [3]. □

Definition 1. *Let k be a field of characteristic zero. Let $X \in \text{SmVar}(k)$ and $i : Z \subset X$ a smooth subvariety of pure codimension d . We consider using proposition 1 the canonical map in $\text{DA}(k)$*

$$\text{Res}_{Z,X} : M(Z)(d)[-2d] \xrightarrow{P_{Z,X}^{-1} \circ \text{Th}(N_{Z/X})^{-1}} M_Z(X) \xrightarrow{c(\mathbb{Z}(X \setminus Z))} M(X \setminus Z)[1].$$

We recall that $c(\mathbb{Z}(X \setminus Z)) : \mathbb{Z}(X, X \setminus Z) \rightarrow \mathbb{Z}(X \setminus Z)[1]$ is the canonical map in $C(\text{SmVar}(k))$.

- Recall that we say that $F \in \text{PSh}(\text{SmVar}(k))$ is \mathbb{A}^1 invariant if for all $X \in \text{SmVar}(k)$, $p^* := F(p) : F(X) \rightarrow F(X \times \mathbb{A}^1)$ is an isomorphism, where $p : X \times \mathbb{A}^1 \rightarrow X$ is the projection.
- Similarly, we say that $F \in \text{PSh}(\text{AnSm}(\mathbb{C}))$ is \mathbb{A}^1 invariant if for all $X \in \text{AnSm}(\mathbb{C})$, $p^* := F(p) : F(X) \rightarrow F(X \times \mathbb{A}^1)$ is an isomorphism, where $p : X \times \mathbb{A}^1 \rightarrow X$ is the projection.

2.4 The logarithmic De Rham complexes

We recall from [2] the following notion :

- For k a field and $X \in \text{Var}(k)$, we consider the embedding in $C(X)$

$$OL_X : \Omega_{X,\log}^\bullet \hookrightarrow \Omega_X^\bullet := \Omega_{X/k}^\bullet,$$

such that, for $X^\circ \subset X$ an open subset and $w \in \Omega_X^p(X^\circ)$, $w \in \Omega_{X,\log}^p(X^\circ)$ if and only if there exists $(n_i)_{1 \leq i \leq s} \in \mathbb{Z}$ and $(f_{i,\alpha_k})_{1 \leq i \leq s, 1 \leq k \leq p} \in \Gamma(X^\circ, O_X)^*$ such that

$$w = \sum_{1 \leq i \leq s} n_i df_{i,\alpha_1}/f_{i,\alpha_1} \wedge \cdots \wedge df_{i,\alpha_p}/f_{i,\alpha_p} \in \Omega_X^p(X^\circ),$$

and for $p = 0$, $\Omega_{X,\log}^0 := \mathbb{Z}$. Let k be a field. We get an embedding in $C(\text{Var}(k))$

$$OL : \Omega_{/k,\log}^\bullet \hookrightarrow \Omega_{/k}^\bullet, \text{ given by, for } X \in \text{Var}(k), \\ OL(X) := OL_X : \Omega_{/k,\log}^\bullet(X) := \Gamma(X, \Omega_{X,\log}^\bullet) \hookrightarrow \Gamma(X, \Omega_X^\bullet) =: \Omega_{/k}^\bullet(X)$$

and its restriction to $\text{SmVar}(k) \subset \text{Var}(k)$.

- For $X \in \text{AnSp}(\mathbb{C})$, we consider the embedding in $C(X)$

$$OL_X : \Omega_{X,\log}^\bullet \hookrightarrow \Omega_X^\bullet := \Omega_{X/\mathbb{C}}^\bullet,$$

such that, for $X^\circ \subset X$ an open subset and $w \in \Omega_X^p(X^\circ)$, $w \in \Omega_{X,\log}^p(X^\circ)$ if and only if there exists $(n_i)_{1 \leq i \leq s} \in \mathbb{Z}$ and $(f_{i,\alpha_k})_{1 \leq i \leq s, 1 \leq k \leq p} \in \Gamma(X^\circ, O_X)^*$ such that

$$w = \sum_{1 \leq i \leq s} n_i df_{i,\alpha_1}/f_{i,\alpha_1} \wedge \cdots \wedge df_{i,\alpha_p}/f_{i,\alpha_p} \in \Omega_X^p(X^\circ),$$

and for $p = 0$, $\Omega_{X,\log}^0 := \mathbb{Z}$. The differential is by definition \mathbb{C} linear. Recall that for $f \in O(X)$, $df = 0$ if and only if f is locally constant with complex values, that is $f|_{X_i^\circ} = \lambda_i \in \mathbb{C}$ for each $1 \leq i \leq r$, where $X = \sqcup_{i=1}^r X_i$ is the decomposition into connected components. We get an embedding in $C(\text{AnSp}(\mathbb{C}))$

$$OL^{an} : \Omega_{\log}^{\bullet,an} \hookrightarrow \Omega^{\bullet,an}, \text{ given by, for } X \in \text{AnSp}(\mathbb{C}), \\ OL^{an}(X) := OL_X : \Omega_{\log}^{\bullet,an}(X) := \Gamma(X, \Omega_{X,\log}^\bullet) \hookrightarrow \Gamma(X, \Omega_X^\bullet) =: \Omega^{\bullet,an}(X)$$

and its restriction to $\text{AnSm}(\mathbb{C}) \subset \text{AnSp}(\mathbb{C})$.

In the analytic case we also consider the following

Definition 2. For $X \in \text{AnSp}(\mathbb{C})$, we consider the morphism in $C(X)$

$$OL_{X,0} : \Omega_{X,\log,0}^\bullet \xrightarrow{q_L} \Omega_{X,\log}^\bullet \xrightarrow{OL_X} \Omega_X^\bullet,$$

where, for $X^\circ \subset X$ an open subset, $p \in \mathbb{N}$,

$$\Omega_{X,\log,0}^p(X^\circ) = \langle df_{i,\alpha_1}/f_{i,\alpha_1} \wedge \cdots \wedge df_{i,\alpha_p}/f_{i,\alpha_p}, f_{i,\alpha_j} \in O(X^\circ)^* \rangle / \\ \langle df_{i,\alpha_1}/f_{i,\alpha_1} \wedge \cdots \wedge d(f_{i,\alpha_j} f'_{i,\alpha_j}) / (f_{i,\alpha_j} f'_{i,\alpha_j}) \wedge \cdots \wedge df_{i,\alpha_p}/f_{i,\alpha_p} \\ - df_{i,\alpha_1}/f_{i,\alpha_1} \wedge \cdots \wedge df_{i,\alpha_j}/f_{i,\alpha_j} \wedge \cdots \wedge df_{i,\alpha_p}/f_{i,\alpha_p} - df_{i,\alpha_1}/f_{i,\alpha_1} \wedge \cdots \wedge df'_{i,\alpha_j}/f'_{i,\alpha_j} \wedge \cdots \wedge df_{i,\alpha_p}/f_{i,\alpha_p} \rangle$$

is the abstract abelian group generated by logarithmic forms : for $f \in O(X^\circ)^*$, $df = 0$ if and only if f is locally constant with integral values (NOT complex values), that is $f|_{X_i^\circ} = \lambda_i \in \mathbb{Z}$ for each $1 \leq i \leq r$, where

$X = \sqcup_{i=1}^r X_i$ is the decomposition into connected components, for $p = 0$, $\Omega_{X, \log, 0}^0 := \mathbb{Z}$, for $p < 0$ of course $\Omega_{X, \log, 0}^p = 0$, and by definition all the differentials of this complex are trivial. Note that $\Omega_{X, \log, 0}^1 = O_X^*$. In particular $OL_{X, 0}$ is NOT injective. We get the morphism in $C(\text{AnSp}(\mathbb{C}))$

$$OL_0^{an} : \Omega_{\log, 0}^{\bullet, an} \xrightarrow{qL} \Omega_{\log}^{\bullet, an} \xrightarrow{OL^{an}} \Omega^{\bullet, an}, \text{ given by, for } X \in \text{AnSp}(\mathbb{C}),$$

$$OL_0^{an}(X) := OL_{X, 0} : \Omega_{\log, 0}^{\bullet, an}(X) := \Gamma(X, \Omega_{X, \log, 0}^{\bullet, an}) \rightarrow \Gamma(X, \Omega_X^{\bullet, an}) =: \Omega^{\bullet, an}(X)$$

and its restriction to $\text{AnSm}(\mathbb{C}) \subset \text{AnSp}(\mathbb{C})$.

Remark 1. Note that for $X \in \text{AnSp}(\mathbb{C})$ and $\partial f \in \Omega^{1, an}(X^{an})$ with $f \in O(X^{an})$, we have $\partial f = (\partial e^f)/e^f \in \Omega_{\log}^{1, an}(X^{an})$, hence, since open balls form a basis of the complex topology on complex analytic manifold,

$$a_{usu}(\Omega_{\log}^{1, an}) = \Omega^{1, an, \partial=0} = a_{usu} \partial O^{an}.$$

On the other hand, for $X \in \text{AnSp}(\mathbb{C})$, $H^1(X, \Omega_{\log, 0}^{1, an}) = \text{Pic}(X)$ is the Picard group, $H^1(X, \Omega^{1, an, \partial=0}) = \mathbb{H}^2(X, F^1 \Omega_X^{\bullet})$ is a complex vector space and

$$OL_{X, 0} : H^1(X, \Omega_{\log, 0}^{1, an}) = \text{Pic}(X) \rightarrow \mathbb{H}^2(X, F^1 \Omega_X^{\bullet}) \rightarrow \mathbb{H}^2(X, \Omega_X^{\bullet})$$

is the cycle class map.

We will also consider :

Definition 3. We consider the (non full) subcategory $\text{AnSm}(\mathbb{C})^{pr} \subset \text{AnSp}(\mathbb{C})$ whose set of objects consists of $X \times \mathbb{A}^{1, an}$, $X \in \text{AnSm}(\mathbb{C})$ and

$$\text{Hom}_{\text{AnSm}(\mathbb{C})^{pr}}(X \times \mathbb{A}^{1, an}, Y \times \mathbb{A}^{1, an}) := \text{Hom}_{\text{AnSp}(\mathbb{C})}(X, Y) \otimes I_{\mathbb{A}^{1, an}}.$$

We have then the morphism of sites

- $\pi : \text{AnSm}(\mathbb{C}) \rightarrow \text{AnSm}(\mathbb{C})^{pr}$, $\pi(X \times \mathbb{A}^{1, an}) := X$, $\pi(f \otimes I) := f$,
- $\rho : \text{AnSm}(\mathbb{C})^{pr} \rightarrow \text{AnSm}(\mathbb{C})$, $\rho(X) := X \times \mathbb{A}^{1, an}$, $\rho(f) := f \otimes I$,

in particular $\rho \circ \pi = I$. We have then, for each $p \in \mathbb{Z}$, a canonical splitting in $\text{PSh}(\text{AnSm}(\mathbb{C})^{pr})$

$$\Omega^{p, an, \partial=0} = \rho_* \Omega^{p, an, \partial=0} \oplus \Omega^{p, e},$$

with

$$\Omega^{p, e}(X \times \mathbb{A}^1) := \left\{ \sum_{n=1}^{\infty} a_n z^n w(x) + w' \wedge dz, a_n \in \mathbb{C}, w \in \Omega^p(X), w' \in \Omega^{p-1}(X \times \mathbb{A}^1) \right\}$$

$$\subset \Omega^{p, an, \partial=0}(X \times \mathbb{A}^1)$$

where z is the coordinate of \mathbb{A}^1 , $a_n r^n \rightarrow 0$ when $n \rightarrow \infty$ for all $r \in \mathbb{R}^+$, and $x \in X$. We have the factorization in $\text{PSh}(\text{AnSm}(\mathbb{C}))$

$$\text{ad}(\pi) := \text{ad}(\pi^*, \pi_*)(-) : \pi^* \Omega^{p, e} \xrightarrow{\subset} \pi^* \Omega^{p, an, \partial=0} := \pi^* \pi_* \Omega^{p, an, \partial=0}$$

$$\xrightarrow{(0, \text{ad}(\pi))} \text{ad}(\pi)(\pi^* \Omega^{p, e}) \xrightarrow{i_{\text{ad}(\pi)}} \Omega^{p, an, \partial=0},$$

and we denote again by abuse

- $\pi^* \Omega^{\bullet, e} := \text{ad}(\pi)(\pi^* \Omega^{\bullet, e}) \subset \Omega^{\bullet, an, \partial=0} \subset \Omega^{\bullet, an}$,
- $\pi^* \Omega^{\bullet, e} := \text{ad}(\pi)(\pi^* \Omega^{\bullet, e}) \cap \Omega_{\log}^{\bullet, an} \subset \Omega^{p, an, \partial=0} \subset \Omega^{\bullet, an}$.

We will consider, referring to definition 2, for each $p \in \mathbb{Z}$, the exact sequence in $\text{PSh}(\text{AnSm}(\mathbb{C}))$

$$0 \rightarrow \Omega_{\log,0}^{p,an} \cap q_L^{-1}(\pi^* \Omega^{p,e}) \xrightarrow{e} \Omega_{\log,0}^{p,an} \xrightarrow{q} \Omega_{\log,0}^{p,an,ne} := \Omega_{\log,0}^{p,an} / (\Omega_{\log,0}^{p,an} \cap q_L^{-1}(\pi^* \Omega^{p,e})) \rightarrow 0.$$

We have then the commutative diagram in $C(\text{AnSm}(\mathbb{C}))$

$$\begin{array}{ccc} \Omega_{\log,0}^{\bullet,an} \cap q_L^{-1}(\pi^* \Omega^{\bullet,e}) & \xrightarrow{OL_0^{an}} & \pi^* \Omega^{\bullet,e} \\ \downarrow e & & \downarrow e \\ \Omega_{\log,0}^{\bullet,an} & \xrightarrow{OL_0^{an}} & \Omega^{\bullet,an} \\ \downarrow q & & \downarrow q \\ \Omega_{\log,0}^{\bullet,an,ne} := \Omega_{\log,0}^{\bullet,an} / (\Omega_{\log,0}^{\bullet,an} \cap q_L^{-1}(\pi^* \Omega^{\bullet,e})) & \xrightarrow{\tilde{O}L_0^{an}} & \Omega^{\bullet,an} / (\pi^* \Omega^{\bullet,e}) \end{array},$$

whose columns are exact with e injective and q surjective, and of course all differentials of $\Omega_{\log,0}^{\bullet,an}$ and of $\pi^* \Omega^{\bullet,e}$ are trivial.

Lemma 1. We have a splitting in $C(\text{AnSm}(\mathbb{C}))$

$$a_{usu} \Omega_{\log,0}^{\bullet,an} = a_{usu} (\Omega_{\log,0}^{\bullet,an} \cap q_L^{-1}(\pi^* \Omega^{\bullet,e})) \oplus a_{usu} \Omega_{\log,0}^{\bullet,an,ne}$$

Proof. We have

$$\pi^* \Omega^\bullet(X) := \varinjlim_{m: X \rightarrow Y \times \mathbb{A}^1} \Omega(m)(\Omega^{\bullet,an}(Y \times \mathbb{A}^1)) \subset \Omega^{\bullet,an}(X),$$

where we denote again by abuse

$$\pi^* \Omega^\bullet := \text{ad}(\pi)(\pi^* \Omega^\bullet) \subset \Omega^{\bullet,an}.$$

Now,

- if $q \circ m : X \xrightarrow{m} Y \times \mathbb{A}^1 \xrightarrow{q} \mathbb{A}^1$ is non constant, we can factor

$$m = (n \times I) \circ m' : X \xrightarrow{m'} Y' \times \mathbb{A}^1 \xrightarrow{n \times I} Y \times \mathbb{A}^1$$

with $\dim Y' = \dim X - 1$ and $m'(X) \subset Y' \times \mathbb{A}^1$ an open subset, and

$$\Omega(m') : \Omega^{\bullet,an}(Y' \times \mathbb{A}^1) \rightarrow \Omega^{\bullet,an}(X)$$

so that we still have a splitting in the image of $\Omega(m')$

- if $q \circ m : X \xrightarrow{m} Y \times \mathbb{A}^1 \xrightarrow{q} \mathbb{A}^1$ is constant with value c ,

$$\Omega(m)(\pi^* \Omega^{\bullet,e})(Y \times \mathbb{A}^1) = 0$$

since $c^* dz = 0$, where $c : Y \times \{c\} \hookrightarrow Y \times \mathbb{A}^1$ is the closed embedding.

We then define

$$(\pi^* \Omega^\bullet)^{nc} \subset \pi^* \Omega^\bullet, (\pi^* \Omega^\bullet)^{nc}(X) := \varinjlim_{m: X \rightarrow Y \times \mathbb{A}^1, q \circ m \text{ nc}} \Omega(m)(\Omega^{\bullet,an}(Y \times \mathbb{A}^1)) \subset \pi^* \Omega^\bullet(X),$$

where nc means non constant. Now consider a splitting a complex vector spaces

$$\Omega^{1,\partial=0}(D^1) = \Omega^{1,\partial=0}(\mathbb{A}^1) \oplus \Omega^{1,na}(D^1).$$

We consider similarly the (non full) subcategory $\text{AnSm}(\mathbb{C})^{pr'} \subset \text{AnSp}(\mathbb{C})$ whose set of objects consists of $X \times D^1$, $X \in \text{AnSm}(\mathbb{C})$ and

$$\text{Hom}_{\text{AnSm}(\mathbb{C})^{pr'}}(X \times D^1, Y \times D^1) := \text{Hom}_{\text{AnSp}(\mathbb{C})}(X, Y) \otimes I_{D^1}.$$

We have then similarly the morphism of sites

- $\pi' : \text{AnSm}(\mathbb{C}) \rightarrow \text{AnSm}(\mathbb{C})^{pr'}$, $\pi'(X \times D^1) := X$, $\pi'(f \otimes I) := f$,
- $\rho' : \text{AnSm}(\mathbb{C})^{pr'} \rightarrow \text{AnSm}(\mathbb{C})$, $\rho'(X) := X \times D^1$, $\rho'(f) := f \otimes I$,

in particular $\rho' \circ \pi' = I$. We define similarly

$$(\pi'^* \Omega^\bullet)^{nc} \subset \pi'^* \Omega^\bullet, (\pi'^* \Omega^\bullet)^{nc}(X) := \varinjlim_{m: X \rightarrow Y \times D^1, q \circ m \text{ nc}} \Omega(m)(\Omega^{\bullet, an}(Y \times D^1)) \subset \pi'^* \Omega^\bullet(X).$$

We have then

$$\begin{aligned} \Omega^{\bullet, an} &= a_{usu}(\pi'^* \Omega^\bullet)^{nc} = a_{usu}(\pi^* \Omega^\bullet)^{nc} \oplus a_{usu} \pi'^* \Omega^{\bullet, nc, na}, \\ \pi'^* \Omega^{\bullet, na}(X) &:= \varinjlim_{m: X \rightarrow Y \times D^1, q \circ m \text{ nc}} \Omega(m)(\Omega^{\bullet-1, an}(Y) \times \Omega^{1, na}(D^1)) \subset \Omega^{\bullet, an}(X) \end{aligned}$$

and thus

$$a_{usu} \Omega_{\log, 0}^{\bullet, an} = a_{usu} q_L^{-1}(\Omega_{\log}^{\bullet, an} \cap (\pi'^* \Omega^\bullet)^{nc}) = a_{usu}(\Omega_{\log, 0}^{\bullet, an} \cap q_L^{-1}(\pi^* \Omega^\bullet)^{nc}) \oplus a_{usu}(\Omega_{\log, 0}^{\bullet, an} \cap q_L^{-1}(\pi'^* \Omega^{\bullet, nc, na})),$$

where the first equalities follows from the fact for an open ball $D^{dx} \subset X$, we have the morphism $m_I : D^{dx} \rightarrow D^{dx-1} \times D^1$ which is the canonical isomorphism. Hence,

$$a_{usu} \Omega_{\log, 0}^{\bullet, an} = a_{usu}(\Omega_{\log, 0}^{\bullet, an} \cap q_L^{-1}(\pi^* \Omega^{\bullet, e})) \oplus \tilde{\Omega}_{\log, 0}^{\bullet, an, ne}$$

with

$$\tilde{\Omega}_{\log, 0}^{\bullet, an, ne}(X) := a_{usu}(\Omega_{\log, 0}^{\bullet, an} \cap q_L^{-1}(\pi^* \rho_* \Omega^\bullet)^{nc}) \oplus a_{usu}(\Omega_{\log, 0}^{\bullet, an} \cap q_L^{-1}(\pi'^* \Omega^{\bullet, nc, na})).$$

□

We have the following :

Lemma 2. (i0) The sheaves $O_{an}^* \in \text{PSh}(\text{AnSm}(\mathbb{C}))$ and $\Omega_{\log}^{1, an} \in \text{PSh}(\text{AnSm}(\mathbb{C}))$ admit transfers compatible with transfers on $\Omega^{1, an} \in \text{PSh}(\text{AnSm}(\mathbb{C}))$.

(i) For each $l \in \mathbb{Z}$, the sheaf $\Omega_{\log}^{l, an} \in \text{PSh}(\text{AnSm}(\mathbb{C}))$ admits transfers compatible with transfers on $\Omega^{l, an} \in \text{PSh}(\text{AnSm}(\mathbb{C}))$, that is $\Omega^{\bullet, an} \in C(\text{Cor AnSm}(\mathbb{C}))$ and the inclusion $OL : \Omega_{\log}^{l, an}[-l] \hookrightarrow \Omega^{\bullet, an}$ in $C(\text{AnSm}(\mathbb{C}))$ is compatible with transfers.

(ii) For each $l \in \mathbb{Z}$, the presheaf $\Omega_{\log, 0}^{l, an, ne} \in \text{PSh}(\text{AnSm}(\mathbb{C}))$ given in definition 3 is \mathbb{A}^1 invariant.

(ii)' For each $l \in \mathbb{Z}$, the sheaf $a_{usu} \Omega_{\log, 0}^{l, an, ne} \in \text{PSh}(\text{AnSm}(\mathbb{C}))$ given in definition 3 is \mathbb{A}^1 invariant, where $a_{usu} : \text{PSh}(\text{AnSp}(\mathbb{C})) \rightarrow \text{Shv}(\text{AnSp}(\mathbb{C}))$ is the sheafification functor.

Proof. (i0): Let $W \subset X' \times X$ with $X, X' \in \text{AnSm}(\mathbb{C})$, X' connected, and $p_{X'} : W \subset X' \times X \rightarrow X'$ finite surjective over X' . Then $M(p_X) : M(X') \hookrightarrow M(W)$ is a finite, where for $Y \in \text{AnSp}(\mathbb{C})$ irreducible, $M(Y) := \text{Frac}(O(Y))$ denote the field of meromorphic function. Since $O(X') \subset M(X')$ is integrally closed, the trace map $Tr_{W/X'} : M(W) \rightarrow M(X')$ sends $O(W)$ to $O(X')$, and the norm map $N_{W/X'} : M(W)^* \rightarrow M(X')^*$ sends $O(W)^*$ to $O(X')^*$.

The sheaf $O_{an}^* \in \text{PSh}(\text{AnSm}(\mathbb{C}))$ admits transfers : for $W \subset X' \times X$ with $X, X' \in \text{AnSm}(\mathbb{C})$ and W finite surjective over X' and $f \in O(X)^*$, $W^* f := N_{W/X'}(p_X^* f)$ where $p_X : W \hookrightarrow X' \times X \rightarrow X$ is the projection and $N_{W/X'} : M(W)^* \rightarrow M(X')^*$ is the norm map. This gives transfers on $\Omega_{\log}^{1, an} \in \text{PSh}(\text{AnSm}(\mathbb{C}))$ compatible with transfers on $\Omega^{1, an} \in \text{PSh}(\text{AnSm}(\mathbb{C}))$: for $W \subset X' \times X$ with $X, X' \in \text{AnSm}(\mathbb{C})$ and W finite surjective over X' and $f \in O(X)^*$,

$$W^* df/f := dW^* f/W^* f = Tr_{W/X'}(p_X^*(df/f)),$$

where we recall $p_X : W \hookrightarrow X' \times X \rightarrow X$ is the projection and $Tr_{W/X'} : O(W) \rightarrow O(X')$ is the trace map. Note that $d(fg)/fg = df/f + dg/g$.

(i): By (i0), we get transfers on

$$\otimes_{\mathbb{Q}}^l \Omega_{\log}^{1,an}, \otimes_O^l \Omega^{1,an} \in \text{PSh}(\text{AnSm}(\mathbb{C}))$$

since $\otimes_{\mathbb{Q}}^l \Omega_{\log}^{1,an} = H^0(\otimes_{\mathbb{Q}}^{L,l} \Omega_{\log}^{1,an})$ and $\otimes_O^l \Omega^{1,an} = H^0(\otimes_O^{L,l} \Omega^{1,an})$. This induces transfers on

$$\wedge_{\mathbb{Q}}^l \Omega_{\log}^{1,an} := \text{coker}(\oplus_{I_2 \subset [1, \dots, l]} \otimes_{\mathbb{Q}}^{l-1} \Omega_{\log}^{1,an} \xrightarrow{\oplus_{I_2 \subset [1, \dots, l]} \Delta_{I_2 := (w \otimes w' \mapsto w \otimes w \otimes w')}} \otimes_{\mathbb{Q}}^l \Omega_{\log}^{1,an}) \in \text{PSh}(\text{AnSm}(\mathbb{C})).$$

and

$$\wedge_O^l \Omega^{1,an} := \text{coker}(\oplus_{I_2 \subset [1, \dots, l]} \otimes_{O_k}^{l-1} \Omega^{1,an} \xrightarrow{\oplus_{I_2 \subset [1, \dots, l]} \Delta_{I_2 := (w \otimes w' \mapsto w \otimes w \otimes w')}} \otimes_O^l \Omega^{1,an}) \in \text{PSh}(\text{AnSm}(\mathbb{C})).$$

(ii): Let $X \in \text{AnSm}(\mathbb{C})$ and, denoting $p : X \times \mathbb{A}^1 \rightarrow X$ the projection,

$$\Omega(p) : \Omega_{\log,0}^{p,an,ne}(X) \rightarrow \Omega_{\log,0}^{p,an,ne}(X \times \mathbb{A}^1).$$

Since $\Omega(p)$ is split injective, it is enough to show that $\Omega(p)$ is surjective. We have

$$\Omega_{\log,0}^{p,an,ne}(X \times \mathbb{A}^1) := \Omega_{\log,0}^{p,an}(X \times \mathbb{A}^1) / q_L^{-1} \Omega^{p,e}(X \times \mathbb{A}^1)$$

The result then follows from the fact that for $X \in \text{AnSp}(\mathbb{C})$, $\Omega^p(X \times \mathbb{A}^1) = \Omega^p(X) \oplus (\Omega^{p-1}(X) \otimes \Omega^1(\mathbb{A}^{1,an}))$ and that $\mathbb{A}^{1,an}$ is simply connected. Indeed, for $w = w' + w'' \wedge dz \in \Omega^p(X \times \mathbb{A}^1)^{\partial=0}$,

$$w(x, z) = \partial \left(\int_{t=0}^{t=z} w''(x, t) dt \right) + w(x, 0) = \partial \left(\int_{t=0}^{t=z} w''(x, t) dt \right)_e + \partial \left(\int_{t=0}^{t=z} w''(x, t) dt \right)(x, 0) + w'(x, 0)$$

and $\partial \left(\int_{t=0}^{t=z} w''(x, t) dt \right)_e \in \Omega^{p,e}(X \times \mathbb{A}^1)$

(ii): Follows from (ii) since open ball form a basis of the complex topology on complex analytic manifolds. \square

We have the following result :

Proposition 2. Consider, for each $p \in \mathbb{Z}$, the presheaf $\Omega_{\log}^{p,an,ne} \in \text{PSh}(\text{AnSm}(\mathbb{C}))$ given in definition 3.

(i) Let $D \in \text{AnSm}(\mathbb{C})$ be an open ball. Then, for each $p, q \in \mathbb{Z}$, $q \neq 0$, $H_{usu}^q(D, \Omega_{D, \log, 0}^p) = 0$.

(i)' Let $D \in \text{AnSm}(\mathbb{C})$ be an open ball. Then, for each $p, q \in \mathbb{Z}$, $q \neq 0$, $H_{usu}^q(D, \Omega_{\log, 0}^{p,an,ne}) = 0$.

(ii) Let $U \in \text{SmVar}(\mathbb{C})$ such that there exists an etale map $e : U \rightarrow \mathbb{A}_{\mathbb{C}}^{d_U}$ such that $e : U \rightarrow e(U)$ is finite etale and $\mathbb{A}_{\mathbb{C}}^{d_U} \setminus e(U) \subset \mathbb{A}_{\mathbb{C}}^{d_U}$ is a divisor. Then, for each $p \in \mathbb{Z}$, $p \neq 0$, $p \neq d_U$, $H_{usu}^p(U^{an}, \Omega_{\log, 0}^{p,an,ne}) = 0$.

(ii)' Let $U \in \text{SmVar}(\mathbb{C})$ such that there exists an etale map $e : U \rightarrow \mathbb{A}_{\mathbb{C}}^{d_U}$ such that $e : U \rightarrow e(U)$ is finite etale and $\mathbb{A}_{\mathbb{C}}^{d_U} \setminus e(U) \subset \mathbb{A}_{\mathbb{C}}^{d_U}$ is a divisor. Then, for each $p, q \in \mathbb{Z}$, $p, q \neq 0$, $p \geq q$, $q \neq d_U$, $H_{usu}^q(U^{an}, \Omega_{\log, 0}^{p,an,ne}) = 0$.

Proof. (i): Let $B = D(0, 1)^N \in \text{AnSm}(\mathbb{C})$ be an open ball. Let $\alpha \in H_{usu}^q(B, \Omega_{B, \log, 0}^p)$, $q \in \mathbb{Z}$, $q \neq 0$. Consider a sequence of sub-balls

$$l_n : B_n := D(0, r_n)^N \hookrightarrow B := D(0, 1)^N, n \in \mathbb{N}, \text{ with } r_n \rightarrow 0 \text{ when } n \rightarrow \infty.$$

Since $(B_n := D(0, r_n))_{n \in \mathbb{N}}$ form a basis of neighborhood of $0 \in B$ and since $\Omega_{B, \log, 0}^p$ is a single presheaf, there exists $n := n_{\alpha} \in \mathbb{N}$ (depending on α) such that

$$l_n^* \alpha = 0 \in H_{usu}^q(B_n, \Omega_{B, \log, 0}^p).$$

But it follows immediately from the definition of the the presheaf $\Omega_{B,\log,0}^p$ considering the isomorphism

$$\phi_n : B_n \xrightarrow{\sim} B, \phi_n(z) := r_n^{-1}z$$

so that $(\phi_n)^{-1} \circ l_n = r_n I$, that

$$l_n^* \alpha = l_n^* \phi_n^{-1*} \phi_n^* \alpha = r_n \phi_n^* \alpha \in H_{usu}^q(B_n, \Omega_{B,\log,0}^p).$$

Hence $\phi_n^* \alpha = 0$. Since ϕ_n is an isomorphism, we get $\alpha = 0$.

(i)': Consider the splitting

$$a_{usu} \Omega_{\log,0}^{\bullet,an} = a_{usu} (\Omega_{\log,0}^{\bullet,an} \cap q_L^{-1}(\pi^* \Omega^{\bullet,e})) \oplus a_{usu} \Omega_{\log,0}^{\bullet,an,ne}$$

given in lemma 1. It gives in particular a canonical isomorphism

$$H_{usu}^q(D, \Omega_{\log,0}^p) = H_{usu}^q(D, \Omega_{\log,0}^p \cap \pi^* \Omega^{p,e}) \oplus H_{usu}^q(D, \Omega_{\log,0}^{p,an,ne}).$$

By (i) we have

$$H_{usu}^q(D, \Omega_{\log,0}^p) = 0 = H_{usu}^q(D, \Omega_{\log,0}^p \cap \pi^* \Omega^{p,e}) \oplus H_{usu}^q(D, \Omega_{\log,0}^{p,an,ne}).$$

Hence $H_{usu}^q(D, \Omega_{\log,0}^{p,an,ne}) = 0$. This proves (i)'. One can also prove (i)' directly using the same arguments as in the proof of (i).

(ii): We proceed by induction on the dimension of U . Let $U \in \text{SmVar}(\mathbb{C})$ such that there exists an etale map $e : U \rightarrow \mathbb{A}_{\mathbb{C}}^{d_U}$ such that $e : U \rightarrow e(U)$ is finite etale and $\mathbb{A}_{\mathbb{C}}^{d_U} \setminus e(U) \subset \mathbb{A}_{\mathbb{C}}^{d_U}$ is a divisor. Denote $D := \mathbb{A}_{\mathbb{C}}^{d_U} \setminus e(U)$ the complementary closed subset. Since $H_{usu}^q(\mathbb{A}_{\mathbb{C}}^{d_U}, \Omega_{\log,0}^{p,an,ne}) = 0$ for all $p, q \in \mathbb{Z}$,

$$H^q E_{usu}(\Omega_{\log,0}^{p,an,ne})(c(\mathbb{Z}(e(U)))) = \partial : H_{usu}^q(e(U), \Omega_{\log,0}^{p,an,ne}) \xrightarrow{\sim} H_{usu,D}^{q+1}(\mathbb{A}_{\mathbb{C}}^{d_U}, \Omega_{\log,0}^{p,an,ne})$$

is an isomorphism for all $p, q \in \mathbb{Z}$. By (i),

$$H^q E_{usu}(\Omega_{\log,0}^{p,an,ne}) \in \text{PSh}(\text{AnSm}(\mathbb{C}))$$

is \mathbb{A}^1 invariant for each $p, q \in \mathbb{Z}$. Indeed, for $X \in \text{AnSm}(\mathbb{C})$ connected, take an open cover $X = \cup_{i \in I} D_i$ by open balls $D_i \simeq D(0, 1)^{d_X}$, and let $D_{\bullet} \in \text{Fun}(\Delta, \text{AnSm}(\mathbb{C}))$ be the associated simplicial complex manifold, with for $J \subset I$, $j_{IJ} : D_I := \cap_{i \in I} D_i \hookrightarrow D_J := \cap_{i \in J} D_i$ the open embedding. Then we get the isomorphism

$$\begin{aligned} p_X^* : H^q E_{usu}(\Omega_{\log,0}^{p,an,ne})(X \times \mathbb{A}^1) &:= H_{usu}^q(X \times \mathbb{A}^1, \Omega_{\log,0}^{p,an,ne}) \xrightarrow{\sim} \\ &\mathbb{H}^q(D_{\bullet} \times \mathbb{A}^1, \Omega_{\log,0}^{p,an,ne}) \xrightarrow{\sim} H^q \Gamma(D_{\bullet} \times \mathbb{A}^1, a_{usu} \Omega_{\log,0}^{p,an,ne}) \\ \xrightarrow{\sim} H^q \Gamma(D_{\bullet}, a_{usu} \Omega_{\log,0}^{p,an,ne}) &\xrightarrow{\sim} \mathbb{H}^q(D_{\bullet}, \Omega_{\log,0}^{p,an,ne}) \xrightarrow{\sim} H_{usu}^q(X, \Omega_{\log,0}^{p,an,ne}) =: H^q E_{usu}(\Omega_{\log,0}^{p,an,ne})(X) \end{aligned}$$

where a_{usu} is the sheaffication functor, $p_X : X \times \mathbb{A}^1 \rightarrow X$ is the projection, and the second and the fourth map is the isomorphism given by (i)', and the middle map is an isomorphism by lemma 2(ii)'. Take a desingularization of the pair $\epsilon = \epsilon_1 : (\widetilde{\mathbb{A}_{\mathbb{C}}^{d_U}}, E) \rightarrow (\mathbb{A}_{\mathbb{C}}^{d_U}, D)$, so that $E = \cup_{i=1}^s E_i \subset \widetilde{\mathbb{A}_{\mathbb{C}}^{d_U}}$ is a normal crossing divisor and $\epsilon : \widetilde{\mathbb{A}_{\mathbb{C}}^{d_U}} \setminus E \xrightarrow{\sim} \mathbb{A}_{\mathbb{C}}^{d_U} \setminus D$. We thus get by proposition 1, for each $p, q \in \mathbb{Z}$, $p, q \neq 0$,

$$H^{q+1} E_{usu}(\Omega_{\log,0}^{p,an,ne})(P_{E_i, \mathbb{A}_{\mathbb{C}}^{d_U}} : \widetilde{\mathbb{A}_{\mathbb{C}}^{d_U}}) : H_{usu, E_i}^{q+1}(\mathbb{A}_{\mathbb{C}}^{d_U}, \Omega_{\log,0}^{p,an,ne}) \xrightarrow{\sim} H_{usu}^{q-1}(E_i \times \mathbb{G}_m, \Omega_{\log,0}^{p,an,ne}).$$

For $p \in \mathbb{Z}$, $p \neq 0, d_U$, we get, using the induction hypothesis, by a finite decreasing induction $\tilde{E}_{ic} \subset \tilde{E}_{ic+1}^N, \dots, E_{i0} \subset E_i$, where

- $E_i^{\circ} := E_i \setminus E_{i0} \subset E_i$ is an open subset such that there exists a finite etale map $e_i : E_i^{\circ} \rightarrow e_i(E_i^{\circ}) \subset \mathbb{A}^{d_U-1}$ with $\mathbb{A}^{d_U-1} \setminus e_i(E_i^{\circ}) \subset \mathbb{A}^{d_U-1}$ a divisor, $\epsilon_2 : (\tilde{E}_i, \tilde{E}_{i0}) \rightarrow (E_i, E_{i0})$ is a desingularization,

- $E_{ik}^o := \tilde{E}_{ik} \setminus E_{ik+1} \subset \tilde{E}_{ik}$ is an open subset such that there exists a finite etale map $e_{ik} : E_{ik}^o \rightarrow e_{ik}(E_{ik}^o) \subset \mathbb{A}^{d_U - k}$ with $\mathbb{A}^{d_U - k} \setminus e_{ik}(E_{ik}^o) \subset \mathbb{A}^{d_U - k}$ a divisor, $\epsilon_k : (\tilde{E}_{ik}^N, \tilde{E}_{ik+1}) \rightarrow (\tilde{E}_{ik}, E_{ik+1})$ is a desingularization

since $d_U > p$ ($\Omega_Y^p = 0$ for $p > d_Y, Y \in \text{AnSm}(\mathbb{C})$),

$$\begin{aligned} H_{usu}^p(e(U), \Omega_{\log,0}^{p,an,ne}) &= H_{usu,D}^{p+1}(\mathbb{A}_{\mathbb{C}}^{d_U}, \Omega_{\log,0}^{p,an,ne}) = H_{usu,E}^{p+1}(\widetilde{\mathbb{A}_{\mathbb{C}}^{d_U}}, \Omega_{\log,0}^{p,an,ne}) \\ &\xrightarrow{\sim} \bigoplus_{i=1}^r H_{usu}^{p-c}(\tilde{E}_{ic-1} \times \mathbb{G}_m^c, \Omega_{\log,0}^{p,an,ne}) = 0 \end{aligned}$$

since $\text{Pic}(E_{ic-1}) = \text{Pic}(E_i) = 0$ by induction as $\epsilon(E_i) \subset \mathbb{A}^{d_U}$ and $\epsilon_k(\tilde{E}_{ik+1}) \subset \tilde{E}_{ik}$ are hypersurfaces of dimension at least two, and for $Y \in \text{AnSm}(\mathbb{C})$,

$$H_{usu}^1(Y, \Omega_{\log,0}^{1,an}) = \text{Pic}(Y), \quad H_{usu}^1(Y, \Omega_{\log,0}^{1,an}) = H_{usu}^1(Y, q_L^{-1}(\pi^* \Omega^{1,e}) \cap \Omega_{\log,0}^{1,an}) \oplus H_{usu}^1(Y, \Omega_{\log,0}^{1,an,ne}),$$

where the second equality follows from lemma 1. We thus have, since $M(U) \simeq M(e(U))$, for each $p \in \mathbb{Z}$, $p \neq 0, d_U$

$$H_{usu}^p(U, \Omega_{\log,0}^{p,an,ne}) = H_{usu}^p(e(U), \Omega_{\log,0}^{p,an,ne}) = 0.$$

(ii)’: Follows from (ii). □

3 Hypersurface De Rham cohomology and its complex periods

We denote $\mathbb{P}^N := \mathbb{P}_{\mathbb{C}}^N := \text{Proj}(\mathbb{C}[z_0, \dots, z_N])$. Let $X = V(f) \subset \mathbb{P}^N$ be a smooth hypersurface. If $N - 1 = 2p$ is even, we consider the decompositions

$$H_{DR}^{N-1}(X) = \mathbb{C}[\Lambda \cap X] \oplus H_{DR,v}^{N-1}(X), \quad H_{\text{sing}}^{N-1}(X^{an}, \mathbb{Q}) = \mathbb{C}[\Lambda \cap X] \oplus H_{\text{sing},v}^{N-1}(X^{an}, \mathbb{Q})$$

of vector spaces, where $\Lambda \subset \mathbb{P}^N$ is a linear subspace of codimension p . If $N - 1 = 2p + 1$ is even, we set

$$H_{DR,v}^{N-1}(X) := H_{DR}^{N-1}(X), \quad H_{\text{sing},v}^{N-1}(X^{an}, \mathbb{Q}) := H_{\text{sing}}^{N-1}(X^{an}, \mathbb{Q}).$$

Let $X = V(f) \subset \mathbb{P}^N$ be a smooth hypersurface. Denote $j : U := \mathbb{P}^N \setminus X \hookrightarrow \mathbb{P}^N$ the open embedding. We consider the algebraic Hodge module

$$j_* \text{Hdg}(O_U, F_b) := ((j_* O_U, F), j_* \mathbb{Q}_{U^{an}}, j_* \alpha(U)) \in \text{HM}(\mathbb{P}^N),$$

where $F^k j_* O_U = \mathbb{C}[x_1, \dots, x_N][\frac{1}{f^k}]$. Recall for each $p \in \mathbb{Z}$,

$$F^p H_{DR}^N(U) := H^N \iota^{\geq p}(\mathbb{H}^N(\mathbb{P}^N, \Omega_{\mathbb{P}^N}^{\bullet \geq p}(\log X))) = H^N \iota^{\geq p} H^N(\mathbb{P}^N, \Omega_{\mathbb{P}^N}^{\bullet} \otimes_{O_{\mathbb{P}^N}} F^{\bullet - p} j_* O_U)$$

and

$$F^p H^{N-1}(X) := H^{N-1} \iota^{\geq p}(\mathbb{H}^{N-1}(X, \Omega_X^{\bullet \geq p})).$$

where

$$\iota^{\geq p} : \Omega_{\mathbb{P}^N}^{\bullet \geq p}(\log X) \hookrightarrow \Omega_{\mathbb{P}^N}^{\bullet}(\log X), \quad \iota^{\geq p} : \Omega_{\mathbb{P}^N}^{\bullet} \otimes_{O_{\mathbb{P}^N}} F^{\bullet - p} j_* O_U \hookrightarrow \Omega_{\mathbb{P}^N}^{\bullet} \otimes_{O_{\mathbb{P}^N}} j_* O_U = j_* \Omega_U^{\bullet}$$

and $\iota^{\geq p} : \Omega_X^{\bullet \geq p} \hookrightarrow \Omega_X^{\bullet}$ are the canonical embeddings. Then the map $\text{Res}_{X, \mathbb{P}^N}$ of definition 1 induces for each $p \in \mathbb{Z}$, canonical isomorphisms

$$H^N \Omega(\text{Res}_{X, \mathbb{P}^N}) : F^{p+1} H_{DR}^N(U) \xrightarrow{\sim} F^p H_{DR}^{N-1}(X)_v, \quad H^N \text{Bti}(\text{Res}_{X, \mathbb{P}^N}) : H_{\text{sing}}^N(U, \mathbb{Q}) \xrightarrow{\sim} H_{\text{sing}}^{N-1}(X, \mathbb{Q})_v.$$

Lemma 3. *Let $X = V(f) \subset \mathbb{P}^N$ be a smooth hypersurface. Denote $j : U := \mathbb{P}^N \setminus X \hookrightarrow \mathbb{P}^N$ the open embedding. Let $B \subset \mathbb{P}^N$, $\phi : B \simeq D(0, 1)^N$ be an open ball such that $\phi(X \cap B) = \{0\} \times D(0, 1)^{N-1}$. Then,*

(i) For $q \in \mathbb{Z}$, $q \neq 0$, $\mathbb{H}_{usu}^q(B, (\Omega_{\mathbb{P}^N}^\bullet \otimes_{\mathcal{O}_{\mathbb{P}^N}} F^{\bullet-p} j_* O_U)^{an}) = 0$.

(ii) For $q \in \mathbb{Z}$, $q \neq 0$, $H_{usu}^q(B, \Omega_{\mathbb{P}^N, an}^{N-p}(\log X)^{\partial=0}) = 0$.

Proof. (i): Since B is Stein, we have for $r, q \in \mathbb{Z}$, $q \neq 0$,

$$H_{usu}^q(B, (\Omega_{\mathbb{P}^N}^r \otimes_{\mathcal{O}_{\mathbb{P}^N}} F^{r-p} j_* O_U)^{an}) = 0.$$

On the other hand the De Rham complex of $(j_* O_{B \setminus X}, F)$ is acyclic (see e.g. [6] corollary 18.7)

(ii): Consider a sequence of sub-balls

$$l_n : B_n := D(0, r_n)^N \hookrightarrow B := D(0, 1)^N, n \in \mathbb{N}, \text{ with } r_n \rightarrow 0 \text{ when } n \rightarrow \infty.$$

Since $(B_n := D(0, r_n))_{n \in \mathbb{N}}$ form a basis of neighborhood of $0 \in B$ and since $\Omega_{\mathbb{P}^N, an}^{N-p}(\log X)^{\partial=0}$ is a single presheaf, there exists $n := n_\alpha \in \mathbb{N}$ (depending on α) such that

$$l_n^* \alpha = 0 \in H^q(B_n, \Omega_{\mathbb{P}^N, an}^{N-p}(\log X)^{\partial=0}).$$

But it follows immediately from the definition of the the presheaf $\Omega_{\mathbb{P}^N, an}^{N-p}(\log X)^{\partial=0}$ considering the isomorphism

$$\phi_n : B_n \xrightarrow{\sim} B, \phi_n(z) := r_n^{-1} z$$

so that $(\phi_n)^{-1} \circ l_n = r_n I$, that

$$l_n^* \alpha = l_n^* \phi_n^{-1*} \phi_n^* \alpha = r_n \phi_n^* \alpha \in H^q(B_n, \Omega_{\mathbb{P}^N, an}^{N-p}(\log X)^{\partial=0}).$$

Hence $\phi_n^* \alpha = 0$. Since ϕ_n is an isomorphism, we get $\alpha = 0$. \square

We have by the local acyclicity of the De Rham complex of $(j_* O_U, F)$ a canonical map $B_\partial :$

Definition 4. Let $X = V(f) \subset \mathbb{P}^N$ be a smooth hypersurface. Denote $j : U := \mathbb{P}^N \setminus X \hookrightarrow \mathbb{P}^N$ the open embedding. Consider the canonical embedding

$$l_\partial : \Omega_{\mathbb{P}^N, an}^{N-p}(\log X)^{\partial=0}[-p] \hookrightarrow (\Omega_{\mathbb{P}^N}^\bullet \otimes_{\mathcal{O}_{\mathbb{P}^N}} F^{\bullet-p} j_* O_U)^{an}$$

Let $\mathbb{P}^{N, an} = \cup_{i=1}^n B_i$ an open cover, with $\phi_i : \mathbb{B}_i \simeq D(0, 1)^N$ open balls such that $\phi_i(X \cap B_i) = \{0\} \times D(0, 1)^{N-1}$. Denote, for $I \subset [1, \dots, n]$, $l_I : B_I \hookrightarrow \mathbb{P}^N$ the open embeddings, with $B_I := \cap_{i \in I} B_i$. Since, for each $I \subset [1, \dots, n]$ and $p \in \mathbb{Z}$,

$$l_I^* l_\partial : l_I^* \Omega_{\mathbb{P}^N, an}^{N-p}(\log X)^{\partial=0}[-p] \hookrightarrow l_I^* (\Omega_{\mathbb{P}^N}^\bullet \otimes_{\mathcal{O}_{\mathbb{P}^N}} F^{\bullet-p} j_* O_U)^{an}$$

is an quasi-isomorphism (see e.g. [6] corollary 18.7), the spectral sequence of the filtration on the total complex of a bi-complex gives, for each $p \in \mathbb{Z}$, a canonical isomorphism, where the first and last map are isomorphism by lemma 3(i) and (ii) respectively,

$$\begin{aligned} B_\partial : F^p H_{DR}(U) &:= H^N l^{\geq p} \mathbb{H}_{zar}^N(\mathbb{P}^N, \Omega_{\mathbb{P}^N}^\bullet \otimes_{\mathcal{O}_{\mathbb{P}^N}} F^{\bullet-p} j_* O_U) \\ &\xrightarrow{H^N((l_i^* = \Omega(l_i))_{1 \leq i \leq n})} H^N l^{\geq p} H^N \Gamma(B_\bullet, (\Omega_{\mathbb{P}^N}^\bullet \otimes_{\mathcal{O}_{\mathbb{P}^N}} F^{\bullet-p} j_* O_U)^{an}) \\ &\rightarrow H^p \Gamma(B_\bullet, \Omega_{\mathbb{P}^N, an}^{N-p}(\log X)^{\partial=0}) \xrightarrow{H^N((l_i^* = \Omega(l_i))_{1 \leq i \leq n})^{-1}} H_{usu}^p(\mathbb{P}^N, \Omega_{\mathbb{P}^N, an}^{N-p}(\log X)^{\partial=0}), \end{aligned}$$

given by, for $w \in \Gamma(U, \Omega_{\mathbb{P}^N}^N)$, inductively by a p step induction where p is the order of the pole of w at X ,

- taking for each $1 \leq i \leq n$, $l_i^* w = \partial w_{1i}$ since $\partial w = 0$ for dimensional reason and by exactness of $l_i^* l_\partial$,

- we then have for each $1 \leq i, j \leq n$, considering $w_{1ij} := l_{ij}^*(w_{1i} - w_{1j})$, $w_{1ij} = \partial w_{2ij}$ since

$$\partial w_{1ij} = l_{ij}^*(\partial w_{1i} - \partial w_{1j}) = l_{ij}^*(l_i^*w - l_j^*w) = 0$$

and by exactness of $l_{ij}^*\iota_\partial$,

- at the final p step, $B(w) := w_{pI} \in H^N(B_\bullet, \Omega_{\mathbb{P}^N, an}^{N-p}(\log X)^{\partial=0})$, $\text{card}I = p$.

By definition $B_\partial(w) = (H^p \iota_\partial)^{-1}(w)$.

Let $X = V(f) \subset \mathbb{P}^N$ be a smooth hypersurface. Denote $U := \mathbb{P}^N \setminus X$. For $l \in \mathbb{Z}$, we consider the factorization in $C(U^{an})$

$$OL_{U^{an}, 0}^l : \Omega_{U^{an}, \log, 0}^l[-l] \xrightarrow{\widehat{OL_{U^{an}, 0}^l[-l]}} \Omega_{\mathbb{P}^N, an}^l(\log X)^{\partial=0}[-l] \xrightarrow{\iota_\partial} (\Omega_{\mathbb{P}^N}^\bullet \otimes_{\mathcal{O}_{\mathbb{P}^N}} F^{\bullet-p} j_* \mathcal{O}_U)^{an} \xrightarrow{\iota^{\geq l}} \Omega_{U^{an}}^\bullet,$$

where

$$\iota_\partial : \Omega_{\mathbb{P}^N, an}^l(\log X)^{\partial=0}[-l] \hookrightarrow (\Omega_{\mathbb{P}^N}^\bullet \otimes_{\mathcal{O}_{\mathbb{P}^N}} F^{\bullet-p} j_* \mathcal{O}_U)^{an}$$

is an equivalence usu local (see [6]), and the factorization in $C(U^{an})$

$$OL_{U^{an}, 0}^l : \Omega_{U^{an}, \log, 0, v}^l[-l] := \Omega_{U^{an}, \log, 0}^l / j^* \Omega_{\mathbb{P}^N, an, \log, 0}^l[-l] \xrightarrow{\widehat{OL_{U^{an}, 0}^l[-l]}} \Omega_{\mathbb{P}^N, an}^l(\log X)^{\partial=0} / j^* \Omega_{\mathbb{P}^N, an}^l[-l] \xrightarrow{\iota^{\geq l} \circ \iota_\partial} \Omega_{U^{an}}^\bullet / j^* \Omega_{\mathbb{P}^N, an}^l.$$

The main result of this section is the following :

Proposition 3. *Let $X = V(f) \subset \mathbb{P}^N$ be a smooth projective hypersurface. Denote $U := \mathbb{P}^N \setminus X$. Let $w \in F^{p+1} H_{DR}^N(U) / j^* H_{DR}^N(\mathbb{P}^N)$. Then the following assertions are equivalent*

- (i) $ev(U)(w) \in H_{\text{sing}}^N(U^{an}, 2i\pi\mathbb{Q})$,
- (ii) $B_\partial(w) \in \widehat{OL_{U^{an}, 0}^p}(H_{usu}^p(U^{an}, \Omega_{U^{an}, \log, 0, v}^{N-p}))$ (see definition 4),
- (iii) $w \in OL_{U^{an}, 0}^N(H_{usu}^p(U^{an}, \Omega_{U^{an}, \log, 0, v}^{N-p}))$.

Proof. (i) implies (ii): Let $\mathbb{P}^{N, an} = \cup_{i=1}^n B_i$ an open cover, with $\phi_i : \mathbb{B}_i \simeq D(0, 1)^N$ open balls such that $\phi_i(X \cap B_i) = \{0\} \times D(0, 1)^{N-1}$. Denote, for $I \subset [1, \dots, n]$, $l_I : B_I \hookrightarrow \mathbb{P}^N$ the open embeddings, with $B_I := \cap_{i \in I} B_i$. Then

$$B_\partial(w) := (w_{p+1, I}) \in H^{p+1} \Gamma(B_\bullet, \Omega_{\mathbb{P}^N, an}^{N-p}(\log X)^{\partial=0}).$$

We have, for each $I \subset [1, \dots, n]$, a decomposition

$$w_{p+1, I} = \frac{a_I}{z_1} dz_1 \wedge w'_I + w''_I \in \Gamma(B_I, \Omega_{\mathbb{P}^N, an}^{N-p}(\log X)^{\partial=0})$$

with $w'_I \in \Gamma(B_I, \Omega_{\mathbb{P}^N, an}^{N-p-1})^{\partial=0}$ and $w''_I \in \Gamma(B_I, \Omega_{\mathbb{P}^N, an}^{N-p})^{\partial=0}$. Consider the evaluation map

$$ev(U)^\bullet : \Gamma(B_\bullet \setminus X^{an}, \Omega_{\mathbb{P}^N, an}^\bullet) \rightarrow C_{\text{diff}}^\bullet(B_\bullet \setminus X^{an}, \mathbb{C}),$$

which is a morphism of bi-complexes. On the other hand, we have the following canonical map

$$BC_\partial : H^p \Gamma(B_\bullet \setminus X^{an}, C_{U^{an}, \text{diff}}^{N-p, \partial}) \xrightarrow{\Gamma(B_\bullet \setminus X^{an}, \iota_\partial)} H^N \Gamma(B_\bullet \setminus X^{an}, C_{U^{an}, \text{diff}}^{\bullet \geq N-p}) \xrightarrow{\Gamma(B_\bullet \setminus X^{an}, \iota^{\geq N-p})} H^N \Gamma(B_\bullet \setminus X^{an}, C_{U^{an}, \text{diff}}^\bullet) \xrightarrow{(H^N(l_i^* =: C_{U^{an}, \text{diff}}^\bullet(l_i))_{1 \leq i \leq n})^{-1}} H_{\text{sing}}^N(U^{an}, \mathbb{Q}) = H^N \Gamma(U^{an}, C_{U^{an}, \text{diff}}^\bullet),$$

where $H^N(l_i^*)$ is an isomorphism by the acyclicity of $C_{U^{an}, \text{diff}}^j \in \text{PSh}(U^{an})$ for each $j \in \mathbb{Z}$, and

$$C_{U^{an}, \text{diff}}^\bullet \hookrightarrow C_{U^{an}, \text{sing}}^\bullet, \text{ for } V \subset U^{an}, C_{\text{diff}}^\bullet(V) \hookrightarrow C_{\text{sing}}^\bullet(V)$$

is the canonical quasi-isomorphism in $C(U^{an})$. We then have the following commutative diagram

$$\begin{array}{ccc} H^N \iota^{\geq p} H^N \Gamma(\mathbb{P}^N, \Omega_{\mathbb{P}^N}^\bullet \otimes_{O_{\mathbb{P}^N}} F^{\bullet-p} j_* O_U) & & \\ \downarrow H^N((l_i^* = \Omega(l_i))_{1 \leq i \leq n}) & \searrow^{B_\partial} & \\ H^N \iota^{\geq p} H^N \Gamma(B_\bullet, (\Omega_{\mathbb{P}^N}^\bullet \otimes_{O_{\mathbb{P}^N}} F^{\bullet-p} j_* O_U)^{an}) & \xrightarrow{B_\partial} & H^p \Gamma(B_\bullet, \Omega_{\mathbb{P}^N, an}^{N-p} (\log X)^{\partial=0}) \\ \downarrow ev(U) & & \downarrow ev(U) \bullet \\ H_{\text{sing}}^N(U^{an}, \mathbb{C}) = H^N \Gamma(U^{an}, C_{U^{an}, \text{diff}}^\bullet \otimes \mathbb{C}) & \xleftarrow{BC_\partial \otimes \mathbb{C}} & H^p \Gamma(B_\bullet \setminus X^{an}, C_{\text{diff}}^{N-p, \partial=0} \otimes \mathbb{C}) \end{array}$$

Since $ev(U)$ is injective by the strictness of the Hodge filtration and since B_∂ is an isomorphism, $ev(U) \bullet$ is injective and

$$\begin{aligned} (BC_\partial \otimes \mathbb{C})' &:= (BC_\partial \otimes \mathbb{C})|_{ev(U) \bullet (H^p \Gamma(B_\bullet, \Omega_{\mathbb{P}^N, an}^{N-p} (\log X)^{\partial=0}))} : \\ &ev(U) \bullet (H^p \Gamma(B_\bullet, \Omega_{\mathbb{P}^N, an}^{N-p} (\log X)^{\partial=0})) \rightarrow H_{\text{sing}}^N(U^{an}, \mathbb{C}) \end{aligned}$$

is injective. For $\gamma_I \in C_{N-p}^{\text{sing}}(B_I \setminus X^{an}, \mathbb{Q})^{\partial=0}$ where $p_1 \circ \phi_I(\gamma_I) \subset D(0, 1)^*$ is a loop around 0, we have by the above diagram

$$ev(U)(w)(\gamma_I) = ev(U)(w_{p+1, I})(\gamma_I) = a_I.$$

Hence if $ev(U)(w) \in H_{\text{sing}}^N(U^{an}, 2i\pi\mathbb{Q})$, then $a_I \in \mathbb{Q}$, using the injectivity of $(BC_\partial \otimes \mathbb{C})'$. Hence

$$w := (w_{p+1, I}) \in \Gamma(B_\bullet \setminus X^{an}, \Omega_{U, \log, 0}^{N-p} / \Omega_{\mathbb{P}^N, an}^{N-p})^{\partial=0}.$$

(ii) is equivalent to (iii): By definition $B_\partial(w) = (H^p \iota_\partial)^{-1}(w)$. On the other hand by definition

$$\iota^{\geq p} \circ \iota_\partial \circ \widehat{OL}_{U^{an}, 0}^{N-p} = OL_{U^{an}, 0}^{N-p}.$$

(iii) implies (i): See [2] proposition 7. □

4 Complex analytic logarithmic de Rham classes

We recall (see section 2) the morphism of site given by the analytification functor

$$\text{An} : \text{AnSm}(\mathbb{C}) \rightarrow \text{SmVar}(\mathbb{C}), X \mapsto \text{An}(X) := X^{an}, (f : X' \rightarrow X) \mapsto \text{An}(f) := f^{an}.$$

and for $X \in \text{SmVar}(\mathbb{C})$, the following commutative diagram

$$\begin{array}{ccc} \text{AnSm}(\mathbb{C}) & \xrightarrow{\text{An}} & \text{SmVar}(\mathbb{C}) \\ \downarrow o_{X^{an}} & & \downarrow o_X \\ X^{an, et} & \xrightarrow{\text{an}_X} & X^{et} \end{array}$$

We have also the morphism of sites (see definition 3)

- $\pi : \text{AnSm}(\mathbb{C}) \rightarrow \text{AnSm}(\mathbb{C})^{pr}$, $\pi(X \times \mathbb{A}^{1, an}) := X$, $\pi(f \otimes I) := f$,
- $\rho : \text{AnSm}(\mathbb{C})^{pr} \rightarrow \text{AnSm}(\mathbb{C})$, $\rho(X) := X \times \mathbb{A}^{1, an}$, $\rho(f) := f \otimes I$,

in particular $\rho \circ \pi = I$. Consider the commutative diagram in $C(\text{AnSm}(\mathbb{C}))$

$$\begin{array}{ccc}
\Omega_{\log,0}^{\bullet,an} \cap q_L^{-1}(\pi^* \Omega^{\bullet,e}) & \xrightarrow{OL_0^{an}} & \pi^* \Omega^{\bullet,e} \\
\downarrow e & & \downarrow e \\
\Omega_{\log,0}^{\bullet,an} & \xrightarrow{OL_0^{an}} & \Omega^{\bullet,an} \\
\downarrow q & & \downarrow q \\
\Omega_{\log,0}^{\bullet,an,ne} := \Omega_{\log,0}^{\bullet,an} / (\Omega_{\log,0}^{\bullet,an} \cap q_L^{-1}(\pi^* \Omega^{\bullet,e})) & \xrightarrow{\bar{O}L_0^{an}} & \Omega^{\bullet,an} / (\pi^* \Omega^{\bullet,e})
\end{array} ,$$

whose columns are exact with e injective and q surjective (see definition 3). We have the factorization in $C(\text{AnSm}(\mathbb{C}))$

$$\begin{aligned}
\text{ad}(\pi) := \text{ad}(\pi^*, \pi_*)(-) : \pi^* \Omega^{\bullet,e} &\xrightarrow{\subset} \pi^* a_{usu} \partial \Omega^{\bullet-1,an} := \pi^* \pi_* \Omega^{\bullet,an, \partial=0} \\
&\xrightarrow{(\text{ad}(\pi), 0)} \text{ad}(\pi)(\pi^* \Omega^{\bullet,e}) \xrightarrow{i_{\text{ad}(\pi)}} \Omega^{\bullet,an, \partial=0},
\end{aligned}$$

and we have denoted again by abuse

- $\pi^* \Omega^{\bullet,e} := \text{ad}(\pi)(\pi^* \Omega^{\bullet,e}) \subset \Omega^{\bullet,an, \partial=0} \subset \Omega^{\bullet,an}$.
- $\pi^* \Omega^{\bullet,e} := \text{ad}(\pi)(\pi^* \Omega^{\bullet,e}) \cap \Omega_{\log}^{\bullet,an} \subset \Omega^{\bullet,an, \partial=0} \subset \Omega^{\bullet,an}$.

Let $X \in \text{SmVar}(\mathbb{C})$. Then, for $k \in \mathbb{Z}$, we have the commutative diagram

$$\begin{array}{ccc}
\mathbb{H}_{usu}^k(X^{an}, \Omega_{\log,0}^{\bullet,an} \cap q_L^{-1}(\pi^* \Omega^{\bullet,e})) & \xrightarrow{H^k OL_0^{an}} & \mathbb{H}_{usu}^k(X^{an}, \text{ad}(\pi)(\pi^* \Omega^{\bullet,e})) \\
\downarrow H^k(e) & & \downarrow H^k(e) \\
\mathbb{H}_{usu}^k(X^{an}, \Omega_{\log,0}^{\bullet,an}) & \xrightarrow{H^k OL_0^{an}} & H_{DR}^k(X) = \mathbb{H}_{usu}^k(X^{an}, \Omega^{\bullet,an}) \\
\downarrow H^k(q) & & \downarrow H^k(q) \\
\mathbb{H}_{usu}^k(X^{an}, \Omega_{\log,0}^{\bullet,an,ne}) & \xrightarrow{H^k \bar{O}L_0^{an}} & \mathbb{H}_{usu}^k(X^{an}, \Omega^{\bullet,an} / (\pi^* \Omega^{\bullet,e}))
\end{array}$$

whose columns are exact, and of course all differentials of $\Omega_{\log,0}^{\bullet,an}$ and of $\pi^* \Omega^{\bullet,e}$ are trivial. Let $Z \subset X$ a closed subset. Then, for $k \in \mathbb{Z}$, we also have the commutative diagram

$$\begin{array}{ccc}
\mathbb{H}_{usu,Z}^k(X^{an}, \Omega_{\log,0}^{\bullet,an} \cap q_L^{-1}(\pi^* \Omega^{\bullet,e})) & \xrightarrow{H^k OL_0^{an}} & \mathbb{H}_{usu,Z}^k(X^{an}, \text{ad}(\pi)(\pi^* \Omega^{\bullet,e})) \\
\downarrow H^k(e) & & \downarrow H^k(e) \\
\mathbb{H}_{usu,Z}^k(X^{an}, \Omega_{\log}^{\bullet,an}) & \xrightarrow{H^k OL_0^{an}} & H_{DR,Z}^k(X) = \mathbb{H}_{usu}^k(X^{an}, \Omega^{\bullet,an}) \\
\downarrow H^k(q) & & \downarrow H^k(q) \\
\mathbb{H}_{usu,Z}^k(X^{an}, \Omega_{\log}^{\bullet,an,ne}) & \xrightarrow{H^k \bar{O}L_0^{an}} & \mathbb{H}_{usu,Z}^k(X^{an}, \Omega^{\bullet,an} / (\pi^* \Omega^{\bullet,e}))
\end{array}$$

whose columns are exact.

Lemma 4. *Let $X \in \text{SmVar}(\mathbb{C})$. For each $k \in \mathbb{Z}$, $k \neq 0$, we have $H^k(e)(\mathbb{H}_{usu}^k(X^{an}, \text{ad}(\pi)(\pi^* \Omega^{\bullet,e}))) = 0$.*

Proof. We have the canonical splitting

$$\mathbb{H}_{usu}^k(X^{an}, \text{ad}(\pi)(\pi^* \Omega^{\bullet,e})) = \bigoplus_{p \in \mathbb{Z}} H_{usu}^{k-p}(X^{an}, \text{ad}(\pi)(\pi^* \Omega^{p,e}))$$

Consider for each $p \in \mathbb{Z}$, the composition in $C(\text{AnSm}(\mathbb{C}))$

$$\text{ad}(\pi) := \text{ad}(\pi^*, \pi_*)(-) : \pi^*(\mathcal{A}^{\bullet \geq p, \bullet, e}) \xrightarrow{\text{ad}(\pi)} \text{ad}(\pi)(\pi^*(\mathcal{A}^{\bullet \geq p, \bullet, e})) \xrightarrow{i_{\text{ad}(\pi)}} \mathcal{A}^{\bullet \geq p, \bullet},$$

where the embedding

$$\Omega^{p, e} \hookrightarrow (\Omega^{\bullet \geq p})^e \hookrightarrow \mathcal{A}^{\bullet \geq p, \bullet, e}$$

in $C(\text{AnSm}(\mathbb{C})^{pr})$ is the Dolbeault resolution. Let $X = \cup_{i=1}^r X_i$ be an open affine cover. Then for each $I \subset [1, \dots, r]$, there exists by Noether normalization lemma a finite map $X_I \rightarrow \mathbb{A}_{\mathbb{C}}^{d_X}$. Hence the presheaves $(\pi^*(\mathcal{A}^{l, r, e}))|_{X_I}$ and $(\pi^*\Omega^{p, e})|_{X_I}$ are sheaves. We have thus, by Poincare lemma on open balls, for each $I \subset [1, \dots, r]$,

$$H_{usu}^k(X_I^{an}, \text{ad}(\pi)(\pi^*\Omega^{p, e})) = H^k\Gamma(X_I^{an}, \text{ad}(\pi)(\pi^*\mathcal{A}^{\bullet \geq p, \bullet, e})),$$

indeed, since open balls form a basis of the complex topology, there exists D^l such that $D^l \subset D^{l-1} \times \mathbb{A}^1$. We have

$$\text{ad}(\pi)(\pi^*\mathcal{A}^{\bullet \geq p, \bullet, e})(X_I^{an}) := \varinjlim_{m: X_I \rightarrow Y \times \mathbb{A}^1} \mathcal{A}^e(m)(\mathcal{A}^{\bullet \geq p, \bullet, e}(Y \times \mathbb{A}^1)) \subset \mathcal{A}^{\bullet \geq p, \bullet}(X^{an})$$

Now, for each $I \subset [1, \dots, r]$,

- if $q \circ m : X_I^{an} \xrightarrow{m} Y \times \mathbb{A}^1 \xrightarrow{q} \mathbb{A}^1$ is non constant, we can factor

$$m = (n \times I) \circ m' : X_I^{an} \xrightarrow{m'} Y' \times \mathbb{A}^1 \xrightarrow{n \times I} Y \times \mathbb{A}^1$$

with $\dim Y' = \dim X - 1$ and $m'(X_I^{an}) \subset Y' \times \mathbb{A}^1$ a dense open subset, and

$$\mathcal{A}^e(m') : (\mathcal{A}^{\bullet \geq p, \bullet, e}(Y' \times \mathbb{A}^1)) \rightarrow \mathcal{A}^{\bullet \geq p, \bullet}(X_I^{an})$$

is injective, and

$$\begin{aligned} H^k \mathcal{A}^e(m)(\mathcal{A}^{\bullet \geq p, \bullet, e}(Y \times \mathbb{A}^1)) &= H^k \mathcal{A}^e(m')(\mathcal{A}^{\bullet \geq p, \bullet, e}(Y' \times \mathbb{A}^1)) \\ &= \mathcal{A}^e(m')(H^k(\mathcal{A}^{\bullet \geq p, \bullet, e})(Y' \times \mathbb{A}^1)) = 0 \end{aligned}$$

by Kunneth formula for $Y' \times \mathbb{A}^1$,

- if $q \circ m : X \xrightarrow{m} Y \times \mathbb{A}^1 \xrightarrow{q} \mathbb{A}^1$ is constant with value c ,

$$H^k \mathcal{A}^e(m)(\mathcal{A}^{\bullet \geq p, \bullet, e}(Y \times \mathbb{A}^1)) = 0$$

since $c^*d\bar{z} = 0$ and $c^*dz = 0$, where $c : Y \times \{c\} \hookrightarrow Y \times \mathbb{A}^1$ is the closed embedding.

Hence, for each $I \subset [1, \dots, r]$ and $k \in \mathbb{Z}$

$$H^k(e)(H_{usu}^k(X_I^{an}, \text{ad}(\pi)(\pi^*\Omega^{p, e}))) = 0.$$

□

By the spectral sequence associated to the bi complex $\Gamma(X_{\bullet}, \mathcal{A}^{\bullet \geq p, \bullet, e})$, we get

$$H^k(e)(H_{usu}^k(X^{an}, \text{ad}(\pi)(\pi^*\Omega^{p, e}))) = 0.$$

for each $k \in \mathbb{Z}$.

Proposition 4. *Let $X \in \text{PSmVar}(\mathbb{C})$. Consider the morphism $\text{an}_X : X^{an} \rightarrow X$ in RTop given by analytical functor.*

(i) The analytic De Rham cohomology class of an algebraic cycle is logarithmic and is of type (d, d) , that is, for $Z \in \mathcal{Z}^d(X)$

$$[Z] := H^{2d}\Omega(\gamma_Z^\vee)([Z]) \subset H^{2d}OL_{X^{an},0}(H_{usu}^d(X^{an}, \Omega_{X^{an},\log,0}^d)) \subset H_{DR}^{2d}(X^{an}) = H_{DR}^{2d}(X).$$

(ii) Conversely, any $w \in H^{2d}OL_{X^{an},0}(H_{usu}^d(X^{an}, \Omega_{X^{an},\log,0}^d))$ is the class of an algebraic cycle $Z \in \mathcal{Z}^d(X) \otimes \mathbb{Q}$, i.e. $w = [Z]$.

(iii) We have $H^jOL_{X^{an}}(H_{usu}^{j-l}(X^{an}, \Omega_{X^{an},\log,0}^l)) = 0$ for $j, l \in \mathbb{Z}$ such that $2l > j$.

Proof. Consider, for $j, l \in \mathbb{Z}$ and $X \in \text{SmVar}(\mathbb{C})$,

$$L^{l,j-l}(X) := H^jOL_{X^{an},0}(H_{usu}^{j-l}(X^{an}, \Omega_{X^{an},\log,0}^l)) := H^jOL_0^{an}(H^{j-l}\text{Hom}(\text{An}^*\mathbb{Z}(X), E_{usu}^\bullet(\Omega_{\log,0}^{l,an}))).$$

Consider also for $j, l \in \mathbb{Z}$, $X \in \text{SmVar}(\mathbb{C})$ and $Z \subset X$ a closed subset,

$$L_Z^{l,j-l}(X) := H^jOL_{X^{an},0}(H_{usu,Z}^{j-l}(X^{an}, \Omega_{X^{an},\log,0}^l)) := H^jOL_0^{an}(H^{j-l}\text{Hom}(\text{An}^*\mathbb{Z}(X, X \setminus Z), E_{usu}^\bullet(\Omega_{\log,0}^{l,an}))).$$

Consider, for $j, l \in \mathbb{Z}$ and $X \in \text{SmVar}(\mathbb{C})$,

$$L^{l,j-l,ne}(X) := H^j(\bar{O}L_0^{an})(H^{j-l}\text{Hom}(\text{An}^*\mathbb{Z}(X), E_{usu}^\bullet(\Omega_{\log,0}^{l,an,ne}))).$$

Consider also for $j, l \in \mathbb{Z}$, $X \in \text{SmVar}(\mathbb{C})$ and $Z \subset X$ a closed subset,

$$L_Z^{l,j-l,ne}(X) := H^j(\bar{O}L_0^{an})(H^{j-l}\text{Hom}(\text{An}^*\mathbb{Z}(X, X \setminus Z), E_{usu}^\bullet(\Omega_{\log,0}^{l,an,ne}))).$$

For each $l, j \in \mathbb{Z}$, the presheaf $H^{j-l}E_{usu}(\Omega_{\log}^{l,an,ne}) \in \text{PSh}(\text{AnSm}(\mathbb{C}))$ is \mathbb{A}^1 invariant by proposition 2(i). Indeed, for $Y \in \text{AnSm}(\mathbb{C})$ connected, take an open cover $Y = \cup_{i \in I} D_i$ by open balls $D_i \simeq D(0, 1)^{d_Y}$, and let $D_\bullet \in \text{Fun}(\Delta, \text{AnSm}(\mathbb{C}))$ be the associated simplicial analytic manifold, with for $J \subset I$, $j_{IJ} : D_I := \cap_{i \in I} D_i \hookrightarrow D_J := \cap_{i \in J} D_i$ the open embedding, we then get the isomorphism

$$\begin{aligned} p_Y^* : H^{j-l}E_{usu}(\Omega_{\log,0}^{l,an,ne})(Y \times \mathbb{A}^1) &:= H_{usu}^{j-l}(Y \times \mathbb{A}^1, \Omega_{\log,0}^{l,an,ne}) \xrightarrow{\sim} \\ &\mathbb{H}^{j-l}(D_\bullet \times \mathbb{A}^1, \Omega_{\log,0}^{l,an,ne}) \xrightarrow{\sim} H^{j-l}\Gamma(D_\bullet \times \mathbb{A}^1, a_{usu}\Omega_{\log,0}^{l,an,ne}) \\ &\xrightarrow{\sim} H^{j-l}\Gamma(D_\bullet, a_{usu}\Omega_{\log,0}^{l,an,ne}) \xrightarrow{\sim} \mathbb{H}^{j-l}(D_\bullet, \Omega_{\log,0}^{l,an,ne}) \xrightarrow{\sim} H_{usu}^{j-l}(Y, \Omega_{\log,0}^{l,an,ne}) =: H^{j-l}E_{usu}(\Omega_{\log,0}^{l,an,ne})(Y) \end{aligned}$$

where $p_Y : X \times \mathbb{A}^1 \rightarrow Y$ is the projection, and the second and the fourth map is the isomorphism given by proposition 2(i), and the middle map is an isomorphism by lemma 2(ii)'. This gives in particular, for $X \in \text{SmVar}(\mathbb{C})$ and $Z \subset X$ a smooth subvariety of (pure) codimension d , by proposition 1 an isomorphism

$$E_{usu}(\Omega_{\log,0}^{l,an,ne})(P_{Z,X}) : L_Z^{l,j-l,ne}(X) \xrightarrow{\sim} L_Z^{l,j-l,ne}(N_{Z/X}) \xrightarrow{\sim} L^{l-d,j-l-d,ne}(Z).$$

(i): Similar to the proof of [2] theorem 2.

(ii): Let $X \in \text{PSmVar}(\mathbb{C})$ and

$$\begin{aligned} \alpha = H^{2d}OL_{X^{an}}(\alpha) \in L^{d,d}(X) : &= H^{2d}OL_{X^{an}}(H_{usu}^d(X^{an}, \Omega_{X^{an},\log,0}^d)) \\ &= H^{2d}OL^{an}(H^d\text{Hom}(\text{An}^*\mathbb{Z}(X), E_{usu}^\bullet(\Omega_{\log,0}^{d,an}))). \end{aligned}$$

Then,

$$\begin{aligned} H^{2d}(q)(\alpha) &= H^{2d}(q) \circ H^{2d}OL_{X^{an},0}(\alpha) = H^{2d}\bar{O}L_0^{an} \circ H^{2d}(q)(\alpha) \\ &\in L^{d,d,ne}(X) := H^{2d}\bar{O}L_0^{an}(H^d\text{Hom}(\text{An}^*\mathbb{Z}(X), E_{usu}^\bullet(\Omega_{\log,0}^{d,an,ne}))). \end{aligned}$$

Up to split X by its connected components, we may assume that X is connected. The case $d = d_X$ is trivial. So let assume $d \neq d_X$. Let $U \subset X$ an open subset such that there exists an etale map $e : U \rightarrow \mathbb{A}_{\mathbb{C}}^{d_U}$

such that $e : U \rightarrow e(U)$ is finite etale and $\mathbb{A}_{\mathbb{C}}^{d_U} \setminus e(U) \subset \mathbb{A}_{\mathbb{C}}^{d_U}$ is a divisor. Denote $j : U \hookrightarrow X$ the open embedding. By proposition 2(ii), we have

$$j^* H^{2d}(q)(\alpha) = 0 \in H_{usu}^d(U^{an}, \Omega_{\log,0}^{d,an,ne}) = 0.$$

Considering a divisor $X \setminus U \subset D \subset X$, we get

$$H^{2d}(q)(\alpha) = H^d E_{usu}^\bullet(\Omega_{\log,0}^{d,an,ne})(\gamma_D^\vee)(\alpha'), \alpha' \in L_D^{l,j-l,ne}(X).$$

Denote $D^o \subset D$ its smooth locus and $l : X^o \hookrightarrow X$ a Zariski open subset such that $X^o \cap D = D^o$. We then have

$$l^* \alpha' \in L_{D^o}^{d,d}(X^o) = L^{d-1,d-1}(D^o)$$

We repeat this procedure with each connected components of D^o instead of X . By a finite induction of d steps, we get

$$H^{2d}(q)(\alpha) = H^d E_{usu}^\bullet(\Omega_{\log,0}^{d,an,ne})(\gamma_Z^\vee)(\alpha'), \alpha' \in L_Z^{d,d,ne}(X) = L^{0,0,ne}(Z^o).$$

with $Z := D_d \subset \dots \subset D \subset X$ a pure codimension d (Zariski) closed subset, thus

$$\alpha = \sum_i n_i [Z_i] + \beta \in H_{DR}^{2d}(X), \beta \in H^{2d}(e)(\mathbb{H}_{usu}^{2d}(X^{an}, \text{ad}(\pi)(\pi^* \Omega^{\bullet,e})))$$

where $n_i \in \mathbb{Q}$ and $(Z_i)_{1 \leq i \leq t} \subset Z$ are the irreducible components of Z . But by lemma 4, we have

$$H^k(e)(\mathbb{H}_{usu}^k(X^{an}, \text{ad}(\pi)(\pi^* \Omega^{\bullet,e}))) = 0.$$

for each $k \in \mathbb{Z}$, $k \neq 0$. Hence $\beta = 0$, that is

$$\alpha = \sum_i n_i [Z_i] \in H_{DR}^{2d}(X).$$

(iii): Let $j < 2l$. Let $X \in \text{PSmVar}(\mathbb{C})$ and $w \in L^{l,j-l}(X)$. By proposition 2(ii)', arguing as in the proof of (ii), there exists $Z \subset X$ a closed subset of pure codimension $j-l$ such that

$$H^j(q)(w) = H^{j-l} E_{usu}^\bullet(\Omega_{\log,0}^{l,an,ne})(\gamma_Z^\vee)(w'), w' \in L_Z^{l,j-l,ne}(X).$$

For $Z' \subset X$ a closed subset of pure codimension c , consider a desingularisation $\epsilon : \tilde{Z}' \rightarrow Z'$ of Z' and denote $n : \tilde{Z}' \xrightarrow{\epsilon} Z' \subset X$. The morphism in $\text{DA}(k)$

$$G_{Z',X} : M(X) \xrightarrow{D(\mathbb{Z}(n))} M(\tilde{Z}')(c)[2c] \xrightarrow{\mathbb{Z}(\epsilon)} M(Z')(c)[2c]$$

where $D : \text{Hom}_{\text{DA}(k)}(M_c(\tilde{Z}'), M_c(X)) \xrightarrow{\sim} \text{Hom}_{\text{DA}(k)}(M(X), M(\tilde{Z}')(c)[2c])$ is the duality isomorphism from the six functor formalism (moving lemma of Suzlin and Voevodsky) and $\mathbb{Z}(n) := \text{ad}(n_l, n^l)(a_X^! \mathbb{Z})$, is given by a morphism in $C(\text{SmVar}(k))$

$$\hat{G}_{Z',X} : \mathbb{Z}_{tr}(X) \rightarrow E_{et}(C_* \mathbb{Z}_{tr}(Z'))(c)[2c].$$

Let $l : X^o \hookrightarrow X$ be an open embedding such that $Z^o := Z \cap X^o$ is the smooth locus of Z . We have the following commutative diagram of abelian groups

$$\begin{array}{ccccccc} 0 & \longrightarrow & H_{usu,Z}^{j-l}(X^{an}, \Omega_{\log,0}^{l,an,ne}) & \xrightarrow{l^*} & H_{usu,Z^o}^{j-l}(X^{o,an}, \Omega_{\log,0}^{l,an,ne}) & \xrightarrow{\partial} & H_{usu,Z \setminus Z^o}^{j-l+1}(X^{an}, \Omega_{\log,0}^{l,an,ne}) \longrightarrow \dots \\ & & \Omega(\hat{G}_{Z,X}) \uparrow & & \Omega(P_{Z^o, X^o}) \uparrow & & \Omega(\hat{G}_{Z,X}) \uparrow \\ 0 & \longrightarrow & H_{usu}^0(Z^{an}, \Omega_{\log,0}^{2l-j,an,ne}) & \xrightarrow{l^*} & H_{usu}^0(Z^{o,an}, \Omega_{\log,0}^{2l-j,an,ne}) & \xrightarrow{\partial} & H_{usu,Z \setminus Z^o}^1(Z^{an}, \Omega_{\log,0}^{2l-j,an,ne}) \longrightarrow \dots \end{array}$$

whose rows are exact sequences. Consider

$$l^*w' = \Omega(P_{Z^o, X^o})(w^o) \in H_{usu, Z^o}^{j-l}(X^{o, an}, \Omega_{\log, 0}^{l, an, ne}), \quad w^o \in H_{et}^0(Z^{o, an}, \Omega_{\log, 0}^{2l-j, an, ne}).$$

Since $\partial l^*w' = 0 \in H_{usu, Z \setminus Z^o}^{j-l+1}(X^{an}, \Omega_{\log, 0}^{l, an, ne})$, we get, considering the smooth locus $(Z \setminus Z^o)^o \subset Z \setminus Z^o$ and $X^{oo} \subset X$ is an open subset such that $X \cap (Z \setminus Z^o)^o = X^{oo} \cap (Z \setminus Z^o)$,

$$\partial w^o = 0 \in H_{usu, Z \setminus Z^o}^1(Z^{an}, \Omega_{\log, 0}^{2l-j, an, ne}),$$

since $H_{usu, (Z \setminus Z^o) \setminus (Z \setminus Z^o)^o}^1(Z^{an}, \Omega_{\log, 0}^{2l-j, an, ne}) = 0$ for dimension reasons, that is

$$H^j(q)(w) = \Omega(\hat{G}_{Z, X})(w'), \quad \text{with } w' \in H_{et}^0(Z^{an}, \Omega_{\log, 0}^{2l-j, an, ne}).$$

Hence $H^j(q)(w) = 0$ since $H^0(Z'^{an}, \Omega_{\log, 0}^k) = 0$ for all $k > 0$ and all $Z' \in \text{PVar}(\mathbb{C})$. But by lemma 4

$$H(e)(\mathbb{H}_{usu}^j(X^{an}, \text{ad}(\pi)(\pi^*\Omega^{\bullet, e}))) = 0.$$

Hence $w = 0$. □

5 Hodge Conjecture for complex smooth projective hypersurfaces

We now state and prove using the result of section 3 and section 4 the main result of this article :

Theorem 1. *Let $X = V(f) \subset \mathbb{P}_{\mathbb{C}}^N$ be a smooth projective hypersurface, with $N = 2p + 1$ odd. Let $\lambda \in F^p H^{2p}(X^{an}, \mathbb{Q})$. Then there exists an algebraic cycle $Z \in \mathbb{Z}^p(X)$ such that $\lambda = [Z]$.*

Proof. Let $U := \mathbb{P}_{\mathbb{C}}^N \setminus X$. Consider $\text{Res}_{X, \mathbb{P}^N} : M(X) \rightarrow M(U)[-1]$ the canonical map given in definition 1. We have the commutative diagram of complex vector spaces :

$$\begin{array}{ccc} H_{DR}^N(U) & \xrightarrow{H^N \Omega(\text{Res}_{X, \mathbb{P}^N})} & H_{DR}^{N-1}(X)_v \\ \uparrow H^N \alpha(U^{an}) = ev(U)^{-1} & & \uparrow H^{N-1} \alpha(X^{an}) = ev(X)^{-1} \\ H_{\text{sing}}^N(U^{an}, \mathbb{C}) & \xrightarrow{(H^N \text{Bti}(\text{Res}_{X, \mathbb{P}^N})) \otimes_{\mathbb{Q}} \mathbb{C}} & H_{\text{sing}}^{N-1}(X^{an}, \mathbb{C})_v \end{array} .$$

whose arrows are isomorphisms. Denote $w(\lambda) := H^{N-1} \alpha(X^{an})(\lambda) \in F^p H_{DR}^{2p}(X)$. We have

$$\lambda = H^{N-1} \text{Bti}(\text{Res}_{X, \mathbb{P}^N})(\alpha) + m[H], \quad \text{with } m \in \mathbb{Q}, \quad \alpha \in H_{\text{sing}}^N(U^{an}, \mathbb{Q}),$$

where $H := X \cap \Lambda \subset X$ is an hyperplane section. Then

$$w(\lambda) = H^{N-1} \Omega(\text{Res}_{X, \mathbb{P}^N})(w(\alpha)) + m[H], \quad \text{with } w(\alpha) := H^N \alpha(U^{an})(\alpha) \in F^{p+1} H_{DR}^N(U).$$

Since $ev(U)(w(\alpha)) = \alpha \in H_{\text{sing}}^N(U^{an}, 2i\pi\mathbb{Q})$, we have by proposition 3,

$$w(\alpha) \in H^N OL_{U^{an}, 0}(H_{usu}^p(U^{an}, \Omega_{U^{an}, \log, 0}^{N-p}))$$

Hence

$$w(\lambda) = H^{N-1} OL_{X^{an}, 0}(\Omega_{\log, 0}^{N-p-1, an}(\text{Res}_{X, \mathbb{P}^N})(w(\alpha))) + m[H] \in OL_{X^{an}, 0}(H_{usu}^p(X^{an}, \Omega_{X^{an}, \log, 0}^p)).$$

Hence by proposition 4(ii), $w(\lambda) = [Z]$ with $Z \in \mathbb{Z}^p(X)$. □

References

- [1] J.Bouali, *The De Rham, complex and p adic Hodge realization functor for relative motives of algebraic varieties over a field of characteristic zero*, arxiv 2203.10875, preprint, 2022.
- [2] J.Bouali, *De Rham logarithmic classes and Tate conjecture*, arxiv preprint 2303.09932, 2023
- [3] D.C.Cisinski, F.Deglise, *Triangulated categories of mixed motives*, arxiv preprint 0912.2110, 2009-arxiv.org.
- [4] F.Lecomte, N.Wach, *Réalisation de Hodge des motifs de Voevodsky*, manuscripta mathematica.141,663-697, Springer-Verlag, Berlin, 2012.
- [5] C.Mazza, V.Voevodsky, C.Weibel, *Lecture Notes on Motivic Cohomology*, Clay mathematics monographs, Vol 2, AMS, Providence RI, 2006.
- [6] C.Voisin, *Théorie de Hodge et géométrie algébrique complexe*, SMF, cours spécialisés, 2002.

LAGA UMR CNRS 7539
Université Paris 13, Sorbonne Paris-Nord, 99 av Jean-Baptiste Clement,
93430 Villetaneuse, France,
bouali@math.univ-paris13.fr