Hodge conjecture for projective hypersurfaces

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Abstract

We show that a Hodge class of a complex smooth projective hypersurface is an analytic logarithmic De Rham class. On the other hand we show that for a complex smooth projective variety an analytic logarithmic De Rham class of type (d, d) is the class of a codimension d algebraic cycle. We deduce the Hodge conjecture for smooth projective hypersurfaces.

Contents

1 Introduction

1.1 Complex analytic De Rham logarithmic classes

In [2], we introduced, for X an algebraic variety over a field, the notion of logarithmic De Rham cohomology classes. For X a complex algebraic variety, we also introduce in this work the notion of logarithmic analytic De Rham cohomology classes. In proposition 4 (section 4) we give an analytic analogue of [2] theorem 2 (ii), that is if X is a smooth complex projective variety,

- a logarithmic analytic De Rham cohomology class of type (d, d) is the class of a codimension d algebraic cycle.
- we have $H^j OL_{X^{an}}(H^{j-l}_{usu}(X^{an}, \Omega^l_{X^{an},\log})) = 0$ for $j, l \in \mathbb{Z}$ such that $2l > j$.

Where $OL_{X^{an}} : \Omega^l_{X^{an},\log}[-l] \hookrightarrow \Omega^\bullet_{X^{an}}$ is the injection of the sub-presheaf of logarithmic analytic l-forms in the analytic de Rham complex and usu denote the usual complex topology. The proof of proposition 4 is motivic and similar to the proof of [2] theorem 2 : a logarithmic analytic class of bidegree (p, q), $p, q \neq 0$, $p \geq q$, $q \neq d_U$ is acyclic for the Zariski topology on each open subset $U \subset X$ such that there exists an etale

map $e : U \to \mathbb{A}^{d_U}_{\mathbb{C}}$ such that $e : U \to e(U)$ is finite etale and $\mathbb{A}^{d_U}_{\mathbb{C}} \backslash e(U) \subset \mathbb{A}^{d_U}_{\mathbb{C}}$ is a divisor (proposition 2), this allows us to proceed by a finite induction using the crucial fact that the purity isomorphism for De Rham cohomology preserve logarithmic analytic classes since the purity isomorphism is motivic (c.f. [3], see proposition 1).

1.2 Hodge conjecture for smooth projective hypersurfaces

Let $X = V(f) \subset \mathbb{P}^N_{\mathbb{C}}$ be a smooth projective hypersurface. Denote $j : U := \mathbb{P}^N_{\mathbb{C}} \backslash X \hookrightarrow \mathbb{P}^N_{\mathbb{C}}$ the open complementary subset. Let $(\Omega^\bullet_{\mathbb{P}^N} \otimes_{O_{\mathbb{P}^N}} F^{\bullet-p} j_* O_U)^{an}$ be the filtered De Rham complex of U. The cone of ι_∂ :

$$\iota_\partial : \Omega^{N-p}_{\mathbb{P}^{N,an}}(\log X)^{\partial=0}[-p] \hookrightarrow (\Omega^\bullet_{\mathbb{P}^N} \otimes_{O_{\mathbb{P}^N}} F^{\bullet-p} j_* O_U)^{an}$$

is locally acyclic for the complex topology, more precisely is acyclic on open balls (see e.g.[6]). As a consequence, we show in proposition 3, using an open cover $\mathbb{P}^N_{\mathbb{C}} = \cup_{i=1}^n B_i$ by open balls $\phi_i : B_i \simeq D(0,1)^N$ such that $\phi_i(B_i \cap X) = 0 \times D(0,1)^{N-1}$, that a Hodge class $\alpha \in F^{N-p} H^N(U^{an}, \mathbb{Q})$ is logarithmic, that is

$$\alpha \in H^N OL_{U^{an}}(H^p_{usu}(U^{an}, \Omega^{N-p}_{U^{an},\log})).$$

We get that, if $N = 2p+1$ and $\lambda \in F^p H^{2p}(X^{an}, \mathbb{Q})$ is a Hodge class,

$$\lambda = Res_{X,\mathbb{P}^N}(\alpha) + m[H] \in H^{2p} OL_{X^{an}}(H^p_{usu}(X^{an}, \Omega^p_{X^{an},\log})),$$

where $Res_{X,\mathbb{P}^N} : H^N(U^{an}, \mathbb{C}) \xrightarrow{\sim} H^{2p}(X^{an}, \mathbb{C})$ is the residu map, using the fact that since the residu map is motivic (see definition 1), it preserves the logaritmic De Rham classes. Proposition 4 (ii) then implies that $\lambda = [Z]$, $Z \in \mathcal{Z}^p(X)$, is the class of an algebraic cycle.

To our knowledge, Hodge's conjecture for hypersurfaces was known for only a few cases : hyperplanes, quadrics and Fermat's hypersurfaces of degree ≤ 21 (Shioda). Our new contribution which allowed us to tackle the general case of hypersurfaces is the introduction of the notion of analytical logarithmic de Rham classes on the one hand and the motivic purity isomorphism on the other hand.

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2 Preliminaries and Notations

2.1 Notations

- Denote by Top the category of topological spaces and RTop the category of ringed spaces.
- Denote by Cat the category of small categories and RCat the category of ringed topos.
- For $\mathcal{S} \in$ Cat and $X \in \mathcal{S}$, we denote $\mathcal{S}/X \in$ Cat the category whose objects are $Y/X := (Y, f)$ with $Y \in \mathcal{S}$ and $f : Y \to X$ is a morphism in $\mathcal{S}$, and whose morphisms $\mathrm{Hom}((Y', f'), (Y, f))$ consists of $g : Y' \to Y$ in $\mathcal{S}$ such that $f \circ g = f'$.
- For $(\mathcal{S}, O_S) \in$ RCat a ringed topos, we denote by
  - $\mathrm{PSh}(\mathcal{S})$ the category of presheaves of O_S modules on $\mathcal{S}$ and $\mathrm{PSh}_{O_S}(\mathcal{S})$ the category of presheaves of O_S modules on $\mathcal{S}$, whose objects are $\mathrm{PSh}_{O_S}(\mathcal{S})^0 := \{(M, m), M \in \mathrm{PSh}(\mathcal{S}), m : M \otimes O_S \to M\}$, together with the forgetful functor $o : \mathrm{PSh}(\mathcal{S}) \to \mathrm{PSh}_{O_S}(\mathcal{S})$,
  - $C(\mathcal{S}) = C(\mathrm{PSh}(\mathcal{S}))$ and $C_{O_S}(\mathcal{S}) = C(\mathrm{PSh}_{O_S}(\mathcal{S}))$ the big abelian category of complexes of presheaves of O_S modules on $\mathcal{S}$,
  - $C_{O_S(2)fil}(\mathcal{S}) := C_{(2)fil}(\mathrm{PSh}_{O_S}(\mathcal{S})) \subset C(\mathrm{PSh}_{O_S}(\mathcal{S}), F, W)$, the big abelian category of (bi)filtered complexes of presheaves of O_S modules on $\mathcal{S}$ such that the filtration is biregular and $\mathrm{PSh}_{O_S(2)fil}(\mathcal{S}) := (\mathrm{PSh}_{O_S}(\mathcal{S}), F, W)$.

- Let $(\mathcal{S}, O_S) \in \mathrm{RCat}$ a ringed topos with topology τ. For $F \in C_{O_S}(\mathcal{S})$, we denote by $k : F \to E_\tau(F)$ the canonical flasque resolution in $C_{O_S}(\mathcal{S})$ (see [1]). In particular for $X \in \mathcal{S}$, $H^*(X, E_\tau(F)) \xrightarrow{\sim} \mathbb{H}^*_\tau(X, F)$.

- For $f : \mathcal{S}' \to \mathcal{S}$ a morphism with $\mathcal{S}, \mathcal{S}' \in \mathrm{RCat}$, endowed with topology τ and τ' respectively, we denote for $F \in C_{O_S}(\mathcal{S})$ and each $j \in \mathbb{Z}$,

  - $f^* := H^j\Gamma(\mathcal{S}, k \circ \mathrm{ad}(f^*, f_*)(F)) : \mathbb{H}^j(\mathcal{S}, F) \to \mathbb{H}^j(\mathcal{S}', f^*F)$,
  - $f^* := H^j\Gamma(\mathcal{S}, k \circ \mathrm{ad}(f^{*mod}, f_*)(F)) : \mathbb{H}^j(\mathcal{S}, F) \to \mathbb{H}^j(\mathcal{S}', f^{*mod}F)$,

  the canonical maps.

- For $m : A \to B$, $A, B \in C(\mathcal{A})$, $\mathcal{A}$ an additive category, we denote $c(A) : \mathrm{Cone}(m : A \to B) \to A[1]$ and $c(B) : B \to \mathrm{Cone}(m : A \to B)$ the canonical maps.

- For $X \in \mathrm{Top}$ and $Z \subset X$ a closed subset, denoting $j : X \backslash Z \hookrightarrow X$ the open complementary, we will consider
$$\Gamma^\vee_Z \mathbb{Z}_X := \mathrm{Cone}(\mathrm{ad}(j_!, j^*)(\mathbb{Z}_X) : j_! j^* \mathbb{Z}_X \hookrightarrow \mathbb{Z}_X) \in C(X)$$
and denote for short $\gamma^\vee_Z := \gamma^\vee_X(\mathbb{Z}_X) := c(\mathbb{Z}_X) : \mathbb{Z}_X \to \Gamma^\vee_Z \mathbb{Z}_X$ the canonical map in $C(X)$.

- Denote by $\mathrm{Sch} \subset \mathrm{RTop}$ the subcategory of schemes (the morphisms are the morphisms of locally ringed spaces). We denote by $\mathrm{PSch} \subset \mathrm{Sch}$ the full subcategory of proper schemes. For a field k, we consider $\mathrm{Sch}\,/k := \mathrm{Sch}\,/\,\mathrm{Spec}\,k$ the category of schemes over $\mathrm{Spec}\,k$. The objects are $X := (X, a_X)$ with $X \in \mathrm{Sch}$ and $a_X : X \to \mathrm{Spec}\,k$ a morphism and the morphisms are the morphisms of schemes $f : X' \to X$ such that $f \circ a_{X'} = a_X$. We then denote by

  - $\mathrm{Var}(k) = \mathrm{Sch}^{ft}\,/k \subset \mathrm{Sch}\,/k$ the full subcategory consisting of algebraic varieties over k, i.e. schemes of finite type over k,
  - $\mathrm{PVar}(k) \subset \mathrm{QPVar}(k) \subset \mathrm{Var}(k)$ the full subcategories consisting of quasi-projective varieties and projective varieties respectively,
  - $\mathrm{PSmVar}(k) \subset \mathrm{SmVar}(k) \subset \mathrm{Var}(k)$, $\mathrm{PSmVar}(k) := \mathrm{PVar}(k) \cap \mathrm{SmVar}(k)$, the full subcategories consisting of smooth varieties and smooth projective varieties respectively.

- Denote by $\mathrm{AnSp}(\mathbb{C}) \subset \mathrm{RTop}$ the subcategory of analytic spaces over $\mathbb{C}$, and by $\mathrm{AnSm}(\mathbb{C}) \subset \mathrm{AnSp}(\mathbb{C})$ the full subcategory of smooth analytic spaces (i.e. complex analytic manifold).

- For $X \in \mathrm{Top}$ and $X = \cup_{i \in I} X_i$ an open cover, we denote $X_\bullet \in \mathrm{Fun}(\Delta, \mathrm{Top})$ the associated simplicial space, with for $J \subset I$, $j_{IJ} : X_I := \cap_{i \in I} X_i \hookrightarrow X_J := \cap_{i \in J} X_i$ the open embedding.

- For $X \in \mathrm{AnSm}(\mathbb{C})$ and $X = \cup_{i \in I} X_i$ an open cover, we denote $X_\bullet \in \mathrm{Fun}(\Delta, \mathrm{AnSm}(\mathbb{C}))$ the associated simplicial complex manifold, with for $J \subset I$, $j_{IJ} : X_I := \cap_{i \in I} X_i \hookrightarrow X_J := \cap_{i \in J} X_i$ the open embedding.

- Let $(X, O_X) \in \mathrm{RTop}$. We consider its De Rham complex $\Omega^\bullet_X := DR(X)(O_X)$.

  - Let $X \in \mathrm{Sch}$. Considering its De Rham complex $\Omega^\bullet_X := DR(X)(O_X)$, we have for $j \in \mathbb{Z}$ its De Rham cohomology $H^j_{DR}(X) := \mathbb{H}^j(X, \Omega^\bullet_X)$.
  - Let $X \in \mathrm{Var}(k)$. Considering its De Rham complex $\Omega^\bullet_X := \Omega^\bullet_{X/k} := DR(X/k)(O_X)$, we have for $j \in \mathbb{Z}$ its De Rham cohomology $H^j_{DR}(X) := \mathbb{H}^j(X, \Omega^\bullet_X)$. The differentials of $\Omega^\bullet_X := \Omega^\bullet_{X/k}$ are by definition k-linear, thus $H^j_{DR}(X) := \mathbb{H}^j(X, \Omega^\bullet_X)$ has a structure of a k vector space.
  - Let $X \in \mathrm{AnSp}(\mathbb{C})$. Considering its De Rham complex $\Omega^\bullet_X := DR(X)(O_X)$, we have for $j \in \mathbb{Z}$ its De Rham cohomology $H^j_{DR}(X) := \mathbb{H}^j(X, \Omega^\bullet_X)$.

- For $X \in \mathrm{AnSp}(\mathbb{C})$, we denote $\alpha(X) : \mathbb{C}_X \hookrightarrow \Omega^\bullet_X$ the embedding in $C(X)$. For $X \in \mathrm{AnSm}(\mathbb{C})$, $\alpha(X) : \mathbb{C}_X \hookrightarrow \Omega^\bullet_X$ is an equivalence usu local by Poincare lemma.
- We denote $\mathbb{I}^n := [0,1]^n \in \mathrm{Diff}(\mathbb{R})$ (with boundary). For $X \in \mathrm{Top}$ and R a ring, we consider its singular cochain complex

$$C^*_{\mathrm{sing}}(X,R) := (\mathbb{Z}\,\mathrm{Hom}_{\mathrm{Top}}(\mathbb{I}^*, X)^\vee) \otimes R$$

and for $l \in \mathbb{Z}$ its singular cohomology $H^l_{\mathrm{sing}}(X,R) := H^n C^*_{\mathrm{sing}}(X,R)$. For $f : X' \to X$ a continuous map with $X, X' \in \mathrm{Top}$, we have the canonical map of complexes

$$f^* : C^*_{\mathrm{sing}}(X,R) \to C^*_{\mathrm{sing}}(X,R), \sigma \mapsto f^*\sigma := (\gamma \mapsto \sigma(f \circ \gamma)).$$

In particular, we get by functoriality the complex

$$C^*_{X,R\,\mathrm{sing}} \in C_R(X),\ (U \subset X) \mapsto C^*_{\mathrm{sing}}(U,R)$$

We recall that

  - For $X \in \mathrm{Top}$ locally contractible, e.g. $X \in \mathrm{CW}$, and R a ring, the inclusion in $C_R(X)$ $c_X : R_X \to C^*_{X,R\,\mathrm{sing}}$ is by definition an equivalence top local and that we get by the small chain theorem, for all $l \in \mathbb{Z}$, an isomorphism $H^l c_X : H^l(X, R_X) \xrightarrow{\sim} H^l_{\mathrm{sing}}(X,R)$.
  - For $X \in \mathrm{Diff}(\mathbb{R})$, the restriction map

$$r_X : \mathbb{Z}\,\mathrm{Hom}_{\mathrm{Diff}(\mathbb{R})}(\mathbb{I}^*, X)^\vee \to C^*_{\mathrm{sing}}(X,R),\ w \mapsto w : (\phi \mapsto w(\phi))$$

    is a quasi-isomorphism by Whitney approximation theorem.

- We will consider the morphism of site given by the analytification functor

$$\mathrm{An} : \mathrm{AnSp}(\mathbb{C}) \to \mathrm{Var}(\mathbb{C}),\ \ X \mapsto \mathrm{An}(X) := X^{an},\ (f : X' \to X) \mapsto \mathrm{An}(f) := f^{an}.$$

For $X \in \mathrm{Var}(\mathbb{C})$, we denote by

$$\mathrm{an}_X := \mathrm{An}_{|X^{et}} := X^{an,et} \to X^{et},\ \mathrm{an}_X(U \to X) := U^{an},$$

its restriction to the small etale site $o_X(X^{et}) \subset \mathrm{Var}(\mathbb{C})$, where $o_X(U \to X) = U$, in particular we have the following commutative diagram

$$\begin{array}{ccc} \mathrm{AnSp}(\mathbb{C}) & \xrightarrow{\mathrm{An}} & \mathrm{Var}(\mathbb{C}) \\ \downarrow{\scriptstyle o_{X^{an}}} & & \downarrow{\scriptstyle o_X} \\ X^{an,et} & \xrightarrow{\mathrm{an}_X} & X^{et} \end{array},$$

where $o_{X^{an}}(U \hookrightarrow X^{an}) = U$. We get the morphism in RTop $\mathrm{an}_X := \mathrm{an}_{X|O(X)} : X^{an} \to X$, $o : O(X) \subset X^{et}$ being the embedding of the Zariski open subsets of X.

2.2 Complex integral periods

Let k be a field of characteristic zero and $X \in \mathrm{SmVar}(k)$ be a smooth variety. Let $X = \cup_{i=1}^s X_i$ be an open affine cover. For each embedding $\sigma : k \hookrightarrow \mathbb{C}$, we have the evaluation period embedding map which is the morphism of bi-complexes

$$ev(X)^\bullet_\bullet : \Gamma(X_\bullet, \Omega^\bullet_{X_\bullet}) \to \mathbb{Z}\,\mathrm{Hom}_{\mathrm{Diff}}(\mathbb{I}^\bullet, X^{an}_{\mathbb{C},\bullet})^\vee \otimes \mathbb{C},$$

$$w^l_I \in \Gamma(X_I, \Omega^l_{X_I}) \mapsto (ev(X)^l_I(w^l_I) : \phi^l_I \in \mathbb{Z}\,\mathrm{Hom}_{\mathrm{Diff}}(\mathbb{I}^l, X^{an}_{\mathbb{C},I})^\vee \otimes \mathbb{C} \mapsto ev^l_I(w^l_I)(\phi^l_I) := \int_{\mathbb{I}^l} \phi^{l*}_I w^l_I)$$

given by integration. By taking all the affine open cover $(j_i : X_i \hookrightarrow X)$ of X, we get for $\sigma : k \hookrightarrow \mathbb{C}$, the evaluation period embedding map

$$ev(X) := \varinjlim_{(j_i : X_i \hookrightarrow X)} ev(X)^\bullet_\bullet : \varinjlim_{(j_i : X_i \hookrightarrow X)} \Gamma(X_\bullet, \Omega^\bullet_{X_\bullet}) \to \varinjlim_{(j_i : X_i \hookrightarrow X)} \mathbb{Z}\,\mathrm{Hom}_{\mathrm{Diff}(\mathbb{R})}(\mathbb{I}^\bullet, X^{an}_{\mathbb{C},\bullet})^\vee \otimes \mathbb{C}$$

It induces in cohomology, for $j \in \mathbb{Z}$, the evaluation period map

$$H^j ev(X) = H^j ev(X)^\bullet_\bullet : H^j_{DR}(X) = H^j \Gamma(X_\bullet, \Omega^\bullet_{X_\bullet}) \to H^j_{\mathrm{sing}}(X^{an}_{\mathbb{C}}, \mathbb{C}) = H^j(\mathrm{Hom}_{\mathrm{Diff}(\mathbb{R})}(\mathbb{I}^\bullet, X^{an}_{\mathbb{C},\bullet})^\vee \otimes \mathbb{C}).$$

which does NOT depend on the choice of the affine open cover by acyclicity of quasi-coherent sheaves on affine noetherian schemes for the left hand side and from Mayer-Vietoris quasi-isomorphism for singular cohomology of topological spaces and Whitney approximation theorem for differential manifolds for the right hand side.

Let $X \in \mathrm{SmVar}(k)$. Then,

$$H^* ev(X_{\mathbb{C}}) = H^* R\Gamma(X^{an}_{\mathbb{C}}, \alpha(X))^{-1} \circ \Gamma(X^{an}_{\mathbb{C}}, E_{zar}(\Omega(\mathrm{an}_X))) : H^*_{DR}(X_{\mathbb{C}}) \xrightarrow{\sim} H^*_{\mathrm{sing}}(X^{an}_{\mathbb{C}}, \mathbb{C})$$

is the canonical isomorphism induced by the analytical functor and the quasi-isomorphism $\alpha(X) : \mathbb{C}_{X^{an}_{\mathbb{C}}} \hookrightarrow \Omega^\bullet_{X^{an}_{\mathbb{C}}}$, which gives the periods elements $H^* ev(X)(H^*_{DR}(X)) \subset H^*_{\mathrm{sing}}(X^{an}_{\mathbb{C}}, \mathbb{C})$.

2.3 Algebraic cycles and motives

For $X \in \mathrm{Sch}$ noetherian irreducible and $d \in \mathbb{N}$, we denote by $\mathcal{Z}^d(X)$ the group of algebraic cycles of codimension d, which is the free abelian group generated by irreducible closed subsets of codimension d.

For $X, X' \in \mathrm{Sch}$ noetherian, with X' irreducible, we denote by $\mathcal{Z}^{fs/X'}(X' \times X) \subset \mathcal{Z}_{d_{X'}}(X' \times X)$ which consist of algebraic cycles $\alpha = \sum_i n_i \alpha_i \in \mathcal{Z}_{d_{X'}}(X' \times X)$ such that, denoting $\mathrm{supp}(\alpha) = \cup_i \alpha_i \subset X' \times X$ its support and $p' : X' \times X \to X'$ the projection, $p'_{|\,\mathrm{supp}(\alpha)} : \mathrm{supp}(\alpha) \to X'$ is finite surjective.

Let k be a field. We denote by $\mathrm{Cor}\,\mathrm{SmVar}(k)$ the category whose objects are $\{X \in \mathrm{SmVar}(k)\}$ and

$$\mathrm{Hom}_{\mathrm{Cor}\,\mathrm{SmVar}(k)}(X', X) := \mathcal{Z}^{fs/X'}(X' \times X)$$

(see [3] for the composition law). We denote by $\mathrm{Tr} : \mathrm{Cor}\,\mathrm{SmVar}(k) \to \mathrm{SmVar}(k)$ the morphism of site given by the embedding $\mathrm{Tr} : \mathrm{SmVar}(k) \hookrightarrow \mathrm{Cor}\,\mathrm{SmVar}(k)$. Let $F \in \mathrm{PSh}(\mathrm{SmVar}(k))$, we say that F admits transfers if $F = \mathrm{Tr}_* F$ with $F \in \mathrm{PSh}(\mathrm{Cor}\,\mathrm{SmVar}(k))$.

Let k be a field. We recall that $\mathrm{DA}(k) := \mathrm{Ho}_{\mathbb{A}^1,et}(C(\mathrm{SmVar}(k)))$ is the derived category of mixed motives.

We have (see e.g. [1]) the Hodge realization functor

$$\mathcal{F}^{Hdg} : \mathrm{DA}(\mathbb{C}) \to D(MHS),\ M \mapsto \mathcal{F}^{Hdg}(M) := \iota^{-1}(\mathcal{F}^{FDR}(M), \mathrm{Bti}^*\, M, \alpha(M)),$$

where $D(MHS) \hookrightarrow D_{2fil}(\mathbb{C}) \times_I D_{fil}(\mathbb{Q})$ is the derived category of mixed Hodge structures.

For $X \in \mathrm{Var}(k)$ and $Z \subset X$ a closed subset, denoting $j : X \backslash Z \hookrightarrow X$ the open complementary, we will consider

$$\Gamma^\vee_Z \mathbb{Z}_X := \mathrm{Cone}(\mathrm{ad}(j_\sharp, j^*)(\mathbb{Z}_X) : j_\sharp j^* \mathbb{Z}_X \hookrightarrow \mathbb{Z}_X) \in C(\mathrm{Var}(k)^{sm}/X)$$

and denote for short

$$\gamma^\vee_Z := \gamma^\vee_Z(\mathbb{Z}_X) := c(\mathbb{Z}_X) : \mathbb{Z}_X \to \Gamma^\vee_Z \mathbb{Z}_X$$

the canonical map in $C(\mathrm{Var}(k)^{sm}/X)$. For $X \in \mathrm{Var}(k)$ and $Z \subset X$ a closed subset, denoting $a_X : X \to \mathrm{Spec}\, k$ the structural map, we have then the motive of X with support in Z defined as

$$M_Z(X) := a_{X!} \Gamma^\vee_Z a^!_X \mathbb{Z} \in \mathrm{DA}(k).$$

If $X \in \mathrm{SmVar}(k)$, we will also consider

$$a_{X\sharp} \Gamma^\vee_Z \mathbb{Z}_X := \mathrm{Cone}(a_{X\sharp} \circ \mathrm{ad}(j_\sharp, j^*)(\mathbb{Z}_X) : \mathbb{Z}(U) \hookrightarrow \mathbb{Z}(X)) =: \mathbb{Z}(X, X \backslash Z) \in C(\mathrm{SmVar}(k)),$$

and denote for short

$$\gamma_Z^\vee := a_{X\sharp}\gamma_Z^\vee(\mathbb{Z}_X) := c(\mathbb{Z}(X)) : \mathbb{Z}(X) \to \mathbb{Z}(X, X\backslash Z)$$

the map in $C(\mathrm{SmVar}(k))$. Then for $X \in \mathrm{SmVar}(k)$ and $Z \subset X$ a closed subset

$$M_Z(X) := a_{X!}\Gamma_Z^\vee a_X^! \mathbb{Z} = a_{X\sharp}\Gamma_Z^\vee \mathbb{Z}_X =: \mathbb{Z}(X, X\backslash Z) \in \mathrm{DA}(k).$$

- Let $(X, Z) \in \mathrm{Sch}^2$ with $X \in \mathrm{Sch}$ a noetherian scheme and $Z \subset X$ a closed subset. We have the deformation $(D_Z X, \mathbb{A}^1_Z) \to \mathbb{A}^1$, $(D_Z X, \mathbb{A}^1_Z) \in \mathrm{Sch}^2$ of (X, Z) by the normal cone $C_{Z/X} \to Z$, i.e. such that

  $$(D_Z X, \mathbb{A}^1_Z)_s = (X, Z),\ s \in \mathbb{A}^1\backslash 0,\ (D_Z X, \mathbb{A}^1_Z)_0 = (C_{Z/X}, Z).$$

  We denote by $i_1 : (X, Z) \hookrightarrow (D_Z X, \mathbb{A}^1_Z)$ and $i_0 : (C_{Z/X}, Z) \hookrightarrow (D_Z X, \mathbb{A}^1_Z)$ the closed embeddings in Sch^2.

- Let k be a field of characteristic zero. Let $X \in \mathrm{SmVar}(k)$. For $Z \subset X$ a closed subset of pure codimension c, consider a desingularisation $\epsilon : \tilde{Z} \to Z$ of Z and denote $n : \tilde{Z} \xrightarrow{\epsilon} Z \subset X$. We have then the morphism in $\mathrm{DA}(k)$

  $$G_{Z,X} : M(X) \xrightarrow{D(\mathbb{Z}(n))} M(\tilde{Z})(c)[2c] \xrightarrow{\mathbb{Z}(\epsilon)} M(Z)(c)[2c]$$

  where $D : \mathrm{Hom}_{\mathrm{DA}(k)}(M_c(\tilde{Z}), M_c(X)) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{DA}(k)}(M(X), M(\tilde{Z})(c)[2c])$ is the duality isomorphism from the six functors formalism (moving lemma of Suzlin and Voevodsky) and $\mathbb{Z}(n) := \mathrm{ad}(n_!, n^!)(a_X^!\mathbb{Z})$, noting that $n_! = n_*$ since n is proper and that $a_X^! = a_X^*[d_X]$ and $a_{\tilde{Z}}^! = a_{\tilde{Z}}^*[d_Z]$ since X, resp. $\tilde{Z}$, are smooth (considering the connected components, we may assume X and $\tilde{Z}$ of pure dimension).

We recall the following facts (see [3] and [1]):

Proposition 1. *Let k be a field a characteristic zero. Let $X \in \mathrm{SmVar}(k)$ and $i : Z \subset X$ a smooth subvariety of pure codimension d. Then $C_{Z/X} = N_{Z/X} \to Z$ is a vector bundle of rank d. The closed embeddings $i_1 : (X, Z) \hookrightarrow (D_Z X, \mathbb{A}^1_Z)$ and $i_0 : (C_{Z/X}, Z) \hookrightarrow (D_Z X, \mathbb{A}^1_Z)$ in $\mathrm{SmVar}^2(k)$ induces isomorphisms of motives $\mathbb{Z}(i_1) : M_Z(X) \xrightarrow{\sim} M_{\mathbb{A}^1_Z}(D_Z X)$ and $\mathbb{Z}(i_0) : M_Z(N_{Z/X}) \xrightarrow{\sim} M_{\mathbb{A}^1_Z}(D_Z X)$ in $\mathrm{DA}(k)$. We get the excision isomorphism in $\mathrm{DA}(k)$*

$$P_{Z,X} := \mathbb{Z}(i_0)^{-1} \circ \mathbb{Z}(i_1) : M_Z(X) \xrightarrow{\sim} M_Z(N_{Z/X}).$$

We have

$$Th(N_{Z/X}) \circ P_{Z,X} \circ \gamma_Z^\vee(\mathbb{Z}_X) = G_{Z,X} := D(\mathbb{Z}(i)) : M(X) \to M(Z)(d)[2d].$$

Proof. See [3]. □

Definition 1. *Let k be a field a characteristic zero. Let $X \in \mathrm{SmVar}(k)$ and $i : Z \subset X$ a smooth subvariety of pure codimension d. We consider using proposition 1 the canonical map in $\mathrm{DA}(k)$*

$$Res_{Z,X} : M(Z)(d)[-2d] \xrightarrow{P_{Z,X}^{-1} \circ Th(N_{Z/X})^{-1}} M_Z(X) \xrightarrow{c(\mathbb{Z}(X\backslash Z))} M(X\backslash Z)[1].$$

We recall that $c(\mathbb{Z}(X\backslash Z)) : \mathbb{Z}(X, X\backslash Z) \to \mathbb{Z}(X\backslash Z)[1]$ is the canonical map in $C(\mathrm{SmVar}(k))$.

- Recall that we say that $F \in \mathrm{PSh}(\mathrm{SmVar}(k))$ is $\mathbb{A}^1$ invariant if for all $X \in \mathrm{SmVar}(k)$, $p^* := F(p) : F(X) \to F(X \times \mathbb{A}^1)$ is an isomorphism, where $p : X \times \mathbb{A}^1 \to X$ is the projection.
- Similarly, we say that $F \in \mathrm{PSh}(\mathrm{AnSm}(\mathbb{C}))$ is $\mathbb{A}^1$ invariant if for all $X \in \mathrm{AnSm}(\mathbb{C})$, $p^* := F(p) : F(X) \to F(X \times \mathbb{A}^1)$ is an isomorphism, where $p : X \times \mathbb{A}^1 \to X$ is the projection.

2.4 The logaritmic De Rham complexes

We recall from [2] the following notion :

- For k a field and $X \in \mathrm{Var}(k)$, we consider the embedding in $C(X)$

$$OL_X : \Omega^\bullet_{X,\log} \hookrightarrow \Omega^\bullet_X := \Omega^\bullet_{X/k},$$

such that, for $X^o \subset X$ an open subset and $w \in \Omega^p_X(X^o)$, $w \in \Omega^p_{X,\log}(X^o)$ if and only if there exists $(n_i)_{1\leq i\leq s} \in \mathbb{Z}$ and $(f_{i,\alpha_k})_{1\leq i\leq s, 1\leq k\leq p} \in \Gamma(X^o, O_X)^*$ such that

$$w = \sum_{1\leq i\leq s} n_i df_{i,\alpha_1}/f_{i,\alpha_1} \wedge \cdots \wedge df_{i,\alpha_p}/f_{i,\alpha_p} \in \Omega^p_X(X^o),$$

and for $p = 0$, $\Omega^0_{X,\log} := \mathbb{Z}$. Let k be a field. We get an embedding in $C(\mathrm{Var}(k))$

$$OL : \Omega^\bullet_{/k,\log} \hookrightarrow \Omega^\bullet_{/k}, \text{ given by, for } X \in \mathrm{Var}(k),$$
$$OL(X) := OL_X : \Omega^\bullet_{/k,\log}(X) := \Gamma(X, \Omega^\bullet_{X,\log}) \hookrightarrow \Gamma(X, \Omega^\bullet_X) =: \Omega^\bullet_{/k}(X)$$

and its restriction to $\mathrm{SmVar}(k) \subset \mathrm{Var}(k)$.

- For $X \in \mathrm{AnSp}(\mathbb{C})$, we consider the embedding in $C(X)$

$$OL_X : \Omega^\bullet_{X,\log} \hookrightarrow \Omega^\bullet_X,$$

such that, for $X^o \subset X$ an open subset and $w \in \Omega^p_X(X^o)$, $w \in \Omega^p_{X,\log}(X^o)$ if and only if there exists $(n_i)_{1\leq i\leq s} \in \mathbb{Z}$ and $(f_{i,\alpha_k})_{1\leq i\leq s, 1\leq k\leq p} \in \Gamma(X^o, O_X)^*$ such that

$$w = \sum_{1\leq i\leq s} n_i df_{i,\alpha_1}/f_{i,\alpha_1} \wedge \cdots \wedge df_{i,\alpha_p}/f_{i,\alpha_p} \in \Omega^p_X(X^o),$$

and for $p = 0$, $\Omega^0_{X,\log} := \mathbb{Z}$. We get an embedding in $C(\mathrm{AnSp}(\mathbb{C}))$

$$OL^{an} : \Omega^{\bullet,an}_{\log} \hookrightarrow \Omega^{\bullet,an}, \text{ given by, for } X \in \mathrm{AnSp}(\mathbb{C}),$$
$$OL^{an}(X) := OL_X : \Omega^{\bullet,an}_{\log}(X) := \Gamma(X, \Omega^\bullet_{X,\log}) \hookrightarrow \Gamma(X, \Omega^\bullet_X) =: \Omega^{\bullet,an}(X)$$

and its restriction to $\mathrm{AnSm}(\mathbb{C}) \subset \mathrm{AnSp}(\mathbb{C})$.

Definition 2. *We consider the (non full) subcategory* $\mathrm{AnSm}(\mathbb{C})^{pr} \subset \mathrm{AnSp}(\mathbb{C})$ *whose set of objects consists of* $X \times \mathbb{A}^{1,an}$, $X \in \mathrm{AnSm}(\mathbb{C})$ *and*

$$\mathrm{Hom}_{\mathrm{AnSm}(\mathbb{C})^{pr}}(X \times \mathbb{A}^{1,an}, Y \times \mathbb{A}^{1,an}) := \mathrm{Hom}_{\mathrm{AnSp}(\mathbb{C})}(X, Y) \otimes I_{\mathbb{A}^{1,an}}.$$

We have then the morphism of sites

- $\pi : \mathrm{AnSm}(\mathbb{C}) \to \mathrm{AnSm}(\mathbb{C})^{pr}$, $\pi(X \times \mathbb{A}^{1,an}) := X$, $\pi(f \otimes I) := f$,
- $\rho : \mathrm{AnSm}(\mathbb{C})^{pr} \to \mathrm{AnSm}(\mathbb{C})$, $\rho(X) := X \times \mathbb{A}^{1,an}$, $\rho(f) := f \otimes I$,

in particular $\rho \circ \pi = I$. *We have then, for each* $p \in \mathbb{Z}$, *a canonical splitting in* $\mathrm{PSh}(\mathrm{AnSm}(\mathbb{C})^{pr})$

$$\Omega^{p,an,\partial=0} = a_{usu}\partial\Omega^{p-1,an} = \rho_* a_{usu}\partial\Omega^{p-1,an} \oplus \Omega^{p,e},$$

with

$$\Omega^{p,e}(X \times \mathbb{A}^1) := \left\{ \sum_{n=1}^{\infty} a_n z^n w(x) + w' \wedge dz, a_n \in \mathbb{C}, w \in \Omega^p(X), w' \in \Omega^{p-1}(X \times \mathbb{A}^1) \right\}$$
$$\subset a_{usu}\partial\Omega^{p-1,an}(X \times \mathbb{A}^1)$$

where z is the coordinate of $\mathbb{A}^1$, $a_n r^n \to 0$ when $n \to \infty$ for all $r \in \mathbb{R}^+$, and $x \in X$. We have the factorization in $\mathrm{PSh}(\mathrm{AnSm}(\mathbb{C}))$

$$\mathrm{ad}(\pi) := \mathrm{ad}(\pi^*, \pi_*)(-) : \pi^*\Omega^{p,e} \xrightarrow{\subset} \pi^* a_{usu}\partial\Omega^{p-1,an} := \pi^*\pi_* a_{usu}\partial\Omega^{p-1,an}$$
$$\xrightarrow{(0,\mathrm{ad}(\pi))} \mathrm{ad}(\pi)(\pi^*\Omega^{p,e}) \xrightarrow{i_{\mathrm{ad}(\pi)}} a_{usu}\partial\Omega^{p-1,an},$$

and we denote again by abuse

$$\pi^*\Omega^{\bullet,e} := \mathrm{ad}(\pi)(\pi^*\Omega^{\bullet,e}) \subset a_{usu}\partial\Omega^{p-1,an} \subset \Omega^{\bullet,an}.$$

We will consider, for each $p \in \mathbb{Z}$, the exact sequence in $\mathrm{PSh}(\mathrm{AnSm}(\mathbb{C}))$

$$0 \to \Omega^{p,an}_{\log} \cap (\pi^*\Omega^{p,e}) \xrightarrow{e} \Omega^{p,an}_{\log} \xrightarrow{q} \Omega^{p,an,ne}_{\log} := \Omega^{p,an}_{\log}/(\Omega^{p,an}_{\log} \cap (\pi^*\Omega^{p,e})) \to 0.$$

Note that for $X \in \mathrm{AnSp}(\mathbb{C})$ and $\partial f \in \Omega^{1,an}(X^{an})$ with $f \in O(X^{an})$, we have $\partial f = (\partial e^f)/e^f \in \Omega^{1,an}_{\log}(X^{an})$, hence, since open balls form a basis of the complex topology on complex analytic manifold,

$$a_{usu}(\Omega^{1,an}_{\log} \otimes \mathbb{C}) = a_{usu}\Omega^{1,an,\partial=0} = a_{usu}\partial O^{an}.$$

We have then the commutative diagram in $C(\mathrm{AnSm}(\mathbb{C}))$

$$\begin{array}{ccc}
\Omega^{\bullet,an}_{\log} \cap (\pi^*\Omega^{\bullet,e}) & \xrightarrow[OL^{an}]{} & \pi^*\Omega^{\bullet,e} \\
\downarrow{\scriptstyle e} & & \downarrow{\scriptstyle e} \\
\Omega^{\bullet,an}_{\log} & \xrightarrow[OL^{an}]{} & \Omega^{\bullet,an} \\
\downarrow{\scriptstyle q} & & \downarrow{\scriptstyle q} \\
\Omega^{\bullet,an,ne}_{\log} := \Omega^{\bullet,an}_{\log}/(\Omega^{\bullet,an}_{\log} \cap (\pi^*\Omega^{\bullet,e})) & \xrightarrow{\bar{O}L^{an}} & \Omega^{\bullet,an}/(\pi^*\Omega^{\bullet,e})
\end{array},$$

whose columns are exact with e injective and q surjective, and of course all differentials of $\Omega^{\bullet,an}_{\log}$ and of $\pi^*\Omega^{\bullet,e}$ are trivial.

Lemma 1. *We have a splitting in* $C(\mathrm{AnSm}(\mathbb{C}))$

$$a_{usu}\Omega^{\bullet,an}_{\log} = a_{usu}(\Omega^{\bullet,an}_{\log} \cap (\pi^*\Omega^{\bullet,e})) \oplus a_{usu}\Omega^{\bullet,an,ne}_{\log}$$

Proof. We have

$$\pi^*\Omega^{\bullet}(X) := \varinjlim_{m: X \to Y \times \mathbb{A}^1} \Omega(m)(\Omega^{\bullet,an}(Y \times \mathbb{A}^1)) \subset \Omega^{\bullet,an}(X),$$

where we denote again by abuse

$$\pi^*\Omega^{\bullet} := \mathrm{ad}(\pi)(\pi^*\Omega^{\bullet}) \subset \Omega^{\bullet,an}.$$

Now,

- if $q \circ m : X \xrightarrow{m} Y \times \mathbb{A}^1 \xrightarrow{q} \mathbb{A}^1$ is non constant, we can factor

  $$m = (n \times I) \circ m' : X \xrightarrow{m'} Y' \times \mathbb{A}^1 \xrightarrow{n \times I} Y \times \mathbb{A}^1$$

  with $\dim Y' = \dim X - 1$ and $m'(X) \subset Y' \times \mathbb{A}^1$ an open subset, and

  $$\Omega(m') : \Omega^{\bullet,an}(Y' \times \mathbb{A}^1) \to \Omega^{\bullet,an}(X)$$

  so that we still have a splitting in the image of $\Omega(m')$

- if $q \circ m : X \xrightarrow{m} Y \times \mathbb{A}^1 \xrightarrow{q} \mathbb{A}^1$ is constant with value c,

$$\Omega(m)(\pi^*\Omega^{\bullet,e})(Y \times \mathbb{A}^1) = 0$$

since $c^*dz = 0$, where $c : Y \times \{c\} \hookrightarrow Y \times \mathbb{A}^1$ is the closed embedding.

We then define

$$(\pi^*\Omega^\bullet)^{nc} \subset \pi^*\Omega^\bullet,\ (\pi^*\Omega^\bullet)^{nc}(X) := \varinjlim_{m:X \to Y \times D^1, q \circ m \,\mathrm{nc}} \Omega(m)(\Omega^{\bullet,an}(Y \times \mathbb{A}^1)) \subset \pi^*\Omega^\bullet(X),$$

where nc means non constant. Now consider a splitting a complex vector spaces

$$\Omega^{1,\partial=0}(D^1) = \Omega^{1,\partial=0}(\mathbb{A}^1) \oplus \Omega^{1,na}(D^1).$$

We consider similarly the (non full) subcategory $\mathrm{AnSm}(\mathbb{C})^{pr'} \subset \mathrm{AnSp}(\mathbb{C})$ whose set of objects consists of $X \times D^1$, $X \in \mathrm{AnSm}(\mathbb{C})$ and

$$\mathrm{Hom}_{\mathrm{AnSm}(\mathbb{C})^{pr}}(X \times D^1, Y \times D^1) := \mathrm{Hom}_{\mathrm{AnSp}(\mathbb{C})}(X, Y) \otimes I_{D^1}.$$

We have then similarly the morphism of sites

- $\pi' : \mathrm{AnSm}(\mathbb{C}) \to \mathrm{AnSm}(\mathbb{C})^{pr'}$, $\pi'(X \times D^1) := X$, $\pi'(f \otimes I) := f$,
- $\rho' : \mathrm{AnSm}(\mathbb{C})^{pr'} \to \mathrm{AnSm}(\mathbb{C})$, $\rho'(X) := X \times D^1$, $\rho'(f) := f \otimes I$,

in particular $\rho' \circ \pi' = I$. We define similarly

$$(\pi'^*\Omega^\bullet)^{nc} \subset \pi'^*\Omega^\bullet,\ (\pi'^*\Omega^\bullet)^{nc}(X) := \varinjlim_{m:X \to Y \times D^1, q \circ m \,\mathrm{nc}} \Omega(m)(\Omega^{\bullet,an}(Y \times D^1)) \subset \pi'^*\Omega^\bullet(X).$$

We have then

$$\Omega^{\bullet,an} = a_{usu}(\pi'^*\Omega^\bullet)^{nc} = a_{usu}(\pi^*\Omega^\bullet)^{nc} \oplus a_{usu}\pi'^*\Omega^{\bullet,nc,na},$$
$$\pi'^*\Omega^{\bullet,na}(X) := \varinjlim_{m:X \to Y \times D^1, q \circ m \,\mathrm{nc}} \Omega(m)(\Omega^{\bullet-1,an}(Y) \times \Omega^{1,na}(D^1)) \subset \Omega^{\bullet,an}(X)$$

and thus

$$a_{usu}\Omega^{\bullet,an}_{\log} = a_{usu}(\Omega^{\bullet,an}_{\log} \cap (\pi'^*\Omega^\bullet)^{nc}) = a_{usu}(\Omega^{\bullet,an}_{\log} \cap (\pi^*\Omega^\bullet)^{nc}) \oplus a_{usu}(\Omega^{\bullet,an}_{\log} \cap \pi'^*\Omega^{\bullet,nc,na})$$

where the first equalities follows from the fact for an open ball $D^{dx} \subset X$, we have the morphism $m_I : D^{dx} \to D^{dx-1} \times D^1$ which is the canonical isomorphism. Hence,

$$a_{usu}\Omega^{\bullet,an}_{\log} = a_{usu}(\Omega^{\bullet,an}_{\log} \cap (\pi^*\Omega^{\bullet,e})) \oplus \tilde{\Omega}^{\bullet,an,ne}_{\log}$$

with

$$\tilde{\Omega}^{\bullet,an,ne}_{\log}(X) := a_{usu}(\Omega^{\bullet,an}_{\log} \cap (\pi^*\rho_*\Omega^\bullet)^{nc}) \oplus a_{usu}(\Omega^{\bullet,an}_{\log} \cap \pi'^*\Omega^{\bullet,nc,na}).$$

□

We have the following :

Lemma 2. *(i0) The sheaves $O^*_{an} \in \mathrm{PSh}(\mathrm{AnSm}(\mathbb{C}))$ and $\Omega^{1,an}_{\log} \in \mathrm{PSh}(\mathrm{AnSm}(\mathbb{C}))$ admit transfers compatible with transfers on $\Omega^{1,an} \in \mathrm{PSh}(\mathrm{AnSm}(\mathbb{C}))$.*

(i) For each $l \in \mathbb{Z}$, the sheaf $\Omega^{l,an}_{\log} \in \mathrm{PSh}(\mathrm{AnSm}(\mathbb{C}))$ admits transfers compatible with transfers on $\Omega^{l,an} \in \mathrm{PSh}(\mathrm{AnSm}(\mathbb{C}))$, that is $\Omega^{\bullet,an} \in C(\mathrm{Cor}\,\mathrm{AnSm}(\mathbb{C}))$ and the inclusion $OL : \Omega^{l,an}_{\log}[-l] \hookrightarrow \Omega^{\bullet,an}$ in $C(\mathrm{AnSm}(\mathbb{C}))$ is compatible with transfers.

(ii) For each $l \in \mathbb{Z}$, the presheaf $\Omega^{l,an,ne}_{\log} \in \mathrm{PSh}(\mathrm{AnSm}(\mathbb{C}))$ given in definition 2 is $\mathbb{A}^1$ invariant.

(ii)' For each $l \in \mathbb{Z}$, the sheaf $a_{usu}\Omega^{l,an,ne}_{\log} \in \mathrm{PSh}(\mathrm{AnSm}(\mathbb{C}))$ given in definition 2 is $\mathbb{A}^1$ invariant, where $a_{usu} : \mathrm{PSh}(\mathrm{AnSp}(\mathbb{C})) \to \mathrm{Shv}(\mathrm{AnSp}(\mathbb{C}))$ is the sheaftification functor.

Proof. (i0):Let $W \subset X' \times X$ with $X, X' \in \mathrm{AnSm}(\mathbb{C})$, X' connected, and $p_{X'} : W \subset X' \times X \to X'$ finite surjective over X'. Then $M(p_X) : M(X') \hookrightarrow M(W)$ is a finite, where for $Y \in \mathrm{AnSp}(\mathbb{C})$ irreducible, $M(Y) := Frac(O(Y))$ denote the field of meromorphic function. Since $O(X') \subset M(X')$ is integrally closed, the trace map $Tr_{W/X'} : M(W) \to M(X')$ sends $O(W)$ to $O(X')$, and the norm map $N_{W/X'} : M(W)^* \to M(X')^*$ sends $O(W)^*$ to $O(X')^*$.

The sheaf $O^*_{an} \in \mathrm{PSh}(\mathrm{AnSm}(\mathbb{C}))$ admits transfers : for $W \subset X' \times X$ with $X, X' \in \mathrm{AnSm}(\mathbb{C})$ and W finite surjective over X' and $f \in O(X)^*$, $W^*f := N_{W/X'}(p_X^* f)$ where $p_X : W \hookrightarrow X' \times X \to X$ is the projection and $N_{W/X'} : M(W)^* \to M(X')^*$ is the norm map. This gives transfers on $\Omega^{1,an}_{\log} \in \mathrm{PSh}(\mathrm{AnSm}(\mathbb{C}))$ compatible with transfers on $\Omega^{1,an} \in \mathrm{PSh}(\mathrm{AnSm}(\mathbb{C}))$: for $W \subset X' \times X$ with $X, X' \in \mathrm{AnSm}(\mathbb{C})$ and W finite surjective over X' and $f \in O(X)^*$,

$$W^*df/f := dW^*f/W^*f = Tr_{W/X'}(p_X^*(df/f)),$$

where we recall $p_X : W \hookrightarrow X' \times X \to X$ is the projection and $Tr_{W/X'} : O(W) \to O(X')$ is the trace map. Note that $d(fg)/fg = df/f + dg/g$.
(i): By (i0), we get transfers on

$$\otimes^l_{\mathbb{Q}}\Omega^{1,an}_{\log},\ \otimes^l_O\Omega^{1,an} \in \mathrm{PSh}(\mathrm{AnSm}(\mathbb{C}))$$

since $\otimes^l_{\mathbb{Q}}\Omega^{1,an}_{\log} = H^0(\otimes^{L,l}_{\mathbb{Q}}\Omega^{1,an}_{\log})$ and $\otimes^l_O\Omega^{1,an} = H^0(\otimes^{L,l}_O\Omega^{1,an})$. This induces transfers on

$$\wedge^l_{\mathbb{Q}}\Omega^{1,an}_{\log} := \mathrm{coker}(\oplus_{I_2 \subset [1,\dots,l]} \otimes^{l-1}_{\mathbb{Q}} \Omega^{1,an}_{\log} \xrightarrow{\oplus_{I_2 \subset [1,\dots,l]} \Delta_{I_2} := (w \otimes w' \mapsto w \otimes w \otimes w')} \otimes^l_{\mathbb{Q}}\Omega^{1,an}_{\log}) \in \mathrm{PSh}(\mathrm{AnSm}(\mathbb{C})).$$

and

$$\wedge^l_O\Omega^{1,an} := \mathrm{coker}(\oplus_{I_2 \subset [1,\dots,l]} \otimes^{l-1}_{O_k} \Omega^{1,an} \xrightarrow{\oplus_{I_2 \subset [1,\dots,l]} \Delta_{I_2} := (w \otimes w' \mapsto w \otimes w \otimes w')}) \otimes^l_O \Omega^{1,an} \in \mathrm{PSh}(\mathrm{AnSm}(\mathbb{C})).$$

(ii): Let $X \in \mathrm{AnSm}(\mathbb{C})$ and, denoting $p : X \times \mathbb{A}^1 \to X$ the projection,

$$\Omega(p) : \Omega^{p,an,ne}_{\log}(X) \to \Omega^{p,an,ne}_{\log}(X \times \mathbb{A}^1).$$

Since $\Omega(p)$ is split injective, it is enough to show that $\Omega(p)$ is surjective. We have

$$\Omega^{p,an,ne}_{\log}(X \times \mathbb{A}^1) := \Omega^{p,an}_{\log}(X \times \mathbb{A}^1)/\Omega^{p,e}(X \times \mathbb{A}^1)$$

The result then follows from the fact that for $X \in \mathrm{AnSp}(\mathbb{C})$, $\Omega^p(X \times \mathbb{A}^1) = \Omega^p(X) \oplus (\Omega^{p-1}(X) \otimes \Omega^1(\mathbb{A}^{1,an}))$ and that $\mathbb{A}^{1,an}$ is simply connected. Indeed, for $w = w' + w'' \wedge dz \in \Omega^p(X \times \mathbb{A}^1)^{\partial=0}$,

$$w(x,z) = \partial(\int_{t=0}^{t=z} w''(x,t)dt) + w(x,0) = \partial(\int_{t=0}^{t=z} w''(x,t)dt)_e + \partial(\int_{t=0}^{t=z} w''(x,t)dt)(x,0) + w'(x,0)$$

and $\partial(\int_{t=0}^{t=z} w''(x,t)dt)_e \in \Omega^{p,e}(X \times \mathbb{A}^1)$
(ii): Follows from (ii) since open ball form a basis of the complex topology on complex analytic manifolds. □

We have the following result :

Proposition 2. *Consider, for each $p \in \mathbb{Z}$, the presheaf $\Omega^{p,an,ne}_{\log} \in \mathrm{PSh}(\mathrm{AnSm}(\mathbb{C}))$ given in definition 2.*

(i) Let $D \in \mathrm{AnSm}(\mathbb{C})$ be an open ball. Then, for each $p, q \in \mathbb{Z}$, $q \neq 0$, $H^q_{usu}(D, \Omega^p_{D,\log}) = 0$.

(i)' Let $D \in \mathrm{AnSm}(\mathbb{C})$ be an open ball. Then, for each $p, q \in \mathbb{Z}$, $q \neq 0$, $H^q_{usu}(D, \Omega^{p,an,ne}_{\log}) = 0$.

(ii) Let $U \in \mathrm{SmVar}(\mathbb{C})$ such that there exists an etale map $e : U \to \mathbb{A}^{d_U}_{\mathbb{C}}$ such that $e : U \to e(U)$ is finite etale and $\mathbb{A}^{d_U}_{\mathbb{C}} \backslash e(U) \subset \mathbb{A}^{d_U}_{\mathbb{C}}$ is a divisor. Then, for each $p \in \mathbb{Z}$, $p \neq 0$, $p \neq d_U$, $H^p_{usu}(U^{an}, \Omega^{p,an,ne}_{\log}) = 0$.

(ii)' Let $U \in \mathrm{SmVar}(\mathbb{C})$ such that there exists an etale map $e : U \to \mathbb{A}^{d_U}_{\mathbb{C}}$ such that $e : U \to e(U)$ is finite etale and $\mathbb{A}^{d_U}_{\mathbb{C}} \backslash e(U) \subset \mathbb{A}^{d_U}_{\mathbb{C}}$ is a divisor. Then, for each $p, q \in \mathbb{Z}$, $p, q \neq 0$, $p \geq q$, $q \neq d_U$, $H^q_{usu}(U^{an}, \Omega^{p,an,ne}_{\log}) = 0$.

Proof. (i):Let $B = D(0,1)^N \in \mathrm{AnSm}(\mathbb{C})$ be an open ball. Let $\alpha \in H^q_{usu}(B, \Omega^p_{B,\log})$, $q \in \mathbb{Z}$, $q \neq 0$. Consider a sequence of sub-balls

$$l_n : B_n := D(0, r_n)^N \hookrightarrow B := D(0,1)^N, n \in \mathbb{N}, \text{ with } r_n \to 0 \text{ when } n \to \infty.$$

Since $(B_n := D(0, r_n))_{n \in \mathbb{N}}$ form a basis of neighborhood of $0 \in B$ and since $\Omega^p_{B,\log}$ is a single presheaf, there exists $n := n_\alpha \in \mathbb{N}$ (depending on α) such that

$$l_n^* \alpha = 0 \in H^q_{usu}(B_n, \Omega^p_{B,\log}).$$

But it follows immediately from the definition of the the presheaf $\Omega^p_{B,\log}$ considering the isomorphism

$$\phi_n : B_n \xrightarrow{\sim} B, \ \phi_n(z) := r_n^{-1} z$$

so that $(\phi_n)^{-1} \circ l_n = r_n I$, that

$$l_n^* \alpha = l_n^* \phi_n^{-1*} \phi_n^* \alpha = r_n \phi_n^* \alpha \in H^q_{usu}(B_n, \Omega^p_{B,\log}).$$

Hence $\phi_n^* \alpha = 0$. Since ϕ_n is an isomorphism, we get $\alpha = 0$.
(i)':Consider the splitting

$$a_{usu} \Omega^{\bullet,an}_{\log} = a_{usu}(\Omega^{\bullet,an}_{\log} \cap (\pi^* \Omega^{\bullet,e})) \oplus a_{usu} \Omega^{\bullet,an,ne}_{\log}$$

given in lemma 1. It gives in particular a canonical isomorphism

$$H^q_{usu}(D, \Omega^p_{\log}) = H^q_{usu}(D, \Omega^p_{\log} \cap \pi^* \Omega^{p,e}) \oplus H^q_{usu}(D, \Omega^{p,an,ne}_{\log}).$$

By (i) we have

$$H^q_{usu}(D, \Omega^p_{\log}) = 0 = H^q_{usu}(D, \Omega^p_{\log} \cap \pi^* \Omega^{p,e}) \oplus H^q_{usu}(D, \Omega^{p,an,ne}_{\log}).$$

Hence $H^q_{usu}(D, \Omega^{p,an,ne}_{\log}) = 0$. This proves (i)'. One can also prove (i)' directly using the same arguments as in the proof of (i).
(ii): We proceed by induction on the dimension of U. Let $U \in \mathrm{SmVar}(\mathbb{C})$ such that there exists an etale map $e : U \to \mathbb{A}^{d_U}_{\mathbb{C}}$. such that $e : U \to e(U)$ is finite etale and $\mathbb{A}^{d_U}_{\mathbb{C}} \backslash e(U) \subset \mathbb{A}^{d_U}_{\mathbb{C}}$ is a divisor Denote $D := \mathbb{A}^{d_U}_{\mathbb{C}} \backslash e(U)$ the complementary closed subset. Since $H^q_{usu}(\mathbb{A}^{d_U}_{\mathbb{C}}, \Omega^{p,an,ne}_{\log}) = 0$ for all $p, q \in \mathbb{Z}$,

$$H^q E_{usu}(\Omega^{p,an,ne}_{\log})(c(\mathbb{Z}(e(U)))) = \partial : H^q_{usu}(e(U), \Omega^{p,an,ne}_{\log}) \xrightarrow{\sim} H^{q+1}_{usu,D}(\mathbb{A}^{d_U}_{\mathbb{C}}, \Omega^{p,an,ne}_{\log})$$

is an isomorphism for all $p, q \in \mathbb{Z}$. By (i),

$$H^q E_{usu}(\Omega^{p,an,ne}_{\log}) \in \mathrm{PSh}(\mathrm{AnSm}(\mathbb{C}))$$

is $\mathbb{A}^1$ invariant for each $p, q \in \mathbb{Z}$. Indeed, for $X \in \mathrm{AnSm}(\mathbb{C})$ connected, take an open cover $X = \cup_{i \in I} D_i$ by open balls $D_i \simeq D(0,1)^{d_X}$, and let $D_\bullet \in \mathrm{Fun}(\Delta, \mathrm{AnSm}(\mathbb{C}))$ be the associated simplicial complex

manifold, with for $J \subset I$, $j_{IJ} : D_I := \cap_{i\in I} D_i \hookrightarrow D_J := \cap_{i\in J} D_i$ the open embedding. Then we get the isomorphism

$$p_X^* : H^q E_{usu}(\Omega_{\log}^{p,an,ne})(X \times \mathbb{A}^1) := H^q_{usu}(X \times \mathbb{A}^1, \Omega_{\log}^{p,an,ne}) \xrightarrow{\sim}$$
$$\mathbb{H}^q(D_\bullet \times \mathbb{A}^1, \Omega_{\log}^{p,an,ne}) \xrightarrow{\sim} H^q\Gamma(D_\bullet \times \mathbb{A}^1, a_{usu}\Omega_{\log}^{p,an,ne})$$
$$\xrightarrow{\sim} H^q\Gamma(D_\bullet, a_{usu}\Omega_{\log}^{p,an,ne}) \xrightarrow{\sim} \mathbb{H}^q(D_\bullet, \Omega_{\log}^{p,an,ne}) \xrightarrow{\sim} H^q_{usu}(X, \Omega_{\log}^{p,an,ne}) =: H^q E_{usu}(\Omega_{\log}^{p,an,ne})(X)$$

where a_{usu} is the sheafification functor, $p_X : X \times \mathbb{A}^1 \to X$ is the projection, and the second and the fourth map is the isomorphism given by (i)', and the middle map is an isomorphism by lemma 2(ii)'. Take a desingularization of the pair $\epsilon = \epsilon_1 : (\widetilde{\mathbb{A}^{d_U}_{\mathbb{C}}}, E) \to (\mathbb{A}^{d_U}_{\mathbb{C}}, D)$, so that $E = \cup_{i=1}^s E_i \subset \widetilde{\mathbb{A}^{d_U}_{\mathbb{C}}}$ is a normal crossing divisor and $\epsilon : \widetilde{\mathbb{A}^{d_U}_{\mathbb{C}}} \backslash E \xrightarrow{\sim} \mathbb{A}^{d_U}_{\mathbb{C}} \backslash D$. We thus get by proposition 1, for each $p, q \in \mathbb{Z}$, $p, q \neq 0$,

$$H^{q+1}E_{usu}(\Omega_{\log}^{p,an,ne})(P_{E_i, \widetilde{\mathbb{A}^{d_U}_{\mathbb{C}}}}) : H^{q+1}_{usu,E_i}(\widetilde{\mathbb{A}^{d_U}_{\mathbb{C}}}, \Omega_{\log}^{p,an,ne}) \xrightarrow{\sim} H^{q-1}_{usu}(E_i \times \mathbb{G}_m, \Omega_{\log}^{p,an,ne}).$$

For $p \in \mathbb{Z}$, $p \neq 0, d_U$, we get, using the induction hypothesis, by a finite decreasing induction $\tilde{E}_{ic} \subset \tilde{E}^N_{ic+1}, \cdots, E_{i0} \subset E_i$, where

- $E_i^o := E_i \backslash E_{i0} \subset E_i$ is an open subset such that there exists a finite etale map $e_i : E_i^o \to e_i(E_i^o) \subset \mathbb{A}^{d_U-1}$ with $\mathbb{A}^{d_U-1} \backslash e_i(E_i^o) \subset \mathbb{A}^{d_U-1}$ a divisor, $\epsilon_2 : (\tilde{E}_i, \tilde{E}_{i0}) \to (E_i, E_{i0})$ is a desingularization,
- $E_{ik}^o := \tilde{E}_{ik} \backslash E_{ik+1} \subset \tilde{E}_{ik}$ is an open subset such that there exists a finite etale map $e_{ik} : E_{ik}^o \to e_{ik}(E_{ik}^o) \subset \mathbb{A}^{d_U-k}$ with $\mathbb{A}^{d_U-k} \backslash e_{ik}(E_{ik}^o) \subset \mathbb{A}^{d_U-k}$ a divisor, $\epsilon_k : (\tilde{E}^N_{ik}, \tilde{E}_{ik+1}) \to (\tilde{E}_{ik}, E_{ik+1})$ is a desingularization

since $d_U > p$ ($\Omega^p_Y = 0$ for $p > d_Y$,$Y \in \mathrm{AnSm}(\mathbb{C})$),

$$H^p_{usu}(e(U), \Omega_{\log}^{p,an,ne}) = H^{p+1}_{usu,D}(\mathbb{A}^{d_U}_{\mathbb{C}}, \Omega_{\log}^{p,an,ne}) = H^{p+1}_{usu,E}(\widetilde{\mathbb{A}^{d_U}_{\mathbb{C}}}, \Omega_{\log}^{p,an,ne})$$
$$\xrightarrow{\sim} \oplus_{i=1}^r H^{p-c}_{usu}(\tilde{E}_{ic-1} \times \mathbb{G}_m^c, \Omega_{\log}^{p,an,ne}) = 0$$

since $\mathrm{Pic}(E_{ic-1}) = \mathrm{Pic}(E_i) = 0$ by induction as $\epsilon(E_i) \subset \mathbb{A}^{d_U}$ and $\epsilon_k(\tilde{E}_{ik+1}) \subset \tilde{E}_{ik}$ are hypersurfaces of dimension at least two, and for $Y \in \mathrm{AnSm}(\mathbb{C})$,

$$H^1_{usu}(Y, \Omega^{1,an}_{\log}) = \mathrm{Pic}(Y),\ H^1_{usu}(Y, \Omega^{1,an}_{\log}) = H^1_{usu}(Y, \pi^*\Omega^{1,e} \cap \Omega^{1,an}_{\log}) \oplus H^1_{usu}(Y, \Omega^{1,an,ne}_{\log}),$$

where the second equality follows from lemma 1. We thus have, since $M(U) \simeq M(e(U))$, for each $p \in \mathbb{Z}$, $p \neq 0, d_U$

$$H^p_{usu}(U, \Omega_{\log}^{p,an,ne}) = H^p_{usu}(e(U), \Omega_{\log}^{p,an,ne}) = 0.$$

(ii)':Follows from (ii). □

3 Hypersurface De Rham cohomology and its complex periods

We denote $\mathbb{P}^N := \mathbb{P}^N_{\mathbb{C}} := \mathrm{Proj}(\mathbb{C}[z_0, \cdots, z_N])$. Let $X = V(f) \subset \mathbb{P}^N$ be a smooth hypersurface. If $N - 1 = 2p$ is even, we consider the decompositions

$$H^{N-1}_{DR}(X) = \mathbb{C}[\Lambda \cap X] \oplus H^{N-1}_{DR,v}(X)\ ,\ H^{N-1}_{\mathrm{sing}}(X^{an}, \mathbb{Q}) = \mathbb{C}[\Lambda \cap X] \oplus H^{N-1}_{\mathrm{sing},v}(X^{an}, \mathbb{Q})$$

of vector spaces, where $\Lambda \subset \mathbb{P}^N$ is a linear subspace of codimension p. If $N - 1 = 2p + 1$ is even, we set

$$H^{N-1}_{DR,v}(X) := H^{N-1}_{DR}(X)\ ,\ H^{N-1}_{\mathrm{sing},v}(X^{an}, \mathbb{Q}) := H^{N-1}_{\mathrm{sing}}(X^{an}, \mathbb{Q}).$$

Let $X = V(f) \subset \mathbb{P}^N$ be a smooth hypersurface. Denote $j : U := \mathbb{P}^N \backslash X \hookrightarrow \mathbb{P}^N$ the open embedding. We consider the algebraic Hodge module

$$j_{*Hdg}(O_U, F_b) := ((j_*O_U, F), j_*\mathbb{Q}_{U^{an}}, j_*\alpha(U)) \in HM(\mathbb{P}^N),$$

where $F^k j_*O_U = \mathbb{C}[x_1, \cdots x_N][\frac{1}{f^k}]$. Recall for each $p \in \mathbb{Z}$,

$$F^p H^N_{DR}(U) := H^N \iota^{\geq p}(\mathbb{H}^N(\mathbb{P}^N, \Omega^{\bullet \geq p}_{\mathbb{P}^N}(\log X))) = H^N \iota^{\geq p} H^N(\mathbb{P}^N, \Omega^\bullet_{\mathbb{P}^N} \otimes_{O_{\mathbb{P}^N}} F^{\bullet - p} j_*O_U)$$

and

$$F^p H^{N-1}(X) := H^{N-1} \iota^{\geq p}(\mathbb{H}^{N-1}(X, \Omega^{\bullet \geq p}_X)).$$

where

$$\iota^{\geq p} : \Omega^{\bullet \geq p}_{\mathbb{P}^N}(\log X) \hookrightarrow \Omega^\bullet_{\mathbb{P}^N}(\log X)\ ,\ \iota^{\geq p} : \Omega^\bullet_{\mathbb{P}^N} \otimes_{O_{\mathbb{P}^N}} F^{\bullet - p} j_*O_U \hookrightarrow \Omega^\bullet_{\mathbb{P}^N} \otimes_{O_{\mathbb{P}^N}} j_*O_U = j_*\Omega^\bullet_U$$

and $\iota^{\geq p} : \Omega^{\bullet \geq p}_X \hookrightarrow \Omega^\bullet_X$ are the canonical embeddings. Then the map $Res_{X,\mathbb{P}^N}$ of definition 1 induces for each $p \in \mathbb{Z}$, canonical isomorphisms

$$H^N\Omega(Res_{X,\mathbb{P}^N}) : F^{p+1}H^N_{DR}(U) \xrightarrow{\sim} F^p H^{N-1}_{DR}(X)_v\ ,\ H^N \operatorname{Bti}(Res_{X,\mathbb{P}^N}) : H^N_{\text{sing}}(U, \mathbb{Q}) \xrightarrow{\sim} H^{N-1}_{\text{sing}}(X, \mathbb{Q})_v.$$

Lemma 3. *Let $X = V(f) \subset \mathbb{P}^N$ be a smooth hypersurface. Denote $j : U := \mathbb{P}^N \backslash X \hookrightarrow \mathbb{P}^N$ the open embedding. Let $B \subset \mathbb{P}^N$, $\phi : B \simeq D(0,1)^N$ be an open ball such that $\phi(X \cap B) = \{0\} \times D(0,1)^{N-1}$. Then,*

(i) For $q \in \mathbb{Z}$, $q \neq 0$, $\mathbb{H}^q_{usu}(B, (\Omega^\bullet_{\mathbb{P}^N} \otimes_{O_{\mathbb{P}^N}} F^{\bullet - p} j_*O_U)^{an}) = 0$.

(ii) For $q \in \mathbb{Z}$, $q \neq 0$, $H^q_{usu}(B, \Omega^{N-p}_{\mathbb{P}^N, an}(\log X)^{\partial = 0}) = 0$.

Proof. (i): Since B is Stein, we have for $r, q \in \mathbb{Z}$, $q \neq 0$,

$$H^q_{usu}(B, (\Omega^r_{\mathbb{P}^N} \otimes_{O_{\mathbb{P}^N}} F^{r-p} j_*O_U)^{an}) = 0.$$

On the other hand the De Rham complex of $(j_*O_{B\backslash X}, F)$ is acyclic (see e.g. [6] corollary 18.7)
(ii): Consider a sequence of sub-balls

$$l_n : B_n := D(0, r_n)^N \hookrightarrow B := D(0,1)^N, n \in \mathbb{N}, \text{ with } r_n \to 0 \text{ when } n \to \infty.$$

Since $(B_n := D(0, r_n))_{n \in \mathbb{N}}$ form a basis of neighborhood of $0 \in B$ and since $\Omega^{N-p}_{\mathbb{P}^N, an}(\log X)^{\partial = 0}$ is a single presheaf, there exists $n := n_\alpha \in \mathbb{N}$ (depending on α) such that

$$l_n^* \alpha = 0 \in H^q(B_n, \Omega^{N-p}_{\mathbb{P}^N, an}(\log X)^{\partial = 0}).$$

But it follows immediately from the definition of the the presheaf $\Omega^{N-p}_{\mathbb{P}^N, an}(\log X)^{\partial = 0}$ considering the isomorphism

$$\phi_n : B_n \xrightarrow{\sim} B,\ \phi_n(z) := r_n^{-1} z$$

so that $(\phi_n)^{-1} \circ l_n = r_n I$, that

$$l_n^* \alpha = l_n^* \phi_n^{-1 *} \phi_n^* \alpha = r_n \phi_n^* \alpha \in H^q(B_n, \Omega^{N-p}_{\mathbb{P}^N, an}(\log X)^{\partial = 0}).$$

Hence $\phi_n^* \alpha = 0$. Since ϕ_n is an isomorphism, we get $\alpha = 0$. □

We have by the local acyclicity of the De Rham complex of (j_*O_U, F) a canonical map B_∂ :

Definition 3. *Let $X = V(f) \subset \mathbb{P}^N$ be a smooth hypersurface. Denote $j : U := \mathbb{P}^N \backslash X \hookrightarrow \mathbb{P}^N$ the open embedding. Consider the canonical embedding*

$$\iota_\partial : \Omega^{N-p}_{\mathbb{P}^{N,an}}(\log X)^{\partial=0}[-p] \hookrightarrow (\Omega^\bullet_{\mathbb{P}^N} \otimes_{O_{\mathbb{P}^N}} F^{\bullet-p} j_* O_U)^{an}$$

Let $\mathbb{P}^{N,an} = \cup_{i=1}^n B_i$ an open cover, with $\phi_i : \mathbb{B}_i \simeq D(0,1)^N$ open balls such that $\phi_i(X \cap B_i) = \{0\} \times D(0,1)^{N-1}$. Denote, for $I \subset [1,\dots,n]$, $l_I : B_I \hookrightarrow \mathbb{P}^N$ the open embeddings, with $B_I := \cap_{i\in I} B_i$. Since, for each $I \subset [1,\dots,n]$ and $p \in \mathbb{Z}$,

$$l_I^* \iota_\partial : l_I^* \Omega^{N-p}_{\mathbb{P}^{N,an}}(\log X)^{\partial=0}[-p] \hookrightarrow l_I^*(\Omega^\bullet_{\mathbb{P}^N} \otimes_{O_{\mathbb{P}^N}} F^{\bullet-p} j_* O_U)^{an}$$

is an quasi-isomorphism (see e.g. [6] corollary 18.7), the spectral sequence of the filtration on the total complex of a bi-complex gives, for each $p \in \mathbb{Z}$, a canonical isomorphism, where the first and last map are isomorphism by lemma 3(i) and (ii) respectively,

$$\begin{aligned} B_\partial : F^p H_{DR}(U) := H^N \iota^{\geq p} \mathbb{H}^N_{zar}(\mathbb{P}^N, \Omega^\bullet_{\mathbb{P}^N} \otimes_{O_{\mathbb{P}^N}} F^{\bullet-p} j_* O_U) \\ \xrightarrow{H^N((l_i^* = \Omega(l_i))_{1\leq i\leq n})} H^N \iota^{\geq p} H^N \Gamma(B_\bullet, (\Omega^\bullet_{\mathbb{P}^N} \otimes_{O_{\mathbb{P}^N}} F^{\bullet-p} j_* O_U)^{an}) \\ \to H^p \Gamma(B_\bullet, \Omega^{N-p}_{\mathbb{P}^{N,an}}(\log X)^{\partial=0}) \xrightarrow{H^N((l_i^* = \Omega(l_i))_{1\leq i\leq n})^{-1}} H^p_{usu}(\mathbb{P}^N, \Omega^{N-p}_{\mathbb{P}^{N,an}}(\log X)^{\partial=0}), \end{aligned}$$

given by, for $w \in \Gamma(U, \Omega^N_{\mathbb{P}^N})$, inductively by a p step induction where p is the order of the pole of w at X,

- *taking for each $1 \leq i \leq n$, $l_i^* w = \partial w_{1i}$ since $\partial w = 0$ for dimensional reason and by exactness of $l_i^* \iota_\partial$,*
- *we then have for each $1 \leq i, j \leq n$, considering $w_{1ij} := l_{ij}^*(w_{1i} - w_{1j})$, $w_{1ij} = \partial w_{2ij}$ since*

  $$\partial w_{1ij} = l_{ij}^*(\partial w_{1i} - \partial w_{1j}) = l_{ij}^*(l_i^* w - l_j^* w) = 0$$

  and by exactness of $l_{ij}^* \iota_\partial$,
- *at the final p step, $B(w) := w_{pI} \in H^N(B_\bullet, \Omega^{N-p}_{\mathbb{P}^{N,an}}(\log X)^{\partial=0})$, $card I = p$.*

By definition $B_\partial(w) = (H^p \iota_\partial)^{-1}(w)$.

Let $X = V(f) \subset \mathbb{P}^N$ be a smooth hypersurface. Denote $U := \mathbb{P}^N \backslash X$. For $l \in \mathbb{Z}$, we consider

- the sequence of embeddings in $C(U^{et})$

  $$OL^l_U : \Omega^l_{U,\log}[-l] \xrightarrow{\widehat{OL^l_U}[-l]} \Omega^l_{\mathbb{P}^N}(\log X)^{\partial=0}[-l] \xrightarrow{\iota_\partial} \Omega^\bullet_{\mathbb{P}^N} \otimes_{O_{\mathbb{P}^N}} F^{\bullet-p} j_* O_U \xrightarrow{\iota^{\geq l}} \Omega^\bullet_U$$

- and the sequence of embeddings in $C(U^{an})$

  $$OL^l_{U^{an}} : \Omega^l_{U^{an},\log}[-l] \xrightarrow{\widehat{OL^l_{U^{an}}}[-l]} \Omega^l_{\mathbb{P}^{N,an}}(\log X)^{\partial=0}[-l] \xrightarrow{\iota_\partial} (\Omega^\bullet_{\mathbb{P}^N} \otimes_{O_{\mathbb{P}^N}} F^{\bullet-p} j_* O_U)^{an} \xrightarrow{\iota^{\geq l}} \Omega^\bullet_{U^{an}},$$

  where

  $$\iota_\partial : \Omega^l_{\mathbb{P}^{N,an}}(\log X)^{\partial=0}[-l] \hookrightarrow (\Omega^\bullet_{\mathbb{P}^N} \otimes_{O_{\mathbb{P}^N}} F^{\bullet-p} j_* O_U)^{an}$$

  is an equivalence usu local (see [6]).

The main result of this section is the following :

Proposition 3. *Let $X = V(f) \subset \mathbb{P}^N$ be a smooth projective hypersurface. Denote $U := \mathbb{P}^N \backslash X$. Let $w \in F^{p+1} H^N_{DR}(U)$. Then the following assertions are equivalent*

(i) $ev(U)(w) \in H^N_{\text{sing}}(U^{an}, 2i\pi\mathbb{Q})$,

(ii) $B_\partial(w) \in \widehat{OL^p_{U^{an}}}(H^p_{usu}(U^{an}, \Omega^{N-p}_{U^{an},\log}))$ *(see definition 3),*

(iii) $w \in OL^N_{U^{an}}(H^p_{usu}(U^{an}, \Omega^{N-p}_{U^{an},\log}))$.

Proof. (i) implies (ii): Let $\mathbb{P}^{N,an} = \cup_{i=1}^n B_i$ an open cover, with $\phi_i : \mathbb{B}_i \simeq D(0,1)^N$ open balls such that $\phi_i(X \cap B_i) = \{0\} \times D(0,1)^{N-1}$. Denote, for $I \subset [1,\dots,n]$, $l_I : B_I \hookrightarrow \mathbb{P}^N$ the open embeddings, with $B_I := \cap_{i\in I} B_i$. Then

$$B_\partial(w) := (w_{p+1,I}) \in H^{p+1}(B_\bullet, \Omega^{N-p}_{\mathbb{P}^{N,an}}(\log X)^{\partial=0}).$$

We have, for each $I \subset [1,\dots n]$, a decomposition

$$w_{p+1,I} = \frac{a_I}{z_1} dz_1 \wedge w'_I + w''_I \in \Gamma(B_I, \Omega^{N-p}_{\mathbb{P}^{N,an}}(\log X)^{\partial=0})$$

with $w'_I \in \Gamma(B_I, \Omega^{N-p-1}_{\mathbb{P}^{N,an}})^{\partial=0}$ and $w''_I \in \Gamma(B_I, \Omega^{N-p}_{\mathbb{P}^{N,an}})^{\partial=0}$. Consider the evaluation map

$$ev(U)^\bullet_\bullet : \Gamma(B_\bullet \backslash X^{an}, \Omega^\bullet_{\mathbb{P}^{N,an}}) \to C^\bullet_{\text{diff}}(B_\bullet \backslash X^{an}, \mathbb{C}),$$

which is a morphism of bi-complexes. On the other hand, we have the following canonical map

$$BC_\partial : H^p\Gamma(B_\bullet \backslash X^{an}, C^{N-p,\partial}_{U^{an},\text{diff}}) \xrightarrow{\Gamma(B_\bullet\backslash X^{an}, \iota_\partial)} H^N\Gamma(B_\bullet \backslash X^{an}, C^{\bullet\geq N-p}_{U^{an},\text{diff}}) \xrightarrow{\Gamma(B_\bullet\backslash X^{an}, \iota^{\geq N-p})}$$
$$H^N\Gamma(B_\bullet \backslash X^{an}, C^\bullet_{U^{an},\text{diff}}) \xrightarrow{(H^N(l_i^* =: C^\bullet_{U^{an},\text{diff}}(l_i))_{1\leq i\leq n})^{-1}} H^N_{\text{sing}}(U^{an}, \mathbb{Q}) = H^N\Gamma(U^{an}, C^\bullet_{U^{an},\text{diff}}),$$

where $H^N(l_i^*)$ is an isomorphism by the acyclicity of $C^j_{U^{an},\text{diff}} \in \text{PSh}(U^{an})$ for each $j \in \mathbb{Z}$, and

$$C^\bullet_{U^{an},\text{diff}} \hookrightarrow C^\bullet_{U^{an},\text{sing}}, \text{ for } V \subset U^{an},\ C^\bullet_{\text{diff}}(V) \hookrightarrow C^\bullet_{\text{sing}}(V)$$

is the canonical quasi-isomorphism in $C(U^{an})$. We then have the following commutative diagram

$$\begin{array}{ccc}
H^N\iota^{\geq p}H^N\Gamma(\mathbb{P}^N, \Omega^\bullet_{\mathbb{P}^N} \otimes_{O_{\mathbb{P}^N}} F^{\bullet-p}j_*O_U) & & \\
\downarrow{\scriptstyle H^N((l_i^* = \Omega(l_i))_{1\leq i\leq n})} & \searrow^{B_\partial} & \\
H^N\iota^{\geq p}H^N\Gamma(B_\bullet, (\Omega^\bullet_{\mathbb{P}^N} \otimes_{O_{\mathbb{P}^N}} F^{\bullet-p}j_*O_U)^{an}) & \xrightarrow{B_\partial} & H^p\Gamma(B_\bullet, \Omega^{N-p}_{\mathbb{P}^{N,an}}(\log X)^{\partial=0}) \\
\downarrow{\scriptstyle ev(U)} & & \downarrow{\scriptstyle ev(U)^\bullet_\bullet} \\
H^N_{\text{sing}}(U^{an}, \mathbb{C}) = H^N\Gamma(U^{an}, C^\bullet_{U^{an},\text{diff}} \otimes \mathbb{C}) & \xleftarrow[BC_\partial\otimes\mathbb{C}]{} & H^p\Gamma(B_\bullet\backslash X^{an}, C^{N-p,\partial=0}_{\text{diff}} \otimes \mathbb{C})
\end{array}$$

Since $ev(U)$ is injective by the strictness of the Hodge filtration and since B_∂ is an isomorphism, $ev(U)^\bullet_\bullet$ is injective and

$$(BC_\partial \otimes \mathbb{C})' := (BC_\partial \otimes \mathbb{C})_{|ev(U)^\bullet_\bullet(H^p\Gamma(B_\bullet, \Omega^{N-p}_{\mathbb{P}^{N,an}}(\log X)^{\partial=0}))} :$$
$$ev(U)^\bullet_\bullet(H^p\Gamma(B_\bullet, \Omega^{N-p}_{\mathbb{P}^{N,an}}(\log X)^{\partial=0})) \to H^N_{\text{sing}}(U^{an}, \mathbb{C})$$

is injective. For $\gamma_I \in C^{\text{sing}}_{N-p}(B_I \backslash X^{an}, \mathbb{Q})^{\partial=0}$ where $p_1 \circ \phi_I(\gamma_I) \subset D(0,1)^*$ is a loop around 0, we have by the above diagram

$$ev(U)(w)(\gamma_I) = ev(U)(w_{p+1,I})(\gamma_I) = a_I.$$

Hence if $ev(U)(w) \in H^N_{\text{sing}}(U^{an}, 2i\pi\mathbb{Q})$, then $a_I \in \mathbb{Q}$, using the injectivity of $(BC_\partial \otimes \mathbb{C})'$.
(ii) is equivalent to (iii): By definition $B_\partial(w) = (H^p\iota_\partial)^{-1}(w)$. On the other hand by definition

$$\iota^{\geq p} \circ \iota_\partial \circ \widehat{OL^{N-p}_{U^{an}}} = OL^{N-p}_{U^{an}}.$$

(iii) implies (i): See [2] proposition 7. □

4 Complex analytic logarithmic de Rham classes

We recall (see section 2) the morphism of site given by the analytification functor

$$\mathrm{An} : \mathrm{AnSm}(\mathbb{C}) \to \mathrm{SmVar}(\mathbb{C}),\ \ X \mapsto \mathrm{An}(X) := X^{an},\ (f : X' \to X) \mapsto \mathrm{An}(f) := f^{an}.$$

and for $X \in \mathrm{SmVar}(\mathbb{C})$, the following commutative diagram

$$\begin{array}{ccc} \mathrm{AnSm}(\mathbb{C}) & \xrightarrow{\mathrm{An}} & \mathrm{SmVar}(\mathbb{C}) \\ \downarrow{\scriptstyle o_{X^{an}}} & & \downarrow{\scriptstyle o_X} \\ X^{an,et} & \xrightarrow{\mathrm{an}_X} & X^{et} \end{array}.$$

We have also the morphism of sites (see definition 2)

- $\pi : \mathrm{AnSm}(\mathbb{C}) \to \mathrm{AnSm}(\mathbb{C})^{pr}$, $\pi(X \times \mathbb{A}^{1,an}) := X$, $\pi(f \otimes I) := f$,
- $\rho : \mathrm{AnSm}(\mathbb{C})^{pr} \to \mathrm{AnSm}(\mathbb{C})$, $\rho(X) := X \times \mathbb{A}^{1,an}$, $\rho(f) := f \otimes I$,

in particular $\rho \circ \pi = I$. Consider the commutative diagram in $C(\mathrm{AnSm}(\mathbb{C}))$

$$\begin{array}{ccc} \Omega^{\bullet,an}_{\log} \cap (\pi^*\Omega^{\bullet,e}) & \xrightarrow[OL^{an}]{} & \pi^*\Omega^{\bullet,e} \\ \downarrow{\scriptstyle e} & & \downarrow{\scriptstyle e} \\ \Omega^{\bullet,an}_{\log} & \xrightarrow[OL^{an}]{} & \Omega^{\bullet,an} \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle q} \\ \Omega^{\bullet,an,ne}_{\log} := \Omega^{\bullet,an}_{\log}/(\Omega^{\bullet,an}_{\log} \cap (\pi^*\Omega^{\bullet,e})) & \xrightarrow{\bar{O}L^{an}} & \Omega^{\bullet,an}/(\pi^*\Omega^{\bullet,e}) \end{array},$$

whose columns are exact with e injective and q surjective (see definition 2). We have the factorization in $C(\mathrm{AnSm}(\mathbb{C}))$

$$\begin{aligned} \mathrm{ad}(\pi) := \mathrm{ad}(\pi^*, \pi_*)(-) : \pi^*\Omega^{\bullet,e} &\xrightarrow{\subset} \pi^* a_{usu}\partial\Omega^{\bullet-1,an} := \pi^*\pi_* a_{usu}\partial\Omega^{\bullet-1,an} \\ &\xrightarrow{(\mathrm{ad}(\pi),0)} \mathrm{ad}(\pi)(\pi^*\Omega^{\bullet,e}) \xrightarrow{i_{\mathrm{ad}(\pi)}} a_{usu}\partial\Omega^{\bullet-1,an}, \end{aligned}$$

and we have denoted again by abuse

$$\pi^*\Omega^{\bullet,e} := \mathrm{ad}(\pi)(\pi^*\Omega^{\bullet,e}) \subset a_{usu}\partial\Omega^{\bullet-1,an} \subset \Omega^{\bullet,an}.$$

Let $X \in \mathrm{SmVar}(\mathbb{C})$. Then, for $k \in \mathbb{Z}$, we have the commutative diagram

$$\begin{array}{ccc} \mathbb{H}^k_{usu}(X^{an}, \Omega^{\bullet,an}_{\log} \cap (\pi^*\Omega^{\bullet,e})) & \xrightarrow{H^k OL^{an}} & \mathbb{H}^k_{usu}(X^{an}, \mathrm{ad}(\pi)(\pi^*\Omega^{\bullet,e})) \\ \downarrow{\scriptstyle H^k(e)} & & \downarrow{\scriptstyle H^k(e)} \\ \mathbb{H}^k_{usu}(X^{an}, \Omega^{\bullet,an}_{\log}) & \xrightarrow{H^k OL^{an}} & H^k_{DR}(X) = \mathbb{H}^k_{usu}(X^{an}, \Omega^{\bullet,an}) \\ \downarrow{\scriptstyle H^k(q)} & & \downarrow{\scriptstyle H^k(q)} \\ \mathbb{H}^k_{usu}(X^{an}, \Omega^{\bullet,an,ne}_{\log}) & \xrightarrow{H^k \bar{O}L^{an}} & \mathbb{H}^k_{usu}(X^{an}, \Omega^{\bullet,an}/(\pi^*\Omega^{\bullet,e})) \end{array}$$

whose columns are exact, and of course all differentials of $\Omega^{\bullet,an}_{\log}$ and of $\pi^*\Omega^{\bullet,e}$ are trivial. Let $Z \subset X$ a closed subset. Then, for $k \in \mathbb{Z}$, we also have the commutative diagram

$$\begin{array}{ccc}
\mathbb{H}^k_{usu,Z}(X^{an}, \Omega^{\bullet,an}_{\log} \cap (\pi^*\Omega^{\bullet,e})) & \xrightarrow{H^kOL^{an}} & \mathbb{H}^k_{usu,Z}(X^{an}, \mathrm{ad}(\pi)(\pi^*\Omega^{\bullet,e})) \\
\downarrow{\scriptstyle H^k(e)} & & \downarrow{\scriptstyle H^k(e)} \\
\mathbb{H}^k_{usu,Z}(X^{an}, \Omega^{\bullet,an}_{\log}) & \xrightarrow{H^kOL^{an}} & H^k_{DR,Z}(X) = \mathbb{H}^k_{usu}(X^{an}, \Omega^{\bullet,an}) \\
\downarrow{\scriptstyle H^k(q)} & & \downarrow{\scriptstyle H^k(q)} \\
\mathbb{H}^k_{usu,Z}(X^{an}, \Omega^{\bullet,an,ne}_{\log}) & \xrightarrow{H^k\bar{O}L^{an}} & \mathbb{H}^k_{usu,Z}(X^{an}, \Omega^{\bullet,an}/(\pi^*\Omega^{\bullet,e}))
\end{array}$$

whose columns are exact.

Lemma 4. *Let $X \in \mathrm{SmVar}(\mathbb{C})$. For each $k \in \mathbb{Z}$, $k \neq 0$, we have $\mathbb{H}^k_{usu}(X^{an}, \mathrm{ad}(\pi)(\pi^*\Omega^{\bullet,e})) = 0$.*

Proof. We have the canonical splitting

$$\mathbb{H}^k_{usu}(X^{an}, \mathrm{ad}(\pi)(\pi^*\Omega^{\bullet,e})) = \oplus_{p\in\mathbb{Z}} H^{k-p}_{usu}(X^{an}, \mathrm{ad}(\pi)(\pi^*\Omega^{p,e}))$$

Consider for each $p \in \mathbb{Z}$, the composition in $C(\mathrm{AnSm}(\mathbb{C}))$

$$\mathrm{ad}(\pi) := \mathrm{ad}(\pi^*, \pi_*)(-) : \pi^*(\mathcal{A}^{\bullet\geq p,\bullet,e}) \xrightarrow{\mathrm{ad}(\pi)} \mathrm{ad}(\pi)(\pi^*(\mathcal{A}^{\bullet\geq p,\bullet,e}) \xrightarrow{i_{\mathrm{ad}(\pi)}} \mathcal{A}^{\bullet\geq p,\bullet},$$

where the embedding

$$\Omega^{p,e} \hookrightarrow (\Omega^{\bullet\geq p})^e \hookrightarrow \mathcal{A}^{\bullet\geq p,\bullet,e}$$

in $C(\mathrm{AnSm}(\mathbb{C})^{pr})$ is the Dolbeault resolution. We have, by Poincare lemma on open balls,

$$H^k_{usu}(X^{an}, \mathrm{ad}(\pi)(\pi^*\Omega^{p,e})) = H^k\Gamma(X^{an}, \mathrm{ad}(\pi)(\pi^*\mathcal{A}^{\bullet\geq p,\bullet,e})),$$

indeed, since open balls form a basis of the complex topology, there exists D^i such that $D^i \subset D^{i-1} \times \mathbb{A}^1$. We have

$$\mathrm{ad}(\pi)(\pi^*\mathcal{A}^{\bullet\geq p,\bullet,e})(X^{an}) := \varinjlim_{m: X \to Y\times\mathbb{A}^1} \mathcal{A}^e(m)(\mathcal{A}^{\bullet\geq p,\bullet,e}(Y \times \mathbb{A}^1)) \subset \mathcal{A}^{\bullet\geq p,\bullet}(X^{an})$$

Now,

- if $q \circ m : X^{an} \xrightarrow{m} Y \times \mathbb{A}^1 \xrightarrow{q} \mathbb{A}^1$ is non constant, we can factor

  $$m = (n \times I) \circ m' : X^{an} \xrightarrow{m'} Y' \times \mathbb{A}^1 \xrightarrow{n\times I} Y \times \mathbb{A}^1$$

  with $\dim Y' = \dim X - 1$ and $m'(X^{an}) \subset Y' \times \mathbb{A}^1$ a dense open subset, and

  $$\mathcal{A}^e(m') : (\mathcal{A}^{\bullet\geq p,\bullet,e}(Y' \times \mathbb{A}^1) \to \mathcal{A}^{\bullet\geq p,\bullet}(X^{an})$$

  is injective, and

  $$\begin{array}{rcl} H^k\mathcal{A}^e(m)(\mathcal{A}^{\bullet\geq p,\bullet,e}(Y \times \mathbb{A}^1)) & = & H^k\mathcal{A}^e(m')(\mathcal{A}^{\bullet\geq p,\bullet,e}(Y' \times \mathbb{A}^1)) \\ & = & \mathcal{A}^e(m')(H^k(\mathcal{A}^{\bullet\geq p,\bullet,e})(Y' \times \mathbb{A}^1)) = 0 \end{array}$$

  by Kunneth formula for $Y' \times \mathbb{A}^1$,

- if $q \circ m : X \xrightarrow{m} Y \times \mathbb{A}^1 \xrightarrow{q} \mathbb{A}^1$ is constant with value c,

  $$H^k\mathcal{A}^e(m)(\mathcal{A}^{\bullet\geq p,\bullet,e})(Y \times \mathbb{A}^1)) = 0$$

  since $c^*d\bar{z} = 0$ and $c^*dz = 0$, where $c : Y \times \{c\} \hookrightarrow Y \times \mathbb{A}^1$ is the closed embedding.

□

Proposition 4. *Let $X \in \mathrm{PSmVar}(\mathbb{C})$. Consider the morphism $\mathrm{an}_X : X^{an} \to X$ in* RTop *given by analytical functor.*

(i) The analytic De Rham cohomology class of an algebraic cycle is logarithmic and is of type (d,d), that is, for $Z \in \mathcal{Z}^d(X)$

$$[Z] := H^{2d}\Omega(\gamma_Z^\vee)([Z]) \subset H^{2d}OL_{X^{an}}(H^d_{usu}(X^{an}, \Omega^d_{X^{an},\log})) \subset H^{2d}_{DR}(X^{an}) = H^{2d}_{DR}(X).$$

(ii) Conversely, any $w \in H^{2d}OL_{X^{an}}(H^d_{usu}(X^{an}, \Omega^d_{X^{an},\log}))$ is the class of an algebraic cycle $Z \in \mathcal{Z}^d(X) \otimes \mathbb{Q}$, i.e. $w = [Z]$.

(iii) We have $H^jOL_{X^{an}}(H^{j-l}_{usu}(X^{an}, \Omega^l_{X^{an},\log})) = 0$ for $j, l \in \mathbb{Z}$ such that $2l > j$.

Proof. Consider, for $j, l \in \mathbb{Z}$ and $X \in \mathrm{SmVar}(\mathbb{C})$,

$$L^{l,j-l}(X) := H^jOL_{X^{an}}(H^{j-l}_{usu}(X^{an}, \Omega^l_{X^{an},\log})) := H^jOL^{an}(H^{j-l}\operatorname{Hom}(\mathrm{An}^*\,\mathbb{Z}(X), E^\bullet_{usu}(\Omega^{l,an}_{\log}))).$$

Consider also for $j, l \in \mathbb{Z}$, $X \in \mathrm{SmVar}(\mathbb{C})$ and $Z \subset X$ a closed subset,

$$L^{l,j-l}_Z(X) := H^jOL_{X^{an}}(H^{j-l}_{usu,Z}(X^{an}, \Omega^l_{X^{an},\log})) := H^jOL^{an}(H^{j-l}\operatorname{Hom}(\mathrm{An}^*\,\mathbb{Z}(X, X\backslash Z), E^\bullet_{usu}(\Omega^{l,an}_{\log}))).$$

Consider, for $j, l \in \mathbb{Z}$ and $X \in \mathrm{SmVar}(\mathbb{C})$,

$$L^{l,j-l,ne}(X) := H^j(\bar{O}L^{an})(H^{j-l}\operatorname{Hom}(\mathrm{An}^*\,\mathbb{Z}(X), E^\bullet_{usu}(\Omega^{l,an,ne}_{\log}))).$$

Consider also for $j, l \in \mathbb{Z}$, $X \in \mathrm{SmVar}(\mathbb{C})$ and $Z \subset X$ a closed subset,

$$L^{l,j-l,ne}_Z(X) := H^j(\bar{O}L^{an})(H^{j-l}\operatorname{Hom}(\mathrm{An}^*\,\mathbb{Z}(X, X\backslash Z), E^\bullet_{usu}(\Omega^{l,an,ne}_{\log}))).$$

For each $l, j \in \mathbb{Z}$, the presheaf $H^{j-l}E_{usu}(\Omega^{l,an,ne}_{\log}) \in \mathrm{PSh}(\mathrm{AnSm}(\mathbb{C}))$ is $\mathbb{A}^1$ invariant by proposition 2(i). Indeed, for $Y \in \mathrm{AnSm}(\mathbb{C})$ connected, take an open cover $Y = \cup_{i\in I} D_i$ by open balls $D_i \simeq D(0,1)^{d_Y}$, and let $D_\bullet \in \mathrm{Fun}(\Delta, \mathrm{AnSm}(\mathbb{C})$ be the associated simplicial analytic manifold, with for $J \subset I$, $j_{IJ} : D_I := \cap_{i\in I} D_i \hookrightarrow D_J := \cap_{i\in J} D_i$ the open embedding, we then get the isomorphism

$$\begin{aligned} p_Y^* : H^{j-l}E_{usu}(\Omega^{l,an,ne}_{\log})(Y \times \mathbb{A}^1) := H^{j-l}_{usu}(Y \times \mathbb{A}^1, \Omega^{l,an,ne}_{\log}) \xrightarrow{\sim} \\ \mathbb{H}^{j-l}(D_\bullet \times \mathbb{A}^1, \Omega^{l,an,ne}_{\log}) \xrightarrow{\sim} H^{j-l}\Gamma(D_\bullet \times \mathbb{A}^1, a_{usu}\Omega^{l,an,ne}_{\log}) \\ \xrightarrow{\sim} H^{j-l}\Gamma(D_\bullet, a_{usu}\Omega^{l,an,ne}_{\log}) \xrightarrow{\sim} \mathbb{H}^{j-l}(D_\bullet, \Omega^{l,an,ne}_{\log}) \xrightarrow{\sim} H^{j-l}_{usu}(Y, \Omega^{l,an,ne}_{\log}) =: H^{j-l}E_{usu}(\Omega^{l,an,ne}_{\log})(Y) \end{aligned}$$

where $p_Y : X \times \mathbb{A}^1 \to Y$ is the projection, and the second and the fourth map is the isomorphism given by proposition 2(i), and the middle map is an isomorphism by lemma 2(ii)'. This gives in particular, for $X \in \mathrm{SmVar}(\mathbb{C})$ and $Z \subset X$ a smooth subvariety of (pure) codimension d, by proposition 1 an isomorphism

$$E_{usu}(\Omega^{l,an,ne}_{\log})(P_{Z,X}) : L^{l,j-l,ne}_Z(X) \xrightarrow{\sim} L^{l,j-l,ne}_Z(N_{Z/X}) \xrightarrow{\sim} L^{l-d,j-l-d,ne}(Z).$$

(i):Similar to the proof of [2] theorem 2.
(ii): Let $X \in \mathrm{PSmVar}(\mathbb{C})$ and

$$\begin{aligned} \alpha = H^{2d}OL_{X^{an}}(\alpha) \in L^{d,d}(X) : &= H^{2d}OL_{X^{an}}(H^d_{usu}(X^{an}, \Omega^d_{X^{an},\log})) \\ &= H^{2d}OL^{an}(H^d\operatorname{Hom}(\mathrm{An}^*\,\mathbb{Z}(X), E^\bullet_{usu}(\Omega^{d,an}_{\log}))). \end{aligned}$$

Then,

$$\begin{aligned} H^{2d}(q)(\alpha) = H^{2d}(q) \circ H^{2d}OL_{X^{an}}(\alpha) = H^{2d}\bar{O}L^{an} \circ H^{2d}(q)(\alpha) \\ \in L^{d,d,ne}(X) := H^{2d}\bar{O}L^{an}(H^d\operatorname{Hom}(\mathrm{An}^*\,\mathbb{Z}(X), E^\bullet_{usu}(\Omega^{d,an,ne}_{\log}))). \end{aligned}$$

Up to split X by its connected components, we may assume that X is connected. The case $d = d_X$ is trivial. So let assume $d \neq d_X$. Let $U \subset X$ an open subset such that there exists an etale map $e : U \to \mathbb{A}^{d_U}_{\mathbb{C}}$ such that $e : U \to e(U)$ is finite etale and $\mathbb{A}^{d_U}_{\mathbb{C}} \backslash e(U) \subset \mathbb{A}^{d_U}_{\mathbb{C}}$ is a divisor Denote $j : U \hookrightarrow X$ the open embedding. By proposition 2(ii), we have

$$j^* H^{2d}(q)(\alpha) = 0 \in H^d_{usu}(U^{an}, \Omega^{d,an,ne}_{\log}) = 0.$$

Considering a divisor $X \backslash U \subset D \subset X$, we get

$$H^{2d}(q)(\alpha) = H^d E^{\bullet}_{usu}(\Omega^{d,an,ne}_{\log})(\gamma_D^{\vee})(\alpha'),\ \alpha' \in L_D^{l,j-l,ne}(X).$$

Denote $D^o \subset D$ its smooth locus and $l : X^o \hookrightarrow X$ a Zariski open subset such that $X^o \cap D = D^o$. We then have

$$l^* \alpha' \in L^{d,d}_{D^o}(X^o) = L^{d-1,d-1}(D^o)$$

We repeat this procedure with each connected components of D^o instead of X. By a finite induction of d steps, we get

$$H^{2d}(q)(\alpha) = H^d E^{\bullet}_{usu}(\Omega^{d,an,ne}_{\log})(\gamma_Z^{\vee})(\alpha'),\ \alpha' \in L_Z^{d,d,ne}(X) = L^{0,0,ne}(Z^o).$$

with $Z := D_d \subset \cdots \subset D \subset X$ a pure codimension d (Zariski) closed subset, thus

$$\alpha = \sum_i n_i [Z_i] + \beta \in H^{2d}_{DR}(X),\ \beta \in H^{2d}(e)(\mathbb{H}^{2d}_{usu}(X^{an}, \mathrm{ad}(\pi)(\pi^* \Omega^{\bullet,e})))$$

where $n_i \in \mathbb{Q}$ and $(Z_i)_{1 \leq i \leq t} \subset Z$ are the irreducible components of Z. But by lemma 4, we have

$$\mathbb{H}^k_{usu}(X^{an}, \mathrm{ad}(\pi)(\pi^* \Omega^{\bullet,e})) = 0.$$

for each $k \in \mathbb{Z}$, $k \neq 0$. Hence $\beta = 0$, that is

$$\alpha = \sum_i n_i [Z_i] \in H^{2d}_{DR}(X).$$

(iii): Let $j < 2l$. Let $X \in \mathrm{PSmVar}(\mathbb{C})$ and $w \in L^{l,j-l}(X)$. By proposition 2(ii)', arguing as in the proof of (ii), there exists $Z \subset X$ a closed subset of pure codimension $j - l$ such that

$$H^j(q)(w) = H^{j-l} E^{\bullet}_{usu}(\Omega^{l,an,ne}_{\log})(\gamma_Z^{\vee})(w'),\ w' \in L_Z^{l,j-l,ne}(X).$$

For $Z' \subset X$ a closed subset of pure codimension c, consider a desingularisation $\epsilon : \tilde{Z}' \to Z'$ of Z' and denote $n : \tilde{Z}' \xrightarrow{\epsilon} Z' \subset X$. The morphism in $\mathrm{DA}(k)$

$$G_{Z',X} : M(X) \xrightarrow{D(\mathbb{Z}(n))} M(\tilde{Z}')(c)[2c] \xrightarrow{\mathbb{Z}(\epsilon)} M(Z')(c)[2c]$$

where $D : \mathrm{Hom}_{\mathrm{DA}(k)}(M_c(\tilde{Z}'), M_c(X)) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{DA}(k)}(M(X), M(\tilde{Z}')(c)[2c])$ is the duality isomorphism from the six functor formalism (moving lemma of Suzlin and Voevodsky) and $\mathbb{Z}(n) := \mathrm{ad}(n_!, n^!)(a_X^! \mathbb{Z})$, is given by a morphism in $C(\mathrm{SmVar}(k))$

$$\hat{G}_{Z',X} : \mathbb{Z}_{tr}(X) \to E_{et}(C_* \mathbb{Z}_{tr}(Z'))(c)[2c].$$

Let $l : X^o \hookrightarrow X$ be an open embedding such that $Z^o := Z \cap X^o$ is the smooth locus of Z. We have the following commutative diagram of abelian groups

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^{j-l}_{usu,Z}(X^{an}, \Omega^{l,an,ne}_{\log}) & \xrightarrow{l^*} & H^{j-l}_{usu,Z^o}(X^{o,an}, \Omega^{l,an,ne}_{\log}) & \xrightarrow{\partial} & H^{j-l+1}_{usu,Z\backslash Z^o}(X^{an}, \Omega^{l,an,ne}_{\log}) & \longrightarrow & \cdots \\
 & & \uparrow{\scriptstyle \Omega(\hat{G}_{Z,X})} & & \uparrow{\scriptstyle \Omega(P_{Z^o,X^o})} & & \uparrow{\scriptstyle \Omega(\hat{G}_{Z,X})} & & \\
0 & \longrightarrow & H^0_{usu}(Z^{an}, \Omega^{2l-j,an,ne}_{\log}) & \xrightarrow{l*} & H^0_{usu}(Z^{o,an}, \Omega^{2l-j,an,ne}_{\log}) & \xrightarrow{\partial} & H^1_{usu,Z\backslash Z^o}(Z^{an}, \Omega^{2l-j,an,ne}_{\log}) & \longrightarrow & \cdots
\end{array}$$

whose rows are exact sequences. Consider

$$l^*w' = \Omega(P_{Z^o,X^o})(w^o) \in H^{j-l}_{usu,Z^o}(X^{o,an}, \Omega^{l,an,ne}_{\log}),\ w^o \in H^0_{et}(Z^{o,an}, \Omega^{2l-j,an,ne}_{\log}).$$

Since $\partial l^*w' = 0 \in H^{j-l+1}_{usu,Z\backslash Z^o}(X^{an}, \Omega^{l,an,ne}_{\log})$, we get, considering the smooth locus $(Z\backslash Z^o)^o \subset Z\backslash Z^o$ and $X^{oo} \subset X$ is an open subset such that $X \cap (Z\backslash Z^o)^o = X^{oo} \cap (Z\backslash Z^o)$,

$$\partial w^o = 0 \in H^1_{usu,Z\backslash Z^o}(Z^{an}, \Omega^{2l-j,an,ne}_{\log}),$$

since $H^1_{usu,(Z\backslash Z^o)\backslash(Z\backslash Z^o)^o}(Z^{an}, \Omega^{2l-j,an,ne}_{\log}) = 0$ for dimension reasons, that is

$$H^j(q)(w) = \Omega(\hat{G}_{Z,X})(w'), \text{ with } w' \in H^0_{et}(Z^{an}, \Omega^{2l-j,an,ne}_{\log}).$$

Hence $H^j(q)(w) = 0$ since $H^0(Z'^{an}, \Omega^k_{\log}) = 0$ for all $k > 0$ and all $Z' \in \mathrm{PVar}(\mathbb{C})$. But by lemma 4

$$\mathbb{H}^j_{usu}(X^{an}, \mathrm{ad}(\pi)(\pi^*\Omega^{\bullet,e})) = 0.$$

Hence $w = 0$. □

5 Hodge Conjecture for complex smooth projective hypersurfaces

We now state and prove using the result of section 3 and section 4 the main result of this article :

Theorem 1. *Let $X = V(f) \subset \mathbb{P}^N_{\mathbb{C}}$ be a smooth projective hypersurface, with $N = 2p+1$ odd. Let $\lambda \in F^pH^{2p}(X^{an}, \mathbb{Q})$. Then there exists an algebraic cycle $Z \in \mathbb{Z}^p(X)$ such that $\lambda = [Z]$.*

Proof. Let $U := \mathbb{P}^N_{\mathbb{C}}\backslash X$. Consider $Res_{X,\mathbb{P}^N} : M(X) \to M(U)[-1]$ the canonical map given in definition 1. We have the commuative diagram of complex vector spaces :

$$\begin{array}{ccc} H^N_{DR}(U) & \xrightarrow{H^N\Omega(Res_{X,\mathbb{P}^N})} & H^{N-1}_{DR}(X)_v \\ {\scriptstyle H^N\alpha(U^{an})=ev(U)^{-1}}\uparrow & & \uparrow{\scriptstyle H^{N-1}\alpha(X^{an})=ev(X)^{-1}} \\ H^N_{\mathrm{sing}}(U^{an}, \mathbb{C}) & \xrightarrow{(H^N \mathrm{Bti}(Res_{X,\mathbb{P}^N}))\otimes_{\mathbb{Q}}\mathbb{C}} & H^{N-1}_{\mathrm{sing}}(X^{an}, \mathbb{C})_v \end{array}.$$

whose arrows are isomorphisms. Denote $w(\lambda) := H^{N-1}\alpha(X^{an})(\lambda) \in F^pH^{2p}_{DR}(X)$. We have

$$\lambda = H^{N-1}\,\mathrm{Bti}(Res_{X,\mathbb{P}^N})(\alpha) + m[H],\ \text{with } m \in \mathbb{Q},\ \alpha \in H^N_{\mathrm{sing}}(U^{an}, \mathbb{Q}),$$

where $H := X \cap \Lambda \subset X$ is an hyperplane section. Then

$$w(\lambda) = H^{N-1}\Omega(Res_{X,\mathbb{P}^N})(w(\alpha)) + m[H],\ \text{with } w(\alpha) := H^N\alpha(U^{an})(\alpha) \in F^{p+1}H^N_{DR}(U).$$

Since $ev(U)(w(\alpha)) = \alpha \in H^N_{\mathrm{sing}}(U^{an}, 2i\pi\mathbb{Q})$, we have by proposition 3,

$$w(\alpha) \in H^NOL_{U^{an}}(H^p_{usu}(U^{an}, \Omega^{N-p}_{U^{an},\log}))$$

Hence

$$w(\lambda) = H^{N-1}OL_{X^{an}}(\Omega^{N-p-1,an}_{\log}(Res_{X,\mathbb{P}^N})(w(\alpha))) + m[H] \in OL_{X^{an}}(H^p_{usu}(X^{an}, \Omega^p_{X^{an},\log})).$$

Hence by proposition 4(ii), $w(\lambda) = [Z]$ with $Z \in \mathcal{Z}^p(X)$. □

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