Hodge conjecture for projective hypersurfaces

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December 14, 2023

Abstract

We show that an Hodge class of a complex smooth projective hypersurface is an analytic logarithmic De Rham class. On the other hand we show that for a complex smooth projective variety an analytic logarithmic De Rham class of of type (d,d) is the class of codimension d algebraic cycle. We deduce the Hodge conjecture for smooth projective hypersurfaces.

Contents

1 Introduction

1.1 Complex analytic De Rham logarithmic classes

In [2], we introduced, for X an algebraic variety over a field, the notion of logarithmic De Rham cohomology classes. For X a complex algebraic variety, we also introduce the notion of logarithmic analytic De Rham cohomology classes. In proposition 4 we give an analytic analogue of [2] theorem 2 (ii), that is if X is a smooth complex projective variety,

- a logarithmic analytic De Rham cohomology class of type (d,d) is the class of a codimension d algebraic cycle.
- we have $H^jOL_{X^{an}}(H^{j-l}_{usu}(X^{an},\Omega^l_{X^{an},\log}))=0$ for $j,l\in\mathbb{Z}$ such that $2l>j$.

The proof of proposition 4 is motivic and similar to the proof of [2] theorem 2 : a logarithmic analytic class of bidegree (p,q), $p,q\neq 0$, $p\geq q$, $q\neq d_U$ is acyclic for the etale topology on each open subset $U\subset X$ such that there exists an etale map $e:U\to\mathbb{A}^{d_U}_{\mathbb{C}}$ such that $e:U\to e(U)$ is finite etale and $\mathbb{A}^{d_U}_{\mathbb{C}}\backslash e(U)\subset\mathbb{A}^{d_U}_{\mathbb{C}}$ is a divisor (proposition 2), this allows us to proceed by a finite induction using the

crucial fact that the purity isomorphism for De Rham cohomology preserve logarithmic analytic classes since the purity isomorphism is motivic (c.f. [3], see proposition 1).

1.2 Hodge conjecture for smooth projective hypersurfaces

Let $X = V(f) \subset \mathbb{P}^N_{\mathbb{C}}$ a smooth projective hypersurface. Denote $j : U := \mathbb{P}^N_{\mathbb{C}} \backslash X \hookrightarrow \mathbb{P}^N_{\mathbb{C}}$ the open complementary subset. Using the fact the filtered De Rham complex of U

$$\iota_\partial : \Omega^{N-p}_{\mathbb{P}^{N,an}}(\log X)^{\partial=0}[-p] \hookrightarrow (\Omega^\bullet_{\mathbb{P}^N} \otimes_{O_{\mathbb{P}^N}} F^{\bullet-p} j_* O_U)^{an}$$

is locally acyclic for the complex topology, more precisely is acyclic on open balls (see e.g.[6]), we show, considering an open cover $\mathbb{P}^N_{\mathbb{C}} = \cup_{i=1}^n B_i$ by open balls $\phi_i : B_i \simeq D(0,1)^N$ such that $\phi_i(B_i \cap X) = 0 \times D(0,1)^N$, that an Hodge class $\alpha \in F^p H^N(U^{an}, \mathbb{Q})$ is logarithmic, that is

$$\alpha \in H^N OL_{U^{an}}(H^p_{usu}(U^{an}, \Omega^{N-p}_{U^{an},\log}))$$

(proposition 3). We get that, if $N = 2p+1$ and $\lambda \in F^p H^{2p}(X^{an}, \mathbb{Q})$ is an Hodge class,

$$\lambda = Res_{X,\mathbb{P}^N}(\alpha) + m[H] \in H^N OL_{X^{an}}(H^p_{usu}(X^{an}, \Omega^{N-p}_{X^{an},\log})),$$

where $Res_{X,\mathbb{P}^N} : H^N(U^{an}, \mathbb{C}) \xrightarrow{\sim} H^{2p}(X^{an}, \mathbb{C})$ is the residu map, using the fact that since the residu map is motivic (see definition 1), it preserve the logaritmic De Rham classes. Proposition 4 (ii) then implies that $\lambda = [Z]$, $Z \in \mathcal{Z}^p(X)$, is the class of an algebraic cycle.

2 Preliminaries and Notations

2.1 Notations

- Denote by Top the category of topological spaces and RTop the category of ringed spaces.
- Denote by Cat the category of small categories and RCat the category of ringed topos.
- For $\mathcal{S} \in$ Cat and $X \in \mathcal{S}$, we denote $\mathcal{S}/X \in$ Cat the category whose objects are $Y/X := (Y, f)$ with $Y \in \mathcal{S}$ and $f : Y \to X$ is a morphism in $\mathcal{S}$, and whose morphisms $\mathrm{Hom}((Y', f'), (Y, f))$ consists of $g : Y' \to Y$ in $\mathcal{S}$ such that $f \circ g = f'$.
- For $(\mathcal{S}, O_S) \in$ RCat a ringed topos, we denote by
  - $\mathrm{PSh}(\mathcal{S})$ the category of presheaves of O_S modules on $\mathcal{S}$ and $\mathrm{PSh}_{O_S}(\mathcal{S})$ the category of presheaves of O_S modules on $\mathcal{S}$, whose objects are $\mathrm{PSh}_{O_S}(\mathcal{S})^0 := \{(M, m), M \in \mathrm{PSh}(\mathcal{S}), m : M \otimes O_S \to M\}$, together with the forgetful functor $o : \mathrm{PSh}(\mathcal{S}) \to \mathrm{PSh}_{O_S}(\mathcal{S})$,
  - $C(\mathcal{S}) = C(\mathrm{PSh}(\mathcal{S}))$ and $C_{O_S}(\mathcal{S}) = C(\mathrm{PSh}_{O_S}(\mathcal{S}))$ the big abelian category of complexes of presheaves of O_S modules on $\mathcal{S}$,
  - $C_{O_S(2)fil}(\mathcal{S}) := C_{(2)fil}(\mathrm{PSh}_{O_S}(\mathcal{S})) \subset C(\mathrm{PSh}_{O_S}(\mathcal{S}), F, W)$, the big abelian category of (bi)filtered complexes of presheaves of O_S modules on $\mathcal{S}$ such that the filtration is biregular and $\mathrm{PSh}_{O_S(2)fil}(\mathcal{S}) := (\mathrm{PSh}_{O_S}(\mathcal{S}), F, W)$.
- Let $(\mathcal{S}, O_S) \in$ RCat a ringed topos with topology τ. For $F \in C_{O_S}(\mathcal{S})$, we denote by $k : F \to E_\tau(F)$ the canonical flasque resolution in $C_{O_S}(\mathcal{S})$ (see [1]). In particular for $X \in \mathcal{S}$, $H^*(X, E_\tau(F)) \xrightarrow{\sim} \mathbb{H}^*_\tau(X, F)$.
- For $f : \mathcal{S}' \to \mathcal{S}$ a morphism with $\mathcal{S}, \mathcal{S}' \in$ RCat, endowed with topology τ and τ' respectively, we denote for $F \in C_{O_S}(\mathcal{S})$ and each $j \in \mathbb{Z}$,
  - $f^* := H^j\Gamma(\mathcal{S}, k \circ \mathrm{ad}(f^*, f_*)(F)) : \mathbb{H}^j(\mathcal{S}, F) \to \mathbb{H}^j(\mathcal{S}', f^*F)$,

– $f^* := H^j\Gamma(\mathcal{S}, k \circ \mathrm{ad}(f^{*mod}, f_*)(F)) : \mathbb{H}^j(\mathcal{S}, F) \to \mathbb{H}^j(\mathcal{S}', f^{*mod}F)$,

the canonical maps.

- For $X \in \mathrm{Top}$ and $Z \subset X$ a closed subset, denoting $j : X\backslash Z \hookrightarrow X$ the open complementary, we will consider
$$\Gamma_Z^\vee \mathbb{Z}_X := \mathrm{Cone}(\mathrm{ad}(j_!, j^*)(\mathbb{Z}_X) : j_!j^*\mathbb{Z}_X \hookrightarrow \mathbb{Z}_X) \in C(X)$$
and denote for short $\gamma_Z^\vee := \gamma_X^\vee(\mathbb{Z}_X) : \mathbb{Z}_X \to \Gamma_Z^\vee \mathbb{Z}_X$ the canonical map in $C(X)$.

- Denote by $\mathrm{Sch} \subset \mathrm{RTop}$ the subcategory of schemes (the morphisms are the morphisms of locally ringed spaces). We denote by $\mathrm{PSch} \subset \mathrm{Sch}$ the full subcategory of proper schemes. For a field k, we consider $\mathrm{Sch}\,/k := \mathrm{Sch}\,/\mathrm{Spec}\,k$ the category of schemes over $\mathrm{Spec}\,k$, that is whose object are $X := (X, a_X)$ with $X \in \mathrm{Sch}$ and $a_X : X \to \mathrm{Spec}\,k$ a morphism and whose objects are morphism of schemes $f : X' \to X$ such that $f \circ a_{X'} = a_X$. We then denote by
  - $\mathrm{Var}(k) = \mathrm{Sch}^{ft}\,/k \subset \mathrm{Sch}\,/k$ the full subcategory consisting of algebraic varieties over k, i.e. schemes of finite type over k,
  - $\mathrm{PVar}(k) \subset \mathrm{QPVar}(k) \subset \mathrm{Var}(k)$ the full subcategories consisting of quasi-projective varieties and projective varieties respectively,
  - $\mathrm{PSmVar}(k) \subset \mathrm{SmVar}(k) \subset \mathrm{Var}(k)$, $\mathrm{PSmVar}(k) := \mathrm{PVar}(k) \cap \mathrm{SmVar}(k)$, the full subcategories consisting of smooth varieties and smooth projective varieties respectively.

- Denote by $\mathrm{AnSp}(\mathbb{C}) \subset \mathrm{RTop}$ the subcategory of analytic spaces over $\mathbb{C}$, and by $\mathrm{AnSm}(\mathbb{C}) \subset \mathrm{AnSp}(\mathbb{C})$ the full subcategory of smooth analytic spaces (i.e. complex analytic manifold).

- Let $(X, O_X) \in \mathrm{RTop}$. We consider its De Rham complex $\Omega_X^\bullet := DR(X)(O_X)$.
  - Let $X \in \mathrm{Sch}$. Considering its De Rham complex $\Omega_X^\bullet := DR(X)(O_X)$, we have for $j \in \mathbb{Z}$ its De Rham cohomology $H^j_{DR}(X) := \mathbb{H}^j(X, \Omega_X^\bullet)$.
  - Let $X \in \mathrm{Var}(k)$. Considering its De Rham complex $\Omega_X^\bullet := \Omega_{X/k}^\bullet := DR(X/k)(O_X)$, we have for $j \in \mathbb{Z}$ its De Rham cohomology $H^j_{DR}(X) := \mathbb{H}^j(X, \Omega_X^\bullet)$. The differentials of $\Omega_X^\bullet := \Omega_{X/k}^\bullet$ are by definition k-linear, thus $H^j_{DR}(X) := \mathbb{H}^j(X, \Omega_X^\bullet)$ has a structure of a k vector space.
  - Let $X \in \mathrm{AnSp}(\mathbb{C})$. Considering its De Rham complex $\Omega_X^\bullet := DR(X)(O_X)$, we have for $j \in \mathbb{Z}$ its De Rham cohomology $H^j_{DR}(X) := \mathbb{H}^j(X, \Omega_X^\bullet)$.

- For $X \in \mathrm{AnSp}(\mathbb{C})$, we denote $\alpha(X) : \mathbb{C}_X \hookrightarrow \Omega_X^\bullet$ the embedding in $C(X)$. For $X \in \mathrm{AnSm}(\mathbb{C})$, $\alpha(X) : \mathbb{C}_X \hookrightarrow \Omega_X^\bullet$ is an equivalence usu local by Poincare lemma.

- We denote $\mathbb{I}^n := [0,1]^n \in \mathrm{Diff}(\mathbb{R})$ (with boundary). For $X \in \mathrm{Top}$ and R a ring, we consider its singular cochain complex
$$C^*_{\mathrm{sing}}(X, R) := (\mathbb{Z}\,\mathrm{Hom}_{\mathrm{Top}}(\mathbb{I}^*, X)^\vee) \otimes R$$
and for $l \in \mathbb{Z}$ its singular cohomology $H^l_{\mathrm{sing}}(X, R) := H^n C^*_{\mathrm{sing}}(X, R)$. For $f : X' \to X$ a continous map with $X, X' \in \mathrm{Top}$, we have the canonical map of complexes
$$f^* : C^*_{\mathrm{sing}}(X, R) \to C^*_{\mathrm{sing}}(X, R), \sigma \mapsto f^*\sigma := (\gamma \mapsto \sigma(f \circ \gamma)).$$
In particular, we get by functoriality the complex
$$C^*_{X, R\,\mathrm{sing}} \in C_R(X),\ (U \subset X) \mapsto C^*_{\mathrm{sing}}(U, R)$$
We recall we have

– For $X \in$ Top locally contractile, e.g. $X \in$ CW, and R a ring, the inclusion in $C_R(X)$ $c_X : R_X \to C^*_{X,R\,\mathrm{sing}}$ is by definition an equivalence top local and that we get by the small chain theorem, for all $l \in \mathbb{Z}$, an isomorphism $H^l c_X : H^l(X, R_X) \xrightarrow{\sim} H^l_{\mathrm{sing}}(X, R)$.

– For $X \in \mathrm{Diff}(\mathbb{R})$, the restriction map

$$r_X : \mathbb{Z}\,\mathrm{Hom}_{\mathrm{Diff}(\mathbb{R})}(\mathbb{I}^*, X)^\vee \to C^*_{\mathrm{sing}}(X, R),\ w \mapsto w : (\phi \mapsto w(\phi))$$

is a quasi-isomorphism by Whitney approximation theorem.

- We will consider the morphism of site given by the analytification functor

$$\mathrm{An} : \mathrm{AnSp}(\mathbb{C}) \to \mathrm{Var}(\mathbb{C}),\ \ X \mapsto \mathrm{An}(X) := X^{an},\ (f : X' \to X) \mapsto \mathrm{An}(f) := f^{an}.$$

For $X \in \mathrm{Var}(\mathbb{C})$, we denote by

$$\mathrm{an}_X := \mathrm{An}_{|X^{et}} := X^{an,et} \to X^{et},\ \mathrm{an}_X(U \to X) := U^{an},$$

its restriction to the small etale site $o_X(X^{et}) \subset \mathrm{Var}(\mathbb{C})$, where $o_X(U \to X) = U$, in particular we have the following commutative diagram

$$\begin{array}{ccc} \mathrm{AnSp}(\mathbb{C}) & \xrightarrow{\mathrm{An}} & \mathrm{Var}(\mathbb{C}) \\ \downarrow{\scriptstyle o_{X^{an}}} & & \downarrow{\scriptstyle o_X} \\ X^{an,et} & \xrightarrow{\mathrm{an}_X} & X^{et} \end{array},$$

where $o_{X^{an}}(U \hookrightarrow X^{an}) = U$. We get the morphism in RTop $\mathrm{an}_X := \mathrm{an}_{X|O(X)} : X^{an} \to X$, $o : O(X) \subset X^{et}$ being the embedding consisting of the Zariski open subsets of X.

2.2 Complex integral periods

Let k a field of characteristic zero.

Let $X \in \mathrm{SmVar}(k)$ a smooth variety. Let $X = \cup_{i=1}^s X_i$ an open affine cover. We have for $\sigma : k \hookrightarrow \mathbb{C}$ an embedding, the evaluation period embedding map which is the morphism of bi-complexes

$$ev(X)^\bullet_\bullet : \Gamma(X_\bullet, \Omega^\bullet_{X_\bullet}) \to \mathbb{Z}\,\mathrm{Hom}_{\mathrm{Diff}}(\mathbb{I}^\bullet, X^{an}_{\mathbb{C},\bullet})^\vee \otimes \mathbb{C},$$

$$w^l_I \in \Gamma(X_I, \Omega^l_{X_I}) \mapsto (ev(X)^l_I(w^l_I) : \phi^l_I \in \mathbb{Z}\,\mathrm{Hom}_{\mathrm{Diff}}(\mathbb{I}^l, X^{an}_{\mathbb{C},I})^\vee \otimes \mathbb{C} \mapsto ev^l_I(w^l_I)(\phi^l_I) := \int_{\mathbb{I}^l} \phi^{l*}_I w^l_I)$$

given by integration. By taking all the affine open cover $(j_i : X_i \hookrightarrow X)$ of X, we get for $\sigma : k \hookrightarrow \mathbb{C}$, the evaluation period embedding map

$$ev(X) := \varinjlim_{(j_i : X_i \hookrightarrow X)} ev(X)^\bullet_\bullet : \varinjlim_{(j_i : X_i \hookrightarrow X)} \Gamma(X_\bullet, \Omega^\bullet_{X_\bullet}) \to \varinjlim_{(j_i : X_i \hookrightarrow X)} \mathbb{Z}\,\mathrm{Hom}_{\mathrm{Diff}(\mathbb{R})}(\mathbb{I}^\bullet, X^{an}_{\mathbb{C},\bullet})^\vee \otimes \mathbb{C}$$

It induces in cohomology, for $j \in \mathbb{Z}$, the evaluation period map

$$H^j ev(X) = H^j ev(X)^\bullet_\bullet : H^j_{DR}(X) = H^j\Gamma(X_\bullet, \Omega^\bullet_{X_\bullet}) \to H^j_{\mathrm{sing}}(X^{an}_{\mathbb{C}}, \mathbb{C}) = H^j(\mathrm{Hom}_{\mathrm{Diff}(\mathbb{R})}(\mathbb{I}^\bullet, X^{an}_{\mathbb{C},\bullet})^\vee \otimes \mathbb{C}).$$

which does NOT depend on the choice of the affine open cover by acyclicity of quasi-coherent sheaves on affine noetherian schemes for the left hand side and from Mayer-Vietoris quasi-isomorphism for singular cohomology of topological spaces and Whitney approximation theorem for differential manifolds for the right hand side.

Let $X \in \mathrm{SmVar}(k)$. Then,

$$H^* ev(X_{\mathbb{C}}) = H^* R\Gamma(X^{an}_{\mathbb{C}}, \alpha(X))^{-1} \circ \Gamma(X^{an}_{\mathbb{C}}, E_{zar}(\Omega(\mathrm{an}_X))) : H^*_{DR}(X_{\mathbb{C}}) \xrightarrow{\sim} H^*_{\mathrm{sing}}(X^{an}_{\mathbb{C}}, \mathbb{C})$$

is the canonical isomorphism induced by the analytical functor and the quasi-isomorphism $\alpha(X) : \mathbb{C}_{X^{an}_{\mathbb{C}}} \hookrightarrow \Omega^\bullet_{X^{an}_{\mathbb{C}}}$, which gives the periods elements $H^* ev(X)(H^*_{DR}(X)) \subset H^*_{\mathrm{sing}}(X^{an}_{\mathbb{C}}, \mathbb{C})$.

2.3 Algebraic cycles and motives

For $X \in \mathrm{Sch}$ noetherian irreducible and $d \in \mathbb{N}$, we denote by $\mathcal{Z}^d(X)$ the group of algebraic cycles of codimension d, which is the free abelian group generated by irreducible closed subsets of codimension d.

For $X, X' \in \mathrm{Sch}$ noetherian, with X' irreducible, we denote $\mathcal{Z}^{fs/X'}(X' \times X) \subset \mathcal{Z}_{d_{X'}}(X' \times X)$ which consist of algebraic cycles $\alpha = \sum_i n_i \alpha_i \in \mathcal{Z}_{d_{X'}}(X' \times X)$ such that, denoting $\mathrm{supp}(\alpha) = \cup_i \alpha_i \subset X' \times X$ its support and $p' : X' \times X \to X'$ the projection, $p'_{|\mathrm{supp}(\alpha)} : \mathrm{supp}(\alpha) \to X'$ is finite surjective.

Let k a field. We denote by $\mathrm{Cor}\,\mathrm{SmVar}(k)$ the category such that the objects are $\{X \in \mathrm{SmVar}(k)\}$ and such that $\mathrm{Hom}_{\mathrm{Cor}\,\mathrm{SmVar}(k)}(X', X) := \mathcal{Z}^{fs/X'}(X' \times X)$. See [3] for the composition law. We denote by $\mathrm{Tr} : \mathrm{Cor}\,\mathrm{SmVar}(k) \to \mathrm{SmVar}(k)$ the morphism of site given by the embedding $\mathrm{Tr} : \mathrm{SmVar}(k) \hookrightarrow \mathrm{Cor}\,\mathrm{SmVar}(k)$. Let $F \in \mathrm{PSh}(\mathrm{SmVar}(k))$. We say that F admits transfers if $F = \mathrm{Tr}_* F$ with $F \in \mathrm{PSh}(\mathrm{Cor}\,\mathrm{SmVar}(k))$. We recall that $\mathrm{DA}(k) := \mathrm{Ho}_{\mathbb{A}^1,et}(C(\mathrm{SmVar}(k)))$ is then the derived category of mixed motives.

We have (see e.g. [1]) the Hodge realization functor

$$\mathcal{F}^{Hdg} : \mathrm{DA}(\mathbb{C}) \to D(MHS),\ M \mapsto \mathcal{F}^{Hdg}(M) := \iota^{-1}(\mathcal{F}^{FDR}(M), \mathrm{Bti}^*\, M, \alpha(M))$$

For $X \in \mathrm{Var}(k)$ and $Z \subset X$ a closed subset, denoting $j : X \backslash Z \hookrightarrow X$ the open complementary, we will consider

$$\Gamma_Z^\vee \mathbb{Z}_X := \mathrm{Cone}(\mathrm{ad}(j_\sharp, j^*)(\mathbb{Z}_X) : \mathbb{Z}_X \hookrightarrow \mathbb{Z}_X) \in C(\mathrm{Var}(k)^{sm}/X)$$

and denote for short $\gamma_Z^\vee := \gamma_Z^\vee(\mathbb{Z}_X) : \mathbb{Z}_X \to \Gamma_Z^\vee \mathbb{Z}_X$ the canonical map in $C(\mathrm{Var}(k)^{sm}/X)$. Denote $a_X : X \to \mathrm{Spec}\, k$ the structural map. For $X \in \mathrm{Var}(k)$ and $Z \subset X$ a closed subset, we have the motive of X with support in Z defined as

$$M_Z(X) := a_{X!}\Gamma_Z^\vee a_X^! \mathbb{Z} \in \mathrm{DA}(k).$$

If $X \in \mathrm{SmVar}(k)$, we will also consider

$$a_{X\sharp}\Gamma_Z^\vee \mathbb{Z}_X := \mathrm{Cone}(a_{X\sharp} \circ \mathrm{ad}(j_\sharp, j^*)(\mathbb{Z}_X) : \mathbb{Z}(U) \hookrightarrow \mathbb{Z}(X)) =: \mathbb{Z}(X, X\backslash Z) \in C(\mathrm{SmVar}(k)).$$

Then for $X \in \mathrm{SmVar}(k)$ and $Z \subset X$ a closed subset

$$M_Z(X) := a_{X!}\Gamma_Z^\vee a_X^! \mathbb{Z} = a_{X\sharp}\Gamma_Z^\vee \mathbb{Z}_X =: \mathbb{Z}(X, X\backslash Z) \in \mathrm{DA}(k).$$

- Let $(X, Z) \in \mathrm{Sch}^2$ with $X \in \mathrm{Sch}$ a noetherian scheme and $Z \subset X$ a closed subset. We have the deformation $(D_Z X, \mathbb{A}^1_Z) \to \mathbb{A}^1$, $(D_Z X, \mathbb{A}^1_Z) \in \mathrm{Sch}^2$ of (X, Z) by the normal cone $C_{Z/X} \to Z$, i.e. such that
$$(D_Z X, \mathbb{A}^1_Z)_s = (X, Z),\ s \in \mathbb{A}^1 \backslash 0,\ (D_Z X, \mathbb{A}^1_Z)_0 = (C_{Z/X}, Z).$$
We denote by $i_1 : (X, Z) \hookrightarrow (D_Z X, \mathbb{A}^1_Z)$ and $i_0 : (C_{Z/X}, Z) \hookrightarrow (D_Z X, \mathbb{A}^1_Z)$ the closed embeddings in Sch^2.

- Let k a field of characteristic zero. Let $X \in \mathrm{SmVar}(k)$. For $Z \subset X$ a closed subset of pure codimension c, consider a desingularisation $\epsilon : \tilde{Z} \to Z$ of Z and denote $n : \tilde{Z} \xrightarrow{\epsilon} Z \subset X$. We have then the morphism in $\mathrm{DA}(k)$
$$G_{Z,X} : M(X) \xrightarrow{D(\mathbb{Z}(n))} M(\tilde{Z})(c)[2c] \xrightarrow{\mathbb{Z}(\epsilon)} M(Z)(c)[2c]$$
where $D : \mathrm{Hom}_{\mathrm{DA}(k)}(M_c(\tilde{Z}), M_c(X)) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{DA}(k)}(M(X), M(\tilde{Z})(c)[2c])$ is the duality isomorphism from the six functors formalism (moving lemma of Suzlin and Voevodsky) and $\mathbb{Z}(n) := \mathrm{ad}(n_!, n^!)(a_X^! \mathbb{Z})$, noting that $n_! = n_*$ since n is proper and that $a_X^! = a_X^*[d_X]$ and $a_{\tilde{Z}}^! = a_{\tilde{Z}}^*[d_Z]$ since X, resp. $\tilde{Z}$, are smooth (considering the connected components, we may assume X and $\tilde{Z}$ of pure dimension).

We recall the following facts (see [3] and [1]):

Proposition 1. *Let k be a field a characteristic zero. Let $X \in \mathrm{SmVar}(k)$ and $i : Z \subset X$ a smooth subvariety of pure codimension d. Then $C_{Z/X} = N_{Z/X} \to Z$ is a vector bundle of rank d. The closed embeddings $i_1 : (X, Z) \hookrightarrow (D_Z X, \mathbb{A}^1_Z)$ and $i_0 : (C_{Z/X}, Z) \hookrightarrow (D_Z X, \mathbb{A}^1_Z)$ in $\mathrm{SmVar}^2(k)$ induces isomorphisms of motives $\mathbb{Z}(i_1) : M_Z(X) \xrightarrow{\sim} M_{\mathbb{A}^1_Z}(D_Z X)$ and $\mathbb{Z}(i_0) : M_Z(N_{Z/X}) \xrightarrow{\sim} M_{\mathbb{A}^1_Z}(D_Z X)$ in $\mathrm{DA}(k)$. We get the excision isomorphism in $\mathrm{DA}(k)$*

$$P_{Z,X} := \mathbb{Z}(i_0)^{-1} \circ \mathbb{Z}(i_1) : M_Z(X) \xrightarrow{\sim} M_Z(N_{Z/X}).$$

We have

$$Th(N_{Z/X}) \circ P_{Z,X} \circ \gamma_Z^{\vee}(\mathbb{Z}_X) = G_{Z,X} := D(\mathbb{Z}(i)) : M(X) \to M(Z)(d)[2d].$$

Proof. See [3]. □

Definition 1. *Let k be a field a characteristic zero. Let $X \in \mathrm{SmVar}(k)$ and $i : Z \subset X$ a smooth subvariety of pure codimension d. We consider using proposition 1 the canonical map in $\mathrm{DA}(k)$*

$$Res_{Z,X} : M(Z)(d)[-2d] \xrightarrow{P_{Z,X}^{-1} \circ Th(N_{Z/X})^{-1}} M_Z(X) \xrightarrow{c(\mathbb{Z}(X \backslash Z))} M(X \backslash Z)[1]$$

where, for $m : A \to B$, $A, B \in C(\mathcal{A})$, $\mathcal{A}$ an additive category, $c(A) : \mathrm{Cone}(m : A \to B) \to A[1]$ is the canonical map.

- Recall that we say that $F \in \mathrm{PSh}(\mathrm{SmVar}(k))$ is $\mathbb{A}^1$ invariant if for all $X \in \mathrm{SmVar}(k)$, $p^* := F(p) : F(X) \to F(X \times \mathbb{A}^1)$ is an isomorphism, where $p : X \times \mathbb{A}^1 \to X$ is the projection.
- Similarly, we say that $F \in \mathrm{PSh}(\mathrm{AnSm}(\mathbb{C}))$ is $\mathbb{D}^1$ invariant if for all $X \in \mathrm{AnSm}(\mathbb{C})$, $p^* := F(p) : F(X) \to F(X \times \mathbb{D}^1)$ is an isomorphism, where $p : X \times \mathbb{D}^1 \to X$ is the projection.

2.4 The logaritmic De Rham complexes

We recall from [2] the following notion :

- For k a field and $X \in \mathrm{Var}(k)$, we consider the embedding in $C(X)$

  $$OL_X : \Omega^{\bullet}_{X,\log} \hookrightarrow \Omega^{\bullet}_X := \Omega^{\bullet}_{X/k},$$

  such that, for $X^o \subset X$ an open subset and $w \in \Omega^p_X(X^o)$, $w \in \Omega^p_{X,\log}(X^o)$ if and only if there exists $(n_i)_{1 \leq i \leq s} \in \mathbb{Z}$ and $(f_{i,\alpha_k})_{1 \leq i \leq s, 1 \leq k \leq p} \in \Gamma(X^o, O_X)^*$ such that

  $$w = \sum_{1 \leq i \leq s} n_i df_{i,\alpha_1}/f_{i,\alpha_1} \wedge \cdots \wedge df_{i,\alpha_p}/f_{i,\alpha_p} \in \Omega^p_X(X^o),$$

  and for $p = 0$, $\Omega^0_{X,\log} := \mathbb{Z}$. Let k a field. We get an embedding in $C(\mathrm{Var}(k))$

  $$OL : \Omega^{\bullet}_{/k,\log} \hookrightarrow \Omega^{\bullet}_{/k}, \text{ given by, for } X \in \mathrm{Var}(k),$$
  $$OL(X) := OL_X : \Omega^{\bullet}_{/k,\log}(X) := \Gamma(X, \Omega^{\bullet}_{X,\log}) \hookrightarrow \Gamma(X, \Omega^{\bullet}_X) =: \Omega^{\bullet}_{/k}(X)$$

  and its restriction to $\mathrm{SmVar}(k) \subset \mathrm{Var}(k)$.

- For $X \in \mathrm{AnSp}(\mathbb{C})$, we consider the embedding in $C(X)$

  $$OL_X : \Omega^{\bullet}_{X,\log} \hookrightarrow \Omega^{\bullet}_X,$$

  such that, for $X^o \subset X$ an open subset and $w \in \Omega^p_X(X^o)$, $w \in \Omega^p_{X,\log}(X^o)$ if and only if there exists $(n_i)_{1 \leq i \leq s} \in \mathbb{Z}$ and $(f_{i,\alpha_k})_{1 \leq i \leq s, 1 \leq k \leq p} \in \Gamma(X^o, O_X)^*$ such that

  $$w = \sum_{1 \leq i \leq s} n_i df_{i,\alpha_1}/f_{i,\alpha_1} \wedge \cdots \wedge df_{i,\alpha_p}/f_{i,\alpha_p} \in \Omega^p_X(X^o),$$

and for $p=0$, $\Omega^0_{X,\log} := \mathbb{Z}$. We get an embedding in $C(\mathrm{AnSp}(\mathbb{C}))$

$$OL^{an} : \Omega^{\bullet,an}_{\log} \hookrightarrow \Omega^{\bullet,an}, \text{ given by, for } X \in \mathrm{AnSp}(\mathbb{C}),$$
$$OL^{an}(X) := OL_X : \Omega^{\bullet,an}_{\log}(X) := \Gamma(X, \Omega^{\bullet}_{X,\log}) \hookrightarrow \Gamma(X, \Omega^{\bullet}_X) =: \Omega^{\bullet,an}(X)$$

and its restriction to $\mathrm{AnSm}(\mathbb{C}) \subset \mathrm{AnSp}(\mathbb{C})$.

Definition 2. *Recall that* $\mathrm{An} : \mathrm{AnSp}(\mathbb{C}) \to \mathrm{Var}(\mathbb{C})$ *denote the analytical functor. We will consider, for each* $p \in \mathbb{Z}$*, the exact sequence in* $\mathrm{PSh}(\mathrm{AnSp}(\mathbb{C}))$

$$0 \to \Omega^{p,an}_{\log} \cap (\mathrm{An}^* \, \partial \, \mathrm{An}_* \, \Omega^{p-1,an}) \xrightarrow{e} \Omega^{p,an}_{\log} \xrightarrow{q} \Omega^{p,an,ne}_{\log} := \Omega^{p,an}_{\log} / (\Omega^{p,an}_{\log} \cap (\mathrm{An}^* \, \partial \, \mathrm{An}_* \, \Omega^{p-1,an})) \to 0,$$

and its restriction in $\mathrm{PSh}(\mathrm{AnSm}(\mathbb{C}))$*. Note that for* $X \in \mathrm{Var}(\mathbb{C})$ *and* $\partial f \in \Omega^{1,an}(X^{an})$ *with* $f \in O(X^{an})$*, we have* $\partial f = (\partial e^f)/e^f \in \Omega^{1,an}_{\log}(X^{an})$*. We have then the commutative diagram in* $C(\mathrm{AnSp}(\mathbb{C}))$

$$\begin{array}{ccc}
\Omega^{\bullet,an}_{\log} \cap (\mathrm{An}^* \, \partial \, \mathrm{An}_* \, \Omega^{\bullet-1,an}) & \xrightarrow[OL^{an}]{} & \mathrm{An}^* \, \partial \, \mathrm{An}_* \, \Omega^{\bullet-1,an} \\
\downarrow e & & \downarrow e \\
\Omega^{\bullet,an}_{\log} & \xrightarrow[OL^{an}]{} & \Omega^{\bullet,an} \\
\downarrow q & & \downarrow q \\
\Omega^{\bullet,an,ne}_{\log} := \Omega^{\bullet,an}_{\log} / (\Omega^{\bullet,an}_{\log} \cap (\mathrm{An}^* \, \partial \, \mathrm{An}_* \, \Omega^{\bullet-1,an})) & \xrightarrow{\bar{OL}^{an}} & \Omega^{\bullet,an} / (\mathrm{An}^* \, \partial \, \mathrm{An}_* \, \Omega^{\bullet-1,an})
\end{array},$$

whose columns are exact with e *injective and* q *surjective.*

We have the following :

Lemma 1. *(i0) The sheaves* $O^*_{an} \in \mathrm{PSh}(\mathrm{AnSm}(\mathbb{C}))$ *and* $\Omega^{1,an}_{\log} \in \mathrm{PSh}(\mathrm{AnSm}(\mathbb{C}))$ *admit transfers compatible with transfers on* $\Omega^{1,an} \in \mathrm{PSh}(\mathrm{AnSm}(\mathbb{C}))$*.*

(i) For each $l \in \mathbb{Z}$*, the sheaf* $\Omega^{l,an}_{\log} \in \mathrm{PSh}(\mathrm{AnSm}(\mathbb{C}))$ *admits transfers compatible with transfers on* $\Omega^{l,an} \in \mathrm{PSh}(\mathrm{AnSm}(\mathbb{C}))$*, that is* $\Omega^{\bullet,an} \in C(\mathrm{Cor}\,\mathrm{AnSm}(\mathbb{C}))$ *and the inclusion* $OL : \Omega^{l,an}_{\log}[-l] \hookrightarrow \Omega^{\bullet,an}$ *in* $C(\mathrm{AnSm}(\mathbb{C}))$ *is compatible with transfers.*

(ii) For each $l \in \mathbb{Z}$*, the sheaf* $\Omega^{l,an,ne}_{\log} \in \mathrm{PSh}(\mathrm{AnSm}(\mathbb{C}))$ *given in definition 2 is* $\mathbb{A}^1$ *invariant.*

Proof. (i0):Let $W \subset X' \times X$ with $X, X' \in \mathrm{AnSm}(\mathbb{C})$, X' connected, and $p_{X'} : W \subset X' \times X \to X'$ finite surjective over X'. Then $M(p_X) : M(X') \hookrightarrow M(W)$ is a finite, where for $Y \in \mathrm{AnSp}(\mathbb{C})$ irreducible, $M(Y) := Frac(O(Y))$ denote the field of meromorphic function. Since $O(X') \subset M(X')$ is integrally closed, the trace map $Tr_{W/X'} : M(W) \to M(X')$ sends $O(W)$ to $O(X')$, and the norm map $N_{W/X'} : M(W)^* \to M(X')^*$ sends $O(W)^*$ to $O(X')^*$.

The sheaf $O^*_{an} \in \mathrm{PSh}(\mathrm{AnSm}(\mathbb{C}))$ admits transfers : for $W \subset X' \times X$ with $X, X' \in \mathrm{AnSm}(\mathbb{C})$ and W finite surjective over X' and $f \in O(X)^*$, $W^*f := N_{W/X'}(p_X^* f)$ where $p_X : W \hookrightarrow X' \times X \to X$ is the projection and $N_{W/X'} : M(W)^* \to M(X')^*$ is the norm map. This gives transfers on $\Omega^{1,an}_{\log} \in \mathrm{PSh}(\mathrm{AnSm}(\mathbb{C}))$ compatible with transfers on $\Omega^{1,an} \in \mathrm{PSh}(\mathrm{AnSm}(\mathbb{C}))$: for $W \subset X' \times X$ with $X, X' \in \mathrm{AnSm}(\mathbb{C})$ and W finite surjective over X' and $f \in O(X)^*$,

$$W^* df/f := dW^* f / W^* f = Tr_{W/X'}(p_X^*(df/f)),$$

where we recall $p_X : W \hookrightarrow X' \times X \to X$ is the projection and $Tr_{W/X'} : O(W) \to O(X')$ is the trace map. Note that $d(fg)/fg = df/f + dg/g$.
(i): By (i0), we get transfers on

$$\otimes^l_{\mathbb{Q}} \Omega^{1,an}_{\log}, \otimes^l_O \Omega^{1,an} \in \mathrm{PSh}(\mathrm{AnSm}(\mathbb{C}))$$

since $\otimes^l_{\mathbb{Q}}\Omega^{1,an}_{\log} = H^0(\otimes^{L,l}_{\mathbb{Q}}\Omega^{1,an}_{\log})$ and $\otimes^l_O\Omega^{1,an} = H^0(\otimes^{L,l}_O\Omega^{1,an})$. This induces transfers on

$$\wedge^l_{\mathbb{Q}}\Omega^{1,an}_{\log} := \mathrm{coker}(\oplus_{I_2\subset[1,\dots,l]}\otimes^{l-1}_{\mathbb{Q}}\Omega^{1,an}_{\log} \xrightarrow{\oplus_{I_2\subset[1,\dots,l]}\Delta_{I_2}:=(w\otimes w'\mapsto w\otimes w\otimes w')} \otimes^l_{\mathbb{Q}}\Omega^{1,an}_{\log}) \in \mathrm{PSh}(\mathrm{AnSm}(\mathbb{C})).$$

and

$$\wedge^l_O\Omega^{1,an} := \mathrm{coker}(\oplus_{I_2\subset[1,\dots,l]}\otimes^{l-1}_{O_k}\Omega^{1,an} \xrightarrow{\oplus_{I_2\subset[1,\dots,l]}\Delta_{I_2}:=(w\otimes w'\mapsto w\otimes w\otimes w')}) \otimes^l_O \Omega^{1,an} \in \mathrm{PSh}(\mathrm{AnSm}(\mathbb{C})).$$

(ii): Follows from the fact that for $X \in \mathrm{AnSp}(\mathbb{C})$, $\Omega^p(X\times\mathbb{A}^1) = \Omega^p(X)\otimes\Omega^1(\mathbb{A}^{1,an})$ and that $\Omega^1(\mathbb{A}^{1,an}) = \partial O(\mathbb{A}^{1,an})$. □

We have the following result :

Proposition 2. *Consider, for each $p \in \mathbb{Z}$, the presheaf $\Omega^{p,an,ne}_{D,\log} \in \mathrm{PSh}(\mathrm{AnSm}(\mathbb{C}))$ given in definition 2.*

(i) Let $D \in \mathrm{AnSm}(\mathbb{C})$ be an open ball. Then, for each $p,q \in \mathbb{Z}$, $q \neq 0$, $H^q_{usu}(D,\Omega^p_{D,\log}) = 0$.

(i)' Let $D \in \mathrm{AnSm}(\mathbb{C})$ be an open ball. Then, for each $p,q \in \mathbb{Z}$, $q \neq 0$, $H^q_{usu}(D,\Omega^{p,an,ne}_{D,\log}) = 0$.

(ii) Let $U \in \mathrm{SmVar}(\mathbb{C})$ such that there exists an etale map $e : U \to \mathbb{A}^{d_U}_{\mathbb{C}}$ such that $e : U \to e(U)$ is finite etale and $\mathbb{A}^{d_U}_{\mathbb{C}}\backslash e(U) \subset \mathbb{A}^{d_U}_{\mathbb{C}}$ is a divisor. Then, for each $p,q \in \mathbb{Z}$, $p,q \neq 0$, $p \geq q$, $q \neq d_U$, $H^q_{usu}(U^{an},\Omega^{p,an,ne}_{\log}) = 0$.

Proof. (i):Let $B = D(0,1)^N \in \mathrm{AnSm}(\mathbb{C})$ be an open ball. Let $\alpha \in H^q_{usu}(B,\Omega^p_{B,\log})$, $q \in \mathbb{Z}$, $q \neq 0$. Consider a sequence of sub-balls

$$l_n : B_n := D(0,r_n)^N \hookrightarrow B := D(0,1)^N, n \in \mathbb{N}, \text{ with } r_n \to 0 \text{ when } n \to \infty.$$

Since $(B_n := D(0,r_n))_{n\in\mathbb{N}}$ form a basis of neighborhood of $0 \in B$ and since $\Omega^p_{B,\log}$ is a single presheaf, there exists $n := n_\alpha \in \mathbb{N}$ (depending on α) such that

$$l_n^*\alpha = 0 \in H^q_{usu}(B_n,\Omega^p_{B,\log}).$$

But it follows immediately from the definition of the the presheaf $\Omega^p_{B,\log}$ considering the isomorphism

$$\phi_n : B_n \xrightarrow{\sim} B,\ \phi_n(z) := r_n^{-1}z$$

so that $(\phi_n)^{-1}\circ l_n = r_n I$, that

$$l_n^*\alpha = l_n^*\phi_n^{-1*}\phi_n^*\alpha = r_n\phi_n^*\alpha \in H^q_{usu}(B_n,\Omega^p_{B,\log}).$$

Hence $\phi_n^*\alpha = 0$. Since ϕ_n is an isomorphism, we get $\alpha = 0$.
(i)':Let $B = D(0,1)^N \in \mathrm{AnSm}(\mathbb{C})$ be an open ball. Let $\alpha \in H^q_{usu}(B,\Omega^{p,an,ne}_{B,\log})$, $q \in \mathbb{Z}$, $q \neq 0$. Consider a sequence of sub-balls

$$l_n : B_n := D(0,r_n)^N \hookrightarrow B := D(0,1)^N, n \in \mathbb{N}, \text{ with } r_n \to 0 \text{ when } n \to \infty.$$

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$$l_n^*\alpha = 0 \in H^q_{usu}(B_n,\Omega^{p,an,ne}_{B,\log}).$$

But it follows immediately from the definition of the the presheaf $\Omega^p_{B,\log}$ considering the isomorphism

$$\phi_n : B_n \xrightarrow{\sim} B,\ \phi_n(z) := r_n^{-1}z$$

so that $(\phi_n)^{-1}\circ l_n = r_n I$, that

$$l_n^*\alpha = l_n^*\phi_n^{-1*}\phi_n^*\alpha = r_n\phi_n^*\alpha \in H^q_{usu}(B_n, \Omega^{p,an,ne}_{B,\log}).$$

Hence $\phi_n^*\alpha = 0$. Since ϕ_n is an isomorphism, we get $\alpha = 0$.
(ii): We proceed by induction on the dimension of U. Let $U \in \mathrm{SmVar}(\mathbb{C})$ such that there exists an etale map $e : U \to \mathbb{A}^{d_U}_{\mathbb{C}}$. such that $e : U \to e(U)$ is finite etale and $\mathbb{A}^{d_U}_{\mathbb{C}}\backslash e(U) \subset \mathbb{A}^{d_U}_{\mathbb{C}}$ is a divisor Denote $D := \mathbb{A}^{d_U}_{\mathbb{C}}\backslash e(U)$ the complementary closed subset. Since $H^q_{usu}(\mathbb{A}^{d_U}_{\mathbb{C}}, \Omega^{p,an,ne}_{\log}) = 0$ for all $p, q \in \mathbb{Z}$,

$$H^qE_{usu}(\Omega^{p,an,ne}_{\log})(c(\mathbb{Z}(e(U)))) = \partial : H^q_{usu}(e(U), \Omega^{p,an,ne}_{\log}) \xrightarrow{\sim} H^{q+1}_{usu,D}(\mathbb{A}^{d_U}_{\mathbb{C}}, \Omega^{p,an,ne}_{\log})$$

is an isomorphism for all $p, q \in \mathbb{Z}$. By (i),

$$H^qE_{usu}(\Omega^{p,an,ne}_{\log}) \in \mathrm{PSh}(\mathrm{AnSm}(\mathbb{C}))$$

is $\mathbb{A}^1$ invariant for each $p, q \in \mathbb{Z}$. Indeed, for $X \in \mathrm{AnSm}(\mathbb{C})$ connected, take an open cover $X = \cup_{i\in I} D_i$ by open balls $D_i \simeq D(0,1)^{d_X}$, and then we get the isomorphism

$$\begin{aligned} p_X^* : H^qE_{usu}(\Omega^{p,an,ne}_{\log})(X \times \mathbb{A}^1) := H^q_{usu}(X \times \mathbb{A}^1, \Omega^{p,an,ne}_{\log}) \xrightarrow{\sim} \\ \mathbb{H}^q(D_\bullet \times \mathbb{A}^1, \Omega^{p,an,ne}_{\log}) \xrightarrow{\sim} H^q\Gamma(D_\bullet \times \mathbb{A}^1, \Omega^{p,an,ne}_{\log}) \\ \xrightarrow{\sim} H^q\Gamma(D_\bullet, \Omega^{p,an,ne}_{\log}) \xrightarrow{\sim} \mathbb{H}^q(D_\bullet, \Omega^{p,an,ne}_{\log}) \xrightarrow{\sim} H^q_{usu}(X, \Omega^{p,an,ne}_{\log}) =: H^qE_{usu}(\Omega^{p,an,ne}_{\log})(X) \end{aligned}$$

where $p_X : X \times \mathbb{A}^1 \to X$ is the projection, and the second and the fourth map is the isomorphism given by (i). Take a desingularization of the pair $\epsilon = \epsilon_1 : (\widetilde{\mathbb{A}^{d_U}_{\mathbb{C}}}, E) \to (\mathbb{A}^{d_U}_{\mathbb{C}}, D)$, so that $E = \cup_{i=1}^s E_i \subset \widetilde{\mathbb{A}^{d_U}_{\mathbb{C}}}$ is a normal crossing divisor and $\epsilon : \widetilde{\mathbb{A}^{d_U}_{\mathbb{C}}}\backslash E \xrightarrow{\sim} \mathbb{A}^{d_U}_{\mathbb{C}}\backslash D$. We thus get by proposition 1, for each $p, q \in \mathbb{Z}$, $p, q \neq 0$,

$$H^{q+1}E_{usu}(\Omega^{p,an,ne}_{\log})(P_{E_i, \widetilde{\mathbb{A}^{d_U}_{\mathbb{C}}}}) : H^{q+1}_{usu,E_i}(\widetilde{\mathbb{A}^{d_U}_{\mathbb{C}}}, \Omega^{p,an,ne}_{\log}) \xrightarrow{\sim} H^q_{usu}(E_i \times \mathbb{G}_m, \Omega^{p,an,ne}_{\log}).$$

For $p, q \in \mathbb{Z}$, $p, q \neq 0$, $p \geq q$, $q \neq d_U$, we get, using the induction hypothesis, by a finite decreasing induction $\tilde{E}_{iq-1} \subset \tilde{E}^N_{iq-2}, \cdots, E_{i0} \subset E_i$, where

- $E^o_i := E_i\backslash E_{i0} \subset E_i$ is an open subset such that there exists a finite etale map $e_i : E^o_i \to e_i(E^o_i) \subset \mathbb{A}^{d_U-1}$ with $\mathbb{A}^{d_U-1}\backslash e_i(E^o_i) \subset \mathbb{A}^{d_U-1}$ a divisor, $\epsilon_2 : (\tilde{E}_i, \tilde{E}_{i0}) \to (E_i, E_{i0})$ is a desingularization,
- $E^o_{ik} := \tilde{E}_{ik}\backslash E_{ik+1} \subset \tilde{E}_{ik}$ is an open subset such that there exists a finite etale map $e_{ik} : E^o_{ik} \to e_{ik}(E^o_{ik}) \subset \mathbb{A}^{d_U-k}$ with $\mathbb{A}^{d_U-k}\backslash e_{ik}(E^o_{ik}) \subset \mathbb{A}^{d_U-k}$ a divisor, $\epsilon_k : (\tilde{E}^N_{ik}, \tilde{E}_{ik+1}) \to (\tilde{E}_{ik}, E_{ik+1})$ is a desingularization

since $d_U > p \geq q$ ($\Omega^p_Y = 0$ for $p > d_Y$,$Y \in \mathrm{AnSm}(\mathbb{C})$),

$$\begin{aligned} H^q_{usu}(e(U), \Omega^{p,an,ne}_{\log}) = H^{q+1}_{usu,D}(\mathbb{A}^{d_U}_{\mathbb{C}}, \Omega^{p,an,ne}_{\log}) = H^{q+1}_{usu,E}(\widetilde{\mathbb{A}^{d_U}_{\mathbb{C}}}, \Omega^{p,an,ne}_{\log}) \\ \xrightarrow{\sim} \oplus_{i=1}^r H^1_{usu}(\tilde{E}_{iq-1} \times \mathbb{G}_m^{q-1}, \Omega^{p,an,ne}_{\log}) = 0 \end{aligned}$$

since $\mathrm{Pic}(E_{iq-1}) = \mathrm{Pic}(E_i) = 0$ by induction as $\epsilon(E_i) \subset \mathbb{A}^{d_U}$ and $\epsilon_k(\tilde{E}_{ik+1}) \subset \tilde{E}_{ik}$ are hypersurfaces of dimension at least two. We thus have, since $M(U) \simeq M(e(U))$, for each $p, q \in \mathbb{Z}$, $p, q \neq 0$, $p \geq q$, $q \neq d_U$

$$H^q_{usu}(U, \Omega^{p,an,ne}_{\log}) = H^q_{usu}(e(U), \Omega^{p,an,ne}_{\log}) = 0.$$

□

3 Hypersurface De Rham cohomology and its complex periods

We denote $\mathbb{P}^N := \mathbb{P}^N_{\mathbb{C}} := \mathrm{Proj}(\mathbb{C}[z_0, \cdots, z_N])$. Let $X = V(f) \subset \mathbb{P}^N$ be a smooth hypersurface. If $N-1 = 2p$ is even, we consider the decompositions

$$H^{N-1}_{DR}(X) = \mathbb{C}[\Lambda \cap X] \oplus H^{N-1}_{DR,v}(X)\ ,\ H^{N-1}_{\mathrm{sing}}(X^{an}, \mathbb{Q}) = \mathbb{C}[\Lambda \cap X] \oplus H^{N-1}_{\mathrm{sing},v}(X^{an}, \mathbb{Q})$$

of vector spaces, where $\Lambda \subset \mathbb{P}^N$ is a linear subspace of codimension p. If $N-1 = 2p+1$ is even, we set

$$H^{N-1}_{DR,v}(X) := H^{N-1}_{DR}(X)\ ,\ H^{N-1}_{\mathrm{sing},v}(X^{an}, \mathbb{Q}) := H^{N-1}_{\mathrm{sing}}(X^{an}, \mathbb{Q}).$$

Let $X = V(f) \subset \mathbb{P}^N$ be a smooth hypersurface. Denote $j : U := \mathbb{P}^N \backslash X \hookrightarrow \mathbb{P}^N$ the open embedding. We consider the algebraic Hodge module

$$j_{*Hdg}(O_U, F_b) := ((j_*O_U, F), j_*\mathbb{Q}_{U^{an}}, j_*\alpha(U)) \in HM(\mathbb{P}^N),$$

where $F^k j_*O_U = \mathbb{C}[x_1, \cdots x_N][\frac{1}{f^k}]$. Recall for each $p \in \mathbb{Z}$,

$$F^p H^N_{DR}(U) := H^N \iota^{\geq p}(\mathbb{H}^N(\mathbb{P}^N, \Omega^{\bullet \geq p}_{\mathbb{P}^N}(\log X))) = H^N \iota^{\geq p} H^N(\mathbb{P}^N, \Omega^\bullet_{\mathbb{P}^N} \otimes_{O_{\mathbb{P}^N}} F^{\bullet - p} j_*O_U)$$

and

$$F^p H^{N-1}(X) := H^{N-1} \iota^{\geq p}(\mathbb{H}^{N-1}(X, \Omega^{\bullet \geq p}_X)).$$

where

$$\iota^{\geq p} : \Omega^{\bullet \geq p}_{\mathbb{P}^N}(\log X) \hookrightarrow \Omega^\bullet_{\mathbb{P}^N}(\log X)\ ,\ \iota^{\geq p} : \Omega^\bullet_{\mathbb{P}^N} \otimes_{O_{\mathbb{P}^N}} F^{\bullet - p} j_*O_U \hookrightarrow \Omega^\bullet_{\mathbb{P}^N} \otimes_{O_{\mathbb{P}^N}} j_*O_U = j_*\Omega^\bullet_U$$

and $\iota^{\geq p} : \Omega^{\bullet \geq p}_X \hookrightarrow \Omega^\bullet_X$ are the canonical embeddings. Then the map $Res_{X,\mathbb{P}^N}$ of definition 1 induces for each $p \in \mathbb{Z}$, canonical isomorphisms

$$H^N\Omega(Res_{X,\mathbb{P}^N}) : F^{p+1}H^N_{DR}(U) \xrightarrow{\sim} F^p H^{N-1}_{DR}(X)_v\ ,\ H^N \mathrm{Bti}(Res_{X,\mathbb{P}^N}) : H^N_{\mathrm{sing}}(U, \mathbb{Q}) \xrightarrow{\sim} H^{N-1}_{\mathrm{sing}}(X, \mathbb{Q})_v.$$

Lemma 2. *Let $X = V(f) \subset \mathbb{P}^N$ be a smooth hypersurface. Denote $j : U := \mathbb{P}^N \backslash X \hookrightarrow \mathbb{P}^N$ the open embedding. Let $B \subset \mathbb{P}^N$, $\phi : B \simeq D(0,1)^N$ be an open ball such that $\phi(X \cap B) = \{0\} \times D(0,1)^{N-1}$. Then,*

(i) For $q \in \mathbb{Z}$, $q \neq 0$, $\mathbb{H}^q_{usu}(B, (\Omega^\bullet_{\mathbb{P}^N} \otimes_{O_{\mathbb{P}^N}} F^{\bullet - p} j_*O_U)^{an}) = 0$.

(ii) For $q \in \mathbb{Z}$, $q \neq 0$, $H^q_{usu}(B, \Omega^{N-p}_{\mathbb{P}^{N,an}}(\log X)^{\partial = 0}) = 0$.

Proof. (i): Since B is Stein, we have for $r, q \in \mathbb{Z}$, $q \neq 0$,

$$H^q_{usu}(B, (\Omega^r_{\mathbb{P}^N} \otimes_{O_{\mathbb{P}^N}} F^{r-p} j_*O_U)^{an}) = 0.$$

On the other hand the De Rham complex of $(j_*O_{B\backslash X}, F)$ is acyclic (see e.g. [6] corollary 18.7)
(ii): Since open balls form a basis of the topology of $\mathbb{P}^{N,an}$, we have

$$a_{usu}\iota : a_{usu}(j_*\Omega^{N-p}_{U^{an},\log} \oplus \partial\Omega^{N-p+1}_{\mathbb{P}^{N,an}}) \xrightarrow{\sim} a_{usu}\Omega^{N-p}_{\mathbb{P}^{N,an}}(\log X)^{\partial=0}$$

where

$$\iota : j_*\Omega^{N-p}_{U^{an},\log} \oplus \partial\Omega^{N-p+1}_{\mathbb{P}^{N,an}} \hookrightarrow \Omega^{N-p}_{\mathbb{P}^{N,an}}(\log X)^{\partial=0}$$

is the canonical embedding. Now,

- Let $B = D(0,1)^N \in \mathrm{AnSm}(\mathbb{C})$ be an open ball. Let $\alpha \in H^q(B, \partial\Omega^{N-p+1}_{\mathbb{P}^{N,an}})$, $q \in \mathbb{Z}$, $q \neq 0$. Consider a sequence of sub-balls

$$l_n : B_n := D(0, r_n)^N \hookrightarrow B := D(0,1)^N, n \in \mathbb{N}, \text{ with } r_n \to 0 \text{ when } n \to \infty.$$

Since $(B_n := D(0, r_n))_{n\in\mathbb{N}}$ form a basis of neighborhood of $0 \in B$ and since $\partial\Omega^{N-p+1}_{\mathbb{P}^{N,an}}$ is a single presheaf, there exists $n := n_\alpha \in \mathbb{N}$ (depending on α) such that

$$l_n^*\alpha = 0 \in H^q(B_n, \partial\Omega^{N-p+1}_{\mathbb{P}^{N,an}}).$$

But it follows immediately from the definition of the the presheaf $\partial\Omega^{N-p+1}_{\mathbb{P}^{N,an}}$ considering the isomorphism

$$\phi_n : B_n \xrightarrow{\sim} B,\ \phi_n(z) := r_n^{-1} z$$

so that $(\phi_n)^{-1} \circ l_n = r_n I$, that

$$l_n^*\alpha = l_n^*\phi_n^{-1*}\phi_n^*\alpha = r_n\phi_n^*\alpha \in H^q(B_n, \partial\Omega^{N-p+1}_{\mathbb{P}^{N,an}}).$$

Hence $\phi_n^*\alpha = 0$. Since ϕ_n is an isomorphism, we get $\alpha = 0$.

- we have for $q \in \mathbb{Z}$, $q \neq 0$, $H^q_{usu}(B, \Omega^{N-p}_{B^{an},\log}) = 0$ by proposition 2 (i). Hence

$$H^q E_{usu}(\Omega^{an}_{\log})(c(\mathbb{Z}(B))) = \partial : H^q_{usu}(B \cap U, \Omega^{N-p}_{U^{an},\log}) \xrightarrow{\sim} H^{q+1}_{usu,B\cap X}(B, \Omega^{N-p}_{B^{an},\log})$$

is an isomorphism. Proposition 2 (i) also gives, using the spectral sequence associated to a double complex, for $q \in \mathbb{Z}$,

$$H^q_{usu}(B, \underline{\mathrm{sing}}_{D^*}\Omega^{N-p}_{B^{an},\log}) = 0.$$

By a finite decreasing induction on N, using purity for complex analytic motives, we get for $q \in \mathbb{Z}$, $q \neq 0$,

$$H^q_{usu}(B \cap U, \Omega^{N-p}_{U^{an},\log}) = H^q_{usu}(B \cap U, \underline{\mathrm{sing}}_{D^*}\Omega^{N-p}_{U^{an},\log}) = 0.$$

□

We have by the local acyclicity of the De Rham complex of (j_*O_U, F) a canonical map :

Definition 3. *Let $X = V(f) \subset \mathbb{P}^N$ be a smooth hypersurface. Denote $j : U := \mathbb{P}^N \backslash X \hookrightarrow \mathbb{P}^N$ the open embedding. Let $\mathbb{P}^{N,an} = \cup_{i=1}^n B_i$ an open cover, with $\phi_i : \mathbb{B}_i \simeq D(0,1)^N$ open balls such that $\phi_i(X \cap B_i) = \{0\} \times D(0,1)^{N-1}$. Denote, for $I \subset [1, \ldots, n]$, $l_I : B_I \hookrightarrow \mathbb{P}^N$ the open embeddings, with $B_I := \cap_{i\in I} B_i$. Since, for each $I \subset [1, \ldots, n]$ and $p \in \mathbb{Z}$,*

$$l_I^*\iota_\partial : l_I^*\Omega^{N-p}_{\mathbb{P}^{N,an}}(\log X)^{\partial=0}[-p] \hookrightarrow l_I^*(\Omega^\bullet_{\mathbb{P}^N} \otimes_{O_{\mathbb{P}^N}} F^{\bullet-p}j_*O_U)^{an}$$

is an quasi-isomorphism (see e.g. [6] corollary 18.7), the spectral sequence of the filtration on the total complex of a bi-complex gives, for each $p \in \mathbb{Z}$, a canonical isomorphism, where the first and last map are isomorphism by lemma 2(i) and (ii) respectively,

$$\begin{aligned} B_\partial : F^pH_{DR}(U) := H^N\iota^{\geq p}\mathbb{H}^N_{zar}(\mathbb{P}^N, \Omega^\bullet_{\mathbb{P}^N} \otimes_{O_{\mathbb{P}^N}} F^{\bullet-p}j_*O_U) \\ \xrightarrow{H^N((l_i^*=\Omega(l_i))_{1\leq i\leq n})} H^N\iota^{\geq p}H^N\Gamma(B_\bullet, (\Omega^\bullet_{\mathbb{P}^N} \otimes_{O_{\mathbb{P}^N}} F^{\bullet-p}j_*O_U)^{an}) \\ \to H^p\Gamma(B_\bullet, \Omega^{N-p}_{\mathbb{P}^{N,an}}(\log X)^{\partial=0}) \xrightarrow{H^N((l_i^*=\Omega(l_i))_{1\leq i\leq n})^{-1}} H^p_{usu}(\mathbb{P}^N, \Omega^{N-p}_{\mathbb{P}^{N,an}}(\log X)^{\partial=0}), \end{aligned}$$

given by, for $w \in \Gamma(U, \Omega^N_{\mathbb{P}^N})$, inductively by a p step induction where p is the order of the pole of w at X,

- *taking for each $1 \leq i \leq n$, $l_i^*w = \partial w_{1i}$ since $\partial w = 0$ for dimensional reason and by exactness of $l_i^*\iota_\partial$,*

- *we then have for each* $1 \leq i, j \leq n$, *considering* $w_{1ij} := l_{ij}^*(w_{1i} - w_{1j})$, $w_{1ij} = \partial w_{2ij}$ *since*

$$\partial w_{1ij} = l_{ij}^*(\partial w_{1i} - \partial w_{1j}) = l_{ij}^*(l_i^* w - l_j^* w) = 0$$

and by exactness of $l_{ij}^* \iota_\partial$,

- *at the final* p *step,* $B(w) := w_{pI} \in H^N(B_\bullet, \Omega^{N-p}_{\mathbb{P}^{N,an}}(\log X)^{\partial=0})$, $cardI = p$.

Let $X = V(f) \subset \mathbb{P}^N$ be a smooth hypersurface. Denote $U := \mathbb{P}^N \backslash X$. For $l \in \mathbb{Z}$, we consider

- the sequence of embeddings in $C(U^{et})$

$$OL_U^l : \Omega^l_{U,\log}[-l] \xrightarrow{\widehat{OL_U^l}[-l]} \Omega^l_{\mathbb{P}^N}(\log X)^{\partial=0}[-l] \xrightarrow{\iota_\partial} \Omega^\bullet_{\mathbb{P}^N} \otimes_{O_{\mathbb{P}^N}} F^{\bullet-p} j_* O_U \xrightarrow{\iota^{\geq l}} \Omega^\bullet_U$$

- and the sequence of embeddings in $C(U^{an})$

$$OL_{U^{an}}^l : \Omega^l_{U^{an},\log}[-l] \xrightarrow{\widehat{OL_{U^{an}}^l}[-l]} \Omega^l_{\mathbb{P}^{N,an}}(\log X)^{\partial=0}[-l] \xrightarrow{\iota_\partial} (\Omega^\bullet_{\mathbb{P}^N} \otimes_{O_{\mathbb{P}^N}} F^{\bullet-p} j_* O_U)^{an} \xrightarrow{\iota^{\geq l}} \Omega^\bullet_{U^{an}},$$

where

$$\iota_\partial : \Omega^l_{\mathbb{P}^{N,an}}(\log X)^{\partial=0}[-l] \hookrightarrow (\Omega^\bullet_{\mathbb{P}^N} \otimes_{O_{\mathbb{P}^N}} F^{\bullet-p} j_* O_U)^{an}$$

is an equivalence usu local (see [6]).

The main result of this section is the following :

Proposition 3. *Let* $X = V(f) \subset \mathbb{P}^N$ *be a smooth projective hypersurface. Denote* $U := \mathbb{P}^N \backslash X$. *Let* $w \in F^{p+1} H^N_{DR}(U)$. *Then the following assertions are equivalent*

(i) $ev(U)(w) \in H^N_{\text{sing}}(U^{an}, 2i\pi\mathbb{Q})$,

(ii) $B_\partial(w) \in \widehat{OL_{U^{an}}^{p+1}}(H^{p+1}_{usu}(U^{an}, \Omega^{N-p}_{U^{an},\log}))$ *(see definition 3),*

(iii) $w \in OL_{U^{an}}^{N-p}(H^{p+1}_{usu}(U^{an}, \Omega^{N-p}_{U^{an},\log})) = OL_{U^{an}}(H^{p+1}_{usu}(U^{an}, \Omega^{N-p}_{U^{an},\log}))$.

Proof. (i) implies (ii): Let $\mathbb{P}^{N,an} = \cup_{i=1}^n B_i$ an open cover, with $\phi_i : \mathbb{B}_i \simeq D(0,1)^N$ open balls such that $\phi_i(X \cap B_i) = \{0\} \times D(0,1)^{N-1}$. Denote, for $I \subset [1, \ldots, n]$, $l_I : B_I \hookrightarrow \mathbb{P}^N$ the open embeddings, with $B_I := \cap_{i \in I} B_i$. Then

$$B_\partial(w) := (w_{p+1,I}) \in H^{p+1}(B_\bullet, \Omega^{N-p}_{\mathbb{P}^{N,an}}(\log X)^{\partial=0}).$$

We have, for each $I \subset [1, \ldots n]$, a decomposition

$$w_{p+1,I} = \frac{a_I}{z_1} dz_1 \wedge w'_I + w''_I \in \Gamma(B_I, \Omega^{N-p}_{\mathbb{P}^{N,an}}(\log X)^{\partial=0})$$

with $w'_I \in \Gamma(B_I, \Omega^{N-p-1}_{\mathbb{P}^N})^{\partial=0}$ and $w''_I \in \Gamma(B_I, \Omega^{N-p}_{\mathbb{P}^N})^{\partial=0}$. Consider the evaluation map

$$ev(U)^\bullet_\bullet : \Gamma(B_\bullet \backslash X, \Omega^\bullet_{\mathbb{P}^N}) \to C^\bullet_{\text{diff}}(B_\bullet \backslash X^{an}, \mathbb{C}),$$

which is a morphism of bi-complexes. On the other hand, we have the following canonical map

$$BC_\partial : H^p\Gamma(B_\bullet \backslash X^{an}, C^{N-p,\partial}_{U^{an},\text{diff}}) \xrightarrow{\Gamma(B_\bullet \backslash X^{an}, \iota_\partial)} = H^N\Gamma(B_\bullet \backslash X^{an}, C^{\bullet \geq N-p}_{U^{an},\text{diff}}) \xrightarrow{\Gamma(B_\bullet \backslash X^{an}, \iota^{\geq N-p})}$$

$$= H^N\Gamma(B_\bullet \backslash X^{an}, C^\bullet_{U^{an},\text{diff}}) \xrightarrow{(H^N(l_i^* =: C^\bullet_{U^{an},\text{diff}}(l_i))_{1 \leq i \leq n})^{-1}} H^N_{\text{sing}}(U^{an}, \mathbb{Q}) = H^N\Gamma(U^{an}, C^\bullet_{U^{an},\text{diff}}),$$

where $H^N(l_i^*)$ is an isomorphism by the acyclicity of $C^{\bullet \geq N-p}_{U^{an},\text{diff}}(B_I \backslash X^{an})$ for each $I \subset [1, \cdots n]$, and

$$C^\bullet_{U^{an},\text{diff}} \hookrightarrow C^\bullet_{U^{an},\text{sing}}, \text{ for } V \subset U^{an}, \ C^\bullet_{\text{diff}}(V) \hookrightarrow C^\bullet_{\text{sing}}(V)$$

is the canonical quasi-isomorphism in $C(U^{an})$. We then have the following commutative diagram

$$\begin{array}{ccc}
H^N\iota^{\geq p}H^N\Gamma(\mathbb{P}^N,\Omega^\bullet_{\mathbb{P}^N}\otimes_{O_{\mathbb{P}^N}}F^{\bullet-p}j_*O_U) & & \\
\downarrow{\scriptstyle H^N((l_i^*=\Omega(l_i))_{1\leq i\leq n})} & \searrow^{B_\partial} & \\
H^N\iota^{\geq p}H^N\Gamma(B_\bullet,(\Omega^\bullet_{\mathbb{P}^N}\otimes_{O_{\mathbb{P}^N}}F^{\bullet-p}j_*O_U)^{an}) & \xrightarrow{B_\partial} & H^p\Gamma(B_\bullet,\Omega^{N-p}_{\mathbb{P}^N,an}(\log X)^{\partial=0}) \\
\downarrow{\scriptstyle ev(U)} & & \downarrow{\scriptstyle ev(U)^\bullet_\bullet} \\
H^N_{\mathrm{sing}}(U^{an},\mathbb{C})=H^N\Gamma(U^{an},C^\bullet_{U^{an},\mathrm{diff}}\otimes\mathbb{C}) & \xleftarrow[BC_\partial\otimes\mathbb{C}]{} & H^p\Gamma(B_\bullet\backslash X^{an},C^{N-p,\partial=0}_{\mathrm{diff}}\otimes\mathbb{C})
\end{array}$$

Since $ev(U)$ is injective by the strictness of the Hodge filtration and since B_∂ is an isomorphism, $ev(U)^\bullet_\bullet$ is injective and

$$\begin{aligned}(BC_\partial\otimes\mathbb{C})' := (BC_\partial\otimes\mathbb{C})_{|ev(U)^\bullet_\bullet(H^p\Gamma(B_\bullet,\Omega^{N-p}_{\mathbb{P}^N,an}(\log X)^{\partial=0}))}:\\ ev(U)^\bullet_\bullet(H^p\Gamma(B_\bullet,\Omega^{N-p}_{\mathbb{P}^N,an}(\log X)^{\partial=0}))\to H^N_{\mathrm{sing}}(U^{an},\mathbb{C})\end{aligned}$$

is injective. For $\gamma_I\in C^{\mathrm{sing}}_{N-p}(B_I\backslash X^{an},\mathbb{Q})^{\partial=0}$ where $p_1\circ\phi_I(\gamma_I)\subset D(0,1)^*$ is a loop around 0, we have by the above diagram

$$ev(U)(w)(\gamma_I)=ev(U)(w_{p+1,I})(\gamma_I)=a_I.$$

Hence if $ev(U)(w)\in H^N_{\mathrm{sing}}(U^{an},2i\pi\mathbb{Q})$, then $a_I\in\mathbb{Q}$, using the injectivity of $(BC_\partial\otimes\mathbb{C})'$.
(ii) is equivalent to (iii): By definition $B_\partial(w)=(H^p\iota_\partial)^{-1}(w)$. On the other hand by definition

$$\iota^{\geq p}\circ\iota_\partial\circ\widehat{OL^{N-p}_{U^{an}}}=OL^{N-p}_{U^{an}}.$$

(iii) implies (i): See [2]. □

4 Complex analytic logarithmic de Rham classes

We recall (see section 2) the morphism of site given by the analytification functor

$$\mathrm{An}:\mathrm{AnSm}(\mathbb{C})\to\mathrm{SmVar}(\mathbb{C}),\ \ X\mapsto\mathrm{An}(X):=X^{an},\ (f:X'\to X)\mapsto\mathrm{An}(f):=f^{an}.$$

and for $X\in\mathrm{SmVar}(\mathbb{C})$, the following commutative diagram

$$\begin{array}{ccc}
\mathrm{AnSm}(\mathbb{C}) & \xrightarrow{\mathrm{An}} & \mathrm{SmVar}(\mathbb{C}) \\
\downarrow{\scriptstyle o_{X^{an}}} & & \downarrow{\scriptstyle o_X} \\
X^{an,et} & \xrightarrow{\mathrm{an}_X} & X^{et}
\end{array}.$$

Consider the commutative diagram in $C(\mathrm{AnSp}(\mathbb{C}))$

$$\begin{array}{ccc}
\Omega^{\bullet,an}_{\log}\cap(\mathrm{An}^*\,\partial\,\mathrm{An}_*\,\Omega^{\bullet-1,an}) & \xrightarrow[OL^{an}]{} & \mathrm{An}^*\,\partial\,\mathrm{An}_*\,\Omega^{\bullet-1,an} \\
\downarrow{\scriptstyle e} & & \downarrow{\scriptstyle e} \\
\Omega^{\bullet,an}_{\log} & \xrightarrow[OL^{an}]{} & \Omega^{\bullet,an} \\
\downarrow{\scriptstyle q} & & \downarrow{\scriptstyle q} \\
\Omega^{\bullet,an,ne}_{\log}:=\Omega^{\bullet,an}_{\log}/(\Omega^{\bullet,an}_{\log}\cap(\mathrm{An}^*\,\partial\,\mathrm{An}_*\,\Omega^{\bullet-1,an})) & \xrightarrow{\bar{OL}^{an}} & \Omega^{\bullet,an}/(\mathrm{An}^*\,\partial\,\mathrm{An}_*\,\Omega^{\bullet-1,an})
\end{array},$$

whose columns are exact with e injective and q surjective (see definition 2). Let $X \in \mathrm{SmVar}(\mathbb{C})$. Then, for $k \in \mathbb{Z}$, we have the commutative diagram

$$\begin{array}{ccc}
\mathbb{H}^k_{usu}(X^{an}, \Omega^{\bullet,an}_{\log} \cap (\mathrm{An}^* \partial \mathrm{An}_* \Omega^{\bullet-1,an})) & \xrightarrow{H^k OL^{an}} & \mathbb{H}^k_{usu}(X^{an}, \mathrm{An}^* \partial \mathrm{An}_* \Omega^{\bullet-1,an}) \\
\downarrow{\scriptstyle H^k(e)} & & \downarrow{\scriptstyle H^k(e)} \\
\mathbb{H}^k_{usu}(X^{an}, \Omega^{\bullet,an}_{\log}) & \xrightarrow{H^k OL^{an}} & H^k_{DR}(X) = \mathbb{H}^k_{usu}(X^{an}, \Omega^{\bullet,an}) \\
\downarrow{\scriptstyle H^k(q)} & & \downarrow{\scriptstyle H^k(q)} \\
\mathbb{H}^k_{usu}(X^{an}, \Omega^{\bullet,an,ne}_{\log}) & \xrightarrow{H^k \bar{O}L^{an}} & \mathbb{H}^k_{usu}(X^{an}, \Omega^{\bullet,an}/(\mathrm{An}^* \partial \mathrm{An}_* \Omega^{\bullet-1,an}))
\end{array}$$

whose columns are exact. Let $Z \subset X$ a closed subset. Then, for $k \in \mathbb{Z}$, we also have the commutative diagram

$$\begin{array}{ccc}
\mathbb{H}^k_{usu,Z}(X^{an}, \Omega^{\bullet,an}_{\log} \cap (\mathrm{An}^* \partial \mathrm{An}_* \Omega^{\bullet-1,an})) & \xrightarrow{H^k OL^{an}} & \mathbb{H}^k_{usu,Z}(X^{an}, \mathrm{An}^* \partial \mathrm{An}_* \Omega^{\bullet-1,an}) \\
\downarrow{\scriptstyle H^k(e)} & & \downarrow{\scriptstyle H^k(e)} \\
\mathbb{H}^k_{usu,Z}(X^{an}, \Omega^{\bullet,an}_{\log}) & \xrightarrow{H^k OL^{an}} & H^k_{DR,Z}(X) = \mathbb{H}^k_{usu}(X^{an}, \Omega^{\bullet,an}) \\
\downarrow{\scriptstyle H^k(q)} & & \downarrow{\scriptstyle H^k(q)} \\
\mathbb{H}^k_{usu,Z}(X^{an}, \Omega^{\bullet,an,ne}_{\log}) & \xrightarrow{H^k \bar{O}L^{an}} & \mathbb{H}^k_{usu,Z}(X^{an}, \Omega^{\bullet,an}/(\mathrm{An}^* \partial \mathrm{An}_* \Omega^{\bullet-1,an}))
\end{array}$$

whose columns are exact.

Proposition 4. *Let $X \in \mathrm{PSmVar}(\mathbb{C})$. Consider the morphism $\mathrm{an}_X : X^{an} \to X$ in* RTop *given by analytical functor.*

(i) The analytic De Rham cohomology class of an algebraic cycle is logarithmic and is of type (d,d), that is, for $Z \in \mathcal{Z}^d(X)$

$$[Z] := H^{2d}\Omega(\gamma_Z^\vee)([Z]) \subset H^{2d} OL_{X^{an}}(H^d_{usu}(X^{an}, \Omega^d_{X^{an},\log})) \subset H^{2d}_{DR}(X^{an}) = H^{2d}_{DR}(X).$$

(ii) Conversely, any $w \in H^{2d} OL_{X^{an}}(H^d_{usu}(X^{an}, \Omega^d_{X^{an},\log}))$ is the class of an algebraic cycle $Z \in \mathcal{Z}^d(X) \otimes \mathbb{Q}$, i.e. $w = [Z]$.

(iii) We have $H^j OL_{X^{an}}(H^{j-l}_{usu}(X^{an}, \Omega^l_{X^{an},\log})) = 0$ for $j, l \in \mathbb{Z}$ such that $2l > j$.

Proof. Consider, for $j, l \in \mathbb{Z}$ and $X \in \mathrm{SmVar}(\mathbb{C})$,

$$L^{l,j-l}(X) := H^j OL_{X^{an}}(H^{j-l}_{usu}(X^{an}, \Omega^l_{X^{an},\log})) := H^j OL^{an}(H^{j-l}\,\mathrm{Hom}(\mathrm{An}^*\,\mathbb{Z}(X), E^\bullet_{usu}(\Omega^{l,an}_{\log}))).$$

Consider also for $j, l \in \mathbb{Z}$, $X \in \mathrm{SmVar}(\mathbb{C})$ and $Z \subset X$ a closed subset,

$$L^{l,j-l}_Z(X) := H^j OL_{X^{an}}(H^{j-l}_{usu,Z}(X^{an}, \Omega^l_{X^{an},\log})) := H^j OL^{an}(H^{j-l}\,\mathrm{Hom}(\mathrm{An}^*\,\mathbb{Z}(X, X\backslash Z), E^\bullet_{usu}(\Omega^{l,an}_{\log}))).$$

Consider, for $j, l \in \mathbb{Z}$ and $X \in \mathrm{SmVar}(\mathbb{C})$,

$$L^{l,j-l,ne}(X) := H^j(\bar{O}L^{an})(H^{j-l}\,\mathrm{Hom}(\mathrm{An}^*\,\mathbb{Z}(X), E^\bullet_{usu}(\Omega^{l,an,ne}_{\log}))).$$

Consider also for $j, l \in \mathbb{Z}$, $X \in \mathrm{SmVar}(\mathbb{C})$ and $Z \subset X$ a closed subset,

$$L^{l,j-l,ne}_Z(X) := H^j(\bar{O}L^{an})(H^{j-l}\,\mathrm{Hom}(\mathrm{An}^*\,\mathbb{Z}(X, X\backslash Z), E^\bullet_{usu}(\Omega^{l,an,ne}_{\log}))).$$

For each $l, j \in \mathbb{Z}$, the presheaf $H^{j-l}E_{usu}(\Omega^{l,an,ne}_{\log}) \in \mathrm{PSh}(\mathrm{AnSm}(\mathbb{C}))$ is $\mathbb{A}^1$ invariant by proposition 2(i). Indeed, for $Y \in \mathrm{AnSm}(\mathbb{C})$ connected, take an open cover $Y = \cup_{i\in I} D_i$ by open balls $D_i \simeq D(0,1)^{d_Y}$, and then we get the isomorphism

$$\begin{aligned} p_Y^* : H^{j-l}E_{usu}(\Omega^{l,an,ne}_{\log})(Y \times \mathbb{A}^1) := H^{j-l}_{usu}(Y \times \mathbb{A}^1, \Omega^{l,an,ne}_{\log}) \xrightarrow{\sim} \\ \mathbb{H}^{j-l}(D_\bullet \times \mathbb{A}^1, \Omega^{l,an,ne}_{\log}) \xrightarrow{\sim} H^{j-l}\Gamma(D_\bullet \times \mathbb{A}^1, \Omega^{l,an,ne}_{\log}) \\ \xrightarrow{\sim} H^{j-l}\Gamma(D_\bullet, \Omega^{l,an,ne}_{\log}) \xrightarrow{\sim} \mathbb{H}^{j-l}(D_\bullet, \Omega^{l,an,ne}_{\log}) \xrightarrow{\sim} H^{j-l}_{usu}(Y, \Omega^{l,an,ne}_{\log}) =: H^{j-l}E_{usu}(\Omega^{l,an,ne}_{\log})(Y) \end{aligned}$$

where $p_Y : X \times \mathbb{A}^1 \to Y$ is the projection, and the second and the fourth map is the isomorphism given by (i). This gives in particular, for $X \in \mathrm{SmVar}(\mathbb{C})$ and $Z \subset X$ a smooth subvariety of (pure) codimension d, by proposition 1 an isomorphism

$$E_{usu}(\Omega^{l,an,ne}_{\log})(P_{Z,X}) : L^{l,j-l,ne}_Z(X) \xrightarrow{\sim} L^{l,j-l,ne}_Z(N_{Z/X}) \xrightarrow{\sim} L^{l-d,j-l-d,ne}(Z).$$

(i):Similar to the proof of [2] theorem 2.
(ii): Let $X \in \mathrm{PSmVar}(\mathbb{C})$ and

$$\begin{aligned} \alpha = H^{2d}OL_{X^{an}}(\alpha) \in L^{d,d}(X) : &= H^{2d}OL_{X^{an}}(H^d_{usu}(X^{an}, \Omega^d_{X^{an},\log})) \\ &= H^{2d}OL^{an}(H^d \mathrm{Hom}(\mathrm{An}^* \mathbb{Z}(X), E^\bullet_{usu}(\Omega^{d,an}_{\log}))). \end{aligned}$$

Then,

$$\begin{aligned} H^{2d}(q)(\alpha) = H^{2d}(q) \circ H^{2d}OL_{X^{an}}(\alpha) = H^{2d}\bar{O}L^{an} \circ H^{2d}(q)(\alpha) \\ \in L^{d,d,ne}(X) := H^{2d}\bar{O}L^{an}(H^d \mathrm{Hom}(\mathrm{An}^* \mathbb{Z}(X), E^\bullet_{usu}(\Omega^{d,an,ne}_{\log}))). \end{aligned}$$

Up to split X by its connected components, we may assume that X is connected. The case $d = d_X$ is trivial. So let assume $d \neq d_X$. Let $U \subset X$ an open subset such that there exists an etale map $e : U \to \mathbb{A}^{d_U}_{\mathbb{C}}$ such that $e : U \to e(U)$ is finite etale and $\mathbb{A}^{d_U}_{\mathbb{C}} \backslash e(U) \subset \mathbb{A}^{d_U}_{\mathbb{C}}$ is a divisor Denote $j : U \hookrightarrow X$ the open embedding. By proposition 2(ii), we have

$$j^* H^{2d}(q)(\alpha) = 0 \in H^d_{usu}(U^{an}, \Omega^{d,an,ne}_{\log}) = 0.$$

Considering a divisor $X \backslash U \subset D \subset X$, we get

$$H^{2d}(q)(\alpha) = H^d E^\bullet_{usu}(\Omega^{d,an,ne}_{\log})(\gamma_D^\vee)(\alpha'),\ \alpha' \in L^{l,j-l,ne}_D(X).$$

Denote $D^o \subset D$ its smooth locus and $l : X^o \hookrightarrow X$ a Zariski open subset such that $X^o \cap D = D^o$. We then have

$$l^* \alpha' \in L^{d,d}_{D^o}(X^o) = L^{d-1,d-1}(D^o)$$

We repeat this procedure with each connected components of D^o instead of X. By a finite induction of d steps, we get

$$H^{2d}(q)(\alpha) = H^d E^\bullet_{usu}(\Omega^{d,an,ne}_{\log})(\gamma_Z^\vee)(\alpha'),\ \alpha' \in L^{d,d,ne}_Z(X) = L^{0,0,ne}(Z^o).$$

with $Z := D_d \subset \cdots \subset D \subset X$ a pure codimension d (Zariski) closed subset, thus

$$\alpha = \sum_i n_i [Z_i] + \beta \in H^{2d}_{DR}(X),\ \beta \in H^{2d}(e)(\mathbb{H}^{2d}_{usu}(X^{an}, \mathrm{An}^* \partial \mathrm{An}_* \Omega^{\bullet-1,an}))$$

where $n_i \in \mathbb{Q}$ and $(Z_i)_{1\leq i\leq t} \subset Z$ are the irreducible components of Z. But since X is projective, the E_1 degenerescence of the spectral sequence associated to the trivial filtration of the De Rham complex shows that $H^{2d}(e)(\mathbb{H}^{2d}_{usu}(X^{an}, \mathrm{An}^* \partial \mathrm{An}_* \Omega^{\bullet-1,an})) = 0$. Hence,

$$\alpha = \sum_i n_i [Z_i] \in H^{2d}_{DR}(X).$$

(iii): Let $j < 2l$. Let $X \in \mathrm{PSmVar}(\mathbb{C})$ and $w \in L^{l,j-l}(X)$. By the proof of (ii) there exists $Z \subset X$ a closed subset of pure codimension $j-l$ such that

$$H^j(q)(w) = H^{j-l}E^{\bullet}_{usu}(\Omega^{l,an,ne}_{\log})(\gamma_Z^{\vee})(w'),\ w' \in L^{l,j-l,ne}_Z(X).$$

For $Z' \subset X$ a closed subset of pure codimension c, consider a desingularisation $\epsilon : \tilde{Z}' \to Z'$ of Z' and denote $n : \tilde{Z}' \xrightarrow{\epsilon} Z' \subset X$. The morphism in $\mathrm{DA}(k)$

$$G_{Z',X} : M(X) \xrightarrow{D(\mathbb{Z}(n))} M(\tilde{Z}')(c)[2c] \xrightarrow{\mathbb{Z}(\epsilon)} M(Z')(c)[2c]$$

where $D : \mathrm{Hom}_{\mathrm{DA}(k)}(M_c(\tilde{Z}'), M_c(X)) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{DA}(k)}(M(X), M(\tilde{Z}')(c)[2c])$ is the duality isomorphism from the six functor formalism (moving lemma of Suzlin and Voevodsky) and $\mathbb{Z}(n) := \mathrm{ad}(n_!, n^!)(a_X^!\mathbb{Z})$, is given by a morphism in $C(\mathrm{SmVar}(k))$

$$\hat{G}_{Z',X} : \mathbb{Z}_{tr}(X) \to E_{et}(C_*\mathbb{Z}_{tr}(Z'))(c)[2c].$$

Let $l : X^o \hookrightarrow X$ be an open embedding such that $Z^o := Z \cap X^o$ is the smooth locus of Z with good reduction. We then have the following commutative diagram of abelian groups

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^{j-l}_{usu,Z}(X^{an}, \Omega^{l,an,ne}_{\log}) & \xrightarrow{l^*} & H^{j-l}_{usu,Z^o}(X^{o,an}, \Omega^{l,an,ne}_{\log}) & \xrightarrow{\partial} & H^{j-l+1}_{usu,Z\setminus Z^o}(X^{an}, \Omega^{l,an,ne}_{\log}) & \longrightarrow & \cdots \\
 & & \uparrow{\scriptstyle \Omega(\hat{G}_{Z,X})} & & \uparrow{\scriptstyle \Omega(P_{Z^o,X^o})} & & \uparrow{\scriptstyle \Omega(\hat{G}_{Z,X})} & & \\
0 & \longrightarrow & H^0_{usu}(Z^{an}, \Omega^{2l-j,an,ne}_{\log}) & \xrightarrow{l*} & H^0_{usu}(Z^{o,an}, \Omega^{2l-j,an,ne}_{\log}) & \xrightarrow{\partial} & H^1_{usu,Z\setminus Z^o}(Z^{an}, \Omega^{2l-j,an,ne}_{\log}) & \longrightarrow & \cdots
\end{array}$$

whose rows are exact sequences. Consider

$$l^*w' = \Omega(P_{Z^o,X^o})(w^o) \in H^{j-l}_{usu,Z^o}(X^{o,an}, \Omega^{l,an,ne}_{\log}),\ w^o \in H^0_{et}(Z^{o,an}, \Omega^{2l-j,an,ne}_{\log}).$$

Since $\partial l^*w' = 0 \in H^{j-l+1}_{usu,Z\setminus Z^o}(X^{an}, \Omega^{l,an,ne}_{\log})$, we get, considering the smooth locus with good reduction $(Z\setminus Z^o)^o \subset Z\setminus Z^o$ and $X^{oo} \subset X$ is an open subset such that $X \cap (Z\setminus Z^o)^o = X^{oo} \cap (Z\setminus Z^o)$,

$$\partial w^o = 0 \in H^1_{usu,Z\setminus Z^o}(Z^{an}, \Omega^{2l-j,an,ne}_{\log}),$$

since $H^1_{usu,(Z\setminus Z^o)\setminus(Z\setminus Z^o)^o}(Z^{an}, \Omega^{2l-j,an,ne}_{\log}) = 0$ for dimension reasons, that is

$$H^j(q)(w) = \Omega(\hat{G}_{Z,X})(w'), \text{ with } w' \in H^0_{et}(Z^{an}, \Omega^{2l-j,an,ne}_{\log}).$$

Hence $H^j(q)(w) = 0$ since $H^0(Z'^{an}, \Omega^k_{\log}) = 0$ for all $k > 0$ and all $Z' \in \mathrm{PVar}(\mathbb{C})$. But since X is projective, the E_1 degenerescence of the spectral sequence associated to the trivial filtration of the De Rham complex shows that $H^j(e)(\mathbb{H}^j_{usu}(X^{an}, \mathrm{An}^*\,\partial\,\mathrm{An}_*\,\Omega^{\bullet-1,an})) = 0$. Hence $w = 0$. □

5 Hodge Conjecture for complex smooth projective hypersurfaces

We now state and prove using the result of section 3 and section 4 the main result of this article :

Theorem 1. *Let $X = V(f) \subset \mathbb{P}^N_{\mathbb{C}}$ be a smooth projective hypersurface, with $N = 2p+1$ odd. Let $\lambda \in F^pH^{2p}(X^{an}, \mathbb{Q})$. Then there exists an algebraic cycle $Z \in \mathcal{Z}^p(X)$ such that $\lambda = [Z]$.*

Proof. Consider $U := \mathbb{P}^N_{\mathbb{C}} \backslash X$. Consider $Res_{X,\mathbb{P}^N} : M(X) \to M(U)[-1]$ the canonical map given in definition 1. We the have the commuative diagram of abelian group

$$\begin{array}{ccc} H^N_{DR}(U) & \xrightarrow{H^N\Omega(Res_{X,\mathbb{P}^N})} & H^{N-1}_{DR}(X)_v \\ {\scriptstyle H^N\alpha(U^{an})=ev(U)^{-1}}\uparrow & & \uparrow{\scriptstyle H^{N-1}\alpha(X^{an})=ev(X)^{-1}} \\ H^N_{\text{sing}}(U^{an},\mathbb{C}) & \xrightarrow{(H^N \text{ Bti}(Res_{X,\mathbb{P}^N}))\otimes_{\mathbb{Q}}\mathbb{C}} & H^{N-1}_{\text{sing}}(X^{an},\mathbb{C})_v \end{array} .$$

whose arrows are isomorphisms. Denote $w(\lambda) := H^{N-1}\alpha(X^{an})(\lambda) \in F^pH^{2p}_{DR}(X)$. We have

$$\lambda = H^{N-1}\,\text{Bti}(Res_{X,\mathbb{P}^N})(\alpha), \text{ with } \alpha \in H^N_{\text{sing}}(U^{an},\mathbb{Q}).$$

Then

$$w(\lambda) = H^{N-1}\Omega(Res_{X,\mathbb{P}^N})(w(\alpha)), \text{ with } w(\alpha) := H^N\alpha(U^{an})(\alpha) \in F^{p+1}H^N_{DR}(U).$$

Since $ev(U)(w(\alpha)) = \alpha \in H^N_{\text{sing}}(U^{an}, 2i\pi\mathbb{Q})$, we have by proposition 3,

$$w(\alpha) \in H^N OL_{U^{an}}(H^{p+1}_{usu}(U^{an}, \Omega^{N-p}_{U^{an},\log}))$$

Hence

$$w(\lambda) = H^{N-1}OL_{X^{an}}(\Omega^{N-p,an}_{\log}(Res_{X,\mathbb{P}^N})(w(\alpha))) \in OL_{X^{an}}(H^p_{usu}(X^{an}, \Omega^p_{X^{an},\log})).$$

Hence by proposition 4(ii), $w(\lambda) = [Z]$ with $Z \in \mathcal{Z}^p(X)$. □

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