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Hodge conjecture for projective hypersurfaces

Johann Bouali

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Abstract

We show that an Hodge class of a complex smooth projective hypersurface is an analytic logarithmic De Rham class. On the other hand we show that for a complex smooth projective variety an analytic logarithmic De Rham class of type (d, d) is the class of codimension d algebraic cycle. We deduce the Hodge conjecture for smooth projective hypersurfaces.

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1 Introduction

1.1 Complex analytic De Rham logarithmic classes

In [2], we introduced, for X an algebraic variety over a field, the notion of logarithmic De Rham cohomology classes. For X a complex algebraic variety, we also introduce the notion of logarithmic analytic De Rham cohomology classes. In proposition 4 we give an analytic analogue of [2] theorem 2 (ii), that is if X is a smooth complex projective variety,

- a logarithmic analytic De Rham cohomology class of type (d, d) is the class of a codimension d algebraic cycle.
- we have $H^j OL_{X^{an}}(H_{usu}^{j-l}(X^{an}, \Omega_{X^{an}, \log}^l)) = 0$ for $j, l \in \mathbb{Z}$ such that $2l > j$.

The proof of proposition 4 is motivic and similar to the proof of [2] theorem 2 : a logarithmic analytic class of bidegree (p, q) , $p, q \neq 0$, $p \geq q$, $q \neq d_U$ is acyclic for the etale topology on each open subset $U \subset X$ such that there exists an etale map $e : U \rightarrow \mathbb{A}_{\mathbb{C}}^{d_U}$ such that $e : U \rightarrow e(U)$ is finite etale and $\mathbb{A}_{\mathbb{C}}^{d_U} \setminus e(U) \subset \mathbb{A}_{\mathbb{C}}^{d_U}$ is a divisor (proposition 2), this allows us to proceed by a finite induction using the

crucial fact that the purity isomorphism for De Rham cohomology preserve logarithmic analytic classes since the purity isomorphism is motivic (c.f. [3], see proposition 1).

1.2 Hodge conjecture for smooth projective hypersurfaces

Let $X = V(f) \subset \mathbb{P}_{\mathbb{C}}^N$ a smooth projective hypersurface. Denote $j : U := \mathbb{P}_{\mathbb{C}}^N \setminus X \hookrightarrow \mathbb{P}_{\mathbb{C}}^N$ the open complementary subset. Using the fact the filtered De Rham complex of U

$$\iota_{\partial} : \Omega_{\mathbb{P}^N, an}^{N-p}(\log X)^{\partial=0}[-p] \hookrightarrow (\Omega_{\mathbb{P}^N}^{\bullet} \otimes_{\mathcal{O}_{\mathbb{P}^N}} F^{\bullet-p} j_* \mathcal{O}_U)^{an}$$

is locally acyclic for the complex topology, more precisely is acyclic on open balls (see e.g. [6]), we show, considering an open cover $\mathbb{P}_{\mathbb{C}}^N = \cup_{i=1}^n B_i$ by open balls $\phi_i : B_i \simeq D(0, 1)^N$ such that $\phi_i(B_i \cap X) = 0 \times D(0, 1)^N$, that an Hodge class $\alpha \in F^p H^N(U^{an}, \mathbb{Q})$ is logarithmic, that is

$$\alpha \in H^N OL_{U^{an}}(H_{usu}^p(U^{an}, \Omega_{U^{an}, \log}^{N-p}))$$

(proposition 3). We get that, if $N = 2p + 1$ and $\lambda \in F^p H^{2p}(X^{an}, \mathbb{Q})$ is an Hodge class,

$$\lambda = Res_{X, \mathbb{P}^N}(\alpha) + m[H] \in H^N OL_{X^{an}}(H_{usu}^p(X^{an}, \Omega_{X^{an}, \log}^{N-p})),$$

where $Res_{X, \mathbb{P}^N} : H^N(U^{an}, \mathbb{C}) \xrightarrow{\sim} H^{2p}(X^{an}, \mathbb{C})$ is the residu map, using the fact that since the residu map is motivic (see definition 1), it preserve the logarithmic De Rham classes. Proposition 4 (ii) then implies that $\lambda = [Z]$, $Z \in \mathcal{Z}^p(X)$, is the class of an algebraic cycle.

2 Preliminaries and Notations

2.1 Notations

- Denote by Top the category of topological spaces and RTop the category of ringed spaces.
- Denote by Cat the category of small categories and RCat the category of ringed topoi.
- For $\mathcal{S} \in \text{Cat}$ and $X \in \mathcal{S}$, we denote $\mathcal{S}/X \in \text{Cat}$ the category whose objects are $Y/X := (Y, f)$ with $Y \in \mathcal{S}$ and $f : Y \rightarrow X$ is a morphism in \mathcal{S} , and whose morphisms $\text{Hom}((Y', f'), (Y, f))$ consists of $g : Y' \rightarrow Y$ in \mathcal{S} such that $f \circ g = f'$.
- For $(\mathcal{S}, \mathcal{O}_{\mathcal{S}}) \in \text{RCat}$ a ringed topos, we denote by
 - $\text{PSh}(\mathcal{S})$ the category of presheaves of $\mathcal{O}_{\mathcal{S}}$ modules on \mathcal{S} and $\text{PSh}_{\mathcal{O}_{\mathcal{S}}}(\mathcal{S})$ the category of presheaves of $\mathcal{O}_{\mathcal{S}}$ modules on \mathcal{S} , whose objects are $\text{PSh}_{\mathcal{O}_{\mathcal{S}}}(\mathcal{S})^0 := \{(M, m), M \in \text{PSh}(\mathcal{S}), m : M \otimes_{\mathcal{O}_{\mathcal{S}}} \rightarrow M\}$, together with the forgetful functor $o : \text{PSh}(\mathcal{S}) \rightarrow \text{PSh}_{\mathcal{O}_{\mathcal{S}}}(\mathcal{S})$,
 - $C(\mathcal{S}) = C(\text{PSh}(\mathcal{S}))$ and $C_{\mathcal{O}_{\mathcal{S}}}(\mathcal{S}) = C(\text{PSh}_{\mathcal{O}_{\mathcal{S}}}(\mathcal{S}))$ the big abelian category of complexes of presheaves of $\mathcal{O}_{\mathcal{S}}$ modules on \mathcal{S} ,
 - $C_{\mathcal{O}_{\mathcal{S}}(2)\text{fil}}(\mathcal{S}) := C_{(2)\text{fil}}(\text{PSh}_{\mathcal{O}_{\mathcal{S}}}(\mathcal{S})) \subset C(\text{PSh}_{\mathcal{O}_{\mathcal{S}}}(\mathcal{S}), F, W)$, the big abelian category of (bi)filtered complexes of presheaves of $\mathcal{O}_{\mathcal{S}}$ modules on \mathcal{S} such that the filtration is biregular and $\text{PSh}_{\mathcal{O}_{\mathcal{S}}(2)\text{fil}}(\mathcal{S}) := (\text{PSh}_{\mathcal{O}_{\mathcal{S}}}(\mathcal{S}), F, W)$.
- Let $(\mathcal{S}, \mathcal{O}_{\mathcal{S}}) \in \text{RCat}$ a ringed topos with topology τ . For $F \in C_{\mathcal{O}_{\mathcal{S}}}(\mathcal{S})$, we denote by $k : F \rightarrow E_{\tau}(F)$ the canonical flasque resolution in $C_{\mathcal{O}_{\mathcal{S}}}(\mathcal{S})$ (see [1]). In particular for $X \in \mathcal{S}$, $H^*(X, E_{\tau}(F)) \xrightarrow{\sim} \mathbb{H}_{\tau}^*(X, F)$.
- For $f : \mathcal{S}' \rightarrow \mathcal{S}$ a morphism with $\mathcal{S}, \mathcal{S}' \in \text{RCat}$, endowed with topology τ and τ' respectively, we denote for $F \in C_{\mathcal{O}_{\mathcal{S}}}(\mathcal{S})$ and each $j \in \mathbb{Z}$,
 - $f^* := H^j \Gamma(\mathcal{S}, k \circ \text{ad}(f^*, f_*)(F)) : \mathbb{H}^j(\mathcal{S}, F) \rightarrow \mathbb{H}^j(\mathcal{S}', f^*F)$,

$$- f^* := H^j \Gamma(\mathcal{S}, k \circ \text{ad}(f^{*mod}, f_*)(F)) : \mathbb{H}^j(\mathcal{S}, F) \rightarrow \mathbb{H}^j(\mathcal{S}', f^{*mod}F),$$

the canonical maps.

- For $X \in \text{Top}$ and $Z \subset X$ a closed subset, denoting $j : X \setminus Z \hookrightarrow X$ the open complementary, we will consider

$$\Gamma_Z^\vee \mathbb{Z}_X := \text{Cone}(\text{ad}(j!, j^*)(\mathbb{Z}_X) : j!j^* \mathbb{Z}_X \hookrightarrow \mathbb{Z}_X) \in C(X)$$

and denote for short $\gamma_Z^\vee := \gamma_X^\vee(\mathbb{Z}_X) : \mathbb{Z}_X \rightarrow \Gamma_Z^\vee \mathbb{Z}_X$ the canonical map in $C(X)$.

- Denote by $\text{Sch} \subset \text{RTop}$ the subcategory of schemes (the morphisms are the morphisms of locally ringed spaces). We denote by $\text{PSch} \subset \text{Sch}$ the full subcategory of proper schemes. For a field k , we consider $\text{Sch}/k := \text{Sch}/\text{Spec } k$ the category of schemes over $\text{Spec } k$, that is whose object are $X := (X, a_X)$ with $X \in \text{Sch}$ and $a_X : X \rightarrow \text{Spec } k$ a morphism and whose objects are morphism of schemes $f : X' \rightarrow X$ such that $f \circ a_{X'} = a_X$. We then denote by

- $\text{Var}(k) = \text{Sch}^{ft}/k \subset \text{Sch}/k$ the full subcategory consisting of algebraic varieties over k , i.e. schemes of finite type over k ,
- $\text{PVar}(k) \subset \text{QPVar}(k) \subset \text{Var}(k)$ the full subcategories consisting of quasi-projective varieties and projective varieties respectively,
- $\text{PSmVar}(k) \subset \text{SmVar}(k) \subset \text{Var}(k)$, $\text{PSmVar}(k) := \text{PVar}(k) \cap \text{SmVar}(k)$, the full subcategories consisting of smooth varieties and smooth projective varieties respectively.

- Denote by $\text{AnSp}(\mathbb{C}) \subset \text{RTop}$ the subcategory of analytic spaces over \mathbb{C} , and by $\text{AnSm}(\mathbb{C}) \subset \text{AnSp}(\mathbb{C})$ the full subcategory of smooth analytic spaces (i.e. complex analytic manifold).

- Let $(X, O_X) \in \text{RTop}$. We consider its De Rham complex $\Omega_X^\bullet := DR(X)(O_X)$.

- Let $X \in \text{Sch}$. Considering its De Rham complex $\Omega_X^\bullet := DR(X)(O_X)$, we have for $j \in \mathbb{Z}$ its De Rham cohomology $H_{DR}^j(X) := \mathbb{H}^j(X, \Omega_X^\bullet)$.
- Let $X \in \text{Var}(k)$. Considering its De Rham complex $\Omega_X^\bullet := \Omega_{X/k}^\bullet := DR(X/k)(O_X)$, we have for $j \in \mathbb{Z}$ its De Rham cohomology $H_{DR}^j(X) := \mathbb{H}^j(X, \Omega_X^\bullet)$. The differentials of $\Omega_X^\bullet := \Omega_{X/k}^\bullet$ are by definition k -linear, thus $H_{DR}^j(X) := \mathbb{H}^j(X, \Omega_X^\bullet)$ has a structure of a k vector space.
- Let $X \in \text{AnSp}(\mathbb{C})$. Considering its De Rham complex $\Omega_X^\bullet := DR(X)(O_X)$, we have for $j \in \mathbb{Z}$ its De Rham cohomology $H_{DR}^j(X) := \mathbb{H}^j(X, \Omega_X^\bullet)$.

- For $X \in \text{AnSp}(\mathbb{C})$, we denote $\alpha(X) : \mathbb{C}_X \hookrightarrow \Omega_X^\bullet$ the embedding in $C(X)$. For $X \in \text{AnSm}(\mathbb{C})$, $\alpha(X) : \mathbb{C}_X \hookrightarrow \Omega_X^\bullet$ is an equivalence usu local by Poincare lemma.

- We denote $\mathbb{I}^n := [0, 1]^n \in \text{Diff}(\mathbb{R})$ (with boundary). For $X \in \text{Top}$ and R a ring, we consider its singular cochain complex

$$C_{\text{sing}}^*(X, R) := (\mathbb{Z} \text{Hom}_{\text{Top}}(\mathbb{I}^*, X)^\vee) \otimes R$$

and for $l \in \mathbb{Z}$ its singular cohomology $H_{\text{sing}}^l(X, R) := H^n C_{\text{sing}}^*(X, R)$. For $f : X' \rightarrow X$ a continuous map with $X, X' \in \text{Top}$, we have the canonical map of complexes

$$f^* : C_{\text{sing}}^*(X, R) \rightarrow C_{\text{sing}}^*(X', R), \sigma \mapsto f^* \sigma := (\gamma \mapsto \sigma(f \circ \gamma)).$$

In particular, we get by functoriality the complex

$$C_{X,R}^* \in C_R(X), (U \subset X) \mapsto C_{\text{sing}}^*(U, R)$$

We recall we have

- For $X \in \text{Top}$ locally contractile, e.g. $X \in \text{CW}$, and R a ring, the inclusion in $C_R(X)$ $c_X : R_X \rightarrow C_{X,R \text{ sing}}^*$ is by definition an equivalence top local and that we get by the small chain theorem, for all $l \in \mathbb{Z}$, an isomorphism $H^l c_X : H^l(X, R_X) \xrightarrow{\sim} H_{\text{sing}}^l(X, R)$.
- For $X \in \text{Diff}(\mathbb{R})$, the restriction map

$$r_X : \mathbb{Z} \text{Hom}_{\text{Diff}(\mathbb{R})}(\mathbb{I}^*, X)^\vee \rightarrow C_{\text{sing}}^*(X, R), w \mapsto w : (\phi \mapsto w(\phi))$$

is a quasi-isomorphism by Whitney approximation theorem.

- We will consider the morphism of site given by the analytification functor

$$\text{An} : \text{AnSp}(\mathbb{C}) \rightarrow \text{Var}(\mathbb{C}), X \mapsto \text{An}(X) := X^{an}, (f : X' \rightarrow X) \mapsto \text{An}(f) := f^{an}.$$

For $X \in \text{Var}(\mathbb{C})$, we denote by

$$\text{an}_X := \text{An}|_{X^{et}} := X^{an,et} \rightarrow X^{et}, \text{an}_X(U \rightarrow X) := U^{an},$$

its restriction to the small etale site $o_X(X^{et}) \subset \text{Var}(\mathbb{C})$, where $o_X(U \rightarrow X) = U$, in particular we have the following commutative diagram

$$\begin{array}{ccc} \text{AnSp}(\mathbb{C}) & \xrightarrow{\text{An}} & \text{Var}(\mathbb{C}), \\ \downarrow o_{X^{an}} & & \downarrow o_X \\ X^{an,et} & \xrightarrow{\text{an}_X} & X^{et} \end{array}$$

where $o_{X^{an}}(U \hookrightarrow X^{an}) = U$. We get the morphism in RTop $\text{an}_X := \text{an}_X|_{O(X)} : X^{an} \rightarrow X$, $o : O(X) \subset X^{et}$ being the embedding consisting of the Zariski open subsets of X .

2.2 Complex integral periods

Let k a field of characteristic zero.

Let $X \in \text{SmVar}(k)$ a smooth variety. Let $X = \cup_{i=1}^s X_i$ an open affine cover. We have for $\sigma : k \hookrightarrow \mathbb{C}$ an embedding, the evaluation period embedding map which is the morphism of bi-complexes

$$ev(X) \bullet : \Gamma(X_\bullet, \Omega_{X_\bullet}^\bullet) \rightarrow \mathbb{Z} \text{Hom}_{\text{Diff}}(\mathbb{I}^\bullet, X_{\mathbb{C},\bullet}^{an})^\vee \otimes \mathbb{C},$$

$$w_I^l \in \Gamma(X_I, \Omega_{X_I}^l) \mapsto (ev(X)_I^l(w_I^l)) : \phi_I^l \in \mathbb{Z} \text{Hom}_{\text{Diff}}(\mathbb{I}^l, X_{\mathbb{C},I}^{an})^\vee \otimes \mathbb{C} \mapsto ev_I^l(w_I^l)(\phi_I^l) := \int_{\mathbb{I}^l} \phi_I^{l*} w_I^l$$

given by integration. By taking all the affine open cover $(j_i : X_i \hookrightarrow X)$ of X , we get for $\sigma : k \hookrightarrow \mathbb{C}$, the evaluation period embedding map

$$ev(X) := \varinjlim_{(j_i : X_i \hookrightarrow X)} ev(X) \bullet : \varinjlim_{(j_i : X_i \hookrightarrow X)} \Gamma(X_\bullet, \Omega_{X_\bullet}^\bullet) \rightarrow \varinjlim_{(j_i : X_i \hookrightarrow X)} \mathbb{Z} \text{Hom}_{\text{Diff}(\mathbb{R})}(\mathbb{I}^\bullet, X_{\mathbb{C},\bullet}^{an})^\vee \otimes \mathbb{C}$$

It induces in cohomology, for $j \in \mathbb{Z}$, the evaluation period map

$$H^j ev(X) = H^j ev(X) \bullet : H_{DR}^j(X) = H^j \Gamma(X_\bullet, \Omega_{X_\bullet}^\bullet) \rightarrow H_{\text{sing}}^j(X_{\mathbb{C}}^{an}, \mathbb{C}) = H^j(\text{Hom}_{\text{Diff}(\mathbb{R})}(\mathbb{I}^\bullet, X_{\mathbb{C},\bullet}^{an})^\vee \otimes \mathbb{C}).$$

which does NOT depend on the choice of the affine open cover by acyclicity of quasi-coherent sheaves on affine noetherian schemes for the left hand side and from Mayer-Vietoris quasi-isomorphism for singular cohomology of topological spaces and Whitney approximation theorem for differential manifolds for the right hand side.

Let $X \in \text{SmVar}(k)$. Then,

$$H^* ev(X_{\mathbb{C}}) = H^* R\Gamma(X_{\mathbb{C}}^{an}, \alpha(X))^{-1} \circ \Gamma(X_{\mathbb{C}}^{an}, E_{zar}(\Omega(\text{an}_X))) : H_{DR}^*(X_{\mathbb{C}}) \xrightarrow{\sim} H_{\text{sing}}^*(X_{\mathbb{C}}^{an}, \mathbb{C})$$

is the canonical isomorphism induced by the analytical functor and the quasi-isomorphism $\alpha(X) : \mathbb{C}_{X_{\mathbb{C}}^{an}} \hookrightarrow \Omega_{X_{\mathbb{C}}^{an}}^\bullet$, which gives the periods elements $H^* ev(X)(H_{DR}^*(X)) \subset H_{\text{sing}}^*(X_{\mathbb{C}}^{an}, \mathbb{C})$.

2.3 Algebraic cycles and motives

For $X \in \text{Sch}$ noetherian irreducible and $d \in \mathbb{N}$, we denote by $\mathcal{Z}^d(X)$ the group of algebraic cycles of codimension d , which is the free abelian group generated by irreducible closed subsets of codimension d .

For $X, X' \in \text{Sch}$ noetherian, with X' irreducible, we denote $\mathcal{Z}^{fs/X'}(X' \times X) \subset \mathcal{Z}_{d_{X'}}(X' \times X)$ which consist of algebraic cycles $\alpha = \sum_i n_i \alpha_i \in \mathcal{Z}_{d_{X'}}(X' \times X)$ such that, denoting $\text{supp}(\alpha) = \cup_i \alpha_i \subset X' \times X$ its support and $p' : X' \times X \rightarrow X'$ the projection, $p'_{|\text{supp}(\alpha)} : \text{supp}(\alpha) \rightarrow X'$ is finite surjective.

Let k a field. We denote by $\text{Cor SmVar}(k)$ the category such that the objects are $\{X \in \text{SmVar}(k)\}$ and such that $\text{Hom}_{\text{Cor SmVar}(k)}(X', X) := \mathcal{Z}^{fs/X'}(X' \times X)$. See [3] for the composition law. We denote by $\text{Tr} : \text{Cor SmVar}(k) \rightarrow \text{SmVar}(k)$ the morphism of site given by the embedding $\text{Tr} : \text{SmVar}(k) \hookrightarrow \text{Cor SmVar}(k)$. Let $F \in \text{PSh}(\text{SmVar}(k))$. We say that F admits transfers if $F = \text{Tr}_* F$ with $F \in \text{PSh}(\text{Cor SmVar}(k))$. We recall that $\text{DA}(k) := \text{Ho}_{\mathbb{A}^1, \text{et}}(C(\text{SmVar}(k)))$ is then the derived category of mixed motives.

We have (see e.g. [1]) the Hodge realization functor

$$\mathcal{F}^{Hdg} : \text{DA}(\mathbb{C}) \rightarrow D(\text{MHS}), M \mapsto \mathcal{F}^{Hdg}(M) := \iota^{-1}(\mathcal{F}^{FDR}(M), \text{Bti}^* M, \alpha(M))$$

For $X \in \text{Var}(k)$ and $Z \subset X$ a closed subset, denoting $j : X \setminus Z \hookrightarrow X$ the open complementary, we will consider

$$\Gamma_Z^\vee \mathbb{Z}_X := \text{Cone}(\text{ad}(j_\#, j^*)(\mathbb{Z}_X) : \mathbb{Z}_X \hookrightarrow \mathbb{Z}_X) \in C(\text{Var}(k)^{\text{sm}}/X)$$

and denote for short $\gamma_Z^\vee := \gamma_Z^\vee(\mathbb{Z}_X) : \mathbb{Z}_X \rightarrow \Gamma_Z^\vee \mathbb{Z}_X$ the canonical map in $C(\text{Var}(k)^{\text{sm}}/X)$. Denote $a_X : X \rightarrow \text{Spec } k$ the structural map. For $X \in \text{Var}(k)$ and $Z \subset X$ a closed subset, we have the motive of X with support in Z defined as

$$M_Z(X) := a_{X!} \Gamma_Z^\vee a_X^! \mathbb{Z} \in \text{DA}(k).$$

If $X \in \text{SmVar}(k)$, we will also consider

$$a_{X\#} \Gamma_Z^\vee \mathbb{Z}_X := \text{Cone}(a_{X\#} \circ \text{ad}(j_\#, j^*)(\mathbb{Z}_X) : \mathbb{Z}(U) \hookrightarrow \mathbb{Z}(X)) =: \mathbb{Z}(X, X \setminus Z) \in C(\text{SmVar}(k)).$$

Then for $X \in \text{SmVar}(k)$ and $Z \subset X$ a closed subset

$$M_Z(X) := a_{X!} \Gamma_Z^\vee a_X^! \mathbb{Z} = a_{X\#} \Gamma_Z^\vee \mathbb{Z}_X =: \mathbb{Z}(X, X \setminus Z) \in \text{DA}(k).$$

- Let $(X, Z) \in \text{Sch}^2$ with $X \in \text{Sch}$ a noetherian scheme and $Z \subset X$ a closed subset. We have the deformation $(D_Z X, \mathbb{A}_Z^1) \rightarrow \mathbb{A}^1$, $(D_Z X, \mathbb{A}_Z^1) \in \text{Sch}^2$ of (X, Z) by the normal cone $C_{Z/X} \rightarrow Z$, i.e. such that

$$(D_Z X, \mathbb{A}_Z^1)_s = (X, Z), s \in \mathbb{A}^1 \setminus 0, (D_Z X, \mathbb{A}_Z^1)_0 = (C_{Z/X}, Z).$$

We denote by $i_1 : (X, Z) \hookrightarrow (D_Z X, \mathbb{A}_Z^1)$ and $i_0 : (C_{Z/X}, Z) \hookrightarrow (D_Z X, \mathbb{A}_Z^1)$ the closed embeddings in Sch^2 .

- Let k a field of characteristic zero. Let $X \in \text{SmVar}(k)$. For $Z \subset X$ a closed subset of pure codimension c , consider a desingularisation $\epsilon : \tilde{Z} \rightarrow Z$ of Z and denote $n : \tilde{Z} \xrightarrow{\epsilon} Z \subset X$. We have then the morphism in $\text{DA}(k)$

$$G_{Z,X} : M(X) \xrightarrow{D(\mathbb{Z}(n))} M(\tilde{Z})(c)[2c] \xrightarrow{\mathbb{Z}(\epsilon)} M(Z)(c)[2c]$$

where $D : \text{Hom}_{\text{DA}(k)}(M_c(\tilde{Z}), M_c(X)) \xrightarrow{\sim} \text{Hom}_{\text{DA}(k)}(M(X), M(\tilde{Z})(c)[2c])$ is the duality isomorphism from the six functors formalism (moving lemma of Suzlin and Voevodsky) and $\mathbb{Z}(n) := \text{ad}(n_!, n^!)(a_X^! \mathbb{Z})$, noting that $n_! = n_*$ since n is proper and that $a_X^! = a_X^*[d_X]$ and $a_Z^! = a_Z^*[d_Z]$ since X , resp. \tilde{Z} , are smooth (considering the connected components, we may assume X and \tilde{Z} of pure dimension).

We recall the following facts (see [3] and [1]):

Proposition 1. *Let k be a field a characteristic zero. Let $X \in \text{SmVar}(k)$ and $i : Z \subset X$ a smooth subvariety of pure codimension d . Then $C_{Z/X} = N_{Z/X} \rightarrow Z$ is a vector bundle of rank d . The closed embeddings $i_1 : (X, Z) \hookrightarrow (D_Z X, \mathbb{A}_Z^1)$ and $i_0 : (C_{Z/X}, Z) \hookrightarrow (D_Z X, \mathbb{A}_Z^1)$ in $\text{SmVar}^2(k)$ induces isomorphisms of motives $\mathbb{Z}(i_1) : M_Z(X) \xrightarrow{\sim} M_{\mathbb{A}_Z^1}(D_Z X)$ and $\mathbb{Z}(i_0) : M_Z(N_{Z/X}) \xrightarrow{\sim} M_{\mathbb{A}_Z^1}(D_Z X)$ in $\text{DA}(k)$. We get the excision isomorphism in $\text{DA}(k)$*

$$P_{Z,X} := \mathbb{Z}(i_0)^{-1} \circ \mathbb{Z}(i_1) : M_Z(X) \xrightarrow{\sim} M_Z(N_{Z/X}).$$

We have

$$\text{Th}(N_{Z/X}) \circ P_{Z,X} \circ \gamma_Z^\vee(\mathbb{Z}_X) = G_{Z,X} := D(\mathbb{Z}(i)) : M(X) \rightarrow M(Z)(d)[2d].$$

Proof. See [3]. □

Definition 1. *Let k be a field a characteristic zero. Let $X \in \text{SmVar}(k)$ and $i : Z \subset X$ a smooth subvariety of pure codimension d . We consider using proposition 1 the canonical map in $\text{DA}(k)$*

$$\text{Res}_{Z,X} : M(Z)(d)[-2d] \xrightarrow{P_{Z,X}^{-1} \circ \text{Th}(N_{Z/X})^{-1}} M_Z(X) \xrightarrow{c(\mathbb{Z}(X \setminus Z))} M(X \setminus Z)[1]$$

where, for $m : A \rightarrow B$, $A, B \in \mathcal{C}(\mathcal{A})$, \mathcal{A} an additive category, $c(A) : \text{Cone}(m : A \rightarrow B) \rightarrow A[1]$ is the canonical map.

- Recall that we say that $F \in \text{PSh}(\text{SmVar}(k))$ is \mathbb{A}^1 invariant if for all $X \in \text{SmVar}(k)$, $p^* := F(p) : F(X) \rightarrow F(X \times \mathbb{A}^1)$ is an isomorphism, where $p : X \times \mathbb{A}^1 \rightarrow X$ is the projection.
- Similarly, we say that $F \in \text{PSh}(\text{AnSm}(\mathbb{C}))$ is \mathbb{D}^1 invariant if for all $X \in \text{AnSm}(\mathbb{C})$, $p^* := F(p) : F(X) \rightarrow F(X \times \mathbb{D}^1)$ is an isomorphism, where $p : X \times \mathbb{D}^1 \rightarrow X$ is the projection.

2.4 The logarithmic De Rham complexes

We recall from [2] the following notion :

- For k a field and $X \in \text{Var}(k)$, we consider the embedding in $C(X)$

$$OL_X : \Omega_{X,\log}^\bullet \hookrightarrow \Omega_X^\bullet := \Omega_{X/k}^\bullet,$$

such that, for $X^\circ \subset X$ an open subset and $w \in \Omega_X^p(X^\circ)$, $w \in \Omega_{X,\log}^p(X^\circ)$ if and only if there exists $(n_i)_{1 \leq i \leq s} \in \mathbb{Z}$ and $(f_{i,\alpha_k})_{1 \leq i \leq s, 1 \leq k \leq p} \in \Gamma(X^\circ, \mathcal{O}_X)^*$ such that

$$w = \sum_{1 \leq i \leq s} n_i df_{i,\alpha_1} / f_{i,\alpha_1} \wedge \cdots \wedge df_{i,\alpha_p} / f_{i,\alpha_p} \in \Omega_X^p(X^\circ),$$

and for $p = 0$, $\Omega_{X,\log}^0 := \mathbb{Z}$. Let k a field. We get an embedding in $C(\text{Var}(k))$

$$OL : \Omega_{/k,\log}^\bullet \hookrightarrow \Omega_{/k}^\bullet, \text{ given by, for } X \in \text{Var}(k), \\ OL(X) := OL_X : \Omega_{/k,\log}^\bullet(X) := \Gamma(X, \Omega_{X,\log}^\bullet) \hookrightarrow \Gamma(X, \Omega_X^\bullet) =: \Omega_{/k}^\bullet(X)$$

and its restriction to $\text{SmVar}(k) \subset \text{Var}(k)$.

- For $X \in \text{AnSp}(\mathbb{C})$, we consider the embedding in $C(X)$

$$OL_X : \Omega_{X,\log}^\bullet \hookrightarrow \Omega_X^\bullet,$$

such that, for $X^\circ \subset X$ an open subset and $w \in \Omega_X^p(X^\circ)$, $w \in \Omega_{X,\log}^p(X^\circ)$ if and only if there exists $(n_i)_{1 \leq i \leq s} \in \mathbb{Z}$ and $(f_{i,\alpha_k})_{1 \leq i \leq s, 1 \leq k \leq p} \in \Gamma(X^\circ, \mathcal{O}_X)^*$ such that

$$w = \sum_{1 \leq i \leq s} n_i df_{i,\alpha_1} / f_{i,\alpha_1} \wedge \cdots \wedge df_{i,\alpha_p} / f_{i,\alpha_p} \in \Omega_X^p(X^\circ),$$

and for $p = 0$, $\Omega_{X, \log}^0 := \mathbb{Z}$. We get an embedding in $C(\text{AnSp}(\mathbb{C}))$

$$OL^{an} : \Omega_{\log}^{\bullet, an} \hookrightarrow \Omega^{\bullet, an}, \text{ given by, for } X \in \text{AnSp}(\mathbb{C}),$$

$$OL^{an}(X) := OL_X : \Omega_{\log}^{\bullet, an}(X) := \Gamma(X, \Omega_{X, \log}^{\bullet, an}) \hookrightarrow \Gamma(X, \Omega_X^{\bullet, an}) =: \Omega^{\bullet, an}(X)$$

and its restriction to $\text{AnSm}(\mathbb{C}) \subset \text{AnSp}(\mathbb{C})$.

Definition 2. Recall that $\text{An} : \text{AnSp}(\mathbb{C}) \rightarrow \text{Var}(\mathbb{C})$ denote the analytical functor. We will consider, for each $p \in \mathbb{Z}$, the exact sequence in $\text{PSh}(\text{AnSp}(\mathbb{C}))$

$$0 \rightarrow \Omega_{\log}^{p, an} \cap (\text{An}^* \partial \text{An}_* \Omega^{p-1, an}) \xrightarrow{e} \Omega_{\log}^{p, an} \xrightarrow{q} \Omega_{\log}^{p, an, ne} := \Omega_{\log}^{p, an} / (\Omega_{\log}^{p, an} \cap (\text{An}^* \partial \text{An}_* \Omega^{p-1, an})) \rightarrow 0,$$

and its restriction in $\text{PSh}(\text{AnSm}(\mathbb{C}))$. Note that for $X \in \text{Var}(\mathbb{C})$ and $\partial f \in \Omega^{1, an}(X^{an})$ with $f \in O(X^{an})$, we have $\partial f = (\partial e^f)/e^f \in \Omega_{\log}^{1, an}(X^{an})$. We have then the commutative diagram in $C(\text{AnSp}(\mathbb{C}))$

$$\begin{array}{ccc} \Omega_{\log}^{\bullet, an} \cap (\text{An}^* \partial \text{An}_* \Omega^{\bullet-1, an}) & \xrightarrow{OL^{an}} & \text{An}^* \partial \text{An}_* \Omega^{\bullet-1, an} \\ \downarrow e & & \downarrow e \\ \Omega_{\log}^{\bullet, an} & \xrightarrow{OL^{an}} & \Omega^{\bullet, an} \\ \downarrow q & & \downarrow q \\ \Omega_{\log}^{\bullet, an, ne} := \Omega_{\log}^{\bullet, an} / (\Omega_{\log}^{\bullet, an} \cap (\text{An}^* \partial \text{An}_* \Omega^{\bullet-1, an})) & \xrightarrow{\bar{O}L^{an}} & \Omega^{\bullet, an} / (\text{An}^* \partial \text{An}_* \Omega^{\bullet-1, an}) \end{array},$$

whose columns are exact with e injective and q surjective.

We have the following :

Lemma 1. (i0) The sheaves $O_{an}^* \in \text{PSh}(\text{AnSm}(\mathbb{C}))$ and $\Omega_{\log}^{1, an} \in \text{PSh}(\text{AnSm}(\mathbb{C}))$ admit transfers compatible with transfers on $\Omega^{1, an} \in \text{PSh}(\text{AnSm}(\mathbb{C}))$.

(i) For each $l \in \mathbb{Z}$, the sheaf $\Omega_{\log}^{l, an} \in \text{PSh}(\text{AnSm}(\mathbb{C}))$ admits transfers compatible with transfers on $\Omega^{l, an} \in \text{PSh}(\text{AnSm}(\mathbb{C}))$, that is $\Omega^{\bullet, an} \in C(\text{Cor AnSm}(\mathbb{C}))$ and the inclusion $OL : \Omega_{\log}^{1, an}[-l] \hookrightarrow \Omega^{\bullet, an}$ in $C(\text{AnSm}(\mathbb{C}))$ is compatible with transfers.

(ii) For each $l \in \mathbb{Z}$, the sheaf $\Omega_{\log}^{l, an, ne} \in \text{PSh}(\text{AnSm}(\mathbb{C}))$ given in definition 2 is \mathbb{A}^1 invariant.

Proof. (i0): Let $W \subset X' \times X$ with $X, X' \in \text{AnSm}(\mathbb{C})$, X' connected, and $p_{X'} : W \subset X' \times X \rightarrow X'$ finite surjective over X' . Then $M(p_X) : M(X') \hookrightarrow M(W)$ is a finite, where for $Y \in \text{AnSp}(\mathbb{C})$ irreducible, $M(Y) := \text{Frac}(O(Y))$ denote the field of meromorphic function. Since $O(X') \subset M(X')$ is integrally closed, the trace map $\text{Tr}_{W/X'} : M(W) \rightarrow M(X')$ sends $O(W)$ to $O(X')$, and the norm map $N_{W/X'} : M(W)^* \rightarrow M(X')^*$ sends $O(W)^*$ to $O(X')^*$.

The sheaf $O_{an}^* \in \text{PSh}(\text{AnSm}(\mathbb{C}))$ admits transfers : for $W \subset X' \times X$ with $X, X' \in \text{AnSm}(\mathbb{C})$ and W finite surjective over X' and $f \in O(X)^*$, $W^*f := N_{W/X'}(p_X^*f)$ where $p_X : W \hookrightarrow X' \times X \rightarrow X$ is the projection and $N_{W/X'} : M(W)^* \rightarrow M(X')^*$ is the norm map. This gives transfers on $\Omega_{\log}^{1, an} \in \text{PSh}(\text{AnSm}(\mathbb{C}))$ compatible with transfers on $\Omega^{1, an} \in \text{PSh}(\text{AnSm}(\mathbb{C}))$: for $W \subset X' \times X$ with $X, X' \in \text{AnSm}(\mathbb{C})$ and W finite surjective over X' and $f \in O(X)^*$,

$$W^*df/f := dW^*f/W^*f = \text{Tr}_{W/X'}(p_X^*(df/f)),$$

where we recall $p_X : W \hookrightarrow X' \times X \rightarrow X$ is the projection and $\text{Tr}_{W/X'} : O(W) \rightarrow O(X')$ is the trace map. Note that $d(fg)/fg = df/f + dg/g$.

(i): By (i0), we get transfers on

$$\otimes_{\mathbb{Q}}^l \Omega_{\log}^{1, an}, \otimes_O^l \Omega^{1, an} \in \text{PSh}(\text{AnSm}(\mathbb{C}))$$

since $\otimes_{\mathbb{Q}}^l \Omega_{\log}^{1,an} = H^0(\otimes_{\mathbb{Q}}^{L,l} \Omega_{\log}^{1,an})$ and $\otimes_O^l \Omega^{1,an} = H^0(\otimes_O^{L,l} \Omega^{1,an})$. This induces transfers on

$$\wedge_{\mathbb{Q}}^l \Omega_{\log}^{1,an} := \text{coker}(\oplus_{I_2 \subset [1, \dots, l]} \otimes_{\mathbb{Q}}^{l-1} \Omega_{\log}^{1,an} \xrightarrow{\oplus_{I_2 \subset [1, \dots, l]} \Delta_{I_2 := (w \otimes w' \mapsto w \otimes w \otimes w')}} \otimes_{\mathbb{Q}}^l \Omega_{\log}^{1,an}) \in \text{PSh}(\text{AnSm}(\mathbb{C})).$$

and

$$\wedge_O^l \Omega^{1,an} := \text{coker}(\oplus_{I_2 \subset [1, \dots, l]} \otimes_{O_k}^{l-1} \Omega^{1,an} \xrightarrow{\oplus_{I_2 \subset [1, \dots, l]} \Delta_{I_2 := (w \otimes w' \mapsto w \otimes w \otimes w')}} \otimes_O^l \Omega^{1,an}) \in \text{PSh}(\text{AnSm}(\mathbb{C})).$$

(ii): Follows from the fact that for $X \in \text{AnSp}(\mathbb{C})$, $\Omega^p(X \times \mathbb{A}^1) = \Omega^p(X) \otimes \Omega^1(\mathbb{A}^{1,an})$ and that $\Omega^1(\mathbb{A}^{1,an}) = \partial \mathcal{O}(\mathbb{A}^{1,an})$. \square

We have the following result :

Proposition 2. Consider, for each $p \in \mathbb{Z}$, the presheaf $\Omega_{D, \log}^{p,an,ne} \in \text{PSh}(\text{AnSm}(\mathbb{C}))$ given in definition 2.

(i) Let $D \in \text{AnSm}(\mathbb{C})$ be an open ball. Then, for each $p, q \in \mathbb{Z}$, $q \neq 0$, $H_{usu}^q(D, \Omega_{D, \log}^p) = 0$.

(i)' Let $D \in \text{AnSm}(\mathbb{C})$ be an open ball. Then, for each $p, q \in \mathbb{Z}$, $q \neq 0$, $H_{usu}^q(D, \Omega_{D, \log}^{p,an,ne}) = 0$.

(ii) Let $U \in \text{SmVar}(\mathbb{C})$ such that there exists an etale map $e : U \rightarrow \mathbb{A}_{\mathbb{C}}^{d_U}$ such that $e : U \rightarrow e(U)$ is finite etale and $\mathbb{A}_{\mathbb{C}}^{d_U} \setminus e(U) \subset \mathbb{A}_{\mathbb{C}}^{d_U}$ is a divisor. Then, for each $p, q \in \mathbb{Z}$, $p, q \neq 0$, $p \geq q$, $q \neq d_U$, $H_{usu}^q(U^{an}, \Omega_{\log}^{p,an,ne}) = 0$.

Proof. (i): Let $B = D(0, 1)^N \in \text{AnSm}(\mathbb{C})$ be an open ball. Let $\alpha \in H_{usu}^q(B, \Omega_{B, \log}^p)$, $q \in \mathbb{Z}$, $q \neq 0$. Consider a sequence of sub-balls

$$l_n : B_n := D(0, r_n)^N \hookrightarrow B := D(0, 1)^N, n \in \mathbb{N}, \text{ with } r_n \rightarrow 0 \text{ when } n \rightarrow \infty.$$

Since $(B_n := D(0, r_n))_{n \in \mathbb{N}}$ form a basis of neighborhood of $0 \in B$ and since $\Omega_{B, \log}^p$ is a single presheaf, there exists $n := n_\alpha \in \mathbb{N}$ (depending on α) such that

$$l_n^* \alpha = 0 \in H_{usu}^q(B_n, \Omega_{B, \log}^p).$$

But it follows immediately from the definition of the the presheaf $\Omega_{B, \log}^p$ considering the isomorphism

$$\phi_n : B_n \xrightarrow{\sim} B, \phi_n(z) := r_n^{-1} z$$

so that $(\phi_n)^{-1} \circ l_n = r_n I$, that

$$l_n^* \alpha = l_n^* \phi_n^{-1*} \phi_n^* \alpha = r_n \phi_n^* \alpha \in H_{usu}^q(B_n, \Omega_{B, \log}^p).$$

Hence $\phi_n^* \alpha = 0$. Since ϕ_n is an isomorphism, we get $\alpha = 0$.

(i)': Let $B = D(0, 1)^N \in \text{AnSm}(\mathbb{C})$ be an open ball. Let $\alpha \in H_{usu}^q(B, \Omega_{B, \log}^{p,an,ne})$, $q \in \mathbb{Z}$, $q \neq 0$. Consider a sequence of sub-balls

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$$l_n^* \alpha = l_n^* \phi_n^{-1*} \phi_n^* \alpha = r_n \phi_n^* \alpha \in H_{usu}^q(B_n, \Omega_{B, \log}^{p, an, ne}).$$

Hence $\phi_n^* \alpha = 0$. Since ϕ_n is an isomorphism, we get $\alpha = 0$.

(ii): We proceed by induction on the dimension of U . Let $U \in \text{SmVar}(\mathbb{C})$ such that there exists an etale map $e : U \rightarrow \mathbb{A}_{\mathbb{C}}^{d_U}$ such that $e : U \rightarrow e(U)$ is finite etale and $\mathbb{A}_{\mathbb{C}}^{d_U} \setminus e(U) \subset \mathbb{A}_{\mathbb{C}}^{d_U}$ is a divisor. Denote $D := \mathbb{A}_{\mathbb{C}}^{d_U} \setminus e(U)$ the complementary closed subset. Since $H_{usu}^q(\mathbb{A}_{\mathbb{C}}^{d_U}, \Omega_{\log}^{p, an, ne}) = 0$ for all $p, q \in \mathbb{Z}$,

$$H^q E_{usu}(\Omega_{\log}^{p, an, ne})(c(\mathbb{Z}(e(U)))) = \partial : H_{usu}^q(e(U), \Omega_{\log}^{p, an, ne}) \xrightarrow{\sim} H_{usu, D}^{q+1}(\mathbb{A}_{\mathbb{C}}^{d_U}, \Omega_{\log}^{p, an, ne})$$

is an isomorphism for all $p, q \in \mathbb{Z}$. By (i),

$$H^q E_{usu}(\Omega_{\log}^{p, an, ne}) \in \text{PSh}(\text{AnSm}(\mathbb{C}))$$

is \mathbb{A}^1 invariant for each $p, q \in \mathbb{Z}$. Indeed, for $X \in \text{AnSm}(\mathbb{C})$ connected, take an open cover $X = \cup_{i \in I} D_i$ by open balls $D_i \simeq D(0, 1)^{d_X}$, and then we get the isomorphism

$$\begin{aligned} p_X^* : H^q E_{usu}(\Omega_{\log}^{p, an, ne})(X \times \mathbb{A}^1) &:= H_{usu}^q(X \times \mathbb{A}^1, \Omega_{\log}^{p, an, ne}) \xrightarrow{\sim} \\ &\mathbb{H}^q(D_{\bullet} \times \mathbb{A}^1, \Omega_{\log}^{p, an, ne}) \xrightarrow{\sim} H^q \Gamma(D_{\bullet} \times \mathbb{A}^1, \Omega_{\log}^{p, an, ne}) \\ \xrightarrow{\sim} H^q \Gamma(D_{\bullet}, \Omega_{\log}^{p, an, ne}) &\xrightarrow{\sim} \mathbb{H}^q(D_{\bullet}, \Omega_{\log}^{p, an, ne}) \xrightarrow{\sim} H_{usu}^q(X, \Omega_{\log}^{p, an, ne}) =: H^q E_{usu}(\Omega_{\log}^{p, an, ne})(X) \end{aligned}$$

where $p_X : X \times \mathbb{A}^1 \rightarrow X$ is the projection, and the second and the fourth map is the isomorphism given by (i). Take a desingularization of the pair $\epsilon = \epsilon_1 : (\mathbb{A}_{\mathbb{C}}^{d_U}, E) \rightarrow (\mathbb{A}_{\mathbb{C}}^{d_U}, D)$, so that $E = \cup_{i=1}^s E_i \subset \widetilde{\mathbb{A}_{\mathbb{C}}^{d_U}}$ is a normal crossing divisor and $\epsilon : \widetilde{\mathbb{A}_{\mathbb{C}}^{d_U}} \setminus E \xrightarrow{\sim} \mathbb{A}_{\mathbb{C}}^{d_U} \setminus D$. We thus get by proposition 1, for each $p, q \in \mathbb{Z}$, $p, q \neq 0$,

$$H^{q+1} E_{usu}(\Omega_{\log}^{p, an, ne})(P_{E_i, \mathbb{A}_{\mathbb{C}}^{\widetilde{d_U}}}) : H_{usu, E_i}^{q+1}(\widetilde{\mathbb{A}_{\mathbb{C}}^{d_U}}, \Omega_{\log}^{p, an, ne}) \xrightarrow{\sim} H_{usu}^q(E_i \times \mathbb{G}_m, \Omega_{\log}^{p, an, ne}).$$

For $p, q \in \mathbb{Z}$, $p, q \neq 0$, $p \geq q$, $q \neq d_U$, we get, using the induction hypothesis, by a finite decreasing induction $\tilde{E}_{i_{q-1}} \subset \tilde{E}_{i_{q-2}}^N, \dots, E_{i_0} \subset E_i$, where

- $E_i^o := E_i \setminus E_{i_0} \subset E_i$ is an open subset such that there exists a finite etale map $e_i : E_i^o \rightarrow e_i(E_i^o) \subset \mathbb{A}^{d_U-1}$ with $\mathbb{A}^{d_U-1} \setminus e_i(E_i^o) \subset \mathbb{A}^{d_U-1}$ a divisor, $\epsilon_2 : (\tilde{E}_i, \tilde{E}_{i_0}) \rightarrow (E_i, E_{i_0})$ is a desingularization,
- $E_{ik}^o := \tilde{E}_{ik} \setminus E_{ik+1} \subset \tilde{E}_{ik}$ is an open subset such that there exists a finite etale map $e_{ik} : E_{ik}^o \rightarrow e_{ik}(E_{ik}^o) \subset \mathbb{A}^{d_U-k}$ with $\mathbb{A}^{d_U-k} \setminus e_{ik}(E_{ik}^o) \subset \mathbb{A}^{d_U-k}$ a divisor, $\epsilon_k : (\tilde{E}_{ik}^N, \tilde{E}_{ik+1}) \rightarrow (\tilde{E}_{ik}, E_{ik+1})$ is a desingularization

since $d_U > p \geq q$ ($\Omega_Y^p = 0$ for $p > d_Y, Y \in \text{AnSm}(\mathbb{C})$),

$$\begin{aligned} H_{usu}^q(e(U), \Omega_{\log}^{p, an, ne}) &= H_{usu, D}^{q+1}(\mathbb{A}_{\mathbb{C}}^{d_U}, \Omega_{\log}^{p, an, ne}) = H_{usu, E}^{q+1}(\widetilde{\mathbb{A}_{\mathbb{C}}^{d_U}}, \Omega_{\log}^{p, an, ne}) \\ &\xrightarrow{\sim} \oplus_{i=1}^r H_{usu}^1(\tilde{E}_{i_{q-1}} \times \mathbb{G}_m^{q-1}, \Omega_{\log}^{p, an, ne}) = 0 \end{aligned}$$

since $\text{Pic}(E_{i_{q-1}}) = \text{Pic}(E_i) = 0$ by induction as $\epsilon(E_i) \subset \mathbb{A}^{d_U}$ and $\epsilon_k(\tilde{E}_{ik+1}) \subset \tilde{E}_{ik}$ are hypersurfaces of dimension at least two. We thus have, since $M(U) \simeq M(e(U))$, for each $p, q \in \mathbb{Z}$, $p, q \neq 0$, $p \geq q$, $q \neq d_U$

$$H_{usu}^q(U, \Omega_{\log}^{p, an, ne}) = H_{usu}^q(e(U), \Omega_{\log}^{p, an, ne}) = 0.$$

□

3 Hypersurface De Rham cohomology and its complex periods

We denote $\mathbb{P}^N := \mathbb{P}_{\mathbb{C}}^N := \text{Proj}(\mathbb{C}[z_0, \dots, z_N])$. Let $X = V(f) \subset \mathbb{P}^N$ be a smooth hypersurface. If $N - 1 = 2p$ is even, we consider the decompositions

$$H_{DR}^{N-1}(X) = \mathbb{C}[\Lambda \cap X] \oplus H_{DR,v}^{N-1}(X), \quad H_{\text{sing}}^{N-1}(X^{an}, \mathbb{Q}) = \mathbb{C}[\Lambda \cap X] \oplus H_{\text{sing},v}^{N-1}(X^{an}, \mathbb{Q})$$

of vector spaces, where $\Lambda \subset \mathbb{P}^N$ is a linear subspace of codimension p . If $N - 1 = 2p + 1$ is even, we set

$$H_{DR,v}^{N-1}(X) := H_{DR}^{N-1}(X), \quad H_{\text{sing},v}^{N-1}(X^{an}, \mathbb{Q}) := H_{\text{sing}}^{N-1}(X^{an}, \mathbb{Q}).$$

Let $X = V(f) \subset \mathbb{P}^N$ be a smooth hypersurface. Denote $j : U := \mathbb{P}^N \setminus X \hookrightarrow \mathbb{P}^N$ the open embedding. We consider the algebraic Hodge module

$$j_{*Hdg}(O_U, F_b) := ((j_*O_U, F), j_*\mathbb{Q}_{U^{an}}, j_*\alpha(U)) \in HM(\mathbb{P}^N),$$

where $F^k j_*O_U = \mathbb{C}[x_1, \dots, x_N][\frac{1}{f^k}]$. Recall for each $p \in \mathbb{Z}$,

$$F^p H_{DR}^N(U) := H^N \iota^{\geq p}(\mathbb{H}^N(\mathbb{P}^N, \Omega_{\mathbb{P}^N}^{\bullet \geq p}(\log X))) = H^N \iota^{\geq p} H^N(\mathbb{P}^N, \Omega_{\mathbb{P}^N}^{\bullet} \otimes_{\mathcal{O}_{\mathbb{P}^N}} F^{\bullet-p} j_*O_U)$$

and

$$F^p H^{N-1}(X) := H^{N-1} \iota^{\geq p}(\mathbb{H}^{N-1}(X, \Omega_X^{\bullet \geq p})).$$

where

$$\iota^{\geq p} : \Omega_{\mathbb{P}^N}^{\bullet \geq p}(\log X) \hookrightarrow \Omega_{\mathbb{P}^N}^{\bullet}(\log X), \quad \iota^{\geq p} : \Omega_{\mathbb{P}^N}^{\bullet} \otimes_{\mathcal{O}_{\mathbb{P}^N}} F^{\bullet-p} j_*O_U \hookrightarrow \Omega_{\mathbb{P}^N}^{\bullet} \otimes_{\mathcal{O}_{\mathbb{P}^N}} j_*O_U = j_*\Omega_U^{\bullet}$$

and $\iota^{\geq p} : \Omega_X^{\bullet \geq p} \hookrightarrow \Omega_X^{\bullet}$ are the canonical embeddings. Then the map Res_{X, \mathbb{P}^N} of definition 1 induces for each $p \in \mathbb{Z}$, canonical isomorphisms

$$H^N \Omega(Res_{X, \mathbb{P}^N}) : F^{p+1} H_{DR}^N(U) \xrightarrow{\sim} F^p H_{DR}^{N-1}(X)_v, \quad H^N \text{Bti}(Res_{X, \mathbb{P}^N}) : H_{\text{sing}}^N(U, \mathbb{Q}) \xrightarrow{\sim} H_{\text{sing}}^{N-1}(X, \mathbb{Q})_v.$$

Lemma 2. *Let $X = V(f) \subset \mathbb{P}^N$ be a smooth hypersurface. Denote $j : U := \mathbb{P}^N \setminus X \hookrightarrow \mathbb{P}^N$ the open embedding. Let $B \subset \mathbb{P}^N$, $\phi : B \simeq D(0, 1)^N$ be an open ball such that $\phi(X \cap B) = \{0\} \times D(0, 1)^{N-1}$. Then,*

$$(i) \text{ For } q \in \mathbb{Z}, q \neq 0, \mathbb{H}_{usu}^q(B, (\Omega_{\mathbb{P}^N}^{\bullet} \otimes_{\mathcal{O}_{\mathbb{P}^N}} F^{\bullet-p} j_*O_U)^{an}) = 0.$$

$$(ii) \text{ For } q \in \mathbb{Z}, q \neq 0, H_{usu}^q(B, \Omega_{\mathbb{P}^N, an}^{N-p}(\log X)^{\partial=0}) = 0.$$

Proof. (i): Since B is Stein, we have for $r, q \in \mathbb{Z}, q \neq 0$,

$$H_{usu}^q(B, (\Omega_{\mathbb{P}^N}^r \otimes_{\mathcal{O}_{\mathbb{P}^N}} F^{r-p} j_*O_U)^{an}) = 0.$$

On the other hand the De Rham complex of $(j_*O_{B \setminus X}, F)$ is acyclic (see e.g. [6] corollary 18.7)

(ii): Since open balls form a basis of the topology of $\mathbb{P}^{N, an}$, we have

$$a_{usu} \iota : a_{usu}(j_*\Omega_{U^{an}, \log}^{N-p} \oplus \partial\Omega_{\mathbb{P}^N, an}^{N-p+1}) \xrightarrow{\sim} a_{usu}\Omega_{\mathbb{P}^N, an}^{N-p}(\log X)^{\partial=0}$$

where

$$\iota : j_*\Omega_{U^{an}, \log}^{N-p} \oplus \partial\Omega_{\mathbb{P}^N, an}^{N-p+1} \hookrightarrow \Omega_{\mathbb{P}^N, an}^{N-p}(\log X)^{\partial=0}$$

is the canonical embedding. Now,

- Let $B = D(0, 1)^N \in \text{AnSm}(\mathbb{C})$ be an open ball. Let $\alpha \in H^q(B, \partial\Omega_{\mathbb{P}^N, an}^{N-p+1})$, $q \in \mathbb{Z}$, $q \neq 0$. Consider a sequence of sub-balls

$$l_n : B_n := D(0, r_n)^N \hookrightarrow B := D(0, 1)^N, n \in \mathbb{N}, \text{ with } r_n \rightarrow 0 \text{ when } n \rightarrow \infty.$$

Since $(B_n := D(0, r_n))_{n \in \mathbb{N}}$ form a basis of neighborhood of $0 \in B$ and since $\partial\Omega_{\mathbb{P}^N, an}^{N-p+1}$ is a single presheaf, there exists $n := n_\alpha \in \mathbb{N}$ (depending on α) such that

$$l_n^* \alpha = 0 \in H^q(B_n, \partial\Omega_{\mathbb{P}^N, an}^{N-p+1}).$$

But it follows immediately from the definition of the the presheaf $\partial\Omega_{\mathbb{P}^N, an}^{N-p+1}$ considering the isomorphism

$$\phi_n : B_n \xrightarrow{\sim} B, \phi_n(z) := r_n^{-1}z$$

so that $(\phi_n)^{-1} \circ l_n = r_n I$, that

$$l_n^* \alpha = l_n^* \phi_n^{-1*} \phi_n^* \alpha = r_n \phi_n^* \alpha \in H^q(B_n, \partial\Omega_{\mathbb{P}^N, an}^{N-p+1}).$$

Hence $\phi_n^* \alpha = 0$. Since ϕ_n is an isomorphism, we get $\alpha = 0$.

- we have for $q \in \mathbb{Z}$, $q \neq 0$, $H_{usu}^q(B, \Omega_{B^{an}, \log}^{N-p}) = 0$ by proposition 2 (i). Hence

$$H^q E_{usu}(\Omega_{\log}^{an})(c(\mathbb{Z}(B))) = \partial : H_{usu}^q(B \cap U, \Omega_{U^{an}, \log}^{N-p}) \xrightarrow{\sim} H_{usu, B \cap X}^{q+1}(B, \Omega_{B^{an}, \log}^{N-p})$$

is an isomorphism. Proposition 2 (i) also gives, using the spectral sequence associated to a double complex, for $q \in \mathbb{Z}$,

$$H_{usu}^q(B, \underline{\text{sing}}_{D^*} \Omega_{B^{an}, \log}^{N-p}) = 0.$$

By a finite decreasing induction on N , using purity for complex analytic motives, we get for $q \in \mathbb{Z}$, $q \neq 0$,

$$H_{usu}^q(B \cap U, \Omega_{U^{an}, \log}^{N-p}) = H_{usu}^q(B \cap U, \underline{\text{sing}}_{D^*} \Omega_{U^{an}, \log}^{N-p}) = 0.$$

□

We have by the local acyclicity of the De Rham complex of $(j_* O_U, F)$ a canonical map :

Definition 3. Let $X = V(f) \subset \mathbb{P}^N$ be a smooth hypersurface. Denote $j : U := \mathbb{P}^N \setminus X \hookrightarrow \mathbb{P}^N$ the open embedding. Let $\mathbb{P}^{N, an} = \cup_{i=1}^n B_i$ an open cover, with $\phi_i : B_i \simeq D(0, 1)^N$ open balls such that $\phi_i(X \cap B_i) = \{0\} \times D(0, 1)^{N-1}$. Denote, for $I \subset [1, \dots, n]$, $l_I : B_I \hookrightarrow \mathbb{P}^N$ the open embeddings, with $B_I := \cap_{i \in I} B_i$. Since, for each $I \subset [1, \dots, n]$ and $p \in \mathbb{Z}$,

$$l_I^* \iota_\partial : l_I^* \Omega_{\mathbb{P}^N, an}^{N-p}(\log X)^{\partial=0}[-p] \hookrightarrow l_I^*(\Omega_{\mathbb{P}^N}^\bullet \otimes_{O_{\mathbb{P}^N}} F^{\bullet-p} j_* O_U)^{an}$$

is an quasi-isomorphism (see e.g. [6] corollary 18.7), the spectral sequence of the filtration on the total complex of a bi-complex gives, for each $p \in \mathbb{Z}$, a canonical isomorphism, where the first and last map are isomorphism by lemma 2(i) and (ii) respectively,

$$\begin{aligned} B_\partial : F^p H_{DR}(U) &:= H^N \iota^{\geq p} \mathbb{H}_{zar}^N(\mathbb{P}^N, \Omega_{\mathbb{P}^N}^\bullet \otimes_{O_{\mathbb{P}^N}} F^{\bullet-p} j_* O_U) \\ &\xrightarrow{H^N((l_i^* = \Omega(l_i))_{1 \leq i \leq n})} H^N \iota^{\geq p} H^N \Gamma(B_\bullet, (\Omega_{\mathbb{P}^N}^\bullet \otimes_{O_{\mathbb{P}^N}} F^{\bullet-p} j_* O_U)^{an}) \\ &\rightarrow H^p \Gamma(B_\bullet, \Omega_{\mathbb{P}^N, an}^{N-p}(\log X)^{\partial=0}) \xrightarrow{H^N((l_i^* = \Omega(l_i))_{1 \leq i \leq n})^{-1}} H_{usu}^p(\mathbb{P}^N, \Omega_{\mathbb{P}^N, an}^{N-p}(\log X)^{\partial=0}), \end{aligned}$$

given by, for $w \in \Gamma(U, \Omega_{\mathbb{P}^N}^N)$, inductively by a p step induction where p is the order of the pole of w at X ,

- taking for each $1 \leq i \leq n$, $l_i^* w = \partial w_{1_i}$ since $\partial w = 0$ for dimensional reason and by exactness of $l_i^* \iota_\partial$,

- we then have for each $1 \leq i, j \leq n$, considering $w_{1ij} := l_{ij}^*(w_{1i} - w_{1j})$, $w_{1ij} = \partial w_{2ij}$ since

$$\partial w_{1ij} = l_{ij}^*(\partial w_{1i} - \partial w_{1j}) = l_{ij}^*(l_i^*w - l_j^*w) = 0$$

and by exactness of $l_{ij}^*\iota_\partial$,

- at the final p step, $B(w) := w_{pI} \in H^N(B_\bullet, \Omega_{\mathbb{P}^N, an}^{N-p}(\log X)^{\partial=0})$, $\text{card}I = p$.

Let $X = V(f) \subset \mathbb{P}^N$ be a smooth hypersurface. Denote $U := \mathbb{P}^N \setminus X$. For $l \in \mathbb{Z}$, we consider

- the sequence of embeddings in $C(U^{et})$

$$OL_U^l : \Omega_{U, \log}^l[-l] \xrightarrow{\widehat{OL_U^l[-l]}} \Omega_{\mathbb{P}^N}^l(\log X)^{\partial=0}[-l] \xrightarrow{\iota_\partial} \Omega_{\mathbb{P}^N}^\bullet \otimes_{O_{\mathbb{P}^N}} F^{\bullet-p} j_* O_U \xrightarrow{\iota^{\geq l}} \Omega_U^\bullet$$

- and the sequence of embeddings in $C(U^{an})$

$$OL_{U^{an}}^l : \Omega_{U^{an}, \log}^l[-l] \xrightarrow{\widehat{OL_{U^{an}}^l[-l]}} \Omega_{\mathbb{P}^N, an}^l(\log X)^{\partial=0}[-l] \xrightarrow{\iota_\partial} (\Omega_{\mathbb{P}^N}^\bullet \otimes_{O_{\mathbb{P}^N}} F^{\bullet-p} j_* O_U)^{an} \xrightarrow{\iota^{\geq l}} \Omega_{U^{an}}^\bullet,$$

where

$$\iota_\partial : \Omega_{\mathbb{P}^N, an}^l(\log X)^{\partial=0}[-l] \hookrightarrow (\Omega_{\mathbb{P}^N}^\bullet \otimes_{O_{\mathbb{P}^N}} F^{\bullet-p} j_* O_U)^{an}$$

is an equivalence usu local (see [6]).

The main result of this section is the following :

Proposition 3. *Let $X = V(f) \subset \mathbb{P}^N$ be a smooth projective hypersurface. Denote $U := \mathbb{P}^N \setminus X$. Let $w \in F^{p+1}H_{DR}^N(U)$. Then the following assertions are equivalent*

(i) $ev(U)(w) \in H_{\text{sing}}^N(U^{an}, 2i\pi\mathbb{Q})$,

(ii) $B_\partial(w) \in \widehat{OL_{U^{an}}^{p+1}}(H_{usu}^{p+1}(U^{an}, \Omega_{U^{an}, \log}^{N-p}))$ (see definition 3),

(iii) $w \in OL_{U^{an}}^{N-p}(H_{usu}^{p+1}(U^{an}, \Omega_{U^{an}, \log}^{N-p})) = OL_{U^{an}}(H_{usu}^{p+1}(U^{an}, \Omega_{U^{an}, \log}^{N-p}))$.

Proof. (i) implies (ii): Let $\mathbb{P}^{N, an} = \cup_{i=1}^n B_i$ an open cover, with $\phi_i : \mathbb{B}_i \simeq D(0, 1)^N$ open balls such that $\phi_i(X \cap B_i) = \{0\} \times D(0, 1)^{N-1}$. Denote, for $I \subset [1, \dots, n]$, $l_I : B_I \hookrightarrow \mathbb{P}^N$ the open embeddings, with $B_I := \cap_{i \in I} B_i$. Then

$$B_\partial(w) := (w_{p+1, I}) \in H^{p+1}(B_\bullet, \Omega_{\mathbb{P}^N, an}^{N-p}(\log X)^{\partial=0}).$$

We have, for each $I \subset [1, \dots, n]$, a decomposition

$$w_{p+1, I} = \frac{a_I}{z_1} dz_1 \wedge w'_I + w''_I \in \Gamma(B_I, \Omega_{\mathbb{P}^N, an}^{N-p}(\log X)^{\partial=0})$$

with $w'_I \in \Gamma(B_I, \Omega_{\mathbb{P}^N}^{N-p-1})^{\partial=0}$ and $w''_I \in \Gamma(B_I, \Omega_{\mathbb{P}^N}^{N-p})^{\partial=0}$. Consider the evaluation map

$$ev(U)^\bullet : \Gamma(B_\bullet \setminus X, \Omega_{\mathbb{P}^N}^\bullet) \rightarrow C_{\text{diff}}^\bullet(B_\bullet \setminus X^{an}, \mathbb{C}),$$

which is a morphism of bi-complexes. On the other hand, we have the following canonical map

$$\begin{aligned} BC_\partial : H^p \Gamma(B_\bullet \setminus X^{an}, C_{U^{an}, \text{diff}}^{N-p, \partial}) &\xrightarrow{\Gamma(B_\bullet \setminus X^{an}, \iota_\partial)} H^N \Gamma(B_\bullet \setminus X^{an}, C_{U^{an}, \text{diff}}^{\bullet \geq N-p}) \xrightarrow{\Gamma(B_\bullet \setminus X^{an}, \iota^{\geq N-p})} \\ &= H^N \Gamma(B_\bullet \setminus X^{an}, C_{U^{an}, \text{diff}}^\bullet) \xrightarrow{(H^N(l_i^* = C_{U^{an}, \text{diff}}^\bullet(l_i))_{1 \leq i \leq n})^{-1}} H_{\text{sing}}^N(U^{an}, \mathbb{Q}) = H^N \Gamma(U^{an}, C_{U^{an}, \text{diff}}^\bullet), \end{aligned}$$

where $H^N(l_i^*)$ is an isomorphism by the acyclicity of $C_{U^{an}, \text{diff}}^{\bullet \geq N-p}(B_I \setminus X^{an})$ for each $I \subset [1, \dots, n]$, and

$$C_{U^{an}, \text{diff}}^\bullet \hookrightarrow C_{U^{an}, \text{sing}}^\bullet, \text{ for } V \subset U^{an}, C_{\text{diff}}^\bullet(V) \hookrightarrow C_{\text{sing}}^\bullet(V)$$

is the canonical quasi-isomorphism in $C(U^{an})$. We then have the following commutative diagram

$$\begin{array}{ccc}
H^N \iota^{\geq p} H^N \Gamma(\mathbb{P}^N, \Omega_{\mathbb{P}^N}^\bullet \otimes_{\mathcal{O}_{\mathbb{P}^N}} F^{\bullet-p} j_* \mathcal{O}_U) & & \\
\downarrow H^N((l_i^* = \Omega(l_i))_{1 \leq i \leq n}) & \searrow^{B_\partial} & \\
H^N \iota^{\geq p} H^N \Gamma(B_\bullet, (\Omega_{\mathbb{P}^N}^\bullet \otimes_{\mathcal{O}_{\mathbb{P}^N}} F^{\bullet-p} j_* \mathcal{O}_U)^{an}) & \xrightarrow{B_\partial} & H^p \Gamma(B_\bullet, \Omega_{\mathbb{P}^N, an}^{N-p}(\log X)^{\partial=0}) \\
\downarrow ev(U) & & \downarrow ev(U) \bullet \\
H_{\text{sing}}^N(U^{an}, \mathbb{C}) = H^N \Gamma(U^{an}, C_{U^{an}, \text{diff}}^\bullet \otimes \mathbb{C}) & \xleftarrow{BC_\partial \otimes \mathbb{C}} & H^p \Gamma(B_\bullet \setminus X^{an}, C_{\text{diff}}^{N-p, \partial=0} \otimes \mathbb{C})
\end{array}$$

Since $ev(U)$ is injective by the strictness of the Hodge filtration and since B_∂ is an isomorphism, $ev(U) \bullet$ is injective and

$$\begin{aligned}
(BC_\partial \otimes \mathbb{C})' &:= (BC_\partial \otimes \mathbb{C})|_{ev(U) \bullet (H^p \Gamma(B_\bullet, \Omega_{\mathbb{P}^N, an}^{N-p}(\log X)^{\partial=0}))} : \\
&ev(U) \bullet (H^p \Gamma(B_\bullet, \Omega_{\mathbb{P}^N, an}^{N-p}(\log X)^{\partial=0})) \rightarrow H_{\text{sing}}^N(U^{an}, \mathbb{C})
\end{aligned}$$

is injective. For $\gamma_I \in C_{N-p}^{\text{sing}}(B_I \setminus X^{an}, \mathbb{Q})^{\partial=0}$ where $p_1 \circ \phi_I(\gamma_I) \subset D(0, 1)^*$ is a loop around 0, we have by the above diagram

$$ev(U)(w)(\gamma_I) = ev(U)(w_{p+1, I})(\gamma_I) = a_I.$$

Hence if $ev(U)(w) \in H_{\text{sing}}^N(U^{an}, 2i\pi\mathbb{Q})$, then $a_I \in \mathbb{Q}$, using the injectivity of $(BC_\partial \otimes \mathbb{C})'$.

(ii) is equivalent to (iii): By definition $B_\partial(w) = (H^p \iota_\partial)^{-1}(w)$. On the other hand by definition

$$\iota^{\geq p} \circ \iota_\partial \circ \widehat{OL_{U^{an}}^{N-p}} = OL_{U^{an}}^{N-p}.$$

(iii) implies (i): See [2]. □

4 Complex analytic logarithmic de Rham classes

We recall (see section 2) the morphism of site given by the analytification functor

$$\text{An} : \text{AnSm}(\mathbb{C}) \rightarrow \text{SmVar}(\mathbb{C}), \quad X \mapsto \text{An}(X) := X^{an}, \quad (f : X' \rightarrow X) \mapsto \text{An}(f) := f^{an}.$$

and for $X \in \text{SmVar}(\mathbb{C})$, the following commutative diagram

$$\begin{array}{ccc}
\text{AnSm}(\mathbb{C}) & \xrightarrow{\text{An}} & \text{SmVar}(\mathbb{C}) \\
\downarrow o_{X^{an}} & & \downarrow o_X \\
X^{an, et} & \xrightarrow{\text{an}_X} & X^{et}
\end{array}$$

Consider the commutative diagram in $C(\text{AnSp}(\mathbb{C}))$

$$\begin{array}{ccc}
\Omega_{\log}^{\bullet, an} \cap (\text{An}^* \partial \text{An}_* \Omega^{\bullet-1, an}) & \xrightarrow{OL^{an}} & \text{An}^* \partial \text{An}_* \Omega^{\bullet-1, an} \\
\downarrow e & & \downarrow e \\
\Omega_{\log}^{\bullet, an} & \xrightarrow{OL^{an}} & \Omega^{\bullet, an} \\
\downarrow q & & \downarrow q \\
\Omega_{\log}^{\bullet, an, ne} := \Omega_{\log}^{\bullet, an} / (\Omega_{\log}^{\bullet, an} \cap (\text{An}^* \partial \text{An}_* \Omega^{\bullet-1, an})) & \xrightarrow{\bar{O}L^{an}} & \Omega^{\bullet, an} / (\text{An}^* \partial \text{An}_* \Omega^{\bullet-1, an})
\end{array} ,$$

whose columns are exact with e injective and q surjective (see definition 2). Let $X \in \text{SmVar}(\mathbb{C})$. Then, for $k \in \mathbb{Z}$, we have the commutative diagram

$$\begin{array}{ccc}
\mathbb{H}_{usu}^k(X^{an}, \Omega_{\log}^{\bullet, an} \cap (\text{An}^* \partial \text{An}_* \Omega^{\bullet-1, an})) & \xrightarrow{H^k OL^{an}} & \mathbb{H}_{usu}^k(X^{an}, \text{An}^* \partial \text{An}_* \Omega^{\bullet-1, an}) \\
\downarrow H^k(e) & & \downarrow H^k(e) \\
\mathbb{H}_{usu}^k(X^{an}, \Omega_{\log}^{\bullet, an}) & \xrightarrow{H^k OL^{an}} & H_{DR}^k(X) = \mathbb{H}_{usu}^k(X^{an}, \Omega^{\bullet, an}) \\
\downarrow H^k(q) & & \downarrow H^k(q) \\
\mathbb{H}_{usu}^k(X^{an}, \Omega_{\log}^{\bullet, an, ne}) & \xrightarrow{H^k \bar{O}L^{an}} & \mathbb{H}_{usu}^k(X^{an}, \Omega^{\bullet, an} / (\text{An}^* \partial \text{An}_* \Omega^{\bullet-1, an}))
\end{array}$$

whose columns are exact. Let $Z \subset X$ a closed subset. Then, for $k \in \mathbb{Z}$, we also have the commutative diagram

$$\begin{array}{ccc}
\mathbb{H}_{usu, Z}^k(X^{an}, \Omega_{\log}^{\bullet, an} \cap (\text{An}^* \partial \text{An}_* \Omega^{\bullet-1, an})) & \xrightarrow{H^k OL^{an}} & \mathbb{H}_{usu, Z}^k(X^{an}, \text{An}^* \partial \text{An}_* \Omega^{\bullet-1, an}) \\
\downarrow H^k(e) & & \downarrow H^k(e) \\
\mathbb{H}_{usu, Z}^k(X^{an}, \Omega_{\log}^{\bullet, an}) & \xrightarrow{H^k OL^{an}} & H_{DR, Z}^k(X) = \mathbb{H}_{usu}^k(X^{an}, \Omega^{\bullet, an}) \\
\downarrow H^k(q) & & \downarrow H^k(q) \\
\mathbb{H}_{usu, Z}^k(X^{an}, \Omega_{\log}^{\bullet, an, ne}) & \xrightarrow{H^k \bar{O}L^{an}} & \mathbb{H}_{usu, Z}^k(X^{an}, \Omega^{\bullet, an} / (\text{An}^* \partial \text{An}_* \Omega^{\bullet-1, an}))
\end{array}$$

whose columns are exact.

Proposition 4. *Let $X \in \text{PSmVar}(\mathbb{C})$. Consider the morphism $\text{an}_X : X^{an} \rightarrow X$ in RTop given by analytical functor.*

(i) *The analytic De Rham cohomology class of an algebraic cycle is logarithmic and is of type (d, d) , that is, for $Z \in \mathcal{Z}^d(X)$*

$$[Z] := H^{2d} \Omega(\gamma_Z^\vee)([Z]) \subset H^{2d} OL_{X^{an}}(H_{usu}^d(X^{an}, \Omega_{X^{an}, \log}^d)) \subset H_{DR}^{2d}(X^{an}) = H_{DR}^{2d}(X).$$

(ii) *Conversely, any $w \in H^{2d} OL_{X^{an}}(H_{usu}^d(X^{an}, \Omega_{X^{an}, \log}^d))$ is the class of an algebraic cycle $Z \in \mathcal{Z}^d(X) \otimes \mathbb{Q}$, i.e. $w = [Z]$.*

(iii) *We have $H^j OL_{X^{an}}(H_{usu}^{j-l}(X^{an}, \Omega_{X^{an}, \log}^l)) = 0$ for $j, l \in \mathbb{Z}$ such that $2l > j$.*

Proof. Consider, for $j, l \in \mathbb{Z}$ and $X \in \text{SmVar}(\mathbb{C})$,

$$L^{l, j-l}(X) := H^j OL_{X^{an}}(H_{usu}^{j-l}(X^{an}, \Omega_{X^{an}, \log}^l)) := H^j OL^{an}(H^{j-l} \text{Hom}(\text{An}^* \mathbb{Z}(X), E_{usu}^\bullet(\Omega_{\log}^{l, an}))).$$

Consider also for $j, l \in \mathbb{Z}$, $X \in \text{SmVar}(\mathbb{C})$ and $Z \subset X$ a closed subset,

$$L_Z^{l, j-l}(X) := H^j OL_{X^{an}}(H_{usu, Z}^{j-l}(X^{an}, \Omega_{X^{an}, \log}^l)) := H^j OL^{an}(H^{j-l} \text{Hom}(\text{An}^* \mathbb{Z}(X, X \setminus Z), E_{usu}^\bullet(\Omega_{\log}^{l, an}))).$$

Consider, for $j, l \in \mathbb{Z}$ and $X \in \text{SmVar}(\mathbb{C})$,

$$L^{l, j-l, ne}(X) := H^j(\bar{O}L^{an})(H^{j-l} \text{Hom}(\text{An}^* \mathbb{Z}(X), E_{usu}^\bullet(\Omega_{\log}^{l, an, ne}))).$$

Consider also for $j, l \in \mathbb{Z}$, $X \in \text{SmVar}(\mathbb{C})$ and $Z \subset X$ a closed subset,

$$L_Z^{l, j-l, ne}(X) := H^j(\bar{O}L^{an})(H^{j-l} \text{Hom}(\text{An}^* \mathbb{Z}(X, X \setminus Z), E_{usu}^\bullet(\Omega_{\log}^{l, an, ne}))).$$

For each $l, j \in \mathbb{Z}$, the presheaf $H^{j-l}E_{usu}(\Omega_{\log}^{l,an,ne}) \in \text{PSh}(\text{AnSm}(\mathbb{C}))$ is \mathbb{A}^1 invariant by proposition 2(i). Indeed, for $Y \in \text{AnSm}(\mathbb{C})$ connected, take an open cover $Y = \cup_{i \in I} D_i$ by open balls $D_i \simeq D(0, 1)^{d_Y}$, and then we get the isomorphism

$$\begin{aligned} p_Y^* : H^{j-l}E_{usu}(\Omega_{\log}^{l,an,ne})(Y \times \mathbb{A}^1) &:= H_{usu}^{j-l}(Y \times \mathbb{A}^1, \Omega_{\log}^{l,an,ne}) \xrightarrow{\sim} \\ &\mathbb{H}^{j-l}(D_{\bullet} \times \mathbb{A}^1, \Omega_{\log}^{l,an,ne}) \xrightarrow{\sim} H^{j-l}\Gamma(D_{\bullet} \times \mathbb{A}^1, \Omega_{\log}^{l,an,ne}) \\ \xrightarrow{\sim} H^{j-l}\Gamma(D_{\bullet}, \Omega_{\log}^{l,an,ne}) &\xrightarrow{\sim} \mathbb{H}^{j-l}(D_{\bullet}, \Omega_{\log}^{l,an,ne}) \xrightarrow{\sim} H_{usu}^{j-l}(Y, \Omega_{\log}^{l,an,ne}) =: H^{j-l}E_{usu}(\Omega_{\log}^{l,an,ne})(Y) \end{aligned}$$

where $p_Y : X \times \mathbb{A}^1 \rightarrow Y$ is the projection, and the second and the fourth map is the isomorphism given by (i). This gives in particular, for $X \in \text{SmVar}(\mathbb{C})$ and $Z \subset X$ a smooth subvariety of (pure) codimension d , by proposition 1 an isomorphism

$$E_{usu}(\Omega_{\log}^{l,an,ne})(P_{Z,X}) : L_Z^{l,j-l,ne}(X) \xrightarrow{\sim} L_Z^{l,j-l,ne}(N_{Z/X}) \xrightarrow{\sim} L^{l-d,j-l-d,ne}(Z).$$

(i): Similar to the proof of [2] theorem 2.

(ii): Let $X \in \text{PSmVar}(\mathbb{C})$ and

$$\begin{aligned} \alpha = H^{2d}OL_{X^{an}}(\alpha) \in L^{d,d}(X) : &= H^{2d}OL_{X^{an}}(H_{usu}^d(X^{an}, \Omega_{X^{an}, \log}^d)) \\ &= H^{2d}OL^{an}(H^d \text{Hom}(\text{An}^* \mathbb{Z}(X), E_{usu}^{\bullet}(\Omega_{\log}^{d,an}))). \end{aligned}$$

Then,

$$\begin{aligned} H^{2d}(q)(\alpha) &= H^{2d}(q) \circ H^{2d}OL_{X^{an}}(\alpha) = H^{2d}\bar{O}L^{an} \circ H^{2d}(q)(\alpha) \\ \in L^{d,d,ne}(X) &:= H^{2d}\bar{O}L^{an}(H^d \text{Hom}(\text{An}^* \mathbb{Z}(X), E_{usu}^{\bullet}(\Omega_{\log}^{d,an,ne}))). \end{aligned}$$

Up to split X by its connected components, we may assume that X is connected. The case $d = d_X$ is trivial. So let assume $d \neq d_X$. Let $U \subset X$ an open subset such that there exists an etale map $e : U \rightarrow \mathbb{A}_{\mathbb{C}}^{d_U}$ such that $e : U \rightarrow e(U)$ is finite etale and $\mathbb{A}_{\mathbb{C}}^{d_U} \setminus e(U) \subset \mathbb{A}_{\mathbb{C}}^{d_U}$ is a divisor. Denote $j : U \hookrightarrow X$ the open embedding. By proposition 2(ii), we have

$$j^* H^{2d}(q)(\alpha) = 0 \in H_{usu}^d(U^{an}, \Omega_{\log}^{d,an,ne}) = 0.$$

Considering a divisor $X \setminus U \subset D \subset X$, we get

$$H^{2d}(q)(\alpha) = H^d E_{usu}^{\bullet}(\Omega_{\log}^{d,an,ne})(\gamma_D^{\vee})(\alpha'), \alpha' \in L_D^{l,j-l,ne}(X).$$

Denote $D^{\circ} \subset D$ its smooth locus and $l : X^{\circ} \hookrightarrow X$ a Zariski open subset such that $X^{\circ} \cap D = D^{\circ}$. We then have

$$l^* \alpha' \in L_{D^{\circ}}^{d,d}(X^{\circ}) = L^{d-1,d-1}(D^{\circ})$$

We repeat this procedure with each connected components of D° instead of X . By a finite induction of d steps, we get

$$H^{2d}(q)(\alpha) = H^d E_{usu}^{\bullet}(\Omega_{\log}^{d,an,ne})(\gamma_Z^{\vee})(\alpha'), \alpha' \in L_Z^{d,d,ne}(X) = L^{0,0,ne}(Z^{\circ}).$$

with $Z := D_d \subset \dots \subset D \subset X$ a pure codimension d (Zariski) closed subset, thus

$$\alpha = \sum_i n_i [Z_i] + \beta \in H_{DR}^{2d}(X), \beta \in H^{2d}(e)(\mathbb{H}_{usu}^{2d}(X^{an}, \text{An}^* \partial \text{An}_* \Omega^{\bullet-1,an}))$$

where $n_i \in \mathbb{Q}$ and $(Z_i)_{1 \leq i \leq t} \subset Z$ are the irreducible components of Z . But since X is projective, the E_1 degenerescence of the spectral sequence associated to the trivial filtration of the De Rham complex shows that $H^{2d}(e)(\mathbb{H}_{usu}^{2d}(X^{an}, \text{An}^* \partial \text{An}_* \Omega^{\bullet-1,an})) = 0$. Hence,

$$\alpha = \sum_i n_i [Z_i] \in H_{DR}^{2d}(X).$$

(iii): Let $j < 2l$. Let $X \in \text{PSmVar}(\mathbb{C})$ and $w \in L^{l,j-l}(X)$. By the proof of (ii) there exists $Z \subset X$ a closed subset of pure codimension $j - l$ such that

$$H^j(q)(w) = H^{j-l} E_{usu}^\bullet(\Omega_{\log}^{l,an,ne})(\gamma_Z^\vee)(w'), w' \in L_Z^{l,j-l,ne}(X).$$

For $Z' \subset X$ a closed subset of pure codimension c , consider a desingularisation $\epsilon : \tilde{Z}' \rightarrow Z'$ of Z' and denote $n : \tilde{Z}' \xrightarrow{\epsilon} Z' \subset X$. The morphism in $\text{DA}(k)$

$$G_{Z',X} : M(X) \xrightarrow{D(\mathbb{Z}(n))} M(\tilde{Z}')(c)[2c] \xrightarrow{\mathbb{Z}(\epsilon)} M(Z')(c)[2c]$$

where $D : \text{Hom}_{\text{DA}(k)}(M_c(\tilde{Z}'), M_c(X)) \xrightarrow{\sim} \text{Hom}_{\text{DA}(k)}(M(X), M(\tilde{Z}')(c)[2c])$ is the duality isomorphism from the six functor formalism (moving lemma of Suzlin and Voevodsky) and $\mathbb{Z}(n) := \text{ad}(n_l, n^1)(a_X^! \mathbb{Z})$, is given by a morphism in $C(\text{SmVar}(k))$

$$\hat{G}_{Z',X} : \mathbb{Z}_{tr}(X) \rightarrow E_{et}(C_* \mathbb{Z}_{tr}(Z'))(c)[2c].$$

Let $l : X^o \hookrightarrow X$ be an open embedding such that $Z^o := Z \cap X^o$ is the smooth locus of Z with good reduction. We then have the following commutative diagram of abelian groups

$$\begin{array}{ccccccc} 0 & \longrightarrow & H_{usu,Z}^{j-l}(X^{an}, \Omega_{\log}^{l,an,ne}) & \xrightarrow{l^*} & H_{usu,Z^o}^{j-l}(X^{o,an}, \Omega_{\log}^{l,an,ne}) & \xrightarrow{\partial} & H_{usu,Z \setminus Z^o}^{j-l+1}(X^{an}, \Omega_{\log}^{l,an,ne}) \longrightarrow \dots \\ & & \Omega(\hat{G}_{Z,X}) \uparrow & & \Omega(P_{Z^o, X^o}) \uparrow & & \Omega(\hat{G}_{Z,X}) \uparrow \\ 0 & \longrightarrow & H_{usu}^0(Z^{an}, \Omega_{\log}^{2l-j,an,ne}) & \xrightarrow{l^*} & H_{usu}^0(Z^{o,an}, \Omega_{\log}^{2l-j,an,ne}) & \xrightarrow{\partial} & H_{usu,Z \setminus Z^o}^1(Z^{an}, \Omega_{\log}^{2l-j,an,ne}) \longrightarrow \dots \end{array}$$

whose rows are exact sequences. Consider

$$l^* w' = \Omega(P_{Z^o, X^o})(w^o) \in H_{usu,Z^o}^{j-l}(X^{o,an}, \Omega_{\log}^{l,an,ne}), w^o \in H_{et}^0(Z^{o,an}, \Omega_{\log}^{2l-j,an,ne}).$$

Since $\partial l^* w' = 0 \in H_{usu,Z \setminus Z^o}^{j-l+1}(X^{an}, \Omega_{\log}^{l,an,ne})$, we get, considering the smooth locus with good reduction $(Z \setminus Z^o)^o \subset Z \setminus Z^o$ and $X^{oo} \subset X$ is an open subset such that $X \cap (Z \setminus Z^o)^o = X^{oo} \cap (Z \setminus Z^o)$,

$$\partial w^o = 0 \in H_{usu,Z \setminus Z^o}^1(Z^{an}, \Omega_{\log}^{2l-j,an,ne}),$$

since $H_{usu,(Z \setminus Z^o) \setminus (Z \setminus Z^o)^o}^1(Z^{an}, \Omega_{\log}^{2l-j,an,ne}) = 0$ for dimension reasons, that is

$$H^j(q)(w) = \Omega(\hat{G}_{Z,X})(w'), \text{ with } w' \in H_{et}^0(Z^{an}, \Omega_{\log}^{2l-j,an,ne}).$$

Hence $H^j(q)(w) = 0$ since $H^0(Z'^{an}, \Omega_{\log}^k) = 0$ for all $k > 0$ and all $Z' \in \text{PVar}(\mathbb{C})$. But since X is projective, the E_1 degenerescence of the spectral sequence associated to the trivial filtration of the De Rham complex shows that $H^j(e)(\mathbb{H}_{usu}^j(X^{an}, \text{An}^* \partial \text{An}_* \Omega^{\bullet-1,an})) = 0$. Hence $w = 0$. \square

5 Hodge Conjecture for complex smooth projective hypersurfaces

We now state and prove using the result of section 3 and section 4 the main result of this article :

Theorem 1. *Let $X = V(f) \subset \mathbb{P}_{\mathbb{C}}^N$ be a smooth projective hypersurface, with $N = 2p + 1$ odd. Let $\lambda \in F^p H^{2p}(X^{an}, \mathbb{Q})$. Then there exists an algebraic cycle $Z \in \mathbb{Z}^p(X)$ such that $\lambda = [Z]$.*

Proof. Consider $U := \mathbb{P}_{\mathbb{C}}^N \setminus X$. Consider $Res_{X, \mathbb{P}^N} : M(X) \rightarrow M(U)[-1]$ the canonical map given in definition 1. We have the commutative diagram of abelian group

$$\begin{array}{ccc}
H_{DR}^N(U) & \xrightarrow{H^N \Omega(Res_{X, \mathbb{P}^N})} & H_{DR}^{N-1}(X)_v \\
\uparrow H^N \alpha(U^{an})=ev(U)^{-1} & & \uparrow H^{N-1} \alpha(X^{an})=ev(X)^{-1} \\
H_{\text{sing}}^N(U^{an}, \mathbb{C}) & \xrightarrow{(H^N \text{Bti}(Res_{X, \mathbb{P}^N})) \otimes_{\mathbb{Q}} \mathbb{C}} & H_{\text{sing}}^{N-1}(X^{an}, \mathbb{C})_v
\end{array}$$

whose arrows are isomorphisms. Denote $w(\lambda) := H^{N-1} \alpha(X^{an})(\lambda) \in F^p H_{DR}^{2p}(X)$. We have

$$\lambda = H^{N-1} \text{Bti}(Res_{X, \mathbb{P}^N})(\alpha), \text{ with } \alpha \in H_{\text{sing}}^N(U^{an}, \mathbb{Q}).$$

Then

$$w(\lambda) = H^{N-1} \Omega(Res_{X, \mathbb{P}^N})(w(\alpha)), \text{ with } w(\alpha) := H^N \alpha(U^{an})(\alpha) \in F^{p+1} H_{DR}^N(U).$$

Since $ev(U)(w(\alpha)) = \alpha \in H_{\text{sing}}^N(U^{an}, 2i\pi\mathbb{Q})$, we have by proposition 3,

$$w(\alpha) \in H^N OL_{U^{an}}(H_{usu}^{p+1}(U^{an}, \Omega_{U^{an}, \log}^{N-p}))$$

Hence

$$w(\lambda) = H^{N-1} OL_{X^{an}}(\Omega_{\log}^{N-p, an}(Res_{X, \mathbb{P}^N})(w(\alpha))) \in OL_{X^{an}}(H_{usu}^p(X^{an}, \Omega_{X^{an}, \log}^p)).$$

Hence by proposition 4(ii), $w(\lambda) = [Z]$ with $Z \in \mathcal{Z}^p(X)$. □

References

- [1] J.Bouali, *The De Rham, complex and p adic Hodge realization functor for relative motives of algebraic varieties over a field of characteristic zero*, arxiv 2203.10875, preprint, 2022.
- [2] J.Bouali, *De Rham logarithmic classes and Tate conjecture*, arxiv preprint 2303.09932, 2023
- [3] D.C.Cisinski, F.Deglise, *Triangulated categories of mixed motives*, arxiv preprint 0912.2110, 2009-arxiv.org.
- [4] F.Lecomte, N.Wach, *Réalisation de Hodge des motifs de Voevodsky*, manuscripta mathematica.141,663-697, Springer-Verlag, Berlin, 2012.
- [5] C.Mazza, V.Voevodsky, C.Weibel, *Lecture Notes on Motivic Cohomology*, Clay mathematics monographs, Vol 2, AMS, Providence RI, 2006.
- [6] C.Voisin, *Théorie de Hodge et géométrie algébrique complexe*, SMF, cours spécialisés, 2002.

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