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Hodge conjecture for projective hypersurfaces

Johann Bouali

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Abstract

We show that an Hodge class of a complex smooth projective hypersurface is an analytic logarithmic De Rham class. On the other hand we show that for a complex smooth projective variety an analytic logarithmic De Rham class of type (d, d) is the class of codimension d algebraic cycle and that there is no an analytic logarithmic De Rham class of type (p, q) , $p \neq q$. We deduce the Hodge conjecture for smooth projective hypersurfaces.

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1 Introduction

1.1 Complex analytic De Rham logarithmic classes

In [2], we introduced, for X an algebraic variety over a field, the notion of logarithmic De Rham cohomology classes. For X a complex algebraic variety, we also introduce the notion of logarithmic analytic De Rham cohomology classes. In proposition 4 we give an analytic analogue of [2] theorem 2, that is if X is a smooth complex projective variety,

- a logarithmic analytic De Rham cohomology class of type (d, d) is the class of a codimension d algebraic cycle,
- there are no non trivial logarithmic analytic De Rham cohomology classes of type (p, q) for $p \neq q$, i.e. $H^{2d}OL_{X^{an}}(H_{usu}^{d-k}(X^{an}, \Omega_{X^{an}, \log}^{d+k})) = 0$ for $k \neq 0$.

The proof of proposition 4 is motivic and similar to the proof of [2] theorem 2 : a logarithmic analytic class of bidegree (p, q) is acyclic for the etale topology on each open subset $U \subset X$ such that there exists an etale map $e : U \rightarrow A_{\mathbb{C}}^{d_U}$ (proposition 2), this allows us to proceed by a finite induction using the crucial fact that the purity isomorphism for De Rham cohomology preserve logarithmic analytic classes since the purity isomorphism is motivic (c.f. [3], see proposition 1). The GAGA comparison theorem for de Rham cohomology of X induce, by proposition 4 and [2] theorem 2, a GAGA comparison result for logarithmic De Rham classes.

1.2 Hodge conjecture for smooth projective hypersurfaces

Let $X = V(f) \subset \mathbb{P}_{\mathbb{C}}^N$ a smooth projective hypersurface. Denote $j : U := \mathbb{P}_{\mathbb{C}}^N \setminus X \hookrightarrow \mathbb{P}_{\mathbb{C}}^N$ the open complementary subset. Using the fact the filtered De Rham complex of U

$$\iota_{\partial} : \Omega_{\mathbb{P}^N, an}^{N-p}(\log X)^{\partial=0}[-p] \hookrightarrow (\Omega_{\mathbb{P}^N}^{\bullet} \otimes_{\mathcal{O}_{\mathbb{P}^N}} F^{\bullet-p} j_* \mathcal{O}_U)^{an}$$

is locally acyclic for the complex topology, more precisely is acyclic on open balls (see e.g.[6]), we show, considering an open cover $\mathbb{P}_{\mathbb{C}}^N = \cup_{i=1}^n B_i$ by open balls $\phi_i : B_i \simeq D(0, 1)^N$ such that $\phi_i(B_i \cap X) = 0 \times D(0, 1)^N$, that an Hodge class $\alpha \in F^p H^N(U^{an}, \mathbb{Q})$ is logarithmic, that is

$$\alpha \in H^N OL_{U^{an}}(H_{usu}^p(U^{an}, \Omega_{U^{an}, \log}^{N-p}))$$

(proposition 3). We get that, if $N = 2p + 1$ and $\lambda \in F^p H^{2p}(X^{an}, \mathbb{Q})$ is an Hodge class,

$$\lambda = Res_{X, \mathbb{P}^N}(\alpha) + m[H] \in H^N OL_{X^{an}}(H_{usu}^p(X^{an}, \Omega_{X^{an}, \log}^{N-p})),$$

where $Res_{X, \mathbb{P}^N} : H^N(U^{an}, \mathbb{C}) \xrightarrow{\sim} H^{2p}(X^{an}, \mathbb{C})$ is the residu map, using the fact that since the residu map is motivic (see definition 1), it preserve the logarithmic De Rham classes. Proposition 4 (ii) then implies that $\lambda = [Z]$, $Z \in \mathcal{Z}^p(X)$, is the class of an algebraic cycle.

2 Preliminaries and Notations

2.1 Notations

- Denote by Top the category of topological spaces and RTop the category of ringed spaces.
- Denote by Cat the category of small categories and RCat the category of ringed topos.
- For $\mathcal{S} \in \text{Cat}$ and $X \in \mathcal{S}$, we denote $\mathcal{S}/X \in \text{Cat}$ the category whose objects are $Y/X := (Y, f)$ with $Y \in \mathcal{S}$ and $f : Y \rightarrow X$ is a morphism in \mathcal{S} , and whose morphisms $\text{Hom}((Y', f'), (Y, f))$ consists of $g : Y' \rightarrow Y$ in \mathcal{S} such that $f \circ g = f'$.
- For $(\mathcal{S}, \mathcal{O}_{\mathcal{S}}) \in \text{RCat}$ a ringed topos, we denote by
 - $\text{PSh}(\mathcal{S})$ the category of presheaves of $\mathcal{O}_{\mathcal{S}}$ modules on \mathcal{S} and $\text{PSh}_{\mathcal{O}_{\mathcal{S}}}(\mathcal{S})$ the category of presheaves of $\mathcal{O}_{\mathcal{S}}$ modules on \mathcal{S} , whose objects are $\text{PSh}_{\mathcal{O}_{\mathcal{S}}}(\mathcal{S})^0 := \{(M, m), M \in \text{PSh}(\mathcal{S}), m : M \otimes \mathcal{O}_{\mathcal{S}} \rightarrow M\}$, together with the forgetful functor $o : \text{PSh}(\mathcal{S}) \rightarrow \text{PSh}_{\mathcal{O}_{\mathcal{S}}}(\mathcal{S})$,
 - $C(\mathcal{S}) = C(\text{PSh}(\mathcal{S}))$ and $C_{\mathcal{O}_{\mathcal{S}}}(\mathcal{S}) = C(\text{PSh}_{\mathcal{O}_{\mathcal{S}}}(\mathcal{S}))$ the big abelian category of complexes of presheaves of $\mathcal{O}_{\mathcal{S}}$ modules on \mathcal{S} ,
 - $C_{\mathcal{O}_{\mathcal{S}}(2)fil}(\mathcal{S}) := C_{(2)fil}(\text{PSh}_{\mathcal{O}_{\mathcal{S}}}(\mathcal{S})) \subset C(\text{PSh}_{\mathcal{O}_{\mathcal{S}}}(\mathcal{S}), F, W)$, the big abelian category of (bi)filtered complexes of presheaves of $\mathcal{O}_{\mathcal{S}}$ modules on \mathcal{S} such that the filtration is biregular and $\text{PSh}_{\mathcal{O}_{\mathcal{S}}(2)fil}(\mathcal{S}) := (\text{PSh}_{\mathcal{O}_{\mathcal{S}}}(\mathcal{S}), F, W)$.
- Let $(\mathcal{S}, \mathcal{O}_{\mathcal{S}}) \in \text{RCat}$ a ringed topos with topology τ . For $F \in C_{\mathcal{O}_{\mathcal{S}}}(\mathcal{S})$, we denote by $k : F \rightarrow E_{\tau}(F)$ the canonical flasque resolution in $C_{\mathcal{O}_{\mathcal{S}}}(\mathcal{S})$ (see [1]). In particular for $X \in \mathcal{S}$, $H^*(X, E_{\tau}(F)) \xrightarrow{\sim} \mathbb{H}_{\tau}^*(X, F)$.

- For $f : \mathcal{S}' \rightarrow \mathcal{S}$ a morphism with $\mathcal{S}, \mathcal{S}' \in \text{RCat}$, endowed with topology τ and τ' respectively, we denote for $F \in C_{O_S}(\mathcal{S})$ and each $j \in \mathbb{Z}$,

$$\begin{aligned} - f^* &:= H^j \Gamma(\mathcal{S}, k \circ \text{ad}(f^*, f_*)(F)) : \mathbb{H}^j(\mathcal{S}, F) \rightarrow \mathbb{H}^j(\mathcal{S}', f^*F), \\ - f^* &:= H^j \Gamma(\mathcal{S}, k \circ \text{ad}(f^{*mod}, f_*)(F)) : \mathbb{H}^j(\mathcal{S}, F) \rightarrow \mathbb{H}^j(\mathcal{S}', f^{*mod}F), \end{aligned}$$

the canonical maps.

- For $X \in \text{Top}$ and $Z \subset X$ a closed subset, denoting $j : X \setminus Z \hookrightarrow X$ the open complementary, we will consider

$$\Gamma_Z^\vee \mathbb{Z}_X := \text{Cone}(\text{ad}(j!, j^*)(\mathbb{Z}_X) : j!j^* \mathbb{Z}_X \hookrightarrow \mathbb{Z}_X) \in C(X)$$

and denote for short $\gamma_Z^\vee := \gamma_X^\vee(\mathbb{Z}_X) : \mathbb{Z}_X \rightarrow \Gamma_Z^\vee \mathbb{Z}_X$ the canonical map in $C(X)$.

- Denote by $\text{Sch} \subset \text{RTop}$ the subcategory of schemes (the morphisms are the morphisms of locally ringed spaces). We denote by $\text{PSch} \subset \text{Sch}$ the full subcategory of proper schemes. For a field k , we consider $\text{Sch}/k := \text{Sch}/\text{Spec } k$ the category of schemes over $\text{Spec } k$, that is whose object are $X := (X, a_X)$ with $X \in \text{Sch}$ and $a_X : X \rightarrow \text{Spec } k$ a morphism and whose objects are morphism of schemes $f : X' \rightarrow X$ such that $f \circ a_{X'} = a_X$. We then denote by

- $\text{Var}(k) = \text{Sch}^{ft}/k \subset \text{Sch}/k$ the full subcategory consisting of algebraic varieties over k , i.e. schemes of finite type over k ,
- $\text{PVar}(k) \subset \text{QPVar}(k) \subset \text{Var}(k)$ the full subcategories consisting of quasi-projective varieties and projective varieties respectively,
- $\text{PSmVar}(k) \subset \text{SmVar}(k) \subset \text{Var}(k)$, $\text{PSmVar}(k) := \text{PVar}(k) \cap \text{SmVar}(k)$, the full subcategories consisting of smooth varieties and smooth projective varieties respectively.

- Denote by $\text{AnSp}(\mathbb{C}) \subset \text{RTop}$ the subcategory of analytic spaces over \mathbb{C} , and by $\text{AnSm}(\mathbb{C}) \subset \text{AnSp}(\mathbb{C})$ the full subcategory of smooth analytic spaces (i.e. complex analytic manifold).

- Let $(X, O_X) \in \text{RTop}$. We consider its De Rham complex $\Omega_X^\bullet := DR(X)(O_X)$.

- Let $X \in \text{Sch}$. Considering its De Rham complex $\Omega_X^\bullet := DR(X)(O_X)$, we have for $j \in \mathbb{Z}$ its De Rham cohomology $H_{DR}^j(X) := \mathbb{H}^j(X, \Omega_X^\bullet)$.
- Let $X \in \text{Var}(k)$. Considering its De Rham complex $\Omega_X^\bullet := \Omega_{X/k}^\bullet := DR(X/k)(O_X)$, we have for $j \in \mathbb{Z}$ its De Rham cohomology $H_{DR}^j(X) := \mathbb{H}^j(X, \Omega_X^\bullet)$. The differentials of $\Omega_X^\bullet := \Omega_{X/k}^\bullet$ are by definition k -linear, thus $H_{DR}^j(X) := \mathbb{H}^j(X, \Omega_X^\bullet)$ has a structure of a k vector space.
- Let $X \in \text{AnSp}(\mathbb{C})$. Considering its De Rham complex $\Omega_X^\bullet := DR(X)(O_X)$, we have for $j \in \mathbb{Z}$ its De Rham cohomology $H_{DR}^j(X) := \mathbb{H}^j(X, \Omega_X^\bullet)$.

- For $X \in \text{AnSp}(\mathbb{C})$, we denote $\alpha(X) : \mathbb{C}_X \hookrightarrow \Omega_X^\bullet$ the embedding in $C(X)$. For $X \in \text{AnSm}(\mathbb{C})$, $\alpha(X) : \mathbb{C}_X \hookrightarrow \Omega_X^\bullet$ is an equivalence usu local by Poincare lemma.

- We denote $\mathbb{I}^n := [0, 1]^n \in \text{Diff}(\mathbb{R})$ (with boundary). For $X \in \text{Top}$ and R a ring, we consider its singular cochain complex

$$C_{\text{sing}}^*(X, R) := (\mathbb{Z} \text{Hom}_{\text{Top}}(\mathbb{I}^*, X)^\vee) \otimes R$$

and for $l \in \mathbb{Z}$ its singular cohomology $H_{\text{sing}}^l(X, R) := H^n C_{\text{sing}}^*(X, R)$. For $f : X' \rightarrow X$ a continuous map with $X, X' \in \text{Top}$, we have the canonical map of complexes

$$f^* : C_{\text{sing}}^*(X, R) \rightarrow C_{\text{sing}}^*(X', R), \sigma \mapsto f^* \sigma := (\gamma \mapsto \sigma(f \circ \gamma)).$$

In particular, we get by functoriality the complex

$$C_{X, R \text{ sing}}^* \in C_R(X), (U \subset X) \mapsto C_{\text{sing}}^*(U, R)$$

We recall we have

- For $X \in \text{Top}$ locally contractile, e.g. $X \in \text{CW}$, and R a ring, the inclusion in $C_R(X)$ $c_X : R_X \rightarrow C_{X,R \text{ sing}}^*$ is by definition an equivalence top local and that we get by the small chain theorem, for all $l \in \mathbb{Z}$, an isomorphism $H^l c_X : H^l(X, R_X) \xrightarrow{\sim} H_{\text{sing}}^l(X, R)$.
- For $X \in \text{Diff}(\mathbb{R})$, the restriction map

$$r_X : \mathbb{Z} \text{Hom}_{\text{Diff}(\mathbb{R})}(\mathbb{I}^*, X)^\vee \rightarrow C_{\text{sing}}^*(X, R), w \mapsto w : (\phi \mapsto w(\phi))$$

is a quasi-isomorphism by Whitney approximation theorem.

- We will consider the morphism of site given by the analytification functor

$$\text{An} : \text{AnSp}(\mathbb{C}) \rightarrow \text{Var}(\mathbb{C}), X \mapsto \text{An}(X) := X^{an}, (f : X' \rightarrow X) \mapsto \text{An}(f) := f^{an}.$$

For $X \in \text{Var}(\mathbb{C})$, we denote by

$$\text{an}_X := \text{An}|_{X^{et}} := X^{an,et} \rightarrow X^{et}, \text{an}_X(U \rightarrow X) := U^{an},$$

its restriction to the small etale site $o_X(X^{et}) \subset \text{Var}(\mathbb{C})$, where $o_X(U \rightarrow X) = U$, in particular we have the following commutative diagram

$$\begin{array}{ccc} \text{AnSp}(\mathbb{C}) & \xrightarrow{\text{An}} & \text{Var}(\mathbb{C}), \\ \downarrow o_{X^{an}} & & \downarrow o_X \\ X^{an,et} & \xrightarrow{\text{an}_X} & X^{et} \end{array}$$

where $o_{X^{an}}(U \hookrightarrow X^{an}) = U$. We get the morphism in RTop $\text{an}_X := \text{an}_X|_{O(X)} : X^{an} \rightarrow X$, $o : O(X) \subset X^{et}$ being the embedding consisting of the Zariski open subsets of X .

2.2 Complex integral periods

Let k a field of characteristic zero.

Let $X \in \text{SmVar}(k)$ a smooth variety. Let $X = \cup_{i=1}^s X_i$ an open affine cover. We have for $\sigma : k \hookrightarrow \mathbb{C}$ an embedding, the evaluation period embedding map which is the morphism of bi-complexes

$$ev(X) \bullet : \Gamma(X_\bullet, \Omega_{X_\bullet}^\bullet) \rightarrow \mathbb{Z} \text{Hom}_{\text{Diff}}(\mathbb{I}^\bullet, X_{\mathbb{C},\bullet}^{an})^\vee \otimes \mathbb{C},$$

$$w_I^l \in \Gamma(X_I, \Omega_{X_I}^l) \mapsto (ev(X)_I^l(w_I^l)) : \phi_I^l \in \mathbb{Z} \text{Hom}_{\text{Diff}}(\mathbb{I}^l, X_{\mathbb{C},I}^{an})^\vee \otimes \mathbb{C} \mapsto ev_I^l(w_I^l)(\phi_I^l) := \int_{\mathbb{I}^l} \phi_I^{l*} w_I^l$$

given by integration. By taking all the affine open cover $(j_i : X_i \hookrightarrow X)$ of X , we get for $\sigma : k \hookrightarrow \mathbb{C}$, the evaluation period embedding map

$$ev(X) := \varinjlim_{(j_i : X_i \hookrightarrow X)} ev(X) \bullet : \varinjlim_{(j_i : X_i \hookrightarrow X)} \Gamma(X_\bullet, \Omega_{X_\bullet}^\bullet) \rightarrow \varinjlim_{(j_i : X_i \hookrightarrow X)} \mathbb{Z} \text{Hom}_{\text{Diff}(\mathbb{R})}(\mathbb{I}^\bullet, X_{\mathbb{C},\bullet}^{an})^\vee \otimes \mathbb{C}$$

It induces in cohomology, for $j \in \mathbb{Z}$, the evaluation period map

$$H^j ev(X) = H^j ev(X) \bullet : H_{DR}^j(X) = H^j \Gamma(X_\bullet, \Omega_{X_\bullet}^\bullet) \rightarrow H_{\text{sing}}^j(X_{\mathbb{C}}^{an}, \mathbb{C}) = H^j(\text{Hom}_{\text{Diff}(\mathbb{R})}(\mathbb{I}^\bullet, X_{\mathbb{C},\bullet}^{an})^\vee \otimes \mathbb{C}).$$

which does NOT depend on the choice of the affine open cover by acyclicity of quasi-coherent sheaves on affine noetherian schemes for the left hand side and from Mayer-Vietoris quasi-isomorphism for singular cohomology of topological spaces and Whitney approximation theorem for differential manifolds for the right hand side.

Let $X \in \text{SmVar}(k)$. Then,

$$H^* ev(X_{\mathbb{C}}) = H^* R\Gamma(X_{\mathbb{C}}^{an}, \alpha(X))^{-1} \circ \Gamma(X_{\mathbb{C}}^{an}, E_{zar}(\Omega(\text{an}_X))) : H_{DR}^*(X_{\mathbb{C}}) \xrightarrow{\sim} H_{\text{sing}}^*(X_{\mathbb{C}}^{an}, \mathbb{C})$$

is the canonical isomorphism induced by the analytical functor and the quasi-isomorphism $\alpha(X) : \mathbb{C}_{X_{\mathbb{C}}^{an}} \hookrightarrow \Omega_{X_{\mathbb{C}}^{an}}^\bullet$, which gives the periods elements $H^* ev(X)(H_{DR}^*(X)) \subset H_{\text{sing}}^*(X_{\mathbb{C}}^{an}, \mathbb{C})$.

2.3 Algebraic cycles and motives

For $X \in \text{Sch}$ noetherian irreducible and $d \in \mathbb{N}$, we denote by $\mathcal{Z}^d(X)$ the group of algebraic cycles of codimension d , which is the free abelian group generated by irreducible closed subsets of codimension d .

For $X, X' \in \text{Sch}$ noetherian, with X' irreducible, we denote $\mathcal{Z}^{fs/X'}(X' \times X) \subset \mathcal{Z}_{d_{X'}}(X' \times X)$ which consist of algebraic cycles $\alpha = \sum_i n_i \alpha_i \in \mathcal{Z}_{d_{X'}}(X' \times X)$ such that, denoting $\text{supp}(\alpha) = \cup_i \alpha_i \subset X' \times X$ its support and $p' : X' \times X \rightarrow X'$ the projection, $p'_{|\text{supp}(\alpha)} : \text{supp}(\alpha) \rightarrow X'$ is finite surjective.

Let k a field. We denote by $\text{Cor SmVar}(k)$ the category such that the objects are $\{X \in \text{SmVar}(k)\}$ and such that $\text{Hom}_{\text{Cor SmVar}(k)}(X', X) := \mathcal{Z}^{fs/X'}(X' \times X)$. See [3] for the composition law. We denote by $\text{Tr} : \text{Cor SmVar}(k) \rightarrow \text{SmVar}(k)$ the morphism of site given by the embedding $\text{Tr} : \text{SmVar}(k) \hookrightarrow \text{Cor SmVar}(k)$. Let $F \in \text{PSh}(\text{SmVar}(k))$. We say that F admits transfers if $F = \text{Tr}_* F$ with $F \in \text{PSh}(\text{Cor SmVar}(k))$. We recall that $\text{DA}(k) := \text{Ho}_{\mathbb{A}^1, \text{et}}(C(\text{SmVar}(k)))$ is then the derived category of mixed motives.

We have (see e.g. [1]) the Hodge realization functor

$$\mathcal{F}^{Hdg} : \text{DA}(\mathbb{C}) \rightarrow D(\text{MHS}), M \mapsto \mathcal{F}^{Hdg}(M) := \iota^{-1}(\mathcal{F}^{FDR}(M), \text{Bti}^* M, \alpha(M))$$

For $X \in \text{Var}(k)$ and $Z \subset X$ a closed subset, denoting $j : X \setminus Z \hookrightarrow X$ the open complementary, we will consider

$$\Gamma_Z^\vee \mathbb{Z}_X := \text{Cone}(\text{ad}(j_\#, j^*)(\mathbb{Z}_X) : \mathbb{Z}_X \hookrightarrow \mathbb{Z}_X) \in C(\text{Var}(k)^{\text{sm}}/X)$$

and denote for short $\gamma_Z^\vee := \gamma_Z^\vee(\mathbb{Z}_X) : \mathbb{Z}_X \rightarrow \Gamma_Z^\vee \mathbb{Z}_X$ the canonical map in $C(\text{Var}(k)^{\text{sm}}/X)$. Denote $a_X : X \rightarrow \text{Spec } k$ the structural map. For $X \in \text{Var}(k)$ and $Z \subset X$ a closed subset, we have the motive of X with support in Z defined as

$$M_Z(X) := a_{X!} \Gamma_Z^\vee a_X^! \mathbb{Z} \in \text{DA}(k).$$

If $X \in \text{SmVar}(k)$, we will also consider

$$a_{X\#} \Gamma_Z^\vee \mathbb{Z}_X := \text{Cone}(a_{X\#} \circ \text{ad}(j_\#, j^*)(\mathbb{Z}_X) : \mathbb{Z}(U) \hookrightarrow \mathbb{Z}(X)) =: \mathbb{Z}(X, X \setminus Z) \in C(\text{SmVar}(k)).$$

Then for $X \in \text{SmVar}(k)$ and $Z \subset X$ a closed subset

$$M_Z(X) := a_{X!} \Gamma_Z^\vee a_X^! \mathbb{Z} = a_{X\#} \Gamma_Z^\vee \mathbb{Z}_X =: \mathbb{Z}(X, X \setminus Z) \in \text{DA}(k).$$

- Let $(X, Z) \in \text{Sch}^2$ with $X \in \text{Sch}$ a noetherian scheme and $Z \subset X$ a closed subset. We have the deformation $(D_Z X, \mathbb{A}_Z^1) \rightarrow \mathbb{A}^1$, $(D_Z X, \mathbb{A}_Z^1) \in \text{Sch}^2$ of (X, Z) by the normal cone $C_{Z/X} \rightarrow Z$, i.e. such that

$$(D_Z X, \mathbb{A}_Z^1)_s = (X, Z), s \in \mathbb{A}^1 \setminus 0, (D_Z X, \mathbb{A}_Z^1)_0 = (C_{Z/X}, Z).$$

We denote by $i_1 : (X, Z) \hookrightarrow (D_Z X, \mathbb{A}_Z^1)$ and $i_0 : (C_{Z/X}, Z) \hookrightarrow (D_Z X, \mathbb{A}_Z^1)$ the closed embeddings in Sch^2 .

- Let k a field of characteristic zero. Let $X \in \text{SmVar}(k)$. For $Z \subset X$ a closed subset of pure codimension c , consider a desingularisation $\epsilon : \tilde{Z} \rightarrow Z$ of Z and denote $n : \tilde{Z} \xrightarrow{\epsilon} Z \subset X$. We have then the morphism in $\text{DA}(k)$

$$G_{Z,X} : M(X) \xrightarrow{D(\mathbb{Z}(n))} M(\tilde{Z})(c)[2c] \xrightarrow{\mathbb{Z}(\epsilon)} M(Z)(c)[2c]$$

where $D : \text{Hom}_{\text{DA}(k)}(M_c(\tilde{Z}), M_c(X)) \xrightarrow{\sim} \text{Hom}_{\text{DA}(k)}(M(X), M(\tilde{Z})(c)[2c])$ is the duality isomorphism from the six functors formalism (moving lemma of Suzlin and Voevodsky) and $\mathbb{Z}(n) := \text{ad}(n_!, n^!)(a_X^! \mathbb{Z})$, noting that $n_! = n_*$ since n is proper and that $a_X^! = a_X^*[d_X]$ and $a_Z^! = a_Z^*[d_Z]$ since X , resp. \tilde{Z} , are smooth (considering the connected components, we may assume X and \tilde{Z} of pure dimension).

We recall the following facts (see [3] and [1]):

Proposition 1. *Let k be a field a characteristic zero. Let $X \in \text{SmVar}(k)$ and $i : Z \subset X$ a smooth subvariety of pure codimension d . Then $C_{Z/X} = N_{Z/X} \rightarrow Z$ is a vector bundle of rank d . The closed embeddings $i_1 : (X, Z) \hookrightarrow (D_Z X, \mathbb{A}_Z^1)$ and $i_0 : (C_{Z/X}, Z) \hookrightarrow (D_Z X, \mathbb{A}_Z^1)$ in $\text{SmVar}^2(k)$ induces isomorphisms of motives $\mathbb{Z}(i_1) : M_Z(X) \xrightarrow{\sim} M_{\mathbb{A}_Z^1}(D_Z X)$ and $\mathbb{Z}(i_0) : M_Z(N_{Z/X}) \xrightarrow{\sim} M_{\mathbb{A}_Z^1}(D_Z X)$ in $\text{DA}(k)$. We get the excision isomorphism in $\text{DA}(k)$*

$$P_{Z,X} := \mathbb{Z}(i_0)^{-1} \circ \mathbb{Z}(i_1) : M_Z(X) \xrightarrow{\sim} M_Z(N_{Z/X}).$$

We have

$$\text{Th}(N_{Z/X}) \circ P_{Z,X} \circ \gamma_Z^\vee(\mathbb{Z}_X) = G_{Z,X} := D(\mathbb{Z}(i)) : M(X) \rightarrow M(Z)(d)[2d].$$

Proof. See [3]. □

Definition 1. *Let k be a field a characteristic zero. Let $X \in \text{SmVar}(k)$ and $i : Z \subset X$ a smooth subvariety of pure codimension d . We consider using proposition 1 the canonical map in $\text{DA}(k)$*

$$\text{Res}_{Z,X} : M(Z)(d)[-2d] \xrightarrow{P_{Z,X}^{-1} \circ \text{Th}(N_{Z/X})^{-1}} M_Z(X) \xrightarrow{c(\mathbb{Z}(X \setminus Z))} M(X \setminus Z)[1]$$

where, for $m : A \rightarrow B$, $A, B \in C(\mathcal{A})$, \mathcal{A} an additive category, $c(A) : \text{Cone}(m : A \rightarrow B) \rightarrow A[1]$ is the canonical map.

- Recall that we say that $F \in \text{PSh}(\text{SmVar}(k))$ is \mathbb{A}^1 invariant if for all $X \in \text{SmVar}(k)$, $p^* := F(p) : F(X) \rightarrow F(X \times \mathbb{A}^1)$ is an isomorphism, where $p : X \times \mathbb{A}^1 \rightarrow X$ is the projection.
- Similarly, we say that $F \in \text{PSh}(\text{AnSm}(\mathbb{C}))$ is \mathbb{D}^1 invariant if for all $X \in \text{AnSm}(\mathbb{C})$, $p^* := F(p) : F(X) \rightarrow F(X \times \mathbb{D}^1)$ is an isomorphism, where $p : X \times \mathbb{D}^1 \rightarrow X$ is the projection.

We will use the following theorem of Voevodsky :

Theorem 1. (Voevodsky) *Let k be a perfect field (e.g. k a field of characteristic zero). Let $F \in \text{PSh}(\text{SmVar}(k), \mathbb{Q})$. If F is \mathbb{A}^1 invariant and admits transfers, then for all $j \in \mathbb{Z}$, $H^j E_{\text{et}}^\bullet(F) \in \text{PSh}(\text{SmVar}(k))$ are \mathbb{A}^1 invariant. That is, if F is \mathbb{A}^1 invariant and admits transfers then F is \mathbb{A}^1 local.*

Proof. By [5], $H^j E_{N_{\text{is}}}^\bullet(F) \in \text{PSh}(\text{SmVar}(k))$ are \mathbb{A}^1 invariant. On the other hand since F takes values in \mathbb{Q} -vector spaces, $H^j E_{N_{\text{is}}}^\bullet(F) = H^j E_{\text{et}}^\bullet(F)$. □

2.4 The logarithmic De Rham complexes

We recall from [2] the following notion :

- For k a field and $X \in \text{Var}(k)$, we consider the embedding in $C(X)$

$$OL_X : \Omega_{X,\log}^\bullet \hookrightarrow \Omega_X^\bullet := \Omega_{X/k}^\bullet,$$

such that, for $X^\circ \subset X$ an open subset and $w \in \Omega_X^p(X^\circ)$, $w \in \Omega_{X,\log}^p(X^\circ)$ if and only if there exists $(n_i)_{1 \leq i \leq s} \in \mathbb{Z}$ and $(f_{i,\alpha_k})_{1 \leq i \leq s, 1 \leq k \leq p} \in \Gamma(X^\circ, \mathcal{O}_X)^*$ such that

$$w = \sum_{1 \leq i \leq s} n_i df_{i,\alpha_1} / f_{i,\alpha_1} \wedge \cdots \wedge df_{i,\alpha_p} / f_{i,\alpha_p} \in \Omega_X^p(X^\circ),$$

and for $p = 0$, $\Omega_{X,\log}^0 := \mathbb{Z}$. Let k a field. We get an embedding in $C(\text{Var}(k))$

$$OL : \Omega_{/k,\log}^\bullet \hookrightarrow \Omega_{/k}^\bullet, \text{ given by, for } X \in \text{Var}(k), \\ OL(X) := OL_X : \Omega_{/k,\log}^\bullet(X) := \Gamma(X, \Omega_{X,\log}^\bullet) \hookrightarrow \Gamma(X, \Omega_X^\bullet) =: \Omega_{/k}^\bullet(X)$$

and its restriction to $\text{SmVar}(k) \subset \text{Var}(k)$.

- For $X \in \text{AnSp}(\mathbb{C})$, we consider the embedding in $C(X)$

$$OL_X : \Omega_{X,\log}^\bullet \hookrightarrow \Omega_X^\bullet,$$

such that, for $X^\circ \subset X$ an open subset and $w \in \Omega_X^p(X^\circ)$, $w \in \Omega_{X,\log}^p(X^\circ)$ if and only if there exists $(n_i)_{1 \leq i \leq s} \in \mathbb{Z}$ and $(f_{i,\alpha_k})_{1 \leq i \leq s, 1 \leq k \leq p} \in \Gamma(X^\circ, \mathcal{O}_X)^*$ such that

$$w = \sum_{1 \leq i \leq s} n_i df_{i,\alpha_1}/f_{i,\alpha_1} \wedge \cdots \wedge df_{i,\alpha_p}/f_{i,\alpha_p} \in \Omega_X^p(X^\circ),$$

and for $p = 0$, $\Omega_{X,\log}^0 := \mathbb{Z}$. We get an embedding in $C(\text{AnSp}(\mathbb{C}))$

$$OL^{an} : \Omega_{\log}^{\bullet,an} \hookrightarrow \Omega^{\bullet,an}, \text{ given by, for } X \in \text{AnSp}(\mathbb{C}),$$

$$OL^{an}(X) := OL_X : \Omega_{\log}^\bullet(X) := \Gamma(X, \Omega_{X,\log}^{\bullet,an}) \hookrightarrow \Gamma(X, \Omega_X^{\bullet,an}) =: \Omega^{\bullet,an}(X)$$

and its restriction to $\text{AnSm}(\mathbb{C}) \subset \text{AnSp}(\mathbb{C})$.

We have the following result :

Proposition 2. (i00) Let $D \in \text{AnSm}(\mathbb{C})$ be an open ball. Then, for each $p, q \in \mathbb{Z}$, $q \neq 0$, $H_{et}^q(D, \Omega_{D,\log}^p) = 0$.

(i0) Let $U \in \text{SmVar}(\mathbb{C})$ such that there exists an etale map $e : U \rightarrow \mathbb{A}_{\mathbb{C}}^{dv}$ such that $\mathbb{A}_{\mathbb{C}}^{dv} \setminus e(U) \subset \mathbb{A}_{\mathbb{C}}^{dv}$ is a divisor. Then, for each $p, q \in \mathbb{Z}$, $q \neq 0$, $p \neq 0$, $H_{et}^q(U, \Omega_{U^{an},\log}^p) = 0$.

(i) Let $U \in \text{SmVar}(\mathbb{C})$ such that there exists an etale map $e : U \rightarrow \mathbb{A}_{\mathbb{C}}^{dv}$ such that $\mathbb{A}_{\mathbb{C}}^{dv} \setminus e(U) \subset \mathbb{A}_{\mathbb{C}}^{dv}$ is a divisor. Then, for each $p, q \in \mathbb{Z}$, $q \neq 0$, $p \neq 0$, $H_{usu}^q(U^{an}, \Omega_{U^{an},\log}^p) = 0$.

(ii) Let $X \in \text{SmVar}(\mathbb{C})$. We still denote $\text{an}_X : X^{an} \xrightarrow{e, \sim} X^{an,et} \xrightarrow{\text{an}_X} X^{et}$ the composite. Consider the map in $C(X^{et})$

$$T(et, usu) = E_{et}(\text{an}_{X^*} k) : E_{et}(\text{an}_{X^*} \Omega_{X^{an},\log}^p) \rightarrow E_{et}(\text{an}_{X^*} E_{usu}(\Omega_{X^{an},\log}^p)).$$

Then, for each $p, q \in \mathbb{Z}$,

$$H^{p+q}T(et, usu)(X) : H_{et}^q(X, \text{an}_{X^*} \Omega_{X^{an},\log}^p) \rightarrow H_{usu}^q(X^{an}, \Omega_{X^{an},\log}^p)$$

is an isomorphism.

(ii)' Let $X \in \text{SmVar}(\mathbb{C})$. Let Z a closed subset. Then, for each $p, q \in \mathbb{Z}$,

$$H_Z^{p+q}T(et, usu)(X) : H_{et,Z}^q(X, \text{an}_{X^*} \Omega_{X^{an},\log}^p) \rightarrow H_{usu,Z}^q(X^{an}, \Omega_{X^{an},\log}^p)$$

is an isomorphism.

Proof. (i00): Let $B = D(0, 1)^N \in \text{AnSm}(\mathbb{C})$ be an open ball. Let $\alpha \in H^q(B, \Omega_{B,\log}^p)$, $q \in \mathbb{Z}$, $q \neq 0$. Consider a sequence of sub-balls

$$l_n : B_n := D(0, r_n)^N \hookrightarrow B := D(0, 1)^N, n \in \mathbb{N}, \text{ with } r_n \rightarrow 0 \text{ when } n \rightarrow \infty.$$

Since $(B_n := D(0, r_n))_{n \in \mathbb{N}}$ form a basis of neighborhood of $0 \in B$ and since $\Omega_{B,\log}^p$ is a single presheaf, there exists $n := n_\alpha \in \mathbb{N}$ (depending on α) such that

$$l_n^* \alpha = 0 \in H^q(B_n, \Omega_{B_n,\log}^p).$$

But it follows immediately from the definition of the the presheaf $\Omega_{B,\log}^p$ considering the isomorphism

$$\phi_n : B_n \xrightarrow{\sim} B, \phi_n(z) := r_n^{-1}z$$

so that $(\phi_n)^{-1} \circ l_n = r_n I$, that

$$l_n^* \alpha = l_n^* \phi_n^{-1*} \phi_n^* \alpha = r_n \phi_n^* \alpha \in H^q(B_n, \Omega_{B, \log}^p).$$

Hence $\phi_n^* \alpha = 0$. Since ϕ_n is an isomorphism, we get $\alpha = 0$.

(i): We prove (i) by induction on the dimension of U . Let $U \in \text{SmVar}(\mathbb{C})$ such that there exists an etale map $e : U \rightarrow \mathbb{A}_{\mathbb{C}}^{d_U}$. Denote $D := \mathbb{A}_{\mathbb{C}}^{d_U} \setminus e(U)$ the complementary divisor. Since $H_{usu}^q(\mathbb{A}_{\mathbb{C}}^{d_U}, \Omega_{\log}^{p, an}) = 0$ for all $p, q \in \mathbb{Z}$,

$$H^q E_{usu} \Omega_{\log}^{p, an}(c(\mathbb{Z}(e(U)))) = \partial : H_{usu}^q(e(U), \Omega_{\log}^{p, an}) \xrightarrow{\sim} H_{usu, D}^{q+1}(\mathbb{A}_{\mathbb{C}}^{d_U}, \Omega_{\log}^{p, an})$$

is an isomorphism for all $p, q \in \mathbb{Z}$. Let $D = \cup_i D_i$ be the irreducible components of D . Let $D_{i0} \subset D_i$ be divisors such that there exist etale maps $e_i : D_i \setminus D_{i0} \rightarrow \mathbb{A}_{\mathbb{C}}^{d_U-1}$ such that $\mathbb{A}_{\mathbb{C}}^{d_U-1} \setminus e_i(D_i \setminus D_{i0}) \subset \mathbb{A}_{\mathbb{C}}^{d_U-1}$ is a divisor. By (i00),

$$H^q E_{usu}(\Omega_{\log}^{p, an}) \in \text{PSh}(\text{AnSm}(\mathbb{C}))$$

is \mathbb{D}^1 invariant for each $q \in \mathbb{Z}$, $q \neq 0$. By the local to global spectral sequence

$$H^{q'} E_{usu}(H^q E_{usu}(\Omega_{\log}^{p, an})) \in \text{PSh}(\text{AnSm}(\mathbb{C}))$$

are also \mathbb{D}^1 invariant for all $q' \in \mathbb{Z}$ (one can also use the fact that if $F \in \text{PSh}(\text{AnSm}(\mathbb{C}))$ is \mathbb{D}^1 invariant and admit transfers then $H^q E_{usu}(F) \in \text{PSh}(\text{AnSm}(\mathbb{C}))$ is also \mathbb{D}^1 invariant all $q' \in \mathbb{Z}$). We thus get by proposition 1 the isomorphism for each $q, q' \in \mathbb{Z}$, $q \neq 0$.

$$\begin{aligned} & H^{q'} E_{usu}(H^q E_{usu}(\Omega_{\log}^{p, an}(P_{D_i \setminus D_{i0}, \mathbb{A}_{\mathbb{C}}^{d_U} \setminus Z_{i0})))) : \\ & H_{usu, D_i \setminus D_{i0}}^{q+q'}(\mathbb{A}_{\mathbb{C}}^{d_U}, \Omega_{\log}^{p, an}) \xrightarrow{\sim} H_{usu}^{q+q'-1}(D_i \setminus D_{i0}, \Omega_{\log}^{p-1, an}). \end{aligned}$$

On the other hand, by induction hypothesis $H_{usu}^{q''}(D_i \setminus D_{i0}, \Omega_{\log}^{p-1, an}) = 0$ for all $q'' \in \mathbb{Z}$, $q'' \neq 0$. Hence,

$$H_{usu, D_i \setminus D_{i0}}^{q''}(\mathbb{A}_{\mathbb{C}}^{d_U}, \Omega_{\log}^{p, an}) = 0$$

for all $q'' \in \mathbb{Z}$, $q'' \neq 1$. By a finite decreasing induction $D_{ir} \subset \dots \subset D_{i0} \subset Z$, $r = d_U - 1$, with $e_r : D_{ir} \rightarrow \mathbb{A}^{d_{D_{ir}}}$ etale such that $\mathbb{A}^{d_{D_{ir}}} \setminus e_r(D_{ir}) \subset \mathbb{A}^{d_{D_{ir}}}$ is a divisor, we get

$$H_{usu}^{q''}(e(U), \Omega_{\log}^{p, an}) = H_{usu, D}^{q''+1}(\mathbb{A}_{\mathbb{C}}^{d_U}, \Omega_{\log}^{p, an}) = 0$$

for each $q'' \in \mathbb{Z}$, $q'' \neq 0$. Let $Z' \subset U$ be a (Zariski) closed subset such that $e' := e|_{U \setminus Z'} : (U \setminus Z') \rightarrow e(U \setminus Z')$ is finite etale. We thus have, since $M(U \setminus Z') \simeq M(e(U \setminus Z'))$, for each $q \in \mathbb{Z}$, $q \neq 0$

$$H_{usu}^q(U \setminus Z', \Omega_{\log}^{p, an}) = H_{usu}^q(e(U \setminus Z'), \Omega_{\log}^{p, an}) = 0.$$

Denote $j : U \setminus Z' \hookrightarrow U$ the open embedding. By the long exact sequence

$$\dots \xrightarrow{j^* := \Omega_{\log}^{p, an}(j)} H_{usu}^q(U \setminus Z', \Omega_{\log}^{p, an}) \xrightarrow{\partial := \Omega_{\log}^{p, an}(c(\mathbb{Z}(U \setminus Z')))} H_{usu, Z'}^{q+1}(U, \Omega_{\log}^{p, an}) \xrightarrow{\Omega_{\log}^{p, an}(\gamma_{Z'})} H_{usu}^{q+1}(U, \Omega_{\log}^{p, an}) \xrightarrow{j^*} \dots,$$

we get by a finite induction on the dimension of Z' , we get $H_{usu}^q(U, \Omega_{\log}^{p, an}) = 0$ for all $q \in \mathbb{Z}$, $q \neq 0$.

(i0): By (i), we have $a_{et} R^q \text{an}_{U^*} \Omega_{U^{an}, \log}^p = 0$, $q \in \mathbb{Z}$, $q \neq 0$. Hence for $q \in \mathbb{Z}$, $q \neq 0$,

$$H_{et}^q(U, \text{an}_{U^*} \Omega_{U^{an}, \log}^p) = \mathbb{H}_{et}^q(U, R \text{an}_{U^*} \Omega_{U^{an}, \log}^p) = H_{usu}^q(U^{an}, \Omega_{U^{an}, \log}^p) = 0,$$

where the last equality follows again by (i).

(ii): By (i), $a_{et} R^q \text{an}_{X^*} \Omega_{X^{an}, \log}^p = 0$, $q \in \mathbb{Z}$, $q \neq 0$.

(ii)': Follows from (ii) applied to X and $X \setminus Z$ by the distinguish triangle in $Ho(C(\text{SmVar}(\mathbb{C})))$

$$\mathbb{Z}(X \setminus Z) \xrightarrow{\mathbb{Z}(j)} \mathbb{Z}(X) \rightarrow \mathbb{Z}(X, X \setminus Z) \rightarrow \mathbb{Z}(X \setminus Z)[1],$$

where $j : X \setminus Z \hookrightarrow X$ is the open embedding. □

3 Hypersurface De Rham cohomology and its complex periods

We denote $\mathbb{P}^N := \mathbb{P}_{\mathbb{C}}^N := \text{Proj}(\mathbb{C}[z_0, \dots, z_N])$. Let $X = V(f) \subset \mathbb{P}^N$ be a smooth hypersurface. If $N - 1 = 2p$ is even, we consider the decompositions

$$H_{DR}^{N-1}(X) = \mathbb{C}[\Lambda \cap X] \oplus H_{DR,v}^{N-1}(X), \quad H_{\text{sing}}^{N-1}(X^{an}, \mathbb{Q}) = \mathbb{C}[\Lambda \cap X] \oplus H_{\text{sing},v}^{N-1}(X^{an}, \mathbb{Q})$$

of vector spaces, where $\Lambda \subset \mathbb{P}^N$ is a linear subspace of codimension p . If $N - 1 = 2p + 1$ is even, we set

$$H_{DR,v}^{N-1}(X) := H_{DR}^{N-1}(X), \quad H_{\text{sing},v}^{N-1}(X^{an}, \mathbb{Q}) := H_{\text{sing}}^{N-1}(X^{an}, \mathbb{Q}).$$

Let $X = V(f) \subset \mathbb{P}^N$ be a smooth hypersurface. Denote $j : U := \mathbb{P}^N \setminus X \hookrightarrow \mathbb{P}^N$ the open embedding. We consider the algebraic Hodge module

$$j_{*Hdg}(O_U, F_b) := ((j_*O_U, F), j_*\mathbb{Q}_{U^{an}}, j_*\alpha(U)) \in HM(\mathbb{P}^N),$$

where $F^k j_*O_U = \mathbb{C}[x_1, \dots, x_N][\frac{1}{f^k}]$. Recall for each $p \in \mathbb{Z}$,

$$F^p H_{DR}^N(U) := H^N \iota^{\geq p}(\mathbb{H}^N(\mathbb{P}^N, \Omega_{\mathbb{P}^N}^{\bullet \geq p}(\log X))) = H^N \iota^{\geq p} H^N(\mathbb{P}^N, \Omega_{\mathbb{P}^N}^{\bullet} \otimes_{\mathcal{O}_{\mathbb{P}^N}} F^{\bullet-p} j_*O_U)$$

and

$$F^p H^{N-1}(X) := H^{N-1} \iota^{\geq p}(\mathbb{H}^{N-1}(X, \Omega_X^{\bullet \geq p})).$$

where

$$\iota^{\geq p} : \Omega_{\mathbb{P}^N}^{\bullet \geq p}(\log X) \hookrightarrow \Omega_{\mathbb{P}^N}^{\bullet}(\log X), \quad \iota^{\geq p} : \Omega_{\mathbb{P}^N}^{\bullet} \otimes_{\mathcal{O}_{\mathbb{P}^N}} F^{\bullet-p} j_*O_U \hookrightarrow \Omega_{\mathbb{P}^N}^{\bullet} \otimes_{\mathcal{O}_{\mathbb{P}^N}} j_*O_U = j_*\Omega_U^{\bullet}$$

and $\iota^{\geq p} : \Omega_X^{\bullet \geq p} \hookrightarrow \Omega_X^{\bullet}$ are the canonical embeddings. Then the map Res_{X, \mathbb{P}^N} of definition 1 induces for each $p \in \mathbb{Z}$, canonical isomorphisms

$$H^N \Omega(Res_{X, \mathbb{P}^N}) : F^{p+1} H_{DR}^N(U) \xrightarrow{\sim} F^p H_{DR}^{N-1}(X)_v, \quad H^N \text{Bti}(Res_{X, \mathbb{P}^N}) : H_{\text{sing}}^N(U, \mathbb{Q}) \xrightarrow{\sim} H_{\text{sing}}^{N-1}(X, \mathbb{Q})_v.$$

Lemma 1. *Let $X = V(f) \subset \mathbb{P}^N$ be a smooth hypersurface. Denote $j : U := \mathbb{P}^N \setminus X \hookrightarrow \mathbb{P}^N$ the open embedding. Let $B \subset \mathbb{P}^N$, $\phi : B \simeq D(0, 1)^N$ be an open ball such that $\phi(X \cap B) = \{0\} \times D(0, 1)^{N-1}$. Then,*

$$(i) \text{ For } q \in \mathbb{Z}, q \neq 0, \mathbb{H}_{usu}^q(B, (\Omega_{\mathbb{P}^N}^{\bullet} \otimes_{\mathcal{O}_{\mathbb{P}^N}} F^{\bullet-p} j_*O_U)^{an}) = 0.$$

$$(ii) \text{ For } q \in \mathbb{Z}, q \neq 0, H_{usu}^q(B, \Omega_{\mathbb{P}^N, an}^{N-p}(\log X)^{\partial=0}) = 0.$$

Proof. (i): Since B is Stein, we have for $r, q \in \mathbb{Z}, q \neq 0$,

$$H_{usu}^q(B, (\Omega_{\mathbb{P}^N}^r \otimes_{\mathcal{O}_{\mathbb{P}^N}} F^{r-p} j_*O_U)^{an}) = 0.$$

On the other hand the De Rham complex of $(j_*O_{B \setminus X}, F)$ is acyclic (see e.g. [6] corollary 18.7)

(ii): Since open balls form a basis of the topology of $\mathbb{P}^{N, an}$, we have

$$a_{usu} \iota : a_{usu}(j_*\Omega_{U^{an}, \log}^{N-p} \oplus \partial\Omega_{\mathbb{P}^N, an}^{N-p+1}) \xrightarrow{\sim} a_{usu}\Omega_{\mathbb{P}^N, an}^{N-p}(\log X)^{\partial=0}$$

where

$$\iota : j_*\Omega_{U^{an}, \log}^{N-p} \oplus \partial\Omega_{\mathbb{P}^N, an}^{N-p+1} \hookrightarrow \Omega_{\mathbb{P}^N, an}^{N-p}(\log X)^{\partial=0}$$

is the canonical embedding. Now,

- Let $B = D(0, 1)^N \in \text{AnSm}(\mathbb{C})$ be an open ball. Let $\alpha \in H^q(B, \partial\Omega_{\mathbb{P}^N, an}^{N-p+1})$, $q \in \mathbb{Z}$, $q \neq 0$. Consider a sequence of sub-balls

$$l_n : B_n := D(0, r_n)^N \hookrightarrow B := D(0, 1)^N, n \in \mathbb{N}, \text{ with } r_n \rightarrow 0 \text{ when } n \rightarrow \infty.$$

Since $(B_n := D(0, r_n))_{n \in \mathbb{N}}$ form a basis of neighborhood of $0 \in B$ and since $\partial\Omega_{\mathbb{P}^N, an}^{N-p+1}$ is a single presheaf, there exists $n := n_\alpha \in \mathbb{N}$ (depending on α) such that

$$l_n^* \alpha = 0 \in H^q(B_n, \partial\Omega_{\mathbb{P}^N, an}^{N-p+1}).$$

But it follows immediately from the definition of the the presheaf $\partial\Omega_{\mathbb{P}^N, an}^{N-p+1}$ considering the isomorphism

$$\phi_n : B_n \xrightarrow{\sim} B, \phi_n(z) := r_n^{-1}z$$

so that $(\phi_n)^{-1} \circ l_n = r_n I$, that

$$l_n^* \alpha = l_n^* \phi_n^{-1*} \phi_n^* \alpha = r_n \phi_n^* \alpha \in H^q(B_n, \Omega_{B, \log}^p).$$

Hence $\phi_n^* \alpha = 0$. Since ϕ_n is an isomorphism, we get $\alpha = 0$.

- we have for $q \in \mathbb{Z}$, $q \neq 0$, $H_{usu}^q(B, \Omega_{B^{an}, \log}^{N-p}) = 0$ by proposition 2 (i00). Hence

$$H^q E_{usu} \Omega_{\log}^{an}(c(\mathbb{Z}(B))) = \partial : H_{usu}^q(B \cap U, \Omega_{U^{an}, \log}^{N-p}) \xrightarrow{\sim} H_{usu, B \cap X}^{q+1}(B, \Omega_{B^{an}, \log}^{N-p})$$

is an isomorphism. By a finite decreasing induction on N , using purity for complex analytic motives, we get $H_{usu}^q(B \cap U, \Omega_{U^{an}, \log}^{N-p}) = 0$.

□

We have by the local acyclicity of the De Rham complex of $(j_* O_U, F)$ a canonical map :

Definition 2. Let $X = V(f) \subset \mathbb{P}^N$ be a smooth hypersurface. Denote $j : U := \mathbb{P}^N \setminus X \hookrightarrow \mathbb{P}^N$ the open embedding. Let $\mathbb{P}^{N, an} = \cup_{i=1}^n B_i$ an open cover, with $\phi_i : B_i \simeq D(0, 1)^N$ open balls such that $\phi_i(X \cap B_i) = \{0\} \times D(0, 1)^{N-1}$. Denote, for $I \subset [1, \dots, n]$, $l_I : B_I \hookrightarrow \mathbb{P}^N$ the open embeddings, with $B_I := \cap_{i \in I} B_i$. Since, for each $I \subset [1, \dots, n]$ and $p \in \mathbb{Z}$,

$$l_I^* \iota_\partial : l_I^* \Omega_{\mathbb{P}^N, an}^{N-p}(\log X)^{\partial=0}[-p] \hookrightarrow l_I^*(\Omega_{\mathbb{P}^N}^\bullet \otimes_{O_{\mathbb{P}^N}} F^{\bullet-p} j_* O_U)^{an}$$

is an quasi-isomorphism (see e.g. [6] corollary 18.7), the spectral sequence of the filtration on the total complex of a bi-complex gives, for each $p \in \mathbb{Z}$, a canonical isomorphism, where the first and last map are isomorphism by lemma 1(i) and (ii) respectively,

$$\begin{aligned} B_\partial : F^p H_{DR}(U) &:= H^N \iota^{\geq p} \mathbb{H}_{zar}^N(\mathbb{P}^N, \Omega_{\mathbb{P}^N}^\bullet \otimes_{O_{\mathbb{P}^N}} F^{\bullet-p} j_* O_U) \\ &\xrightarrow{H^N((l_i^* = \Omega(l_i))_{1 \leq i \leq n})} H^N \iota^{\geq p} H^N \Gamma(B_\bullet, (\Omega_{\mathbb{P}^N}^\bullet \otimes_{O_{\mathbb{P}^N}} F^{\bullet-p} j_* O_U)^{an}) \\ &\rightarrow H^p \Gamma(B_\bullet, \Omega_{\mathbb{P}^N, an}^{N-p}(\log X)^{\partial=0}) \xrightarrow{H^N((l_i^* = \Omega(l_i))_{1 \leq i \leq n})^{-1}} H_{usu}^p(\mathbb{P}^N, \Omega_{\mathbb{P}^N, an}^{N-p}(\log X)^{\partial=0}), \end{aligned}$$

given by, for $w \in \Gamma(U, \Omega_{\mathbb{P}^N}^N)$, inductively by a p step induction where p is the order of the pole of w at X ,

- taking for each $1 \leq i \leq n$, $l_i^* w = \partial w_{1i}$ since $\partial w = 0$ for dimensional reason and by exactness of $l_i^* \iota_\partial$,
- we then have for each $1 \leq i, j \leq n$, considering $w_{1ij} := l_{ij}^*(w_{1i} - w_{1j})$, $w_{1ij} = \partial w_{2ij}$ since

$$\partial w_{1ij} = l_{ij}^*(\partial w_{1i} - \partial w_{1j}) = l_{ij}^*(l_i^* w - l_j^* w) = 0$$

and by exactness of $l_{ij}^* \iota_\partial$,

- at the final p step, $B(w) := w_{pI} \in H^N(B_\bullet, \Omega_{\mathbb{P}^N, an}^{N-p}(\log X)^{\partial=0})$, $\text{card} I = p$.

Let $X = V(f) \subset \mathbb{P}^N$ be a smooth hypersurface. Denote $U := \mathbb{P}^N \setminus X$. For $l \in \mathbb{Z}$, we consider

- the sequence of embeddings in $C(U^{et})$

$$OL_U^l : \Omega_{U, \log}^l[-l] \xrightarrow{\widehat{OL_U^l[-l]}} \Omega_{\mathbb{P}^N}^l(\log X)^{\partial=0}[-l] \xrightarrow{\iota_\partial} \Omega_{\mathbb{P}^N}^\bullet \otimes_{\mathcal{O}_{\mathbb{P}^N}} F^{\bullet-p} j_* \mathcal{O}_U \xrightarrow{\iota^{\geq l}} \Omega_U^\bullet$$

- and the sequence of embeddings in $C(U^{an})$

$$OL_{U^{an}}^l : \Omega_{U^{an}, \log}^l[-l] \xrightarrow{\widehat{OL_{U^{an}}^l[-l]}} \Omega_{\mathbb{P}^N, an}^l(\log X)^{\partial=0}[-l] \xrightarrow{\iota_\partial} (\Omega_{\mathbb{P}^N}^\bullet \otimes_{\mathcal{O}_{\mathbb{P}^N}} F^{\bullet-p} j_* \mathcal{O}_U)^{an} \xrightarrow{\iota^{\geq l}} \Omega_{U^{an}}^\bullet,$$

where

$$\iota_\partial : \Omega_{\mathbb{P}^N, an}^l(\log X)^{\partial=0}[-l] \hookrightarrow (\Omega_{\mathbb{P}^N}^\bullet \otimes_{\mathcal{O}_{\mathbb{P}^N}} F^{\bullet-p} j_* \mathcal{O}_U)^{an}$$

is an equivalence *usu local* (see [6]).

The main result of this section is the following :

Proposition 3. *Let $X = V(f) \subset \mathbb{P}^N$ be a smooth projective hypersurface. Denote $U := \mathbb{P}^N \setminus X$. Let $w \in F^{p+1} H_{DR}^N(U)$. Then the following assertions are equivalent*

(i) $ev(U)(w) \in H_{\text{sing}}^N(U^{an}, 2i\pi\mathbb{Q})$,

(ii) $B_\partial(w) \in \widehat{OL_{U^{an}}^{p+1}}(H_{usu}^{p+1}(U^{an}, \Omega_{U^{an}, \log}^{N-p}))$ (see definition 2),

(iii) $w \in OL_{U^{an}}^{N-p}(H_{usu}^{p+1}(U^{an}, \Omega_{U^{an}, \log}^{N-p})) = OL_{U^{an}}(H_{usu}^{p+1}(U^{an}, \Omega_{U^{an}, \log}^{N-p}))$.

Proof. (i) implies (ii): Let $\mathbb{P}^{N, an} = \cup_{i=1}^n B_i$ an open cover, with $\phi_i : \mathbb{B}_i \simeq D(0, 1)^N$ open balls such that $\phi_i(X \cap B_i) = \{0\} \times D(0, 1)^{N-1}$. Denote, for $I \subset [1, \dots, n]$, $l_I : B_I \hookrightarrow \mathbb{P}^N$ the open embeddings, with $B_I := \cap_{i \in I} B_i$. Then

$$B_\partial(w) := (w_{p+1, I}) \in H^{p+1}(B_\bullet, \Omega_{\mathbb{P}^N, an}^{N-p}(\log X)^{\partial=0}).$$

We have, for each $I \subset [1, \dots, n]$, a decomposition

$$w_{p+1, I} = \frac{a_I}{z_1} dz_1 \wedge w'_I + w''_I \in \Gamma(B_I, \Omega_{\mathbb{P}^N, an}^{N-p}(\log X)^{\partial=0})$$

with $w'_I \in \Gamma(B_I, \Omega_{\mathbb{P}^N}^{N-p-1})^{\partial=0}$ and $w''_I \in \Gamma(B_I, \Omega_{\mathbb{P}^N}^{N-p})^{\partial=0}$. Consider the evaluation map

$$ev(U)^\bullet : \Gamma(B_\bullet \setminus X, \Omega_{\mathbb{P}^N}^\bullet) \rightarrow C_{\text{diff}}^\bullet(B_\bullet \setminus X^{an}, \mathbb{C}),$$

which is a morphism of bi-complexes. On the other hand, we have the following canonical map

$$\begin{aligned} BC_\partial &: H^p \Gamma(B_\bullet \setminus X^{an}, C_{U^{an}, \text{diff}}^{N-p, \partial}) \xrightarrow{\Gamma(B_\bullet \setminus X^{an}, \iota_\partial)} H^N \Gamma(B_\bullet \setminus X^{an}, C_{U^{an}, \text{diff}}^{\bullet \geq N-p}) \xrightarrow{\Gamma(B_\bullet \setminus X^{an}, \iota^{\geq N-p})} \\ &= H^N \Gamma(B_\bullet \setminus X^{an}, C_{U^{an}, \text{diff}}^\bullet) \xrightarrow{(H^N(l_i^* = C_{U^{an}, \text{diff}}^\bullet(l_i))_{1 \leq i \leq n})^{-1}} H_{\text{sing}}^N(U^{an}, \mathbb{Q}) = H^N \Gamma(U^{an}, C_{U^{an}, \text{diff}}^\bullet), \end{aligned}$$

where $H^N(l_i^*)$ is an isomorphism by the acyclicity of $C_{U^{an}, \text{diff}}^{\bullet \geq N-p}(B_I \setminus X^{an})$ for each $I \subset [1, \dots, n]$, and

$$C_{U^{an}, \text{diff}}^\bullet \hookrightarrow C_{U^{an}, \text{sing}}^\bullet, \text{ for } V \subset U^{an}, C_{\text{diff}}^\bullet(V) \hookrightarrow C_{\text{sing}}^\bullet(V)$$

is the canonical quasi-isomorphism in $C(U^{an})$. We then have the following commutative diagram

$$\begin{array}{ccc}
H^N \iota^{\geq p} H^N \Gamma(\mathbb{P}^N, \Omega_{\mathbb{P}^N}^\bullet \otimes_{\mathcal{O}_{\mathbb{P}^N}} F^{\bullet-p} j_* \mathcal{O}_U) & & \\
\downarrow H^N((l_i^* = \Omega(l_i))_{1 \leq i \leq n}) & \searrow^{B_\partial} & \\
H^N \iota^{\geq p} H^N \Gamma(B_\bullet, (\Omega_{\mathbb{P}^N}^\bullet \otimes_{\mathcal{O}_{\mathbb{P}^N}} F^{\bullet-p} j_* \mathcal{O}_U)^{an}) & \xrightarrow{B_\partial} & H^p \Gamma(B_\bullet, \Omega_{\mathbb{P}^N, an}^{N-p}(\log X)^{\partial=0}) \\
\downarrow ev(U) & & \downarrow ev(U) \bullet \\
H_{\text{sing}}^N(U^{an}, \mathbb{C}) = H^N \Gamma(U^{an}, C_{U^{an}, \text{diff}}^\bullet \otimes \mathbb{C}) & \xleftarrow{BC_\partial \otimes \mathbb{C}} & H^p \Gamma(B_\bullet \setminus X^{an}, C_{\text{diff}}^{N-p, \partial=0} \otimes \mathbb{C})
\end{array}$$

Since $ev(U)$ is injective by the strictness of the Hodge filtration and since B_∂ is an isomorphism, $ev(U) \bullet$ is injective and

$$\begin{aligned}
(BC_\partial \otimes \mathbb{C})' &:= (BC_\partial \otimes \mathbb{C})|_{ev(U) \bullet (H^p \Gamma(B_\bullet, \Omega_{\mathbb{P}^N, an}^{N-p}(\log X)^{\partial=0}))} : \\
ev(U) \bullet (H^p \Gamma(B_\bullet, \Omega_{\mathbb{P}^N, an}^{N-p}(\log X)^{\partial=0})) &\rightarrow H_{\text{sing}}^N(U^{an}, \mathbb{C})
\end{aligned}$$

is injective. For $\gamma_I \in C_{N-p}^{\text{sing}}(B_I \setminus X^{an}, \mathbb{Q})^{\partial=0}$ where $p_1 \circ \phi_I(\gamma_I) \subset D(0, 1)^*$ is a loop around 0, we have by the above diagram

$$ev(U)(w)(\gamma_I) = ev(U)(w_{p+1, I})(\gamma_I) = a_I.$$

Hence if $ev(U)(w) \in H_{\text{sing}}^N(U^{an}, 2i\pi\mathbb{Q})$, then $a_I \in \mathbb{Q}$, using the injectivity of $(BC_\partial \otimes \mathbb{C})'$.

(ii) is equivalent to (iii): By definition $B_\partial(w) = (H^p \iota_\partial)^{-1}(w)$. On the other hand by definition $(\iota^{\geq p} \circ \iota_\partial \circ \widehat{OL}_{U^{an}}^{N-p} = \widehat{OL}_{U^{an}}^{N-p})$.

(iii) implies (i): See [2]. □

4 Complex GAGA for logarithmic de Rham classes

We recall (see section 2) the morphism of site given by the analytification functor

$$\text{An} : \text{AnSm}(\mathbb{C}) \rightarrow \text{SmVar}(\mathbb{C}), \quad X \mapsto \text{An}(X) := X^{an}, \quad (f : X' \rightarrow X) \mapsto \text{An}(f) := f^{an}.$$

and for $X \in \text{SmVar}(\mathbb{C})$, the following commutative diagram

$$\begin{array}{ccc}
\text{AnSm}(\mathbb{C}) & \xrightarrow{\text{An}} & \text{SmVar}(\mathbb{C}) \\
\downarrow o_{X^{an}} & & \downarrow o_X \\
X^{an, et} & \xrightarrow{\text{an}_X} & X^{et}
\end{array}$$

We have the following :

Lemma 2. (i0) The sheaves $O_{an}^* \in \text{PSh}(\text{AnSm}(\mathbb{C}))$ and $\Omega_{\log}^{1, an} \in \text{PSh}(\text{AnSm}(\mathbb{C}))$ admit transfers compatible with transfers on $\Omega^{1, an} \in \text{PSh}(\text{AnSm}(\mathbb{C}))$.

(i) For each $l \in \mathbb{Z}$, the sheaf $\Omega_{\log}^{l, an} \in \text{PSh}(\text{AnSm}(\mathbb{C}))$ admits transfers compatible with transfers on $\Omega^{l, an} \in \text{PSh}(\text{AnSm}(\mathbb{C}))$, that is $\Omega^{\bullet, an} \in C(\text{Cor AnSm}(\mathbb{C}))$ and the inclusion $OL : \Omega_{\log}^{l, an}[-l] \hookrightarrow \Omega^{\bullet, an}$ in $C(\text{AnSm}(\mathbb{C}))$ is compatible with transfers.

(ii) For each $l \in \mathbb{Z}$, the sheaf $\text{An}_* \Omega_{\log}^{l, an} \in \text{PSh}(\text{SmVar}(\mathbb{C}))$ is \mathbb{A}^1 invariant.

Proof. (i0): Let $W \subset X' \times X$ with $X, X' \in \text{AnSm}(\mathbb{C})$, X' connected, and $p_{X'} : W \subset X' \times X \rightarrow X'$ finite surjective over X' . Then $M(p_X) : M(X') \hookrightarrow M(W)$ is a finite, where for $Y \in \text{AnSp}(\mathbb{C})$ irreducible, $M(Y) := \text{Frac}(O(Y))$ denote the field of meromorphic function. Since $O(X') \subset M(X')$ is integrally closed, the trace map $Tr_{W/X'} : M(W) \rightarrow M(X')$ sends $O(W)$ to $O(X')$, and the norm map $N_{W/X'} : M(W)^* \rightarrow M(X')^*$ sends $O(W)^*$ to $O(X')^*$.

The sheaf $O_{an}^* \in \text{PSh}(\text{AnSm}(\mathbb{C}))$ admits transfers : for $W \subset X' \times X$ with $X, X' \in \text{AnSm}(\mathbb{C})$ and W finite surjective over X' and $f \in O(X)^*$, $W^*f := N_{W/X'}(p_X^*f)$ where $p_X : W \hookrightarrow X' \times X \rightarrow X$ is the projection and $N_{W/X'} : M(W)^* \rightarrow M(X')^*$ is the norm map. This gives transfers on $\Omega_{\log}^{1,an} \in \text{PSh}(\text{AnSm}(\mathbb{C}))$ compatible with transfers on $\Omega^{1,an} \in \text{PSh}(\text{AnSm}(\mathbb{C}))$: for $W \subset X' \times X$ with $X, X' \in \text{AnSm}(\mathbb{C})$ and W finite surjective over X' and $f \in O(X)^*$,

$$W^*df/f := dW^*f/W^*f = Tr_{W/X'}(p_X^*(df/f)),$$

where we recall $p_X : W \hookrightarrow X' \times X \rightarrow X$ is the projection and $Tr_{W/X'} : O(W) \rightarrow O(X')$ is the trace map. Note that $d(fg)/fg = df/f + dg/g$.

(i): By (i0), we get transfers on

$$\otimes_{\mathbb{Q}}^l \Omega_{\log}^{1,an}, \otimes_O^l \Omega^{1,an} \in \text{PSh}(\text{AnSm}(\mathbb{C}))$$

since $\otimes_{\mathbb{Q}}^l \Omega_{\log}^{1,an} = H^0(\otimes_{\mathbb{Q}}^{L,l} \Omega_{\log}^{1,an})$ and $\otimes_O^l \Omega^{1,an} = H^0(\otimes_O^{L,l} \Omega^{1,an})$. This induces transfers on

$$\wedge_{\mathbb{Q}}^l \Omega_{\log}^{1,an} := \text{coker}(\oplus_{I_2 \subset [1, \dots, l]} \otimes_{\mathbb{Q}}^{l-1} \Omega_{\log}^{1,an} \xrightarrow{\oplus_{I_2 \subset [1, \dots, l]} \Delta_{I_2 := (w \otimes w' \mapsto w \otimes w \otimes w')}} \otimes_{\mathbb{Q}}^l \Omega_{\log}^{1,an}) \in \text{PSh}(\text{AnSm}(\mathbb{C})).$$

and

$$\wedge_O^l \Omega^{1,an} := \text{coker}(\oplus_{I_2 \subset [1, \dots, l]} \otimes_O^{l-1} \Omega^{1,an} \xrightarrow{\oplus_{I_2 \subset [1, \dots, l]} \Delta_{I_2 := (w \otimes w' \mapsto w \otimes w \otimes w')}} \otimes_O^l \Omega^{1,an}) \in \text{PSh}(\text{AnSm}(\mathbb{C})).$$

(ii): Follows from the fact that for $X \in \text{AnSp}(\mathbb{C})$, $O^*(X \times \mathbb{A}^1) = O^*(X)$. Indeed, let $h = h(x, z) \in O(X \times \mathbb{A}^1)^*$. Then for $x \in X$, $z \in \mathbb{A}^1 \mapsto h(x, z)$ is constant since it is an holomorphic function on \mathbb{A}^1 with no zero. Hence $h = h(x)$. \square

Proposition 4. *Let $X \in \text{PSmVar}(\mathbb{C})$. Consider the morphism $\text{an}_X : X^{an} \rightarrow X$ in RTop given by analytical functor.*

(i) *The analytic De Rham cohomology class of an algebraic cycle is logarithmic and is of type (d, d) , that is, for $Z \in \mathcal{Z}^d(X)$*

$$[Z] := H^{2d} \Omega(\gamma_Z^\vee)([Z]) \subset H^{2d} OL_{X^{an}}(H_{usu}^d(X^{an}, \Omega_{X^{an}, \log}^d)) \subset H_{DR}^{2d}(X^{an}) = H_{DR}^{2d}(X).$$

(ii) *Conversely, any $w \in H^{2d} OL_{X^{an}}(H_{usu}^d(X^{an}, \Omega_{X^{an}, \log}^d))$ is the class of an algebraic cycle $Z \in \mathcal{Z}^d(X) \otimes \mathbb{Q}$, i.e. $w = [Z]$.*

(iii) *We have $H^j OL_{X^{an}}(H_{usu}^{j-l}(X^{an}, \Omega_{X^{an}, \log}^l)) = 0$ for $j, l \in \mathbb{Z}$ such that $2l \neq j$. That is, we only have analytic logarithmic classes in bidegree (d, d) for $d \in \mathbb{N}$, in particular there is no non trivial logarithmic classes for odd degree De Rham cohomology $H_{DR}^{2d+1}(X^{an}) = H_{DR}^{2d+1}(X)$.*

(iv) *For each $j \in \mathbb{Z}$, the isomorphism*

$$\text{an}_X^* : H_{DR}^j(X) = \mathbb{H}_{et}^j(X, \Omega_X^\bullet) \xrightarrow{\sim} \mathbb{H}_{usu}^j(X^{an}, \Omega_{X^{an}}^\bullet) = H_{DR}^j(X^{an})$$

and its inverse preserve logarithmic classes, that is for each $j, l \in \mathbb{Z}$,

$$\text{an}_X^*(H^j(OL_X)(H_{et}^{j-l}(X, \Omega_{X, \log}^l))) = H^j OL_{X^{an}}(H_{usu}^{j-l}(X^{an}, \Omega_{X^{an}, \log}^l)).$$

Proof. Consider, for $j, l \in \mathbb{Z}$ and $X \in \text{SmVar}(\mathbb{C})$,

$$L^{l,j-l}(X) := H^j OL_{X^{an}}(H_{usu}^{j-l}(X^{an}, \Omega_{X^{an}, \log}^l)).$$

Consider also for $j, l \in \mathbb{Z}$, $X \in \text{SmVar}(\mathbb{C})$ with good reduction and $Z \subset X$ a closed subset,

$$L_Z^{l,j-l}(X) := H^j OL_{X^{an}}(H_{usu,Z}^{j-l}(X^{an}, \Omega_{X^{an}, \log}^l)).$$

We still denote $\text{an}_X : X^{an} \xrightarrow{e, \sim} X^{an, et} \xrightarrow{\text{an}_X} X^{et}$ the composite, with $e(U \rightarrow X^{an}) := e(U)$. By proposition 2(ii) and (ii)', we have

$$L^{l,j-l}(X) = H^j OL_{X^{an}}(H_{et}^{j-l}(X, \text{an}_{X^*} \Omega_{X^{an}, \log}^l)) = H^j OL^{an}(H^{j-l} \text{Hom}(\mathbb{Z}(X), E_{et}^\bullet(\text{An}_* \Omega_{\log}^{l, an}))).$$

and

$$L_Z^{l,j-l}(X) = H^j OL_{X^{an}}(H_{et,Z}^{j-l}(X, \text{an}_{X^*} \Omega_{X^{an}, \log}^l)) = H^j OL^{an}(H^{j-l} \text{Hom}(\mathbb{Z}(X, X \setminus Z), E_{et}^\bullet(\text{An}_* \Omega_{\log}^{l, an}))),$$

for the second equalities, note that

$$(\text{An}_* \Omega_{\log}^{l, an})|_{X^{et}} := o_X^* \text{An}_* \Omega_{\log}^{l, an} = \text{an}_{X^*} o_{X^{an, et}} \Omega_{\log}^{l, an} = \text{an}_{X^*} \Omega_{X^{an}, \log}^l,$$

where $o_X : X^{et} \subset \text{SmVar}(\mathbb{C})$ is the small etale site $o_X : X^{et} \hookrightarrow \text{SmVar}(\mathbb{C})/X \xrightarrow{o_X} \text{SmVar}(\mathbb{C})$, $o_X(U \rightarrow X) = U$. For each $l \in \mathbb{Z}$, the presheaves $\text{An}_* \Omega_{\log}^{l, an} \in \text{PSh}(\text{SmVar}(\mathbb{C}))$ admit transfers and are \mathbb{A}^1 invariant by lemma 2. Hence by theorem 1, they are \mathbb{A}^1 local since they are \mathbb{A}^1 invariant and admit transfers. This gives in particular, for $Z \subset X$ a smooth subvariety of (pure) codimension d with good reduction, by proposition 1 an isomorphism

$$E_{et} \text{An}_* \Omega_{\log}^{l, an}(P_{Z,X}) : L_Z^{l,j-l}(X) \xrightarrow{\sim} L_Z^{l,j-l}(N_{Z/X}) \xrightarrow{\sim} L^{l-d, j-l-d}(Z).$$

Let

$$\alpha = OL_{X^{an}}(\alpha) \in L^{l,j-l}(X) := H^j OL_{X^{an}}(H_{usu}^{j-l}(X^{an}, \Omega_{X^{an}, \log}^l)) = H^{j-l} \text{Hom}(\mathbb{Z}(X), E_{et}^\bullet(\text{An}_* \Omega_{\log}^{l, an})).$$

with $j-l \neq 0$. Let $U \subset X$ an open subset such that there exists an etale map $e : U \rightarrow \mathbb{A}_{\mathbb{C}}^{d_U}$. Denote $j : U \hookrightarrow X$ the open embedding. By proposition 2(i), we have

$$j^* \alpha = 0 \in H_{usu}^{j-l}(U^{an}, \Omega_{X^{an}, \log}^l) = 0.$$

Considering a divisor $X \setminus U \subset D \subset X$, we get

$$\alpha = H^{j-l} E_{et}^\bullet(\text{An}_* \Omega_{\log}^{l, an})(\gamma_D^\vee)(\alpha), \alpha \in L_D^{l,j-l}(X).$$

- Proof of (i): Similar to the proof of [2] theorem 2.
- Proof of (ii): By assumption we have $w \in L^{d,d}(X)$. Let $U \subset X$ an open subset such that there exists an etale map $e : U \rightarrow \mathbb{A}_{\mathbb{C}}^{d_U}$. Considering a divisor $X \setminus U \subset D \subset X$, we get

$$w = H^d E_{et}^\bullet(\text{An}_* \Omega_{\log}^{d, an})(\gamma_D^\vee)(w), \alpha \in L_D^{d,d}(X).$$

Denote $D^\circ \subset D$ its smooth locus with good reduction and $l : X^\circ \hookrightarrow X$ a Zariski open subset such that $X^\circ \cap D = D^\circ$. We then have

$$l^* w \in L_{D^\circ}^{d,d}(X^\circ) = L^{d-1, d-1}(D^\circ)$$

We repeat this procedure with each connected components of D° instead of X . By a finite induction of d steps, we get

$$w = H^d E_{et}^\bullet(\text{An}_* \Omega_{\log}^{d, an})(\gamma_Z^\vee)(w), w \in L_Z^{d,d}(X) = L^{0,0}(Z^\circ).$$

with $Z := D_d \subset \dots \subset D \subset X$ a pure codimension d (Zariski) closed subset, thus $\alpha = \sum_i n_i [Z_i] \in H_{DR}^{2d}(X)$, where $n_i \in \mathbb{Q}$ and $(Z_i)_{1 \leq i \leq t} \subset Z$ are the irreducible components of Z ,

- Proof of (iii) part 1: Let $j > 2l$. Let $w \in L^{l,j-l}(X)$. By the proof of (ii) there exists $Z \subset X$ a closed subset of pure codimension l such that

$$w = H^{j-l} E_{et}^\bullet(\text{An}_* \Omega_{\log}^{l,an})(\gamma_Z^\vee)(w), \quad w \in L_Z^{l,j-l}(X).$$

By a finite induction of $d_X - l$ steps, restricting to the smooth locus with good reduction of closed subsets of Z , $w = 0$ since $L_{Z'}^{l,j-l}(X) = 0$ for all $Z' \in \text{SmVar}(k)$ with good reduction such that $\text{codim}(Z', X) \geq l$.

- Proof of (iii) part 2: Let $j < 2l$. Let $w \in L^{l,j-l}(X)$. By the proof of (ii) there exists $Z \subset X$ a closed subset of pure codimension $j - l$ such that

$$w = H^{j-l} E_{et}^\bullet(\text{An}_* \Omega_{\log}^{l,an})(\gamma_Z^\vee)(w), \quad w \in L_Z^{l,j-l}(X).$$

For $Z' \subset X$ a closed subset of pure codimension c , consider a desingularisation $\epsilon : \tilde{Z}' \rightarrow Z'$ of Z' and denote $n : \tilde{Z}' \xrightarrow{\epsilon} Z' \subset X$. The morphism in $\text{DA}(k)$

$$G_{Z',X} : M(X) \xrightarrow{D(\mathbb{Z}(n))} M(\tilde{Z}')(c)[2c] \xrightarrow{\mathbb{Z}(\epsilon)} M(Z')(c)[2c]$$

where $D : \text{Hom}_{\text{DA}(k)}(M_c(\tilde{Z}'), M_c(X)) \xrightarrow{\sim} \text{Hom}_{\text{DA}(k)}(M(X), M(\tilde{Z}')(c)[2c])$ is the duality isomorphism from the six functor formalism (moving lemma of Suzlin and Voevodsky) and $\mathbb{Z}(n) := \text{ad}(n!, n^1)(a_X^! \mathbb{Z})$, is given by a morphism in $C(\text{SmVar}(k))$

$$\hat{G}_{Z',X} : \mathbb{Z}_{tr}(X) \rightarrow E_{et}(C_* \mathbb{Z}_{tr}(Z'))(c)[2c].$$

Let $l : X^\circ \hookrightarrow X$ be an open embedding such that $Z^\circ := Z \cap X^\circ$ is the smooth locus of Z with good reduction. We then have the following commutative diagram of abelian groups

$$\begin{array}{ccccccc} 0 & \longrightarrow & H_{et,Z}^{j-l}(X, \text{an}_{X^*} \Omega_{X^{an}, \log}^l) & \xrightarrow{l^*} & H_{et,Z^\circ}^{j-l}(X^\circ, \text{an}_{X^*} \Omega_{X^{an}, \log}^l) & \xrightarrow{\partial} & H_{et,Z \setminus Z^\circ}^{j-l+1}(X, \text{an}_{X^*} \Omega_{X^{an}, \log}^l) \longrightarrow \dots \\ & & \Omega(\hat{G}_{Z,X}) \uparrow & & \Omega(P_{Z^\circ, X^\circ}) \uparrow & & \Omega(\hat{G}_{Z,X}) \uparrow \\ 0 & \longrightarrow & H_{et}^0(Z, \text{an}_{Z^*} \Omega_{Z^{an}, \log}^{2l-j}) & \xrightarrow{l^*} & H_{et}^0(Z^\circ, \Omega_{Z^\circ, an, \log}^{2l-j}) & \xrightarrow{\partial} & H_{et, Z \setminus Z^\circ}^1(Z, \Omega_{Z^{an}, \log}^{2l-j}) \longrightarrow \dots \end{array}$$

whose rows are exact sequences. Consider

$$l^* w = \Omega(P_{Z^\circ, X^\circ})(w^\circ) \in H_{et, Z^\circ}^{j-l}(X^\circ, \text{an}_{X^*} \Omega_{X^{an}, \log}^l), \quad w^\circ \in H_{et}^0(Z^\circ, \text{an}_{Z^*} \Omega_{Z^\circ, an, \log}^{2l-j}).$$

Since $\partial l^* w = 0 \in H_{et, Z \setminus Z^\circ}^{j-l+1}(X, \text{an}_{X^*} \Omega_{X^{an}, \log}^l)$, we get, considering the smooth locus with good reduction $(Z \setminus Z^\circ)^\circ \subset Z \setminus Z^\circ$ and $X^{\circ\circ} \subset X$ is an open subset such that $X \cap (Z \setminus Z^\circ)^\circ = X^{\circ\circ} \cap (Z \setminus Z^\circ)$,

$$\partial w^\circ = 0 \in H_{et, Z \setminus Z^\circ}^1(Z, \text{an}_{Z^*} \Omega_{Z^{an}, \log}^{2l-j}),$$

since $H_{et, (Z \setminus Z^\circ)^\circ \setminus (Z \setminus Z^\circ)^\circ}^1(Z, \text{an}_{Z^*} \Omega_{Z^{an}, \log}^{2l-j}) = 0$ for dimension reasons, that is

$$w = \Omega(\hat{G}_{Z,X})(w), \quad \text{with } w \in H_{et}^0(Z, \text{an}_{Z^*} \Omega_{Z^{an}, \log}^{2l-j}).$$

Hence $w = 0$ since $H^0(Z', \Omega_{Z', an, \log}^k) = 0$ for all $k > 0$ and all $Z' \in \text{PVar}(K)$.

(iv): Follows from (ii), (iii) and [2] theorem 2. □

5 Hodge Conjecture for complex smooth projective hypersurfaces

We now state and prove using the result of section 3 and section 4 the main result of this article :

Theorem 2. *Let $X = V(f) \subset \mathbb{P}_{\mathbb{C}}^N$ be a smooth projective hypersurface, with $N = 2p + 1$ odd. Let $\lambda \in F^p H^{2p}(X^{an}, \mathbb{Q})$. Then there exists an algebraic cycle $Z \in \mathbb{Z}^p(X)$ such that $\lambda = [Z]$.*

Proof. Consider $U := \mathbb{P}_{\mathbb{C}}^N \setminus X$. Consider $Res_{X, \mathbb{P}^N} : M(X) \rightarrow M(U)[-1]$ the canonical map given in definition 1. We then have the commutative diagram of abelian group

$$\begin{array}{ccc}
 H_{DR}^N(U) & \xrightarrow{H^N \Omega(Res_{X, \mathbb{P}^N})} & H_{DR}^{N-1}(X)_v \\
 \uparrow H^N \alpha(U^{an}) = ev(U)^{-1} & & \uparrow H^{N-1} \alpha(X^{an}) = ev(X)^{-1} \\
 H_{sing}^N(U^{an}, \mathbb{C}) & \xrightarrow{(H^N \text{Bti}(Res_{X, \mathbb{P}^N})) \otimes_{\mathbb{Q}} \mathbb{C}} & H_{sing}^{N-1}(X^{an}, \mathbb{C})_v
 \end{array}$$

whose arrows are isomorphisms. Denote $w(\lambda) := H^{N-1} \alpha(X^{an})(\lambda) \in F^p H_{DR}^{2p}(X)$. We have

$$\lambda = H^{N-1} \text{Bti}(Res_{X, \mathbb{P}^N})(\alpha), \text{ with } \alpha \in H_{sing}^N(U^{an}, \mathbb{Q}).$$

Then

$$w(\lambda) = H^{N-1} \Omega(Res_{X, \mathbb{P}^N})(w(\alpha)), \text{ with } w(\alpha) := H^N \alpha(U^{an})(\alpha) \in F^{p+1} H_{DR}^N(U).$$

Since $ev(U)(w(\alpha)) = \alpha \in H_{sing}^N(U^{an}, 2i\pi\mathbb{Q})$, we have by proposition 3,

$$w(\alpha) \in H^N OL_{U^{an}}(H_{usu}^{p+1}(U^{an}, \Omega_{U^{an}, \log}^{N-p}))$$

Hence

$$w(\lambda) = H^{N-1} OL_{X^{an}}(\Omega_{\log}^{N-p, an}(Res_{X, \mathbb{P}^N})(w(\alpha))) \in OL_{X^{an}}(H_{usu}^p(X^{an}, \Omega_{X^{an}, \log}^p)).$$

Hence by proposition 4(ii), $w(\lambda) = [Z]$ with $Z \in \mathbb{Z}^p(X)$. □

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