



# Hodge conjecture for projective hypersurfaces

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# Hodge conjecture for projective hypersurfaces

Johann Bouali

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## Abstract

We show that an Hodge class of a complex smooth projective hypersurface is an analytic logarithmic De Rham class. On the other hand we show that for a complex smooth projective variety an analytic logarithmic De Rham class of type  $(d, d)$  is the class of codimension  $d$  algebraic cycle and that there is no analytic logarithmic De Rham class of type  $(p, q)$ ,  $p \neq q$ . We deduce the Hodge conjecture for smooth projective hypersurfaces.

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## 1 Introduction

### 1.1 Complex analytic De Rham logarithmic classes

In [2], we introduced, for  $X$  an algebraic variety over a field, the notion of logarithmic De Rham cohomology classes. For  $X$  a complex algebraic variety, we also introduce the notion of logarithmic analytic De Rham cohomology classes. In proposition 4 we give an analytic analogue of [2] theorem 2, that is if  $X$  is a smooth complex projective variety,

- a logarithmic analytic De Rham cohomology class of type  $(d, d)$  is the class of a codimension  $d$  algebraic cycle,
- there are no non trivial logarithmic analytic De Rham cohomology classes of type  $(p, q)$  for  $p \neq q$ , i.e.  $H^{2d}OL_{X^{an}}(H_{usu}^{d-k}(X^{an}, \Omega_{X^{an}, \log}^{d+k})) = 0$  for  $k \neq 0$ .

The proof of proposition 4 is motivic and similar to the proof of [2] theorem 2 : a logarithmic analytic class of bidegree  $(p, q)$  is acyclic for the etale topology on each open subset  $U \subset X$  such that there exists an etale map  $e : U \rightarrow A_{\mathbb{C}}^{d_U}$  (proposition 2), this allows us to proceed by a finite induction using the crucial fact that the purity isomorphism for De Rham cohomology preserve logarithmic analytic classes since the purity isomorphism is motivic (c.f. [3], see proposition 1). The GAGA comparison theorem for de Rham cohomology of  $X$  induce, by proposition 4 and [2] theorem 2, a GAGA comparison result for logarithmic De Rham classes.

## 1.2 Hodge conjecture for smooth projective hypersurfaces

Let  $X = V(f) \subset \mathbb{P}_{\mathbb{C}}^N$  a smooth projective hypersurface. Denote  $j : U := \mathbb{P}_{\mathbb{C}}^N \hookrightarrow \mathbb{P}_{\mathbb{C}}^N$  the open complementary subset. Using the fact the filtered De Rham complex of  $U$

$$\iota_{\partial} : \Omega_{\mathbb{P}_{N,an}}^{N-p}(\log X)^{\partial=0}[-p] \hookrightarrow (\Omega_{\mathbb{P}_N}^{\bullet} \otimes_{O_{\mathbb{P}_N}} F^{\bullet-p} j_* O_U)^{an}$$

is locally acyclic for the complex topology, more precisely is acyclic on open balls (see e.g.[6]), we show, considering an open cover  $\mathbb{P}_{\mathbb{C}}^N = \cup_{i=1}^n B_i$  by open balls  $\phi_i : B_i \simeq D(0, 1)^N$  such that  $\phi_i(B_i \cap X) = 0 \times D(0, 1)^N$ , that an Hodge class  $\alpha \in F^p H^N(U^{an}, \mathbb{Q})$  is logarithmic, that is

$$\alpha \in H^N OL_{U^{an}}(H_{usu}^p(U^{an}, \Omega_{U^{an}, \log}^{N-p}))$$

(proposition 3). We get that, if  $N = 2p + 1$  and  $\lambda \in F^p H^{2p}(X^{an}, \mathbb{Q})$  is an Hodge class,

$$\lambda = Res_{X, \mathbb{P}^N}(\alpha) + m[H] \in H^N OL_{X^{an}}(H_{usu}^p(X^{an}, \Omega_{X^{an}, \log}^{N-p})),$$

where  $Res_{X, \mathbb{P}^N} : H^N(U^{an}, \mathbb{C}) \xrightarrow{\sim} H^{2p}(X^{an}, \mathbb{C})$  is the residu map, using the fact that since the residu map is motivic (see definition 1), it preserve the logarithmic De Rham classes. Proposition 4 (ii) then implies that  $\lambda = [Z]$ ,  $Z \in \mathcal{Z}^p(X)$ , is the class of an algebraic cycle.

## 2 Preliminaries and Notations

### 2.1 Notations

- Denote by Top the category of topological spaces and RTop the category of ringed spaces.
- Denote by Cat the category of small categories and RCat the category of ringed topos.
- For  $\mathcal{S} \in \text{Cat}$  and  $X \in \mathcal{S}$ , we denote  $\mathcal{S}/X \in \text{Cat}$  the category whose objects are  $Y/X := (Y, f)$  with  $Y \in \mathcal{S}$  and  $f : Y \rightarrow X$  is a morphism in  $\mathcal{S}$ , and whose morphisms  $\text{Hom}((Y', f'), (Y, f))$  consists of  $g : Y' \rightarrow Y$  in  $\mathcal{S}$  such that  $f \circ g = f'$ .
- For  $(\mathcal{S}, O_{\mathcal{S}}) \in \text{RCat}$  a ringed topos, we denote by
  - $\text{PSh}(\mathcal{S})$  the category of presheaves of  $O_{\mathcal{S}}$  modules on  $\mathcal{S}$  and  $\text{PSh}_{O_{\mathcal{S}}}(\mathcal{S})$  the category of presheaves of  $O_{\mathcal{S}}$  modules on  $\mathcal{S}$ , whose objects are  $\text{PSh}_{O_{\mathcal{S}}}(\mathcal{S})^0 := \{(M, m), M \in \text{PSh}(\mathcal{S}), m : M \otimes O_{\mathcal{S}} \rightarrow M\}$ , together with the forgetful functor  $o : \text{PSh}(\mathcal{S}) \rightarrow \text{PSh}_{O_{\mathcal{S}}}(\mathcal{S})$ ,
  - $C(\mathcal{S}) = C(\text{PSh}(\mathcal{S}))$  and  $C_{O_{\mathcal{S}}}(\mathcal{S}) = C(\text{PSh}_{O_{\mathcal{S}}}(\mathcal{S}))$  the big abelian category of complexes of presheaves of  $O_{\mathcal{S}}$  modules on  $\mathcal{S}$ ,
  - $C_{O_{\mathcal{S}}(2)fil}(\mathcal{S}) := C_{(2)fil}(\text{PSh}_{O_{\mathcal{S}}}(\mathcal{S})) \subset C(\text{PSh}_{O_{\mathcal{S}}}(\mathcal{S}), F, W)$ , the big abelian category of (bi)filtered complexes of presheaves of  $O_{\mathcal{S}}$  modules on  $\mathcal{S}$  such that the filtration is biregular and  $\text{PSh}_{O_{\mathcal{S}}(2)fil}(\mathcal{S}) := (\text{PSh}_{O_{\mathcal{S}}}(\mathcal{S}), F, W)$ .
- Let  $(\mathcal{S}, O_{\mathcal{S}}) \in \text{RCat}$  a ringed topos with topology  $\tau$ . For  $F \in C_{O_{\mathcal{S}}}(\mathcal{S})$ , we denote by  $k : F \rightarrow E_{\tau}(F)$  the canonical flasque resolution in  $C_{O_{\mathcal{S}}}(\mathcal{S})$  (see [1]). In particular for  $X \in \mathcal{S}$ ,  $H^*(X, E_{\tau}(F)) \xrightarrow{\sim} \mathbb{H}_{\tau}^*(X, F)$ .

- For  $f : \mathcal{S}' \rightarrow \mathcal{S}$  a morphism with  $\mathcal{S}, \mathcal{S}' \in \mathbf{RCat}$ , endowed with topology  $\tau$  and  $\tau'$  respectively, we denote for  $F \in C_{O_S}(\mathcal{S})$  and each  $j \in \mathbb{Z}$ ,

$$\begin{aligned} - f^* &:= H^j \Gamma(\mathcal{S}, k \circ \text{ad}(f^*, f_*)(F)) : \mathbb{H}^j(\mathcal{S}, F) \rightarrow \mathbb{H}^j(\mathcal{S}', f^* F), \\ - f^* &:= H^j \Gamma(\mathcal{S}, k \circ \text{ad}(f^{*mod}, f_*)(F)) : \mathbb{H}^j(\mathcal{S}, F) \rightarrow \mathbb{H}^j(\mathcal{S}', f^{*mod} F), \end{aligned}$$

the canonical maps.

- For  $X \in \mathbf{Top}$  and  $Z \subset X$  a closed subset, denoting  $j : X \setminus Z \hookrightarrow X$  the open complementary, we will consider

$$\Gamma_Z^\vee \mathbb{Z}_X := \text{Cone}(\text{ad}(j_!, j^*)(\mathbb{Z}_X) : j_! j^* \mathbb{Z}_X \hookrightarrow \mathbb{Z}_X) \in C(X)$$

and denote for short  $\gamma_Z^\vee := \gamma_X^\vee(\mathbb{Z}_X) : \mathbb{Z}_X \rightarrow \Gamma_Z^\vee \mathbb{Z}_X$  the canonical map in  $C(X)$ .

- Denote by  $\mathbf{Sch} \subset \mathbf{RTop}$  the subcategory of schemes (the morphisms are the morphisms of locally ringed spaces). We denote by  $\mathbf{PSch} \subset \mathbf{Sch}$  the full subcategory of proper schemes. For a field  $k$ , we consider  $\mathbf{Sch}/k := \mathbf{Sch}/\mathbf{Spec} k$  the category of schemes over  $\mathbf{Spec} k$ , that is whose object are  $X := (X, a_X)$  with  $X \in \mathbf{Sch}$  and  $a_X : X \rightarrow \mathbf{Spec} k$  a morphism and whose objects are morphism of schemes  $f : X' \rightarrow X$  such that  $f \circ a_{X'} = a_X$ . We then denote by
  - $\mathbf{Var}(k) = \mathbf{Sch}^{ft}/k \subset \mathbf{Sch}/k$  the full subcategory consisting of algebraic varieties over  $k$ , i.e. schemes of finite type over  $k$ ,
  - $\mathbf{PVar}(k) \subset \mathbf{QPVar}(k) \subset \mathbf{Var}(k)$  the full subcategories consisting of quasi-projective varieties and projective varieties respectively,
  - $\mathbf{PSmVar}(k) \subset \mathbf{SmVar}(k) \subset \mathbf{Var}(k)$ ,  $\mathbf{PSmVar}(k) := \mathbf{PVar}(k) \cap \mathbf{SmVar}(k)$ , the full subcategories consisting of smooth varieties and smooth projective varieties respectively.
- Denote by  $\mathbf{AnSp}(\mathbb{C}) \subset \mathbf{RTop}$  the subcategory of analytic spaces over  $\mathbb{C}$ , and by  $\mathbf{AnSm}(\mathbb{C}) \subset \mathbf{AnSp}(\mathbb{C})$  the full subcategory of smooth analytic spaces (i.e. complex analytic manifold).
- Let  $(X, O_X) \in \mathbf{RTop}$ . We consider its De Rham complex  $\Omega_X^\bullet := DR(X)(O_X)$ .
  - Let  $X \in \mathbf{Sch}$ . Considering its De Rham complex  $\Omega_X^\bullet := DR(X)(O_X)$ , we have for  $j \in \mathbb{Z}$  its De Rham cohomology  $H_{DR}^j(X) := \mathbb{H}^j(X, \Omega_X^\bullet)$ .
  - Let  $X \in \mathbf{Var}(k)$ . Considering its De Rham complex  $\Omega_X^\bullet := \Omega_{X/k}^\bullet := DR(X/k)(O_X)$ , we have for  $j \in \mathbb{Z}$  its De Rham cohomology  $H_{DR}^j(X) := \mathbb{H}^j(X, \Omega_X^\bullet)$ . The differentials of  $\Omega_X^\bullet := \Omega_{X/k}^\bullet$  are by definition  $k$ -linear, thus  $H_{DR}^j(X) := \mathbb{H}^j(X, \Omega_X^\bullet)$  has a structure of a  $k$  vector space.
  - Let  $X \in \mathbf{AnSp}(\mathbb{C})$ . Considering its De Rham complex  $\Omega_X^\bullet := DR(X)(O_X)$ , we have for  $j \in \mathbb{Z}$  its De Rham cohomology  $H_{DR}^j(X) := \mathbb{H}^j(X, \Omega_X^\bullet)$ .
- For  $X \in \mathbf{AnSp}(\mathbb{C})$ , we denote  $\alpha(X) : \mathbb{C}_X \hookrightarrow \Omega_X^\bullet$  the embedding in  $C(X)$ . For  $X \in \mathbf{AnSm}(\mathbb{C})$ ,  $\alpha(X) : \mathbb{C}_X \hookrightarrow \Omega_X^\bullet$  is an equivalence usu local by Poincare lemma.
- We denote  $\mathbb{I}^n := [0, 1]^n \in \mathbf{Diff}(\mathbb{R})$  (with boundary). For  $X \in \mathbf{Top}$  and  $R$  a ring, we consider its singular cochain complex

$$C_{\text{sing}}^*(X, R) := (\mathbb{Z} \text{Hom}_{\mathbf{Top}}(\mathbb{I}^*, X)^\vee) \otimes R$$

and for  $l \in \mathbb{Z}$  its singular cohomology  $H_{\text{sing}}^l(X, R) := H^n C_{\text{sing}}^*(X, R)$ . For  $f : X' \rightarrow X$  a continous map with  $X, X' \in \mathbf{Top}$ , we have the canonical map of complexes

$$f^* : C_{\text{sing}}^*(X, R) \rightarrow C_{\text{sing}}^*(X', R), \sigma \mapsto f^* \sigma := (\gamma \mapsto \sigma(f \circ \gamma)).$$

In particular, we get by functoriality the complex

$$C_{X, R \text{ sing}}^* \in C_R(X), (U \subset X) \mapsto C_{\text{sing}}^*(U, R)$$

We recall we have

- For  $X \in \text{Top}$  locally contractile, e.g.  $X \in \text{CW}$ , and  $R$  a ring, the inclusion in  $C_R(X)$   $c_X : R_X \rightarrow C_{X,R}^{\text{sing}}$  is by definition an equivalence top local and that we get by the small chain theorem, for all  $l \in \mathbb{Z}$ , an isomorphism  $H^l c_X : H^l(X, R_X) \xrightarrow{\sim} H_{\text{sing}}^l(X, R)$ .
- For  $X \in \text{Diff}(\mathbb{R})$ , the restriction map

$$r_X : \mathbb{Z} \text{Hom}_{\text{Diff}(\mathbb{R})}(\mathbb{I}^*, X)^\vee \rightarrow C_{\text{sing}}^*(X, R), w \mapsto w : (\phi \mapsto w(\phi))$$

is a quasi-isomorphism by Whitney approximation theorem.

- We will consider the morphism of site given by the analytification functor

$$\text{An} : \text{AnSp}(\mathbb{C}) \rightarrow \text{Var}(\mathbb{C}), X \mapsto \text{An}(X) := X^{an}, (f : X' \rightarrow X) \mapsto \text{An}(f) := f^{an}.$$

For  $X \in \text{Var}(\mathbb{C})$ , we denote by

$$\text{an}_X := \text{An}|_{X^{et}} : X^{an,et} \rightarrow X^{et}, \text{an}_X(U \rightarrow X) := U^{an},$$

its restriction to the small etale site  $o_X(X^{et}) \subset \text{Var}(\mathbb{C})$ , where  $o_X(U \rightarrow X) = U$ , in particular we have the following commutative diagram

$$\begin{array}{ccc} \text{AnSp}(\mathbb{C}) & \xrightarrow{\text{An}} & \text{Var}(\mathbb{C}) \\ \downarrow o_{X^{an}} & & \downarrow o_X \\ X^{an,et} & \xrightarrow{\text{an}_X} & X^{et} \end{array}$$

where  $o_{X^{an}}(U \hookrightarrow X^{an}) = U$ . We get the morphism in  $\text{RTop}$   $\text{an}_X := \text{an}_X|_{O(X)} : X^{an} \rightarrow X$ ,  $o : O(X) \subset X^{et}$  being the embedding consisting of the Zariski open subsets of  $X$ .

## 2.2 Complex integral periods

Let  $k$  a field of characteristic zero.

Let  $X \in \text{SmVar}(k)$  a smooth variety. Let  $X = \cup_{i=1}^s X_i$  an open affine cover. We have for  $\sigma : k \hookrightarrow \mathbb{C}$  an embedding, the evaluation period embedding map which is the morphism of bi-complexes

$$\begin{aligned} ev(X) \bullet : \Gamma(X_\bullet, \Omega_{X_\bullet}^\bullet) &\rightarrow \mathbb{Z} \text{Hom}_{\text{Diff}}(\mathbb{I}^\bullet, X_{\mathbb{C},\bullet}^{an})^\vee \otimes \mathbb{C}, \\ w_I^l \in \Gamma(X_I, \Omega_{X_I}^l) &\mapsto (ev(X)_I^l(w_I^l)) : \phi_I^l \in \mathbb{Z} \text{Hom}_{\text{Diff}}(\mathbb{I}^l, X_{\mathbb{C},I}^{an})^\vee \otimes \mathbb{C} \mapsto ev_I^l(w_I^l)(\phi_I^l) := \int_{\mathbb{I}^l} \phi_I^{l*} w_I^l \end{aligned}$$

given by integration. By taking all the affine open cover  $(j_i : X_i \hookrightarrow X)$  of  $X$ , we get for  $\sigma : k \hookrightarrow \mathbb{C}$ , the evaluation period embedding map

$$ev(X) := \varinjlim_{(j_i : X_i \hookrightarrow X)} ev(X) \bullet : \varinjlim_{(j_i : X_i \hookrightarrow X)} \Gamma(X_\bullet, \Omega_{X_\bullet}^\bullet) \rightarrow \varinjlim_{(j_i : X_i \hookrightarrow X)} \mathbb{Z} \text{Hom}_{\text{Diff}(\mathbb{R})}(\mathbb{I}^\bullet, X_{\mathbb{C},\bullet}^{an})^\vee \otimes \mathbb{C}$$

It induces in cohomology, for  $j \in \mathbb{Z}$ , the evaluation period map

$$H^j ev(X) = H^j ev(X) \bullet : H_{DR}^j(X) = H^j \Gamma(X_\bullet, \Omega_{X_\bullet}^\bullet) \rightarrow H_{\text{sing}}^j(X_{\mathbb{C}}^{an}, \mathbb{C}) = H^j(\text{Hom}_{\text{Diff}(\mathbb{R})}(\mathbb{I}^\bullet, X_{\mathbb{C},\bullet}^{an})^\vee \otimes \mathbb{C}).$$

which does NOT depend on the choice of the affine open cover by acyclicity of quasi-coherent sheaves on affine noetherian schemes for the left hand side and from Mayer-Vietoris quasi-isomorphism for singular cohomology of topological spaces and Whitney approximation theorem for differential manifolds for the right hand side.

Let  $X \in \text{SmVar}(k)$ . Then,

$$H^* ev(X_{\mathbb{C}}) = H^* R\Gamma(X_{\mathbb{C}}^{an}, \alpha(X))^{-1} \circ \Gamma(X_{\mathbb{C}}^{an}, E_{zar}(\Omega(\text{an}_X))) : H_{DR}^*(X_{\mathbb{C}}) \xrightarrow{\sim} H_{\text{sing}}^*(X_{\mathbb{C}}^{an}, \mathbb{C})$$

is the canonical isomorphism induced by the analytical functor and the quasi-isomorphism  $\alpha(X) : \mathbb{C}_{X_{\mathbb{C}}^{an}} \hookrightarrow \Omega_{X_{\mathbb{C}}^{an}}^\bullet$ , which gives the periods elements  $H^* ev(X)(H_{DR}^*(X)) \subset H_{\text{sing}}^*(X_{\mathbb{C}}^{an}, \mathbb{C})$ .

### 2.3 Algebraic cycles and motives

For  $X \in \text{Sch}$  noetherian irreducible and  $d \in \mathbb{N}$ , we denote by  $\mathcal{Z}^d(X)$  the group of algebraic cycles of codimension  $d$ , which is the free abelian group generated by irreducible closed subsets of codimension  $d$ .

For  $X, X' \in \text{Sch}$  noetherian, with  $X'$  irreducible, we denote  $\mathcal{Z}^{fs/X'}(X' \times X) \subset \mathcal{Z}_{d_{X'}}(X' \times X)$  which consist of algebraic cycles  $\alpha = \sum_i n_i \alpha_i \in \mathcal{Z}_{d_{X'}}(X' \times X)$  such that, denoting  $\text{supp}(\alpha) = \cup_i \alpha_i \subset X' \times X$  its support and  $p' : X' \times X \rightarrow X'$  the projection,  $p'|_{\text{supp}(\alpha)} : \text{supp}(\alpha) \rightarrow X'$  is finite surjective.

Let  $k$  a field. We denote by  $\text{Cor SmVar}(k)$  the category such that the objects are  $\{X \in \text{SmVar}(k)\}$  and such that  $\text{Hom}_{\text{Cor SmVar}(k)}(X', X) := \mathcal{Z}^{fs/X'}(X' \times X)$ . See [3] for the composition law. We denote by  $\text{Tr} : \text{Cor SmVar}(k) \rightarrow \text{SmVar}(k)$  the morphism of site given by the embedding  $\text{Tr} : \text{SmVar}(k) \hookrightarrow \text{Cor SmVar}(k)$ . Let  $F \in \text{PSh}(\text{SmVar}(k))$ . We say that  $F$  admits transfers if  $F = \text{Tr}_* F$  with  $F \in \text{PSh}(\text{Cor SmVar}(k))$ . We recall that  $\text{DA}(k) := \text{Ho}_{\mathbb{A}^1, \text{et}}(C(\text{SmVar}(k)))$  is then the derived category of mixed motives.

We have (see e.g. [1]) the Hodge realization functor

$$\mathcal{F}^{Hdg} : \text{DA}(\mathbb{C}) \rightarrow D(MHS), M \mapsto \mathcal{F}^{Hdg}(M) := \iota^{-1}(\mathcal{F}^{FDR}(M), \text{Bti}^* M, \alpha(M))$$

For  $X \in \text{Var}(k)$  and  $Z \subset X$  a closed subset, denoting  $j : X \setminus Z \hookrightarrow X$  the open complementary, we will consider

$$\Gamma_Z^\vee \mathbb{Z}_X := \text{Cone}(\text{ad}(j_\#, j^*)(\mathbb{Z}_X) : \mathbb{Z}_X \hookrightarrow \mathbb{Z}_X) \in C(\text{Var}(k)^{sm}/X)$$

and denote for short  $\gamma_Z^\vee := \gamma_Z^\vee(\mathbb{Z}_X) : \mathbb{Z}_X \rightarrow \Gamma_Z^\vee \mathbb{Z}_X$  the canonical map in  $C(\text{Var}(k)^{sm}/X)$ . Denote  $a_X : X \rightarrow \text{Spec } k$  the structural map. For  $X \in \text{Var}(k)$  and  $Z \subset X$  a closed subset, we have the motive of  $X$  with support in  $Z$  defined as

$$M_Z(X) := a_{X!} \Gamma_Z^\vee a_X^! \mathbb{Z} \in \text{DA}(k).$$

If  $X \in \text{SmVar}(k)$ , we will also consider

$$a_{X\#} \Gamma_Z^\vee \mathbb{Z}_X := \text{Cone}(a_{X\#} \circ \text{ad}(j_\#, j^*)(\mathbb{Z}_X) : \mathbb{Z}(U) \hookrightarrow \mathbb{Z}(X)) =: \mathbb{Z}(X, X \setminus Z) \in C(\text{SmVar}(k)).$$

Then for  $X \in \text{SmVar}(k)$  and  $Z \subset X$  a closed subset

$$M_Z(X) := a_{X!} \Gamma_Z^\vee a_X^! \mathbb{Z} = a_{X\#} \Gamma_Z^\vee \mathbb{Z}_X =: \mathbb{Z}(X, X \setminus Z) \in \text{DA}(k).$$

- Let  $(X, Z) \in \text{Sch}^2$  with  $X \in \text{Sch}$  a noetherian scheme and  $Z \subset X$  a closed subset. We have the deformation  $(D_Z X, \mathbb{A}_Z^1) \rightarrow \mathbb{A}^1$ ,  $(D_Z X, \mathbb{A}_Z^1) \in \text{Sch}^2$  of  $(X, Z)$  by the normal cone  $C_{Z/X} \rightarrow Z$ , i.e. such that

$$(D_Z X, \mathbb{A}_Z^1)_s = (X, Z), s \in \mathbb{A}^1 \setminus 0, (D_Z X, \mathbb{A}_Z^1)_0 = (C_{Z/X}, Z).$$

We denote by  $i_1 : (X, Z) \hookrightarrow (D_Z X, \mathbb{A}_Z^1)$  and  $i_0 : (C_{Z/X}, Z) \hookrightarrow (D_Z X, \mathbb{A}_Z^1)$  the closed embeddings in  $\text{Sch}^2$ .

- Let  $k$  a field of characteristic zero. Let  $X \in \text{SmVar}(k)$ . For  $Z \subset X$  a closed subset of pure codimension  $c$ , consider a desingularisation  $\epsilon : \tilde{Z} \rightarrow Z$  of  $Z$  and denote  $n : \tilde{Z} \xrightarrow{\epsilon} Z \subset X$ . We have then the morphism in  $\text{DA}(k)$

$$G_{Z,X} : M(X) \xrightarrow{D(\mathbb{Z}(n))} M(\tilde{Z})(c)[2c] \xrightarrow{\mathbb{Z}(\epsilon)} M(Z)(c)[2c]$$

where  $D : \text{Hom}_{\text{DA}(k)}(M_c(\tilde{Z}), M_c(X)) \xrightarrow{\sim} \text{Hom}_{\text{DA}(k)}(M(X), M(\tilde{Z})(c)[2c])$  is the duality isomorphism from the six functors formalism (moving lemma of Suzlin and Voevodsky) and  $\mathbb{Z}(n) := \text{ad}(n_!, n^!)(a_X^! \mathbb{Z})$ , noting that  $n_! = n_*$  since  $n$  is proper and that  $a_X^! = a_X^*[d_X]$  and  $a_Z^! = a_Z^*[d_Z]$  since  $X$ , resp.  $\tilde{Z}$ , are smooth (considering the connected components, we may assume  $X$  and  $\tilde{Z}$  of pure dimension).

We recall the following facts (see [3] and [1]):

**Proposition 1.** *Let  $k$  be a field of characteristic zero. Let  $X \in \text{SmVar}(k)$  and  $i : Z \subset X$  a smooth subvariety of pure codimension  $d$ . Then  $C_{Z/X} = N_{Z/X} \rightarrow Z$  is a vector bundle of rank  $d$ . The closed embeddings  $i_1 : (X, Z) \hookrightarrow (D_Z X, \mathbb{A}_Z^1)$  and  $i_0 : (C_{Z/X}, Z) \hookrightarrow (D_Z X, \mathbb{A}_Z^1)$  in  $\text{SmVar}^2(k)$  induces isomorphisms of motives  $\mathbb{Z}(i_1) : M_Z(X) \xrightarrow{\sim} M_{\mathbb{A}_Z^1}(D_Z X)$  and  $\mathbb{Z}(i_0) : M_Z(N_{Z/X}) \xrightarrow{\sim} M_{\mathbb{A}_Z^1}(D_Z X)$  in  $\text{DA}(k)$ . We get the excision isomorphism in  $\text{DA}(k)$*

$$P_{Z,X} := \mathbb{Z}(i_0)^{-1} \circ \mathbb{Z}(i_1) : M_Z(X) \xrightarrow{\sim} M_Z(N_{Z/X}).$$

We have

$$\text{Th}(N_{Z/X}) \circ P_{Z,X} \circ \gamma_Z^\vee(\mathbb{Z}_X) = G_{Z,X} := D(\mathbb{Z}(i)) : M(X) \rightarrow M(Z)(d)[2d].$$

*Proof.* See [3]. □

**Definition 1.** *Let  $k$  be a field of characteristic zero. Let  $X \in \text{SmVar}(k)$  and  $i : Z \subset X$  a smooth subvariety of pure codimension  $d$ . We consider using proposition 1 the canonical map in  $\text{DA}(k)$*

$$\text{Res}_{Z,X} : M(Z)(d)[-2d] \xrightarrow{P_{Z,X}^{-1} \circ \text{Th}(N_{Z/X})^{-1}} M_Z(X) \xrightarrow{c(\mathbb{Z}(X \setminus Z))} M(X \setminus Z)[1]$$

where, for  $m : A \rightarrow B$ ,  $A, B \in C(\mathcal{A})$ ,  $\mathcal{A}$  an additive category,  $c(A) : \text{Cone}(m : A \rightarrow B) \rightarrow A[1]$  is the canonical map.

- Recall that we say that  $F \in \text{PSh}(\text{SmVar}(k))$  is  $\mathbb{A}^1$  invariant if for all  $X \in \text{SmVar}(k)$ ,  $p^* := F(p) : F(X) \rightarrow F(X \times \mathbb{A}^1)$  is an isomorphism, where  $p : X \times \mathbb{A}^1 \rightarrow X$  is the projection.
- Similarly, we say that  $F \in \text{PSh}(\text{AnSm}(\mathbb{C}))$  is  $\mathbb{D}^1$  invariant if for all  $X \in \text{AnSm}(\mathbb{C})$ ,  $p^* := F(p) : F(X) \rightarrow F(X \times \mathbb{D}^1)$  is an isomorphism, where  $p : X \times \mathbb{D}^1 \rightarrow X$  is the projection.

We will use the following theorem of Voevodsky :

**Theorem 1.** (Voevodsky) *Let  $k$  be a perfect field (e.g.  $k$  a field of characteristic zero). Let  $F \in \text{PSh}(\text{SmVar}(k), \mathbb{Q})$ . If  $F$  is  $\mathbb{A}^1$  invariant and admits transfers, then for all  $j \in \mathbb{Z}$ ,  $H^j E_{\text{et}}^\bullet(F) \in \text{PSh}(\text{SmVar}(k))$  are  $\mathbb{A}^1$  invariant. That is, if  $F$  is  $\mathbb{A}^1$  invariant and admits transfers then  $F$  is  $\mathbb{A}^1$  local.*

*Proof.* By [5],  $H^j E_{N_{\text{is}}}^\bullet(F) \in \text{PSh}(\text{SmVar}(k))$  are  $\mathbb{A}^1$  invariant. On the other hand since  $F$  takes values in  $\mathbb{Q}$ -vector spaces,  $H^j E_{N_{\text{is}}}^\bullet(F) = H^j E_{\text{et}}^\bullet(F)$ . □

## 2.4 The logarithmic De Rham complexes

We recall from [2] the following notion :

- For  $k$  a field and  $X \in \text{Var}(k)$ , we consider the embedding in  $C(X)$

$$OL_X : \Omega_{X,\log}^\bullet \hookrightarrow \Omega_X^\bullet := \Omega_{X/k}^\bullet,$$

such that, for  $X^\circ \subset X$  an open subset and  $w \in \Omega_X^p(X^\circ)$ ,  $w \in \Omega_{X,\log}^p(X^\circ)$  if and only if there exists  $(n_i)_{1 \leq i \leq s} \in \mathbb{Z}$  and  $(f_{i,\alpha_k})_{1 \leq i \leq s, 1 \leq k \leq p} \in \Gamma(X^\circ, \mathcal{O}_X)^*$  such that

$$w = \sum_{1 \leq i \leq s} n_i df_{i,\alpha_1} / f_{i,\alpha_1} \wedge \cdots \wedge df_{i,\alpha_p} / f_{i,\alpha_p} \in \Omega_X^p(X^\circ),$$

and for  $p = 0$ ,  $\Omega_{X,\log}^0 := \mathbb{Z}$ . Let  $k$  a field. We get an embedding in  $C(\text{Var}(k))$

$$OL : \Omega_{/k,\log}^\bullet \hookrightarrow \Omega_{/k}^\bullet, \text{ given by, for } X \in \text{Var}(k), \\ OL(X) := OL_X : \Omega_{/k,\log}^\bullet(X) := \Gamma(X, \Omega_{X,\log}^\bullet) \hookrightarrow \Gamma(X, \Omega_X^\bullet) =: \Omega_{/k}^\bullet(X)$$

and its restriction to  $\text{SmVar}(k) \subset \text{Var}(k)$ .

- For  $X \in \text{AnSp}(\mathbb{C})$ , we consider the embedding in  $C(X)$

$$OL_X : \Omega_{X,\log}^\bullet \hookrightarrow \Omega_X^\bullet,$$

such that, for  $X^\circ \subset X$  an open subset and  $w \in \Omega_X^p(X^\circ)$ ,  $w \in \Omega_{X,\log}^p(X^\circ)$  if and only if there exists  $(n_i)_{1 \leq i \leq s} \in \mathbb{Z}$  and  $(f_{i,\alpha_k})_{1 \leq i \leq s, 1 \leq k \leq p} \in \Gamma(X^\circ, O_X)^*$  such that

$$w = \sum_{1 \leq i \leq s} n_i df_{i,\alpha_1} / f_{i,\alpha_1} \wedge \cdots \wedge df_{i,\alpha_p} / f_{i,\alpha_p} \in \Omega_X^p(X^\circ),$$

and for  $p = 0$ ,  $\Omega_{X,\log}^0 := \mathbb{Z}$ . We get an embedding in  $C(\text{AnSp}(\mathbb{C}))$

$$OL^{an} : \Omega_{\log}^{\bullet,an} \hookrightarrow \Omega^{\bullet,an}, \text{ given by, for } X \in \text{AnSp}(\mathbb{C}),$$

$$OL^{an}(X) := OL_X : \Omega_{\log}^\bullet(X) := \Gamma(X, \Omega_{X,\log}^{\bullet,an}) \hookrightarrow \Gamma(X, \Omega_X^{\bullet,an}) =: \Omega^{\bullet,an}(X)$$

and its restriction to  $\text{AnSm}(\mathbb{C}) \subset \text{AnSp}(\mathbb{C})$ .

We have the following result :

**Proposition 2.** (i00) Let  $D \in \text{AnSm}(\mathbb{C})$  be an open ball. Then, for each  $p, q \in \mathbb{Z}$ ,  $q \neq 0$ ,  $H_{et}^q(D, \Omega_{D,\log}^p) = 0$ .

(i0) Let  $U \in \text{SmVar}(\mathbb{C})$  such that there exists an etale map  $e : U \rightarrow \mathbb{A}_{\mathbb{C}}^{d_U}$  such that  $\mathbb{A}_{\mathbb{C}}^{d_U} \setminus e(U) \subset \mathbb{A}_{\mathbb{C}}^{d_U}$  is a divisor. Then, for each  $p, q \in \mathbb{Z}$ ,  $q \neq 0$ ,  $p \neq 0$ ,  $H_{et}^q(U, \Omega_{U^{an},\log}^p) = 0$ .

(i) Let  $U \in \text{SmVar}(\mathbb{C})$  such that there exists an etale map  $e : U \rightarrow \mathbb{A}_{\mathbb{C}}^{d_U}$  such that  $\mathbb{A}_{\mathbb{C}}^{d_U} \setminus e(U) \subset \mathbb{A}_{\mathbb{C}}^{d_U}$  is a divisor. Then, for each  $p, q \in \mathbb{Z}$ ,  $q \neq 0$ ,  $p \neq 0$ ,  $H_{usu}^q(U^{an}, \Omega_{U^{an},\log}^p) = 0$ .

(ii) Let  $X \in \text{SmVar}(\mathbb{C})$ . We still denote  $\text{an}_X : X^{an} \xrightarrow{e,\sim} X^{an,et} \xrightarrow{\text{an}_X} X^{et}$  the composite. Consider the map in  $C(X^{et})$

$$T(et, usu) = E_{et}(\text{an}_X^* k) : E_{et}(\text{an}_X^* \Omega_{X^{an},\log}^p) \rightarrow E_{et}(\text{an}_X^* E_{usu}(\Omega_{X^{an},\log}^p)).$$

Then, for each  $p, q \in \mathbb{Z}$ ,

$$H^{p+q}T(et, usu)(X) : H_{et}^q(X, \text{an}_X^* \Omega_{X^{an},\log}^p) \rightarrow H_{usu}^q(X^{an}, \Omega_{X^{an},\log}^p)$$

is an isomorphism.

(ii)' Let  $X \in \text{SmVar}(\mathbb{C})$ . Let  $Z$  a closed subset. Then, for each  $p, q \in \mathbb{Z}$ ,

$$H_Z^{p+q}T(et, usu)(X) : H_{et,Z}^q(X, \text{an}_X^* \Omega_{X^{an},\log}^p) \rightarrow H_{usu,Z}^q(X^{an}, \Omega_{X^{an},\log}^p)$$

is an isomorphism.

*Proof.* (i00): Let  $B = D(0, 1)^N \in \text{AnSm}(\mathbb{C})$  be an open ball. Let  $\alpha \in H^q(B, \Omega_{B,\log}^p)$ ,  $q \in \mathbb{Z}$ ,  $q \neq 0$ . Consider a sequence of sub-balls

$$l_n : B_n := D(0, r_n)^N \hookrightarrow B := D(0, 1)^N, n \in \mathbb{N}, \text{ with } r_n \rightarrow 0 \text{ when } n \rightarrow \infty.$$

Since  $(B_n := D(0, r_n))_{n \in \mathbb{N}}$  form a basis of neighborhood of  $0 \in B$  and since  $\Omega_{B,\log}^p$  is a single presheaf, there exists  $n := n_\alpha \in \mathbb{N}$  (depending on  $\alpha$ ) such that

$$l_n^* \alpha = 0 \in H^q(B_n, \Omega_{B_n,\log}^p).$$

But it follows immediately from the definition of the the presheaf  $\Omega_{B,\log}^p$  considering the isomorphism

$$\phi_n : B_n \xrightarrow{\sim} B, \phi_n(z) := r_n^{-1}z$$



so that  $(\phi_n)^{-1} \circ l_n = r_n I$ , that

$$l_n^* \alpha = l_n^* \phi_n^{-1*} \phi_n^* \alpha = r_n \phi_n^* \alpha \in H^q(B_n, \Omega_{B, \log}^p).$$

Hence  $\phi_n^* \alpha = 0$ . Since  $\phi_n$  is an isomorphism, we get  $\alpha = 0$ .

(i): We prove (i) by induction on the dimension of  $U$ . Let  $U \in \text{SmVar}(\mathbb{C})$  such that there exists an etale map  $e : U \rightarrow \mathbb{A}_{\mathbb{C}}^{d_U}$ . Denote  $D := \mathbb{A}_{\mathbb{C}}^{d_U} \setminus e(U)$  the complementary divisor. Since  $H_{usu}^q(\mathbb{A}_{\mathbb{C}}^{d_U}, \Omega_{\log}^{p, an}) = 0$  for all  $p, q \in \mathbb{Z}$ ,

$$H^q E_{usu} \Omega_{\log}^{p, an}(c(\mathbb{Z}(e(U)))) = \partial : H_{usu}^q(e(U), \Omega_{\log}^{p, an}) \xrightarrow{\sim} H_{usu, D}^{q+1}(\mathbb{A}_{\mathbb{C}}^{d_U}, \Omega_{\log}^{p, an})$$

is an isomorphism for all  $p, q \in \mathbb{Z}$ . Let  $D = \cup_i D_i$  be the irreducible components of  $D$ . Let  $D_{i0} \subset D_i$  be divisors such that there exist etale maps  $e_i : D_i \setminus D_{i0} \rightarrow \mathbb{A}_{\mathbb{C}}^{d_U-1}$  such that  $\mathbb{A}_{\mathbb{C}}^{d_U-1} \setminus e_i(D_i \setminus D_{i0}) \subset \mathbb{A}_{\mathbb{C}}^{d_U-1}$  is a divisor. By (i00),

$$H^q E_{usu}(\Omega_{\log}^{p, an}) \in \text{PSh}(\text{AnSm}(\mathbb{C}))$$

is  $\mathbb{D}^1$  invariant for each  $q \in \mathbb{Z}$ ,  $q \neq 0$ . By the local to global spectral sequence

$$H^{q'} E_{usu}(H^q E_{usu}(\Omega_{\log}^{p, an})) \in \text{PSh}(\text{AnSm}(\mathbb{C}))$$

are also  $\mathbb{D}^1$  invariant for all  $q' \in \mathbb{Z}$  (one can also use the fact that if  $F \in \text{PSh}(\text{AnSm}(\mathbb{C}))$  is  $\mathbb{D}^1$  invariant and admit transfers then  $H^q E_{usu}(F) \in \text{PSh}(\text{AnSm}(\mathbb{C}))$  is also  $\mathbb{D}^1$  invariant all  $q' \in \mathbb{Z}$ ). We thus get by proposition 1 the isomorphism for each  $q, q' \in \mathbb{Z}$ ,  $q \neq 0$ .

$$\begin{aligned} & H^{q'} E_{usu}(H^q E_{usu}(\Omega_{\log}^{p, an}(P_{D_i \setminus D_{i0}, \mathbb{A}_{\mathbb{C}}^{d_U \setminus Z_{i0}})))) : \\ & H_{usu, D_i \setminus D_{i0}}^{q+q'}(\mathbb{A}_{\mathbb{C}}^{d_U}, \Omega_{\log}^{p, an}) \xrightarrow{\sim} H_{usu}^{q+q'-1}(D_i \setminus D_{i0}, \Omega_{\log}^{p-1, an}). \end{aligned}$$

On the other hand, by induction hypothesis  $H_{usu}^{q''}(D_i \setminus D_{i0}, \Omega_{\log}^{p-1, an}) = 0$  for all  $q'' \in \mathbb{Z}$ ,  $q'' \neq 0$ . Hence,

$$H_{usu, D_i \setminus D_{i0}}^{q''}(\mathbb{A}_{\mathbb{C}}^{d_U}, \Omega_{\log}^{p, an}) = 0$$

for all  $q'' \in \mathbb{Z}$ ,  $q'' \neq 1$ . By a finite decreasing induction  $D_{ir} \subset \dots \subset D_{i0} \subset Z$ ,  $r = d_U - 1$ , with  $e_r : D_{ir} \rightarrow \mathbb{A}^{d_{D_{ir}}}$  etale such that  $\mathbb{A}^{d_{D_{ir}}} \setminus e_r(D_{ir}) \subset \mathbb{A}^{d_{D_{ir}}}$  is a divisor, we get

$$H_{usu}^{q''}(e(U), \Omega_{\log}^{p, an}) = H_{usu, D}^{q''+1}(\mathbb{A}_{\mathbb{C}}^{d_U}, \Omega_{\log}^{p, an}) = 0$$

for each  $q'' \in \mathbb{Z}$ ,  $q'' \neq 0$ . Let  $Z' \subset U$  be a (Zariski) closed subset such that  $e' := e|_{U \setminus Z'} : (U \setminus Z') \rightarrow e(U \setminus Z')$  is finite etale. We thus have, since  $M(U \setminus Z') \simeq M(e(U \setminus Z'))$ , for each  $q \in \mathbb{Z}$ ,  $q \neq 0$

$$H_{usu}^q(U \setminus Z', \Omega_{\log}^{p, an}) = H_{usu}^q(e(U \setminus Z'), \Omega_{\log}^{p, an}) = 0.$$

Denote  $j : U \setminus Z' \hookrightarrow U$  the open embedding. By the long exact sequence

$$\dots \xrightarrow{j^* := \Omega_{\log}^{p, an}(j)} H_{usu}^q(U \setminus Z', \Omega_{\log}^{p, an}) \xrightarrow{\partial := \Omega_{\log}^{p, an}(c(\mathbb{Z}(U \setminus Z')))} H_{usu, Z'}^{q+1}(U, \Omega_{\log}^{p, an}) \xrightarrow{\Omega_{\log}^{p, an}(\gamma_{Z'}^\vee)} H_{usu}^{q+1}(U, \Omega_{\log}^{p, an}) \xrightarrow{j^*} \dots,$$

we get by a finite induction on the dimension of  $Z'$ , we get  $H_{usu}^q(U, \Omega_{\log}^{p, an}) = 0$  for all  $q \in \mathbb{Z}$ ,  $q \neq 0$ .

(i0): By (i), we have  $a_{et} R^q \text{an}_{U*} \Omega_{U^{an}, \log}^p = 0$ ,  $q \in \mathbb{Z}$ ,  $q \neq 0$ . Hence for  $q \in \mathbb{Z}$ ,  $q \neq 0$ ,

$$H_{et}^q(U, \text{an}_{U*} \Omega_{U^{an}, \log}^p) = \mathbb{H}_{et}^q(U, R \text{an}_{U*} \Omega_{U^{an}, \log}^p) = H_{usu}^q(U^{an}, \Omega_{U^{an}, \log}^p) = 0,$$

where the last equality follows again by (i).

(ii): By (i),  $a_{et} R^q \text{an}_{X*} \Omega_{X^{an}, \log}^p = 0$ ,  $q \in \mathbb{Z}$ ,  $q \neq 0$ .

(ii)': Follows from (ii) applied to  $X$  and  $X \setminus Z$  by the distinguish triangle in  $Ho(C(\text{SmVar}(\mathbb{C})))$

$$\mathbb{Z}(X \setminus Z) \xrightarrow{\mathbb{Z}(j)} \mathbb{Z}(X) \rightarrow \mathbb{Z}(X, X \setminus Z) \rightarrow \mathbb{Z}(X \setminus Z)[1],$$

where  $j : X \setminus Z \hookrightarrow X$  is the open embedding. □

### 3 Hypersurface De Rham cohomology and its complex periods

We denote  $\mathbb{P}^N := \mathbb{P}_{\mathbb{C}}^N := \text{Proj}(\mathbb{C}[z_0, \dots, z_N])$ . Let  $X = V(f) \subset \mathbb{P}^N$  be a smooth hypersurface. If  $N - 1 = 2p$  is even, we consider the decompositions

$$H_{DR}^{N-1}(X) = \mathbb{C}[\Lambda \cap X] \oplus H_{DR,v}^{N-1}(X), \quad H_{\text{sing}}^{N-1}(X^{an}, \mathbb{Q}) = \mathbb{C}[\Lambda \cap X] \oplus H_{\text{sing},v}^{N-1}(X^{an}, \mathbb{Q})$$

of vector spaces, where  $\Lambda \subset \mathbb{P}^N$  is a linear subspace of codimension  $p$ . If  $N - 1 = 2p + 1$  is even, we set

$$H_{DR,v}^{N-1}(X) := H_{DR}^{N-1}(X), \quad H_{\text{sing},v}^{N-1}(X^{an}, \mathbb{Q}) := H_{\text{sing}}^{N-1}(X^{an}, \mathbb{Q}).$$

Let  $X = V(f) \subset \mathbb{P}^N$  be a smooth hypersurface. Denote  $j : U := \mathbb{P}^N \setminus X \hookrightarrow \mathbb{P}^N$  the open embedding. We consider the algebraic Hodge module

$$j_{*Hdg}(O_U, F_b) := ((j_*O_U, F), j_*\mathbb{Q}_{U^{an}}, j_*\alpha(U)) \in HM(\mathbb{P}^N),$$

where  $F^k j_*O_U = \mathbb{C}[x_1, \dots, x_N][\frac{1}{f^k}]$ . Recall for each  $p \in \mathbb{Z}$ ,

$$F^p H_{DR}^N(U) := H^N \iota^{\geq p}(\mathbb{H}^N(\mathbb{P}^N, \Omega_{\mathbb{P}^N}^{\bullet \geq p}(\log X))) = H^N \iota^{\geq p} H^N(\mathbb{P}^N, \Omega_{\mathbb{P}^N}^{\bullet} \otimes_{O_{\mathbb{P}^N}} F^{\bullet-p} j_*O_U)$$

and

$$F^p H^{N-1}(X) := H^{N-1} \iota^{\geq p}(\mathbb{H}^{N-1}(X, \Omega_X^{\bullet \geq p})).$$

where

$$\iota^{\geq p} : \Omega_{\mathbb{P}^N}^{\bullet \geq p}(\log X) \hookrightarrow \Omega_{\mathbb{P}^N}^{\bullet}(\log X), \quad \iota^{\geq p} : \Omega_{\mathbb{P}^N}^{\bullet} \otimes_{O_{\mathbb{P}^N}} F^{\bullet-p} j_*O_U \hookrightarrow \Omega_{\mathbb{P}^N}^{\bullet} \otimes_{O_{\mathbb{P}^N}} j_*O_U = j_*\Omega_U^{\bullet}$$

and  $\iota^{\geq p} : \Omega_X^{\bullet \geq p} \hookrightarrow \Omega_X^{\bullet}$  are the canonical embeddings. Then the map  $\text{Res}_{X, \mathbb{P}^N}$  of definition 1 induces for each  $p \in \mathbb{Z}$ , canonical isomorphisms

$$H^N \Omega(\text{Res}_{X, \mathbb{P}^N}) : F^{p+1} H_{DR}^N(U) \xrightarrow{\sim} F^p H_{DR}^{N-1}(X)_v, \quad H^N \text{Bti}(\text{Res}_{X, \mathbb{P}^N}) : H_{\text{sing}}^N(U, \mathbb{Q}) \xrightarrow{\sim} H_{\text{sing}}^{N-1}(X, \mathbb{Q})_v.$$

**Lemma 1.** *Let  $X = V(f) \subset \mathbb{P}^N$  be a smooth hypersurface. Denote  $j : U := \mathbb{P}^N \setminus X \hookrightarrow \mathbb{P}^N$  the open embedding. Let  $B \subset \mathbb{P}^N$ ,  $\phi : B \simeq D(0, 1)^N$  be an open ball such that  $\phi(X \cap B) = \{0\} \times D(0, 1)^{N-1}$ . Then,*

$$(i) \text{ For } q \in \mathbb{Z}, q \neq 0, \mathbb{H}_{usu}^q(B, (\Omega_{\mathbb{P}^N}^{\bullet} \otimes_{O_{\mathbb{P}^N}} F^{\bullet-p} j_*O_U)^{an}) = 0.$$

$$(ii) \text{ For } q \in \mathbb{Z}, q \neq 0, H_{usu}^q(B, \Omega_{\mathbb{P}^N, an}^{N-p}(\log X)^{\partial=0}) = 0.$$

*Proof.* (i): Since  $B$  is Stein, we have for  $r, q \in \mathbb{Z}, q \neq 0$ ,

$$H_{usu}^q(B, (\Omega_{\mathbb{P}^N}^r \otimes_{O_{\mathbb{P}^N}} F^{r-p} j_*O_U)^{an}) = 0.$$

On the other hand the De Rham complex of  $(j_*O_{B \setminus X}, F)$  is acyclic (see e.g. [6] corollary 18.7)

(ii): Since open balls form a basis of the topology of  $\mathbb{P}^{N, an}$ , we have

$$a_{usu} \iota : a_{usu}(j_*\Omega_{U^{an}, \log}^{N-p} \oplus \partial\Omega_{\mathbb{P}^N, an}^{N-p+1}) \xrightarrow{\sim} a_{usu}\Omega_{\mathbb{P}^N, an}^{N-p}(\log X)^{\partial=0}$$

where

$$\iota : j_*\Omega_{U^{an}, \log}^{N-p} \oplus \partial\Omega_{\mathbb{P}^N, an}^{N-p+1} \hookrightarrow \Omega_{\mathbb{P}^N, an}^{N-p}(\log X)^{\partial=0}$$

is the canonical embedding. Now,

- Let  $B = D(0, 1)^N \in \text{AnSm}(\mathbb{C})$  be an open ball. Let  $\alpha \in H^q(B, \partial\Omega_{\mathbb{P}^N, an}^{N-p+1})$ ,  $q \in \mathbb{Z}$ ,  $q \neq 0$ . Consider a sequence of sub-balls

$$l_n : B_n := D(0, r_n)^N \hookrightarrow B := D(0, 1)^N, n \in \mathbb{N}, \text{ with } r_n \rightarrow 0 \text{ when } n \rightarrow \infty.$$

Since  $(B_n := D(0, r_n))_{n \in \mathbb{N}}$  form a basis of neighborhood of  $0 \in B$  and since  $\partial\Omega_{\mathbb{P}^N, an}^{N-p+1}$  is a single presheaf, there exists  $n := n_\alpha \in \mathbb{N}$  (depending on  $\alpha$ ) such that

$$l_n^* \alpha = 0 \in H^q(B_n, \partial\Omega_{\mathbb{P}^N, an}^{N-p+1}).$$

But it follows immediately from the definition of the the presheaf  $\partial\Omega_{\mathbb{P}^N, an}^{N-p+1}$  considering the isomorphism

$$\phi_n : B_n \xrightarrow{\sim} B, \phi_n(z) := r_n^{-1}z$$

so that  $(\phi_n)^{-1} \circ l_n = r_n I$ , that

$$l_n^* \alpha = l_n^* \phi_n^{-1*} \phi_n^* \alpha = r_n \phi_n^* \alpha \in H^q(B_n, \Omega_{B, \log}^p).$$

Hence  $\phi_n^* \alpha = 0$ . Since  $\phi_n$  is an isomorphism, we get  $\alpha = 0$ .

- we have for  $q \in \mathbb{Z}$ ,  $q \neq 0$ ,  $H_{usu}^q(B, \Omega_{B^{an}, \log}^{N-p}) = 0$  by proposition 2 (i00). Hence

$$H^q E_{usu} \Omega_{\log}^{an}(c(\mathbb{Z}(B))) = \partial : H_{usu}^q(B \cap U, \Omega_{U^{an}, \log}^{N-p}) \xrightarrow{\sim} H_{usu, B \cap X}^{q+1}(B, \Omega_{B^{an}, \log}^{N-p})$$

is an isomorphism. By a finite decreasing induction on  $N$ , using purity for complex analytic motives, we get  $H_{usu}^q(B \cap U, \Omega_{U^{an}, \log}^{N-p}) = 0$ .

□

We have by the local acyclicity of the De Rham complex of  $(j_* O_U, F)$  a canonical map :

**Definition 2.** Let  $X = V(f) \subset \mathbb{P}^N$  be a smooth hypersurface. Denote  $j : U := \mathbb{P}^N \setminus X \hookrightarrow \mathbb{P}^N$  the open embedding. Let  $\mathbb{P}^{N, an} = \cup_{i=1}^n B_i$  an open cover, with  $\phi_i : B_i \simeq D(0, 1)^N$  open balls such that  $\phi_i(X \cap B_i) = \{0\} \times D(0, 1)^{N-1}$ . Denote, for  $I \subset [1, \dots, n]$ ,  $l_I : B_I \hookrightarrow \mathbb{P}^N$  the open embeddings, with  $B_I := \cap_{i \in I} B_i$ . Since, for each  $I \subset [1, \dots, n]$  and  $p \in \mathbb{Z}$ ,

$$l_I^* \iota_\partial : l_I^* \Omega_{\mathbb{P}^N, an}^{N-p}(\log X)^{\partial=0}[-p] \hookrightarrow l_I^*(\Omega_{\mathbb{P}^N}^\bullet \otimes_{O_{\mathbb{P}^N}} F^{\bullet-p} j_* O_U)^{an}$$

is an quasi-isomorphism (see e.g. [6] corollary 18.7), the spectral sequence of the filtration on the total complex of a bi-complex gives, for each  $p \in \mathbb{Z}$ , a canonical isomorphism, where the first and last map are isomorphism by lemma 1(i) and (ii) respectively,

$$\begin{aligned} B_\partial : F^p H_{DR}(U) &:= H^N \iota^{\geq p} \mathbb{H}_{zar}^N(\mathbb{P}^N, \Omega_{\mathbb{P}^N}^\bullet \otimes_{O_{\mathbb{P}^N}} F^{\bullet-p} j_* O_U) \\ &\xrightarrow{H^N((l_i^* = \Omega(l_i))_{1 \leq i \leq n})} H^N \iota^{\geq p} H^N \Gamma(B_\bullet, (\Omega_{\mathbb{P}^N}^\bullet \otimes_{O_{\mathbb{P}^N}} F^{\bullet-p} j_* O_U)^{an}) \\ &\rightarrow H^p \Gamma(B_\bullet, \Omega_{\mathbb{P}^N, an}^{N-p}(\log X)^{\partial=0}) \xrightarrow{H^N((l_i^* = \Omega(l_i))_{1 \leq i \leq n})^{-1}} H_{usu}^p(\mathbb{P}^N, \Omega_{\mathbb{P}^N, an}^{N-p}(\log X)^{\partial=0}), \end{aligned}$$

given by, for  $w \in \Gamma(U, \Omega_{\mathbb{P}^N}^N)$ , inductively by a  $p$  step induction where  $p$  is the order of the pole of  $w$  at  $X$ ,

- taking for each  $1 \leq i \leq n$ ,  $l_i^* w = \partial w_{1i}$  since  $\partial w = 0$  for dimensional reason and by exactness of  $l_i^* \iota_\partial$ ,
- we then have for each  $1 \leq i, j \leq n$ , considering  $w_{1ij} := l_{ij}^*(w_{1i} - w_{1j})$ ,  $w_{1ij} = \partial w_{2ij}$  since

$$\partial w_{1ij} = l_{ij}^*(\partial w_{1i} - \partial w_{1j}) = l_{ij}^*(l_i^* w - l_j^* w) = 0$$

and by exactness of  $l_{ij}^* \iota_\partial$ ,

- at the final  $p$  step,  $B(w) := w_{pI} \in H^N(B_\bullet, \Omega_{\mathbb{P}^N, an}^{N-p}(\log X)^{\partial=0})$ ,  $\text{card} I = p$ .

Let  $X = V(f) \subset \mathbb{P}^N$  be a smooth hypersurface. Denote  $U := \mathbb{P}^N \setminus X$ . For  $l \in \mathbb{Z}$ , we consider

- the sequence of embeddings in  $C(U^{et})$

$$OL_U^l : \Omega_{U, \log}^l[-l] \xrightarrow{\widehat{OL_U^l}[-l]} \Omega_{\mathbb{P}^N}^l(\log X)^{\partial=0}[-l] \xrightarrow{\iota_\partial} \Omega_{\mathbb{P}^N}^\bullet \otimes_{O_{\mathbb{P}^N}} F^{\bullet-p} j_* O_U \xrightarrow{\iota^{\geq l}} \Omega_U^\bullet$$

- and the sequence of embeddings in  $C(U^{an})$

$$OL_{U^{an}}^l : \Omega_{U^{an}, \log}^l[-l] \xrightarrow{\widehat{OL_{U^{an}}^l}[-l]} \Omega_{\mathbb{P}^N, an}^l(\log X)^{\partial=0}[-l] \xrightarrow{\iota_\partial} (\Omega_{\mathbb{P}^N}^\bullet \otimes_{O_{\mathbb{P}^N}} F^{\bullet-p} j_* O_U)^{an} \xrightarrow{\iota^{\geq l}} \Omega_{U^{an}}^\bullet,$$

where

$$\iota_\partial : \Omega_{\mathbb{P}^N, an}^l(\log X)^{\partial=0}[-l] \hookrightarrow (\Omega_{\mathbb{P}^N}^\bullet \otimes_{O_{\mathbb{P}^N}} F^{\bullet-p} j_* O_U)^{an}$$

is an equivalence *usu local* (see [6]).

The main result of this section is the following :

**Proposition 3.** *Let  $X = V(f) \subset \mathbb{P}^N$  be a smooth projective hypersurface. Denote  $U := \mathbb{P}^N \setminus X$ . Let  $w \in F^{p+1} H_{DR}^N(U)$ . Then the following assertions are equivalent*

- (i)  $ev(U)(w) \in H_{\text{sing}}^N(U^{an}, 2i\pi\mathbb{Q})$ ,
- (ii)  $B_\partial(w) \in \widehat{OL_{U^{an}}^{p+1}}(H_{usu}^{p+1}(U^{an}, \Omega_{U^{an}, \log}^{N-p}))$  (see definition 2),
- (iii)  $w \in OL_{U^{an}}^{N-p}(H_{usu}^{p+1}(U^{an}, \Omega_{U^{an}, \log}^{N-p})) = OL_{U^{an}}(H_{usu}^{p+1}(U^{an}, \Omega_{U^{an}, \log}^{N-p}))$ .

*Proof.* (i) implies (ii): Let  $\mathbb{P}^{N, an} = \cup_{i=1}^n B_i$  an open cover, with  $\phi_i : \mathbb{B}_i \simeq D(0, 1)^N$  open balls such that  $\phi_i(X \cap B_i) = \{0\} \times D(0, 1)^{N-1}$ . Denote, for  $I \subset [1, \dots, n]$ ,  $l_I : B_I \hookrightarrow \mathbb{P}^N$  the open embeddings, with  $B_I := \cap_{i \in I} B_i$ . Then

$$B_\partial(w) := (w_{p+1, I}) \in H^{p+1}(B_\bullet, \Omega_{\mathbb{P}^N, an}^{N-p}(\log X)^{\partial=0}).$$

We have, for each  $I \subset [1, \dots, n]$ , a decomposition

$$w_{p+1, I} = \frac{a_I}{z_1} dz_1 \wedge w'_I + w''_I \in \Gamma(B_I, \Omega_{\mathbb{P}^N, an}^{N-p}(\log X)^{\partial=0})$$

with  $w'_I \in \Gamma(B_I, \Omega_{\mathbb{P}^N}^{N-p-1})^{\partial=0}$  and  $w''_I \in \Gamma(B_I, \Omega_{\mathbb{P}^N}^{N-p})^{\partial=0}$ . Consider the evaluation map

$$ev(U)^\bullet : \Gamma(B_\bullet \setminus X, \Omega_{\mathbb{P}^N}^\bullet) \rightarrow C_{\text{diff}}^\bullet(B_\bullet \setminus X^{an}, \mathbb{C}),$$

which is a morphism of bi-complexes. On the other hand, we have the following canonical map

$$\begin{aligned} BC_\partial : H^p \Gamma(B_\bullet \setminus X^{an}, C_{U^{an}, \text{diff}}^{N-p, \partial}) &\xrightarrow{\Gamma(B_\bullet \setminus X^{an}, \iota_\partial)} H^N \Gamma(B_\bullet \setminus X^{an}, C_{U^{an}, \text{diff}}^{\bullet \geq N-p}) \xrightarrow{\Gamma(B_\bullet \setminus X^{an}, \iota^{\geq N-p})} \\ &= H^N \Gamma(B_\bullet \setminus X^{an}, C_{U^{an}, \text{diff}}^\bullet) \xrightarrow{(H^N(l_i^* =: C_{U^{an}, \text{diff}}^\bullet(l_i))_{1 \leq i \leq n})^{-1}} H_{\text{sing}}^N(U^{an}, \mathbb{Q}) = H^N \Gamma(U^{an}, C_{U^{an}, \text{diff}}^\bullet), \end{aligned}$$

where  $H^N(l_i^*)$  is an isomorphism by the acyclicity of  $C_{U^{an}, \text{diff}}^{\bullet \geq N-p}(B_I \setminus X^{an})$  for each  $I \subset [1, \dots, n]$ , and

$$C_{U^{an}, \text{diff}}^\bullet \hookrightarrow C_{U^{an}, \text{sing}}^\bullet, \text{ for } V \subset U^{an}, \quad C_{\text{diff}}^\bullet(V) \hookrightarrow C_{\text{sing}}^\bullet(V)$$

is the canonical quasi-isomorphism in  $C(U^{an})$ . We then have the following commutative diagram

$$\begin{array}{ccc}
H^N \iota^{\geq p} H^N \Gamma(\mathbb{P}^N, \Omega_{\mathbb{P}^N}^\bullet \otimes_{O_{\mathbb{P}^N}} F^{\bullet-p} j_* O_U) & & \\
\downarrow H^N((l_i^* = \Omega(l_i))_{1 \leq i \leq n}) & \searrow B_\partial & \\
H^N \iota^{\geq p} H^N \Gamma(B_\bullet, (\Omega_{\mathbb{P}^N}^\bullet \otimes_{O_{\mathbb{P}^N}} F^{\bullet-p} j_* O_U)^{an}) & \xrightarrow{B_\partial} & H^p \Gamma(B_\bullet, \Omega_{\mathbb{P}^N, an}^{N-p} (\log X)^{\partial=0}) \\
\downarrow ev(U) & & \downarrow ev(U)_\bullet \\
H_{\text{sing}}^N(U^{an}, \mathbb{C}) = H^N \Gamma(U^{an}, C_{U^{an}, \text{diff}}^\bullet \otimes \mathbb{C}) & \xleftarrow{BC_\partial \otimes \mathbb{C}} & H^p \Gamma(B_\bullet \setminus X^{an}, C_{\text{diff}}^{N-p, \partial=0} \otimes \mathbb{C})
\end{array}$$

Since  $ev(U)$  is injective by the strictness of the Hodge filtration and since  $B_\partial$  is an isomorphism,  $ev(U)_\bullet$  is injective and

$$\begin{aligned}
(BC_\partial \otimes \mathbb{C})' &:= (BC_\partial \otimes \mathbb{C})|_{ev(U)_\bullet(H^p \Gamma(B_\bullet, \Omega_{\mathbb{P}^N, an}^{N-p} (\log X)^{\partial=0}))} : \\
ev(U)_\bullet(H^p \Gamma(B_\bullet, \Omega_{\mathbb{P}^N, an}^{N-p} (\log X)^{\partial=0})) &\rightarrow H_{\text{sing}}^N(U^{an}, \mathbb{C})
\end{aligned}$$

is injective. For  $\gamma_I \in C_{N-p}^{\text{sing}}(B_I \setminus X^{an}, \mathbb{Q})^{\partial=0}$  where  $p_1 \circ \phi_I(\gamma_I) \subset D(0, 1)^*$  is a loop around 0, we have by the above diagram

$$ev(U)(w)(\gamma_I) = ev(U)(w_{p+1, I})(\gamma_I) = a_I.$$

Hence if  $ev(U)(w) \in H_{\text{sing}}^N(U^{an}, 2i\pi\mathbb{Q})$ , then  $a_I \in \mathbb{Q}$ , using the injectivity of  $(BC_\partial \otimes \mathbb{C})'$ .

(ii) is equivalent to (iii): By definition  $B_\partial(w) = (H^p \iota_\partial)^{-1}(w)$ . On the other hand by definition  $(\iota^{\geq p} \circ \iota_\partial \circ \widehat{OL_{U^{an}}^{N-p}} = OL_{U^{an}}^{N-p}$ .

(iii) implies (i): See [2]. □

## 4 Complex GAGA for logarithmic de Rham classes

We recall (see section 2) the morphism of site given by the analytification functor

$$\text{An} : \text{AnSm}(\mathbb{C}) \rightarrow \text{SmVar}(\mathbb{C}), \quad X \mapsto \text{An}(X) := X^{an}, \quad (f : X' \rightarrow X) \mapsto \text{An}(f) := f^{an}.$$

and for  $X \in \text{SmVar}(\mathbb{C})$ , the following commutative diagram

$$\begin{array}{ccc}
\text{AnSm}(\mathbb{C}) & \xrightarrow{\text{An}} & \text{SmVar}(\mathbb{C}) \\
\downarrow o_{X^{an}} & & \downarrow o_X \\
X^{an, et} & \xrightarrow{\text{an}_X} & X^{et}
\end{array}$$

We have the following :

**Lemma 2.** (i0) The sheaves  $O_{an}^* \in \text{PSh}(\text{AnSm}(\mathbb{C}))$  and  $\Omega_{\log}^{1, an} \in \text{PSh}(\text{AnSm}(\mathbb{C}))$  admit transfers compatible with transfers on  $\Omega^{1, an} \in \text{PSh}(\text{AnSm}(\mathbb{C}))$ .

(i) For each  $l \in \mathbb{Z}$ , the sheaf  $\Omega_{\log}^{l, an} \in \text{PSh}(\text{AnSm}(\mathbb{C}))$  admits transfers compatible with transfers on  $\Omega^{l, an} \in \text{PSh}(\text{AnSm}(\mathbb{C}))$ , that is  $\Omega^{\bullet, an} \in C(\text{Cor AnSm}(\mathbb{C}))$  and the inclusion  $OL : \Omega_{\log}^{l, an}[-l] \hookrightarrow \Omega^{\bullet, an}$  in  $C(\text{AnSm}(\mathbb{C}))$  is compatible with transfers.

(ii) For each  $l \in \mathbb{Z}$ , the sheaf  $\text{An}_* \Omega_{\log}^{l, an} \in \text{PSh}(\text{SmVar}(\mathbb{C}))$  is  $\mathbb{A}^1$  invariant.

*Proof.* (i0): Let  $W \subset X' \times X$  with  $X, X' \in \text{AnSm}(\mathbb{C})$ ,  $X'$  connected, and  $p_{X'} : W \subset X' \times X \rightarrow X'$  finite surjective over  $X'$ . Then  $M(p_X) : M(X') \hookrightarrow M(W)$  is a finite, where for  $Y \in \text{AnSp}(\mathbb{C})$  irreducible,  $M(Y) := \text{Frac}(O(Y))$  denote the field of meromorphic function. Since  $O(X') \subset M(X')$  is integrally closed, the trace map  $\text{Tr}_{W/X'} : M(W) \rightarrow M(X')$  sends  $O(W)$  to  $O(X')$ , and the norm map  $N_{W/X'} : M(W)^* \rightarrow M(X')^*$  sends  $O(W)^*$  to  $O(X')^*$ .

The sheaf  $O_{an}^* \in \text{PSh}(\text{AnSm}(\mathbb{C}))$  admits transfers : for  $W \subset X' \times X$  with  $X, X' \in \text{AnSm}(\mathbb{C})$  and  $W$  finite surjective over  $X'$  and  $f \in O(X)^*$ ,  $W^*f := N_{W/X'}(p_X^*f)$  where  $p_X : W \hookrightarrow X' \times X \rightarrow X$  is the projection and  $N_{W/X'} : M(W)^* \rightarrow M(X')^*$  is the norm map. This gives transfers on  $\Omega_{\log}^{1,an} \in \text{PSh}(\text{AnSm}(\mathbb{C}))$  compatible with transfers on  $\Omega^{1,an} \in \text{PSh}(\text{AnSm}(\mathbb{C}))$  : for  $W \subset X' \times X$  with  $X, X' \in \text{AnSm}(\mathbb{C})$  and  $W$  finite surjective over  $X'$  and  $f \in O(X)^*$ ,

$$W^*df/f := dW^*f/W^*f = \text{Tr}_{W/X'}(p_X^*(df/f)),$$

where we recall  $p_X : W \hookrightarrow X' \times X \rightarrow X$  is the projection and  $\text{Tr}_{W/X'} : O(W) \rightarrow O(X')$  is the trace map. Note that  $d(fg)/fg = df/f + dg/g$ .

(i): By (i0), we get transfers on

$$\otimes_{\mathbb{Q}}^l \Omega_{\log}^{1,an}, \otimes_O^l \Omega^{1,an} \in \text{PSh}(\text{AnSm}(\mathbb{C}))$$

since  $\otimes_{\mathbb{Q}}^l \Omega_{\log}^{1,an} = H^0(\otimes_{\mathbb{Q}}^{L,l} \Omega_{\log}^{1,an})$  and  $\otimes_O^l \Omega^{1,an} = H^0(\otimes_O^{L,l} \Omega^{1,an})$ . This induces transfers on

$$\wedge_{\mathbb{Q}}^l \Omega_{\log}^{1,an} := \text{coker}(\oplus_{I_2 \subset [1, \dots, l]} \otimes_{\mathbb{Q}}^{l-1} \Omega_{\log}^{1,an} \xrightarrow{\oplus_{I_2 \subset [1, \dots, l]} \Delta_{I_2} := (w \otimes w' \mapsto w \otimes w \otimes w')} \otimes_{\mathbb{Q}}^l \Omega_{\log}^{1,an}) \in \text{PSh}(\text{AnSm}(\mathbb{C})).$$

and

$$\wedge_O^l \Omega^{1,an} := \text{coker}(\oplus_{I_2 \subset [1, \dots, l]} \otimes_{O_k}^{l-1} \Omega^{1,an} \xrightarrow{\oplus_{I_2 \subset [1, \dots, l]} \Delta_{I_2} := (w \otimes w' \mapsto w \otimes w \otimes w')} \otimes_O^l \Omega^{1,an}) \in \text{PSh}(\text{AnSm}(\mathbb{C})).$$

(ii): Follows from the fact that for  $X \in \text{AnSp}(\mathbb{C})$ ,  $O^*(X \times \mathbb{A}^1) = O^*(X)$ . Indeed, let  $h = h(x, z) \in O(X \times \mathbb{A}^1)^*$ . Then for  $x \in X$ ,  $z \in \mathbb{A}^1 \mapsto h(x, z)$  is constant since it is an holomorphic function on  $\mathbb{A}^1$  with no zero. Hence  $h = h(x)$ .  $\square$

**Proposition 4.** *Let  $X \in \text{PSmVar}(\mathbb{C})$ . Consider the morphism  $\text{an}_X : X^{an} \rightarrow X$  in  $\text{RTop}$  given by analytical functor.*

(i) *The analytic De Rham cohomology class of an algebraic cycle is logarithmic and is of type  $(d, d)$ , that is, for  $Z \in \mathcal{Z}^d(X)$*

$$[Z] := H^{2d} \Omega(\gamma_Z^\vee)([Z]) \subset H^{2d} OL_{X^{an}}(H_{usu}^d(X^{an}, \Omega_{X^{an}, \log}^d)) \subset H_{DR}^{2d}(X^{an}) = H_{DR}^{2d}(X).$$

(ii) *Conversely, any  $w \in H^{2d} OL_{X^{an}}(H_{usu}^d(X^{an}, \Omega_{X^{an}, \log}^d))$  is the class of an algebraic cycle  $Z \in \mathcal{Z}^d(X) \otimes \mathbb{Q}$ , i.e.  $w = [Z]$ .*

(iii) *We have  $H^j OL_{X^{an}}(H_{usu}^{j-l}(X^{an}, \Omega_{X^{an}, \log}^l)) = 0$  for  $j, l \in \mathbb{Z}$  such that  $2l \neq j$ . That is, we only have analytic logarithmic classes in bidegree  $(d, d)$  for  $d \in \mathbb{N}$ , in particular there is no non trivial logarithmic classes for odd degree De Rham cohomology  $H_{DR}^{2d+1}(X^{an}) = H_{DR}^{2d+1}(X)$ .*

(iv) *For each  $j \in \mathbb{Z}$ , the isomorphism*

$$\text{an}_X^* : H_{DR}^j(X) = \mathbb{H}_{et}^j(X, \Omega_X^\bullet) \xrightarrow{\sim} \mathbb{H}_{usu}^j(X^{an}, \Omega_{X^{an}}^\bullet) = H_{DR}^j(X^{an})$$

*and its inverse preserve logarithmic classes, that is for each  $j, l \in \mathbb{Z}$ ,*

$$\text{an}_X^*(H^j(OL_X)(H_{et}^{j-l}(X, \Omega_{X, \log}^l))) = H^j OL_{X^{an}}(H_{usu}^{j-l}(X^{an}, \Omega_{X^{an}, \log}^l)).$$

*Proof.* Consider, for  $j, l \in \mathbb{Z}$  and  $X \in \text{SmVar}(\mathbb{C})$ ,

$$L^{l,j-l}(X) := H^j OL_{X^{an}}(H_{usu}^{j-l}(X^{an}, \Omega_{X^{an}, \log}^l)).$$

Consider also for  $j, l \in \mathbb{Z}$ ,  $X \in \text{SmVar}(\mathbb{C})$  with good reduction and  $Z \subset X$  a closed subset,

$$L_Z^{l,j-l}(X) := H^j OL_{X^{an}}(H_{usu,Z}^{j-l}(X^{an}, \Omega_{X^{an}, \log}^l)).$$

We still denote  $\text{an}_X : X^{an} \xrightarrow{e, \sim} X^{an, et} \xrightarrow{\text{an}_X} X^{et}$  the composite, with  $e(U \rightarrow X^{an}) := e(U)$ . By proposition 2(ii) and (ii)', we have

$$L^{l,j-l}(X) = H^j OL_{X^{an}}(H_{et}^{j-l}(X, \text{an}_{X*} \Omega_{X^{an}, \log}^l)) = H^j OL^{an}(H^{j-l} \text{Hom}(\mathbb{Z}(X), E_{et}^\bullet(\text{An}_* \Omega_{\log}^{l, an}))).$$

and

$$L_Z^{l,j-l}(X) = H^j OL_{X^{an}}(H_{et,Z}^{j-l}(X, \text{an}_{X*} \Omega_{X^{an}, \log}^l)) = H^j OL^{an}(H^{j-l} \text{Hom}(\mathbb{Z}(X, X \setminus Z), E_{et}^\bullet(\text{An}_* \Omega_{\log}^{l, an}))),$$

for the second equalities, note that

$$(\text{An}_* \Omega_{\log}^{l, an})|_{X^{et}} := o_X * \text{An}_* \Omega_{\log}^{l, an} = \text{an}_{X*} o_{X^{an, et}} \Omega_{\log}^{l, an} = \text{an}_{X*} \Omega_{X^{an}, \log}^l,$$

where  $o_X : X^{et} \subset \text{SmVar}(\mathbb{C})$  is the small etale site  $o_X : X^{et} \hookrightarrow \text{SmVar}(\mathbb{C})/X \xrightarrow{o_X} \text{SmVar}(\mathbb{C})$ ,  $o_X(U \rightarrow X) = U$ . For each  $l \in \mathbb{Z}$ , the presheaves  $\text{An}_* \Omega_{\log}^{l, an} \in \text{PSh}(\text{SmVar}(\mathbb{C}))$  admit transfers and are  $\mathbb{A}^1$  invariant by lemma 2. Hence by theorem 1, they are  $\mathbb{A}^1$  local since they are  $\mathbb{A}^1$  invariant and admit transfers. This gives in particular, for  $Z \subset X$  a smooth subvariety of (pure) codimension  $d$  with good reduction, by proposition 1 an isomorphism

$$E_{et} \text{An}_* \Omega_{\log}^{l, an}(P_{Z,X}) : L_Z^{l,j-l}(X) \xrightarrow{\sim} L_Z^{l,j-l}(N_{Z/X}) \xrightarrow{\sim} L^{l-d, j-l-d}(Z).$$

Let

$$\alpha = OL_{X^{an}}(\alpha) \in L^{l,j-l}(X) := H^j OL_{X^{an}}(H_{usu}^{j-l}(X^{an}, \Omega_{X^{an}, \log}^l)) = H^{j-l} \text{Hom}(\mathbb{Z}(X), E_{et}^\bullet(\text{An}_* \Omega_{\log}^{l, an})).$$

with  $j-l \neq 0$ . Let  $U \subset X$  an open subset such that there exists an etale map  $e : U \rightarrow \mathbb{A}_{\mathbb{C}}^{d_U}$ . Denote  $j : U \hookrightarrow X$  the open embedding. By proposition 2(i), we have

$$j^* \alpha = 0 \in H_{usu}^{j-l}(U^{an}, \Omega_{X^{an}, \log}^l) = 0.$$

Considering a divisor  $X \setminus U \subset D \subset X$ , we get

$$\alpha = H^{j-l} E_{et}^\bullet(\text{An}_* \Omega_{\log}^{l, an})(\gamma_D^\vee)(\alpha), \alpha \in L_D^{l,j-l}(X).$$

- Proof of (i): Similar to the proof of [2] theorem 2.
- Proof of (ii): By assumption we have  $w \in L^{d,d}(X)$ . Let  $U \subset X$  an open subset such that there exists an etale map  $e : U \rightarrow \mathbb{A}_{\mathbb{C}}^{d_U}$ . Considering a divisor  $X \setminus U \subset D \subset X$ , we get

$$w = H^d E_{et}^\bullet(\text{An}_* \Omega_{\log}^{d, an})(\gamma_D^\vee)(w), \alpha \in L_D^{d,d}(X).$$

Denote  $D^\circ \subset D$  its smooth locus with good reduction and  $l : X^\circ \hookrightarrow X$  a Zariski open subset such that  $X^\circ \cap D = D^\circ$ . We then have

$$l^* w \in L_{D^\circ}^{d,d}(X^\circ) = L^{d-1, d-1}(D^\circ)$$

We repeat this procedure with each connected components of  $D^\circ$  instead of  $X$ . By a finite induction of  $d$  steps, we get

$$w = H^d E_{et}^\bullet(\text{An}_* \Omega_{\log}^{d, an})(\gamma_Z^\vee)(w), w \in L_Z^{d,d}(X) = L^{0,0}(Z^\circ).$$

with  $Z := D_d \subset \dots \subset D \subset X$  a pure codimension  $d$  (Zariski) closed subset, thus  $\alpha = \sum_i n_i [Z_i] \in H_{DR}^{2d}(X)$ , where  $n_i \in \mathbb{Q}$  and  $(Z_i)_{1 \leq i \leq t} \subset Z$  are the irreducible components of  $Z$ ,

- Proof of (iii) part 1: Let  $j > 2l$ . Let  $w \in L^{l,j-l}(X)$ . By the proof of (ii) there exists  $Z \subset X$  a closed subset of pure codimension  $l$  such that

$$w = H^{j-l} E_{et}^\bullet(\text{An}_* \Omega_{\log}^{l,an})(\gamma_Z^\vee)(w), \quad w \in L_Z^{l,j-l}(X).$$

By a finite induction of  $d_X - l$  steps, restricting to the smooth locus with good reduction of closed subsets of  $Z$ ,  $w = 0$  since  $L_{Z'}^{l,j-l}(X) = 0$  for all  $Z' \in \text{SmVar}(k)$  with good reduction such that  $\text{codim}(Z', X) \geq l$ .

- Proof of (iii) part 2: Let  $j < 2l$ . Let  $w \in L^{l,j-l}(X)$ . By the proof of (ii) there exists  $Z \subset X$  a closed subset of pure codimension  $j - l$  such that

$$w = H^{j-l} E_{et}^\bullet(\text{An}_* \Omega_{\log}^{l,an})(\gamma_Z^\vee)(w), \quad w \in L_Z^{l,j-l}(X).$$

For  $Z' \subset X$  a closed subset of pure codimension  $c$ , consider a desingularisation  $\epsilon : \tilde{Z}' \rightarrow Z'$  of  $Z'$  and denote  $n : \tilde{Z}' \xrightarrow{\epsilon} Z' \subset X$ . The morphism in  $\text{DA}(k)$

$$G_{Z',X} : M(X) \xrightarrow{D(\mathbb{Z}(n))} M(\tilde{Z}')(c)[2c] \xrightarrow{\mathbb{Z}(\epsilon)} M(Z')(c)[2c]$$

where  $D : \text{Hom}_{\text{DA}(k)}(M_c(\tilde{Z}'), M_c(X)) \xrightarrow{\sim} \text{Hom}_{\text{DA}(k)}(M(X), M(\tilde{Z}')(c)[2c])$  is the duality isomorphism from the six functor formalism (moving lemma of Suzlin and Voevodsky) and  $\mathbb{Z}(n) := \text{ad}(n!, n^1)(a_X^! \mathbb{Z})$ , is given by a morphism in  $C(\text{SmVar}(k))$

$$\hat{G}_{Z',X} : \mathbb{Z}_{tr}(X) \rightarrow E_{et}(C_* \mathbb{Z}_{tr}(Z'))(c)[2c].$$

Let  $l : X^\circ \hookrightarrow X$  be an open embedding such that  $Z^\circ := Z \cap X^\circ$  is the smooth locus of  $Z$  with good reduction. We then have the following commutative diagram of abelian groups

$$\begin{array}{ccccccc} 0 & \longrightarrow & H_{et,Z}^{j-l}(X, \text{an}_{X*} \Omega_{X^{an},\log}^l) & \xrightarrow{l^*} & H_{et,Z^\circ}^{j-l}(X^\circ, \text{an}_{X*} \Omega_{X^{an},\log}^l) & \xrightarrow{\partial} & H_{et,Z \setminus Z^\circ}^{j-l+1}(X, \text{an}_{X*} \Omega_{X^{an},\log}^l) \longrightarrow \dots \\ & & \uparrow \Omega(\hat{G}_{Z,X}) & & \uparrow \Omega(P_{Z^\circ, X^\circ}) & & \uparrow \Omega(\hat{G}_{Z,X}) \\ 0 & \longrightarrow & H_{et}^0(Z, \text{an}_{Z*} \Omega_{Z^{an},\log}^{2l-j}) & \xrightarrow{l^*} & H_{et}^0(Z^\circ, \Omega_{Z^\circ,an,\log}^{2l-j}) & \xrightarrow{\partial} & H_{et,Z \setminus Z^\circ}^1(Z, \Omega_{Z^{an},\log}^{2l-j}) \longrightarrow \dots \end{array}$$

whose rows are exact sequences. Consider

$$l^* w = \Omega(P_{Z^\circ, X^\circ})(w^\circ) \in H_{et,Z^\circ}^{j-l}(X^\circ, \text{an}_{X*} \Omega_{X^{an},\log}^l), \quad w^\circ \in H_{et}^0(Z^\circ, \text{an}_{Z*} \Omega_{Z^\circ,an,\log}^{2l-j}).$$

Since  $\partial l^* w = 0 \in H_{et,Z \setminus Z^\circ}^{j-l+1}(X, \text{an}_{X*} \Omega_{X^{an},\log}^l)$ , we get, considering the smooth locus with good reduction  $(Z \setminus Z^\circ)^\circ \subset Z \setminus Z^\circ$  and  $X^{\circ\circ} \subset X$  is an open subset such that  $X \cap (Z \setminus Z^\circ)^\circ = X^{\circ\circ} \cap (Z \setminus Z^\circ)$ ,

$$\partial w^\circ = 0 \in H_{et,Z \setminus Z^\circ}^1(Z, \text{an}_{Z*} \Omega_{Z^{an},\log}^{2l-j}),$$

since  $H_{et,(Z \setminus Z^\circ)^\circ \setminus (Z \setminus Z^\circ)^\circ}^1(Z, \text{an}_{Z*} \Omega_{Z^{an},\log}^{2l-j}) = 0$  for dimension reasons, that is

$$w = \Omega(\hat{G}_{Z,X})(w), \quad \text{with } w \in H_{et}^0(Z, \text{an}_{Z*} \Omega_{Z^{an},\log}^{2l-j}).$$

Hence  $w = 0$  since  $H^0(Z', \Omega_{Z',an,\log}^k) = 0$  for all  $k > 0$  and all  $Z' \in \text{PVar}(K)$ .

(iv): Follows from (ii), (iii) and [2] theorem 2. □



## 5 Hodge Conjecture for complex smooth projective hypersurfaces

We now state and prove using the result of section 3 and section 4 the main result of this article :

**Theorem 2.** *Let  $X = V(f) \subset \mathbb{P}_{\mathbb{C}}^N$  be a smooth projective hypersurface, with  $N = 2p + 1$  odd. Let  $\lambda \in F^p H^{2p}(X^{an}, \mathbb{Q})$ . Then there exists an algebraic cycle  $Z \in \mathbb{Z}^p(X)$  such that  $\lambda = [Z]$ .*

*Proof.* Consider  $U := \mathbb{P}_{\mathbb{C}}^N \setminus X$ . Consider  $Res_{X, \mathbb{P}^N} : M(X) \rightarrow M(U)[-1]$  the canonical map given in definition 1. We have the commutative diagram of abelian group

$$\begin{array}{ccc} H_{DR}^N(U) & \xrightarrow{H^N \Omega(Res_{X, \mathbb{P}^N})} & H_{DR}^{N-1}(X)_v \\ \uparrow H^N \alpha(U^{an}) = ev(U)^{-1} & & \uparrow H^{N-1} \alpha(X^{an}) = ev(X)^{-1} \\ H_{\text{sing}}^N(U^{an}, \mathbb{C}) & \xrightarrow{(H^N \text{Bti}(Res_{X, \mathbb{P}^N})) \otimes \mathbb{C}} & H_{\text{sing}}^{N-1}(X^{an}, \mathbb{C})_v \end{array} .$$

whose arrows are isomorphisms. Denote  $w(\lambda) := H^{N-1} \alpha(X^{an})(\lambda) \in F^p H_{DR}^{2p}(X)$ . We have

$$\lambda = H^{N-1} \text{Bti}(Res_{X, \mathbb{P}^N})(\alpha), \text{ with } \alpha \in H_{\text{sing}}^N(U^{an}, \mathbb{Q}).$$

Then

$$w(\lambda) = H^{N-1} \Omega(Res_{X, \mathbb{P}^N})(w(\alpha)), \text{ with } w(\alpha) := H^N \alpha(U^{an})(\alpha) \in F^{p+1} H_{DR}^N(U).$$

Since  $ev(U)(w(\alpha)) = \alpha \in H_{\text{sing}}^N(U^{an}, 2i\pi\mathbb{Q})$ , we have by proposition 3,

$$w(\alpha) \in H^N OL_{U^{an}}(H_{usu}^{p+1}(U^{an}, \Omega_{U^{an}, \log}^{N-p}))$$

Hence

$$w(\lambda) = H^{N-1} OL_{X^{an}}(\Omega_{\log}^{N-p, an}(Res_{X, \mathbb{P}^N})(w(\alpha))) \in OL_{X^{an}}(H_{usu}^p(X^{an}, \Omega_{X^{an}, \log}^p)).$$

Hence by proposition 4(ii),  $w(\lambda) = [Z]$  with  $Z \in \mathbb{Z}^p(X)$ . □

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