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Hodge conjecture for projective hypersurfaces

Johann Bouali

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Abstract

We show that an Hodge class of a complex smooth projective hypersurface is an analytic logarithmic De Rham class. On the other hand we show that for a complex smooth projective variety an analytic logarithmic De Rham class of type (d, d) is the class of codimension d algebraic cycle and that there is no an analytic logarithmic De Rham class of type (p, q) , $p \neq q$. We deduce the Hodge conjecture for smooth projective hypersurfaces.

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1 Introduction

1.1 Complex analytic De Rham logarithmic classes

In [2], we introduced, for X an algebraic variety over a field, the notion of logarithmic De Rham cohomology classes. For X a complex algebraic variety, we also introduce the notion of logarithmic analytic De Rham cohomology classes. In proposition 4 we give an analytic analogue of [2] theorem 2, that is if X is a smooth complex projective variety,

- a logarithmic analytic De Rham cohomology class of type (d, d) is the class of a codimension d algebraic cycle,
- there are no non trivial logarithmic analytic De Rham cohomology classes of type (p, q) for $p \neq q$, i.e. $H^{2d}OL_{X^{an}}(H_{usu}^{d-k}(X^{an}, \Omega_{X^{an}, \log}^{d+k})) = 0$ for $k \neq 0$.

The proof of proposition 4 is motivic and similar to the proof of [2] theorem 2 : a logarithmic analytic class of bidegree (p, q) is acyclic for the etale topology on each open subset $U \subset X$ such that there exists an etale map $e : U \rightarrow A_{\mathbb{C}}^{d_U}$ (proposition 2), this allows us to proceed by a finite induction using the crucial fact that the purity isomorphism for De Rham cohomology preserve logarithmic analytic classes since the purity isomorphism is motivic (c.f. [3], see proposition 1). The GAGA comparison theorem for de Rham cohomology of X induce, by proposition 4 and [2] theorem 2, a GAGA comparison result for logarithmic De Rham classes.

1.2 Hodge conjecture for smooth projective hypersurfaces

Let $X = V(f) \subset \mathbb{P}_{\mathbb{C}}^N$ a smooth projective hypersurface. Denote $j : U := \mathbb{P}_{\mathbb{C}}^N \setminus X \hookrightarrow \mathbb{P}_{\mathbb{C}}^N$ the open complementary subset. Using the fact the filtered De Rham complex of U

$$\iota_{\partial} : \Omega_{\mathbb{P}^N, an}^{N-p}(\log X)^{\partial=0}[-p] \hookrightarrow (\Omega_{\mathbb{P}^N}^{\bullet} \otimes_{\mathcal{O}_{\mathbb{P}^N}} F^{\bullet-p} j_* \mathcal{O}_U)^{an}$$

is locally acyclic for the complex topology, more precisely is acyclic on open balls (see e.g.[6]), we show, considering an open cover $\mathbb{P}_{\mathbb{C}}^N = \cup_{i=1}^n B_i$ by open balls $\phi_i : B_i \simeq D(0, 1)^N$ such that $\phi_i(B_i \cap X) = 0 \times D(0, 1)^N$, that an Hodge class $\alpha \in F^p H^N(U^{an}, \mathbb{Q})$ is logarithmic, that is

$$\alpha \in H^N OL_{U^{an}}(H_{usu}^p(U^{an}, \Omega_{U^{an}, \log}^{N-p}))$$

(proposition 3). We get that, if $N = 2p + 1$ and $\lambda \in F^p H^{2p}(X^{an}, \mathbb{Q})$ is an Hodge class,

$$\lambda = Res_{X, \mathbb{P}^N}(\alpha) + m[H] \in H^N OL_{X^{an}}(H_{usu}^p(X^{an}, \Omega_{X^{an}, \log}^{N-p})),$$

where $Res_{X, \mathbb{P}^N} : H^N(U^{an}, \mathbb{C}) \xrightarrow{\sim} H^{2p}(X^{an}, \mathbb{C})$ is the residu map, using the fact that since the residu map is motivic (see definition 1), it preserve the logarithmic De Rham classes. Proposition 4 (ii) then implies that $\lambda = [Z]$, $Z \in \mathcal{Z}^p(X)$, is the class of an algebraic cycle.

2 Preliminaries and Notations

2.1 Notations

- Denote by Top the category of topological spaces and RTop the category of ringed spaces.
- Denote by Cat the category of small categories and RCat the category of ringed topos.
- For $\mathcal{S} \in \text{Cat}$ and $X \in \mathcal{S}$, we denote $\mathcal{S}/X \in \text{Cat}$ the category whose objects are $Y/X := (Y, f)$ with $Y \in \mathcal{S}$ and $f : Y \rightarrow X$ is a morphism in \mathcal{S} , and whose morphisms $\text{Hom}((Y', f'), (Y, f))$ consists of $g : Y' \rightarrow Y$ in \mathcal{S} such that $f \circ g = f'$.
- For $(\mathcal{S}, \mathcal{O}_{\mathcal{S}}) \in \text{RCat}$ a ringed topos, we denote by
 - $\text{PSh}(\mathcal{S})$ the category of presheaves of $\mathcal{O}_{\mathcal{S}}$ modules on \mathcal{S} and $\text{PSh}_{\mathcal{O}_{\mathcal{S}}}(\mathcal{S})$ the category of presheaves of $\mathcal{O}_{\mathcal{S}}$ modules on \mathcal{S} , whose objects are $\text{PSh}_{\mathcal{O}_{\mathcal{S}}}(\mathcal{S})^0 := \{(M, m), M \in \text{PSh}(\mathcal{S}), m : M \otimes \mathcal{O}_{\mathcal{S}} \rightarrow M\}$, together with the forgetful functor $o : \text{PSh}(\mathcal{S}) \rightarrow \text{PSh}_{\mathcal{O}_{\mathcal{S}}}(\mathcal{S})$,
 - $C(\mathcal{S}) = C(\text{PSh}(\mathcal{S}))$ and $C_{\mathcal{O}_{\mathcal{S}}}(\mathcal{S}) = C(\text{PSh}_{\mathcal{O}_{\mathcal{S}}}(\mathcal{S}))$ the big abelian category of complexes of presheaves of $\mathcal{O}_{\mathcal{S}}$ modules on \mathcal{S} ,
 - $C_{\mathcal{O}_{\mathcal{S}}(2)fil}(\mathcal{S}) := C_{(2)fil}(\text{PSh}_{\mathcal{O}_{\mathcal{S}}}(\mathcal{S})) \subset C(\text{PSh}_{\mathcal{O}_{\mathcal{S}}}(\mathcal{S}), F, W)$, the big abelian category of (bi)filtered complexes of presheaves of $\mathcal{O}_{\mathcal{S}}$ modules on \mathcal{S} such that the filtration is biregular and $\text{PSh}_{\mathcal{O}_{\mathcal{S}}(2)fil}(\mathcal{S}) := (\text{PSh}_{\mathcal{O}_{\mathcal{S}}}(\mathcal{S}), F, W)$.
- Let $(\mathcal{S}, \mathcal{O}_{\mathcal{S}}) \in \text{RCat}$ a ringed topos with topology τ . For $F \in C_{\mathcal{O}_{\mathcal{S}}}(\mathcal{S})$, we denote by $k : F \rightarrow E_{\tau}(F)$ the canonical flasque resolution in $C_{\mathcal{O}_{\mathcal{S}}}(\mathcal{S})$ (see [1]). In particular for $X \in \mathcal{S}$, $H^*(X, E_{\tau}(F)) \xrightarrow{\sim} \mathbb{H}_{\tau}^*(X, F)$.

- For $f : \mathcal{S}' \rightarrow \mathcal{S}$ a morphism with $\mathcal{S}, \mathcal{S}' \in \text{RCat}$, endowed with topology τ and τ' respectively, we denote for $F \in C_{O_{\mathcal{S}}}(\mathcal{S})$ and each $j \in \mathbb{Z}$,

$$\begin{aligned} - f^* &:= H^j \Gamma(\mathcal{S}, k \circ \text{ad}(f^*, f_*)(F)) : \mathbb{H}^j(\mathcal{S}, F) \rightarrow \mathbb{H}^j(\mathcal{S}', f^*F), \\ - f^* &:= H^j \Gamma(\mathcal{S}, k \circ \text{ad}(f^{*mod}, f_*)(F)) : \mathbb{H}^j(\mathcal{S}, F) \rightarrow \mathbb{H}^j(\mathcal{S}', f^{*mod}F), \end{aligned}$$

the canonical maps.

- For $X \in \text{Top}$ and $Z \subset X$ a closed subset, denoting $j : X \setminus Z \hookrightarrow X$ the open complementary, we will consider

$$\Gamma_Z^\vee \mathbb{Z}_X := \text{Cone}(\text{ad}(j!, j^*)(\mathbb{Z}_X) : j!j^* \mathbb{Z}_X \hookrightarrow \mathbb{Z}_X) \in C(X)$$

and denote for short $\gamma_Z^\vee := \gamma_X^\vee(\mathbb{Z}_X) : \mathbb{Z}_X \rightarrow \Gamma_Z^\vee \mathbb{Z}_X$ the canonical map in $C(X)$.

- Denote by $\text{Sch} \subset \text{RTop}$ the subcategory of schemes (the morphisms are the morphisms of locally ringed spaces). We denote by $\text{PSch} \subset \text{Sch}$ the full subcategory of proper schemes. For a field k , we consider $\text{Sch}/k := \text{Sch}/\text{Spec } k$ the category of schemes over $\text{Spec } k$, that is whose object are $X := (X, a_X)$ with $X \in \text{Sch}$ and $a_X : X \rightarrow \text{Spec } k$ a morphism and whose objects are morphism of schemes $f : X' \rightarrow X$ such that $f \circ a_{X'} = a_X$. We then denote by

- $\text{Var}(k) = \text{Sch}^{ft}/k \subset \text{Sch}/k$ the full subcategory consisting of algebraic varieties over k , i.e. schemes of finite type over k ,
- $\text{PVar}(k) \subset \text{QPVar}(k) \subset \text{Var}(k)$ the full subcategories consisting of quasi-projective varieties and projective varieties respectively,
- $\text{PSmVar}(k) \subset \text{SmVar}(k) \subset \text{Var}(k)$, $\text{PSmVar}(k) := \text{PVar}(k) \cap \text{SmVar}(k)$, the full subcategories consisting of smooth varieties and smooth projective varieties respectively.

- Denote by $\text{AnSp}(\mathbb{C}) \subset \text{RTop}$ the subcategory of analytic spaces over \mathbb{C} , and by $\text{AnSm}(\mathbb{C}) \subset \text{AnSp}(\mathbb{C})$ the full subcategory of smooth analytic spaces (i.e. complex analytic manifold).

- Let $(X, O_X) \in \text{RTop}$. We consider its De Rham complex $\Omega_X^\bullet := DR(X)(O_X)$.

- Let $X \in \text{Sch}$. Considering its De Rham complex $\Omega_X^\bullet := DR(X)(O_X)$, we have for $j \in \mathbb{Z}$ its De Rham cohomology $H_{DR}^j(X) := \mathbb{H}^j(X, \Omega_X^\bullet)$.
- Let $X \in \text{Var}(k)$. Considering its De Rham complex $\Omega_X^\bullet := \Omega_{X/k}^\bullet := DR(X/k)(O_X)$, we have for $j \in \mathbb{Z}$ its De Rham cohomology $H_{DR}^j(X) := \mathbb{H}^j(X, \Omega_X^\bullet)$. The differentials of $\Omega_X^\bullet := \Omega_{X/k}^\bullet$ are by definition k -linear, thus $H_{DR}^j(X) := \mathbb{H}^j(X, \Omega_X^\bullet)$ has a structure of a k vector space.
- Let $X \in \text{AnSp}(\mathbb{C})$. Considering its De Rham complex $\Omega_X^\bullet := DR(X)(O_X)$, we have for $j \in \mathbb{Z}$ its De Rham cohomology $H_{DR}^j(X) := \mathbb{H}^j(X, \Omega_X^\bullet)$.

- For $X \in \text{AnSp}(\mathbb{C})$, we denote $\alpha(X) : \mathbb{C}_X \hookrightarrow \Omega_X^\bullet$ the embedding in $C(X)$. For $X \in \text{AnSm}(\mathbb{C})$, $\alpha(X) : \mathbb{C}_X \hookrightarrow \Omega_X^\bullet$ is an equivalence usu local by Poincare lemma.

- We denote $\mathbb{I}^n := [0, 1]^n \in \text{Diff}(\mathbb{R})$ (with boundary). For $X \in \text{Top}$ and R a ring, we consider its singular cochain complex

$$C_{\text{sing}}^*(X, R) := (\mathbb{Z} \text{Hom}_{\text{Top}}(\mathbb{I}^*, X)^\vee) \otimes R$$

and for $l \in \mathbb{Z}$ its singular cohomology $H_{\text{sing}}^l(X, R) := H^n C_{\text{sing}}^*(X, R)$. For $f : X' \rightarrow X$ a continuous map with $X, X' \in \text{Top}$, we have the canonical map of complexes

$$f^* : C_{\text{sing}}^*(X, R) \rightarrow C_{\text{sing}}^*(X', R), \sigma \mapsto f^* \sigma := (\gamma \mapsto \sigma(f \circ \gamma)).$$

In particular, we get by functoriality the complex

$$C_{X, R \text{ sing}}^* \in C_R(X), (U \subset X) \mapsto C_{\text{sing}}^*(U, R)$$

We recall we have

- For $X \in \text{Top}$ locally contractile, e.g. $X \in \text{CW}$, and R a ring, the inclusion in $C_R(X)$ $c_X : R_X \rightarrow C_{X,R \text{ sing}}^*$ is by definition an equivalence top local and that we get by the small chain theorem, for all $l \in \mathbb{Z}$, an isomorphism $H^l c_X : H^l(X, R_X) \xrightarrow{\sim} H_{\text{sing}}^l(X, R)$.
- For $X \in \text{Diff}(\mathbb{R})$, the restriction map

$$r_X : \mathbb{Z} \text{Hom}_{\text{Diff}(\mathbb{R})}(\mathbb{I}^*, X)^\vee \rightarrow C_{\text{sing}}^*(X, R), w \mapsto w : (\phi \mapsto w(\phi))$$

is a quasi-isomorphism by Whitney approximation theorem.

- We will consider the morphism of site given by the analytification functor

$$\text{An} : \text{AnSp}(\mathbb{C}) \rightarrow \text{Var}(\mathbb{C}), X \mapsto \text{An}(X) := X^{an}, (f : X' \rightarrow X) \mapsto \text{An}(f) := f^{an}.$$

For $X \in \text{Var}(\mathbb{C})$, we denote by

$$\text{an}_X := \text{An}|_{X^{et}} := X^{an,et} \rightarrow X^{et}, \text{an}_X(U \rightarrow X) := U^{an},$$

its restriction to the small etale site $o_X(X^{et}) \subset \text{Var}(\mathbb{C})$, where $o_X(U \rightarrow X) = U$, in particular we have the following commutative diagram

$$\begin{array}{ccc} \text{AnSp}(\mathbb{C}) & \xrightarrow{\text{An}} & \text{Var}(\mathbb{C}), \\ \downarrow o_{X^{an}} & & \downarrow o_X \\ X^{an,et} & \xrightarrow{\text{an}_X} & X^{et} \end{array}$$

where $o_{X^{an}}(U \hookrightarrow X^{an}) = U$. We get the morphism in RTop $\text{an}_X := \text{an}_X|_{O(X)} : X^{an} \rightarrow X$, $o : O(X) \subset X^{et}$ being the embedding consisting of the Zariski open subsets of X .

2.2 Complex integral periods

Let k a field of characteristic zero.

Let $X \in \text{SmVar}(k)$ a smooth variety. Let $X = \cup_{i=1}^s X_i$ an open affine cover. We have for $\sigma : k \hookrightarrow \mathbb{C}$ an embedding, the evaluation period embedding map which is the morphism of bi-complexes

$$ev(X) \bullet : \Gamma(X_\bullet, \Omega_{X_\bullet}^\bullet) \rightarrow \mathbb{Z} \text{Hom}_{\text{Diff}}(\mathbb{I}^\bullet, X_{\mathbb{C},\bullet}^{an})^\vee \otimes \mathbb{C},$$

$$w_I^l \in \Gamma(X_I, \Omega_{X_I}^l) \mapsto (ev(X)_I^l(w_I^l)) : \phi_I^l \in \mathbb{Z} \text{Hom}_{\text{Diff}}(\mathbb{I}^l, X_{\mathbb{C},I}^{an})^\vee \otimes \mathbb{C} \mapsto ev_I^l(w_I^l)(\phi_I^l) := \int_{\mathbb{I}^l} \phi_I^{l*} w_I^l$$

given by integration. By taking all the affine open cover $(j_i : X_i \hookrightarrow X)$ of X , we get for $\sigma : k \hookrightarrow \mathbb{C}$, the evaluation period embedding map

$$ev(X) := \varinjlim_{(j_i : X_i \hookrightarrow X)} ev(X) \bullet : \varinjlim_{(j_i : X_i \hookrightarrow X)} \Gamma(X_\bullet, \Omega_{X_\bullet}^\bullet) \rightarrow \varinjlim_{(j_i : X_i \hookrightarrow X)} \mathbb{Z} \text{Hom}_{\text{Diff}(\mathbb{R})}(\mathbb{I}^\bullet, X_{\mathbb{C},\bullet}^{an})^\vee \otimes \mathbb{C}$$

It induces in cohomology, for $j \in \mathbb{Z}$, the evaluation period map

$$H^j ev(X) = H^j ev(X) \bullet : H_{DR}^j(X) = H^j \Gamma(X_\bullet, \Omega_{X_\bullet}^\bullet) \rightarrow H_{\text{sing}}^j(X_{\mathbb{C}}^{an}, \mathbb{C}) = H^j(\text{Hom}_{\text{Diff}(\mathbb{R})}(\mathbb{I}^\bullet, X_{\mathbb{C},\bullet}^{an})^\vee \otimes \mathbb{C}).$$

which does NOT depend on the choice of the affine open cover by acyclicity of quasi-coherent sheaves on affine noetherian schemes for the left hand side and from Mayer-Vietoris quasi-isomorphism for singular cohomology of topological spaces and Whitney approximation theorem for differential manifolds for the right hand side.

Let $X \in \text{SmVar}(k)$. Then,

$$H^* ev(X_{\mathbb{C}}) = H^* R\Gamma(X_{\mathbb{C}}^{an}, \alpha(X))^{-1} \circ \Gamma(X_{\mathbb{C}}^{an}, E_{zar}(\Omega(\text{an}_X))) : H_{DR}^*(X_{\mathbb{C}}) \xrightarrow{\sim} H_{\text{sing}}^*(X_{\mathbb{C}}^{an}, \mathbb{C})$$

is the canonical isomorphism induced by the analytical functor and the quasi-isomorphism $\alpha(X) : \mathbb{C}_{X_{\mathbb{C}}^{an}} \hookrightarrow \Omega_{X_{\mathbb{C}}^{an}}^\bullet$, which gives the periods elements $H^* ev(X)(H_{DR}^*(X)) \subset H_{\text{sing}}^*(X_{\mathbb{C}}^{an}, \mathbb{C})$.

2.3 Algebraic cycles and motives

For $X \in \text{Sch}$ noetherian irreducible and $d \in \mathbb{N}$, we denote by $\mathcal{Z}^d(X)$ the group of algebraic cycles of codimension d , which is the free abelian group generated by irreducible closed subsets of codimension d .

For $X, X' \in \text{Sch}$ noetherian, with X' irreducible, we denote $\mathcal{Z}^{fs/X'}(X' \times X) \subset \mathcal{Z}_{d_{X'}}(X' \times X)$ which consist of algebraic cycles $\alpha = \sum_i n_i \alpha_i \in \mathcal{Z}_{d_{X'}}(X' \times X)$ such that, denoting $\text{supp}(\alpha) = \cup_i \alpha_i \subset X' \times X$ its support and $p' : X' \times X \rightarrow X'$ the projection, $p'_{|\text{supp}(\alpha)} : \text{supp}(\alpha) \rightarrow X'$ is finite surjective.

Let k a field. We denote by $\text{Cor SmVar}(k)$ the category such that the objects are $\{X \in \text{SmVar}(k)\}$ and such that $\text{Hom}_{\text{Cor SmVar}(k)}(X', X) := \mathcal{Z}^{fs/X'}(X' \times X)$. See [3] for the composition law. We denote by $\text{Tr} : \text{Cor SmVar}(k) \rightarrow \text{SmVar}(k)$ the morphism of site given by the embedding $\text{Tr} : \text{SmVar}(k) \hookrightarrow \text{Cor SmVar}(k)$. Let $F \in \text{PSh}(\text{SmVar}(k))$. We say that F admits transfers if $F = \text{Tr}_* F$ with $F \in \text{PSh}(\text{Cor SmVar}(k))$. We recall that $\text{DA}(k) := \text{Ho}_{\mathbb{A}^1, \text{et}}(C(\text{SmVar}(k)))$ is then the derived category of mixed motives.

We have (see e.g. [1]) the Hodge realization functor

$$\mathcal{F}^{Hdg} : \text{DA}(\mathbb{C}) \rightarrow D(\text{MHS}), M \mapsto \mathcal{F}^{Hdg}(M) := \iota^{-1}(\mathcal{F}^{FDR}(M), \text{Bti}^* M, \alpha(M))$$

For $X \in \text{Var}(k)$ and $Z \subset X$ a closed subset, denoting $j : X \setminus Z \hookrightarrow X$ the open complementary, we will consider

$$\Gamma_Z^\vee \mathbb{Z}_X := \text{Cone}(\text{ad}(j_\#, j^*)(\mathbb{Z}_X) : \mathbb{Z}_X \hookrightarrow \mathbb{Z}_X) \in C(\text{Var}(k)^{\text{sm}}/X)$$

and denote for short $\gamma_Z^\vee := \gamma_Z^\vee(\mathbb{Z}_X) : \mathbb{Z}_X \rightarrow \Gamma_Z^\vee \mathbb{Z}_X$ the canonical map in $C(\text{Var}(k)^{\text{sm}}/X)$. Denote $a_X : X \rightarrow \text{Spec } k$ the structural map. For $X \in \text{Var}(k)$ and $Z \subset X$ a closed subset, we have the motive of X with support in Z defined as

$$M_Z(X) := a_{X!} \Gamma_Z^\vee a_X^! \mathbb{Z} \in \text{DA}(k).$$

If $X \in \text{SmVar}(k)$, we will also consider

$$a_{X\#} \Gamma_Z^\vee \mathbb{Z}_X := \text{Cone}(a_{X\#} \circ \text{ad}(j_\#, j^*)(\mathbb{Z}_X) : \mathbb{Z}(U) \hookrightarrow \mathbb{Z}(X)) =: \mathbb{Z}(X, X \setminus Z) \in C(\text{SmVar}(k)).$$

Then for $X \in \text{SmVar}(k)$ and $Z \subset X$ a closed subset

$$M_Z(X) := a_{X!} \Gamma_Z^\vee a_X^! \mathbb{Z} = a_{X\#} \Gamma_Z^\vee \mathbb{Z}_X =: \mathbb{Z}(X, X \setminus Z) \in \text{DA}(k).$$

- Let $(X, Z) \in \text{Sch}^2$ with $X \in \text{Sch}$ a noetherian scheme and $Z \subset X$ a closed subset. We have the deformation $(D_Z X, \mathbb{A}_Z^1) \rightarrow \mathbb{A}^1$, $(D_Z X, \mathbb{A}_Z^1) \in \text{Sch}^2$ of (X, Z) by the normal cone $C_{Z/X} \rightarrow Z$, i.e. such that

$$(D_Z X, \mathbb{A}_Z^1)_s = (X, Z), s \in \mathbb{A}^1 \setminus 0, (D_Z X, \mathbb{A}_Z^1)_0 = (C_{Z/X}, Z).$$

We denote by $i_1 : (X, Z) \hookrightarrow (D_Z X, \mathbb{A}_Z^1)$ and $i_0 : (C_{Z/X}, Z) \hookrightarrow (D_Z X, \mathbb{A}_Z^1)$ the closed embeddings in Sch^2 .

- Let k a field of characteristic zero. Let $X \in \text{SmVar}(k)$. For $Z \subset X$ a closed subset of pure codimension c , consider a desingularisation $\epsilon : \tilde{Z} \rightarrow Z$ of Z and denote $n : \tilde{Z} \xrightarrow{\epsilon} Z \subset X$. We have then the morphism in $\text{DA}(k)$

$$G_{Z,X} : M(X) \xrightarrow{D(\mathbb{Z}(n))} M(\tilde{Z})(c)[2c] \xrightarrow{\mathbb{Z}(\epsilon)} M(Z)(c)[2c]$$

where $D : \text{Hom}_{\text{DA}(k)}(M_c(\tilde{Z}), M_c(X)) \xrightarrow{\sim} \text{Hom}_{\text{DA}(k)}(M(X), M(\tilde{Z})(c)[2c])$ is the duality isomorphism from the six functors formalism (moving lemma of Suzlin and Voevodsky) and $\mathbb{Z}(n) := \text{ad}(n_!, n^!)(a_X^! \mathbb{Z})$, noting that $n_! = n_*$ since n is proper and that $a_X^! = a_X^*[d_X]$ and $a_Z^! = a_Z^*[d_Z]$ since X , resp. \tilde{Z} , are smooth (considering the connected components, we may assume X and \tilde{Z} of pure dimension).

We recall the following facts (see [3] and [1]):

Proposition 1. *Let k be a field a characteristic zero. Let $X \in \text{SmVar}(k)$ and $i : Z \subset X$ a smooth subvariety of pure codimension d . Then $C_{Z/X} = N_{Z/X} \rightarrow Z$ is a vector bundle of rank d . The closed embeddings $i_1 : (X, Z) \hookrightarrow (D_Z X, \mathbb{A}_Z^1)$ and $i_0 : (C_{Z/X}, Z) \hookrightarrow (D_Z X, \mathbb{A}_Z^1)$ in $\text{SmVar}^2(k)$ induces isomorphisms of motives $\mathbb{Z}(i_1) : M_Z(X) \xrightarrow{\sim} M_{\mathbb{A}_Z^1}(D_Z X)$ and $\mathbb{Z}(i_0) : M_Z(N_{Z/X}) \xrightarrow{\sim} M_{\mathbb{A}_Z^1}(D_Z X)$ in $\text{DA}(k)$. We get the excision isomorphism in $\text{DA}(k)$*

$$P_{Z,X} := \mathbb{Z}(i_0)^{-1} \circ \mathbb{Z}(i_1) : M_Z(X) \xrightarrow{\sim} M_Z(N_{Z/X}).$$

We have

$$\text{Th}(N_{Z/X}) \circ P_{Z,X} \circ \gamma_Z^\vee(\mathbb{Z}_X) = G_{Z,X} := D(\mathbb{Z}(i)) : M(X) \rightarrow M(Z)(d)[2d].$$

Proof. See [3]. □

Definition 1. *Let k be a field a characteristic zero. Let $X \in \text{SmVar}(k)$ and $i : Z \subset X$ a smooth subvariety of pure codimension d . We consider using proposition 1 the canonical map in $\text{DA}(k)$*

$$\text{Res}_{Z,X} : M(Z)(d)[-2d] \xrightarrow{P_{Z,X}^{-1} \circ \text{Th}(N_{Z/X})^{-1}} M_Z(X) \xrightarrow{c(\mathbb{Z}(X \setminus Z))} M(X \setminus Z)[1]$$

where, for $m : A \rightarrow B$, $A, B \in C(\mathcal{A})$, \mathcal{A} an additive category, $c(A) : \text{Cone}(m : A \rightarrow B) \rightarrow A[1]$ is the canonical map.

We will use the following theorem of Voevodsky :

Theorem 1. (Voevodsky) *Let k be a perfect field (e.g. k a field of characteristic zero). Let $F \in \text{PSh}(\text{SmVar}(k), \mathbb{Q})$. If F is \mathbb{A}^1 invariant and admits transfers, then for all $j \in \mathbb{Z}$, $H^j E_{\text{et}}^\bullet(F) \in \text{PSh}(\text{SmVar}(k))$ are \mathbb{A}^1 invariant. That is, if F is \mathbb{A}^1 invariant and admits transfers then F is \mathbb{A}^1 local.*

Proof. By [5], $H^j E_{N_{\text{is}}}^\bullet(F) \in \text{PSh}(\text{SmVar}(k))$ are \mathbb{A}^1 invariant. On the other hand since F takes values in \mathbb{Q} -vector spaces, $H^j E_{N_{\text{is}}}^\bullet(F) = H^j E_{\text{et}}^\bullet(F)$. □

2.4 The logarithmic De Rham complexes

We recall from [2] the following notion :

- For k a field and $X \in \text{Var}(k)$, we consider the embedding in $C(X)$

$$OL_X : \Omega_{X,\log}^\bullet \hookrightarrow \Omega_X^\bullet := \Omega_{X/k}^\bullet,$$

such that, for $X^\circ \subset X$ an open subset and $w \in \Omega_X^p(X^\circ)$, $w \in \Omega_{X,\log}^p(X^\circ)$ if and only if there exists $(n_i)_{1 \leq i \leq s} \in \mathbb{Z}$ and $(f_{i,\alpha_k})_{1 \leq i \leq s, 1 \leq k \leq p} \in \Gamma(X^\circ, \mathcal{O}_X)^*$ such that

$$w = \sum_{1 \leq i \leq s} n_i df_{i,\alpha_1} / f_{i,\alpha_1} \wedge \cdots \wedge df_{i,\alpha_p} / f_{i,\alpha_p} \in \Omega_X^p(X^\circ),$$

and for $p = 0$, $\Omega_{X,\log}^0 := \mathbb{Z}$. Let k a field. We get an embedding in $C(\text{Var}(k))$

$$OL : \Omega_{/k,\log}^\bullet \hookrightarrow \Omega_{/k}^\bullet, \text{ given by, for } X \in \text{Var}(k), \\ OL(X) := OL_X : \Omega_{/k,\log}^\bullet(X) := \Gamma(X, \Omega_{X,\log}^\bullet) \hookrightarrow \Gamma(X, \Omega_X^\bullet) =: \Omega_{/k}^\bullet(X)$$

and its restriction to $\text{SmVar}(k) \subset \text{Var}(k)$.

- For $X \in \text{AnSp}(\mathbb{C})$, we consider the embedding in $C(X)$

$$OL_X : \Omega_{X,\log}^\bullet \hookrightarrow \Omega_X^\bullet,$$

such that, for $X^\circ \subset X$ an open subset and $w \in \Omega_X^p(X^\circ)$, $w \in \Omega_{X,\log}^p(X^\circ)$ if and only if there exists $(n_i)_{1 \leq i \leq s} \in \mathbb{Z}$ and $(f_{i,\alpha_k})_{1 \leq i \leq s, 1 \leq k \leq p} \in \Gamma(X^\circ, O_X)^*$ such that

$$w = \sum_{1 \leq i \leq s} n_i df_{i,\alpha_1}/f_{i,\alpha_1} \wedge \cdots \wedge df_{i,\alpha_p}/f_{i,\alpha_p} \in \Omega_X^p(X^\circ),$$

and for $p = 0$, $\Omega_{X,\log}^0 := \mathbb{Z}$. We get an embedding in $C(\text{AnSp}(\mathbb{C}))$

$$\begin{aligned} OL^{an} : \Omega_{\log}^{\bullet,an} &\hookrightarrow \Omega^{\bullet,an}, \text{ given by, for } X \in \text{AnSp}(\mathbb{C}), \\ OL^{an}(X) := OL_X : \Omega_{\log}^{\bullet}(X) &:= \Gamma(X, \Omega_{X,\log}^{\bullet,an}) \hookrightarrow \Gamma(X, \Omega_X^{\bullet,an}) =: \Omega^{\bullet,an}(X) \end{aligned}$$

and its restriction to $\text{AnSm}(\mathbb{C}) \subset \text{AnSp}(\mathbb{C})$.

We have the following result :

Proposition 2. (i00) *Let $D \in \text{AnSm}(\mathbb{C})$ be an open ball. Then, for each $p, q \in \mathbb{Z}$, $q \neq 0$, $H_{et}^q(D, \Omega_{D,\log}^p) = 0$.*

(i0) *Let $U \in \text{SmVar}(\mathbb{C})$ such that there exists an etale map $e : U \rightarrow \mathbb{A}_{\mathbb{C}}^{d_U}$. Then, for each $p, q \in \mathbb{Z}$, $q \neq 0$, $p \neq 0$, $H_{et}^q(U, \Omega_{U^{an},\log}^p) = 0$.*

(i) *Let $U \in \text{SmVar}(\mathbb{C})$ such that there exists an etale map $e : U \rightarrow \mathbb{A}_{\mathbb{C}}^{d_U}$. Then, for each $p, q \in \mathbb{Z}$, $q \neq 0$, $p \neq 0$, $H_{usu}^q(U^{an}, \Omega_{U^{an},\log}^p) = 0$.*

(ii) *Let $X \in \text{SmVar}(\mathbb{C})$. We still denote $\text{an}_X : X^{an} \xrightarrow{e, \sim} X^{an,et} \xrightarrow{\text{an}_X} X^{et}$ the composite. Consider the map in $C(X^{et})$*

$$T(et, usu) = E_{et}(\text{an}_X^* k) : E_{et}(\text{an}_X^* \Omega_{X^{an},\log}^p) \rightarrow E_{et}(\text{an}_X^* E_{usu}(\Omega_{X^{an},\log}^p)).$$

Then, for each $p, q \in \mathbb{Z}$,

$$H^{p+q}T(et, usu)(X) : H_{et}^q(X, \text{an}_X^* \Omega_{X^{an},\log}^p) \rightarrow H_{usu}^q(X^{an}, \Omega_{X^{an},\log}^p)$$

is an isomorphism.

(ii)' *Let $X \in \text{SmVar}(\mathbb{C})$. Let Z a closed subset. Then, for each $p, q \in \mathbb{Z}$,*

$$H_Z^{p+q}T(et, usu)(X) : H_{et,Z}^q(X, \text{an}_X^* \Omega_{X^{an},\log}^p) \rightarrow H_{usu,Z}^q(X^{an}, \Omega_{X^{an},\log}^p)$$

is an isomorphism.

Proof. (i00): Let $B = D(0, 1)^N \in \text{AnSm}(\mathbb{C})$ be an open ball. Let $\alpha \in H^q(B, \Omega_{B,\log}^p)$, $q \in \mathbb{Z}$, $q \neq 0$. Consider a sequence of sub-balls

$$l_n : B_n := D(0, r_n)^N \hookrightarrow B := D(0, 1)^N, n \in \mathbb{N}, \text{ with } r_n \rightarrow 0 \text{ when } n \rightarrow \infty.$$

Since $(B_n := D(0, r_n))_{n \in \mathbb{N}}$ form a basis of neighborhood of $0 \in B$ and since $\Omega_{B,\log}^p$ is a single presheaf, there exists $n := n_\alpha \in \mathbb{N}$ (depending on α) such that

$$l_n^* \alpha = 0 \in H^q(B_n, \Omega_{B_n,\log}^p).$$

But it follows immediately from the definition of the the presheaf $\Omega_{B,\log}^p$ considering the isomorphism

$$\phi_n : B_n \xrightarrow{\sim} B, \phi_n(z) := r_n^{-1}z$$

so that $(\phi_n)^{-1} \circ l_n = r_n I$, that

$$l_n^* \alpha = l_n^* \phi_n^{-1*} \phi_n^* \alpha = r_n \phi_n^* \alpha \in H^q(B_n, \Omega_{B_n,\log}^p).$$

Hence $\phi_n^* \alpha = 0$. Since ϕ_n is an isomorphism, we get $\alpha = 0$.

(i): Let $U \in \text{SmVar}(\mathbb{C})$ such that there exists an etale map $e : U \rightarrow \mathbb{A}_{\mathbb{C}}^{d_U}$. Since $\Omega_{U'}$ is a trivial vector bundle for $U' \rightarrow U^{an}$ etale, we have a pairing

$$w_1 = \sum_{I \subset [1, \dots, d_U], \text{card} I = p} a_I (\wedge_{k \in I} d\tilde{z}_k), \quad w_2 = \sum_{I \subset [1, \dots, d_U], \text{card} I = p} b_I (\wedge_{k \in I} d\tilde{z}_k) \in \Omega_{U^{an}, et}^p(U'),$$

$$\text{where } \tilde{z}_k := z_k \circ e \circ e', \quad \langle w_1, w_2 \rangle := \sum_{I \subset [1, \dots, d_U], \text{card} I = p} a_I b_I,$$

Hence we have for each $p \in \mathbb{N}$, $p \neq 0$, a canonical splitting

$$\Omega_{U^{an}}^p = \Omega_{U^{an}, O \log}^p \oplus (\Omega_{U^{an}, O \log}^p)^\perp \in \text{PSh}_{O_{U^{an}, et}}(U^{an}, et).$$

with

$$\Omega_{U^{an}, O \log}^p(U') := \langle O(U') df_1 / f_1 \wedge \dots \wedge df_p / f_p, (f_k)_{1 \leq k \leq p} \in O(U')^* \rangle \subset \Omega_{U^{an}}^p(U') := \Omega_{U'}^p(U'),$$

where for $F \subset \Omega_{U^{an}}^p(U')$ a sub $O(U')$ module, $F^\perp \subset \Omega_{U^{an}}^p(U')$ denotes the orthogonal sub $O(U')$ module for the pairing $\langle -, - \rangle$, so that $\Omega_{U^{an}}^p(U') = F \oplus F^\perp$ and so if $\mathcal{F} \subset \Omega_{U^{an}}^p$ is a sub O -module presheaf, $\Omega_{U^{an}}^p = \mathcal{F} \oplus \mathcal{F}^\perp$. On the other hand since for $U' \rightarrow U$ etale, U' is affine and hence U'^{an} is Stein for $U' \rightarrow U$ etale, we have

$$H_{usu}^q(U^{an}, \Omega_{U^{an}}^p) = 0,$$

for all $q \in \mathbb{Z}$, $q \neq 0$. Hence,

$$H_{usu}^q(U^{an}, \Omega_{U^{an}, O \log}^p) = H_{usu}^q(U^{an}, (\Omega_{U^{an}, O \log}^p)^\perp) = 0,$$

for all $q \in \mathbb{Z}$, $q \neq 0$. Note that in contrast to the algebraic case, the exact sequences

$$0 \rightarrow \mathbb{Q} \rightarrow O_{U^{an}} \rightarrow O_{U^{an}}^{nc} \rightarrow 0, \quad 0 \rightarrow \Omega_{U^{an}, \log}^p \rightarrow \Omega_{U^{an}, O \log}^p \rightarrow \Omega_{U^{an}, O^{nc} \log}^p \rightarrow 0,$$

with $O_{U^{an}}^{nc} := O_{U^{an}} / \mathbb{Q}$ and $\Omega_{U^{an}, O^{nc} \log}^p := \Omega_{U^{an}, O \log}^p / \Omega_{U^{an}, \log}^p$ do NOT split. We then consider the canonical map

$$m : H_{usu}^q(U^{an}, \Omega_{U^{an}, \log}^p) \otimes O(U^{an}) \rightarrow H_{usu}^q(U^{an}, \Omega_{U^{an}, O \log}^p), \quad m(w \otimes h) := hw$$

Let $U^{an} = \cup_{i=1}^n B_i$ an open cover by open balls. Denote $j_i : B_i \hookrightarrow U^{an}$ the open embeddings. We then have the following commutative diagram

$$\begin{array}{ccc} H_{usu}^q(U^{an}, \Omega_{U^{an}, \log}^p) \otimes O(U^{an}) & \xrightarrow{m: w \otimes h \rightarrow hw} & H_{usu}^q(U^{an}, \Omega_{U^{an}, O \log}^p) \\ \downarrow (j_i)^*, \simeq & & \downarrow (j_i)^*, \simeq \\ \mathbb{H}_{usu}^q(D_\bullet, \Omega_{U^{an}, \log}^p) \otimes O(U^{an}) & \xrightarrow{m: w \otimes h \rightarrow hw} & \mathbb{H}_{usu}^q(D_\bullet, \Omega_{U^{an}, O \log}^p) \\ \uparrow k & & \uparrow k \\ H^q \Gamma(D_\bullet, \Omega_{U^{an}, \log}^p) \otimes O(U^{an}) & \xrightarrow{m: w \otimes h \rightarrow hw} & H^q \Gamma(D_\bullet, \Omega_{U^{an}, O \log}^p) \end{array}$$

where the left and right arrows of the lower square are isomorphisms by (i00). On the other hand the lower map is clearly an isomorphism. Hence the upper map is an isomorphism. Since $H_{usu}^q(U^{an}, \Omega_{U^{an}, O \log}^p) = 0$, we get $H_{usu}^q(U^{an}, \Omega_{U^{an}, \log}^p) = 0$.

(i0): By (i), we have $R^q \text{an}_{U^*} \Omega_{U^{an}, \log}^p = 0$, $q \in \mathbb{Z}$, $q \neq 0$. Hence for $q \in \mathbb{Z}$, $q \neq 0$,

$$H_{et}^q(U, \text{an}_{U^*} \Omega_{U^{an}, \log}^p) = \mathbb{H}_{et}^q(U, R \text{an}_{U^*} \Omega_{U^{an}, \log}^p) = H_{usu}^q(U^{an}, \Omega_{U^{an}, \log}^p) = 0,$$

where the last equality follows again by (i).

(ii): Take an open affine cover $X = \cup_{1 \leq i \leq r} X_i$ of X such that for each $1 \leq i \leq r$ such that there exists an etale map $e_i : X_i \rightarrow \mathbb{A}_{\mathbb{C}}^{d_{X_i}}$ with $e_i : X_i \rightarrow e_i(X_i)$ finite etale. Denote, for each $1 \leq i \leq r$, $j_i : X_i \hookrightarrow X$ the open embedding. Consider then the commutative diagram of abelian groups

$$\begin{array}{ccc} \mathbb{H}_{et}^q(X_{\bullet}, \text{an}_{X^*} \Omega_{X^{an}, \log}^p) & \xrightarrow{H^{p+q}T(et, usu)(X_{\bullet})} & \mathbb{H}_{usu}^q(X_{\bullet}^{an}, \Omega_{X^{an}, \log}^p), \\ \oplus_{1 \leq i \leq r} E_{et} \Omega(j_i) \uparrow & & \oplus_{1 \leq i \leq r} E_{usu} \Omega(j_i) \uparrow \\ H_{et}^q(X, \text{an}_{X^*} \Omega_{X^{an}, \log}^p) & \xrightarrow{H^{p+q}T(et, usu)(X)} & H_{usu}^q(X^{an}, \Omega_{X^{an}, \log}^p) \end{array}$$

where the vertical maps are isomorphisms since the Zariski topology is finer than the etale topology. By (i0) and (i), the upper map is an isomorphism (using the spectral sequence associated to the trivial filtration of complexes). Hence the lower map is also an isomorphism. This proves (ii).

(ii)': Follows from (ii) applied to X and $X \setminus Z$ by the distinguish triangle in $Ho(C(\text{SmVar}(\mathbb{C})))$

$$\mathbb{Z}(X \setminus Z) \xrightarrow{\mathbb{Z}(j)} \mathbb{Z}(X) \rightarrow \mathbb{Z}(X, X \setminus Z) \rightarrow \mathbb{Z}(X \setminus Z)[1],$$

where $j : X \setminus Z \hookrightarrow X$ is the open embedding. □

3 Hypersurface De Rham cohomology and its complex periods

We denote $\mathbb{P}^N := \mathbb{P}_{\mathbb{C}}^N := \text{Proj}(\mathbb{C}[z_0, \dots, z_N])$. Let $X = V(f) \subset \mathbb{P}^N$ be a smooth hypersurface. If $N - 1 = 2p$ is even, we consider the decompositions

$$H_{DR}^{N-1}(X) = \mathbb{C}[\Lambda \cap X] \oplus H_{DR, v}^{N-1}(X), \quad H_{\text{sing}}^{N-1}(X^{an}, \mathbb{Q}) = \mathbb{C}[\Lambda \cap X] \oplus H_{\text{sing}, v}^{N-1}(X^{an}, \mathbb{Q})$$

of vector spaces, where $\Lambda \subset \mathbb{P}^N$ is a linear subspace of codimension p . If $N - 1 = 2p + 1$ is even, we set

$$H_{DR, v}^{N-1}(X) := H_{DR}^{N-1}(X), \quad H_{\text{sing}, v}^{N-1}(X^{an}, \mathbb{Q}) := H_{\text{sing}}^{N-1}(X^{an}, \mathbb{Q}).$$

Let $X = V(f) \subset \mathbb{P}^N$ be a smooth hypersurface. Denote $j : U := \mathbb{P}^N \setminus X \hookrightarrow \mathbb{P}^N$ the open embedding. We consider the algebraic Hodge module

$$j_* \text{Hdg}(O_U, F_b) := ((j_* O_U, F), j_* \mathbb{Q}_{U^{an}}, j_* \alpha(U)) \in \text{HM}(\mathbb{P}^N),$$

where $F^k j_* O_U = \mathbb{C}[x_1, \dots, x_N][\frac{1}{f^k}]$. Recall for each $p \in \mathbb{Z}$,

$$F^p H_{DR}^N(U) := H^N \iota^{\geq p}(\mathbb{H}^N(\mathbb{P}^N, \Omega_{\mathbb{P}^N}^{\bullet \geq p}(\log X))) = H^N \iota^{\geq p} H^N(\mathbb{P}^N, \Omega_{\mathbb{P}^N}^{\bullet} \otimes_{O_{\mathbb{P}^N}} F^{\bullet - p} j_* O_U)$$

and

$$F^p H^{N-1}(X) := H^{N-1} \iota^{\geq p}(\mathbb{H}^{N-1}(X, \Omega_X^{\bullet \geq p})).$$

where

$$\iota^{\geq p} : \Omega_{\mathbb{P}^N}^{\bullet \geq p}(\log X) \hookrightarrow \Omega_{\mathbb{P}^N}^{\bullet}(\log X), \quad \iota^{\geq p} : \Omega_{\mathbb{P}^N}^{\bullet} \otimes_{O_{\mathbb{P}^N}} F^{\bullet - p} j_* O_U \hookrightarrow \Omega_{\mathbb{P}^N}^{\bullet} \otimes_{O_{\mathbb{P}^N}} j_* O_U = j_* \Omega_U^{\bullet}$$

and $\iota^{\geq p} : \Omega_X^{\bullet \geq p} \hookrightarrow \Omega_X^{\bullet}$ are the canonical embeddings. Then the map $\text{Res}_{X, \mathbb{P}^N}$ of definition 1 induces for each $p \in \mathbb{Z}$, canonical isomorphisms

$$H^N \Omega(\text{Res}_{X, \mathbb{P}^N}) : F^{p+1} H_{DR}^N(U) \xrightarrow{\sim} F^p H_{DR}^{N-1}(X)_v, \quad H^N \text{Bti}(\text{Res}_{X, \mathbb{P}^N}) : H_{\text{sing}}^N(U, \mathbb{Q}) \xrightarrow{\sim} H_{\text{sing}}^{N-1}(X, \mathbb{Q})_v.$$

Lemma 1. Let $X = V(f) \subset \mathbb{P}^N$ be a smooth hypersurface. Denote $j : U := \mathbb{P}^N \setminus X \hookrightarrow \mathbb{P}^N$ the open embedding. Let $B \subset \mathbb{P}^N$, $\phi : B \simeq D(0,1)^N$ be an open ball such that $\phi(X \cap B) = \{0\} \times D(0,1)^{N-1}$. Then,

(i) For $q \in \mathbb{Z}$, $q \neq 0$, $\mathbb{H}_{usu}^q(B, (\Omega_{\mathbb{P}^N}^\bullet \otimes_{\mathcal{O}_{\mathbb{P}^N}} F^{\bullet-p} j_* \mathcal{O}_U)^{an}) = 0$.

(ii) For $q \in \mathbb{Z}$, $q \neq 0$, $H_{usu}^q(B, \Omega_{\mathbb{P}^N, an}^{N-p}(\log X)^{\partial=0}) = 0$.

Proof. (i): Since B is Stein, we have for $r, q \in \mathbb{Z}$, $q \neq 0$,

$$H_{usu}^q(B, (\Omega_{\mathbb{P}^N}^r \otimes_{\mathcal{O}_{\mathbb{P}^N}} F^{r-p} j_* \mathcal{O}_U)^{an}) = 0.$$

On the other hand the De Rham complex of $(j_* \mathcal{O}_{B \setminus X}, F)$ is acyclic (see e.g. [6] corollary 18.7)

(ii): Since open balls form a basis of the topology of $\mathbb{P}^{N, an}$, we have

$$a_{usu} \iota : a_{usu}(\Omega_{U^{an}, \log}^{N-p} \oplus \partial \Omega_{\mathbb{P}^N, an}^{N-p+1}) \xrightarrow{\sim} a_{usu} \Omega_{\mathbb{P}^N, an}^{N-p}(\log X)^{\partial=0}$$

where

$$\iota : \Omega_{U^{an}, \log}^{N-p} \oplus \partial \Omega_{\mathbb{P}^N, an}^{N-p+1} \hookrightarrow \Omega_{\mathbb{P}^N, an}^{N-p}(\log X)^{\partial=0}$$

is the canonical embedding. On the other hand we have for $q \in \mathbb{Z}$, $q \neq 0$, $H_{usu}^q(B, \Omega_{\mathbb{P}^N, an}^{N-p+1}) = 0$ since B is Stein, and $H_{usu}^q(B, \Omega_{U^{an}, \log}^{N-p}) = 0$ by proposition 2 (i00). \square

We have by the local acyclicity of the De Rham complex of $(j_* \mathcal{O}_U, F)$ a canonical map :

Definition 2. Let $X = V(f) \subset \mathbb{P}^N$ be a smooth hypersurface. Denote $j : U := \mathbb{P}^N \setminus X \hookrightarrow \mathbb{P}^N$ the open embedding. Let $\mathbb{P}^{N, an} = \cup_{i=1}^n B_i$ an open cover, with $\phi_i : \mathbb{B}_i \simeq D(0,1)^N$ open balls such that $\phi_i(X \cap B_i) = \{0\} \times D(0,1)^{N-1}$. Denote, for $I \subset [1, \dots, n]$, $l_I : B_I \hookrightarrow \mathbb{P}^N$ the open embeddings, with $B_I := \cap_{i \in I} B_i$. Since, for each $I \subset [1, \dots, n]$ and $p \in \mathbb{Z}$,

$$l_I^* \iota_{\partial} : l_I^* \Omega_{\mathbb{P}^N, an}^{N-p}(\log X)^{\partial=0}[-p] \hookrightarrow l_I^*(\Omega_{\mathbb{P}^N}^\bullet \otimes_{\mathcal{O}_{\mathbb{P}^N}} F^{\bullet-p} j_* \mathcal{O}_U)^{an}$$

is an quasi-isomorphism (see e.g. [6] corollary 18.7), the spectral sequence of the filtration on the total complex of a bi-complex gives, for each $p \in \mathbb{Z}$, a canonical isomorphism, where the first and last map are isomorphism by lemma 1(i) and (ii) respectively,

$$\begin{aligned} B_{\partial} : F^p H_{DR}(U) &:= H^N \iota^{\geq p} \mathbb{H}_{zar}^N(\mathbb{P}^N, \Omega_{\mathbb{P}^N}^\bullet \otimes_{\mathcal{O}_{\mathbb{P}^N}} F^{\bullet-p} j_* \mathcal{O}_U) \\ &\xrightarrow{H^N((l_i^* = \Omega(l_i))_{1 \leq i \leq n})} H^N \iota^{\geq p} H^N \Gamma(B_{\bullet}, (\Omega_{\mathbb{P}^N}^\bullet \otimes_{\mathcal{O}_{\mathbb{P}^N}} F^{\bullet-p} j_* \mathcal{O}_U)^{an}) \\ &\rightarrow H^p \Gamma(B_{\bullet}, \Omega_{\mathbb{P}^N, an}^{N-p}(\log X)^{\partial=0}) \xrightarrow{H^N((l_i^* = \Omega(l_i))_{1 \leq i \leq n})^{-1}} H_{usu}^p(\mathbb{P}^N, \Omega_{\mathbb{P}^N, an}^{N-p}(\log X)^{\partial=0}), \end{aligned}$$

given by, for $w \in \Gamma(U, \Omega_{\mathbb{P}^N}^N)$, inductively by a p step induction where p is the order of the pole of w at X ,

- taking for each $1 \leq i \leq n$, $l_i^* w = \partial w_{1i}$ since $\partial w = 0$ for dimensional reason and by exactness of $l_i^* \iota_{\partial}$,
- we then have for each $1 \leq i, j \leq n$, considering $w_{1ij} := l_{ij}^*(w_{1i} - w_{1j})$, $w_{1ij} = \partial w_{2ij}$ since

$$\partial w_{1ij} = l_{ij}^*(\partial w_{1i} - \partial w_{1j}) = l_{ij}^*(l_i^* w - l_j^* w) = 0$$

and by exactness of $l_{ij}^* \iota_{\partial}$,

- at the final p step, $B(w) := w_{pI} \in H^N(B_{\bullet}, \Omega_{\mathbb{P}^N, an}^{N-p}(\log X)^{\partial=0})$, $\text{card} I = p$.

Let $X = V(f) \subset \mathbb{P}^N$ be a smooth hypersurface. Denote $U := \mathbb{P}^N \setminus X$. For $l \in \mathbb{Z}$, we consider

- the sequence of embeddings in $C(U^{et})$

$$OL_U^l : \Omega_{U,\log}^l[-l] \xrightarrow{\widehat{OL_U^l}[-l]} \Omega_{\mathbb{P}^N}^l(\log X)^{\partial=0}[-l] \xrightarrow{\iota_\partial} \Omega_{\mathbb{P}^N}^\bullet \otimes_{O_{\mathbb{P}^N}} F^{\bullet-p} j_* O_U \xrightarrow{\iota^{\geq l}} \Omega_U^\bullet$$

- and the sequence of embeddings in $C(U^{an})$

$$OL_{U^{an}}^l : \Omega_{U^{an},\log}^l[-l] \xrightarrow{\widehat{OL_{U^{an}}^l}[-l]} \Omega_{\mathbb{P}^{N,an}}^l(\log X)^{\partial=0}[-l] \xrightarrow{\iota_\partial} (\Omega_{\mathbb{P}^N}^\bullet \otimes_{O_{\mathbb{P}^N}} F^{\bullet-p} j_* O_U)^{an} \xrightarrow{\iota^{\geq l}} \Omega_{U^{an}}^\bullet,$$

where

$$\iota_\partial : \Omega_{\mathbb{P}^{N,an}}^l(\log X)^{\partial=0}[-l] \hookrightarrow (\Omega_{\mathbb{P}^N}^\bullet \otimes_{O_{\mathbb{P}^N}} F^{\bullet-p} j_* O_U)^{an}$$

is an equivalence *usu* local (see [6]).

The main result of this section is the following :

Proposition 3. *Let $X = V(f) \subset \mathbb{P}^N$ be a smooth projective hypersurface. Denote $U := \mathbb{P}^N \setminus X$. Let $w \in F^{p+1} H_{DR}^N(U)$. Then the following assertions are equivalent*

(i) $ev(U)(w) \in H_{\text{sing}}^N(U^{an}, 2i\pi\mathbb{Q}),$

(ii) $B_\partial(w) \in \widehat{OL_{U^{an}}^{p+1}}(H_{usu}^{p+1}(U^{an}, \Omega_{U^{an},\log}^{N-p}))$ (see definition 2),

(iii) $w \in OL_{U^{an}}^{N-p}(H_{usu}^{p+1}(U^{an}, \Omega_{U^{an},\log}^{N-p})) = OL_{U^{an}}(H_{usu}^{p+1}(U^{an}, \Omega_{U^{an},\log}^{N-p})).$

Proof. (i) implies (ii): Let $\mathbb{P}^{N,an} = \cup_{i=1}^n B_i$ an open cover, with $\phi_i : \mathbb{B}_i \simeq D(0,1)^N$ open balls such that $\phi_i(X \cap B_i) = \{0\} \times D(0,1)^{N-1}$. Denote, for $I \subset [1, \dots, n]$, $l_I : B_I \hookrightarrow \mathbb{P}^N$ the open embeddings, with $B_I := \cap_{i \in I} B_i$. Then

$$B_\partial(w) := (w_{p+1,I}) \in H^{p+1}(B_\bullet, \Omega_{\mathbb{P}^{N,an}}^{N-p}(\log X)^{\partial=0}).$$

We have, for each $I \subset [1, \dots, n]$, a decomposition

$$w_{p+1,I} = \frac{a_I}{z_1} dz_1 \wedge w'_I + w''_I \in \Gamma(B_I, \Omega_{\mathbb{P}^{N,an}}^{N-p}(\log X)^{\partial=0})$$

with $w'_I \in \Gamma(B_I, \Omega_{\mathbb{P}^N}^{N-p-1})^{\partial=0}$ and $w''_I \in \Gamma(B_I, \Omega_{\mathbb{P}^N}^{N-p})^{\partial=0}$. Consider the evaluation map

$$ev(U)^\bullet : \Gamma(B_\bullet \setminus X, \Omega_{\mathbb{P}^N}^\bullet) \rightarrow C_{\text{diff}}^\bullet(B_\bullet \setminus X^{an}, \mathbb{C}),$$

which is a morphism of bi-complexes. On the other hand, we have the following canonical map

$$\begin{aligned} BC_\partial : H^p \Gamma(B_\bullet, C_{U^{an},\text{diff}}^{N-p,\partial}) &\xrightarrow{\Gamma(B_\bullet \setminus X^{an}, \iota_\partial)} H^N \Gamma(B_\bullet \setminus X^{an}, C_{U^{an},\text{diff}}^{\bullet \geq N-p}) \xrightarrow{\Gamma(B_\bullet \setminus X^{an}, \iota^{\geq N-p})} \\ &= H^N \Gamma(B_\bullet \setminus X^{an}, C_{U^{an},\text{diff}}^\bullet) \xrightarrow{(H^N(l_i^* =: C_{U^{an},\text{diff}}^\bullet(l_i))_{1 \leq i \leq n})^{-1}} H_{\text{sing}}^N(U^{an}, \mathbb{Q}) = H^N \Gamma(U^{an}, C_{U^{an},\text{diff}}^\bullet), \end{aligned}$$

where $H^N(l_i^*)$ is an isomorphism by the acyclicity of $C_{U^{an},\text{diff}}^{\bullet \geq N-p}(B_I \setminus X^{an})$ for each $I \subset [1, \dots, n]$, and

$$C_{U^{an},\text{diff}}^\bullet \hookrightarrow C_{U^{an},\text{sing}}^\bullet, \text{ for } V \subset U^{an}, C_{\text{diff}}^\bullet(V) \hookrightarrow C_{\text{sing}}^\bullet(V)$$

is the canonical quasi-isomorphism in $C(U^{an})$. We then have the following commutative diagram

$$\begin{array}{ccc} H^N \iota^{\geq p} H^N \Gamma(\mathbb{P}^N, \Omega_{\mathbb{P}^N}^\bullet \otimes_{O_{\mathbb{P}^N}} F^{\bullet-p} j_* O_U) & & \\ \downarrow H^N((l_i^* = \Omega(l_i))_{1 \leq i \leq n}) & \searrow^{B_\partial} & \\ H^N \iota^{\geq p} H^N \Gamma(B_\bullet, (\Omega_{\mathbb{P}^N}^\bullet \otimes_{O_{\mathbb{P}^N}} F^{\bullet-p} j_* O_U)^{an}) & \xrightarrow{B_\partial} & H^p \Gamma(B_\bullet, \Omega_{\mathbb{P}^{N,an}}^{N-p}(\log X)^{\partial=0}) \\ \downarrow ev(U) & & \downarrow ev(U)^\bullet \\ H_{\text{sing}}^N(U^{an}, \mathbb{C}) = H^N \Gamma(U^{an}, C_{U^{an},\text{diff}}^\bullet \otimes \mathbb{C}) & \xleftarrow{BC_\partial \otimes \mathbb{C}} & H^p \Gamma(B_\bullet \setminus X^{an}, C_{\text{diff}}^{N-p,\partial=0} \otimes \mathbb{C}) \end{array}$$

Since $ev(U)$ is injective by the strictness of the Hodge filtration and since B_∂ is an isomorphism, $ev(U)_\bullet$ is injective and

$$(BC_\partial \otimes \mathbb{C})' := (BC_\partial \otimes \mathbb{C})|_{ev(U)_\bullet(H^p\Gamma(B_\bullet, \Omega_{\mathbb{P}^N, an}^{N-p}(\log X)^{\partial=0}))} : \\ ev(U)_\bullet(H^p\Gamma(B_\bullet, \Omega_{\mathbb{P}^N, an}^{N-p}(\log X)^{\partial=0})) \rightarrow H_{\text{sing}}^N(U^{an}, \mathbb{C})$$

is injective. For $\gamma_I \in C_{N-p}^{\text{sing}}(B_I \setminus X^{an}, \mathbb{Q})^{\partial=0}$ where $p_1 \circ \phi_I(\gamma_I) \subset D(0, 1)^*$ is a loop around 0, we have by the above diagram

$$ev(U)(w)(\gamma_I) = ev(U)(w_{p+1, I})(\gamma_I) = a_I.$$

Hence if $ev(U)(w) \in H_{\text{sing}}^N(U^{an}, 2i\pi\mathbb{Q})$, then $a_I \in \mathbb{Q}$, using the injectivity of $(BC_\partial \otimes \mathbb{C})'$.

(ii) is equivalent to (iii): By definition $B_\partial(w) = (H^p\iota_\partial)^{-1}(w)$. On the other hand by definition $(\iota^{\geq p} \circ \iota_\partial \circ \widehat{OL}_{U^{an}}^{N-p} = OL_{U^{an}}^{N-p}$.

(iii) implies (i): See [2]. □

4 Complex GAGA for logarithmic de Rham classes

We recall (see section 2) the morphism of site given by the analytification functor

$$\text{An} : \text{AnSm}(\mathbb{C}) \rightarrow \text{SmVar}(\mathbb{C}), \quad X \mapsto \text{An}(X) := X^{an}, \quad (f : X' \rightarrow X) \mapsto \text{An}(f) := f^{an}.$$

and for $X \in \text{SmVar}(\mathbb{C})$, the following commutative diagram

$$\begin{array}{ccc} \text{AnSm}(\mathbb{C}) & \xrightarrow{\text{An}} & \text{SmVar}(\mathbb{C}) \\ \downarrow o_{X^{an}} & & \downarrow o_X \\ X^{an, et} & \xrightarrow{\text{an}_X} & X^{et} \end{array}$$

We have the following :

Lemma 2. (i0) The sheaves $O_{an}^* \in \text{PSh}(\text{AnSm}(\mathbb{C}))$ and $\Omega_{\log}^{1, an} \in \text{PSh}(\text{AnSm}(\mathbb{C}))$ admit transfers compatible with transfers on $\Omega^{1, an} \in \text{PSh}(\text{AnSm}(\mathbb{C}))$.

(i) For each $l \in \mathbb{Z}$, the sheaf $\Omega_{\log}^{l, an} \in \text{PSh}(\text{AnSm}(\mathbb{C}))$ admits transfers compatible with transfers on $\Omega^{l, an} \in \text{PSh}(\text{AnSm}(\mathbb{C}))$, that is $\Omega^{\bullet, an} \in C(\text{Cor AnSm}(\mathbb{C}))$ and the inclusion $OL : \Omega_{\log}^{l, an}[-l] \hookrightarrow \Omega^{\bullet, an}$ in $C(\text{AnSm}(\mathbb{C}))$ is compatible with transfers.

(ii) For each $l \in \mathbb{Z}$, the sheaf $\text{An}_* \Omega_{\log}^{l, an} \in \text{PSh}(\text{SmVar}(\mathbb{C}))$ is \mathbb{A}^1 invariant.

Proof. (i0): Let $W \subset X' \times X$ with $X, X' \in \text{AnSm}(\mathbb{C})$, X' connected, and $p_{X'} : W \subset X' \times X \rightarrow X'$ finite surjective over X' . Then $M(p_X) : M(X') \hookrightarrow M(W)$ is a finite, where for $Y \in \text{AnSp}(\mathbb{C})$ irreducible, $M(Y) := \text{Frac}(O(Y))$ denote the field of meromorphic function. Since $O(X') \subset M(X')$ is integrally closed, the trace map $\text{Tr}_{W/X'} : M(W) \rightarrow M(X')$ sends $O(W)$ to $O(X')$, and the norm map $N_{W/X'} : M(W)^* \rightarrow M(X')^*$ sends $O(W)^*$ to $O(X')^*$.

The sheaf $O_{an}^* \in \text{PSh}(\text{AnSm}(\mathbb{C}))$ admits transfers : for $W \subset X' \times X$ with $X, X' \in \text{AnSm}(\mathbb{C})$ and W finite surjective over X' and $f \in O(X)^*$, $W^*f := N_{W/X'}(p_X^*f)$ where $p_X : W \hookrightarrow X' \times X \rightarrow X$ is the projection and $N_{W/X'} : M(W)^* \rightarrow M(X')^*$ is the norm map. This gives transfers on $\Omega_{\log}^{1, an} \in \text{PSh}(\text{AnSm}(\mathbb{C}))$ compatible with transfers on $\Omega^{1, an} \in \text{PSh}(\text{AnSm}(\mathbb{C}))$: for $W \subset X' \times X$ with $X, X' \in \text{AnSm}(\mathbb{C})$ and W finite surjective over X' and $f \in O(X)^*$,

$$W^*df/f := dW^*f/W^*f = \text{Tr}_{W/X'}(p_X^*(df/f)),$$

where we recall $p_X : W \hookrightarrow X' \times X \rightarrow X$ is the projection and $Tr_{W/X'} : O(W) \rightarrow O(X')$ is the trace map. Note that $d(fg)/fg = df/f + dg/g$.

(i): By (i0), we get transfers on

$$\otimes_{\mathbb{Q}}^l \Omega_{\log}^{1,an}, \otimes_O^l \Omega^{1,an} \in \text{PSh}(\text{AnSm}(\mathbb{C}))$$

since $\otimes_{\mathbb{Q}}^l \Omega_{\log}^{1,an} = H^0(\otimes_{\mathbb{Q}}^{L,l} \Omega_{\log}^{1,an})$ and $\otimes_O^l \Omega^{1,an} = H^0(\otimes_O^{L,l} \Omega^{1,an})$. This induces transfers on

$$\wedge_{\mathbb{Q}}^l \Omega_{\log}^{1,an} := \text{coker}(\oplus_{I_2 \subset [1, \dots, l]} \otimes_{\mathbb{Q}}^{l-1} \Omega_{\log}^{1,an} \xrightarrow{\oplus_{I_2 \subset [1, \dots, l]} \Delta_{I_2 := (w \otimes w' \mapsto w \otimes w \otimes w')}} \otimes_{\mathbb{Q}}^l \Omega_{\log}^{1,an}) \in \text{PSh}(\text{AnSm}(\mathbb{C})).$$

and

$$\wedge_O^l \Omega^{1,an} := \text{coker}(\oplus_{I_2 \subset [1, \dots, l]} \otimes_{O_k}^{l-1} \Omega^{1,an} \xrightarrow{\oplus_{I_2 \subset [1, \dots, l]} \Delta_{I_2 := (w \otimes w' \mapsto w \otimes w \otimes w')}} \otimes_O^l \Omega^{1,an}) \in \text{PSh}(\text{AnSm}(\mathbb{C})).$$

(ii): Follows from the fact that for $X \in \text{AnSp}(\mathbb{C})$, $O^*(X \times \mathbb{A}^1) = O^*(X)$. Indeed, let $h = h(x, z) \in O(X \times \mathbb{A}^1)^*$. Then for $x \in X$, $z \in \mathbb{A}^1 \mapsto h(x, z)$ is constant since it is an holomorphic function on \mathbb{A}^1 with no zero. Hence $h = h(x)$. \square

Proposition 4. *Let $X \in \text{PSmVar}(\mathbb{C})$. Consider the morphism $\text{an}_X : X^{an} \rightarrow X$ in RTop given by analytical functor.*

(i) *The analytic De Rham cohomology class of an algebraic cycle is logarithmic and is of type (d, d) , that is, for $Z \in \mathcal{Z}^d(X)$*

$$[Z] := H^{2d} \Omega(\gamma_Z^\vee)([Z]) \subset H^{2d} OL_{X^{an}}(H_{usu}^d(X^{an}, \Omega_{X^{an}, \log}^d)) \subset H_{DR}^{2d}(X^{an}) = H_{DR}^{2d}(X).$$

(ii) *Conversely, any $w \in H^{2d} OL_{X^{an}}(H_{usu}^d(X^{an}, \Omega_{X^{an}, \log}^d))$ is the class of an algebraic cycle $Z \in \mathcal{Z}^d(X) \otimes \mathbb{Q}$, i.e. $w = [Z]$.*

(iii) *We have $H^j OL_{X^{an}}(H_{usu}^{j-l}(X^{an}, \Omega_{X^{an}, \log}^l)) = 0$ for $j, l \in \mathbb{Z}$ such that $2l \neq j$. That is, we only have analytic logarithmic classes in bidegree (d, d) for $d \in \mathbb{N}$, in particular there is no non trivial logarithmic classes for odd degree De Rham cohomology $H_{DR}^{2d+1}(X^{an}) = H_{DR}^{2d+1}(X)$.*

(iv) *For each $j \in \mathbb{Z}$, the isomorphism*

$$\text{an}_X^* : H_{DR}^j(X) = \mathbb{H}_{et}^j(X, \Omega_X^\bullet) \xrightarrow{\sim} \mathbb{H}_{usu}^j(X^{an}, \Omega_{X^{an}}^\bullet) = H_{DR}^j(X^{an})$$

and its inverse preserve logarithmic classes, that is for each $j, l \in \mathbb{Z}$,

$$\text{an}_X^*(H^j(OL_X)(H_{et}^{j-l}(X, \Omega_{X, \log}^l))) = H^j OL_{X^{an}}(H_{usu}^{j-l}(X^{an}, \Omega_{X^{an}, \log}^l)).$$

Proof. Consider, for $j, l \in \mathbb{Z}$ and $X \in \text{SmVar}(\mathbb{C})$,

$$L^{l, j-l}(X) := H^j OL_{X^{an}}(H_{usu}^{j-l}(X^{an}, \Omega_{X^{an}, \log}^l)).$$

Consider also for $j, l \in \mathbb{Z}$, $X \in \text{SmVar}(\mathbb{C})$ with good reduction and $Z \subset X$ a closed subset,

$$L_Z^{l, j-l}(X) := H^j OL_{X^{an}}(H_{usu, Z}^{j-l}(X^{an}, \Omega_{X^{an}, \log}^l)).$$

We still denote $\text{an}_X : X^{an} \xrightarrow{e, \sim} X^{an, et} \xrightarrow{\text{an}_X} X^{et}$ the composite, with $e(U \rightarrow X^{an}) := e(U)$. By proposition 2(ii) and (ii)', we have

$$L^{l, j-l}(X) = H^j OL_{X^{an}}(H_{et}^{j-l}(X, \text{an}_{X^*} \Omega_{X^{an}, \log}^l)) = H^j OL^{an}(H^{j-l} \text{Hom}(\mathbb{Z}(X), E_{et}^\bullet(\text{An}_* \Omega_{\log}^{l, an}))).$$

and

$$L_Z^{l,j-l}(X) = H^j OL_{X^{an}}(H_{et,Z}^{j-l}(X, \text{an}_{X^*} \Omega_{X^{an}, \log}^l)) = H^j OL^{an}(H^{j-l} \text{Hom}(\mathbb{Z}(X, X \setminus Z), E_{et}^\bullet(\text{An}_* \Omega_{\log}^{l,an}))),$$

for the second equalities, note that

$$(\text{An}_* \Omega_{\log}^{l,an})|_{X^{et}} := o_{X^*} \text{An}_* \Omega_{\log}^{l,an} = \text{an}_{X^*} o_{X^{an,et}} \Omega_{\log}^{l,an} = \text{an}_{X^*} \Omega_{X^{an}, \log}^l,$$

where $o_X : X^{et} \subset \text{SmVar}(\mathbb{C})$ is the small etale site $o_X : X^{et} \hookrightarrow \text{SmVar}(\mathbb{C})/X \xrightarrow{o_X} \text{SmVar}(\mathbb{C})$, $o_X(U \rightarrow X) = U$. For each $l \in \mathbb{Z}$, the presheaves $\text{An}_* \Omega_{\log}^{l,an} \in \text{PSh}(\text{SmVar}(\mathbb{C}))$ admit transfers and are \mathbb{A}^1 invariant by lemma 2. Hence by theorem 1, they are \mathbb{A}^1 local since they are \mathbb{A}^1 invariant and admit transfers. This gives in particular, for $Z \subset X$ a smooth subvariety of (pure) codimension d with good reduction, by proposition 1 an isomorphism

$$E_{et} \text{An}_* \Omega_{\log}^{l,an}(P_{Z,X}) : L_Z^{l,j-l}(X) \xrightarrow{\sim} L_Z^{l,j-l}(N_{Z/X}) \xrightarrow{\sim} L^{l-d,j-l-d}(Z).$$

Let

$$\alpha = OL_{X^{an}}(\alpha) \in L^{l,j-l}(X) := H^j OL_{X^{an}}(H_{usu}^{j-l}(X^{an}, \Omega_{X^{an}, \log}^l)) = H^{j-l} \text{Hom}(\mathbb{Z}(X), E_{et}^\bullet(\text{An}_* \Omega_{\log}^{l,an})).$$

with $j-l \neq 0$. Let $U \subset X$ an open subset such that there exists an etale map $e : U \rightarrow \mathbb{A}_{\mathbb{C}}^{d_U}$. Denote $j : U \hookrightarrow X$ the open embedding. By proposition 2(i), we have

$$j^* \alpha = 0 \in H_{usu}^{j-l}(U^{an}, \Omega_{X^{an}, \log}^l) = 0.$$

Considering a divisor $X \setminus U \subset D \subset X$, we get

$$\alpha = H^{j-l} E_{et}^\bullet(\text{An}_* \Omega_{\log}^{l,an})(\gamma_D^\vee)(\alpha), \alpha \in L_D^{l,j-l}(X).$$

- Proof of (i): Similar to the proof of [2] theorem 2.
- Proof of (ii): By assumption we have $w \in L^{d,d}(X)$. Let $U \subset X$ an open subset such that there exists an etale map $e : U \rightarrow \mathbb{A}_{\mathbb{C}}^{d_U}$. Considering a divisor $X \setminus U \subset D \subset X$, we get

$$w = H^d E_{et}^\bullet(\text{An}_* \Omega_{\log}^{d,an})(\gamma_D^\vee)(w), \alpha \in L_D^{d,d}(X).$$

Denote $D^\circ \subset D$ its smooth locus with good reduction and $l : X^\circ \hookrightarrow X$ a Zariski open subset such that $X^\circ \cap D = D^\circ$. We then have

$$l^* w \in L_{D^\circ}^{d,d}(X^\circ) = L^{d-1,d-1}(D^\circ)$$

We repeat this procedure with each connected components of D° instead of X . By a finite induction of d steps, we get

$$w = H^d E_{et}^\bullet(\text{An}_* \Omega_{\log}^{d,an})(\gamma_Z^\vee)(w), w \in L_Z^{d,d}(X) = L^{0,0}(Z^\circ).$$

with $Z := D_d \subset \dots \subset D \subset X$ a pure codimension d (Zariski) closed subset, thus $\alpha = \sum_i n_i [Z_i] \in H_{DR}^{2d}(X)$, where $n_i \in \mathbb{Q}$ and $(Z_i)_{1 \leq i \leq t} \subset Z$ are the irreducible components of Z ,

- Proof of (iii) part 1: Let $j > 2l$. Let $w \in L^{l,j-l}(X)$. By the proof of (ii) there exists $Z \subset X$ a closed subset of pure codimension l such that

$$w = H^{j-l} E_{et}^\bullet(\text{An}_* \Omega_{\log}^{l,an})(\gamma_Z^\vee)(w), w \in L_Z^{l,j-l}(X).$$

By a finite induction of $d_X - l$ steps, restricting to the smooth locus with good reduction of closed subsets of Z , $w = 0$ since $L_{Z'}^{l,j-l}(X) = 0$ for all $Z' \in \text{SmVar}(k)$ with good reduction such that $\text{codim}(Z', X) \geq l$.

- Proof of (iii) part 2: Let $j < 2l$. Let $w \in L^{l,j-l}(X)$. By the proof of (ii) there exists $Z \subset X$ a closed subset of pure codimension $j - l$ such that

$$w = H^{j-l} E_{et}^\bullet(\text{An}_* \Omega_{\log}^{l,an})(\gamma_Z^\vee)(w), \quad w \in L_Z^{l,j-l}(X).$$

For $Z' \subset X$ a closed subset of pure codimension c , consider a desingularisation $\epsilon : \tilde{Z}' \rightarrow Z'$ of Z' and denote $n : \tilde{Z}' \xrightarrow{\epsilon} Z' \subset X$. The morphism in $\text{DA}(k)$

$$G_{Z',X} : M(X) \xrightarrow{D(\mathbb{Z}(n))} M(\tilde{Z}')(c)[2c] \xrightarrow{\mathbb{Z}(\epsilon)} M(Z')(c)[2c]$$

where $D : \text{Hom}_{\text{DA}(k)}(M_c(\tilde{Z}'), M_c(X)) \xrightarrow{\sim} \text{Hom}_{\text{DA}(k)}(M(X), M(\tilde{Z}')(c)[2c])$ is the duality isomorphism from the six functor formalism (moving lemma of Suzlin and Voevodsky) and $\mathbb{Z}(n) := \text{ad}(n_!, n^!)(a_X^! \mathbb{Z})$, is given by a morphism in $C(\text{SmVar}(k))$

$$\hat{G}_{Z',X} : \mathbb{Z}_{tr}(X) \rightarrow E_{et}(C_* \mathbb{Z}_{tr}(Z'))(c)[2c].$$

Let $l : X^o \hookrightarrow X$ be an open embedding such that $Z^o := Z \cap X^o$ is the smooth locus of Z with good reduction. We then have the following commutative diagram of abelian groups

$$\begin{array}{ccccccc} 0 & \longrightarrow & H_{et,Z}^{j-l}(X, \text{an}_{X^*} \Omega_{X^{an},\log}^l) & \xrightarrow{l^*} & H_{et,Z^o}^{j-l}(X^o, \text{an}_{X^*} \Omega_{X^{an},\log}^l) & \xrightarrow{\partial} & H_{et,Z \setminus Z^o}^{j-l+1}(X, \text{an}_{X^*} \Omega_{X^{an},\log}^l) \longrightarrow \dots \\ & & \Omega(\hat{G}_{Z,X}) \uparrow & & \Omega(P_{Z^o, X^o}) \uparrow & & \Omega(\hat{G}_{Z,X}) \uparrow \\ 0 & \longrightarrow & H_{et}^0(Z, \text{an}_{Z^*} \Omega_{Z^{an},\log}^{2l-j}) & \xrightarrow{l^*} & H_{et}^0(Z^o, \Omega_{Z^o,an,\log}^{2l-j}) & \xrightarrow{\partial} & H_{et,Z \setminus Z^o}^1(Z, \Omega_{Z^{an},\log}^{2l-j}) \longrightarrow \dots \end{array}$$

whose rows are exact sequences. Consider

$$l^* w = \Omega(P_{Z^o, X^o})(w^o) \in H_{et,Z^o}^{j-l}(X^o, \text{an}_{X^*} \Omega_{X^{an},\log}^l), \quad w^o \in H_{et}^0(Z^o, \text{an}_{Z^*} \Omega_{Z^o,an,\log}^{2l-j}).$$

Since $\partial l^* w = 0 \in H_{et,Z \setminus Z^o}^{j-l+1}(X, \text{an}_{X^*} \Omega_{X^{an},\log}^l)$, we get, considering the smooth locus with good reduction $(Z \setminus Z^o)^o \subset Z \setminus Z^o$ and $X^{oo} \subset X$ is an open subset such that $X \cap (Z \setminus Z^o)^o = X^{oo} \cap (Z \setminus Z^o)$,

$$\partial w^o = 0 \in H_{et,Z \setminus Z^o}^1(Z, \text{an}_{Z^*} \Omega_{Z^{an},\log}^{2l-j}),$$

since $H_{et,(Z \setminus Z^o)^o}^1(Z, \text{an}_{Z^*} \Omega_{Z^{an},\log}^{2l-j}) = 0$ for dimension reasons, that is

$$w = \Omega(\hat{G}_{Z,X})(w), \quad \text{with } w \in H_{et}^0(Z, \text{an}_{Z^*} \Omega_{Z^{an},\log}^{2l-j}).$$

Hence $w = 0$ since $H^0(Z', \Omega_{Z'^{an},\log}^k) = 0$ for all $k > 0$ and all $Z' \in \text{PVar}(K)$.

(iv): Follows from (ii), (iii) and [2] theorem 2. □

5 Hodge Conjecture for complex smooth projective hypersurfaces

We now state and prove using the result of section 3 and section 4 the main result of this article :

Theorem 2. *Let $X = V(f) \subset \mathbb{P}_{\mathbb{C}}^N$ be a smooth projective hypersurface, with $N = 2p + 1$ odd. Let $\lambda \in F^p H^{2p}(X^{an}, \mathbb{Q})$. Then there exists an algebraic cycle $Z \in \mathbb{Z}^p(X)$ such that $\lambda = [Z]$.*

Proof. Consider $U := \mathbb{P}_{\mathbb{C}}^N \setminus X$. Consider $Res_{X, \mathbb{P}^N} : M(X) \rightarrow M(U)[-1]$ the canonical map given in definition 1. We have the commutative diagram of abelian group

$$\begin{array}{ccc}
H_{DR}^N(U) & \xrightarrow{H^N \Omega(Res_{X, \mathbb{P}^N})} & H_{DR}^{N-1}(X)_v \\
\uparrow H^N \alpha(U^{an}) = ev(U)^{-1} & & \uparrow H^{N-1} \alpha(X^{an}) = ev(X)^{-1} \\
H_{\text{sing}}^N(U^{an}, \mathbb{C}) & \xrightarrow{(H^N \text{Bti}(Res_{X, \mathbb{P}^N})) \otimes_{\mathbb{Q}} \mathbb{C}} & H_{\text{sing}}^{N-1}(X^{an}, \mathbb{C})_v
\end{array}$$

whose arrows are isomorphisms. Denote $w(\lambda) := H^{N-1} \alpha(X^{an})(\lambda) \in F^p H_{DR}^{2p}(X)$. We have

$$\lambda = H^{N-1} \text{Bti}(Res_{X, \mathbb{P}^N})(\alpha), \text{ with } \alpha \in H_{\text{sing}}^N(U^{an}, \mathbb{Q}).$$

Then

$$w(\lambda) = H^{N-1} \Omega(Res_{X, \mathbb{P}^N})(w(\alpha)), \text{ with } w(\alpha) := H^N \alpha(U^{an})(\alpha) \in F^{p+1} H_{DR}^N(U).$$

Since $ev(U)(w(\alpha)) = \alpha \in H_{\text{sing}}^N(U^{an}, 2i\pi\mathbb{Q})$, we have by proposition 3,

$$w(\alpha) \in H^N OL_{U^{an}}(H_{usu}^{p+1}(U^{an}, \Omega_{U^{an}, \log}^{N-p}))$$

Hence

$$w(\lambda) = H^{N-1} OL_{X^{an}}(\Omega_{\log}^{N-p, an}(Res_{X, \mathbb{P}^N})(w(\alpha))) \in OL_{X^{an}}(H_{usu}^p(X^{an}, \Omega_{X^{an}, \log}^p)).$$

Hence by proposition 4(ii), $w(\lambda) = [Z]$ with $Z \in \mathcal{Z}^p(X)$. □

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