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Toughness Properties of Arbitrarily Partitionable Graphs

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Abstract

Drawing inspiration from a well-known conjecture of Chvátal on a toughness threshold guaranteeing graph Hamiltonicity, we investigate toughness properties of so-called arbitrarily partitionable (AP) graphs, which are those graphs that can be partitioned into arbitrarily many connected graphs with arbitrary orders, and can be perceived as a weakening of Hamiltonian and traceable graphs. In particular, we provide constructions of non-AP graphs with toughness about $\frac{5}{4}$, *i.e.*, in which, when removing the vertices of any cut-set S , the number of resulting connected components is at most about $\frac{4}{5}|S|$. We also consider side related questions on graphs that can be partitioned arbitrarily into only a few connected graphs (with arbitrary orders). Among other things, we prove that not all 1-tough graphs can always be partitioned into four connected graphs this way. As going along, we also raise several other questions and problems of interest on the topic.

Keywords: arbitrarily partitionable graph; toughness; Hamiltonicity; split graph.

1. Introduction

We start with some definitions. For an $n \geq 1$, we refer to a graph of order n as an n -graph, and to a partition $\pi = (\lambda_1, \dots, \lambda_p)$ of n (*i.e.*, $\lambda_1 + \dots + \lambda_p = n$) as an n -partition. Now, a *realisation* (V_1, \dots, V_p) of an n -partition $\pi = (\lambda_1, \dots, \lambda_p)$ in an n -graph G is a partition of the vertex set $V(G)$ of G into p parts V_1, \dots, V_p where V_i is a *connected part* (*i.e.*, $G[V_i]$ is connected) of size λ_i for every $i \in \{1, \dots, p\}$. We now say G is *arbitrarily partitionable* (AP for short) if all n -partitions are realisable in G , or, in other words, if G can be partitioned into arbitrarily many connected graphs, regardless of their orders.

AP graphs have been introduced independently by Barth, Baudon, and Puech in [1], and by Horňák and Woźniak in [23]. Since then, several aspects of AP graphs have been investigated in literature, covering structural aspects, algorithmic aspects, and several variants. For more details, we refer the interested reader to several works on the topic, see *e.g.* [2, 9, 13, 16, 24, 30], and, in particular, to the more recent of these investigations.

Among others, an interesting aspect lies in that AP graphs are quite connected to other fundamental notions of graph theory. For instance, it can be noted that any realisation of an n -partition $(2, \dots, 2)$ or $(1, 2, \dots, 2)$ in an n -graph forms a perfect matching or a quasi-perfect matching (*i.e.*, a set of $\lfloor n/2 \rfloor$ pairwise disjoint edges). Another illustration arises from the facts that every path is obviously AP and that adding edges to an AP graph preserves APness, from which we deduce that any Hamiltonian or traceable graph (*i.e.*, admitting a cycle or a path traversing all vertices exactly once) is AP.

This latter fact is actually one of the main motivations behind our investigations in the current work. Indeed, put differently, APness can be perceived as a weaker form of Hamiltonicity and traceability, and, as such, an interesting direction is wondering whether known sufficient conditions for Hamiltonicity/traceability can be weakened to APness. A perfect illustration is the pioneer result by Marczyk in [27], stating that the AP property

holds in any connected n -graph admitting a (quasi-)perfect matching and satisfying that the degree sum of any two non-adjacent vertices is at least $n - 2$ (thereby weakening to APness a famous result of Ore from [29], stating that this last quantity being at least $n - 1$ guarantees traceability, and that it guarantees even Hamiltonicity if it is at least n). Since this result, several authors have provided such positive and negative results, showing that some sufficient conditions for Hamiltonicity and traceability weaken to APness, while some others do not. For instance, Marczyk's original result was improved (to decreased bounds on the degree sums) by Marczyk in [28], and by Horňák, Marczyk, Schiermeyer, and Woźniak in [22]. In [9], Bensmail provided a similar condition for APness in graphs where the degree sum of any three pairwise non-adjacent vertices is large enough. On a different line, Kalinowski, Pilśniak, Schiermeyer, and Woźniak proved in [24] that graphs with sufficiently many edges are AP, thereby weakening another condition for Hamiltonicity and traceability. In [13], Bensmail and Li considered other various conditions for Hamiltonicity and traceability, including the square operation and forbidden induced patterns.

In this work, we explore, in the context of AP graphs, another studied aspect of Hamiltonian and traceable graphs, being the notion of **graph toughness**, which is a refinement over the fundamental notion of graph connectivity. In brief, the toughness is a measure of how disconnected (in terms of number of resulting connected components) a graph may become upon removing some of its vertices. More formally, for a graph G , we denote by $c(G)$ the number of connected components of G . Now, a *cut-set* $S \subset V(G)$ of G is a set of vertices such that $c(G - S) > 1$, or, in other words, whose removal leaves G not connected. Assuming now S is a cut-set of G , we define as $\tau(S)$ the quantity $\frac{|S|}{c(G-S)}$, and as $\tau(G)$ the *toughness* of G , being the smallest value of $\tau(S)$ over all cut-sets S of G . In particular, a *tough cut-set* of G is a cut-set S such that $\tau(S) = \tau(G)$. Last, we say G is t -*tough* for some $t \geq 0$ if we have $t \leq \tau(G)$. Note that the toughness of every non-complete graph is well defined (since the definitions above apply only for graphs admitting cut-sets), and that a t -tough graph is also t' -tough for all $t' \leq t$. Also, a graph has toughness 0 if and only if it is not connected; this apart, the toughness of a graph can be arbitrarily large.

The notion of graph toughness, as mentioned earlier, was introduced to express sufficient conditions for Hamiltonicity and traceability. More precisely, it is Chvátal whom considered this notion first, and raised that, perhaps, all tough enough graphs are Hamiltonian.

Conjecture 1.1 (Chvátal [17]). *There is some $t_0 > 0$ such that every t_0 -tough graph is Hamiltonian.*

As a first guess towards Conjecture 1.1, Chvátal asked whether all graphs with toughness more than $\frac{3}{2}$ are Hamiltonian, which was first disproved by Thomassen (see [14]). After that, as Enomoto, Jackson, Katerinis, and Saito [19] proved that every 2-tough graph admits a 2-factor (*i.e.*, disjoint cycles covering all vertices) and that, for every $\varepsilon > 0$, there is a $(2 - \varepsilon)$ -tough graph without this property, it was believed that, perhaps, 2-tough graphs are all Hamiltonian. Again, this was refuted by Bauer, Broersma, and Veldman, who proved in [8] that for every $\varepsilon > 0$ there is a $(\frac{9}{4} - \varepsilon)$ -tough graph that is not traceable. This is where the main investigations towards Conjecture 1.1 stand to date, although many related results (some of which will be recalled later on in this work, to guide our investigations) can be found in literature, in particular for restricted graph classes.

Although graph toughness has not been considered as such in the context of AP graphs, a few results and investigations actually deal with connectivity properties of these graphs, and, as such, are not so distant from that field. In particular, since investigations on AP trees (see *e.g.* [2]), it is known that removing a cut-vertex (*i.e.*, a cut-set of size 1) from an AP graph results in at most four connected components. This contrasts with more general,

larger cut-sets, as Baudon, Foucaud, Przybyło, and Woźniak proved that removing a cut-set of size at least 2 from an AP graph might result in a graph with arbitrarily many connected components [6]. In the language of graph toughness, this implies $\tau(G)$ can be arbitrarily close to 0, for an AP graph G . It was proved, however, that if one strengthens the definition of AP graphs by a bit, then some of these phenomena do not hold any more [3]. This very concern was studied further by other authors in the recent work [16].

In this work, we thus initiate the study of graph toughness in the very context of AP graphs, regarding these graphs as a weaker form of Hamiltonian graphs. We start in Section 2 by raising a few remarks on a well-known theorem of Tutte on perfect matchings, from which we get to explaining why the most common proof arguments for proving non-APness cannot be applied to construct non-AP graphs with “large” toughness. In that section, we also get to introducing other results and problems related to Tutte’s Theorem, which we think are of independent interest. From Section 3 on, we then focus on the following straight analogue of Chvátal’s Conjecture 1.1:

Conjecture 1.2. *There is some $t^* > 0$ such that every t^* -tough graph is AP.*

Of course, if Conjecture 1.1 was to be proved, then it would imply Conjecture 1.2 holds too, since Hamiltonicity implies APness. In the same line as in the work [25] of Kratsch, Lehel, and Müller, in Section 3 we start by investigating Conjecture 1.2 for split graphs. In particular, we get to constructing non-AP split graphs with toughness tending to $\frac{5}{4}$. Using that construction, we then get to exhibiting non-AP graphs with slightly larger toughness in Section 4, through exploiting ideas of Bauer, Broersma, and Veldman from [8], that allowed them to establish, assuming it holds, the best known lower bound on t_0 in Conjecture 1.1 to date. On the way, we also observe that the very construction of Bauer, Broersma, and Veldman of non-traceable graphs with toughness about $\frac{9}{4}$ provides AP graphs. In Section 5, we move to a slightly different problem, where all these questions are investigated in graphs that are, in some sense, AP only for partitions into a few connected graphs. We finish off in Section 6 with a few more questions and directions for further work on the topic.

2. Around Tutte’s Theorem for perfect matchings

Previous studies on AP graphs have shown that, in general, graphs are more likely to not be AP because of partitions with spectrum¹ of size 1, and in particular because of partitions of the form $(\lambda, \dots, \lambda)$ for $\lambda \geq 2$ being small. A notable case is thus when $\lambda = 2$, corresponding to perfect matchings. Although there are non-AP graphs having perfect matchings, still, for these reasons, building non-AP graphs with certain properties is more easily done through building graphs with these properties admitting no perfect matchings.

Unfortunately, this approach is not quite promising in our context, in which we strive to build non-AP graphs with large toughness. This can *e.g.* be seen through the following famous condition, due to Tutte, for graphs to admit perfect matchings.

Theorem 2.1 (Tutte [31]). *A graph G with even order has a perfect matching if and only if for every $S \subseteq V(G)$ the graph $G - S$ has at most $|S|$ connected components with odd order.*

From Tutte’s Theorem, we directly get that every graph of even order with sufficiently large toughness admits perfect matchings. More precisely, note that a particular toughness value is 1, as 1-tough graphs can more simply be perceived and defined as those graphs

¹Following works of Ravoux on AP graphs [30], the *spectrum* of a partition is the set of its values.

in which, when removing any $k \geq 1$ vertices, we end up with a graph having at most k connected components. In conjunction with Tutte's Theorem, this yields:

Observation 2.2. *Every 1-tough graph of even order admits a perfect matching.*

Regarding earlier remarks, we thus get that, if one aims to build non-AP graphs with large toughness, *i.e.*, more than 1, then one has to consider graphs that are not AP because of partitions different from $(2, \dots, 2)$, while, generally speaking, admitting no realisation of $(2, \dots, 2)$ is a common reason why a non-AP graph is not AP.

Let us also add that Observation 2.2 is best possible, in the sense that, for every $\varepsilon > 0$, there are $(1 - \varepsilon)$ -tough graphs of even order with no perfect matchings [19].

In the next sections, we will exhibit non-AP graphs with toughness more than 1, and this will be done by making sure there is some $\lambda \geq 3$ such that these graphs cannot be partitioned following $(\lambda, \dots, \lambda)$. The main reasons why we focus on such partitions generalise concerns raised above. In particular, note that, intuitively to the least, when partitioning a graph following some partition π , there are more partition possibilities, and thus more chances that a realisation exists, 1) the more distinct values π contains, and 2) the larger the values of π are. For these reasons, we will give a special focus to partitions $(\lambda, \dots, \lambda)$ for some $\lambda \geq 3$ being small ($\lambda = 3$ being thus the most promising case).

Due to these concerns, we wonder whether there is a way to generalise Tutte's Theorem from perfect matchings, thus to partition $(2, \dots, 2)$, to any partition $(\lambda, \dots, \lambda)$ with $\lambda \geq 3$. A problem is that building a realisation of $(2, \dots, 2)$ is much more constrained and different from building one of $(\lambda, \dots, \lambda)$ for some $\lambda \geq 3$, which makes unclear how a proper generalisation should read. An attempt we suggest is based on the following interpretation of one direction of Tutte's Theorem. Let G be a graph and S be a cut-set of G . If C is a connected component of $G - S$ with odd order, then, in a perfect matching of G , it must be that at least one vertex of C is matched to a vertex outside C , and, more precisely, by the definition of a cut-set, this outside vertex must belong to S . From this, we get right away that $|S|$ must be at least the number of connected components with odd order of $G - S$.

For the sake of simplicity, let us focus on $\lambda = 3$ throughout the following explanations. When considering realisations of $(3, \dots, 3)$ in G , note that, if $|V(C)|$ is not a multiple of 3, then there must be at least one part X of size 3 containing both vertices in C and vertices outside C . And this part X can be of multiple forms:

- if $|X \cap V(C)| = 1$, then either
 - X contains one vertex of C and two vertices of S , or
 - X contains one vertex of C , one vertex of S , and one vertex from a connected component of $G - S$ different from C ;
- if $|X \cap V(C)| = 2$, then X contains two vertices of C and one vertex of S .

These arguments generalise naturally to any partition $(\lambda, \dots, \lambda)$ with $\lambda \geq 3$, though, obviously, the possibilities increase as λ rises, and a general description as above, involving the order of C modulo λ , gets more and more tedious to establish properly (in particular, note that, for large values of λ , any part containing vertices of C can contain multiple vertices of S , and, for each connected component C' of $G - S$ different from C , multiple vertices of C'). Still, there is an obvious, strong way of generalising one direction of Tutte's Theorem to partitions of the form $(\lambda, \dots, \lambda)$ for any $\lambda \geq 2$, which reads as follows.

Set $\lambda \geq 2$. For a graph G with a cut-set S , let $C(G, S, \lambda)$ be the following graph:

- for every connected component C of $G - S$, if $|V(C)| \not\equiv 0 \pmod{\lambda}$, then we add a clique K_C of order $|V(C)| \pmod{\lambda}$ to the graph;
- we add a clique K of order $|S|$ to the graph;
- we make all vertices of K universal; that is, for every connected component C of $G - S$ with $|V(C)| \not\equiv 0 \pmod{\lambda}$, we add to the graph all possible edges between vertices of K and vertices of K_C .

Note that $|V(G)| \equiv |V(C(G, S, \lambda))| \pmod{\lambda}$; in particular, if $|V(G)|$ is divisible by λ , then so is $|V(C(G, S, \lambda))|$. Now, by arguments above, it should be clear that the following generalisation of one direction of Tutte's Theorem holds:

Observation 2.3. *Let $\lambda \geq 2$, and G be a graph with order multiple of λ . If, for some cut-set S of G , graph $C(G, S, \lambda)$ admits no realisation of $(\lambda, \dots, \lambda)$, then so does not G .*

At this point, it remains unclear, however, whether the other direction of Tutte's Theorem can be generalised the same way. That is, we wonder whether, assuming a graph G admits no realisation of some partition $(\lambda, \dots, \lambda)$ for some $\lambda \geq 3$, there is necessarily a cut-set S of G such that $C(G, S, \lambda)$ shares the same property. Although we were able to check this for small graphs, proving this seems more demanding than proving Tutte's Theorem.

Due to these thoughts, in the next sections we will also wonder about toughness thresholds guaranteeing a graph admits a realisation of $(\lambda, \dots, \lambda)$ for any $\lambda \geq 3$. Indeed, as a weaker version of Conjecture 1.2 one could also wonder about the following:

Conjecture 2.4. *For every $\lambda \geq 3$, there is some $t_\lambda^- > 0$ such that every t_λ^- -tough graph admits a realisation of $(\lambda, \dots, \lambda)$.*

3. On the toughness of AP and non-AP split graphs

Recall that a graph G is a *split graph* if its vertex set can be partitioned into two sets K, I such that K is a clique (*i.e.*, $G[K]$ is complete) and I is an independent set (*i.e.*, $G[I]$ is empty). For convenience, we say the vertices of K are *black* and those of I are *white*.

In this section, we wonder about the toughness of AP split graphs. In particular, we wonder about the maximum toughness of a non-AP split graph, and about a toughness threshold above which a split graph is necessarily AP. These concerns are primarily motivated by the work [25] of Kratsch, Lehel, and Müller, in which they answered these questions for Hamiltonicity (thereby understanding Conjecture 1.1 completely for split graphs). In particular, they proved that every $\frac{3}{2}$ -tough split graph is Hamiltonian, and that, for every $\varepsilon > 0$, there exist non-Hamiltonian split graphs with toughness $\frac{3}{2} - \varepsilon$.

The former of these two results directly implies the following in our context.

Corollary 3.1. *Every $\frac{3}{2}$ -tough split graph is AP.*

Regarding the latter result of Kratsch *et al.*, it turns out that the construction of split graphs they provide yields graphs that are traceable. Thus, contrarily to their former result, their latter one does not hold as is for non-APness, and we thus have to come up with another construction of non-AP split graphs with “large” toughness.

As mentioned in previous Section 2, one of the most common reasons why a graph is not AP is because it admits no perfect matchings. However, as seen through previous Observation 2.2, having toughness at least 1 guarantees the existence of perfect matchings.

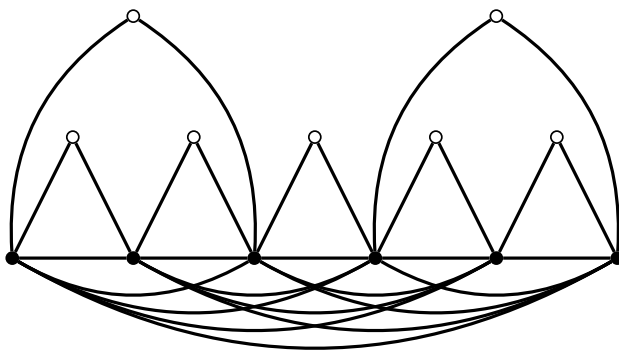


Figure 1: A split graph used to prove Theorem 3.3. The clique contains the black vertices, while the independent set contains the white ones.

Thus, the main difficulty here is that we have to come up with graphs that are not AP because of partitions different from $(2, \dots, 2)$, which, in general, are easier to realise.

Before proceeding, let us mention *en passant* that, in the context of split graphs, the toughness threshold of 1 for perfect matchings is actually best possible. For that, we first need to consider the split graph depicted in Figure 1, which can be used as follows:

Observation 3.2. *Any split graph G of even order containing the graph H in Figure 1 (so that the white vertices remain of degree 2) admits no perfect matching.*

Proof. This follows from Theorem 2.1 since removing the six black vertices of H results in seven connected components, all of which have odd order, namely 1. This implies G has a cut-set S such that $G - S$ has strictly more than $|S|$ connected components of odd order; by Tutte's Theorem, we thus get that G has no perfect matching. \square

Now, for any $k \geq 1$, consider the following construction of a split graph W_k . Start from $k + 2$ disjoint copies H_1, \dots, H_{k+2} of the graph H in Figure 1, then add all possible edges joining black vertices from two distinct H_i 's, and, last, just add k universal (black) vertices to the graph to get W_k . Let us denote by Q the clique formed by these k new vertices. Note that Q and the black vertices of H_1, \dots, H_{k+2} form a clique on $k + 6(k + 2)$ vertices. Actually, W_k is a split graph, the clique K containing Q and the $6(k + 2)$ black vertices of H_1, \dots, H_{k+2} , and the independent set I containing the $7(k + 2)$ white (degree-2) vertices of H_1, \dots, H_{k+2} . In particular, note that $|V(W_k)| = 13(k + 2) + k = 14k + 26$ is even.

This construction has the following properties:

Theorem 3.3. *For every $k \geq 1$, graph W_k admits no perfect matching, and thus is not AP. Furthermore, $\tau(W_k)$ is non-decreasing and tends to 1 as k grows to infinity.*

Proof. Assume, towards a contradiction, that, for some $k \geq 1$, graph $G = W_k$ admits a perfect matching M . Since $|Q| = k$ and W_k contains $k + 2$ copies H_1, \dots, H_{k+2} of H (the graph depicted in Figure 1), there must be $i, j \in \{1, \dots, k + 2\}$ with $i \neq j$ such that no edge of M consists both of a vertex of Q and of a vertex of H_i or H_j . So the graph G' induced by the vertices covered by the edges of M containing at least one vertex of H_i or H_j is a split graph with even order, containing exactly two copies of H , namely H_i and H_j , in which all white vertices have degree 2. In particular, this means the restriction of M to G' is a perfect matching of G' . This stands as a clear contradiction to Observation 3.2; thus, M cannot exist, and G cannot admit perfect matchings, a contradiction.

Now, for any $k \geq 1$, consider a tough cut-set S of $G = W_k$. Since the vertices of Q are universal, clearly S must contain Q . Now, focus on any copy H_i of H in $G - Q$. Since H_i

contains six black vertices all of which have degree at least 7 (in $G - Q$), while all white vertices have degree 2, and only black vertices are connected to the rest of $G - Q$, it should be clear that, besides Q , set S must contain black vertices of H_i only. Now, from H_i 's point of view, it can be observed that removing any black vertex cannot isolate any white vertex, removing any two black vertices isolates at most one white vertex, removing any three black vertices isolates at most three white vertices, removing any four black vertices isolates at most four white vertices, removing any five black vertices isolates at most five white vertices, and removing all six black vertices isolates all seven white vertices. Besides, in any case but the last one, some black vertices of H_i are not removed. Also, note that it is not possible to remove black vertices so that exactly six white vertices get isolated.

We now get to distinguishing whether S contains all black vertices of G .

- If S contains all black vertices, then $G - S$ contains all white vertices. Then

$$\tau(S) = \frac{|S|}{c(G - S)} = \frac{k + 6(k + 2)}{7(k + 2)} = \frac{7k + 12}{7k + 14}$$

which tends to 1 as k grows to infinity.

- If S does not contain all black vertices, then $G - S$ necessarily contains a connected component over all black vertices (and all white vertices adjacent to these) not in S . For every $i \in \{0, \dots, 6\}$, we denote by n_i the number of H_i 's of which S contains exactly i black vertices. Then $|S| = k + n_1 + 2n_2 + 3n_3 + 4n_4 + 5n_5 + 6n_6$, and by arguments above we have

$$\tau(S) = \frac{|S|}{c(G - S)} \geq \frac{k + n_1 + 2n_2 + 3n_3 + 4n_4 + 5n_5 + 6n_6}{n_2 + 3n_3 + 4n_4 + 5n_5 + 7n_6 + 1}.$$

If the right-hand term of the inequality is at least 1, then we have our conclusion. Otherwise, it means $n_6 > k + n_1 + n_2 + 1$, implying, since there are $k + 2$ H_i 's, that S must actually contain all black vertices of G . This is a contradiction.

This concludes the proof. □

We now consider split graphs that are not AP because of partitions different from $(2, \dots, 2)$. This is done through the following construction of a graph G_k . For any $k \geq 1$, start from $4k + 3$ (black) vertices v_1, \dots, v_{4k+3} forming a clique, and, to each v_i of these $4k + 3$ vertices, attach a pendant (white) vertex w_i . Last, add k universal (black) vertices u_1, \dots, u_k to the graph (being our G_k), forming a clique Q on k vertices. Note that $|V(G_k)| = 2(4k + 3) + k = 9k + 6$ is a multiple of 3. Also, G_k is a split graph with clique K containing Q and all v_i 's, and with independent set I containing all w_i 's.

Theorem 3.4. *For every $k \geq 1$, graph G_k admits no realisation of $(3, \dots, 3)$, and thus is not AP. Furthermore, $\tau(G_k)$ is non-decreasing and tends to $\frac{5}{4} = 1.25$ as k grows to infinity.*

Proof. We claim that, for any $k \geq 1$, graph $G = G_k$ admits no realisation of $\pi = (3, \dots, 3)$. Indeed, towards a contradiction, assume this is wrong, and let \mathcal{R} be a realisation of π in G . Note that any part of \mathcal{R} containing some u_i can contain at most two v_i 's and w_i 's. Similarly any part containing some v_i can contain at most two other v_i 's, and at most one pair $\{v_j, w_j\}$ for some $j \neq i$. Since there are only k u_i 's but $4k + 3$ v_i 's and w_i 's, we deduce there are at least $2k + 3$ pairs $\{v_i, w_i\}$ such that v_i and w_i do not belong to the same part X as some vertex of Q . In particular, any such part X is of the $\{w_i, v_i, v_j\}$, where $j \neq i$,

since, in $G - Q$, vertex w_i is only adjacent to v_i , while v_i is only adjacent to w_i and all other v_i 's. For the same reasons, since $N(w_j) = Q \cup \{v_j\}$, this implies w_j must be covered by a part containing a vertex of Q . Since there are at least $2k + 3$ parts X of the form above, and the parts containing the vertices of Q cover only $3k$ vertices of G (k of which being those in Q), we deduce there must be at least one particular part $X = \{w_i, v_i, v_j\}$ such that the part Y containing w_j does not contain any vertex of Q . Thus, $G[Y]$ cannot be connected, which contradicts that \mathcal{R} is a realisation of π . Thus, G is not AP.

Let now S be any tough cut-set of $G = G_k$, for some $k \geq 1$. Note that S must contain Q , since the u_i 's are universal. Now, since the w_i 's have degree 1 in $G - Q$, note that S cannot contain any w_i , and so, besides u_i 's, set S contains only v_i 's.

- If S contains $x \in \{1, \dots, 4k + 2\}$ vertices in $\{v_1, \dots, v_{4k+3}\}$, then note that $c(G - S) = x + 1$ since $G - S$ has a connected component containing all v_i 's (and their adjacent w_i 's) not in S . Precisely, we have

$$\tau(S) = \frac{|S|}{c(G - S)} = \frac{k + x}{x + 1},$$

which function, regardless of k , is non-increasing and thus reaches its minimum for $x = 4k + 2$, for which value we have

$$\tau(S) = \frac{k + (4k + 2)}{(4k + 2) + 1} = \frac{5k + 2}{4k + 3}$$

which tends to $\frac{5}{4}$ as k grows to infinity.

- If S contains all $4k + 3$ v_i 's, then $G - S$ consists in all w_i 's being isolated, and thus $c(G - S) = 4k + 3$. Hence,

$$\tau(S) = \frac{k + (4k + 3)}{4k + 3} = \frac{5k + 3}{4k + 3}$$

which, again, tends to $\frac{5}{4}$ as k grows to infinity.

This concludes the proof. □

4. General graphs, and the construction of Bauer, Broersma, and Veldman

Before proceeding with our main concern in this section, we first observe we can generalise previous Theorem 3.4 to get a similar result (for non-split graphs) regarding other partitions with spectrum of size 1, *i.e.*, of the form $(\lambda, \dots, \lambda)$ for any $\lambda \geq 3$. In particular, upcoming Theorem 4.1 provides a little step towards Conjecture 2.4.

For some fixed $\lambda \geq 3$ and any $k \geq 1$, consider the following graph G_k^λ . Start from $2k(\lambda - 1) + \lambda$ vertices $v_1, \dots, v_{2k(\lambda-1)+\lambda}$ forming a clique, and, to each v_i , attach a new clique W_i on $\lambda - 2$ vertices (that is, W_i is a clique of order $\lambda - 2$, and we have all possible edges between v_i and vertices of W_i). Last, add k universal vertices u_1, \dots, u_k to the graph, forming a clique Q , to achieve the construction of G_k^λ . Note that

$$|V(G_k^\lambda)| = (\lambda - 1)(2k(\lambda - 1) + \lambda) + k = \lambda(2k(\lambda - 1) + \lambda - 2k - 1) + 3k.$$

In particular, $|V(G_k^\lambda)|$ is divisible by λ whenever k is.

Theorem 4.1. *For every $\lambda \geq 3$ and $k \equiv 0 \pmod{\lambda}$, graph G_k^λ admits no realisation of $(\lambda, \dots, \lambda)$, and thus is not AP. Furthermore, $\tau(G_k^\lambda)$ is non-increasing and tends to $\frac{2\lambda-1}{2\lambda-2} > 1$ as k grows to infinity.*

Proof. The proof is essentially the same as that of Theorem 3.4. For k being a multiple of λ , the order of any $G = G_k^\lambda$ is a multiple of λ . Suppose a realisation of $(\lambda, \dots, \lambda)$ in G exists. Then at most k connected parts can contain the u_i 's, and these at most k parts can contain vertices of at most $k(\lambda - 1)$ sets of the form $W_i \cup \{v_i\}$ for some $i \in \{1, \dots, 2k(\lambda - 1) + \lambda\}$. Assume, w.l.o.g., this is the case for all such $W_i \cup \{v_i\}$'s with index in $\{1, \dots, k(\lambda - 1)\}$. Then, there can be at most $k(\lambda - 1)$ indexes $i \notin \{1, \dots, k(\lambda - 1)\}$ such that vertices of $W_i \cup \{v_i\}$ belong to some connected part together with some v_j with $j \in \{1, \dots, k(\lambda - 1)\}$. Indeed, note that for every such i , we have that all vertices of W_i must belong to the same part as v_i (since v_i is a cut-vertex in $G - Q$), and thus, apart from vertices in $W_i \cup \{v_i\}$, the part must contain exactly one other vertex, being some other v_j . Still, w.l.o.g., we can assume all i 's with this property lie in $\{k(\lambda - 1) + 1, \dots, 2k(\lambda - 1)\}$. We now deduce that the graph G' induced by the W_i 's and the v_i 's with index $i \geq 2k(\lambda - 1) + 1$ admits a realisation of $(\lambda, \dots, \lambda)$, which is impossible since all v_i 's are cut-vertices in G' , and the W_i 's have size $\lambda - 2$. So G cannot admit a realisation of $(\lambda, \dots, \lambda)$, and thus cannot be AP.

Regarding the toughness of $G = G_k^\lambda$ (for some $\lambda \geq 3$ and $k \equiv 0 \pmod{\lambda}$), again we have that any cut-set S must contain the k vertices of Q . Now, regarding $G - Q$, again if we assume S is tough, then there is not point having any vertex of some W_i in S . So, besides Q , we have that S only contains v_i 's. If S contains all v_i 's, then

$$\tau(S) = \frac{k + 2k(\lambda - 1) + \lambda}{2k(\lambda - 1) + \lambda} = \frac{k(2\lambda - 1) + \lambda}{k(2\lambda - 2) + \lambda},$$

which tends to $\frac{2\lambda-1}{2\lambda-2}$ as k grows to infinity. Now, if S contains only x v_i 's for some $x \in \{1, \dots, 2k(\lambda - 1) + \lambda - 1\}$, then, again, the worst-case scenario is when $x = 2k(\lambda - 1) + \lambda - 1$, in which case

$$\tau(S) = \frac{k + 2k(\lambda - 1) + \lambda - 1}{2k(\lambda - 1) + \lambda} = \frac{k(2\lambda - 1) + \lambda - 1}{k(2\lambda - 2) + \lambda}$$

which again tends to $\frac{2\lambda-1}{2\lambda-2}$ as k grows to infinity. □

We now move to our main concern in this section. Namely, we consider, in our context, the following construction introduced in [8] by Bauer, Broersma, and Veldman (building upon previous ideas from [7]), from which they established one of the most significant results towards Conjecture 1.1 to date. Let $\ell \geq 1$ and $m \geq 1$ be two positive integers. For a graph H with two vertices x and y , the graph $G(H, x, y, \ell, m)$ is obtained as follows:

- start from m disjoint copies H_1, \dots, H_m of H ;
- then, assuming that, for every $i \in \{1, \dots, m\}$, the copies of vertices x and y in H_i are denoted x_i and y_i , add all possible edges between x_i 's and y_i 's (that is, any x_i becomes adjacent to every other x_j and every y_j , including y_i , and *vice versa*);
- last, add ℓ universal vertices to obtain $G(H, x, y, \ell, m)$.

Bauer *et al.* proved that, provided H , x , y , ℓ , and m satisfy some properties, we can infer that $G(H, x, y, \ell, m)$ is not Hamiltonian, and sometimes not even traceable, all the while having large toughness. In particular, for the graph L depicted in Figure 2 (a) (for which, throughout, we deal with its vertices using the terminology from the figure), they proved:

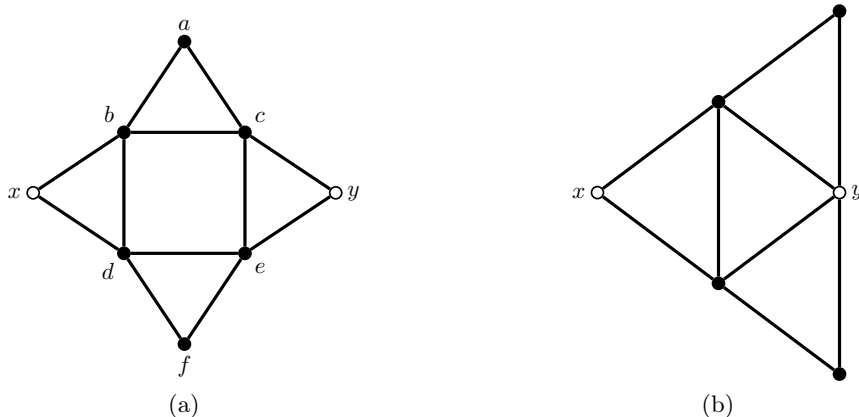


Figure 2: Two graphs introduced in [8]. One of their main properties of interest is that, for each of the two graphs, there is no Hamiltonian path starting and ending in the two white vertices x and y .

Theorem 4.2 (Bauer, Broersma, Veldman [8]). *For every $\ell \geq 2$ and $m \geq 1$, we have*

$$\tau(G(L, x, y, \ell, m)) = \frac{\ell + 4m}{2m + 1}.$$

If $m \geq 2\ell + 3$ then $G(L, x, y, \ell, m)$ is not traceable, while if $m \geq 2\ell + 1$ then $G(L, x, y, \ell, m)$ is not Hamiltonian. So, there are non-traceable graphs with toughness tending to $\frac{9}{4}$.

It seems legitimate to wonder whether the construction of Bauer *et al.* brings something in the context of AP graphs. We prove it is not the case as is; indeed:

Theorem 4.3. *For every $\ell, m \geq 1$, graph $G(L, x, y, \ell, m)$ is AP.*

Proof. Assume ℓ and m are fixed, and set $G = G(L, x, y, \ell, m)$. We denote by L_1, \dots, L_m the m copies of L in G , and, for every $i \in \{1, \dots, m\}$, we denote by $x_i, a_i, b_i, c_i, d_i, e_i, f_i, y_i$ the copies of x, a, b, c, d, e, f, y , respectively, in L_i . Also, we denote by K the clique formed by the ℓ universal vertices of G . Last, we set $n = |V(G)|$.

Let π be an n -partition. Our goal is to build a realisation \mathcal{R} of π in G . For that, we will proceed along the following ideas. First, we will pick consecutive connected parts along L_1, \dots, L_m following that order, in the sense that if, when treating some L_i , we miss vertices to form a connected part, then we will also pick vertices from the next copy L_{i+1} . To that aim, we will make sure, when adding the last vertices of L_i and the first ones of L_{i+1} to a part S , that the vertices of L_i in S induce a connected graph, the vertices of L_{i+1} in S induce a connected graph, and that S contains both y_i and x_{i+1} to guarantee the vertices of both L_i and L_{i+1} in S , altogether, induce a connected graph. Eventually, once all vertices of L_1, \dots, L_m are exhausted, we will then use freely the vertices of K to form connected parts, which can be done at will since K is a clique of universal vertices.

For technical reasons, we first need to take care of the possible connected parts of size 4 of \mathcal{R} , which we do as follows. Note that L admits realisations of $(4, 4)$; for instance, one can consider the two connected parts $\{x, b, a, c\}$ and $\{d, e, f, y\}$. Starting from $i = 1$, as long as \mathcal{R} requires at least two more connected parts of size 4, we then pick two connected parts of size 4 in L_i this way, and then proceed with L_{i+1} . Thus, we can assume we get to the point where all vertices of L_1, \dots, L_{i-1} are covered by connected parts of size 4, and either $i - 1 = m$ and it now suffices to partition K arbitrarily to get the desired \mathcal{R} , or it remains to partition L_i, \dots, L_m and at most one connected part of size 4 must be picked.

When starting dealing with L_i , the last part S we have built when treating L_{i-1} might be missing vertices. We denote by $\delta \geq 0$ the number of such vertices missing in S (to reach

the desired size). As described above, in case $\delta \geq 1$ we assume that S contains y_{i-1} so that L_i contains a vertex, x_i , adjacent to a vertex in S . In case $\delta = 0$, we set $S = \{\}$.

We now consider a few cases:

- If $\delta = 0$, then every connected part built earlier on has reached its desired size. Note that $P = (a_i, c_i, b_i, x_i, d_i, f_i, e_i, y_i)$ is a Hamiltonian path of L_i . In that case, we then pick as many remaining connected parts as possible following P , with the additional care that if there remains exactly one connected part of size 4 to be picked, then we pick it first. This apart, we can pick any remaining connected parts along P . In particular, the last vertices of P might be part of a partial connected part S , that contains y_i as desired, missing $\delta \geq 1$ vertices. In case the last connected part is actually complete, when treating L_{i+1} , we would thus have $S = \{\}$ and $\delta = 0$.
- If $\delta \geq 1$, then we add the following vertices to S .
 - If $\delta = 1$, then we add x_i to S , thereby forming a connected part (due to the edge $y_{i-1}x_i$ and the fact that $S - \{x_i\}$ induces a connected graph by hypothesis) with the desired size. Note then that $P = (a_i, c_i, b_i, d_i, f_i, e_i, y_i)$ forms a Hamiltonian path of what remains of L_i . We can then pick remaining connected parts along P , with the last part, containing y_i , being possibly partial.
 - If $\delta = 2$, then we add x_i and b_i to S to complete the part. We now consider the other connected parts that remain to be picked; in particular, note that it is not possible, since $\delta = 2$, that there remains one of size 4 to pick.
 - * If there remains at least one part of size 1 to be picked, then we pick $\{a_i\}$. Then $P = (d_i, f_i, e_i, c_i, y_i)$ is a Hamiltonian path in what remains to be picked in L_i , and we can again pick remaining connected parts along P , the last one, containing y_i , being possible partial.
 - * If there remains at least one part of size 2 to be picked, then we pick $\{a_i, c_i\}$. Then $P = (d_i, f_i, e_i, y_i)$ is a Hamiltonian path of what remains of L_i , and we can proceed as in the previous case.
 - * If there remains at least one part of size 3 to be picked, then we pick $\{d_i, e_i, f_i\}$. Then $P = (a_i, c_i, y_i)$ is a Hamiltonian path covering the other vertices of L_i , and again we can be done.
 - * If there remains at least one part of size x at least 5 to be picked, then we pick the first x vertices of $(a_i, c_i, d_i, e_i, f_i, y_i)$ to form a connected part. In case we had $x = 5$, then we start picking another remaining connected part from $S = \{y_i\}$ and proceed as earlier. If we had $x = 6$, then, actually, the connected part has reached the desired size, and we continue the process with L_{i+1} from an empty part. If we had $x \geq 7$, then the resulting connected part is partial, and will be completed later on.
 - If $\delta \geq 3$, then we add to S the first vertices of $(x_i, b_i, a_i, d_i, f_i, e_i, c_i, y_i)$, following that order, until either S reaches the desired size, in which case we then continue picking remaining connected parts along the ordering, or we attain a connected part containing y_i , which, if partial, will be completed later on. In all cases, it can be checked that all parts built along that process in L_i are indeed connected.

When adding y_m to a connected part S , we can then add, if necessary, any vertices of K to S so that it reaches the desired size. If, then, other connected parts remain to be picked, then it suffices to pick arbitrary parts with the desired sizes covering the remaining

vertices of K , to achieve the construction of \mathcal{R} . Thus, π is indeed realisable in G , and from this we can conclude that G is indeed AP. \square

Bauer, Broersma, and Veldman also employed the previous construction $G(H, x, y, \ell, m)$ with the building block in Figure 2 (b) as H to establish that, for every $\varepsilon > 0$, there exist chordal non-traceable graphs with toughness $\frac{7}{4} - \varepsilon$ (see [8]). Through a similar proof scheme as in the proof of Theorem 4.3, similarly we can prove these graphs are always AP. Thus, here as well, unfortunately we do not get anything new regarding our concerns.

Despite Theorem 4.3, it is possible to use the construction of Bauer, Broersma, and Veldman, together with one of our constructions from Section 3, to improve Theorem 3.4 by a bit. Our upcoming result relies on the following:

Lemma 4.4. *Let H be a graph with $|V(H)| \equiv 0 \pmod{3}$ admitting no realisation of $(3, \dots, 3)$. Then, for every $\ell \equiv 0 \pmod{3}$ and $m \geq 2\ell + 1$, and any two vertices x and y of H , graph $G(H, x, y, \ell, m)$ admits no realisation of $(3, \dots, 3)$, and thus is not AP.*

Proof. Set $G = G(H, x, y, \ell, m)$ for some $\ell \equiv 0 \pmod{3}$, $m \geq 2\ell + 1$, and any two vertices x and y of H . As in the definition, we denote by H_1, \dots, H_m the m copies of H in G , by z_i the copy of any vertex z of H in the i th copy H_i in G , and by K the clique of G formed by the ℓ universal vertices. Note that we have $|V(G)| \equiv 0 \pmod{3}$.

Towards a contradiction to the claim, assume G admits a realisation \mathcal{R} of $(3, \dots, 3)$. Since $m \geq 2\ell + 1$, all parts of \mathcal{R} contain three vertices each, and $|K| = \ell$, note that there is necessarily an $i \in \{1, \dots, m\}$ such that no vertex of H_i belongs to a connected part together with a vertex of K . Since H_i admits no realisation of $(3, \dots, 3)$ but the order of H_i is a multiple of 3, this means there must be two distinct connected parts, X and Y , of \mathcal{R} , such that $x_i \in X$ and $y_i \in Y$, and X contains one or two vertices (not in K) outside H_i , while similarly Y contains one or two vertices (not in K) outside H_i . More precisely, since $|V(H_i)| \equiv 0 \pmod{3}$, it must be, say, that x_i is the only vertex of H_i in X , while Y contains two vertices of H_i , one being y_i and the second one being some y'_i with $y'_i \notin \{x_i, y_i\}$, such that $y_i y'_i$ is an edge (since Y is a connected part, and the unique vertex of Y outside H_i cannot be a neighbour of y'_i by construction). Then, due to the edge $x_i y_i$, we have that $\{x_i, y_i, y'_i\}$ is a connected part of size 3, and, together with the other parts of \mathcal{R} covering the other vertices of H_i , we get that H_i , and thus H , admits a realisation of $(3, \dots, 3)$, a contradiction. Thus, G cannot admit a realisation of $(3, \dots, 3)$. \square

Now, we have:

Theorem 4.5. *If $\ell \geq 3$ with $\ell \equiv 0 \pmod{3}$, then $G(H, w_1, w_2, \ell, 2\ell + 1)$, where H denotes the graph G_ℓ from Theorem 3.4, admits no realisation of $(3, \dots, 3)$, and thus is not AP. Furthermore, we have*

$$\tau(G(H, w_1, w_2, \ell, 2\ell + 1)) = \frac{10\ell^2 + 8\ell + 1}{8\ell^2 + 6\ell + 2} > \frac{5}{4}.$$

Proof. Set $G = G(H, w_1, w_2, \ell, 2\ell + 1)$, where, in H , recall w_1 and w_2 are two vertices of degree 1 in the graph obtained upon removing all ℓ universal vertices of H . The first part of the statement follows directly from Lemma 4.4, since, by Theorem 3.4, there is no realisation of $(3, \dots, 3)$ in H , as $\ell \equiv 0 \pmod{3}$ and m is basically $2\ell + 1$ in the current case. Thus, we can now focus on proving the second part of the statement.

We denote by K the clique formed by the ℓ universal vertices of G . Also, for every $i \in \{1, \dots, 2\ell + 1\}$, we denote by H_i the i th copy of H in G , and by K_i the clique formed by

the ℓ universal vertices of H_i . For every vertex z of H (being either some u_j , v_j , or w_j), we here denote by z^i its copy in H_i for every $i \in \{1, \dots, 2\ell + 1\}$.

Let S be a tough cut-set of G . Clearly, $K \subset S$. For every $i \in \{1, \dots, 2\ell + 1\}$, we set $S_i = S \cap V(H_i)$ and $s_i = |S_i|$. For every $i \in \{1, \dots, 2\ell + 1\}$, we also denote by \mathcal{C}_i the set of the connected components of $G - S$ that contain only vertices of $V(H_i) \setminus \{w_1^i, w_2^i\}$ (i.e., containing neither w_1^i nor w_2^i) and set $c_i = |\mathcal{C}_i|$. Then

$$\tau(G) = \tau(S) = \frac{\ell + \sum_{i=1}^{2\ell+1} s_i}{r + \sum_{i=1}^{2\ell+1} c_i},$$

where $r = 0$ if $w_1^i, w_2^i \in S$ for every $i \in \{1, \dots, 2\ell + 1\}$, and $r = 1$ otherwise. In particular,

$$\tau(G) = \tau(S) \geq \frac{\ell + \sum_{i=1}^{2\ell+1} s_i}{1 + \sum_{i=1}^{2\ell+1} c_i}.$$

Recall that H contains exactly $4\ell + 3$ v_i 's and w_i 's. In particular, since, for every $i \in \{1, \dots, 2\ell + 1\}$, the largest independent set of $H_i - \{w_1^i, w_2^i\}$ has size $4\ell + 2$ (achieved by the set containing $w_3^i, \dots, w_{4\ell+3}^i$ and, say, v_1^i), then $c_i \leq 4\ell + 2$. Now, for similar arguments as in the proof of Theorem 3.4, it can be noted that, if $c_i \geq 1$, then $s_i \geq \ell + c_i$. Indeed, recall that any cut-set of H must contain the ℓ universal vertices, and that, essentially, having any v_i in the cut-set yields a connected component containing w_i only (assuming w_i is not also part of the cut-set). A particular case here is when $c_i = 4\ell + 2$, which requires S_i to contain K_i , $\{v_3^i, \dots, v_{4\ell+3}^i\}$, and $\{w_1^i, w_2^i\}$; hence, even in that case, $s_i \geq \ell + 4\ell + 3 > \ell + 4\ell + 2 = \ell + c_i$. However, we note that if there is an i such that $c_i = 4\ell + 2$ (and thus $s_i \geq 5\ell + 3$), then we can obtain a cut-set of G tougher than S , a contradiction, by, locally, just replacing S_i with another cut-set S_i' of H_i achieving $s_i = 5\ell + 1$ and $c_i = 4\ell + 1$. For these reasons, we can actually assume further all c_i 's are at most $4\ell + 1$.

Now, since $s_i \geq \ell + c_i$ for every $i \in \{1, \dots, 2\ell + 1\}$, we have

$$\tau(G) = \tau(S) \geq \frac{\ell + \sum_{i=1}^{2\ell+1} s_i}{1 + \sum_{i=1}^{2\ell+1} c_i} \geq \frac{\ell + \ell(2\ell + 1) + \sum_{i=1}^{2\ell+1} c_i}{1 + \sum_{i=1}^{2\ell+1} c_i}.$$

Since $\ell \geq 3$, note that this lower bound on $\tau(S)$ is a non-increasing function of $\sum_{i=1}^{2\ell+1} c_i$ and thus it reaches its minimum when $c_i = 4\ell + 1$ for every $i \in \{1, \dots, 2\ell + 1\}$. Thus

$$\tau(G) = \tau(S) \geq \frac{\ell + \ell(2\ell + 1) + (2\ell + 1)(4\ell + 1)}{1 + (2\ell + 1)(4\ell + 1)} = \frac{10\ell^2 + 8\ell + 1}{8\ell^2 + 6\ell + 2}.$$

Let now S^* be the set of vertices of G containing K , all K_i 's, and, for every $i \in \{1, \dots, 2\ell + 1\}$ and $j \in \{3, \dots, 4\ell + 3\}$, vertex v_j^i . Then, it can be checked that $G - S^*$ contains a connected component containing all w_1^i 's, all w_2^i 's, all v_1^i 's, and all v_2^i 's, while vertex w_j^i is isolated for every $i \in \{1, \dots, 2\ell + 1\}$ and $j \in \{3, \dots, 4\ell + 3\}$. Thus, $|S^*| = \ell + \ell(2\ell + 1) + (2\ell + 1)(4\ell + 1)$, while $c(G - S^*) = 1 + (2\ell + 1)(4\ell + 1)$. So,

$$\tau(G) \leq \tau(S^*) = \frac{\ell + \ell(2\ell + 1) + (2\ell + 1)(4\ell + 1)}{1 + (2\ell + 1)(4\ell + 1)} = \frac{10\ell^2 + 8\ell + 1}{8\ell^2 + 6\ell + 2},$$

and we hence have

$$\tau(G) = \frac{10\ell^2 + 8\ell + 1}{8\ell^2 + 6\ell + 2}$$

as claimed. Since $\ell \geq 3$, this is more than $\frac{5}{4}$. \square

5. Partitioning into few connected parts

For any $k \geq 1$, we say an n -graph G is k -AP if every n -partition $(\lambda_1, \dots, \lambda_k)$ containing exactly k elements is realisable in G . Now, still for any $k \geq 1$, we say G is AP+ k if for any set (v_1, \dots, v_k) of k distinct vertices of G , and for any n -partition $\pi = (\lambda_1, \dots, \lambda_p)$ into $p \geq k$ parts, there is a realisation (V_1, \dots, V_p) of π in G such that $v_i \in V_i$ for every $i \in \{1, \dots, k\}$. Last, for any $k, k' \geq 1$ with $k \geq k'$, we say G is k -AP+ k' if, for any set $(v_1, \dots, v_{k'})$ of k' distinct vertices of G , and for any n -partition $\pi = (\lambda_1, \dots, \lambda_k)$ into k parts, there is a realisation (V_1, \dots, V_k) of π in G such that $v_i \in V_i$ for every $i \in \{1, \dots, k'\}$. In brief, k -APness means we can partition a graph into k connected graphs (regardless of their orders), AP+ k ness means we can partition into at least k connected graphs even under the requirement that k of these must each contain a given vertex, and k -AP+ k' ness is a combination of these two properties (we can partition a graph into a fixed number k of connected graphs, a fixed number k' of which must each contain one particular vertex).

The introduction of these notions was mostly motivated by perhaps one of the most fundamental results when it comes to connected partitions, being the following result proved independently by Gyóri and Lovász, which, following our terminology, reads as follows:

Theorem 5.1 (Gyóri [21], and Lovász [26]). *For every $k \geq 1$, k -connectedness and k -AP+ k ness are equivalent notions.*

Note that k -AP+ k' ness is a particular case of k -APness and of AP+ k' ness. AP+ k ness was introduced and studied more recently in a few works [4, 5], and similarly for k -APness [18]. Complexity aspects behind all these notions were also investigated *e.g.* in [11].

k -APness being a weaker form of APness, one could as well wonder about Conjecture 1.2 for this notion. This yields the following refinement:

Conjecture 5.2. *For every $k \geq 1$, there is some $t_k^* > 0$ such that every t_k^* -tough graph is k -AP.*

Note that, similarly as for Conjecture 1.2, even for more restricted Conjecture 5.2, we mainly get into the problem of constructing non-AP graphs with toughness more than 1, which, again, is not so obvious, recall our arguments from Section 2. Still, in the context of Conjecture 5.2, Theorem 5.1 here yields a positive answer for all values of k :

Observation 5.3. *Conjecture 5.2 holds for every $k \geq 1$.*

Proof. We claim that, for any $k \geq 1$, Conjecture 5.2 holds for $t_k^* = \frac{k}{2}$. Indeed, let G be a $\frac{k}{2}$ -tough graph. It is well-known that, for any $t > 0$, any t -tough graph is $2t$ -connected (as the existence of any cut-set of size at most $2t - 1$ would imply the toughness is at most $\frac{2t-1}{2} < t$, a contradiction). Thus, G is k -connected, and, by Theorem 5.1, we have that G is k -AP (and even k -AP+ k), as desired. \square

Given that Conjecture 5.2 actually holds, a next question could be to determine, for any $k \geq 1$, the smallest $t_k^* > 0$ such that every t_k^* -tough graph is k -AP. Due to the nature of the next results below, we choose to state that question formally as follows.

Question 5.4. For every $k \geq 1$, what is the smallest $t_k^* \geq 0$ such that every graph G with toughness more than t_k^* , i.e., $\tau(G) > t_k^*$, is k -AP?

From now on, every mention to some parameter t_k^* is with respect to Question 5.4, not with respect to Conjecture 5.2. By arguments in the proof of Observation 5.3, we get that $t_k^* < \frac{k}{2}$ for every $k \geq 1$. Going farther in general does not seem that obvious. For small values of $k \geq 1$, however, we can at least raise a few remarks. First, clearly, a graph is 1-AP if and only if it is connected; thus, $t_1^* = 0$. Regarding 2-APness, by Observation 5.3 we have $t_2^* < 1$. Here as well, we can provide an exact value:

Theorem 5.5. We have $t_2^* = \frac{1}{2}$.

Proof. Indeed, let G be a graph with $\tau(G) > \frac{1}{2}$. If G is 2-connected, then G is 2-AP by Theorem 5.1. Otherwise, G has a cut-vertex, which, when removed from G , yields at least two connected components. Thus, in that case, $\tau(G) \leq \frac{1}{2}$, a contradiction. \square

Regarding t_3^* , we note that $K_{2,4}$, the complete bipartite graph with parts of size 2 and 4, has toughness $\frac{2}{4} = \frac{1}{2}$ and admits no perfect matching (realisation of $(2, 2, 2)$); thus $t_3^* \geq \frac{1}{2}$. On the other hand, Observation 5.3 tells us $t_3^* < \frac{3}{2}$. In a private communication, Woźniak suggested that, perhaps, all 1-tough graphs are 3-AP, and thus that $t_3^* < 1$ (see [12] for an earlier mention), which, we think, is an interesting question on its own.

Conjecture 5.6 (Woźniak, see e.g. [12]). All 1-tough graphs are 3-AP; that is, $t_3^* < 1$.

We are not able to come up with arguments answering Conjecture 5.6 in this work, but, in the same line as some results we provided earlier, we can prove that, in general, 1-tough graphs are not 4-AP. Regarding Question 5.4 above, this is sort of similar to considering the other direction, being to wonder, for some $t > 0$, what is the largest $k \geq 1$ such that t -tough graphs are k -AP. Thus, for all 1-tough graphs, we prove they are “at best” 3-AP.

Theorem 5.7. There are arbitrarily large 1-tough graphs that are not 4-AP; that is, $t_4^* \geq 1$.

Proof. For any $k \geq 0$, consider the following construction. Start from five disjoint cliques K_1, \dots, K_5 on $4k + 3$ vertices. For every $i \in \{1, \dots, 5\}$, select any one vertex of K_i , and denote it by u_i . Now turn $\{u_1, \dots, u_5\}$ into a clique, that is, add all possible edges between the u_i 's. Last, add a universal vertex v , and let us call G the resulting graph.

Set $n = |V(G)|$; note that $n = 5(4k+3)+1 = 20k+16$. Thus, $\pi = (5k+4, 5k+4, 5k+4, 5k+4)$ is an n -partition. We claim π is not realisable in G . Assume this is wrong, and suppose G admits a realisation $\mathcal{R} = (V_1, V_2, V_3, V_4)$ of π . As a general observation, note that if some V_i contains some u_j , then, because $|V_i| = 5k + 4 > 4k + 3$ (recall $k \geq 0$), the vertices of K_j must be covered by V_i and the part containing v , which two parts might be the same.

Now, since there are five u_i 's, note that at least one part, say V_1 w.l.o.g., must contain at least two of the u_i 's. If V_1 contains at least three u_i 's, say u_1, u_2 , and u_3 w.l.o.g., then, by a remark above, note that all vertices of K_1, K_2 , and K_3 , must be covered by at most two V_i 's (V_1 and another V_i containing v). However, these at most two parts cover $2(5k+4) = 10k+8$ vertices of G , while, in total, v and the vertices of K_1, K_2 , and K_3 form a set of $3(4k+3) + 1 = 12k+10 > 10k+8$ vertices (recall $k \geq 0$). Thus, this is impossible; so, the V_i 's must contain at most two of the u_i 's each.

We get a similar conclusion if two of the V_i 's each contain two of the u_i 's, say V_1 contains u_1 and u_2 while V_2 contains u_3 and u_4 without loss of generality. Indeed, then, all vertices of K_1, K_2, K_3 , and K_4 , and v , must be covered by at most three parts. At most three parts cover at most $3(5k+4) = 15k+12$ vertices of G , while, here, they must cover at least $4(4k+3) + 1 = 16k+13 > 15k+12$ (recall $k \geq 0$). Thus, again, this is impossible.

So, we must have, say, that V_1 contains u_1 and u_2 , that V_2 contains u_3 , that V_3 contains u_4 , and that V_4 contains u_5 . Then, since $|V_2| = |V_3| = |V_4| = 5k + 4 > 4k + 3$, and because each of V_2 , V_3 , and V_4 contains only one distinct of the u_i 's, we deduce that v must belong to all of V_2 , V_3 , and V_4 , which is another contradiction. Indeed, note that any V_i containing only one of the u_i 's must contain v , which does not comply with the fact that, in the present case, all of V_2 , V_3 , and V_4 have this property.

Thus, G cannot admit any realisation of π , and G is not 4-AP.

It remains to prove that G is 1-tough. Consider S , any tough cut-set of G . Since v is universal, we must have v in S . Now, since the independence number of $G - v$ is 5, we have $c(G - S) \leq 5$. For every $x \in \{2, 3, 4, 5\}$, clearly the minimum number of vertices of $G - v$ that must be removed from $G - v$ to get x connected components is $x - 1$ (one should remove exactly $x - 1$ of the u_i 's). Thus, regardless of x , we have $\tau(G) = \tau(S) = \frac{1+(x-1)}{x} = 1$. \square

6. Conclusions and perspectives

In this work, we have initiated the study of the toughness of AP and non-AP graphs, being motivated mainly by Chvátal's Conjecture 1.1 and by the fact that AP graphs can be perceived as a weakening of Hamiltonian graphs. In a natural way, we thus raised Conjecture 1.2 as an adapted version of Conjecture 1.1. As a first step towards that conjecture, we proved Theorems 3.4 and 4.5, stating, in brief, that the threshold t^* hypothesised in Conjecture 1.2, if it exists, is at least $\frac{5}{4}$. We also studied side questions throughout, for instance generalisations of Tutte's Theorem in Section 2, and, in Section 5, through Conjecture 5.2 and Question 5.4, a restriction of Conjecture 1.2 to graph partitions into a limited, constant number of connected parts only.

Most of the results we provided in the current work actually stand as very first steps towards more general questions. In particular, there remain many aspects of interest which could deserve to be studied further; for instance:

- One of our main results, Theorem 4.5, relies on a construction of Bauer, Broersma, and Veldman from [8], combined together with a graph with certain properties we exhibited through Theorem 3.4. More precisely, the main properties of interest of the latter graph is that it admits no realisation of $(3, \dots, 3)$ and has "large" toughness. Through the same ideas, it would probably be possible to improve upon Theorem 4.5 by coming up with examples of graphs admitting no realisation of $(3, \dots, 3)$ and having larger toughness. Let us mention that, using computer programs, we were not able to spot any small graph (with at most 12 vertices or so) having these properties. These concerns could thus be investigated further on.
- Quite similarly, it can be noticed that Lemma 4.4 could be generalised to graphs admitting no realisation of some $(\lambda, \dots, \lambda)$ for any $\lambda \geq 3$. While we feel confident, for the reasons exposed in Sections 2 and 3, that, towards improving upon Theorem 4.5, one should consider graphs that cannot be partitioned following $(3, \dots, 3)$, perhaps this refinement could be worth investigating further too. Recall that this would also improve upon Theorem 4.1 we provided. On a different note, although constructing graphs with no perfect matchings and large toughness is not interesting enough for our concerns, as explained in Section 2, perhaps one way to go could be to consider partitioning graphs following partitions with spectrum of size more than 1. Regarding all the arguments we provided, perhaps one should consider partitions with spectrum $\{2, 3\}$, or, in other words, partitions into paths of order 2 and 3.

- Regarding split graphs, note that we are still far from understanding Conjecture 1.2 for these. For now, we roughly know that, for these graphs, the lowest t^* lies somewhere in between about $\frac{5}{4}$ (Theorem 3.4) and $\frac{3}{2}$ (Corollary 3.1). We believe tightening these bounds closer could be an interesting challenge. Perhaps a significant result one could invoke here, is the fact that the APness of split graphs relies solely on the realisability of partitions with spectrum lying in $\{1, 2, 3\}$, as proved by Broersma, Kratsch, and Woeginger [15]. That is, if a split n -graph G is not AP, then there is an n -partition containing only 1's, 2's, and 3's that is not realisable in G .
- To progress towards Conjecture 1.2, one could also consider other classes of graphs that have been considered in the context of Conjecture 1.1. For instance, [20] is a recent reference with an up-to-date listing of classes of graphs for which Conjecture 1.1 is known to hold. Although this means Conjecture 1.2 also holds for these classes of graphs, one could wonder whether these results can be improved even further in our context. Following our investigations in Section 3, one could more particularly wonder about superclasses of split graphs, such as chordal graphs and $2K_2$ -free graphs, which have received quite some attention in this context.
- Note that an interesting aspect behind the constructions provided by Kratsch, Lehel, and Müller in [25] and by Bauer, Broersma, and Veldman in [8], which we also got to employ in ours, is the use of universal vertices, which are very important in the definition of graph toughness since these vertices necessarily belong to all cut-sets. In the context of AP graphs, the behaviour of universal vertices has also been investigated, notably in [10], wherein it was proved, among other things, that deciding whether an n -partition is realisable in an n -graph with about a third universal vertices is NP-complete. This leads to the more general question of whether this problem remains NP-complete for graphs with “large” toughness. Note that this general problem was indeed proved to be NP-complete when restricted to several classes of graphs, see *e.g.* [11] and the pointers there, all of which seem to have very low toughness.
- Regarding the thoughts from Section 2, we wonder whether the way we proposed to generalise Tutte’s Theorem from perfect matchings to partitions of the form $(\lambda, \dots, \lambda)$ is plausible. Recall that, in Observation 2.3, we indeed proposed one generalisation of one of the two directions. Although the other direction might seem a bit strong, we were not able to come up with any counterexample. So we wonder whether it also holds, or, more generally speaking, whether Tutte’s Theorem indeed generalises to this setting. If this were to be the case, then it is likely that this could have implications on the study of AP graphs, for the reasons provided in Sections 2 and 3 (in particular, due to the fact that, very generally speaking, if a graph is not AP, then it is likely that it is because of a non-realisable partition with spectrum of size 1). Let us, here, also remind about Conjecture 2.4.
- Last, we think our related investigations in Section 5 are of interest, in particular due to their connection with Theorem 5.1 of Györi and Lovász. One appealing question is Conjecture 5.6, which we believe might be true. This apart, more general Question 5.4 could also be worth studying further.

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