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Research Article

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On a system of multi-component Ginzburg-Landau vortices

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Abstract: We study the asymptotic behavior of solutions for n -component Ginzburg-Landau equations as $\varepsilon \rightarrow 0$. We prove that the minimizers converge locally in any C^k -norm to a solution of a system of generalized harmonic map equations.

Keywords: n -component Ginzburg-Landau equations, semi-local gauge field model, asymptotic behavior of solutions

MSC 2020: 35B40, 35J60, 35Q60

1 Introduction

The classical Ginzburg-Landau model describes the macroscopic theory for phenomena in superconductivity at low temperatures. It is derived from the Helmholtz free energy, which consists of a complex order parameter u and the magnetic potential A [4]. In the physical literature, $|u|^2$ represents the density of the superconducting electron pairs in the superconducting material. The state $|u|^2 = 0$ implies that the material remains in normal conducting state, whereas the material is superconducting if $|u|^2 = 1$.

The Ginzburg-Landau theory also gives a good insight into understanding various topological defects arising from cosmology. The superconducting state is achieved at the vacuum level of the potential, and it can be regarded as a broken symmetry during a phase transition. For instance, this idea is realized by the (special) relativistic extension of the planar Ginzburg-Landau theory, called the Abelian-Higgs model [8]. This model describes a charged scalar field ψ interacting with a $U(1)$ gauge field A and allows vortex solutions that are charged magnetically but electrically neutral.

The Abelian-Higgs model is considered in the $(2+1)$ -dimensional Minkowski space $\mathbb{R}^{1,2}$ with the metric $\text{diag}(-1, 1, 1)$. The metric is used to raise or lower indices. The Lagrangian density for the model is defined as follows:

$$\mathcal{L}_0 = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}D_\mu u \overline{D}^\mu u - \frac{1}{8\varepsilon^2}(|u|^2 - 1)^2. \quad (1.1)$$

Here, $\mu, \nu = 0, 1, 2$, $\varepsilon > 0$ is the Higgs coupling constant, $u : \mathbb{R}^{1,2} \rightarrow \mathbb{C}$ is the Higgs field, and $A_\mu : \mathbb{R}^{1,2} \rightarrow \mathbb{R}$ is $U(1)$ gauge fields. In addition, $F_{\mu\nu} = F_{\mu\nu}^A = \partial_\mu A_\nu - \partial_\nu A_\mu$ represents the electromagnetic field and $D_\mu u = D_\mu^A u = \partial_\mu u + iA_\mu u$ is the covariant derivative. The Lagrangian \mathcal{L}_0 is invariant under the local $U(1)$ gauge transform

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$$u \mapsto e^{i\alpha}u, \quad A_\mu \mapsto A_\mu - \partial_\mu\alpha$$

for any smooth function $\alpha : \mathbb{R}^{1,2} \rightarrow \mathbb{R}$. The Euler-Lagrange equations are

$$\begin{cases} D_\mu D^\mu u = \frac{1}{\varepsilon^2}(|u|^2 - 1)u \\ \partial^\nu F_{\mu\nu} = \frac{i}{2\varepsilon^2}[u\overline{D_\mu u} - \overline{u}D_\mu u], \end{cases} \quad (1.2)$$

which are the (special) relativistic Ginzburg-Landau equations.

As a direct generalization of (1.1), Vachaspati and Achucarro proposed a new model in [12], which contains n scalar fields:

$$\Psi = (u_1, \dots, u_n), \quad \text{where } u_k : \mathbb{R}^{1,2} \rightarrow \mathbb{C} \quad \text{for } k = 1, \dots, n.$$

The Lagrangian \mathcal{L}_0 is modified as follows:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}D_\mu\Psi\overline{D^\mu\Psi} - \frac{1}{8\varepsilon^2}(|\Psi|^2 - n)^2. \quad (1.3)$$

It is easy to show that \mathcal{L} is invariant under the local U(1) gauge transformation

$$\Psi \mapsto e^{i\alpha}\Psi, \quad A_\mu \mapsto A_\mu - \partial_\mu\alpha$$

for any smooth function $\alpha : \mathbb{R}^{1,2} \rightarrow \mathbb{R}$. Indeed, if we write $\Phi = e^{i\alpha}\Psi$ and $B_\mu = A_\mu - \partial_\mu\alpha$, then the invariance of \mathcal{L} follows from the following two identities:

$$\begin{aligned} F_{\mu\nu}^B &= \partial_\mu(A_\nu - \partial_\nu\alpha) - \partial_\nu(A_\mu - \partial_\mu\alpha) = F_{\mu\nu}^A, \\ D_\mu^B\Phi &= \partial_\mu\Phi + iB_\mu\Phi = e^{i\alpha}D_\mu^A\Psi. \end{aligned}$$

It is also invariant under the global SU(n) gauge transformation

$$\Psi \mapsto e^{i\alpha_k\tau^k}\Psi,$$

where $\alpha_k \in \mathbb{R}$ for $k = 1, \dots, n^2 - 1$ and $\{\tau^k\}_{k=1}^{n^2-1}$ are the generators of the Lie algebra $\mathfrak{su}(n)$. The Euler-Lagrange equations are obtained as follows:

$$\begin{cases} D_\mu D^\mu u_k = \frac{1}{\varepsilon^2}\left(\sum_{j=1}^n |u_j|^2 - n\right)u_k, \\ \partial^\nu F_{\mu\nu} = \frac{i}{2\varepsilon^2}\sum_{j=1}^n (u_j\overline{D_\mu u_j} - \overline{u_j}D_\mu u_j), \end{cases} \quad (1.4)$$

which we will refer to as n -component Ginzburg-Landau equations.

Model (1.3) is useful in the study of some issues in cosmology, for instance, the formation of cosmic strings that have both local and global natures [12]. So, the string solutions for (1.3) are called semilocal. The semilocal string solutions for (1.3) are similar to those for (1.1) which reflect the local gauge transformation. But they have additional features that have some resemblance to global defects. For further physical implications of model (1.3), one may refer to the study by Hindmarsh [7] and Vachaspati and Achucarro [12].

In this article, we are interested in static solutions for (1.2) and (1.4). In particular, we assume that the electromagnetic fields vanish, that is, $A_\mu = 0$ for $\mu = 0, 1, 2$. Then, the main topic is to study the asymptotic behavior of solutions for (1.2) and (1.4) as $\varepsilon \rightarrow 0$. For the case (1.2), there have been lots of research on this topic after the seminal work of Bethuel-Brezis-Helein [1,2]. More specifically, let $\Omega \subset \mathbb{R}^2$ be a smooth, bounded, simply connected domain. Given a smooth map

$$g : \partial\Omega \rightarrow S^1 = \{z \in \mathbb{C} : |z| = 1\} \quad \text{with } d = \deg(g, \partial\Omega),$$

the Ginzburg-Landau equations (1.2) with $A_\mu = 0$ reduce to

$$\begin{cases} -\Delta u = \frac{1}{\varepsilon^2}u(1 - |u|^2) & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega. \end{cases} \quad (1.5)$$

The associated energy functional is

$$E_{\varepsilon,\Omega}^b(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 dx + \frac{1}{4\varepsilon^2} \int_{\Omega} (1 - |u|^2)^2 dx. \quad (1.6)$$

Then, $E_{\varepsilon,\Omega}^b$ has a minimizer $u_{\varepsilon,g}^b$ in

$$H_g^1(\Omega; \mathbb{C}) = \{u \in H^1(\Omega; \mathbb{C}) : u = g \text{ on } \partial\Omega\},$$

that is,

$$E_{\varepsilon,\Omega}^b(u_{\varepsilon,g}^b) = \inf\{E_{\varepsilon,\Omega}^b(u) : u \in H_g^1(\Omega; \mathbb{C})\}. \quad (1.7)$$

We often write H_g^1 instead of $H_g^1(\Omega; \mathbb{C})$ if there is no confusion.

The asymptotic behavior of solutions of (1.5) as $\varepsilon \rightarrow 0$ has attracted lots of interest for three decades. The behavior of global minimizers for $u_{\varepsilon,g}^b$ was studied in detail by Bethuel et al. [1,2]. If $d = 0$, $u_{\varepsilon,g}^b$ converges to a harmonic map that minimizes

$$\int_{\Omega} |\nabla u|^2 dx$$

over the space

$$H_g^1(\Omega; S^1) = \{u \in W^{1,2}(\Omega, S^1) : u = g \text{ on } \partial\Omega\}.$$

This problem has a solution u_0^b satisfying

$$\begin{cases} -\Delta u_0^b = u_0^b |\nabla u_0^b|^2 & \text{on } \Omega, \\ u_0^b = g & \text{on } \partial\Omega, \\ |u_0^b| = 1 & \text{on } \Omega. \end{cases} \quad (1.8)$$

When $d \neq 0$, the analysis is more delicate because $H_g^1(\Omega; S^1) = \emptyset$. There exists a set $\{a_1, \dots, a_d\} \subset \Omega$ such that, up to a subsequence, $u_{\varepsilon,g}^b$ converges to the map u_* that satisfies the harmonic map equation [2,6]

$$\begin{cases} -\Delta u_* = u_* |\nabla u_*|^2 & \text{on } \Omega \setminus \{a_1, \dots, a_d\}, \\ u_* = g & \text{on } \partial\Omega, \\ |u_*| = 1 & \text{on } \Omega \setminus \{a_1, \dots, a_d\}. \end{cases} \quad (1.9)$$

Moreover, we have

$$u_*(x) = \frac{x - a_1}{|x - a_1|} \dots \frac{x - a_d}{|x - a_d|} e^{i\psi(x)}, \quad (1.10)$$

where ψ is harmonic on Ω and the singularities a_1, \dots, a_d of u_* minimize the associated renormalized energy. We refer to [3,5,10,11] for the study of the Ginzburg-Landau model with and without a magnetic field that describes superconductivity.

Now, we turn to the n -component Ginzburg-Landau equations (1.4). If we assume $A_\mu = 0$, then (1.4) is reduced to

$$\begin{cases} -\Delta u_i = \frac{1}{\varepsilon^2} u_i \left(n - \sum_{j=1}^n |u_j|^2 \right) & \text{in } \Omega, \\ u_i = g_i & \text{on } \partial\Omega \end{cases} \quad (1.11)$$

for each $i = 1, \dots, n$. One may regard (1.11) as a direct extension of (1.5) to n -component equations. Here, $g_1, \dots, g_n : \partial\Omega \rightarrow S^1$ are smooth maps such that

$$\deg(g_j) := \deg(g_j, \partial\Omega) = d_j \in \mathbb{N} \cup \{0\} \quad \text{for } j = 1, \dots, n. \quad (1.12)$$

The system (1.11) is the Euler-Lagrange equations of the functional

$$E_{\varepsilon, \Omega}(u_1, \dots, u_n) = \frac{1}{2} \int_{\Omega} \sum_{j=1}^n |\nabla u_j|^2 dx + \frac{1}{4\varepsilon^2} \int_{\Omega} \left(n - \sum_{j=1}^n |u_j|^2 \right)^2 dx \quad (1.13)$$

for a pair of maps $(u_1, \dots, u_n) \in H_{g_1}^1(\Omega; \mathbb{C}) \times \dots \times H_{g_n}^1(\Omega; \mathbb{C})$. For simplicity, we write E_ε^b and E_ε instead of $E_{\varepsilon, \Omega}^b$ and $E_{\varepsilon, \Omega}$ if there is no confusion on the domain Ω . It is easy to check that

$$\inf\{E_\varepsilon(u_1, \dots, u_n) : (u_1, \dots, u_n) \in H_{g_1}^1(\Omega; \mathbb{C}) \times \dots \times H_{g_n}^1(\Omega; \mathbb{C})\} \quad (1.14)$$

is achieved by some $(u_{1,\varepsilon}, \dots, u_{n,\varepsilon}) \in H_{g_1}^1(\Omega; \mathbb{C}) \times \dots \times H_{g_n}^1(\Omega; \mathbb{C})$. We also denote by $u_{j,\varepsilon}^b$ a minimizer for E_ε^b on $H_{g_j}^1(\Omega; \mathbb{C})$.

The purpose of this article is to study the asymptotic behavior of minimizers $(u_{1,\varepsilon}, \dots, u_{n,\varepsilon})$ as $\varepsilon \rightarrow 0$. In this study, there is a remarkable difference between (1.5) and (1.11). First, we consider the possible limit equation, which could be an n -component generalization of (1.9) in some sense. For (1.5), if $d \neq 0$, then the limit function u_* in (1.10) has d singularities. However, even for the case $(d_1, \dots, d_n) \neq (0, \dots, 0)$, the limit functions for (1.11) turn out to have no singularities. This difference is related to the nonexistence of singular harmonic map in $H_g^1(\Omega; S^1)$. In fact, we have $H_g^1(\Omega; S^1) = \emptyset$ for the single equation (1.5). For the system (1.11), it is natural to define a function space analogously:

$$\mathcal{X}(g_1, \dots, g_n; \Omega) = \left\{ (u_1, \dots, u_n) \in H_{g_1}^1(\Omega; \mathbb{C}) \times \dots \times H_{g_n}^1(\Omega; \mathbb{C}) : \sum_{j=1}^n |u_j|^2 = n \quad \text{a.e. on } \Omega \right\}.$$

We write $\mathcal{X}(g_1, \dots, g_n)$ if there is no confusion on domains.

The asymptotic behavior of minimizers for E_ε is closely related to the maps in $\mathcal{X}(g_1, \dots, g_n)$. Since it is expected that $|u_{1,\varepsilon}|^2 + \dots + |u_{n,\varepsilon}|^2 \rightarrow n$ as $\varepsilon \rightarrow 0$, the limit functions, say, (u_1^*, \dots, u_n^*) will satisfy $|u_1^*|^2 + \dots + |u_n^*|^2 = n$ a.e. on Ω . Thus, it is important to know whether $\mathcal{X}(g_1, \dots, g_n)$ is nonempty or not. We can see that if $\Psi = (u_1, \dots, u_n) \in \mathcal{X}(g_1, \dots, g_n)$, then

$$\tilde{\Psi} = \left(\frac{u_1}{\sqrt{n}}, \dots, \frac{u_n}{\sqrt{n}} \right) \in S^{2n-1}.$$

Since the homotopy group $\pi_1(S^{2n-1})$ is trivial, there is no topological obstruction, and we expect that the limit problem will not have singularities. More precisely, the next theorem tells us that $\mathcal{X}(g_1, \dots, g_n)$ is nonempty if $n \geq 2$.

Theorem 1.1. *If $d_j \geq 0$ for all $1 \leq j \leq n$, then $\mathcal{X}(g_1, \dots, g_n) \neq \emptyset$. Furthermore, if*

$$I(u_1, \dots, u_n) = \sum_{j=1}^n \int_{\Omega} |\nabla u_j|^2 dx, \quad (1.15)$$

the minimization problem

$$\beta(g_1, \dots, g_n) = \inf\{I(u_1, \dots, u_n) : (u_1, \dots, u_n) \in \mathcal{X}(g_1, \dots, g_n)\} \quad (1.16)$$

is achieved in $\mathcal{X}(g_1, \dots, g_n)$.

Proof. We choose smooth functions $v_j \in H_{g_j}^1(\Omega; \mathbb{C})$ such that $\inf_{\Omega} (|v_1|^2 + \dots + |v_n|^2) > 0$ on Ω . Set

$$w_j = \frac{\sqrt{n} v_j}{(\sum_{k=1}^n |v_k|^2)^{1/2}} \quad \text{for each } j = 1, \dots, n.$$

Then, $(w_1, \dots, w_n) \in \mathcal{X}(g_1, \dots, g_n) \neq \emptyset$. Moreover,

$$\int_{\Omega} |\nabla w_j|^2 dx \leq C \int_{\Omega} \left(\frac{|\nabla v_j|^2}{\sum_k |v_k|^2} + \frac{|v_j|^2 (\sum_k \nabla |v_k|^2)^2}{(\sum_k |v_k|^2)^3} \right) dx \leq C.$$

Thus, $\beta(g_1, \dots, g_n) < \infty$ and it is achieved in $\mathcal{X}(g_1, \dots, g_n)$ by a standard variational argument. \square

The associated Euler-Lagrange equations for (1.15) are

$$\begin{cases} -\Delta u_j = \frac{1}{n} u_j \sum_{k=1}^n |\nabla u_k|^2 & \text{in } \Omega, \quad u_j = g_j \quad \text{on } \partial\Omega, \\ \sum_{j=1}^n |u_j|^2 = n & \text{in } \Omega. \end{cases} \quad (1.17)$$

Indeed, if (u_1, \dots, u_n) is a critical point of I , then we are led to

$$0 = i'(0) = - \sum_{j=1}^n \int_{\Omega} \left[\Delta u_j + \frac{1}{n} u_j \sum_{k=1}^n |\nabla u_k|^2 \right] \psi_j dx$$

by using the fact $\nabla \left(\sum_{j=1}^n |u_j|^2 \right) = 0$. Here, we set $i(t) = I(u_1(t), \dots, u_n(t))$ for $(\psi_1, \dots, \psi_n) \in H_0^1(\Omega) \times \dots \times H_0^1(\Omega)$ and

$$u_j(t) = \frac{n(u_j + t\psi_j)}{\sqrt{n \sum_{k=1}^n |u_k + t\psi_k|^2}} \in \mathcal{X}(g_1, \dots, g_n).$$

We will refer to (1.17) as n -component harmonic map equations. The condition $|u_*| = 1$ in (1.9) makes its zeros singular, whereas the condition $|u_1|^2 + \dots + |u_n|^2 = n$ in (1.17) does not force its zeros to be singular since it does not mean $|u_1| = \dots = |u_n| = 1$. We also note from (1.17) that u_1, \dots, u_n share the same singularities if any.

Now, we are ready to state the first main result of this article.

Theorem 1.2. *Let $(u_{1,\varepsilon}, \dots, u_{n,\varepsilon})$ be a minimizer for (1.14). Then, as $\varepsilon \rightarrow 0$, up to a subsequence,*

$$(u_{1,\varepsilon}, \dots, u_{n,\varepsilon}) \rightarrow (u_1^*, \dots, u_n^*) \text{ in } C^{1,\alpha}(\bar{\Omega}) \times \dots \times C^{1,\alpha}(\bar{\Omega}). \quad (1.18)$$

Here, (u_1^*, \dots, u_n^*) is a minimizer of I defined by (1.15) and satisfies a system (1.17) of harmonic map equations. Moreover, for any positive integer k ,

$$(u_{1,\varepsilon}, \dots, u_{n,\varepsilon}) \rightarrow (u_1^*, \dots, u_n^*) \text{ in } C_{\text{loc}}^k(\Omega) \times \dots \times C_{\text{loc}}^k(\Omega), \quad (1.19)$$

$$\frac{1}{\varepsilon^2} \left(n - \sum_{j=1}^n |u_{j,\varepsilon}|^2 \right) \rightarrow \frac{1}{n} \sum_{j=1}^n |\nabla u_j^*|^2 \text{ in } C_{\text{loc}}^k(\Omega) \times \dots \times C_{\text{loc}}^k(\Omega). \quad (1.20)$$

The key ingredient of the proof is the uniform boundedness of the energy. We will show that although the degrees of the minimizers $(u_{1,\varepsilon}, \dots, u_{n,\varepsilon})$ are not 0, our energy $E_\varepsilon(u_{1,\varepsilon}, \dots, u_{n,\varepsilon})$ is still bounded as $\varepsilon \rightarrow 0$. This situation is very interesting since the energy of the Ginzburg-Landau energy in (1.6) tends to infinity as $\varepsilon \rightarrow 0$. Indeed, it is known that $E_\varepsilon^b(u_{\varepsilon,g}^b)$ grows logarithmically in the limit $\varepsilon \rightarrow 0$ if $\deg(g, \partial\Omega) \neq 0$. See Sandier's study [9] for optimal lower bounds for the energy E_ε^b and many applications.

The convergences (1.19) and (1.20) imply that u_1^*, \dots, u_n^* are smooth. The condition $|u_1^*|^2 + \dots + |u_n^*|^2 = n$ makes each u_j^* smooth at its zeros. It is an interesting question whether $|u_j^*| = 1$ on Ω for some j . This is not true if $d_j > 0$. As the following theorem says, we are not sure that this can happen even for the case $d_j = 0$.

Theorem 1.3. *Let $(u_{1,\varepsilon}, \dots, u_{n,\varepsilon})$ be a minimizing sequence for (1.13).*

(i) *If $d_j > 0$ for some $1 \leq j \leq n$, then we have*

$$\frac{1}{\varepsilon^2} \int_{\Omega} (1 - |u_{j,\varepsilon}|^2)^2 dx \rightarrow \infty.$$

(ii) Suppose that $d_j = 0$ for all $j = 1, \dots, n$. If $\alpha(g_1, \dots, g_n) > \beta(g_1, \dots, g_n)$, then there is some $k \in \{1, \dots, n\}$ such that

$$\frac{1}{\varepsilon^2} \int_{\Omega} (1 - |u_{k,\varepsilon}|^2)^2 dx \rightarrow \infty.$$

Here, $\alpha(g_1, \dots, g_n)$ is defined by Remark 4.2.

(iii) For $n = 2$, if either one of d_1 and d_2 is positive or $\alpha(g_1, g_2) > \beta(g_1, g_2)$, then

$$\frac{1}{\varepsilon^2} \int_{\Omega} (1 - |u_{j,\varepsilon}|^2)^2 dx \rightarrow \infty \quad \text{for each } j = 1, 2.$$

This article is organized as follows. In Sections 2 and 3, we prove Theorem 1.2. The basic idea is based on the study by Bethuel et al. [1]. We deal with interior and boundary estimates, respectively, in Sections 2 and 3. We will focus on how the argument in [1] for single equation (1.5) can be generalized to the system (1.11) nontrivially. In Section 4, we prove Theorem 1.3. We consider another minimization problem for (1.15) on a smaller space and see how it is related to the question that $|u_j^*| = 1$ for some j . We also study some additional properties of solutions for (1.11) and (1.17).

We list some notations and facts that are used hereafter. The vector ν stands for the outward unit normal vector field on a given domain. We write $B_r(x)$ or $B(x, r)$ to denote the ball of radius r centered at point x . We often use the following fact: there exists $\gamma_0 = \gamma_0(\Omega) > 0$ such that

$$\text{meas}(\Omega \cap B_r(x)) \geq \gamma_0 r^2 \quad \forall x \in \Omega, \quad \forall r \leq 1. \quad (1.21)$$

2 Interior estimates

In this section, we will prove Theorem 1.2. In what follows, we often use the following lemma.

Lemma 2.1. [1, Lemma A.1] *Let $-\Delta u = f$ in an open set $\Omega \subset \mathbb{R}^2$. Given any $K \Subset \Omega$, we have*

$$\|\nabla u\|_{L^\infty(K)}^2 \leq C_K \|u\|_{L^\infty(\Omega)} (\|u\|_{L^\infty(\Omega)} + \|f\|_{L^\infty(\Omega)}),$$

where C depends only on Ω and K . In addition, if $u = 0$ on $\partial\Omega$, then

$$\|\nabla u\|_{L^\infty(K)}^2 \leq C \|u\|_{L^\infty(\Omega)} \|f\|_{L^\infty(\Omega)},$$

where C depends only on Ω .

The next lemma provides L^∞ -estimates for $u_{j,\varepsilon}$ and their gradients for $j = 1, \dots, n$.

Lemma 2.2. *Let $(u_{1,\varepsilon}, \dots, u_{n,\varepsilon})$ be any solution of the system (1.11). Then, for $0 < \varepsilon < 1$, we have*

$$\sum_{j=1}^n |u_{j,\varepsilon}|^2 < n \quad \text{on } \Omega, \quad (2.1)$$

$$\|\nabla u_{j,\varepsilon}\|_{L^\infty(\Omega)} \leq \frac{C_0}{\varepsilon} \quad \text{for } j = 1, \dots, n. \quad (2.2)$$

Here, C_0 is a constant that depends only on g_j and Ω .

Proof. Let $f = n - \sum_{j=1}^n |u_{j,\varepsilon}|^2$. Then,

$$\Delta f \leq \frac{2}{\varepsilon^2} \left(\sum_{j=1}^n |u_{j,\varepsilon}|^2 \right) f - 2 \sum_{j=1}^n |\nabla u_{j,\varepsilon}|^2 \leq \frac{2}{\varepsilon^2} \left(\sum_{j=1}^n |u_{j,\varepsilon}|^2 \right) f \quad \text{in } \Omega.$$

Since $f = 0$ on $\partial\Omega$, we have $f > 0$ in Ω by the strong maximum principle.

To show (2.2), let us decompose $u_{j,\varepsilon} = u_{j,\varepsilon}^1 + u_{j,\varepsilon}^2$, where

$$\begin{cases} -\Delta u_{j,\varepsilon}^1 = \frac{1}{\varepsilon^2} u_{j,\varepsilon} (n - \sum_{j=1}^n |u_{j,\varepsilon}|^2) & \text{in } \Omega, \quad u_{j,\varepsilon}^1 = 0 \quad \text{on } \partial\Omega, \\ -\Delta u_{j,\varepsilon}^2 = 0 & \text{in } \Omega, \quad u_{j,\varepsilon}^2 = g_j \quad \text{on } \partial\Omega. \end{cases}$$

Then, $\|\nabla u_{j,\varepsilon}^1\|_{\infty} \leq C/\varepsilon$ by Lemma 2.1 and $\|\nabla u_{j,\varepsilon}^2\|_{\infty} \leq C$ by elliptic estimates for $j = 1, \dots, n$. \square

In the remaining part of this article, we simply write

$$H_{g_1}^1 \times \dots \times H_{g_n}^1 = H_{g_1}^1(\Omega; \mathbb{C}) \times \dots \times H_{g_n}^1(\Omega; \mathbb{C})$$

as long as there is no confusion.

Proposition 2.3. *Let $(u_{1,\varepsilon}, \dots, u_{n,\varepsilon})$ be a minimizer for the problem (1.14). Then,*

$$(u_{1,\varepsilon}, \dots, u_{n,\varepsilon}) \rightarrow (u_1^*, \dots, u_n^*) \quad \text{in } H_{g_1}^1 \times \dots \times H_{g_n}^1 \quad \text{as } \varepsilon \rightarrow 0,$$

where (u_1^*, \dots, u_n^*) is a minimizer for (1.16). Moreover,

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon^2} \int_{\Omega} \left(n - \sum_{j=1}^n |u_{j,\varepsilon}|^2 \right)^2 dx = 0. \quad (2.3)$$

Proof. Let $(\tilde{u}_1, \dots, \tilde{u}_n) \in H_{g_1}^1 \times \dots \times H_{g_n}^1$ be any minimizer for (1.16). Since $\sum_{j=1}^n |\tilde{u}_j|^2 = n$ a.e. on Ω and $E_{\varepsilon}(u_{1,\varepsilon}, \dots, u_{n,\varepsilon}) \leq E_{\varepsilon}(\tilde{u}_1, \dots, \tilde{u}_n)$, we have

$$\frac{1}{2} \int_{\Omega} \sum_{j=1}^n |\nabla u_{j,\varepsilon}|^2 dx + \frac{1}{4\varepsilon^2} \int_{\Omega} \left(n - \sum_{j=1}^n |u_{j,\varepsilon}|^2 \right)^2 dx \leq \frac{1}{2} \int_{\Omega} \sum_{j=1}^n |\nabla \tilde{u}_j|^2 dx. \quad (2.4)$$

Hence,

$$\{(u_{1,\varepsilon}, \dots, u_{n,\varepsilon})\} \text{ is uniformly bounded in } H^1(\Omega; \mathbb{C}) \times \dots \times H^1(\Omega; \mathbb{C}) \quad (2.5)$$

and there exists an n -tuple $(u_1^*, \dots, u_n^*) \in H_{g_1}^1 \times \dots \times H_{g_n}^1$ such that, up to a subsequence,

$$(u_{1,\varepsilon}, \dots, u_{n,\varepsilon}) \rightarrow (u_1^*, \dots, u_n^*)$$

weakly in $H^1 \times \dots \times H^1$ and strongly in $L^p \times \dots \times L^p$ for all $p \geq 1$. Moreover, $\sum_{j=1}^n |u_j^*|^2 = n$ a.e. on Ω by (2.4) and thus $(u_1^*, \dots, u_n^*) \in \mathcal{X}(g_1, \dots, g_n)$. Since $E_{\varepsilon}(u_{1,\varepsilon}, \dots, u_{n,\varepsilon}) \leq E_{\varepsilon}(u_1^*, \dots, u_n^*)$, we obtain

$$\int_{\Omega} \sum_{j=1}^n |\nabla u_{j,\varepsilon}|^2 dx \leq \int_{\Omega} \sum_{j=1}^n |\nabla u_j^*|^2 dx.$$

This implies that

$$\int_{\Omega} \sum_{j=1}^n |\nabla u_{j,\varepsilon} - \nabla u_j^*|^2 dx \leq 2 \int_{\Omega} \sum_{j=1}^n |\nabla u_j^*|^2 dx - 2 \int_{\Omega} \sum_{j=1}^n \nabla u_{j,\varepsilon} \cdot \nabla u_j^* dx \rightarrow 0.$$

So, $(u_{1,\varepsilon}, \dots, u_{n,\varepsilon}) \rightarrow (u_1^*, \dots, u_n^*)$ strongly in $H^1 \times \dots \times H^1$. Taking $\liminf_{\varepsilon \rightarrow 0}$ on (2.4), we are led to

$$\int_{\Omega} \sum_{j=1}^n |\nabla u_j^*|^2 dx \leq \int_{\Omega} \sum_{j=1}^n |\nabla \tilde{u}_j|^2 dx,$$

and we conclude that (u_1^*, \dots, u_n^*) is also a solution for the minimization problem (1.16). Moreover, (2.3) is a direct consequence of (2.4). \square

Lemma 2.4. $\sum_{j=1}^n |u_{j,\varepsilon}|^2 \rightarrow n$ uniformly on $\bar{\Omega}$.

Proof. Fix $x_0 \in \Omega$. By (2.2), we have that for $|x - x_0| < \varepsilon \rho_\varepsilon$,

$$|u_{j,\varepsilon}(x)|^2 \leq (|u_{j,\varepsilon}(x_0)| + C_0 \rho_\varepsilon)^2 \quad \text{for } j = 1, \dots, n.$$

Since $|u_{j,\varepsilon}| \leq \sqrt{n}$ for all $j = 1, \dots, n$, we obtain

$$n - \sum_{j=1}^n |u_{j,\varepsilon}(x)|^2 \geq n - \sum_{j=1}^n |u_{j,\varepsilon}(x_0)|^2 - 2n\sqrt{n}C_0\rho_\varepsilon - nC_0^2\rho_\varepsilon^2 \geq \frac{1}{2} \left(n - \sum_{j=1}^n |u_{j,\varepsilon}(x_0)|^2 \right),$$

where the last inequality holds if and only if

$$a_\varepsilon(x_0) := \frac{1}{2} \left(n - \sum_{j=1}^n |u_{j,\varepsilon}(x_0)|^2 \right) \geq 2n\sqrt{n}C_0\rho_\varepsilon + nC_0^2\rho_\varepsilon^2,$$

in other words,

$$0 < \rho_\varepsilon \leq \frac{-n\sqrt{n} + \sqrt{n^3 + na_\varepsilon(x_0)}}{nC_0} =: b_\varepsilon(x_0).$$

By taking $\rho_\varepsilon = b_\varepsilon$, we deduce from (1.21) and (2.3) that

$$o(1) = \frac{1}{\varepsilon^2} \int_{B_{\varepsilon b_\varepsilon}(x_0) \cap \Omega} \left(n - \sum_{j=1}^n |u_{j,\varepsilon}|^2 \right)^2 dx \geq \gamma_0 a_\varepsilon^2(x_0) b_\varepsilon^2(x_0).$$

Hence, $a_\varepsilon(x_0) \rightarrow 0$ uniformly. \square

Lemma 2.5. $\|(u_{1,\varepsilon}, \dots, u_{n,\varepsilon})\|_{H_{\text{loc}}^2 \times \dots \times H_{\text{loc}}^2} \leq C$.

Proof. We note that

$$\Delta \left(\sum_{j=1}^n |\nabla u_{j,\varepsilon}|^2 \right) = 2 \sum_{j=1}^n |D^2 u_{j,\varepsilon}|^2 + 2 \sum_{j=1}^n \nabla u_{j,\varepsilon} \cdot \nabla (\Delta u_{j,\varepsilon}).$$

Hence, we have

$$\sum_{j=1}^n |D^2 u_{j,\varepsilon}|^2 = \frac{1}{2} \sum_{j=1}^n \Delta (|\nabla u_{j,\varepsilon}|^2) + \frac{1}{\varepsilon^2} \sum_{j=1}^n |\nabla u_{j,\varepsilon}|^2 \left(n - \sum_{i=1}^n |u_{i,\varepsilon}|^2 \right) - \frac{2}{\varepsilon^2} \left(\sum_{j=1}^n u_{j,\varepsilon} \cdot \nabla u_{j,\varepsilon} \right)^2.$$

Since $\sum_{j=1}^n |u_{j,\varepsilon}|^2 \rightarrow n$ uniformly on $\bar{\Omega}$, at least one of $u_{j,\varepsilon}$ satisfies that $|u_{j,\varepsilon}| \geq 1/2$ on Ω for all small ε . If $|u_{k,\varepsilon}| \geq 1/2$ for some $k \in \{1, \dots, n\}$, then

$$\frac{1}{\varepsilon^2} \left(n - \sum_{i=1}^n |u_{i,\varepsilon}|^2 \right) = \frac{|\Delta u_{k,\varepsilon}|}{|u_{k,\varepsilon}|} \leq 2|\Delta u_{k,\varepsilon}|.$$

Hence, we can write

$$\sum_{j=1}^n |D^2 u_{j,\varepsilon}|^2 \leq \frac{1}{2} \sum_{j=1}^n \Delta (|\nabla u_{j,\varepsilon}|^2) + 2 \sum_{j=1}^n |\nabla u_{j,\varepsilon}|^2 |\Delta u_{k,\varepsilon}|,$$

which implies that

$$\frac{1}{2} \sum_{j=1}^n |D^2 u_{j,\varepsilon}|^2 \leq \frac{1}{2} \sum_{j=1}^n \Delta(|\nabla u_{j,\varepsilon}|^2) + C \sum_{j=1}^n |\nabla u_{j,\varepsilon}|^4. \quad (2.6)$$

Fix x_0 and $r = \text{dist}(x_0, \partial\Omega)/4$. Let ζ be a smooth function such that $\zeta = 1$ on $B_r(x_0)$, $\text{supp}\zeta \subset B_{2r}(x_0)$, and $0 \leq \zeta \leq 1$. Multiplying (2.6) by ζ^2 and using the Sobolev embedding $W^{1,1}(\Omega) \hookrightarrow L^2(\Omega)$, we are led to

$$\begin{aligned} \frac{1}{2} \int_{B_{2r}(x_0)} \zeta^2 \sum_{j=1}^n |D^2 u_{j,\varepsilon}|^2 dx &\leq C \int_{B_{2r}(x_0)} \sum_{j=1}^n |\nabla u_{j,\varepsilon}|^2 dx + C \int_{B_{2r}(x_0)} \sum_{j=1}^n (\zeta |\nabla u_{j,\varepsilon}|^2)^2 dx \\ &\leq C \int_{B_{2r}(x_0)} \sum_{j=1}^n |\nabla u_{j,\varepsilon}|^2 dx + C \int_{B_{2r}(x_0)} \zeta \sum_{j=1}^n |\nabla u_{j,\varepsilon}| |D^2 u_{j,\varepsilon}| dx. \end{aligned}$$

In the sequel, by (2.5),

$$\frac{1}{2} \int_{B_{2r}(x_0)} \zeta^2 \sum_{j=1}^n |D^2 u_{j,\varepsilon}|^2 dx \leq C \int_{\Omega} \sum_{j=1}^n |\nabla u_{j,\varepsilon}|^2 dx \leq C.$$

Now, the standard covering argument shows that each $u_{j,\varepsilon}$ is uniformly bounded in $H_{\text{loc}}^2(\Omega)$. \square

Let

$$f_\varepsilon = \frac{1}{\varepsilon^2} \left(n - \sum_{j=1}^n |u_{j,\varepsilon}|^2 \right). \quad (2.7)$$

Then, we can rewrite (1.11) as follows:

$$\begin{cases} -\Delta u_{j,\varepsilon} = f_\varepsilon u_{j,\varepsilon} & \text{in } \Omega, \\ u_{j,\varepsilon} = g_j & \text{on } \partial\Omega. \end{cases} \quad (2.8)$$

A simple calculation provides

$$\begin{cases} -\varepsilon^2 \Delta f_\varepsilon + 2 \sum_{j=1}^n |u_{j,\varepsilon}|^2 f_\varepsilon = 2 \sum_{j=1}^n |\nabla u_{j,\varepsilon}|^2 & \text{in } \Omega, \\ f_\varepsilon = 0 & \text{on } \partial\Omega. \end{cases} \quad (2.9)$$

Given a compact set $K \subset \Omega$, we define

$$\mathcal{A}_K = \{\tilde{K} \subset \Omega \mid K \Subset \tilde{K} \Subset \Omega\}.$$

For $\tilde{K} \in \mathcal{A}_K$, we define a set $\mathcal{F}(K, \tilde{K})$ of smooth functions by

$$\mathcal{F}(K, \tilde{K}) = \{\varphi \in C^\infty(\Omega) \mid 0 \leq \varphi \leq 1, \varphi \equiv 1 \text{ on } K \text{ and } \text{supp}\varphi \subset \tilde{K}\}.$$

In what follows, C_K , or simply C , denotes a generic constant depending on a compact set K but independent of ε . For simplicity, we also often write $\|(u_1, \dots, u_n)\|_{X \times \dots \times X}$ as $\|(u_1, \dots, u_n)\|_X$ for a function space X .

Lemma 2.6. *For any $p \geq 1$, we have $\|(u_{1,\varepsilon}, \dots, u_{n,\varepsilon})\|_{W_{\text{loc}}^{2,p} \times \dots \times W_{\text{loc}}^{2,p}} \leq C$ and $\|f_\varepsilon\|_{C_{\text{loc}}^0} \leq C$.*

Proof. By Lemma 2.4, we may assume that $\sum_{j=1}^n |u_{j,\varepsilon}|^2 \geq 1$ on Ω . Given $K \Subset \Omega$, we denote $K = K_0$ and choose compact sets $K_j \in \mathcal{A}_{K_{j-1}}$ for $j = 1, 2, 3$ and $\varphi \in \mathcal{F}(K_3, K_2)$. Set $F_\varepsilon = \varphi f_\varepsilon$. Then,

$$\begin{aligned} -\varepsilon^2 \Delta F_\varepsilon &= -\varepsilon^2 f_\varepsilon \Delta \varphi - 2\varepsilon^2 \nabla \varphi \cdot \nabla f_\varepsilon - \varepsilon^2 \varphi \Delta f_\varepsilon \\ &= -\left(n - \sum_{j=1}^n |u_{j,\varepsilon}|^2 \right) \Delta \varphi + 4 \sum_{j=1}^n u_{j,\varepsilon} (\nabla \varphi \cdot \nabla u_{j,\varepsilon}) - 2F_\varepsilon \sum_{j=1}^n |u_{j,\varepsilon}|^2 + 2\varphi \sum_{j=1}^n |\nabla u_{j,\varepsilon}|^2 \\ &\leq -2F_\varepsilon + C_K \left(1 + \sum_{j=1}^n |\nabla u_{j,\varepsilon}|^2 \right). \end{aligned}$$

Hence,

$$-\varepsilon^2 \Delta F_\varepsilon + 2F_\varepsilon \leq C_K \left(1 + \sum_{j=1}^n |\nabla u_{j,\varepsilon}|^2 \right) \quad \text{on } K_3. \quad (2.10)$$

For $p > 2$, multiplying (2.10) by F_ε^{p-1} , we are led to

$$\begin{aligned} & \varepsilon^2(p-1) \int_{K_3} |\nabla F_\varepsilon|^2 F_\varepsilon^{p-2} dx + 2 \int_{K_3} F_\varepsilon^p dx \\ & \leq C_K \left(\int_{K_3} \left(1 + \sum_{j=1}^n |\nabla u_{j,\varepsilon}|^2 \right)^p dx \right)^{\frac{1}{p}} \left(\int_{K_3} F_\varepsilon^p dx \right)^{\frac{p-1}{p}} \\ & \leq C_K \|(\mathbf{u}_{1,\varepsilon}, \dots, \mathbf{u}_{n,\varepsilon})\|_{H^2(K_3)} \|F_\varepsilon\|_{L^p(K_3)}^{p-1} \leq C_K \|F_\varepsilon\|_{L^p(K_3)}^{p-1}. \end{aligned}$$

Here, we used Lemma 2.5 in the last inequality. Then, we deduce from Young's inequality that $\|F_\varepsilon\|_{L^p(K_3)} \leq C_K$. This implies by (2.8) that $\|\Delta u_{j,\varepsilon}\|_{L^p(K_2)} \leq C_K$ for any $p > 2$ and $j = 1, \dots, n$. Therefore,

$$\|(\mathbf{u}_{1,\varepsilon}, \dots, \mathbf{u}_{n,\varepsilon})\|_{W^{2,p}(K_1)} \leq C_K$$

by the interior regularity. Moreover, $\|(\mathbf{u}_{1,\varepsilon}, \dots, \mathbf{u}_{n,\varepsilon})\|_{C^1(K_1)} \leq C_K$ by the Sobolev embedding.

Now choose $\tilde{\phi} \in \mathcal{F}(K, K_1)$ and set $\tilde{F}_\varepsilon = \tilde{\phi} f_\varepsilon$. Then, as above, we have

$$-\varepsilon^2 \Delta \tilde{F}_\varepsilon + 2\tilde{F}_\varepsilon \leq C_K \left(1 + \sum_{j=1}^n |\nabla u_{j,\varepsilon}|^2 \right) \quad \text{on } K_1.$$

By applying the maximum principle to this inequality, we are led to

$$\max_{K_1} \tilde{F}_\varepsilon \leq C_K \left(1 + \sum_{j=1}^n \|\nabla u_{j,\varepsilon}\|_{L^\infty(K_1)}^2 \right) \leq C_K.$$

In particular, we conclude that $\|f_\varepsilon\|_{L^\infty(K)} \leq C_K$. □

Lemma 2.7. *Given a nonnegative integer k and a real number $p \geq 1$, we have*

$$\|(\mathbf{u}_{1,\varepsilon}, \dots, \mathbf{u}_{n,\varepsilon})\|_{W_{\text{loc}}^{k+2,p} \times \dots \times W_{\text{loc}}^{k+2,p}} \leq C \quad \text{and} \quad \|f_\varepsilon\|_{C_{\text{loc}}^k} \leq C. \quad (2.11)$$

Proof. We use an induction on k . The case $k = 0$ follows from Lemma 2.6. Suppose that (2.11) is true for k . By the Sobolev embedding, $\|(\mathbf{u}_{1,\varepsilon}, \dots, \mathbf{u}_{n,\varepsilon})\|_{C_{\text{loc}}^{k+1}} \leq C$. By the induction hypothesis, we note from (2.9) that

$$-\varepsilon^2 \Delta \partial^k f_\varepsilon = \partial^k \left(-2 \sum_{j=1}^n |u_{j,\varepsilon}|^2 f_\varepsilon + 2 \sum_{j=1}^n |\nabla u_{j,\varepsilon}|^2 \right) = O(1)$$

on every compact set. Then, by applying Lemma 2.1, we obtain that

$$\|\partial^{k+1} f_\varepsilon\|_{L_{\text{loc}}^\infty} \leq C\varepsilon^{-1}. \quad (2.12)$$

By (2.8) and (2.12), on every compact set,

$$-\Delta \partial^{k+1} u_{j,\varepsilon} = \partial^{k+1} (f_\varepsilon u_{j,\varepsilon}) = O(\varepsilon^{-1}).$$

Hence, by Lemma 2.1 again,

$$\|\partial^{k+2} u_{j,\varepsilon}\|_{L_{\text{loc}}^\infty} \leq C\varepsilon^{-1/2}. \quad (2.13)$$

By (2.12) and (2.13), we obtain

$$-\varepsilon^2 \Delta \partial^{k+1} f_\varepsilon = \partial^{k+1} \left(-2 \sum_{j=1}^n |u_{j,\varepsilon}|^2 f_\varepsilon + 2 \sum_{j=1}^n |\nabla u_{j,\varepsilon}|^2 \right) = O(\varepsilon^{-1})$$

on every compact set. Applying Lemma 2.1 to this equation, we are led to

$$\|\partial^{k+2} f_\varepsilon\|_{L_{\text{loc}}^\infty} \leq C\varepsilon^{-2}. \quad (2.14)$$

Now, given $K \Subset \Omega$, we denote $K = K_0$ and choose compact sets $K_j \in \mathcal{A}_{K_{j-1}}$ for $j = 1, 2, 3$ and $\varphi \in \mathcal{F}(K_2, K_3)$. Set $F_\varepsilon = \varphi \partial^{k+1} f_\varepsilon$. It follows from (2.12) and (2.14) that

$$-\varepsilon^2 \Delta F_\varepsilon = -\varepsilon^2 \partial^{k+1} f_\varepsilon \Delta \varphi - 2\varepsilon^2 \nabla \varphi \cdot \nabla \partial^{k+1} f_\varepsilon - \varepsilon^2 \varphi \Delta \partial^{k+1} f_\varepsilon = O(1) - \varepsilon^2 \varphi \Delta \partial^{k+1} f_\varepsilon.$$

On the other hand, by (2.9),

$$\begin{aligned} -\varepsilon^2 \varphi \Delta \partial^{k+1} f_\varepsilon &= -2 \sum_{j=1}^n |u_{j,\varepsilon}|^2 F_\varepsilon - 2\varphi \sum_{i+l=k+1, l < k+1} \partial^i \left(\sum_{j=1}^n |u_{j,\varepsilon}|^2 \right) \partial^l f_\varepsilon + 2\varphi \partial^{k+1} \left(\sum_{j=1}^n |\nabla u_{j,\varepsilon}|^2 \right) \\ &\leq -2F_\varepsilon + C_K \left(1 + \sum_{j=1}^n |D^{k+2} u_{j,\varepsilon}| \right). \end{aligned}$$

Hence,

$$-\varepsilon^2 \Delta F_\varepsilon + 2F_\varepsilon \leq C_K \left(1 + \sum_{j=1}^n |D^{k+2} u_{j,\varepsilon}| \right) \quad \text{on } K_3. \quad (2.15)$$

For $p > 2$, multiplying (2.15) by F_ε^{p-1} , we are led to

$$\begin{aligned} &\varepsilon^2 (p-1) \int_{K_3} |\nabla F_\varepsilon|^2 F_\varepsilon^{p-2} dx + 2 \int_{K_3} F_\varepsilon^p dx \\ &\leq C_K \left(\int_{K_3} \left(1 + \sum_{j=1}^n |D^{k+2} u_{j,\varepsilon}| \right)^p dx \right)^{\frac{1}{p}} \left(\int_{K_3} F_\varepsilon^p dx \right)^{\frac{p-1}{p}} \\ &\leq C_K \|(\mathbf{u}_{1,\varepsilon}, \dots, \mathbf{u}_{n,\varepsilon})\|_{W^{k+2,p}(K_3)} \|F_\varepsilon\|_{L^p(K_3)}^{p-1} \leq C_K \|F_\varepsilon\|_{L^p(K_3)}^{p-1}, \end{aligned}$$

which implies that $\|F_\varepsilon\|_{L^p(K_3)} \leq C_K$. As a consequence, by (2.8),

$$\|\Delta \partial^{k+1} u_{j,\varepsilon}\|_{L^p(K_2)} \leq C_K \quad \text{for } j = 1, \dots, n.$$

Therefore,

$$\|(\mathbf{u}_{1,\varepsilon}, \dots, \mathbf{u}_{n,\varepsilon})\|_{W^{k+3,p}(K_1)} \leq C_K. \quad (2.16)$$

In particular, $\|\partial^{k+2}(\mathbf{u}_{1,\varepsilon}, \dots, \mathbf{u}_{n,\varepsilon})\|_{C^0(K_1)} \leq C_K$.

Now choose $\tilde{\varphi} \in \mathcal{F}(K, K_1)$ and set $\tilde{F}_\varepsilon = \tilde{\varphi} \partial^{k+1} f_\varepsilon$. As above, we obtain

$$-\varepsilon^2 \Delta \tilde{F}_\varepsilon + 2\tilde{F}_\varepsilon \leq C_K \left(1 + \sum_{j=1}^n |D^{k+2} u_{j,\varepsilon}| \right) \quad \text{on } K_1.$$

Then, we deduce from the maximum principle that

$$\max_{K_1} \tilde{F}_\varepsilon \leq C_K \left(1 + \sum_{j=1}^n \|D^{k+2} u_{j,\varepsilon}\|_{L^\infty(K_1)} \right) \leq C_K.$$

In the sequel, we conclude that

$$\|\partial^{k+1} f_\varepsilon\|_{L^\infty(K)} \leq C_K. \quad (2.17)$$

Therefore, (2.11) is true for $k+1$ by (2.16) and (2.17). This completes the proof. \square

3 Boundary estimates and Proof of Theorem 1.2

Lemma 3.1. For any solution of problem (1.11), there is a constant C such that for all $\varepsilon > 0$,

$$\int_{\partial\Omega} \sum_{j=1}^n \left| \frac{\partial u_{j,\varepsilon}}{\partial \nu} \right|^2 d\sigma \leq C. \quad (3.1)$$

Here, ν is the outward unit normal vector field on $\partial\Omega$.

Proof. For simplicity, we drop the subscript ε . Let us multiply (1.11) by $V \cdot \nabla u_i$, where $V = (V_1, V_2)$ is a smooth vector field on Ω such that $V = \nu$ on $\partial\Omega$. By integrating them by part, we obtain

$$(\text{LHS}) = \int_{\Omega} \Delta u_i (V \cdot \nabla u_i) dx = \int_{\partial\Omega} \left| \frac{\partial u_i}{\partial \nu} \right|^2 d\sigma - \frac{1}{2} \int_{\partial\Omega} |\nabla u_i|^2 d\sigma + O(1)$$

since ∇u_i is uniformly bounded in $L^2(\Omega)$. On the other hand,

$$(\text{RHS}) = \frac{1}{2\varepsilon^2} \int_{\Omega} \left(n - \sum_{j=1}^n |u_j|^2 \right) (V \cdot \nabla |u_i|^2) dx.$$

Adding these for each $i = 1, \dots, n$, we have

$$\begin{aligned} & \sum_{i=1}^n \left(\int_{\partial\Omega} \left| \frac{\partial u_i}{\partial \nu} \right|^2 d\sigma - \frac{1}{2} \int_{\partial\Omega} |\nabla u_i|^2 d\sigma \right) \\ &= \frac{1}{2\varepsilon^2} \int_{\Omega} \left(n - \sum_{j=1}^n |u_j|^2 \right) \sum_{i=1}^n (V \cdot \nabla |u_i|^2) dx + O(1) \\ &= \frac{1}{4\varepsilon^2} \int_{\Omega} \left(n - \sum_{j=1}^n |u_j|^2 \right)^2 (\nabla \cdot V) dx + O(1) = O(1), \end{aligned}$$

where the last inequality comes from (2.4). Combining these identities, we conclude that

$$\sum_{i=1}^n \int_{\partial\Omega} \left| \frac{\partial u_i}{\partial \nu} \right|^2 d\sigma = \sum_{i=1}^n \int_{\partial\Omega} \left| \frac{\partial g_i}{\partial \tau} \right|^2 d\sigma + O(1) = O(1).$$

Here, τ is the tangential vector field on $\partial\Omega$. □

In this section, the estimates of solutions for problem (1.11) would be proved up to the boundary.

Lemma 3.2. $\|(u_{1,\varepsilon}, \dots, u_{n,\varepsilon})\|_{H^2(\Omega) \times \dots \times H^2(\Omega)} \leq C$.

Proof. In view of Lemma 2.5, it suffices to prove uniform H^2 -boundedness near each $x_0 \in \partial\Omega$. We may assume $x_0 = 0$ and change local coordinates $(x_1, x_2) \rightarrow (y_1, y_2) = (x_1, x_2 - h(x_1))$, where h represents $\partial\Omega$ locally and $h'(0) = 0$. We set $\tilde{u}_{i,\varepsilon}(y_1, y_2) = u_{i,\varepsilon}(x_1, x_2)$ and $\tilde{g}_i(y_1, y_2) = g_i(x_1, x_2)$ on $U = \{(y_1, y_2) : y_2 > 0\} \cap B_r(0)$ for some $r > 0$ and rewrite (1.11) as follows:

$$\begin{cases} -L\tilde{u}_{i,\varepsilon} = \frac{1}{\varepsilon^2} \tilde{u}_{i,\varepsilon} \left(n - \sum_{j=1}^n |\tilde{u}_{j,\varepsilon}|^2 \right) & \text{on } U, \\ \tilde{u}_{i,\varepsilon} = \tilde{g}_{i,\varepsilon} & \text{on } \{y_2 = 0\} \cap \partial U. \end{cases}$$

Here,

$$L = \sum_{k,l=1}^2 \frac{\partial}{\partial y_l} \left(a_{kl} \frac{\partial}{\partial y_k} \right) \text{ with } a_{11} = 1, a_{12} = a_{21} = -h', \text{ and } a_{22} = 1 + (h')^2$$

is a strongly elliptic operator if r is small enough. For simplicity, we write $\tilde{u}_{j,\varepsilon}$ as u_j in what follows.

Let $A = \frac{1}{2} \sum_{j=1}^n |\nabla u_j|^2$. By a direct calculation,

$$\begin{aligned} L[A] &= \sum_{j=1}^n \sum_{k,l,r=1}^2 (a_{kl}(u_j)_{y_k y_r} (u_j)_{y_l y_r} + (u_j)_{y_r} \cdot L[(u_j)_{y_r}]) \\ &\geq \alpha \sum_{j=1}^n |D^2 u_j|^2 + \sum_{j=1}^n \sum_{r=1}^2 (u_j)_{y_r} \cdot L[(u_j)_{y_r}], \end{aligned}$$

where α is the ellipticity constant of L . We note that

$$\begin{aligned} \sum_{j=1}^n \sum_{r=1}^2 (u_j)_{y_r} \cdot L[(u_j)_{y_r}] &= \sum_{j=1}^n \sum_{k,l,r=1}^2 (u_j)_{y_r} \cdot \{L[(u_j)_{y_r}] - [(a_{kl})_{y_r} (u_j)_{y_k}]_{y_l}\} \\ &= -\frac{1}{\varepsilon^2} \sum_{j=1}^n |\nabla u_j|^2 \left(n - \sum_{i=1}^n |u_i|^2 \right) + \frac{2}{\varepsilon^2} \sum_{r=1}^2 \left(\sum_{j=1}^n u_j \cdot (u_j)_{y_r} \right) \left(\sum_{i=1}^n u_i \cdot (u_i)_{y_r} \right) \\ &\quad - \sum_{j=1}^n \sum_{k,l,r=1}^2 (u_j)_{y_r} [(a_{kl})_{y_l y_r} (u_j)_{y_k} + (a_{kl})_{y_r} (u_j)_{y_k y_l}] \\ &\geq -\frac{1}{\varepsilon^2} \sum_{j=1}^n |\nabla u_j|^2 \left(n - \sum_{i=1}^n |u_i|^2 \right) - C \sum_{j=1}^n (|\nabla u_j|^2 + |D^2 u_j|^2). \end{aligned}$$

By Lemma 2.4, given $y \in U$, if ε is small, then we can find $k = k(y)$ such that $|u_k(y)| \geq 1/2$. Then,

$$|L[u_k](y)| = \frac{1}{\varepsilon^2} |u_k(y)| \left(n - \sum_{j=1}^n |u_j(y)|^2 \right) \geq \frac{1}{2\varepsilon^2} \left(n - \sum_{j=1}^n |u_j(y)|^2 \right).$$

Hence,

$$\frac{1}{\varepsilon^2} \left(n - \sum_{j=1}^n |u_j(y)|^2 \right) \leq 2|L[u_k](y)| = 2|(a_{kl}(y)(u_k)_{y_k}(y))_{y_l}| \leq C \sum_{j=1}^n (|\nabla u_j(y)| + |D^2 u_j(y)|).$$

As a consequence, by Young's inequality, we are led to

$$-L[A] + \frac{\alpha}{2} \sum_{j=1}^n |D^2 u_j|^2 \leq C \left(1 + \sum_{j=1}^n |\nabla u_j|^4 \right).$$

Let us choose $\zeta \in \mathcal{F}(B_{r/2}(0), B_r(0))$. Then,

$$\frac{\alpha}{2} \sum_{j=1}^n \int_U \zeta^2 |D^2 u_j|^2 dx \leq C + C \sum_{j=1}^n \int_U \zeta^2 |\nabla u_j|^4 dx + \int_U \zeta^2 L[A] dx. \quad (3.2)$$

By integrating by parts, we obtain

$$\begin{aligned} \int_U \zeta^2 L[A] dx &= \int_U \zeta^2 [a_{kl} A_{y_l}]_{y_k} dx \\ &= \int_U AL[\zeta^2] dx + 2 \int_{\{y_2=0\}} a_{12} A(\zeta^2)_{y_1} + \int_{\{y_2=0\}} a_{22} A(\zeta^2)_{y_2} + \int_{\{y_2=0\}} \zeta^2 A(a_{12})_{y_1} - \int_{\{y_2=0\}} \zeta^2 a_{22} A_{y_2}. \end{aligned}$$

The first four terms are uniformly bounded by Lemma 3.1. Furthermore,

$$\begin{aligned} \int_{\{y_2=0\}} \zeta^2 a_{22} A_{y_2} &= \sum_{j=1}^n \left(\int_{\{y_2=0\}} \zeta^2 a_{22} (u_j)_{y_1} \cdot (u_j)_{y_1 y_2} + \zeta^2 a_{22} (u_j)_{y_2} \cdot (u_j)_{y_2 y_2} \right) \\ &= \sum_{j=1}^n ((I) + (II)). \end{aligned}$$

By Lemma 3.1,

$$(I) = - \int_{\{y_2=0\}} ((\zeta^2 a_{22})_{y_1} (u_j)_{y_1} \cdot (u_j)_{y_2} + \zeta^2 a_{22} (g_j)_{y_1 y_1} \cdot (u_j)_{y_2}) = O(1).$$

Since $Lu_j = 0$ on $\{y_2 = 0\}$ and $a_{11} = 1$, we have

$$\begin{aligned} (II) &= - \int_{\{y_2=0\}} \zeta^2 ((u_j)_{y_2} \cdot (g_j)_{y_1 y_1} + (a_{12})_{y_1} |(u_j)_{y_2}|^2 + (a_{21})_{y_2} (u_j)_{y_2} \cdot (g_j)_{y_1} + (a_{22})_{y_2} |(u_j)_{y_2}|^2) - \int_{\{y_2=0\}} \zeta^2 a_{12} (|(u_j)_{y_2}|^2)_{y_1} \\ &= O(1) + \int_{\{y_2=0\}} (\zeta^2 a_{12})_{y_1} |(u_j)_{y_2}|^2 = O(1). \end{aligned}$$

Here, we also used Lemma 3.1. In the sequel, we can rewrite (3.2) as follows:

$$\frac{\alpha}{2} \sum_{j=1}^n \int_U \zeta^2 |D^2 u_j|^2 dx \leq C + C \sum_{j=1}^n \int_U \zeta^2 |\nabla u_j|^4 dx.$$

Now, using this inequality and employing the same argument of the proof of Lemma 2.5, we can show

$$\sum_{j=1}^n \int_{B_{r/2}(x_0)} |D^2 u_j|^2 dx \leq C.$$

This completes the proof. □

Proposition 3.3. *For any $p \geq 1$, we have*

$$\|(u_{1,\varepsilon}, \dots, u_{n,\varepsilon})\|_{W^{2,p}(\Omega) \times \dots \times W^{2,p}(\Omega)} \leq C_p.$$

Moreover, if f_ε is defined by (2.7), then $\|f_\varepsilon\|_{C^0(\Omega)} \leq C$.

Proof. By Lemma 3.2 and the Sobolev embedding,

$$\|(u_{1,\varepsilon}, \dots, u_{n,\varepsilon})\|_{W^{1,p}(\Omega) \times \dots \times W^{1,p}(\Omega)} \leq C \quad \forall p \geq 1.$$

We keep the notation (2.7). By Lemma 2.4 and (2.9), we obtain

$$-2\varepsilon^2 \Delta f_\varepsilon + f_\varepsilon \leq 4 \sum_{j=1}^n |\nabla u_{j,\varepsilon}|^2 \quad \text{on } \Omega. \quad (3.3)$$

For $q > 1$, since $f_\varepsilon = 0$ on $\partial\Omega$, multiplying this equation by f_ε^{q-1} , we have

$$\int_\Omega f_\varepsilon^q dx \leq 4 \sum_{j=1}^n \int_\Omega |\nabla u_{j,\varepsilon}|^2 f_\varepsilon^{q-1} dx \leq 4 \sum_{j=1}^n \|\nabla u_{j,\varepsilon}\|_{2q}^2 \|f_\varepsilon\|_q^{q-1},$$

which implies that $\|f_\varepsilon\|_q \leq C$ for some $C = C(n, q) > 0$. Hence, each $\Delta u_{j,\varepsilon}$ is uniformly bounded in $L^q(\Omega)$ for any $q \geq 1$. By applying the elliptic regularity to (2.8), we obtain

$$\|(u_{1,\varepsilon}, \dots, u_{n,\varepsilon})\|_{C^{1,\alpha}(\Omega) \times \dots \times C^{1,\alpha}(\Omega)} \leq C \quad \forall \alpha \in (0, 1).$$

By the maximum principle, we obtain from (3.3) that $\|f_\varepsilon\|_{C^0(\Omega)} \leq C$. In particular, the right-hand side of (2.8) is uniformly bounded in $C^0(\Omega)$. Hence, each $u_{j,\varepsilon}$ is also uniformly bounded in $W^{2,p}(\Omega)$ for any $p \geq 1$. \square

Proof of Theorem 1.2. The proof of (1.19) follows readily from Propositions 2.3 and 3.3. To show (1.20), let f_ε be defined by (2.7) and set

$$h_\varepsilon = \sum_{j=1}^n |u_{j,\varepsilon}|^2 \quad \text{and} \quad f_* = \frac{1}{n} \sum_{j=1}^n |\nabla u_j^*|^2.$$

Then, $-\Delta u_j^* = u_j^* f_*$ and $\|h_\varepsilon - n\|_{C_{\text{loc}}^k(\Omega)} = \varepsilon^2 \|f_\varepsilon\|_{C_{\text{loc}}^k(\Omega)} \rightarrow 0$. We note that

$$-\sum_{j=1}^n (u_{j,\varepsilon} \Delta u_{j,\varepsilon} - u_j^* \Delta u_j^*) = n(f_\varepsilon - f_*) + f_\varepsilon(h_\varepsilon - n).$$

Hence, as $\varepsilon \rightarrow 0$,

$$\|f_\varepsilon - f_*\|_{C_{\text{loc}}^k(\Omega)} \leq \frac{1}{n} \left(\|f_\varepsilon(h_\varepsilon - n)\|_{C_{\text{loc}}^k(\Omega)} + \sum_{j=1}^n \|u_{j,\varepsilon} \Delta u_{j,\varepsilon} - u_j^* \Delta u_j^*\|_{C_{\text{loc}}^k(\Omega)} \right) \rightarrow 0.$$

This completes the proof. \square

4 Further properties of solutions

In this section, we study some properties of solutions of the n -component Ginzburg-Landau equations (1.11) and the generalized harmonic map equations (1.17). First, the next proposition tells us that if g_j is a rotation of g_1 for each j and ε is not so small, then u_j is a rotation of u_1 for any solution pair (u_1, \dots, u_n) of (1.11). Consequently, each of u_j and u_k is a rotation of the other.

Proposition 4.1. *Let (u_1, \dots, u_n) be a solution of (1.11). Assume that, for each j , there exists $\gamma_j \in \mathbb{R}$ such that $g_j = e^{i\gamma_j} g_1$. Then, $u_j = e^{i\gamma_j} u_1$ for all $\varepsilon > \sqrt{n/\lambda_1}$. Here, λ_1 is the first eigenvalue of $-\Delta$ on Ω with the Dirichlet boundary condition.*

Proof. Set $\tilde{u}_j = e^{-i\gamma_j} u_j$. Then, \tilde{u}_j satisfies

$$\begin{aligned} -\Delta \tilde{u}_j &= \frac{1}{\varepsilon^2} \tilde{u}_j \left(n - \sum_{k=1}^n |\tilde{u}_k|^2 \right) \quad \text{in } \Omega, \\ \tilde{u}_j &= g_1 \quad \text{on } \partial\Omega. \end{aligned}$$

If $w_j = \tilde{u}_j - u_1$, then

$$\begin{aligned} -\Delta w_j + \frac{1}{\varepsilon^2} \sum_{k=1}^n |u_k|^2 w_j &= \frac{n}{\varepsilon^2} w_j, \quad \text{in } \Omega, \\ w_j &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

Hence, it follows that

$$\int_{\Omega} |\nabla w_j|^2 + \frac{1}{\varepsilon^2} \int_{\Omega} \sum_{k=1}^n |u_k|^2 |w_j|^2 = \frac{n}{\varepsilon^2} \int_{\Omega} |w_j|^2 \leq \frac{n}{\lambda_1 \varepsilon^2} \int_{\Omega} |\nabla w_j|^2.$$

If $\varepsilon > \sqrt{n/\lambda_1}$, then w_j is a constant. Since $w_j = 0$ on $\partial\Omega$, we have $w_j = 0$. \square

It is not clear whether the conclusion of Proposition 4.1 is valid for arbitrary ε .

Remark 4.2. Suppose that $d_j = 0$ for all $1 \leq j \leq n$. Since $H_{g_j}^1(\Omega; S^1) \neq \emptyset$, the minimization problem

$$\alpha(g_1, \dots, g_n) = \inf\{I(u_1, \dots, u_n) : u_j \in H_{g_j}^1(\Omega; S^1)\}$$

is achieved by a unique pair (u_1^0, \dots, u_n^0) satisfying

$$\begin{cases} -\Delta u_j^0 = u_j^0 |\nabla u_j^0|^2 & \text{on } \Omega, \\ |u_j^0| = 1 & \text{on } \Omega, \\ u_j^0 = g_j & \text{on } \partial\Omega. \end{cases} \quad (4.1)$$

Since $d_j = 0$ for each $j = 1, \dots, n$, we have that $u_j^0 = e^{i\varphi_j}$ for a harmonic function φ_j in Ω . In general, we have $\alpha(g_1, \dots, g_n) \geq \beta(g_1, \dots, g_n)$ since $(u_1^0, \dots, u_n^0) \in \mathcal{X}(g_1, \dots, g_n)$. An interesting question is whether $\alpha(g_1, \dots, g_n) = \beta(g_1, \dots, g_n)$ or not. The next proposition provides us a necessary condition that they are equal.

Proposition 4.3. *Assume that $d_j = 0$ for all $j \in \{1, \dots, n\}$. If $\alpha(g_1, \dots, g_n) = \beta(g_1, \dots, g_n)$ and it is achieved by (u_1, \dots, u_n) , then $|\nabla u_j| = |\nabla u_k|$ for all $j, k \in \{1, \dots, n\}$.*

Proof. By assumption, $(u_1, \dots, u_n) \in H_{g_1}^1(\Omega; S^1) \times \dots \times H_{g_n}^1(\Omega; S^1)$ and

$$\beta = I(u_1, \dots, u_n).$$

Since $|u_j| = 1$, we can write $u_j = e^{i\varphi_j}$. By plugging this in (1.17), we are led to

$$-i\Delta\varphi_j = \left(\frac{1}{n} - 1\right)|\nabla\varphi_j|^2 + \frac{1}{n} \sum_{k \neq j}^n |\nabla\varphi_k|^2 \quad \text{in } \Omega.$$

Thus, for all $j = 1, \dots, n$,

$$\Delta\varphi_j = 0 \quad \text{and} \quad |\nabla\varphi_j|^2 = \frac{1}{n-1} \sum_{k \neq j}^n |\nabla\varphi_k|^2.$$

In particular,

$$|\nabla\varphi_j|^2 - |\nabla\varphi_1|^2 = \frac{1}{n-1} (|\nabla\varphi_1|^2 - |\nabla\varphi_j|^2),$$

which implies that $|\nabla\varphi_j|^2 = |\nabla\varphi_1|^2$, or equivalently $|\nabla u_j|^2 = |\nabla u_1|^2$ for all $1 \leq j \leq n$. \square

Remark 4.4. Let us consider the case $n = 2$ for simplicity with $d_1 = d_2 = 0$. We can write $u_j = e^{i\varphi_j}$ and $g_j = e^{i\tilde{\varphi}_j}$ for $j = 1, 2$. According to the proof of Proposition 4.3, if $\alpha(g_1, g_2) = \beta(g_1, g_2)$, then it is necessary that

$$\begin{cases} \Delta\varphi_1 = 0 = \Delta\varphi_2 & \text{on } \Omega, \\ |\nabla\varphi_1| = |\nabla\varphi_2| & \text{on } \Omega, \\ \varphi_1 = \tilde{\varphi}_1, \varphi_2 = \tilde{\varphi}_2 & \text{on } \partial\Omega. \end{cases} \quad (4.2)$$

Hence, if there are no solutions for (4.2), we may conclude that $\alpha(g_1, g_2) > \beta(g_1, g_2)$. For instance, if $\tilde{\varphi}_1$ is a constant and $\tilde{\varphi}_2$ is not a constant function, then there are no solution of (4.2). Another example is the case that $\tilde{\varphi}_1 = \lambda\tilde{\varphi}_2$ with $|\lambda| \neq 1$. Indeed, if $\tilde{\varphi}_1 = \lambda\tilde{\varphi}_2$, then

$$-\Delta\left(\frac{\varphi_2}{\lambda}\right) = 0 \quad \text{on } \Omega \quad \text{and} \quad \frac{\varphi_2}{\lambda} = \tilde{\varphi}_1 \quad \text{on } \partial\Omega.$$

Since harmonic maps with the same boundary values are equal, we have $\varphi_1 = \varphi_2/\lambda$. Thus, $|\nabla\varphi_1| = |\nabla\varphi_2|$ only when $|\lambda| = 1$.

As a final subject of this section, we prove Theorem 1.3.

Proof of Theorem 1.3.

(i) Suppose to the contrary that there is $k \in \{1, \dots, n\}$ such that $d_k > 0$ and

$$\frac{1}{\varepsilon^2} \int_{\Omega} (1 - |u_{k,\varepsilon}|^2)^2 dx \leq C.$$

By Theorem 1.2, we have $u_{k,\varepsilon} \rightarrow u_k^* \in H^1(\Omega)$ with $|u_k^*| = 1$. This implies that $u_k^* \in H_{g_k}^1(\Omega; S^1)$, which is impossible since $H_{g_k}^1(\Omega; S^1) = \emptyset$ if $d_k > 0$.

(ii) Suppose that $d_1 = \dots = d_n = 0$. If

$$\frac{1}{\varepsilon^2} \int_{\Omega} (1 - |u_{j,\varepsilon}|^2)^2 dx \leq C \quad \text{for all } j = 1, \dots, n,$$

then $u_{j,\varepsilon} \rightarrow u_j^*$ in H^1 and $|u_j^*| = 1$ for each $j = 1, \dots, n$. Therefore, we have $\alpha(g_1, \dots, g_n) = \beta(g_1, \dots, g_n)$ that contradicts the assumption.

(iii) If we set

$$X_\varepsilon^2 = \frac{1}{\varepsilon^2} \int_{\Omega} (1 - |u_{1,\varepsilon}|^2)^2 dx \quad \text{and} \quad Y_\varepsilon^2 = \frac{1}{\varepsilon^2} \int_{\Omega} (1 - |u_{2,\varepsilon}|^2)^2 dx,$$

we are led by (2.4) and the Cauchy-Schwartz inequality that

$$\begin{aligned} C &\geq \frac{1}{\varepsilon^2} \int_{\Omega} (2 - |u_{1,\varepsilon}|^2 - |u_{2,\varepsilon}|^2)^2 dx \\ &= X_\varepsilon^2 + Y_\varepsilon^2 + \frac{2}{\varepsilon^2} \int_{\Omega} (1 - |u_{1,\varepsilon}|^2)(1 - |u_{2,\varepsilon}|^2) dx \\ &\geq (X_\varepsilon - Y_\varepsilon)^2. \end{aligned}$$

Hence, both X_ε and Y_ε are either bounded or unbounded. By (i) and (ii), at least one of X_ε and Y_ε is unbounded and we obtain the desired conclusion. \square

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