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Yoan Gérard

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# 1 Encoding impredicative hierarchy of type universes 2 with variables

3 Yoan Gérard ✉

4 Mines Paris - PSL, Centre de Recherche en Informatique

5 Université Paris-Saclay, Laboratoire Méthodes Formelles, ENS Paris-Saclay

## 6 — Abstract —

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7 Logical frameworks can be used to translate proofs from a proof system to another one. For this  
8 purpose, we should be able to encode the theory of the proof system in the logical framework. The  
9 Lambda Pi calculus modulo theory is one of these logical frameworks. Powerful theories such as pure  
10 type systems with an infinite hierarchy of universes have been encoded, leading to partial encodings  
11 of proof systems such as Coq, Matita or Agda. In order to fully represent systems such as Coq and  
12 Lean, we introduce a representation of an infinite universe hierarchy with an impredicative universe  
13 and universe variables where universe equivalence is equality, and implement it as a terminating and  
14 confluent rewrite system.

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16 of computation → Type theory

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19 **Supplementary Material** An implementation is available at [https://gitlab.crans.org/geran/  
20 dedukti-level-implementation](https://gitlab.crans.org/geran/dedukti-level-implementation)

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## 23 **1** Introduction

24 The formalization of mathematical theorems and the verification of softwares are done  
25 in several tools, and many logical systems and theories are developed as the research on  
26 proof-checking makes progress. Interoperability is then a big challenge which aims to avoid  
27 the redevelopment of the same proof in each system. Instead of developing translators from  
28 each system to another one, *logical frameworks* propose to define theories in a common  
29 language, which makes translation easier. Thus, the logical framework should be expressive  
30 enough and work should be done to define the wanted theories in the framework.

31 In this paper, our goal is to show how to define the universe levels of the theory of the  
32 COQ proof system in one of these frameworks, the  $\lambda\Pi$ -calculus modulo rewriting. Since  
33 a lot of theories are expressed as extensions of *Pure Type Systems*, the first part of this  
34 introduction will define them. Then, we will present the  $\lambda\Pi$ -calculus modulo rewriting and  
35 the type system behind COQ, in particular the universe levels.

## 36 Pure Type Systems

37 A lot of theories are based on extensions of Church's simply-typed  $\lambda$ -calculus (STLC). In [5],  
38 Barendregt introduced the  $\lambda$ -cube which classifies type systems depending on the possibility  
39 to quantify on types or terms to build new types or new terms. It captures systems such as  
40 System F (with type polymorphism),  $\lambda\omega$  (with type operators),  $\lambda\Pi$  (with dependent types),  
41 and the Calculus of Constructions (CC) which allows all these quantifications.



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## 23:2 Encoding impredicative hierarchy of type universes with variables

42 More generally, these constructions can be extended, leading to more powerful systems  
43 called *Pure Type Systems* [6, 8].

44 ► **Definition 1.** A *Pure Type System (PTS)* is defined by a set of sorts  $\mathcal{S}$  (that we will also  
45 call *universes*), a set of axioms  $\mathcal{A} \subseteq \mathcal{S}^2$  and a set of rules  $\mathcal{R} \subseteq \mathcal{S}^3$ .

46  $\mathcal{A}$  describes the sorts typing ( $s_1$  has the type  $s_2$  when  $(s_1, s_2) \in \mathcal{A}$ ), and  $\mathcal{R}$  describes the  
47 possible quantifications and their typing rules. The terms are the following, where  $s \in \mathcal{S}$  and  
48  $x$  is an element of a countable set of variables  $\mathcal{X}$ .

49  $t := s \mid x \mid \Pi x : t \cdot t \mid (\lambda x : t \cdot t) \mid t t$

50 and the typing rules are given in Figure 1.

$$\begin{array}{c}
 51 \quad (\text{EMPTY}) \frac{}{[] \text{ WF}} \quad (\text{DECL}) \frac{\Gamma \vdash A : s \quad x \notin \Gamma}{\Gamma, x : A \text{ WF}} \quad (\text{VAR}) \frac{\Gamma \text{ WF} \quad (x : A) \in \Gamma}{\Gamma \vdash x : A} \\
 52 \quad (\text{SORT}) \frac{}{\vdash s_1 : s_2} (s_1, s_2) \in \mathcal{A} \quad (\text{PROD}) \frac{\Gamma \vdash A : s_1 \quad \Gamma, x : A \vdash B : s_2}{\Gamma \vdash \Pi x : A \cdot B : s_3} (s_1, s_2, s_3) \in \mathcal{R} \\
 53 \quad (\text{APP}) \frac{\Gamma \vdash t : \Pi x : A \cdot B \quad \Gamma, \vdash u : A}{\Gamma \vdash t u : B[x := u]} \quad (\text{ABS}) \frac{\Gamma, x : A \vdash t : B \quad \Gamma \vdash \Pi x : A \cdot B : s}{\Gamma \vdash \lambda x \cdot t : \Pi x : A \cdot B} \\
 54 \quad (\text{CONV}) \frac{\Gamma \vdash B : s \quad \Gamma \vdash t : A \quad A \equiv_{\beta} B}{\Gamma \vdash t : B} s \in \mathcal{S}
 \end{array}$$

■ **Figure 1** Typing rules

55 ► **Definition 2 (Functional and full PTS).** A *PTS* is said to be *functional* if  $\mathcal{A}$  and  $\mathcal{R}$  are  
56 *functional relations* from  $\mathcal{S}$  and  $\mathcal{S} \times \mathcal{S}$  to  $\mathcal{S}$ , that is to say  $(s_1, s_2) \in \mathcal{A} \wedge (s_1, s_3) \in \mathcal{A} \implies$   
57  $s_2 = s_3$  and  $(s_1, s_2, s_3) \in \mathcal{R} \wedge (s_1, s_2, s_4) \in \mathcal{R} \implies s_3 = s_4$ .

58 A *PTS* is called *full* if  $\mathcal{A}$  and  $\mathcal{R}$  are *total functions* from  $\mathcal{S}$  and  $\mathcal{S} \times \mathcal{S}$  to  $\mathcal{S}$ .

### 59 The $\lambda\Pi$ -calculus modulo rewriting

60  $\lambda\Pi$ , the extension of STLC with dependent types, is the language of the *Edinburgh Logical*  
61 *Framework* (ELF) [22]. However, computation plays an essential role in type theories, then  
62 in modern proof assistant, and  $\lambda\Pi$  is not well-suited for this. To address this point, the  
63  $\lambda\Pi$ -calculus modulo rewriting ( $\lambda\Pi/\equiv$ ) [12] extends  $\lambda\Pi$  by allowing user-defined higher-order  
64 rewrite rules [16, 30] that can be used to define functions but also types. Types are then  
65 identified modulo  $\beta$  and these rewrite rules.

66 In order to have good properties such as the decidability of the type-checking, the rewrite  
67 rules introduced should preserve typing and form a confluent and strongly normalizing rewrite  
68 system, which adds some restrictions and requires more efforts to show that these properties  
69 are respected.

70 ► **Remark 3.** In the rest of this article, we will use the syntax  $u \longrightarrow v$  (where  $u$  may contains  
71 free variables used for matching) to define a rewrite rule and the syntax  $u \hookrightarrow v$  to indicate  
72 that the term  $u$  rewrites to the term  $v$ .

73 The  $\lambda\Pi/\equiv$  can express CC and its subtheories [10], and, in [15], Cousineau and Dowek  
74 show how to embed functional PTS (with a possibly infinite number of symbols and rules):

- 75 1. for each sort  $s$ , symbols  $U_s : \text{Type}$  and  $\text{EL}_s : U_s \rightarrow \text{Type}$ ,
- 76 2. for each axiom  $(s_1, s_2)$ , a symbol  $u_{s_1} : U_{s_2}$  and a rewrite rule  $\text{EL}_{s_2}(s_1) \longrightarrow U_{s_1}$ ,

77 3. for each rule  $(s_1, s_2, s_3)$ , a symbol  $\pi_{(s_1, s_2)}: (x: \mathbf{U}_{s_1}) \rightarrow (\mathbf{El}_{s_1}(x) \rightarrow \mathbf{U}_{s_2}) \rightarrow \mathbf{U}_{s_3}$  and a  
 78 rewrite rule  $\mathbf{El}_{s_3}(\pi_{(s_1, s_2)}(A, B)) \longrightarrow (x: \mathbf{El}_{s_1}(A)) \rightarrow \mathbf{El}_{s_2}(B)$ .  
 79  $\mathbf{u}_s$  corresponds to the sort  $s$  as a term,  $\mathbf{U}_s$  to the sort  $s$  as a type, and  $\mathbf{El}_s$  associates a sort  
 80 (as term) of type  $s$  to its corresponding type, hence the rewrite rule added for each axiom,  
 81 and  $\pi_{(s_1, s_2)}AB$  is the term corresponding to the types of the function from  $A$  to  $B(A)$ , hence  
 82 the rewrite rule added to obtain the type associated to this term.

83 Several systems have been encoded in  $\lambda\Pi/\equiv$ : HOL-LIGHT [31, 2], AGDA [20], MATITA  
 84 [2], but also parts of COQ, on which we will come back to later. Besides, since there exist  
 85 multiple implementations of the  $\lambda\Pi/\equiv$  such as DEDUKTI [3], LAMBDAPI [24], or KONTROLI  
 86 [18], these embeddings have been effectively implemented leading to translations from the  
 87 proofs systems to these implementations, but also to translations from these implementations  
 88 of  $\lambda\Pi/\equiv$  to proof assistants [31, 32].

### 89 Coq's type system

90 The theory of COQ is based on CC extended with an infinite hierarchy of universes and an  
 91 impredicative universe Prop. It corresponds to a slightly different version of the following  
 92 PTS (where Prop is denoted as  $\text{Type}_0$ ).

93 ► **Definition 4** (Impredicative max). We define  $\text{imax}: \mathbb{N} \rightarrow \mathbb{N} \rightarrow \mathbb{N}$  by  $\text{imax}(i, 0) = 0$  and  
 94  $\text{imax}(i, j + 1) = \max(i, j + 1)$ .

95 ► **Definition 5** ( $CC^\infty$ ).  $CC^\infty$  is the full PTS defined with an infinite sequence of sorts  $\text{Type}_i$   
 96 indexed on  $\mathbb{N}$ , the axioms  $(\text{Type}_i, \text{Type}_{i+1})$ , and the rules  $(\text{Type}_i, \text{Type}_j, \text{Type}_{\text{imax}(i, j)})$ .

97 ► **Remark 6.** One can also define a predicative PTS where the products from  $\text{Type}_i$  to  $\text{Type}_j$   
 98 are elements of  $\text{Type}_{\max(i, j)}$  instead of  $\text{Type}_{\text{imax}(i, j)}$ . This latter is the building on AGDA  
 99 proof system while  $CC^\infty$  is the one of COQ but also of LEAN or MATITA.

100 In both cases, the sorts are characterized by *levels* indexed on  $\mathbb{N}$ ; the functions  $\mathcal{R}$  and  $\mathcal{A}$   
 101 can be defined in the  $\lambda\Pi/\equiv$ , and we can adapt the general embedding of Cousineau and  
 102 Dowek to a finite embedding. For that, we define the natural numbers  $\mathbb{N}$  with the successor  
 103 function  $\mathbf{s}, \mathbf{max}: \mathbb{N} \rightarrow \mathbb{N} \rightarrow \mathbb{N}$ ,  $\mathbf{imax}: \mathbb{N} \rightarrow \mathbb{N} \rightarrow \mathbb{N}$ , and

- 104 ■ symbols  $\mathbf{U}: \mathbb{N} \rightarrow \text{Type}$  and  $\mathbf{El}: (i: \mathbb{N}) \rightarrow \mathbf{U}(i) \rightarrow \text{Type}$ ,
- 105 ■ a symbol  $\mathbf{u}: (i: \mathbb{N}) \rightarrow \mathbf{U}(\mathbf{s}(i))$  and a rewrite rule  $\mathbf{El} \_ i \longrightarrow \mathbf{U} i$ ,
- 106 ■ a symbol  $\pi: (i: \mathbb{N}) \rightarrow (j: \mathbb{N}) \rightarrow (A: \mathbf{U} i) \rightarrow (\mathbf{El} i A \rightarrow \mathbf{U} j): \mathbf{U} (\mathbf{imax} i j)$  and a rewrite rule  
 107  $\mathbf{El} \_ (\pi i j A B) \longrightarrow (x: \mathbf{El} i A) \rightarrow \mathbf{El} j (B x)$ .

108 COQ extend  $CC^\infty$  with other features. Some of them have been encoded, leading to a  
 109 partial translator from COQ to DEDUKTI [11, 17], and to the sharing of developements of the  
 110 GeoCoq library [7], a formalization of geometry, to other proof assistants [21]. Extensions  
 111 such as inductive types [13, 28] and cumulativity [27] have been widely covered: Burel and  
 112 Boespflug in [11], then Férey in his thesis [17] proposes embeddings of inductive constructions  
 113 and cumulativity have been studied by Assaf [1, 2], Férey [17] and Thiré [32].

114 In this paper, we are interested in another feature, the level variables, which permits to  
 115 extend  $CC^\infty$  with floating universes [25] or with universe polymorphism [29, 23, 14].

### 116 Level variables

117 We extend the syntax of the levels with variables.

118 ► **Definition 7** (Levels). *A level is a term of the grammar*

$$119 \quad t := 0 \mid s(t) \mid \max(t, t) \mid \text{imax}(t, t) \mid x$$

120 *where  $x$  is an element of a countable set of variables  $\mathcal{X}$ . We denote by  $\mathcal{L}$  the set of the levels,*  
 121 *and we say that  $t$  is a concrete level if  $t$  does not contain any variable.*

122 ► **Definition 8** (Valuation). *A valuation is a function  $\sigma: \mathcal{X} \rightarrow \mathbb{N}$ .*

123 ► **Definition 9.** *Let  $\sigma: \mathcal{X} \rightarrow \mathbb{N}$  be a valuation. We define inductively the value of a level  $t$*   
 124 *over  $\sigma$ , denoted as  $\llbracket t \rrbracket_\sigma$  with*

$$125 \quad \llbracket 0 \rrbracket_\sigma = 0 \quad \llbracket s(t) \rrbracket_\sigma = s(\llbracket t \rrbracket_\sigma) \quad \llbracket x \rrbracket_\sigma = \sigma(x)$$

$$126 \quad \llbracket \max(t_1, t_2) \rrbracket_\sigma = \max(\llbracket t_1 \rrbracket_\sigma, \llbracket t_2 \rrbracket_\sigma) \quad \llbracket \text{imax}(t_1, t_2) \rrbracket_\sigma = \text{imax}(\llbracket t_1 \rrbracket_\sigma, \llbracket t_2 \rrbracket_\sigma)$$

127 We use the same symbol  $s$ ,  $\max$ , and  $\text{imax}$  for the syntax of the levels and the functions  
 128 of the natural numbers. However, levels are abstract terms and are interpreted through  
 129 valuations. Besides, the concrete levels can clearly be identified as the natural numbers which  
 130 justifies the use of the same symbol and permits to see the interpretation as a function that  
 131 *concretizes* a level, turning it into a concrete level.

132 ► **Definition 10** (Level comparison). *Let  $t_1, t_2 \in \mathcal{L}$ . We say that  $t_1 \leq_{\mathcal{L}} t_2$  if for all valuations*  
 133  *$\sigma$ ,  $\llbracket t_1 \rrbracket_\sigma \leq \llbracket t_2 \rrbracket_\sigma$ . In the same way, we say that  $t_1 =_{\mathcal{L}} t_2$  if for all valuations  $\sigma$ ,  $\llbracket t_1 \rrbracket_\sigma = \llbracket t_2 \rrbracket_\sigma$ .*  
 134 *Hence  $t_1 =_{\mathcal{L}} t_2$  if and only if  $t_1 \leq_{\mathcal{L}} t_2$  and  $t_2 \leq_{\mathcal{L}} t_1$ .*

135 With level variables, the equivalence is no more the syntactic equality and the above  
 136 embedding does not reflect it anymore:  $\max(x, y)$  and  $\max(y, x)$  are not convertible and adding  
 137 rules for that would lead to a non-terminating system. In the same way, commutativity and  
 138 equivalences such as  $\max(x, x) =_{\mathcal{L}} x$  or  $\max(s(x), x) =_{\mathcal{L}} s(x)$  are hard to express, and the  
 139 impredicativity introduces other:  $\text{imax}(x, x) =_{\mathcal{L}} x$ ,  $\max(\text{imax}(x, y), x) =_{\mathcal{L}} \max(x, y)$ , etc.

140 And yet, a correct embedding of the levels should reflect these equivalences. For instance,  
 141 a term of the universe  $\text{Type}_x$  is also a term of the universe  $\text{Type}_{\text{imax}(x, x)}$ , and then we should  
 142 be able to identify the universes such as  $\text{Type}_x$  and  $\text{Type}_{\text{imax}(x, x)}$ . This paper presents a  
 143 new embedding that faithfully represents levels with variables.

## 144 Related work

145 Some solutions have been studied in the predicative case. The big issue is the associativity and  
 146 commutativity of the  $\max$  symbol. In [20], Genestier solved this problem to encode AGDA's  
 147 universe polymorphism. For that, he used rewriting modulo associativity and commutativity  
 148 (AC). The idea, also mentioned in a draft of Voevodsky [33], is to represent each level  
 149 as  $\max(n, n_1 + x_1, \dots, n_k + x_k)$  where  $n \geq \max(n_1, \dots, n_k)$ . Besides, if there exists  $i \neq j$   
 150 such that  $x_j = x_i$ , we simplify the term and keep only  $\max(n_i, n_j) + x_i$ . Then, we obtain a  
 151 minimal representation of terms of the  $\max$ -successor algebra.

152 Blanqui gives another presentation of this algebra in [9], with an encoding without  
 153 matching modulo AC. However, this solution requires to keep the level in some AC canonical  
 154 form, and can then require the modification of the  $\lambda\Pi/\equiv$  type-checker.

155 The  $\text{imax}$ -successor algebra is less studied and we do not know easy ways to reflect its  
 156 equalities. A confluent encoding is proposed in [4], but it does not fully reflect the equalities;  
 157 for instance, the levels  $\max(\text{imax}(x, y), x)$  and  $\max(x, y)$  are not convertible. Besides, Férey  
 158 designed a non-confluent encoding of universe polymorphism in [17].

## 159 Contribution and outline

160 We introduce a new representation for the levels, using the idea presented above in the  
 161 predicative case: find a set of subterms such that any level can be expressed as a maximum  
 162 of subterms. They should be easily comparable to simplify  $\max(u, v)$  into  $u$  if  $u \leq_{\mathcal{L}} v$  and  
 163 obtain minimal representations, and they should ensure the uniqueness property:

$$164 \quad \max(u_1, \dots, u_n) =_{\mathcal{L}} \max(v_1, \dots, v_m) \iff \{u_1, \dots, u_n\} = \{v_1, \dots, v_m\}.$$

165 Intuitively, the subterms should be very basic and simple: a subterm  $u$  must not be equivalent  
 166 to a maximum of other subterms. With this representation, we obtain a deep understanding  
 167 of the imax-successor algebra, and an easy procedure-decision for the level inequality problem.

168 In the Section 2, we study the semantic of the imax operator and establish a suitable  
 169 set of subterms. Then, in the Section 3, we introduce the minimal representation and  
 170 show that equivalent terms have the same minimal representation. And the Section 4 is  
 171 dedicated to the implementation of this representation into the  $\lambda\Pi/ \equiv$  as a first-order  
 172 confluent and terminating rewrite system. An implementation in DEDUKTI is available on  
 173 <https://gitlab.crans.org/geran/dedukti-level-implementation>.

## 174 2 Universe representation in impredicative hierarchy

175 In this section, we study the imax operators and its interaction with max and the successor  
 176 and establish semantic equalities that permits to simplify the levels in order to find a set of  
 177 sublevels for the desired representation.

### 178 2.1 Levels as maximum

179 The very first step is to show that any level can be expressed as a maximum of levels that do  
 180 not contain any max, that is the principle of our idea of representation. The successor can be  
 181 distributed over max, the two next propositions show how to distribute imax over max.

182 ► **Proposition 11.** *For all  $u, v, w \in \mathcal{L}$ ,  $\text{imax}(u, \max(v, w)) =_{\mathcal{L}} \max(\text{imax}(u, v), \text{imax}(u, w))$ .*

183 **Proof.** Let  $\sigma$  be a valuation,  $t = \text{imax}(u, \max(v, w))$ ,  $t_1 = \text{imax}(u, v)$  and  $t_2 = \text{imax}(u, w)$ .

- 184 ■ If  $\llbracket v \rrbracket_{\sigma} = \llbracket w \rrbracket_{\sigma} = 0$ , then  $\llbracket \max(t_1, t_2) \rrbracket_{\sigma} = 0 = \llbracket t \rrbracket_{\sigma}$ .
- 185 ■ If  $\llbracket v \rrbracket_{\sigma} \neq 0$  and  $\llbracket w \rrbracket_{\sigma} = 0$ , then  $\llbracket \max(t_1, t_2) \rrbracket_{\sigma} = \max(\llbracket u \rrbracket_{\sigma}, \llbracket v \rrbracket_{\sigma}) = \llbracket t \rrbracket_{\sigma}$ .
- 186 ■ If  $\llbracket v \rrbracket_{\sigma} = 0$  and  $\llbracket w \rrbracket_{\sigma} \neq 0$ , then  $\llbracket \max(t_1, t_2) \rrbracket_{\sigma} = \max(\llbracket u \rrbracket_{\sigma}, \llbracket w \rrbracket_{\sigma}) = \llbracket t \rrbracket_{\sigma}$ .
- 187 ■ Else,  $\llbracket \max(t_1, t_2) \rrbracket_{\sigma} = \max(\llbracket u \rrbracket_{\sigma}, \llbracket v \rrbracket_{\sigma}, \llbracket w \rrbracket_{\sigma}) = \llbracket t \rrbracket_{\sigma}$ .

188 ◀

189 ► **Proposition 12.** *For all  $u, v, w \in \mathcal{L}$ ,  $\text{imax}(\max(u, v), w) =_{\mathcal{L}} \max(\text{imax}(u, w), \text{imax}(v, w))$ .*

190 **Proof.** Let  $\sigma$  be a valuation.

- 191 ■ If  $\llbracket w \rrbracket_{\sigma} = 0$ , then  $\llbracket \max(\text{imax}(u, w), \text{imax}(v, w)) \rrbracket_{\sigma} = 0 = \llbracket \text{imax}(\max(u, v), w) \rrbracket_{\sigma}$ .
- 192 ■ Else,  $\llbracket \max(\text{imax}(u, w), \text{imax}(v, w)) \rrbracket_{\sigma} = \max(\llbracket u \rrbracket_{\sigma}, \llbracket v \rrbracket_{\sigma}, \llbracket w \rrbracket_{\sigma}) = \llbracket \text{imax}(\max(u, v), w) \rrbracket_{\sigma}$ .

193 ◀

194 Then, any level can then be expressed as a maximum of levels without max. Note that  
 195 for this, we consider that max takes a set of levels as argument. We obtain this theorem.

196 ► **Theorem 13.** *For all  $t \in \mathcal{L}$ , there exists  $u_1, \dots, u_n$  in the grammar  $t := 0 \mid s(t) \mid \text{imax}(t, t) \mid$   
 197  $x$  such that  $t =_{\mathcal{L}} \max(u_1, \dots, u_n)$ .*

198 **2.2 Simplification of the levels**

199 We can now focus on levels without maximum. The uniqueness property sought for the  
200 representation requires the subterms to be very basic, and then search to simplify the levels.

201 The main issue is  $\text{imax}$ : its asymmetry complicates its interaction with other symbols.  
202 The previous equalities show how to remove the interaction between  $\text{imax}$  and  $\text{max}$ , now, we  
203 will study the interactions between  $\text{imax}$  and the other symbols. The goal is to restrict the  
204 localisation of the  $\text{imax}$  symbols to specific parts of the levels in order to understand and  
205 control their influence on the levels semantic.

206 Firstly, we recall these equalities that are direct consequences of the semantic of  $\text{imax}$ .  
207 They permit to deal with 0 and the successor.

208 ► **Proposition 14.** *For all  $u, v \in \mathcal{L}$ ,  $\text{imax}(u, 0) =_{\mathcal{L}} 0$  and  $\text{imax}(u, s(v)) = \text{max}(u, s(v))$ .*

209 And we show how to remove  $\text{imax}$  symbol in second argument of  $\text{imax}$ .

210 ► **Proposition 15.** *For all  $u, v, w \in \mathcal{L}$ ,  $\text{imax}(u, \text{imax}(v, w)) =_{\mathcal{L}} \text{max}(\text{imax}(u, w), \text{imax}(v, w))$ .*

211 **Proof.** Let  $\sigma$  be a valuation.

212 ■ If  $\llbracket w \rrbracket_{\sigma} = 0$ , then  $\llbracket \text{max}(\text{imax}(u, w), \text{imax}(u, w)) \rrbracket_{\sigma} = 0 = \llbracket \text{imax}(u, \text{imax}(v, w)) \rrbracket_{\sigma}$ .

213 ■ Else,  $\llbracket \text{max}(\text{imax}(u, w), \text{imax}(v, w)) \rrbracket_{\sigma} = \text{max}(\llbracket u \rrbracket_{\sigma}, \llbracket v \rrbracket_{\sigma}, \llbracket w \rrbracket_{\sigma}) = \llbracket \text{imax}(u, \text{imax}(v, w)) \rrbracket_{\sigma}$ .

214 ◀

215 Thus, we can consider that the second argument of a  $\text{imax}$  is always a variable. It is  
216 more complicated to directly enforce the form of its first argument, but we can obtain one  
217 restriction by distributing the successor over the  $\text{imax}$ . However, we cannot do it as directly  
218 as we distribute the successor over the  $\text{max}$ , as shown in the next example.

219 ► **Example 16.** We show that  $s(\text{imax}(y, x)) \neq_{\mathcal{L}} \text{imax}(s(y), s(x))$  by considering a valuation  
220  $\sigma$  such that  $\sigma(x) = 0$  and  $\sigma(y) = 1$ .

221 ► **Proposition 17.** *For all  $u, v, w \in \mathcal{L}$ ,  $s(\text{imax}(v, w)) =_{\mathcal{L}} \text{max}(s(w), \text{imax}(s(v), w))$ .*

222 **Proof.** Let  $\sigma$  be a valuation.

223 ■ If  $\llbracket w \rrbracket_{\sigma} = 0$ , then  $\llbracket s(\text{imax}(v, w)) \rrbracket_{\sigma} = s(0) = \llbracket \text{max}(s(w), \text{imax}(s(v), w)) \rrbracket_{\sigma}$ .

224 ■ Else  $\llbracket s(\text{imax}(v, w)) \rrbracket_{\sigma} = s(\text{max}(\llbracket v \rrbracket_{\sigma}, \llbracket w \rrbracket_{\sigma})) = \llbracket \text{max}(s(w), \text{imax}(s(v), w)) \rrbracket_{\sigma}$ .

225 ◀

226 Finally, all of these propositions lead to this grammar restriction.

227 ► **Theorem 18.** *For all  $u \in \mathcal{L}$ , there exists  $u_1, \dots, u_n$  in the grammar  $t := s^k(x) \mid s^k(0) \mid$   
228  $\text{imax}(t, x)$  such that  $u =_{\mathcal{L}} \text{max}(u_1, \dots, u_n)$ .*

229 ► **Remark 19.** For all  $t$  in the grammar of Theorem 18, there exists  $x_1, \dots, x_n \in \mathcal{X}$ , and  
230  $v = s^k(0)$  or  $v = s^k(x)$  such that  $t = \text{imax}(\text{imax}(\text{imax}(\dots \text{imax}(v, x_1), x_2) \dots), x_{n-1}), x_n)$ .

231 **2.3 Introducing new levels**

232 Here, we continue the simplification of the levels under  $\text{max}$  in order to find simple enough  
233 terms to reach the uniqueness property. Indeed, the terms of the grammar of Theorem 18  
234 are still not enough.

235 ► **Example 20.** Let us consider  $t = \text{max}(\text{imax}(x, y), \text{imax}(y, x))$ . Then,  $t =_{\mathcal{L}} \text{max}(x, y)$ .

236 To better understand the issue, we should have a deep understanding of the semantic of  
 237  $\text{imax}(x, y)$ . It means that we always consider the value of  $y$ , but we only consider the value  
 238 of  $x$  if  $y$  is not zero: the position of the variables is essential in the levels. But, taking into  
 239 account  $\text{imax}(y, x)$  leads to offset this, and the value of  $x$  and  $y$  are always considered.

240 Here, we come to a solution that consists of introducing new symbols that permit to  
 241 represent the current levels, without taking into account the order of the variables.

242 ► **Definition 21** (Sublevels). *We define new symbols  $\mathcal{A}$  and  $\mathcal{B}$ .  $\mathcal{A}$  takes as argument a set  
 243 of variables, a variable and an integer, and  $\mathcal{B}$  takes as argument a set of variables and an  
 244 integer. They have the following semantic.*

$$245 \quad \llbracket \mathcal{A}(E, x, S) \rrbracket_{\sigma} = \begin{cases} 0 & \text{if there exists } y \in E \text{ such that } \sigma(y) = 0 \\ \llbracket x \rrbracket_{\sigma} + S & \text{else} \end{cases}$$

$$246 \quad \llbracket \mathcal{B}(E, S) \rrbracket_{\sigma} = \begin{cases} 0 & \text{if there exists } y \in E \text{ such that } \sigma(y) = 0 \\ S & \text{else} \end{cases}$$

247 In the next proposition, we show that any level of the grammar of Theorem 18 can be  
 248 written as a maximum of terms of this new grammar.

249 ► **Proposition 22.** *If  $t = \text{imax}(\text{imax}(\text{imax}(\dots \text{imax}(s^k(y), x_1) \dots)), x_{n-1}), x_n)$ , then*

$$250 \quad t =_{\mathcal{L}} \max(\mathcal{A}(\emptyset, x_n, 0), \mathcal{A}(\{x_n\}, x_{n-1}, 0), \dots, \mathcal{A}(\{x_2, \dots, x_n\}, x_1, 0), \mathcal{A}(\{x_1, \dots, x_n\}, y, k)).$$

251 *And if  $t = \text{imax}(\text{imax}(\text{imax}(\dots \text{imax}(s^k(0), x_1) \dots)), x_{n-1}), x_n)$ , then*

$$252 \quad t =_{\mathcal{L}} \max(\mathcal{A}(\emptyset, x_n, 0), \mathcal{A}(\{x_n\}, x_{n-1}, 0), \dots, \mathcal{A}(\{x_2, \dots, x_n\}, x_1, 0), \mathcal{B}(\{x_1, \dots, x_n\}, k)).$$

253 **Proof.** We show the result for the first case. Let  $\sigma$  be a valuation. If for all  $1 \leq i \leq n$ ,  
 254  $\sigma(x_i) \neq 0$ , then

$$255 \quad \llbracket t \rrbracket_{\sigma} = \max(\sigma(x_n), \dots, \sigma(x_1), k + \sigma(y))$$

$$256 \quad \forall 1 < i \leq n, \llbracket \mathcal{A}(\{x_n, \dots, x_{i+1}\}, x_i, 0) \rrbracket_{\sigma} = \sigma(x_i)$$

$$257 \quad \llbracket \mathcal{A}(\{x_n, \dots, x_1\}, y, k) \rrbracket_{\sigma} = k + \sigma(y)$$

258 and else, we take the largest  $1 \leq i \leq n$  such that  $\sigma(x_i) = 0$ , then

$$259 \quad \llbracket t \rrbracket_{\sigma} = \max(\sigma(x_n), \dots, \sigma(x_{i+1})) \quad \forall 1 < j \leq i, \llbracket \mathcal{A}(\{x_n, \dots, x_{j+1}\}, x_j, 0) \rrbracket_{\sigma} = 0$$

$$260 \quad \llbracket \mathcal{A}(\{x_n, \dots, x_1\}, y, k) \rrbracket_{\sigma} = 0 \quad \forall i < j \leq n, \llbracket \mathcal{A}(\{x_n, \dots, x_{j+1}\}, x_j, 0) \rrbracket_{\sigma} = \sigma(x_j)$$

261 hence the equality.

262 And the second case is very similar. Let  $\sigma$  be a valuation. If for all  $1 \leq i \leq n$ ,  $\sigma(x_i) \neq 0$ ,  
 263 then

$$264 \quad \llbracket t \rrbracket_{\sigma} = \max(\sigma(x_n), \dots, \sigma(x_1), k)$$

$$265 \quad \forall 1 < i \leq n, \llbracket \mathcal{A}(\{x_n, \dots, x_{i+1}\}, x_i, 0) \rrbracket_{\sigma} = \sigma(x_i)$$

$$266 \quad \llbracket \mathcal{B}(\{x_n, \dots, x_1\}, k) \rrbracket_{\sigma} = k$$

267 and else, we take the largest  $1 \leq i \leq n$  such that  $\sigma(x_i) = 0$ , then

$$268 \quad \llbracket t \rrbracket_{\sigma} = \max(\sigma(x_n), \dots, \sigma(x_{i+1})) \quad \forall 1 < j \leq i, \llbracket \mathcal{A}(\{x_n, \dots, x_{j+1}\}, x_j, 0) \rrbracket_{\sigma} = 0$$

$$269 \quad \llbracket \mathcal{B}(\{x_n, \dots, x_1\}, k) \rrbracket_{\sigma} = 0 \quad \forall i < j \leq n, \llbracket \mathcal{A}(\{x_n, \dots, x_{j+1}\}, x_j, 0) \rrbracket_{\sigma} = \sigma(x_j)$$

270 hence the equality. ◀



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271 The two equivalences are quite similar, the difference being in the very last subterm of  
 272 the max which is a  $\mathcal{A}$  in the first case (we have to take into account  $s^k(y)$  that is to say  
 273  $k + y$ ) and a  $\mathcal{B}$  in the second one ( $k$  is taken into account with a  $\mathcal{B}$ ).

274 The sublevels that we search for our representation are elements of this grammar. We  
 275 just have two slightly modifications, two simplifications to make. The first one is illustrated  
 276 by this example.

277 ► **Example 23.** With  $t_1 = \mathcal{A}(\emptyset, x, 0)$  and  $t_2 = \mathcal{A}(\{x\}, x, 0)$ , we have  $t_1 =_{\mathcal{L}} t_2$  since for all  
 278 valuation  $\sigma$ ,  $\llbracket t_1 \rrbracket_{\sigma} = \sigma(x) = \llbracket t_2 \rrbracket_{\sigma}$ .

279 The issue here is the fact that the second argument of a  $\mathcal{A}$  symbol does not necessarily  
 280 appear in its first argument. This creates equalities as shown in the next proposition.

281 ► **Proposition 24.** *Let  $x \in \mathcal{X}$ ,  $E \subset \mathcal{X} \setminus \{x\}$  and  $S \in \mathbb{N}$ . Then*

$$282 \quad \mathcal{A}(E, x, S) =_{\mathcal{L}} \max(\mathcal{A}(E \cup \{x\}, x, S), \mathcal{B}(E, S)).$$

283 **Proof.** Let  $\sigma$  be a valuation,  $t = \mathcal{A}(E, x, S)$ ,  $u = \mathcal{A}(E \cup \{x\}, x, S)$  and  $v = \mathcal{B}(E, S)$ .

284 ■ If there exists  $y \in E$  such that  $\sigma(y) = 0$ , then  $\llbracket t \rrbracket_{\sigma} = \llbracket u \rrbracket_{\sigma} = \llbracket v \rrbracket_{\sigma} = 0$ .

285 ■ Else, if  $\sigma(x) = 0$ , then  $\llbracket t \rrbracket_{\sigma} = S$ ,  $\llbracket u \rrbracket_{\sigma} = 0$  and  $\llbracket v \rrbracket_{\sigma} = S$ .

286 ■ Else,  $\sigma(x) \neq 0$ , and then  $\llbracket t \rrbracket_{\sigma} = \sigma(x) + S$ ,  $\llbracket u \rrbracket_{\sigma} = \sigma(x) + S$  and  $\llbracket v \rrbracket_{\sigma} = S$ .

287 Hence the result. ◀

288 Then, we will only consider elements of the form  $\mathcal{A}(E, x, S)$  such that  $x \in E$ .

289 The second modification is related to the representation of 0. Indeed, for all  $E \subset \mathcal{X}$ ,  
 290  $\mathcal{B}(E, 0) =_{\mathcal{L}} 0$ , and we should then keep at most one of them. However, since we already have  
 291  $0 =_{\mathcal{L}} \max(\emptyset)$ , we can remove all of them.

292 And we end up with this set of sublevels which permits to express any level of  $\mathcal{L}$ .

293 ► **Definition 25.** *We denote by  $\mathcal{L}_s$  the set of the sublevels that check the following conditions.*

294 1.  $\mathcal{A}(E, x, S) \in \mathcal{L}_s \iff x \in E$ ,

295 2.  $\mathcal{B}(E, S) \in \mathcal{L}_s \iff S > 0$ .

296 ► **Theorem 26.** *Let  $t \in \mathcal{L}$ . Then there exists  $u_1, \dots, u_n \in \mathcal{L}_s$  such that  $t =_{\mathcal{L}} \max(u_1, \dots, u_n)$ .*

297 ► **Remark 27.** By convenience, we will note  $u_i \in t$  if there exists  $u_1, \dots, u_n \in \mathcal{L}_s$  such that  
 298  $t =_{\mathcal{L}} \max(u_1, \dots, u_n)$ .

### 299 **3 A minimal representation**

300 In the previous section, we find a set  $\mathcal{L}_s$  and showed that any level can be represented  
 301 as a maximum of elements of  $\mathcal{L}_s$ . The goal of this one is to show that any level has a  
 302 minimal representation as maximum of elements of  $\mathcal{L}_s$  and that this representation is unique.  
 303 Intuitively, the sublevels of a minimal representation should be incomparable (else one of  
 304 the sublevels could be removed).

305 ► **Definition 28 (Minimal representation).** *Let  $t$  be a term. We say that  $t$  is a minimal  
 306 representation if and only if there exists  $u_1, \dots, u_n \in \mathcal{L}_s$  such that*

307 1.  $t = \max(u_1, \dots, u_n)$ ,

308 2. for all  $i \neq j$ ,  $u_i$  and  $u_j$  are incomparable.

309 We denote by  $\mathcal{L}_r$  the set of the minimal representations.

310 Of course, any level  $t$  has a minimal representation. We can express  $t$  as a maximum  
 311 of elements of  $\mathcal{L}_s$  (by Theorem 26) and we remove elements if they are comparable. The  
 312 challenging part is its uniqueness. To show it, we study the core of the definition of a minimal  
 313 representation: the sublevel comparison.

314 ► **Theorem 29** (Sublevels comparison). *Elements of  $\mathcal{L}_s$  are compared as follows.*

$$315 \quad \mathcal{A}(E, x, S) \not\leq_{\mathcal{L}} \mathcal{B}(F, K) \tag{1}$$

$$316 \quad \mathcal{B}(E, S) \leq_{\mathcal{L}} \mathcal{B}(F, K) \iff F \subset E \wedge S \leq K \tag{2}$$

$$317 \quad \mathcal{B}(E, S) \leq_{\mathcal{L}} \mathcal{A}(F, x, K) \iff (F \subset E \wedge S \leq K + 1) \tag{3}$$

$$318 \quad \mathcal{A}(E, x, S) \leq_{\mathcal{L}} \mathcal{A}(F, y, K) \iff F \subset E \wedge x = y \wedge S \leq K \tag{4}$$

319 **Proof.** With  $\sigma$  such that  $\sigma(x) = K + 1$  and  $\sigma(y) = 1$  if  $y \neq x$ , we show the first case. Indeed,  
 320  $\llbracket \mathcal{A}(E, x, S) \rrbracket_{\sigma} = K + 1 + S > K = \llbracket \mathcal{B}(F, K) \rrbracket_{\sigma}$  hence  $\mathcal{A}(E, x, S) \not\leq_{\mathcal{L}} \mathcal{B}(F, K)$ . The cases 2, 3  
 321 and 4 correspond to Propositions 30–32 proved below. ◀

322 ► **Proposition 30.** *Let  $E, F \subset \mathcal{X}$ ,  $x \in E$ ,  $y \in F$  and  $S, K \in \mathbb{N}$ . Then*

$$323 \quad \mathcal{A}(E, x, S) \leq_{\mathcal{L}} \mathcal{A}(F, y, K) \iff F \subset E \wedge x = y \wedge S \leq K.$$

324 **Proof.** We note  $t_1 = \mathcal{A}(E, x, S)$  and  $t_2 = \mathcal{A}(F, y, K)$ . Let us suppose  $F \subset E$ ,  $x = y$  and  
 325  $S \leq K$ . Let  $\sigma$  be a valuation.

326 ■ If there exists  $y \in F$  such that  $\sigma(y) = 0$ , then  $\llbracket t_2 \rrbracket_{\sigma} = 0$  and since  $F \subset E$ ,  $\llbracket t_1 \rrbracket_{\sigma} = 0$ .

327 ■ Else,  $\llbracket t_1 \rrbracket_{\sigma} \leq \sigma(x) + S \leq \sigma(x) + K = \llbracket t_2 \rrbracket_{\sigma}$ .

328 In both cases,  $\llbracket t_1 \rrbracket_{\sigma} \leq \llbracket t_2 \rrbracket_{\sigma}$  hence  $t_1 \leq_{\mathcal{L}} t_2$ .

329 Now, we show the other implication by contraposition.

330 ■ If there exists  $z \in F$  such that  $z \notin E$ , we take  $\sigma$  such that  $\sigma(z) = 0$  and for all  $j \neq z$ ,  
 331  $\sigma(j) = 1$ . We note that  $z \neq x$  (since  $z \notin E$  and  $x \in E$ ) hence  $\sigma(x) = 1$ . Then,  
 332  $\llbracket t_1 \rrbracket_{\sigma} = S + 1 > 0 = \llbracket t_2 \rrbracket_{\sigma}$ .

333 ■ If  $x \neq y$  we take  $\sigma$  such that  $\sigma(x) = K + 2$ ,  $\sigma(y) = 1$  and for all  $z \neq x$  and  $z \neq y$ ,  $\sigma(z) = 1$ .  
 334 Then,  $\llbracket t_1 \rrbracket_{\sigma} = K + S + 2 > K + 1 = \llbracket t_2 \rrbracket_{\sigma}$ .

335 ■ If  $S > K$  we take  $\sigma$  such that for all  $z$ ,  $\sigma(z) = 1$ . Then,  $\llbracket t_1 \rrbracket_{\sigma} = S + 1 > K + 1 = \llbracket t_2 \rrbracket_{\sigma}$ .  
 336 ◀

337 ► **Proposition 31.** *Let  $E, F \subset \mathcal{X}$  and  $S, K \in \mathbb{N}$ . Then*

$$338 \quad \mathcal{B}(E, S) \leq_{\mathcal{L}} \mathcal{B}(F, K) \iff F \subset E \wedge S \leq K.$$

339 **Proof.** We note  $t_1 = \mathcal{B}(E, S)$  and  $t_2 = \mathcal{B}(F, K)$ . Let us suppose  $F \subset E$  and  $S \leq K$ . Let  $\sigma$   
 340 be a valuation.

341 ■ If there exists  $y \in F$  such that  $\sigma(y) = 0$ , then  $\llbracket t_2 \rrbracket_{\sigma} = 0$  and since  $F \subset E$ ,  $\llbracket t_1 \rrbracket_{\sigma} = 0$ .

342 ■ Else,  $\llbracket t_1 \rrbracket_{\sigma} \leq K \leq S = \llbracket t_2 \rrbracket_{\sigma}$ .

343 In both cases,  $\llbracket t_1 \rrbracket_{\sigma} \leq \llbracket t_2 \rrbracket_{\sigma}$  hence  $t_1 \leq_{\mathcal{L}} t_2$ .

344 Now, we show the other implication by contraposition.

345 ■ If there exists  $y \in F$  such that  $y \notin E$ , we take  $\sigma$  such that  $\sigma(y) = 0$  and for all  $z \neq y$ ,  
 346  $\sigma(z) = 1$ . Then,  $\llbracket t_1 \rrbracket_{\sigma} = S > 0 = \llbracket t_2 \rrbracket_{\sigma}$ .

347 ■ If  $S < K$  we take  $\sigma$  such that for all  $y$ ,  $\sigma(y) = 1$ . Then,  $\llbracket t_2 \rrbracket_{\sigma} = K > S = \llbracket t_1 \rrbracket_{\sigma}$ .  
 348 ◀

349 ► **Proposition 32.** *Let  $E, F \subset \mathcal{X}$ ,  $x \in E$  and  $K, S \in \mathbb{N}$ . Then*

$$350 \quad \mathcal{B}(E, S) \leq_{\mathcal{L}} \mathcal{A}(F, x, K) \iff (F \subset E \wedge S \leq K + 1).$$

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351 **Proof.** We note  $t_1 = \mathcal{B}(E, S)$  and  $t_2 = \mathcal{A}(F, x, K)$ . Let us suppose  $F \subset E$  and  $S \leq K + 1$ .  
 352 Let  $\sigma$  be a valuation.

353 ■ If there exists  $y \in F$  such that  $\sigma(y) = 0$ , then  $\llbracket t_2 \rrbracket_\sigma = 0$  and since  $F \subset E$ ,  $\llbracket t_1 \rrbracket_\sigma = 0$ .

354 ■ Else,  $\sigma(x) \geq 1$  (because  $x \in F$ ) and then  $\llbracket t_2 \rrbracket_\sigma = \sigma(x) + K \geq 1 + K \geq S \geq \llbracket t_1 \rrbracket_\sigma$ .

355 In both cases,  $\llbracket t_1 \rrbracket_\sigma \leq \llbracket t_2 \rrbracket_\sigma$  hence  $t_1 \leq_{\mathcal{L}} t_2$ .

356 Now, we show the other implication by contraposition. First, we note that  $S > 0$ .

357 ■ If there exists  $y \in F$  such that  $y \notin E$ , we take  $\sigma$  such that  $\sigma(y) = 0$  and for all  $z \neq y$ ,  
 358  $\sigma(z) = 1$ . Then,  $\llbracket t_1 \rrbracket_\sigma = K > 0 = \llbracket t_2 \rrbracket_\sigma$ .

359 ■ If  $S > K + 1$  we take  $\sigma$  such that for all  $y$ ,  $\sigma(y) = 1$ . Then,  $\llbracket t_1 \rrbracket_\sigma = S > K + 1 = \llbracket t_2 \rrbracket_\sigma$ .

360 ◀

361 As a corollary, we get that the sublevel equivalence is a syntactic equality, which is quite  
 362 natural; the uniqueness property would be impossible otherwise.

363 ► **Corollary 33.** *Let  $t_1, t_2 \in \mathcal{L}_s$ . Then  $t_1 =_{\mathcal{L}} t_2 \iff t_1 = t_2$ .*

364 **Proof.** We have  $t_1 =_{\mathcal{L}} t_2 \iff t_1 \leq_{\mathcal{L}} t_2 \wedge t_2 \leq_{\mathcal{L}} t_1$ , and we conclude with the Theorem 29. ◀

365 And we can show the uniqueness of the minimal representation. First, we show that two  
 366 equivalent minimal representations have the same  $\mathcal{A}$ .

367 ► **Proposition 34.** *Let  $t_1, t_2 \in \mathcal{L}_r$  such that  $t_1 =_{\mathcal{L}} t_2$ . Then*

$$368 \quad \mathcal{A}(E, x, S) \in t_1 \iff \mathcal{A}(E, x, S) \in t_2.$$

369 **Proof.** Let us note

$$370 \quad t_1 = \max(\mathcal{A}(E_0, x_0, S_0), \dots, \mathcal{A}(E_n, x_n, S_n), \mathcal{B}(G_0, T_0), \dots, \mathcal{B}(G_p, T_p))$$

$$371 \quad t_2 = \max(\mathcal{A}(F_0, y_0, K_0), \dots, \mathcal{A}(F_m, y_m, K_m), \mathcal{B}(H_0, L_0), \dots, \mathcal{B}(H_q, L_q)).$$

372 Let  $\mathcal{A}(E, x, S)$  be an sublevel of  $t_1$ . We consider  $\sigma$  such that

$$373 \quad \sigma(y) = \begin{cases} \max(S_0, \dots, S_n, K_0, \dots, K_m, T_0, \dots, T_p, L_0, \dots, L_q) + 1 & \text{if } y = x \\ 1 & \text{if } y \in E \setminus \{x\} \\ 0 & \text{else} \end{cases}$$

374 We have  $\llbracket t_1 \rrbracket_\sigma = \llbracket \mathcal{A}(E, x, S) \rrbracket_\sigma = S + \sigma(x)$  and then  $\llbracket t_2 \rrbracket_\sigma = S + \sigma(x)$ . Then, there exists  
 375  $\mathcal{A}(F, y, K)$  in  $t_2$  such that  $F \subset E \cup \{x\} = E$  (else  $F$  contains a variable  $z$  such that  $\sigma(z) = 0$ )  
 376 and  $\sigma(y) + K = \sigma(x) + S$  or there exists  $\mathcal{B}(F, K)$  in  $t_2$  such that  $K = \sigma(x) + S$ . Since  
 377  $\sigma(x) > \max(S_0, \dots, S_n, K_0, \dots, K_m)$ , we deduce that it is the first case and  $y = x$ .

378 Then, there is  $(F, x, S)$  in  $t_2$  with  $F \subset E$  and  $\sigma(y) + K = \sigma(x) + S$ . If  $F \subsetneq E$ , then by  
 379 the same reasoning, we show that there exists  $\mathcal{A}(G, x, S) \in t_1$  with  $G \subset F \subsetneq E$ . But, by  
 380 Definition 28, it is impossible to have  $\mathcal{A}(E, x, S)$  and  $\mathcal{A}(G, x, S)$  in  $t_1$  with  $G \subset E$  since they  
 381 are comparable.

382 Then  $E = F$  and  $\mathcal{A}(E, x, S)$  is also an element of  $t_2$ . ◀

383 And we show the same for the  $\mathcal{B}$ .

384 ► **Proposition 35.** *Let  $t_1, t_2 \in \mathcal{L}_r$  such that  $t_1 =_{\mathcal{L}} t_2$ . Then*

$$385 \quad \mathcal{B}(E, S) \in t_1 \iff \mathcal{B}(E, S) \in t_2.$$

386 **Proof.** Let us note

$$387 \quad t_1 = \max(\mathcal{A}(E_0, x_0, S_0), \dots, \mathcal{A}(E_n, x_n, S_n), \mathcal{B}(G_0, T_0), \dots, \mathcal{B}(G_p, T_p))$$

$$388 \quad t_2 = \max(\mathcal{A}(F_0, y_0, K_0), \dots, \mathcal{A}(F_m, y_m, K_m), \mathcal{B}(H_0, L_0), \dots, \mathcal{B}(H_q, L_q)).$$

389 We show the result by induction on  $E$ . Let  $\mathcal{B}(E, S)$  be a sublevel of  $t_1$ . If  $E = \emptyset$ , we consider  
 390  $\sigma$  the zero function. Then,  $\llbracket t_1 \rrbracket_\sigma = S$ , hence  $\llbracket t_2 \rrbracket_\sigma = S$ . Since  $S > 0$ , it follows that  $\mathcal{B}(\emptyset, S)$   
 391 is a sublevel of  $t_2$ .

392 In the induction case, we consider  $\sigma$  such that  $\sigma(x) = 1$  if  $x \in E$  and  $\sigma(x) = 0$  otherwise,  
 393 hence  $\llbracket t_1 \rrbracket_\sigma = S$ . Then,  $\llbracket t_2 \rrbracket_\sigma = S$  and since  $S > 0$ , there exists  $\mathcal{A}(F, x, K)$  in  $t_2$  such that  
 394  $F \subset E$  and  $\sigma(x) + K = S$  or there exists  $\mathcal{B}(F, S)$  in  $t_2$  such that  $F \subset E$  and  $K = S$ .

395 In the first case, we have  $x \in F \subset E$ , then  $\sigma(x) = 1$  and  $K = S - 1$ . Then, by  
 396 Proposition 34,  $\mathcal{A}(F, x, S - 1) \in t_1$  which is impossible by Definition 28 since it would be  
 397 comparable with  $\mathcal{B}(E, S) \in t_1$ .

398 Then, we have  $\mathcal{B}(F, S) \in t_2$ . If  $F \subsetneq E$ , we apply the induction hypothesis and obtain  
 399  $\mathcal{B}(F, S) \in t_1$ , impossible because it would be comparable with  $\mathcal{B}(E, S)$ .

400 Then  $E = F$  and  $\mathcal{B}(E, S)$  is also an element of  $t_2$ . ◀

401 We immediately obtain that equivalence of minimal representations is the syntactic  
 402 equality.

403 ▶ **Proposition 36.** For all  $t_1, t_2 \in \mathcal{L}_r$ ,  $t_1 =_{\mathcal{L}} t_2 \iff t_1 = t_2$ .

404 **Proof.** The reverse implication is trivial and the first one is a consequences of the Proposi-  
 405 tions 34 and 35. ◀

406 And finally, we obtain the main theorem: the existence and uniqueness of a minimal  
 407 representation for each level. First, we show the intuitive property that the minimal  
 408 representation of a maximum of sublevels is formed with elements of these sublevels.

409 ▶ **Proposition 37.** For all  $u_1, \dots, u_n \in \mathcal{L}_s$ , there exists a unique  $\{v_1, \dots, v_m\} \subseteq \{u_1, \dots, u_n\}$   
 410 such that  $\max(u_1, \dots, u_n) = \max(v_1, \dots, v_m)$  and  $\max(v_1, \dots, v_m) \in \mathcal{L}_r$ .

411 **Proof.** We apply the following procedure. We consider  $E = \{u_1, \dots, u_n\}$ . While there exists  
 412  $u, v \in E$  such that  $u \leq_{\mathcal{L}} v$ , we remove  $u$  from  $E$ . The Proposition 36 permits to obtain the  
 413 uniqueness of this minimal representation. ◀

414 ▶ **Theorem 38 (Representation).** For all  $t \in \mathcal{L}$ , there exists a unique  $\{u_1, \dots, u_n\} \subset \mathcal{L}_s$  such  
 415 that  $t =_{\mathcal{L}} \max(u_1, \dots, u_n)$ . We say that  $\max(u_1, \dots, u_n)$  is the minimal representation of  $t$   
 416 and we denote it as  $\text{repr}(t)$ .

417 **Proof.** By Theorem 26, there exists  $u_1, \dots, u_n \in \mathcal{L}_s$  such that  $t =_{\mathcal{L}} \max(u_1, \dots, u_n)$ , and by  
 418 Proposition 37, there exists a unique minimal representation of  $\max(u_1, \dots, u_n)$ . ◀

419 Having a unique minimal representation is useful to have a faithful embedding of  $\mathcal{L}$  in the  
 420  $\lambda\Pi/\equiv$ , which is done in Section 4. Moreover, this representation also gives us an easy way to  
 421 compare two terms. Indeed, a sublevel can be compared to a level using the representation.

422 ▶ **Lemma 39.** Let  $u, v_1, \dots, v_n \in \mathcal{L}_s$ . Then  $u \leq_{\mathcal{L}} \max(v_1, \dots, v_n)$  if and only if there exists  
 423  $i$  such that  $u \leq_{\mathcal{L}} v_i$ .

424 **Proof.** The reverse implication is trivial. We show the direct implication by contraposition.  
 425 We suppose that for all  $i$ ,  $u \not\leq_{\mathcal{L}} v_i$ .

426 If  $u = \mathcal{B}(E, S)$ , we consider  $\sigma$  such that  $\sigma(x) = 1$  if  $x \in E$  and 0 otherwise. Then, for all  
 427  $v_i$ , we have either

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- 428 ■  $v_i = \mathcal{A}(F, x, K)$  or  $v_i = \mathcal{B}(F, K)$  and  $F \not\subseteq E$  hence  $\llbracket v_i \rrbracket_\sigma = 0 < S = \llbracket u \rrbracket_\sigma$ ,  
 429 ■ or  $v_i = \mathcal{A}(F, x, K)$  with  $F \subseteq E$  and  $K < S - 1$  hence  $\llbracket v_i \rrbracket_\sigma = K + 1 < S = \llbracket u \rrbracket_\sigma$ ,  
 430 ■ or  $v_i = \mathcal{B}(F, K)$  with  $K < S - 1$  hence  $\llbracket v_i \rrbracket_\sigma = K < S = \llbracket t \rrbracket_\sigma$ .

431 Then  $u \not\leq_{\mathcal{L}} \max(v_1, \dots, v_n)$ .

432 If  $u = \mathcal{A}(E, x, S)$ , then, each  $v_i$  is of the form  $\mathcal{A}(F_i, x_i, K_i)$  or  $\mathcal{B}(F_i, K_i)$ , we consider  $M$   
 433 the maximum of these  $K_i$  and  $\sigma$  such that  $\sigma(x) = M + 2$ ,  $\sigma(y) = 1$  if  $y \in E \setminus \{x\}$  and 0  
 434 otherwise. Then, for all  $v_i$ , we have either

- 435 ■  $v_i = \mathcal{B}(F, K)$  hence  $\llbracket v_i \rrbracket_\sigma \leq K < S + M + 2 = \llbracket u \rrbracket_\sigma$ ,  
 436 ■ or  $v_i = \mathcal{A}(F, y, K)$  and  $F \not\subseteq E$  hence  $\llbracket v_i \rrbracket_\sigma = 0 < S = \llbracket u \rrbracket_\sigma$ ,  
 437 ■ or  $v_i = \mathcal{A}(F, y, K)$  with  $F \subseteq E$  and  $x \neq y$ , hence  $\llbracket v_i \rrbracket_\sigma = K + 1 < S + M + 2 = \llbracket u \rrbracket_\sigma$ ,  
 438 ■ or  $v_i = \mathcal{A}(F, x, K)$  with  $F \subseteq E$  and  $K < S$ , hence  $\llbracket v_i \rrbracket_\sigma = K + M + 2 < S + M + 2 = \llbracket u \rrbracket_\sigma$ .  
 439 Then  $u \not\leq_{\mathcal{L}} \max(v_1, \dots, v_n)$ . ◀

440 And we use this lemma to compare two levels, for instance by comparing each sublevel  
 441 of the minimal representation of the first one to the second one. More generally, two  
 442 representations are compared in the following way.

443 ► **Theorem 40.** *Let  $u_1, \dots, u_n, v_1, \dots, v_m \in \mathcal{L}_s$ . Then,  $\max(u_1, \dots, u_n) \leq_{\mathcal{L}} \max(v_1, \dots, v_m)$   
 444 if and only if for all  $i$ , there exists  $j$  such that  $u_i \leq_{\mathcal{L}} v_j$ .*

445 **Proof.** If  $\max(u_1, \dots, u_n) \leq_{\mathcal{L}} \max(v_1, \dots, v_m)$ , then for all  $i$ ,  $u_i \leq_{\mathcal{L}} \max(v_1, \dots, v_m)$  and by  
 446 the Lemma 39, there exists  $v_j$  such that  $u_i \leq_{\mathcal{L}} v_j$ . The reverse implication is trivial. ◀

447 One can note that the Lemma 39 gives us a new proof of the uniqueness property stated  
 448 in Proposition 36.

449 **Proof.** Let  $u = \max(u_1, \dots, u_n)$  and  $v = \max(v_1, \dots, v_m)$  be two minimal representations  
 450 such that  $u =_{\mathcal{L}} v$ . We want to show that for all  $i$ , there exists  $j$  such that  $u_i = v_j$ .

451 We have  $u_i \leq_{\mathcal{L}} u \leq_{\mathcal{L}} v$ , hence by Lemma 39, there exists  $v_j$  such that  $u_i \leq_{\mathcal{L}} v_j$ . In  
 452 the same way, there exists  $u_k$  such that  $v_j \leq_{\mathcal{L}} u_k$ . Then, by Definition 28,  $i = k$  (because  
 453  $u_1, \dots, u_n$  are incomparable), and then  $u_i =_{\mathcal{L}} v_j$  hence  $u_i = v_j$  by Corollary 33. ◀

454 This shows that there is a link between the Lemma 39 and the uniqueness property. In  
 455 fact, this lemma should be understood as an *independence* lemma. Indeed, if we consider  
 456  $\max(u_1, \dots, u_n)$  as a linear combination of  $u_1, \dots, u_n$ , then this lemma states that the only  
 457 way to be smaller than a linear combination is to depend on and be smaller than one of the  
 458 elements of this combination.

459 This analogy provides a new point of view on our work:  $\mathcal{L}_s$  is a ‘linearly independent’  
 460 family (uniqueness of the minimal representation) which generates at least  $\mathcal{L}$  (for instance,  
 461  $\max(\mathcal{A}(\{x\}, y, 0)$  or  $\max(\mathcal{B}(\{x\}, 1))$  are not equivalent to any level).

### 462 **4 A rewriting system for this universe representation**

463 This section is dedicated to the implementation of this representation in the  $\lambda\Pi/\equiv$ .

#### 464 **4.1 Basic tools**

465 Here, we define the very basic term that we will use. The booleans and the natural numbers,  
 466 to begin with, are necessary.

467 ► **Definition 41** (Booleans). We define a type  $B$ , with constructors  $\text{true}: B$ ,  $\text{false}: B$  and  
 468 functions  $\text{and}: B \rightarrow B$ ,  $\text{or}: B \rightarrow B$  and  $\text{not}: B \rightarrow B$ .  $B$  is interpreted as the Booleans,  $\text{true}$ ,  
 469  $\text{false}$ ,  $\text{and}$ ,  $\text{or}$  and  $\text{not}$  as  $\top$ ,  $\perp$ , the conjunction, the disjunction and the negation.

470 ► **Definition 42** (Natural numbers). We define a type  $N$ , with constructors  $0: N$ ,  $s: N \rightarrow N$   
 471 and functions  $+: N \rightarrow N \rightarrow N$ ,  $<=_N: N \rightarrow N \rightarrow B$ ,  $=_N: N \rightarrow N \rightarrow B$ ,  $<_N: N \rightarrow N \rightarrow B$  (with infix  
 472 notation) and  $\text{max}_N: N \rightarrow N \rightarrow N$ .  $N$  is interpreted as  $\mathbb{N}$ ,  $0$  as  $0$ ,  $s$ ,  $+$ ,  $<=_N$ ,  $=_N$  and  $\text{max}_N$  as  $0$ ,  
 473  $s$ ,  $+$ ,  $\leq$ ,  $=$  and  $\text{max}$ .

474 Moreover, we define a *if-then-else* structure. It is not really necessary, but will be very  
 475 convenient to facilitate the writing of some rules.

476 ► **Definition 43.** Let  $T$  be a type. We define the function  $\text{ite}_T: B \rightarrow T \rightarrow T \rightarrow T$  such that  
 477  $\forall u, v \in T$ ,  $\text{ite}_T \text{ true } u \ v \hookrightarrow u$  and  $\text{ite}_T \text{ false } u \ v \hookrightarrow v$ . For convenience reasons, we will  
 478 denote  $\text{ite}_T b \ u \ v$  by *if  $b$  then  $u$  else  $v$* .

479 And finally, we show how to define a type of sets for all ordered types. It will be used for  
 480 set of natural numbers (to define the sublevels), but also for set of sublevels (to define the  
 481 representations).

482 ► **Definition 44** (Sets). Let  $T$  be a type equipped with a total order function  $\text{leq}_T: T \rightarrow$   
 483  $T \rightarrow B$ . Then, we define a type  $S[T]$  corresponding to the finite set of elements of  $T$  with a  
 484 constructor  $\{\}_T: S[T]$  and the functions (all with infix notation)  $\ll_T: S[T] \rightarrow T \rightarrow S[T]$ ,  
 485  $++_T: S[T] \rightarrow S[T] \rightarrow S[T]$ ,  $\text{in}_T: T \rightarrow S[T] \rightarrow B$ ,  $\text{sub}_T: S[T] \rightarrow S[T] \rightarrow B$  and  
 486  $=_{S[T]}: S[T] \rightarrow S[T] \rightarrow B$ .

487  $S[T]$  is interpreted as the set of finite subsets of  $T$ ,  $\{\}_T$  as the empty set,  $\ll_T$  as the  
 488 function that adds an element to a set,  $++_T$ ,  $\text{in}_T$ ,  $\text{sub}_T$  and  $=_{S[T]}$  as  $\cup$ ,  $\in$ ,  $\subseteq$  and  $=$ .

489 For convenience reasons, we will denote the term  $\{\}_T \ll_T x_1 \ll_T \dots \ll_T x_n$  by  $\{x_1, \dots, x_n\}_T$ .  
 490 In particular,  $\{x\}_T$  will represent a singleton.

491  $S[T]$  is implemented as a sorted list of elements of  $T$  with the constructors  $\{\}_T: S[T]$   
 492 and  $::_T: T \rightarrow S[T] \rightarrow S[T]$ , and the function  $\ll_T$  adds an element to a list while keeping  
 493 the uniqueness and order properties. Then, a set will be built through  $\{\}_T$  and  $\ll_T$  and the  
 494 constructor  $::_T$  will only be used in patterns of rewrite rules.

495 ► **Definition 45** (Set order). Let  $T$  be a type with a total order function  $\text{leq}_T$ . Then, we  
 496 define a total order  $\text{leq}_{S[T]}$  by considering the lexicographic order on sorted word of  $T$ .  
 497 Besides, we define the corresponding strict total order  $\text{lq}_{S[T]}$ .

498 We do not give the rules for these elements since they are quite basic. However, the  
 499 DEDUKTI implementation is available if necessary.

## 500 4.2 Level encoding

501 In order to implement the sets of sublevels and to compare two sublevels, we should be able  
 502 to compare and sort level variables. That is why we use a deep encoding where each variable  
 503 is encoded as a natural number. We denote by  $\gamma: \mathcal{X} \rightarrow \mathbb{N}$  a bijection that associates each  
 504 variable to a natural number.

505 ► **Definition 46.** We define  $L$ , the type of the levels together with the constructors  $0_L: L$ ,  
 506  $s_L: L \rightarrow L$ ,  $\text{max}_L: L \rightarrow L \rightarrow L$ ,  $\text{imax}_L: L \rightarrow L \rightarrow L$  and  $\text{var}_L: N \rightarrow L$ . We define inductively a  
 507 translation function  $|\cdot|: \mathcal{L} \rightarrow L$  with

$$508 \quad |0| = 0_L \quad |s(t)| = s_L |t| \quad |x| = \text{var}_L |\gamma(x)|_N$$

$$509 \quad |\text{max}(u, v)| = \text{max}_L |u| |v| \quad |\text{imax}(u, v)| = \text{imax}_L |u| |v|$$

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510 ► **Definition 47.** We define  $L_S$ , the type of the sublevels, with the constructors  $\mathbf{a}: S[N] \rightarrow$   
 511  $N \rightarrow N \rightarrow L_S$  and  $\mathbf{b}: S[N] \rightarrow N \rightarrow L_S$ . We define a translation function  $|\cdot|_s: \mathcal{L}_s \rightarrow L_S$  by  
 512  $|\mathcal{A}(E, x, S)|_s = \mathbf{a} |E| |x| |S|$  and  $|\mathcal{B}(E, S)|_s = \mathbf{b} |E| |S|$ .

513 In order to define  $S[L_S]$ , we need a total order on  $L_S$ . We consider the order where all the  
 514  $\mathbf{a}$  are before the  $\mathbf{b}$ , and the  $\mathbf{a}$  (respectively) the  $\mathbf{b}$  are sorted according to the lexicographic  
 515 order:

- 516 ■  $\mathbf{a} E x S \leq \mathbf{b} F K$ ,
- 517 ■  $\mathbf{a} E x S \leq \mathbf{a} |F| |y| |K|$  is the lexicographic order between  $(E, x, S)$  and  $(F, y, K)$  (which  
 518 is a total order since  $N$  and  $S[N]$  are equipped with a total order by Definition 45),
- 519 ■  $\mathbf{b} E S \leq \mathbf{b} F K$  is the lexicographic order between  $(E, S)$  and  $(F, K)$ .

520 ► **Definition 48** (Total order on sublevels). We define a total  $\text{leq}_{L_S}$  on  $L_S$  (and its corresponding  
 521 strict order  $\text{1q}_{L_S}$ ). Indeed, we consider these rewrite rules.

$$\begin{aligned}
 522 \quad & (\mathbf{a} E x S) \text{leq}_{L_S} (\mathbf{b} F K) \longrightarrow \text{true} \\
 523 \quad & (\mathbf{a} E x S) \text{leq}_{L_S} (\mathbf{a} F y K) \longrightarrow \text{or} (E \text{1q}_{S[N]} F) \\
 & \quad \quad \quad \text{and} (E =_{S[N]} F) ((\text{or} (x <_N y) (\text{and} (x =_N y) (S <=_N K)))) \\
 524 \quad & \quad \quad \quad \text{and} (E =_{S[N]} F) ((\text{or} (x <_N y) (\text{and} (x =_N y) (S <=_N K)))) \\
 525 \quad & \mathbf{b} E S \text{leq}_{L_S} \mathbf{b} F K \longrightarrow \text{or} (E \text{1q}_{S[N]} F, \text{and}(E =_{S[N]} F, S <=_N K))
 \end{aligned}$$

526 And we can then define the representations.

527 ► **Definition 49.** We define a function  $\text{max}_{L_S}: S[L_S] \rightarrow L$  that embeds a set of  $L_S$  into a  $L$  and a  
 528 translation function  $|\cdot|_r: \mathcal{L}_r \rightarrow L$  defined by  $|\text{max}(u_1, \dots, u_n)|_r = \text{max}_{L_S}(\{|u_1|_s, \dots, |u_n|_s\}_T)$ .

529 Now, we have defined all the types and the elements of  $\mathcal{L}_r$  have translations in  $\lambda\Pi/\equiv$ .  
 530 The next step is to provide rewrite rules that transform a level into its minimal representation.  
 531 The cases of  $0_L$  and of variables are easy.

$$532 \quad 0_L \longrightarrow \text{max}_{L_S}(\{\}_{L_S}) \quad \text{var}_L(x) \longrightarrow \text{max}_{L_S}(\{\mathbf{a}(\{x\}_N, x, 0)\}_{L_S})$$

533 For the other cases, and in particular for the  $\text{max}$ , the sublevel comparison will be useful,  
 534 so we define it.

535 ► **Definition 50** (Sublevels comparison). We define  $<=_L: L_S \rightarrow L_S \rightarrow B$  interpreted as  $\leq_{\mathcal{L}}$   
 536 with these rewrite rules.

$$537 \quad \mathbf{a}(E, x, S) <=_L \mathbf{b}(F, K) \longrightarrow \text{false} \tag{1}$$

$$538 \quad \mathbf{b}(E, S) <=_L \mathbf{b}(F, K) \longrightarrow \text{and}(F \text{sub}_N E, S <=_N K) \tag{2}$$

$$539 \quad \mathbf{b}(E, \mathbf{s}_L(S)) <=_L \mathbf{a}(F, y, K) \longrightarrow \text{and}(F \text{sub}_N E, S <=_N K + 1) \tag{3}$$

$$540 \quad \mathbf{a}(E, x, S) <=_L \mathbf{a}(F, y, K) \longrightarrow \text{and}(F \text{sub}_N E, \text{and}(x =_N y, S <=_N K)) \tag{4}$$

541 ► **Proposition 51.** Let  $u, v \in \mathcal{L}_s$ . Then,  $u \leq_{\mathcal{L}} v \iff |u| <=_L |v| \hookrightarrow^* \text{true}$ .

542 **Proof.** Each rule (i) corresponds to the case i of the Theorem 29. ◀

543 And now, we can give the rules to normalize the other levels.

### 4.3 The successor

We show how to compute the minimal representation of  $s(\max(u_1, \dots, u_n))$ .

► **Definition 52.** We define  $s_s$  on the sublevels by  $s_s(\mathcal{A}(E, x, S)) = \mathcal{A}(E, x, S + 1)$  and  $s_s(\mathcal{B}(E, S)) = \mathcal{B}(E, S + 1)$ . Then  $s_s(u) =_{\mathcal{L}} s(u)$ .

► **Proposition 53.**  $\text{repr}(s(\max(\emptyset))) = \max(\mathcal{B}(\emptyset), s(0))$  and for all  $n > 0$  and  $t = \max(u_1, \dots, u_n) \in \mathcal{L}_r$ ,  $\text{repr}(s(t)) = \max(s_s(u_1), \dots, s_s(u_n))$ .

We create a function  $s_{L_s} : \mathcal{S}[L_s] \rightarrow \mathcal{S}[L_s]$  corresponding to  $s_s$ .

$$s_{L_s}(\{ \}_{L_s}) \longrightarrow \{ \}_{L_s} \quad s_{L_s}(\mathbf{b}(E, S) ::_{L_s} q) \longrightarrow s_{L_s}(q) \ll_{L_s} \mathbf{b}(E, \mathbf{s}(S))$$

$$s_{L_s}(\mathbf{a}(E, x, S) ::_{L_s} q) \longrightarrow s_{L_s}(q) \ll_{L_s} \mathbf{a}(E, x, \mathbf{s}(S))$$

And we compute the minimal representation of a successor according to Proposition 53.

$$s_L(\max_{L_s}(\{ \}_{L_s})) \longrightarrow \max_{L_s}(\mathbf{b}(\{ \}_{L_s}, \mathbf{s}(0))) \quad s_L(\max_{L_s}(u ::_{L_s} q)) \longrightarrow \max_{L_s}(s_{L_s}(u ::_{L_s} q))$$

► **Proposition 54.** For all  $t \in \mathcal{L}_r$ ,  $s_L(|t|_r) \hookrightarrow^* |\text{repr}(s(t))|_r$ .

### 4.4 The maximum

For the maximum, we define  $\max_s$ , that adds a sublevel to a minimal representation.

► **Definition 55.** We define inductively  $\max_s : \mathcal{P}(\mathcal{L}_s) \times \mathcal{L}_s \rightarrow \mathcal{P}(\mathcal{L}_s)$  inductively defined by  $\max_s(\emptyset, u) = \{u\}$  and

$$\max_s(\{u_1, \dots, u_n\} \cup \{u\}, v) = \begin{cases} \{u_1, \dots, u_n, u\} & \text{if } v \leq_{\mathcal{L}} u, \\ \{u_1, \dots, u_n, v\} & \text{if } u \leq_{\mathcal{L}} v, \\ \max_s(\{u_1, \dots, u_n\}, v) \cup \{u\} & \text{else.} \end{cases}$$

► **Proposition 56.** For all minimal representations  $t = \max(u_1, \dots, u_n) \in \mathcal{L}_r$  and  $v \in \mathcal{L}_s$ ,  $\text{repr}(\max(u_1, \dots, u_n, v)) = \max_s(\{u_1, \dots, u_n\}, v)$ .

We implement  $\max_s$  as a function  $\text{maxhelper} : \mathcal{S}[L_s] \rightarrow L_s \rightarrow \mathcal{S}[L_s]$  with these rewrite rules.

$$\text{maxhelper}(\{ \}_{L_s}, u) \longrightarrow \{u\}_{L_s}$$

$$\text{maxhelper}(u ::_{L_s} E, v) \longrightarrow \text{if } v \leq_{L_s} u \text{ then } E \ll_{L_s} u$$

$$\text{else if } u \leq_{L_s} v \text{ then } E \ll_{L_s} v \text{ else } \text{maxhelper}(E, v) \ll_{L_s} u$$

And we compute the minimal representation for  $\max_L$  according to Proposition 56.

$$\max_L(\max_{L_s}(E), \max_{L_s}(\{ \}_{L_s})) \longrightarrow \max_{L_s}(E)$$

$$\max_L(\max_{L_s}(E), \max_{L_s}(u ::_{L_s} F)) \longrightarrow \max_L(\text{maxhelper}(E, u), \max_{L_s}(F))$$

► **Proposition 57.** For all  $t_1, t_2 \in \mathcal{L}_r$ ,  $\max_L(|t_1|_r, |t_2|_r) \hookrightarrow^* |\text{repr}(\max(t_1, t_2))|_r$ .

### 4.5 The impredicative maximum

To begin, we study  $\text{imax}(u, v)$  where  $u, v \in \mathcal{L}_s$ .

► **Proposition 58.** Let  $f(E, S)$  be either  $\mathcal{A}(E, x, S)$  or  $\mathcal{B}(E, S)$  and  $g(F, K)$  be either  $\mathcal{A}(F, x, K)$  or  $\mathcal{B}(F, K)$ . Then

$$\text{imax}(f(E, S), g(F, K)) =_{\mathcal{L}} \max(f(E \cup F, S), g(F, K)).$$



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577 **Proof.** We note  $u = f(E, x, S)$  and  $v = g(F, K)$ . Let  $\sigma$  be a valuation.  
578 ■ If there exists  $y \in F$  such that  $\sigma(y) = 0$ , then  $\llbracket \text{imax}(u, v) \rrbracket_\sigma = 0 = \llbracket \max(f(E \cup F, S), v) \rrbracket_\sigma$ .  
579 ■ Else,  $\llbracket v \rrbracket_\sigma = K > 0$  and then  $\llbracket \text{imax}(u, v) \rrbracket_\sigma = \max(\llbracket u \rrbracket_\sigma, K)$ . Besides, since  $\sigma(y) \neq 0$  for all  
580  $y \in F$ , there exists  $y \in E \cup F$  such that  $\sigma(y) = 0$  if and only if there exists  $y \in E$  such  
581 that  $\sigma(y) = 0$ , hence  $\llbracket u \rrbracket_\sigma = \llbracket f(E \cup F, S) \rrbracket_\sigma$ .  
582 Hence the result. ◀

583 We implement it as a function  $\text{imax}_{L_S} : L_S \rightarrow L_S \rightarrow L$ .

584  $\text{imax}_{L_S}(\mathbf{a}(E, x, S), \mathbf{b}(F, K)) \longrightarrow \max_L(\max_{L_S}(\mathbf{a}(E \text{ ++}_N F, x, S)), \max_{L_S}(\mathbf{b}(F, K)))$   
585  $\text{imax}_{L_S}(\mathbf{b}(E, S), \mathbf{b}(F, K)) \longrightarrow \max_L(\max_{L_S}(\mathbf{b}(E \text{ ++}_N F, S)), \max_{L_S}(\mathbf{b}(F, K)))$   
586  $\text{imax}_{L_S}(\mathbf{b}(E, S), \mathbf{a}(F, x, K)) \longrightarrow \max_L(\max_{L_S}(\mathbf{b}(E \text{ ++}_N F, S)), \max_{L_S}(\mathbf{a}(F, x, K)))$   
587  $\text{imax}_{L_S}(\mathbf{a}(E, x, S), \mathbf{a}(F, y, K)) \longrightarrow \max_L(\max_{L_S}(\mathbf{a}(E \text{ ++}_N F, x, S)), \max_{L_S}(\mathbf{a}(F, y, K)))$   
588

589 Then, following the equalities  $\text{imax}(0, t) =_{\mathcal{L}} t$  and  $\text{imax}(t, 0) =_{\mathcal{L}} 0$ , and Propositions 11  
590 and 12, we define  $\text{imax\_aux} : L_S \rightarrow L \rightarrow L$  and add these rewrite rules.

591  $\text{imax}_L(\max_{L_S}(\{\}_L), t) \longrightarrow t$   
592  $\text{imax}_L(\max_{L_S}(u ::_{L_S} q), t) \longrightarrow \max_L(\text{imax\_aux}(u, t), \text{imax}_L(q, t))$   
593  $\text{imax\_aux}(u, \max_{L_S}(\{\}_L)) \longrightarrow \max_{L_S}(\{\}_L)$   
594  $\text{imax\_aux}(u, \max_{L_S}(v ::_{L_S} q)) \longrightarrow \max_L(\text{imax}_{L_S}(u, v), \text{imax\_aux}(u, q))$

595 ► **Proposition 59.** For all  $t_1, t_2 \in \mathcal{L}_r$ ,  $\text{imax}_L(|t_1|_r, |t_2|_r) \hookrightarrow^* |\text{repr}(\text{imax}(t_1, t_2))|_r$ .

596 **Proof.** By Propositions 57 and 58, for all  $u, v \in \mathcal{L}_s$ ,  $\text{imax}_{L_S}(|u|_s, |v|_s) \hookrightarrow^* |\text{repr}(\text{imax}(u, v))|_s$ .  
597 Then, by induction on  $t \in \mathcal{L}_r$  and using Proposition 57, we show that for all  $u \in \mathcal{L}_s$ ,  
598  $\text{imax\_aux}(|u|_s, |t|_r) \hookrightarrow^* |\text{repr}(\text{imax}(u, t))|_r$ . And finally, we show the result by induction on  
599  $t_2$  using Propositions 11 and 12 and the equivalences  $\text{imax}(0, t) =_{\mathcal{L}} t$  and  $\text{imax}(t, 0) =_{\mathcal{L}} 0$ . ◀

### 600 4.6 Implementing the substitution

601 Since we use a deep encoding,  $\beta$ -reduction cannot be used for substitution. Then, we  
602 implement a substitution function. First, we implement  $\text{eval}_{L_S} : L_S \rightarrow N \rightarrow N \rightarrow L$  for the  
603 sublevels following the semantic given in the Definition 21.

604  $\text{eval}_{L_S}(\mathbf{b}(E, S), y, n) \longrightarrow \text{if and}(y \text{ in}_N E, n =_N 0) \text{ then } \max_{L_S}(\{\}_L)$   
605  $\text{else } \max_{L_S}(\mathbf{b}(E \setminus_N y, S))$   
606  $\text{eval}_{L_S}(\mathbf{a}(E, x, S), y, n) \longrightarrow \text{if and}(y \text{ in}_N E, n =_N 0) \text{ then } \max_{L_S}(\{\}_L)$   
607  $\text{else if } x =_N y \text{ then } \max_{L_S}(\mathbf{b}(\{\}_N, S + n))$   
608  $\text{else } \max_{L_S}(\mathbf{a}(E \setminus_N y, x, S))$

609 Then, we create a function  $\text{eval}_L : L \rightarrow N \rightarrow N \rightarrow L$  that evaluate a level using the fact that  
610  $[\max(u_1, \dots, u_n)]\{x \mapsto n\} = \max([u_1]\{x \mapsto n\}, \dots, [u_n]\{x \mapsto n\})$ .

611  $\text{eval}_L(\{\}_L, y, n) \longrightarrow \{\}_L \quad \text{eval}_L(u ::_{L_S} q, y, n) \longrightarrow \max_L(\text{eval}_{L_S}(u, y, n), \text{eval}_L(q, y, n))$

612 ► **Proposition 60.** Let  $t \in \mathcal{L}$ . Then, the normal form of  $|\llbracket t \rrbracket\{x \mapsto u\}|$  is  $\text{eval}_L(|t|, |u|)$ .

## 613 4.7 Properties of the rewrite system

614 The rewrite system that we designed have strong properties. First, one can note that it  
615 is does not use any higher-order rewrite rule, hence it can be implemented in a first order  
616 system.

617 ▶ **Theorem 61.** *The rewrite system is confluent and strongly normalizing.*

618 **Proof.** The termination has been proved with two termination checkers,  $\mathsf{T}\mathsf{T}_2$  [26] and  
619  $\mathsf{SizeChangeTool}$  [19], and the confluence with  $\mathsf{CSI}$  [34]. ◀

620 And of course, we show that it is sound relatively to the minimal representation.

621 ▶ **Theorem 62 (Soundness).** *Let  $t \in \mathcal{L}$ . Then, the normal form of  $|t|$  is  $|\mathsf{repr}(t)|_r$ .*

622 **Proof.** We show that  $t \hookrightarrow^* |\mathsf{repr}(t)|_r$  by induction on  $t$ :  $\mathsf{var}_L(x) \hookrightarrow \mathsf{max}_{L_S}(\{\mathsf{a}(\{x\}_N, 0, x)\}_{L_S})$ ,  
623  $0 \hookrightarrow \mathsf{max}_{L_S} \{\}_{L_S}$ , and we show the cases  $s$ ,  $\mathsf{max}$  and  $\mathsf{imax}$  using Propositions 54, 57, and 59.  
624 Since no rewrite rule can be applied to  $\mathsf{max}_{L_S}$ , the normal form of  $|t|$  is  $|\mathsf{repr}(t)|_r$ . ◀

625 Theorems 38 and 62 give us that the translations of two equivalent levels are convertible,  
626 and even more, they have the same normal form. In other words, this embedding faithfully  
627 represent the level equivalences.

628 Besides, one could note some drawbacks of this embedding. First, we do not have a back  
629 translation from  $\lambda\Pi/\equiv$  to  $L$ . There are two main reason for this.

- 630 1.  $\mathsf{a}$  and  $\mathsf{b}$  are not exact translations of  $\mathcal{A}$  and  $\mathcal{B}$ .
- 631 2.  $\mathcal{L}_r$  and  $\mathcal{L}$  are not equivalent.

632 The first reason is linked to the restrictions that we added to  $\mathcal{A}$  and  $\mathcal{B}$ . Here, it is possible to  
633 write terms  $\mathsf{b} E 0$  or  $\mathsf{a} E x k$  where  $x$  is not an element of  $E$ . In the same way, it is possible  
634 to write  $\mathsf{max}_{L_S} L$  while two sublevels of  $L$  are comparable.

635 A solution could be to add a dependent term as argument, to check these conditions.  
636 For instance,  $\mathsf{a}$  would have the type  $(E: \mathsf{S}[N]) \rightarrow (x: N) \rightarrow \mathsf{Prf} (x \mathsf{in}_N E) \rightarrow N \rightarrow L_S$  where  
637  $\mathsf{Prf}: \mathsf{B} \rightarrow \mathsf{Type}$  represents the proof of a proposition. We declare  $\mathsf{I}: \mathsf{Prf} \mathsf{true}$  and we use  
638  $\mathsf{a} E x k \mathsf{I}$  using the fact that  $x \mathsf{in}_N E$  reduces to  $\mathsf{true}$  if and only if  $x$  is an element of  $E$ .

639 The second reason is not related to the embedding, but to the representation that we  
640 introduced. Indeed, we already noted that some minimal representations are not equivalent  
641 to any level ( $\mathsf{max}(\mathcal{A}(\{x\}, y, 0))$  or  $\mathsf{max}(\mathcal{B}(\{x\}, 1))$  as examples). Then, it could be a good  
642 idea to find a characterization of the elements of  $\mathcal{L}_r$  that actually correspond to levels.

## 643 5 Conclusion

644 We introduced a new representation of the levels of the impredicative PTS where equivalent  
645 levels have the same representation. It provides us an easy procedure decision for the  
646 inequality problem in the  $\mathsf{imax}$ -successor algebra, and it permits us to get a sound encoding  
647 of these levels in the  $\lambda\Pi/\equiv$ , in the sense that equivalent levels have convertible translations.  
648 Moreover, this encoding corresponds to a first-order, confluent and strongly normalizing  
649 rewrite system, and in particular it permits to decide level equality.

650 This encoding of the levels permits to encode  $\mathsf{CC}^\infty$  with universe polymorphism. Besides,  
651 we still have to study how this encoding behaves well together with encodings of inductive  
652 types or cumulativity in order to get a better encoding of  $\mathsf{COQ}$ . The ideas mentioned at the  
653 end of Section 4 are also interesting. In particular, the characterization of the elements of  $\mathcal{L}_r$   
654 that are actually levels would lead to a better understanding of the  $\mathsf{imax}$ -successor grammar.

655 Finally, this idea of representation, with the linear algebra analogy, could certainly be  
 656 adapted to some sets of terms built over a maximum, a supremum or other similar operations.  
 657 In addition to providing decision procedures, it would permit to define a concept similar  
 658 to the basis of a vector space on these spaces on these sets, and then it seems to be an  
 659 interesting direction to explore.

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