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## Complex-Source Beam Representation of the Fields Radiated by a Gaussian Window

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### Abstract

Frame based decomposition of fields provides a rigorous, flexible and effective tool to represent planar source field distributions as summations of Gaussian windows. The field radiated by each window can be approximated as a Gaussian beam in its paraxial region. In this context, beam shooting and bouncing techniques have been proposed as an alternative to corresponding ray methods for describing wave propagation in complex environments. Complex-Source beams, which are closely related to Gaussian beams, provide rigorous solutions for several canonical problems. The objective of this paper is to investigate relations between Gaussian beams radiated by frame windows and Complex-Source beams. Such an approach would open the door to extending the applicability of frame decomposition methods, enhancing Gaussian beam shooting by means of solutions provided by Complex-Source beams.

### 1 Introduction

Gaussian beams (GBs) have initially been used to describe the propagation of fields localized around an axis in free space, as in beam waveguides and more generally quasi-optical systems. Propagating fields are then expressed as summations of Gaussian Laguerre modes. Gaussian beam modes, and especially the fundamental one, exhibit a number of interesting properties: spatial and spectral (i.e. in terms of directions of propagation) localization, no caustics, ray-like formulas for propagation, reflection and refraction modelization.

To take advantage of these properties in more general configurations involving non paraxial propagation, a number of works aimed at describing fields radiated by large apertures as superpositions of tilted and translated GB fields have been presented. Such superpositions were derived by discretizing windowed Fourier transforms of aperture field distributions. Frame theory provided a rigorous mathematical framework to such decompositions [1].

Gaussian beam shooting algorithms could then be developed in various contexts (UWB and time domain applications, inverse scattering, RCS computations, etc.), taking advantage of the flexibility offered by varying the frame oversampling factor [2].

GB shooting and bouncing in complex environments can be performed very efficiently, based on the localization of fields close to the beam axes. Local paraxial reflexion and refraction operators make beam tracking as simple as ray tracking in multipath environments. Diffraction effects are however not fully captured by such algorithms.

On the contrary, Complex Source Beams (CSBs) are often used as incident field propagators to solve canonical diffraction problems. As they are exact solutions of Maxwell's equations and easily achieved by just defining a real-valued source coordinate to a complex one [3], any analytical solution for a line- or point source in front of a canonical object such as a wedge or a cone can be easily extended the exact solution of the boundary-value problem to the case of an incident CSB [4]. Moreover, it is well known that in a paraxial approximation a CSB represents a Gaussian Beam [3].

The aim of this work is to pave the way for algorithms combining both techniques (frame based GB shooting and bouncing, and CSB diffraction results). To this end, we have to establish the degree of accuracy which can be obtained by substituting CSBs to the astigmatic GBs radiated by frame windows. We shall address this problem for single beam propagation, especially for highly tilted beams. After a brief review of frame based GB shooting and of CSB formulas, we present the basic relation between their propagated fields, and first numerical results comparing their respective source fields and radiated fields.

### 2 Mathematical descriptions

#### 2.1 Frame based Gaussian beam shooting

We simplify the presentation by considering the so called 2D problem. The source field is assumed to be distributed in the  $xOy$  plane, with no variation along  $y$ . The source field has only one  $y$ -component. Hence the source field will only be described along the  $Ox$  axis and the radiated field in the  $xOz$  plane.

GB shooting from aperture field distributions is initialized with the representation of the source field distribution as a summation of weighted Gaussian windows translated in

the phase space, that is both the space and the spectrum (directions of the wave-vectors, which identify directions of propagation). In the following we shall denote  $w$  the "mother" Gaussian window, centered at origin in space and spectrum:

$$w(x) = \sqrt{\sqrt{2}/L} e^{-\pi \frac{x^2}{L^2}}. \quad (1)$$

The frame is then defined as  $\{w_{mn}, (m, n) \in \mathbb{Z}^2\}$ , where:

$$w_{mn}(x) = w(x - m\bar{x})e^{-jn\bar{k}_x x}. \quad (2)$$

The integer frame index  $m$  and the frame parameter  $\bar{x}$  (or  $n$  and  $\bar{k}_x$ ) are the spatial (or spectral, respectively) translation index and translation step. This set is a frame if and only if it is redundant, i.e.  $\bar{k}_x \bar{x} = 2\pi\nu$  with  $\nu < 1$  being the oversampling factor.

The field radiated by a field distribution proportional to the  $w_{mn}$  window can be derived from the plane wave spectrum (PWS) integral

$$B_{mn}(x, z) = A_f \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{w}_{mn}(k_x) e^{-j(k_x x + k_z z)} dk_x, \quad (3)$$

where  $\tilde{w}_{mn}$  is the Fourier transform of  $w_{mn}$ ,  $A_f$  is a complex field amplitude factor, and the omitted time factor is  $e^{+j\omega t}$ .

If the width parameter of the mother window,  $L$ , is larger than a few wavelengths, a paraxial approximation of this integral can be performed, leading to the expression of a paraxial GB with its axis tilted according to the spectral translation of the window, or equivalently to the linear phase variation of the spatial window. The direction of the beam axis is given by the angle  $\theta_n$ , defined by  $k \sin \theta_n = n\bar{k}_x$ . A coordinate system  $(O, \hat{\mathbf{x}}_{mn}^\circ, \hat{\mathbf{z}}_{mn}^\circ)$  related to the beam is then introduced, with  $\hat{\mathbf{z}}_{mn}^\circ$  the unit vector in the beam axis direction. In this coordinate system, the paraxial approximation of the GB field, which will be denoted by  $GB_{mn}$ , can be expressed as

$$GB_{mn}(x^\circ, z^\circ) = A_f \left(\frac{\sqrt{2}}{L}\right)^{1/2} \sqrt{\frac{jb_n}{z_{mn}^\circ + jb_n}} e^{-jk(z_{mn}^\circ + \frac{x_{mn}^{\circ 2}}{2(z_{mn}^\circ + jb_n)})} \quad (4)$$

with  $b_n = L^2 \cos^2 \theta_n / \lambda$ .

## 2.2 Uniform Complex-Source Beams

In the context of wave propagation, a Complex-Source Beam (CSB) is obtained by a simple analytical continuation of a real valued point-source coordinate to a complex one. We start from the solution  $\Phi(\mathbf{R})$  with  $\mathbf{R} = x\hat{\mathbf{x}} + z\hat{\mathbf{z}}$  of the two-dimensional Helmholtz equation (time factor  $e^{+j\omega t}$ ) for a line source of amplitude  $A_c$  located at  $\mathbf{R}' = x'\hat{\mathbf{x}} + z'\hat{\mathbf{z}}$  in the free space:

$$\Phi_1(\mathbf{R}) = A_c \frac{-j}{4} H_0^{(2)}(k|\mathbf{R} - \mathbf{R}'|) \approx A_c \frac{e^{-jk|\mathbf{R} - \mathbf{R}'| - j\pi/4}}{\sqrt{8\pi k|\mathbf{R} - \mathbf{R}'|}}. \quad (5)$$

Here,  $H_0^{(2)}$  denotes the zero-order Hankel function of the second kind, while the accuracy of the approximate expression increases for increasing values of  $|\mathbf{R} - \mathbf{R}'|$ , where

the wave number  $k$  is related to the wavelength by  $\lambda$  by  $k = 2\pi/\lambda$ . By setting  $\mathbf{R}' = \mathbf{R}_w - j\mathbf{b}_c$  the line-source field turns into a CSB [3]. Moreover, by using a sufficiently large  $b_c$  the approximation in (5) is fine even for  $R = 0$ . As an example, but without losing generality we set  $\mathbf{R}' = -jb_c \hat{\mathbf{z}}$  and obtain in a paraxial (i.e., nearby the  $z$ -axis) approximation for  $z \geq 0$  a beam which on the  $z$ -axis is propagating in the positive  $z$ -direction:

$$\Phi_1(x, z) = A_c \frac{e^{kb_c - j\pi/4}}{\sqrt{8\pi k(z + jb_c)}} e^{-jkz - j \frac{kx^2}{2(z + \frac{b_c^2}{z})} - \frac{x^2}{2\frac{b_c}{k} \left(1 + \frac{z}{b_c}\right)}}. \quad (6)$$

Note that for  $z < 0$ , the causal solution is propagating in the negative  $z$  direction, but exponentially damped according to  $e^{-kb_c}$ . Moreover,  $\Phi$  is regular only outside of the waist, which is centered at  $z = 0$  with the length given by:

$$w_c = \frac{b_c}{k}. \quad (7)$$

Thus, the representation (5) delivers just one half of a full beam which moreover is non-regular in the waist. However, as has been shown [5] by simply adding to  $\Phi_1$

$$\Phi_2(\mathbf{R}) = A_c \frac{-j}{4} H_0^{(1)}(k|\mathbf{R} - \mathbf{R}'|), \quad (8)$$

with same  $\mathbf{R}' = \mathbf{R}_w - j\mathbf{b}$ , we obtain a full beam with an amplitude  $\sim e^{kb}$ , propagating in the direction  $\mathbf{b}$  which moreover is regular everywhere. This can be easily seen since  $H_0^{(1)}(z) + H_0^{(2)}(z) = 2J_0(z)$  yields the Bessel function of the first kind of order 0 which exhibits no singularities. We therefore call

$$\Phi(\mathbf{R}) = 2A_c \frac{-j}{4} J_0(k|\mathbf{R} - \mathbf{R}'|) \quad (9)$$

(with a complex source coordinate) a uniform CSB. Note that  $\Phi$  also includes a full CSB propagating in the direction  $-\mathbf{b}$ , which however is of amplitude  $e^{-kb}$  and thus practically negligible if  $b$  is chosen large enough.

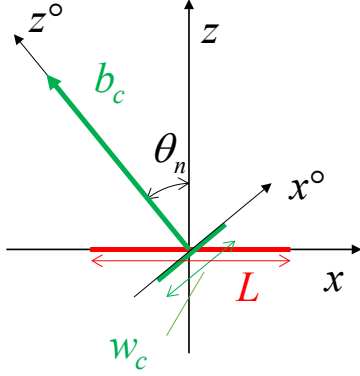
## 2.3 Basic Relation between GB and CSB in the Paraxial Region

To compare the CSB to the Gaussian beam, we first write (4) in a different form:

$$GB_{mn}(x^\circ, z^\circ) = A_f e^{j\pi/4} \left(\frac{\sqrt{2}}{L}\right)^{1/2} \sqrt{\frac{b_n}{z_{mn}^\circ + jb_n}} e^{-jkz_{mn}^\circ - j \frac{kx_{mn}^{\circ 2}}{2\left(\frac{z_{mn}^\circ}{b_n} + \frac{b_n^2}{z_{mn}^\circ}\right)} - \frac{x_{mn}^{\circ 2}}{2\frac{b_n}{k} \left(1 + \frac{z_{mn}^\circ}{b_n}\right)}}. \quad (10)$$

Obviously, there is a close relation between a Gaussian Beam as represented by (10) in a coordinate system  $x^\circ, z^\circ$  and a Complex Source Beam as represented by (6). To this end, we just have to express the latter in the coordinate system  $x^\circ, z^\circ$ , set  $b_c = b_n$ ,  $x = x_{mn}$ ,  $z = z_{mn}$ , and:

$$A_c = A_f j e^{-kb_n} \left(\frac{\sqrt{2}}{L}\right)^{1/2} \sqrt{8\pi k b_n}. \quad (11)$$



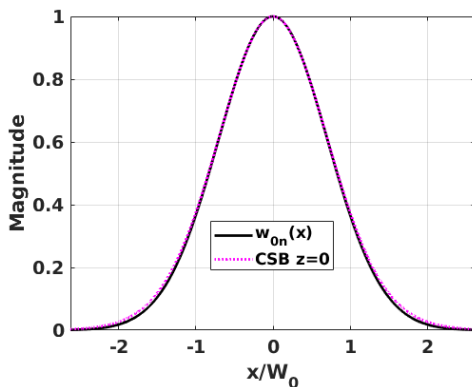
**Figure 1.** Basic geometry.

### 3 First Results

The following results provide comparisons in cases of practical interest for frame based GB shooting and bouncing algorithms used to analyze fields in large environments or multi-reflector systems. The frame parameters chosen for these comparisons are:  $L = 10\lambda$  and  $\nu = 0.25$ . With this choice, and considering that Gaussian windows (in space or spectrum) can be truncated at the relative threshold level of  $10^{-3}$ , the spectrum of  $w_{mn}$  is in the visible domain  $[-k_0, k_0]$  if and only if  $n \leq n_{max}$  with  $n_{max} = 17$ . We shall thus present results both for non tilted GBs ( $n = 0$ ) and for  $n = n_{max}$ , so that the plane wave spectrum integral involves only propagating waves.

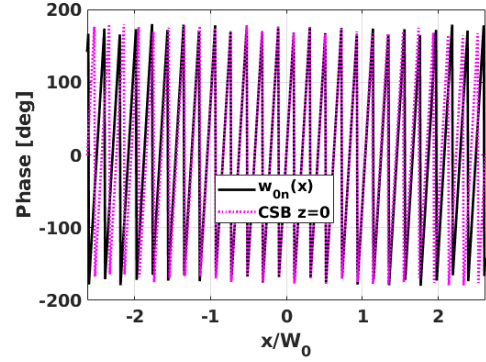
#### 3.1 Source field comparison between frame window sources and uniform CSBs

Figures 2 and 3 compare magnitudes and phases of a single frame window and of a uniform CSB, centered at origin, in the source plane  $z = 0$ .



**Figure 2.** Gaussian window and uniform CSB magnitudes along  $z = 0$ . Half width  $W_0 = L/\sqrt{\pi}$ .

The case when  $n = 0$  (no spectral translation of the frame window) is not presented as there is no visible difference with the CSB with  $\mathbf{R}' = -jb_c\hat{\mathbf{z}}$ . We present the case when



**Figure 3.** Gaussian window and uniform CSB phases along  $z = 0$ . Half width  $W_0 = L/\sqrt{\pi}$ .

the frame window spectral translation index is  $n = n_{max}$ , and we compare it with the uniform CSB field defined by:  $\mathbf{R}' = -jb_n\hat{\mathbf{z}}_{mn}$ , using notations introduced in parts 2.1 and 2.2. The frame window source distribution and the uniform CSB compare very well, with a slight discrepancy visible for  $x > 0.5L$ , which could be observed for any  $L$  value.

#### 3.2 Comparisons between CSBs and Frame-Window Radiation

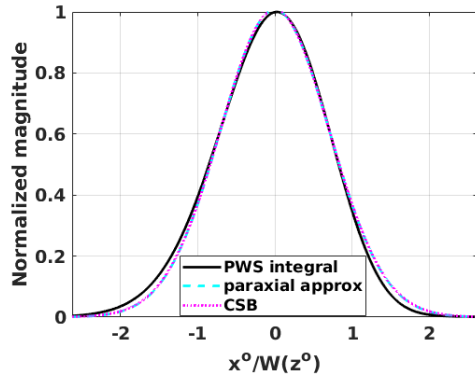
In this part we compare results obtained for frame-window radiated fields through PWS integral (3) and the paraxial approximation (4) with the CSB radiated field given by (5) (exact expression).

The comparison is performed on a line transverse to the considered beam axis (constant  $z^\circ$ ), and on a width which is expressed as a function of the usual paraxial Gaussian beam width parameter, defined in the beam related coordinate system and corresponding to the distance to the beam axis at which the field is divided by  $e$  relative to its norm on the axis:  $W(z^\circ) = \sqrt{\frac{2((z^\circ)^2 + b_n^2)}{kb_n}}$ , down to values of the order of  $10^{-4}$  relative to on axis norm. The chosen  $n$  value is  $n_{max}$ .

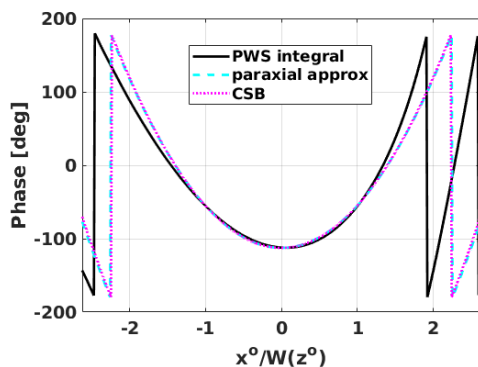
In Figures 4 and 5 the distance  $z^\circ$  is taken equal to the collimation distance  $b_n$ . In Figures 6 and 7 this distance is taken equal to  $30b_n$ . All field magnitudes are normalized to the maximum of the reference field magnitude, computed by the PWS integral.

As expected, at distance  $b_n$  the GB paraxial approximation and the CSB fields look identical (magnitude and phase), and perfectly symmetric with respect to the beam axis, while the reference solution is visibly not symmetric. This discrepancy is noticeable at distances from the axis larger than  $W(z^\circ)$ . At distance  $z^\circ = 30b_n$  along the beam axis, the maximum of the reference solution magnitude is not exactly localized on the beam axis (GB or CSB), but the CSB phase appears to fit very well with the reference solution, contrary to the GB phase at some distance from the beam

axis. This confirms that CSBs provide accurate expressions for beam fields in regions where the paraxial approximation is not valid. Yet, if radiated by a linearly varying phase source window, a beam cannot be substituted with a single CSB without altering the radiated field magnitude.



**Figure 4.** Normalized radiated field along  $z^o = b_n$ .

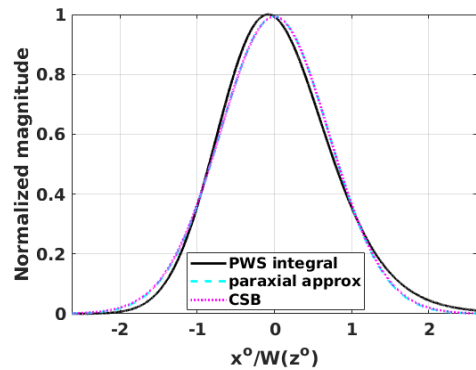


**Figure 5.** Gaussian window and uniform CSB phases along  $z^o = b_n$ .

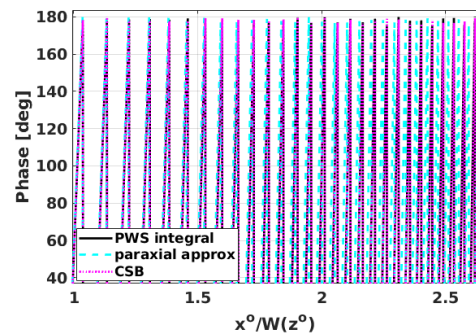
## 4 Conclusion

In order to combine the interesting features of frame-based GB shooting and of CSBs, the problem of representing the fields radiated by frame window source distributions as a function of one or several CSBs has to be addressed. The first results presented in this paper suggest two different directions for future work:

- substitute the field radiated by each frame window distribution with a single CSB and explore the accuracy of such an algorithm in scenarios involving obstacles;
- find combinations of several CSBs which would more accurately represent the radiation of each single frame window distribution.



**Figure 6.** Normalized radiated field along  $z^o = 30b_n$ .



**Figure 7.** Gaussian window and uniform CSB phases along  $z^o = 30b_n$  (zoom).

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