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Abstract

This paper proposes a degradation model for a two-unit series system, where the components exhibit dependence and are subject to imperfect maintenance. The interdependence between both components is captured using the trivariate reduction method. The system failure occurs when the degradation level of either component exceeds a predetermined threshold. The main motivation of this work is to develop a preventive maintenance strategy for this system, incorporating periodic imperfect maintenance actions in order to extend its useful lifetime. These actions aim to reduce the accumulated degradation level of each component from its installation in a fixed percentage, following the *Arithmetic Reduction of Degradation model* of infinite order. Another goal is to derive the distribution of time to the system failure, which provides crucial insights into the system's reliability and performance, especially in the case of bivariate degradation. Additionally, a cost model for this maintenance strategy is developed and several numerical examples are presented to illustrate the practical implications. The maintenance decision variables are optimised in order to obtain the minimal expected cost rate within a finite time horizon.

Notation list

FHT	*First Hitting Time*
ARD(∞)	*Arithmetic Reduction of Degradation of infinite order*
$X_i(t)$	*Degradation of component i in absence of preventive maintenance actions*
$\mathbf{X}$(t)	*Baseline degradation process*
$W_0(t)$	*Degradation process shared by both components*
$W_i(t)$	*Intrinsic degradation process for component i, $i = 1, 2$.*

$Y_i(t)$	*Degradation of maintained component* i, $i = 1, 2$.
$\mathbf{Y}(\mathrm{t})$	*Degradation of the maintained system*
μ_i	*Drift parameter,* $i = 1, 2$
σ_i	*Diffusion parameter,* $i = 0, 1, 2$
$\Lambda(t)$	*General time scale*
$\theta(t)$	*Pearson correlation coefficient between* $Y_1(t)$ *and* $Y_2(t)$
$\alpha_i(t)$	*Gamma process shape parameter* $i = 0, 1, 2$.
b	*Gamma process scale parameter*
L_i	*Failure threshold of component* i
ρ	*Preventive maintenance actions efficiency*
T	*Time between preventive maintenance actions*
T_f	*Time to the system failure*
$\bar{F}_{T_f}$	*Survival function of* T_f
$\tilde{T}_f$	*First hitting time of the unmaintained process*
T_R	*Time to a system replacement*
$C_I(\rho)$	*Preventive maintenance cost*
C_d	*Downtime cost*
C_c	*Replacement corrective cost*
$r_i(\cdot)$	*Reward function of component* i
pdf	*Probability density function*

1 Introduction

Initial research in degradation works focus on describing univariate models. Typical mathematical tools to model the degradation evolution are the stochastic processes. Many stochastic-process-based models have been introduced to describe the degradation evolution based, for example, on Wiener processes ([25], [7]), gamma processes ([27], [3], [5]) or Inverse Gaussian processes ([9]). Although the degradation evolution in unitary systems has been extensively studied, the research on modeling multi-component degrading systems is more limited. In practice, industrial systems have a complex structure with more interrelated parts or components that influence the system performance. Availability and reliability measures are highly dependent on the number of the components in the system. Besides, for multi-component systems, their components may show dependence. For instance, in lighting systems consisting on many LED lamps [24], the system's performance characteristics (PCs) present a likely dependence because of the common usage history. Another example can be found in rubidium discharge lamps [18], whose degradation can be described by the rubidium consumption or the decreasing intensity of the lamp.

Several works on degradation modelling of complex systems with dependent components have been developed in the literature [6], [13], [21], [28]. They can be roughly divided into three groups [10]:

- Multivariate-based methods. The components are dependent with a known multivariate distribution [13].
- Copula-based methods. A copula function separates the marginal distributions from the dependency structure of a given multivariate distribution [18].
- Degradation rate interaction methods. The deterioration of one component affects the deterioration of the other components [1].

Multivariate-based methods consist in building multivariate stochastic processes by the superposition of independent univariate stochastic processes. Among the multivariate-based methods, the trivariate reduction method ([13]) has attracted great attention in the last years. A broad definition of the trivariate reduction method is as follows. Given three independent stochastic processes $\{X_1(t), t \geq 0\}$, $\{X_2(t), t \geq 0\}$ and $\{X_3(t), t \geq 0\}$, the dependent processes $\{X(t), t \geq 0\}$ and $\{Y(t), t \geq 0\}$ are formed as

$$X(t) = X_1(t) + X_3(t), \quad Y(t) = X_2(t) + X_3(t). \tag{1}$$

The use of this trivariate reduction method is not new in the degrading systems literature. For example, in [13] and [20], the bivariate reduction method given by Eq. (1) is used from three independent gamma processes. An early attempt of using Eq.(1) with Wiener processes is given by [8]. Recently, in [26] and [30], the trivariate reduction method with univariate Wiener processes is used.

Once the behaviour of a degrading system is modelled, the time to the system failure can be assessed and different maintenance actions can be planned in order to enlarge the remaining useful system life. These maintenance actions can be perfect (when the system is also restored to "as good as new" condition) or imperfect (where the system is not restored to a new one). Although there are a large number of papers dealing with multivariate degradation processes, developing imperfect maintenance models for them remains an open problem [15]. Early attempts can be found in [26], where a multivariate degradation process with time-variant covariates and imperfect maintenance effects is analyzed. Analogously, in [30], repairs are performed in a two-component system where the repairs rebalance the system. In consequence, developing imperfect maintenance policies in the context of multivariate dependent degradation is an important issue in this work [11], [14] . In practice, as the system degrades, its performance would decrease as well which results in increasing operating cost as it is shown in [2]. In the same way, the repair cost can increase along with the maintenance efficiency. In this case, decision makers should take into account these factors when making maintenance policies in order to minimize the associated costs. Hence, a maintenance policy is developed in this paper considering a deterioration-dependent operating cost and a repair-efficiency-dependent cost.

For stochastic-process-based models, failure time of a degrading system can be derived by considering the time at which the degradation path of a component first reaches a threshold level. The work from [29] provides a systematic literature review on dependent failure behaviour in risk and reliability. For unitary degrading systems, the system reliability has been widely analyzed. If a Wiener process is used to model the degradation of an unitary system, it is well known that the first hitting time to reach a threshold follows an Inverse Gaussian distribution. If a gamma process is used, the first hitting time distribution is obtained by using the upper incomplete gamma function. However, as [23] claims, the results for the multi-component system reliability are scarce in the stochastic-process-based model context. In a bivariate gamma distribution context, the first hitting time to exceed a threshold can be approximated by using the bivariate Birnbaum–Saunders distribution [19]. In this sense, in [4] and [23], the first hitting time of a bivariate Wiener process is developed. However, these two works do not incorporate maintenance actions when evaluating the first hitting time.

In this paper, we focus on the derivation of the system reliability in two-component degrading systems subject to imperfect maintenance actions. The model developed in this paper can be inspired for example by the real-world system considered in [13]. In that paper, the intervention scheduling of a railway track is discussed based on the observation of two dependent randomly increasing deterioration indicators, modelled through a bivariate gamma process. On the other hand, the results obtained are useful from an applied engineering point of view, since the reliability of a maintained system is developed. This reliability is crucial to study the RUL (Remaining Useful Lifetime) of a degrading

system [12]. The increasing expectation and variance in the deterioration level of the components imply the convenience of applying a maintenance policy. Other practical aspect considered in this paper is the deterioration-dependent operation cost. In practice, as the system degrades, its performance would decrease as well which results in increasing operating cost (see [2]).

The challenges proposed in this article lie in various important questions: how to model the evolution of multivariate dependent degradations of the degrading system, how to integrate imperfect maintenance actions, how to derive the failure time and how to design a maintenance policy. In response to these questions, in this paper, we focus on a two dependent component degrading system. The degradation evolution of each component is modelled using Wiener and Gamma processes. The dependence between components is described using the trivariate reduction method. Furthermore, imperfect maintenance actions are periodically performed reducing the accumulated deterioration level of each component from its installation. Following the idea of the Arithmetic Reduction of Degradation of infinite order (ARD(∞)) model introduced in [14] and more recently developed in [26], the maintenance actions reduce the level of degradation of an amount which is proportional to its value. Assuming that the system fails when the deterioration level of a component exceeds a failure threshold, the failure time distribution of the maintained system is derived in this paper.

It is necessary for an ARD imperfect maintenance that preventive maintenance actions are performed at planned times, but these times need not to be periodic. However, for optimizing the preventive maintenance strategy, the periodic assumption is necessary since the periodicity is one of the criteria to optimize. This is not contradictory at all with condition-based maintenance (CBM). In fact, we are implementing both: condition-based maintenance if the system requires it, that is, if it is too degraded (i.e. the failure level is exceeded) or only the imperfect maintenance policy at periodic inspection times if the system is in a good condition. In other words, the imperfect repair is always included through ARD at periodic times unless the system is failed; in this case, we performed a corrective repair which consists in resetting the system degradation to 0. Furthermore, an optimal maintenance policy is analyzed considering deterioration-dependent operation cost and a repair-efficiency-dependent cost and assuming a finite horizon.

The remainder of this paper is organised as follows. The stochastic model describing the system degradation evolution and the imperfect maintenance strategy are presented in Section 2. The distribution of the first hitting time is studied in Section 3. Section 4 develops the cost model formulation taking into account the imperfect maintenance strategy previously defined. Numerical examples are given in Section 5 to evaluate the performance of this maintenance policy. Finally, Section 6 concludes and provides possible future research lines.

2 System degradation and imperfect maintenance modelling

The assumptions of this work are presented in Section 2.1. In Section 2.2, the degradation modelling is developed. Section 2.3 is devoted to the development of the imperfect preventive maintenance.

2.1 General assumptions

In this paper, we deal with a system with two dependent degrading components under the following general assumptions.

1. This system consists of two components subject to an internal degradation. Two models are developed to describe the degradation evolution of each component: the gamma-based model and the Wiener-based model.

2. The degradation of both components is correlated and this correlation is modelled using the trivariate reduction technique.

3. We assume that the system fails when the degradation level of a component first reaches a failure threshold. Let L_1 and L_2 be the respective failure thresholds of both components.

4. The system failure is dormant, that is, it can only be discovered through inspections.

5. The system is inspected each T time units. In these inspection times, if the system is working, imperfect preventive maintenance actions are performed. We assume that these imperfect maintenance actions (PM) reduce the accumulated degradation of each component in a ρ% with $0 \leq \rho < 1$. On the contrary, if the system is failed in an inspection time, the system is replaced by a new one (corrective replacement).

6. Corrective replacement implies a cost of C_c monetary units.

7. Imperfect preventive maintenance actions imply a cost of $C_I(\rho)$ monetary units. This cost increases with the maintenance efficiency parameter ρ.

8. Since failures are only detected at inspections, a system downtime cost of C_d monetary units per unit time is incurred.

9. A reward is obtained when the system is working. This reward depends on the degradation level of the component and it decreases as the component degradation increases. We denote by $r_i(x)$ the reward given by component i when its degradation is x. This function $r_i(x)$ is decreasing in x. We assume that, as long as the system is failed, no reward is produced in the system.

10. The system operates for a finite time period T_{max}.

11. The maintenance duration is negligible.

The development of these assumptions will be developed in this section and the subsequent ones.

2.2 Degradation modelling of the system

Based on the above-listed assumptions, we propose a bivariate stochastic degradation model with imperfect maintenance and we analyze some of its properties.

Firstly, the evolution of the degradation of the two components without maintenance is described. For $i = 1, 2$, let $\{X_i(t),\ t \geq 0\}$ be the degradation level of component i at time t. Starting from three independent univariate processes $\{W_i(t),\ t \geq 0\}$, for $i = 0, 1, 2$ and following the trivariate reduction technique, the degradation of component i is modelled as

$$X_i(t) = W_i(t) + W_0(t), \quad i = 1, 2, \tag{2}$$

where $W_0(t)$ stands for the common degradation part of both components. $W_0(t)$ represents different random factors affecting the two components simultaneously (the influence of the environment, for instance). $W_i(t)$ stands for the particular degradation process for each component describing the uncertainty within components.

The evolution of $X_i(t)$ is analyzed considering univariate Wiener processes and univariate gamma processes for W_i and W_0 respectively in Eq.(2). We call these models as *Wiener-based model* and *gamma-based model* respectively.

2.2.1 Wiener-based model

By using the same approach shown in [8] and [26], processes $W_i(t)$ $(i = 1, 2)$ and $W_0(t)$ are given by

$$W_i(t) = \mu_i \Lambda(t) + \sigma_i B_i(\Lambda(t)), \quad W_0(t) = \sigma_0 B_0(\Lambda(t)), \tag{3}$$

where B_0 and B_i (i=1,2) are independent standard Brownian processes, hence

$$\begin{aligned} X_1(t) &= \mu_1 \Lambda(t) + \sigma_1 B_1(\Lambda(t)) + \sigma_0 B_0(\Lambda(t)) \\ X_2(t) &= \mu_2 \Lambda(t) + \sigma_2 B_2(\Lambda(t)) + \sigma_0 B_0(\Lambda(t)). \end{aligned} \tag{4}$$

Parameter μ_i is also known as the drift parameter, which indicates the rate of degradation of W_i and σ_i denotes the diffusion coefficient of W_i for $i = 0, 1, 2$. $\Lambda(t)$ is a non decreasing function in t, with $\Lambda(0) = 0$, which captures the non-linearity in the random variables $X_1(t)$ and $X_2(t)$. The marginal processes of $\mathbf{X}(t) = (X_1(t), X_2(t))$ are univariate normal with expectation and variance

$$\mathbb{E}[X_i(t)] = \mu_i \Lambda(t) \quad Var(X_i(t)) = \left(\sigma_0^2 + \sigma_i^2\right) \Lambda(t) \quad i = 1, 2.$$

When $\Lambda(t) = t$, process $X_i(t)$ has a linear mean path $\mu_i t$.

The Pearson correlation coefficient between $X_1(t)$ and $X_2(t)$ is equal to

$$\frac{\sigma_0^2}{\sqrt{\sigma_0^2 + \sigma_1^2}\sqrt{\sigma_0^2 + \sigma_2^2}}.$$

This correlation coefficient is positive and it does not depend on t. It cannot reach 1 unless $\sigma_1^2 = \sigma_2^2 = 0$. When the common noise is dominant ($\sigma_0^2 >> \sigma_i^2$), the correlation between $X_1(t)$ and $X_2(t)$ would be stronger. When the common noise is negligible, the two Wiener processes evolve almost independently.

2.2.2 Gamma-based model

By using gamma processes to model the evolution of $W_0(t)$, $W_1(t)$ and $W_2(t)$, the degradation of each component is given by

$$\begin{aligned} X_1(t) &= W_1(t) + W_0(t) \\ X_2(t) &= W_2(t) + W_0(t), \end{aligned}$$

where $W_i(t)$ are independent gamma processes with shape parameter $\alpha_i(t)$ and same scale parameter b for $i = 0, 1, 2$. We assume that $\alpha_i(t)$ are right continuous, non decreasing real valued functions. Process $\{X_i(t), t \geq 0\}$ is univariate gamma distributed with shape parameter $\alpha_i(t) + \alpha_0(t)$ and scale parameter b. The expectation and variance of these processes are equal to

$$\mathbb{E}[X_i(t)] = (\alpha_i(t) + \alpha_0(t))b, \quad Var(X_i(t)) = (\alpha_i(t) + \alpha_0(t))b^2.$$

The Pearson correlation coefficient between $X_1(t)$ and $X_2(t)$ in the gamma-based model depends on t and it is equal to

$$\frac{\alpha_0(t)}{\sqrt{\alpha_1(t) + \alpha_0(t)}\sqrt{\alpha_2(t) + \alpha_0(t)}}.$$

It means that, when $\alpha_0(t)$ is dominant, the correlation between $X_1(t)$ and $X_2(t)$ is stronger. When $\alpha_0(t)$ is negligible, $X_1(t)$ and $X_2(t)$ evolve almost independently.

Next section describes the implementation of the imperfect preventive maintenance policy given in Assumption 5.

2.3 Imperfect maintenance policy

In absence of corrective replacements (see Assumption 5), each T time units, an imperfect PM action is performed and the overall degradation if the components is reduced in a $\rho\%$ with $0 \leq \rho \leq 1$. Let $\mathbf{Y}(t) = (Y_1(t), Y_2(t))$ be the process that describes the degradation of the maintained system and $\{Y_i(t),\ t \geq 0\}$ be the degradation level of the maintained component i, for $i = 1, 2$. Assuming

independent increments for $X_i(t)$, the piece-wise evolution of $\{Y_i(t), t \geq 0\}$ is as follows.

In t with $0 \leq t < T$, the degradation level of the i-th component is given by

$$Y_i(t) = X_i(t).$$

At time T, the first imperfect PM action is performed and the degradation of the two components is reduced in a $\rho\%$. Denoting by T^+ the instant of time just after the first PM action, we get that

$$Y_i(T^+) = (1-\rho)X_i(T).$$

When $T < t < 2T$, the evolution of the degradation of component i is given by

$$Y_i(t) = Y_i(T^+) + X_i(t) - X_i(T).$$

At time $2T^-$ (resp. $2T^+$), just before (resp. after) the second imperfect PM action, the degradation level of the maintained component i is given by

$$\begin{aligned} Y_i(2T^-) &= Y_i(T^+) + X_i(2T) - X_i(T) \\ Y_i(2T^+) &= (1-\rho)\left(Y_i(T^+) + X_i(2T) - X_i(T)\right) \end{aligned}$$

In a general setting, just before (resp. after) the n-th imperfect PM action, the degradation of component i is given by

$$\begin{aligned} Y_i(nT^-) &= Y_i\left((n-1)T^+\right) + X_i(nT) - X_i((n-1)T) \\ Y_i(nT^+) &= (1-\rho)\left(Y_i((n-1)T^+) + X_i(nT) - X_i((n-1)T)\right). \end{aligned}$$

After some straightforward calculus, we get that

$$Y_i(nT^+) = \sum_{j=1}^{n} (1-\rho)^{n-j+1} \left(X_i(jT) - X_i((j-1)T)\right). \tag{5}$$

Hence, the effect of the n-th imperfect maintenance action is given by

$$\begin{aligned} Y_i(nT^+) - Y_i(nT^-) &= (1-\rho)Y_i(nT^-) - Y_i(nT^-) \\ &= -\rho Y_i(nT^-) \\ &= -\rho\left(Y_i\left((n-1)T^+\right) + X_i(nT) - X_i((n-1)T)\right) \\ &= -\rho \sum_{j=1}^{n} (1-\rho)^{n-j} \left(X_i(jT) - X_i((j-1)T)\right). \end{aligned}$$

Finally, for $nT \leq t < (n+1)T$, the degradation of component i at time t is given by

$$\begin{aligned} Y_i(t) &= Y_i(nT^+) + (X_i(t) - X_i(nT)) \\ &= \sum_{j=1}^{n} (1-\rho)^{n-j+1} \left(X_i(jT) - X_i((j-1)T)\right) \\ &+ \left(X_i(t) - X_i(nT)\right). \end{aligned} \tag{6}$$

The distribution of the degradation level of the maintained system is next analyzed for the two particular cases under consideration: the Wiener-based case and the gamma-based case.

2.3.1 Wiener-based model

Using the additivity property of the normal distribution, the degradation after the n-th PM action is normally distributed with expectation

$$\mathbb{E}\left[Y_i(nT^+)\right] = \mu_i \sum_{j=1}^{n} (1-\rho)^{n-j+1} \Delta\Lambda(jT),$$

and variance

$$Var\left(Y_i(nT^+)\right) \quad = \quad \left(\sigma_0^2 + \sigma_i^2\right) \sum_{j=1}^{n} (1-\rho)^{2(n-j+1)} \Delta\Lambda(jT),$$

where $\Delta\Lambda(jT)$ denotes the increment of the function Λ

$$\Delta\Lambda(jT) = \Lambda(jT) - \Lambda((j-1)T). \tag{7}$$

The degradation $Y_i(t)$ at time t for the maintained component i, with $nT \leq t < (n+1)T$, is normally distributed with expectation and variance given by

$$\begin{aligned} \mathbb{E}\left[Y_i(t)\right] &= \mathbb{E}\left[Y_i(nT^+)\right] + \mu_i(\Lambda(t) - \Lambda(nT)) \\ Var\left(Y_i(t)\right) &= Var\left(Y_i(nT^+)\right) + \left(\sigma_0^2 + \sigma_i^2\right)(\Lambda(t) - \Lambda(nT)). \end{aligned}$$

Remark 1. *It is easy to check that* $\mathbb{E}\left[Y_i(t)\right]$ *and* $Var\left(Y_i(t)\right)$ *are decreasing with respect to* ρ *for* $i = 1, 2$.

Example 1. *Using a Wiener-based model, realizations of the evolution of the degradation in absence of maintenance are given in Figure 1 assuming*

$$X_1(t) \quad = \quad 3.5\Lambda(t) + 4.5B_1(\Lambda(t)) + 4B_0(\Lambda(t)) \tag{8}$$

$$X_2(t) \quad = \quad 4\Lambda(t) + 3.5B_2(\Lambda(t)) + 4B_0(\Lambda(t)), \tag{9}$$

for different $\Lambda(t)$. *Figures 2 and 3 show the evolution of the degradation incorporating imperfect PM actions. These imperfect PM actions are performed each* $T = 5$ *time units with an efficiency equals to* $\rho = 0.5$.

Remark 2. *Notice that, if* $\rho = 1$, *the expected degradation at time* t *is given by*

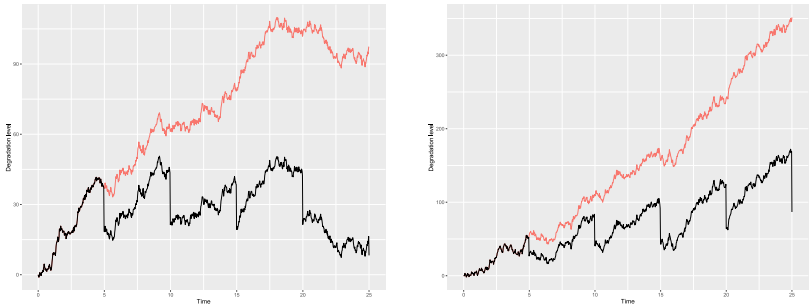
$$\mathbb{E}[Y_i(t)] = \mu_i \left(\Lambda(t) - \Lambda(nT)\right), \qquad nT \leq t < (n+1)T.$$

In the case of a corrective replacement (the replacement of a component by a new one) performed at time nT, *the expected degradation is equal to*

$$\mathbb{E}[Y_i(t)] = \mu_i \Lambda(t - nT), \quad nT \leq t < (n+1)T.$$

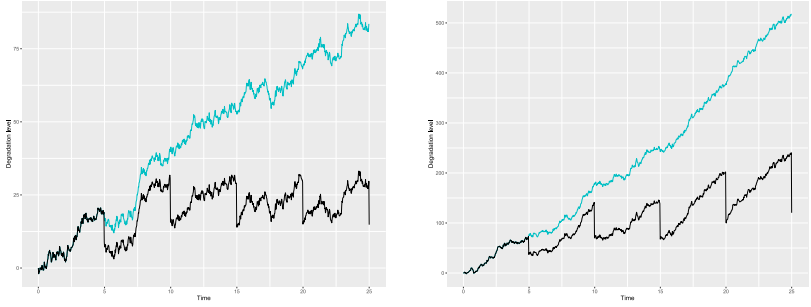
(a) $\Lambda(t) = t$ (b) $\Lambda(t) = 2t^{1.3}$

Figure 1: Realization of processes $X_1(t)$ (red) and $X_2(t)$ (blue).



(a) $\Lambda(t) = t$ (b) $\Lambda(t) = 2t^{1.3}$

Figure 2: Realization of the process $Y_1(t)$ (black) and corresponding $X_1(t)$ (red).



(a) $\Lambda(t) = t$ (b) $\Lambda(t) = 2t^{1.3}$

Figure 3: Realization of the process $Y_2(t)$ (black) and corresponding $X_2(t)$ (blue).

It is straightforward to check that, if $\Lambda(\cdot)$ is convex and $\Lambda(0) = 0$, then

$$\Lambda(t) - \Lambda(nT) \geq \Lambda(t - nT),$$

hence the expected degradation after a PM action with the maximum efficiency ($\rho = 1$) is greater than the expected degradation after a complete replacement of the component. The PM action resets the accumulated degradation to zero but not the system age. This is because $\mathbb{E}[Y_i(t)] \neq \mathbb{E}[Y_i(t - nT)]$ when $nT \leq t < (n+1)T$ and a CM action is performed at time nT, except for the linear case $\Lambda(t) = t$.

At time t, with $nT \leq t < (n+1)T$, the covariance of $Y_1(t)$ and $Y_2(t)$ is given by

$$Cov(Y_1(t), Y_2(t)) = \sigma_0^2 \left(\sum_{i=1}^{n} (1-\rho)^{2(n-i+1)} \Delta\Lambda(iT) + \Lambda(t) - \Lambda(nT) \right),$$

hence the Pearson correlation coefficient between $Y_1(t)$ and $Y_2(t)$ is positive and equal to

$$\theta = \frac{\sigma_0^2}{\sqrt{\sigma_0^2 + \sigma_1^2}\sqrt{\sigma_0^2 + \sigma_2^2}}, \tag{10}$$

which is constant with time, and also the same as this between $X_1(t)$ and $X_2(t)$. This coefficient θ is also constant in n, which means that the maintenance actions have not affected the correlation between both components.

Next, we shall prove some results on the monotonicity of the expectation and variance with respect to the time. Firstly, the following definition is given.

Definition 1. *We say that Λ has increasing increments if*

$$\Lambda(t_1 + T) - \Lambda(t_1) \leq \Lambda(t_2 + T) - \Lambda(t_2),$$

whenever $T > 0$ and $t_1 \leq t_2$.

Remark 3. *If Λ is convex, then it has increasing increments.*

Lemma 1. *In the Wiener-based model, for fixed T, $n \geq 1$, $0 < \rho \leq 1$ and $\mu_i \geq 0$ for $i = 1, 2$, if Λ has increasing increment we get that*

$$\begin{aligned} \mathbb{E}[Y_i(nT^+)] &\leq \mathbb{E}[Y_i((n+1)T^+)], \\ Var(Y_i(nT^+)) &\leq Var(Y_i((n+1)T^+)). \end{aligned}$$

Proof. We get that the difference between expectations is equal to

$$\begin{aligned} \mathbb{E}[Y_i((n+1)T^+)] - \mathbb{E}[Y_i(nT^+)] &= \sum_{j=1}^{n} \mu_i (1-\rho)^{n-j+1} (\Delta\Lambda((j+1)T) - \Delta\Lambda(jT)) \\ &+ \mu_i \Delta\Lambda(T)(1-\rho)^{n+1}. \end{aligned}$$

If $\Lambda(\cdot)$ has increasing increments, then

$$\Delta\Lambda((j+1)T) - \Delta\Lambda(jT) > 0,$$

hence

$$\mathbb{E}[Y_i((n+1)T^+)] \geq \mathbb{E}[Y_i(nT^+)].$$

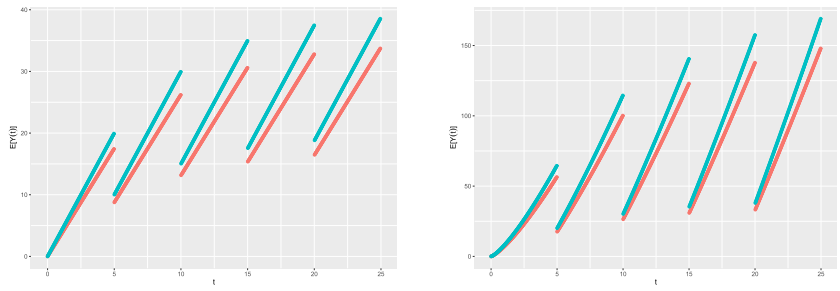
For the variances, the reasoning is analogous since

$$\begin{aligned}
&\frac{Var(Y_i((n+1)T^+)) - Var(Y_i(nT^+))}{(\sigma_0^2 + \sigma_i^2)} \\
&= \sum_{j=1}^{n} (1-\rho)^{2(n-j+1))} (\Delta\Lambda((j+1)T) - \Delta\Lambda(jT)) \\
&+ \Delta\Lambda(T)(1-\rho)^{2(n+1)},
\end{aligned}$$

and the result holds. □

Lemma 1 is true whatever the value of ρ. It means that the degradation increases in mean and in variance despite the PM. If $\Lambda(\cdot)$ has increasing increments, the growth of the degradation is such that no maintenance will be able to compensate it.

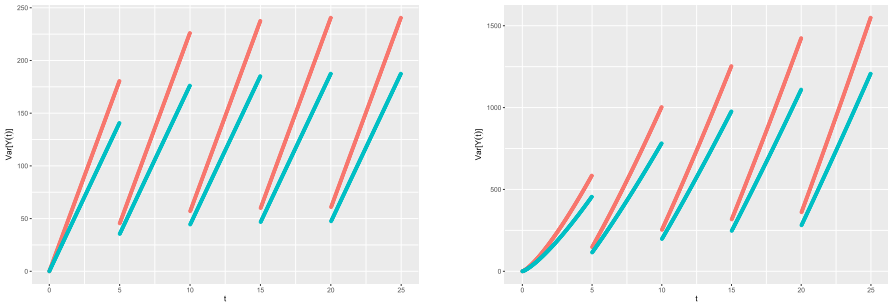
This property is illustrated in Figures 4 and 5, which show respectively the expectations and variances of $Y_1(t)$ and $Y_2(t)$ versus t. These plots have been obtained assuming the Wiener-based model given in Eqs.(8) and (9). PM actions are performed each $T = 5$ time units with maintenance efficiency $\rho = 0.5$.



(a) $\Lambda(t) = t$ (b) $\Lambda(t) = 2t^{1.3}$

Figure 4: Expectation of processes $Y_1(t)$ (red) and $Y_2(t)$ (blue) versus t.

The aim of the following lemma is to give a result analogous to Lemma 1 when the degradation levels are compared inside PM intervals instead of just after maintenance.



(a) $\Lambda(t) = t$ (b) $\Lambda(t) = 2t^{1.3}$

Figure 5: Variance of processes $Y_1(t)$ (red) and $Y_2(t)$ (blue) versus t.

Lemma 2. *Let t_1 and t_2 with $t_1 < t_2$. If $\Lambda(\cdot)$ has increasing increments and t_1 and t_2 fulfil*

$$t_1 - \lfloor t_1/T \rfloor T = t_2 - \lfloor t_2/T \rfloor T,$$

then

$$\mathbb{E}[Y_i(t_1)] \leq \mathbb{E}[Y_i(t_2)], \quad Var(Y_i(t_1)) \leq Var(Y_i(t_2)).$$

Proof. If $\lfloor t_1/T \rfloor = \lfloor t_2/T \rfloor = n$, the result holds since $t_1 = t_2$. Then,

$$\mathbb{E}[Y_i(t_1)] = \mathbb{E}[Y_i(t_2)].$$

If

$$\lfloor t_1/T \rfloor = n_1 < \lfloor t_2/T \rfloor = n_2,$$

then

$$\begin{aligned}
\mathbb{E}[Y_i(t_1)] &= \mu_i \left(\sum_{j=1}^{n_1} (1-\rho)^{n_1-j+1} \Delta\Lambda(jT) + (\Lambda(t_1) - \Lambda(n_1 T)) \right) \\
\mathbb{E}[Y_i(t_2)] &= \mu_i \left(\sum_{j=1}^{n_2} (1-\rho)^{n_2-j+1} \Delta\Lambda(jT) + (\Lambda(t_2) - \Lambda(n_2 T)) \right),
\end{aligned}$$

The inequality $\mathbb{E}[Y_i(t_1)] \leq \mathbb{E}[Y_i(t_2)]$ is simply fulfilled since

$$\mu_i \sum_{j=1}^{n_1} (1-\rho)^{n_1-j+1} \Delta\Lambda(jT) \leq \mu_i \sum_{j=1}^{n_2} (1-\rho)^{n_2-j+1} \Delta\Lambda(jT),$$

and thanks to the increasing increments assumption for $\Lambda(\cdot)$. Let $x = t_1 - \lfloor t_1/T \rfloor T = t_2 - \lfloor t_2/T \rfloor T$, then

$$\begin{aligned}
\Lambda(t_1) - \Lambda(n_1 T) &= \Lambda(x + n_1 T) - \Lambda(n_1 T) \\
&\leq \Lambda(x + n_2 T) - \Lambda(n_2 T) = \Lambda(t_2) - \Lambda(n_2 T).
\end{aligned}$$

The reasoning for the monotonicity of the variance is analogous.

□

2.3.2 Gamma-based model

In $0 \leq t < T$, no imperfect PM action is performed. Hence $Y_i(t)$ is a gamma process with parameters $\alpha_i(t) + \alpha_0(t)$ and b. At time T, the first PM action is performed, hence at time t with $T \leq t < 2T$, the degradation level of the maintained component i is equal to

$$Y_i(t) = (1-\rho)X_i(T) + X_i(t) - X_i(T).$$

Since $(1-\rho)X_i(T)$ is gamma distributed with shape parameter $\alpha_i(T)+\alpha_0(T)$ and scale parameter $b(1-\rho)$, and $X_i(t) - X_i(T)$ is gamma distributed with parameters $\alpha_i(t)+\alpha_0(t)-\alpha_i(T)-\alpha_0(T)$ and b, then $Y_i(t)$ is no longer gamma distributed. The evolution of the degradation level of the maintained system $\{\mathbf{Y}(t),\ t \geq 0\}$ is bivariate gamma distributed in $[0,T]$, but this distribution is not kept beyond T. In a general setting, we get that

$$\begin{aligned} Y_i(t) &= Y_i(nT^+) + X_i(t) - X_i(nT) \\ &= \sum_{j=1}^{n} (1-\rho)^{n-j+1} \Delta X_i(jT) + X_i(t) - X_i(nT), \end{aligned} \tag{11}$$

where

$$\Delta X_i(jT) = X_i(jT) - X_i((j-1)T).$$

Variable $(1-\rho)^{n-j+1}\Delta X_i(jT)$ is gamma distributed with shape parameter

$$\alpha_i(jT) + \alpha_0(jT) - \alpha_i((j-1)T) - \alpha_0((j-1)T)$$

and scale parameter $b(1-\rho)^{n-j+1}$. Hence the sum

$$\sum_{j=1}^{n} (1-\rho)^{n-j+1} \Delta X_i(jT) + X_i(t) - X_i(nT)$$

is no longer gamma distributed if $\rho \neq 0$.

Example 2. *Figure 6 shows the realization of two non homogeneous gamma degradation processes $X_1(t)$ and $X_2(t)$ obtained from the corresponding initial processes $W_0(t)$, $W_1(t)$ and $W_2(t)$ with shape parameters $\alpha_0(t) = t$, $\alpha_1(t) = 1.3t^{1.2}$ and $\alpha_2(t) = 1.8t^{1.5}$, respectively, and common scale parameter $b = 1$. Imperfect PM actions are performed each $T = 5$ time units with a maintenance efficiency equals to $\rho = 0.5$. The evolution of the degradation in the maintained system is plotted in Figure 7.*

The behaviour of the expectation and variance of the deterioration in the maintained system are next obtained.

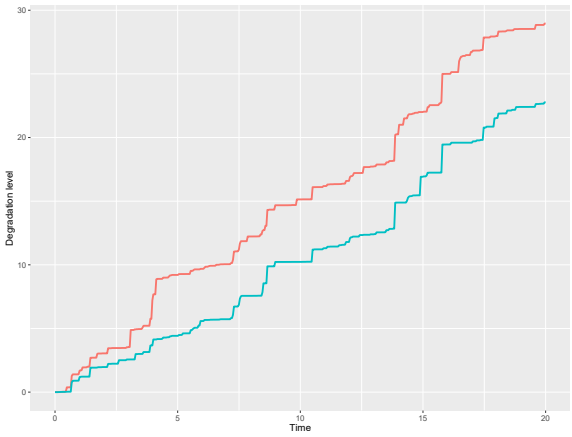
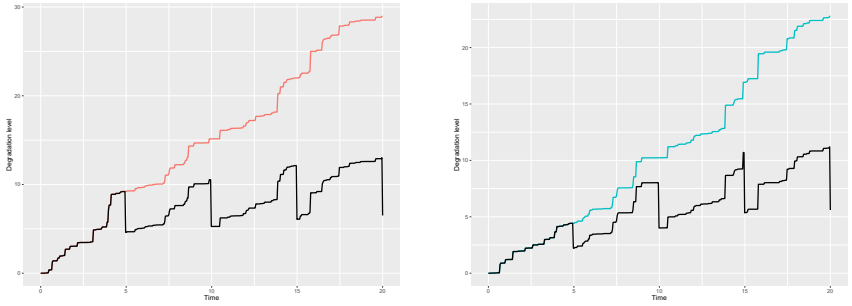


Figure 6: Realization of the gamma processes $X_1(t)$ (red) and $X_2(t)$ (blue).



(a) Process $Y_1(t)$ (b) Process $Y_2(t)$

Figure 7: Realization of processes $Y_1(t)$ and $Y_2(t)$ (black) with corresponding $X_1(t)$ (red) and $X_2(t)$ (blue).

For $nT \leq t < (n+1)T$, the expectation and the variance of $Y_i(t)$, for $i = 1, 2$, are given by

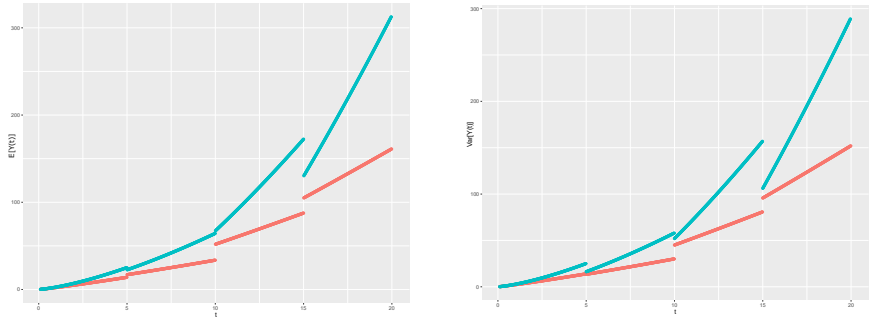
$$\begin{aligned}
\mathbb{E}[Y_i(t)] &= \sum_{j=1}^{n} (1-\rho)^{n-j+1} \left(\Delta\alpha_i(jT) + \Delta\alpha_0(jT)\right) b \\
&+ \left(\alpha_i(t) + \alpha_0(t) - \alpha_i(nT) - \alpha_0(nT)\right) b \\
Var(Y_i(t)) &= \sum_{j=1}^{n} (1-\rho)^{2(n-j+1)} \left(\Delta\alpha_i(jT) + \Delta\alpha_0(jT)\right) b^2 \\
&+ \left(\alpha_i(t) + \alpha_0(t) - \alpha_i(nT) - \alpha_0(nT)\right) b^2
\end{aligned}$$

where for $j = 1, 2, \ldots,$

$$\begin{aligned}
\Delta\alpha_i(jT) &= \alpha_i(jT) - \alpha_i((j-1)T), \\
\Delta\alpha_0(jT) &= \alpha_0(jT) - \alpha_0((j-1)T).
\end{aligned}$$

Remark 4. *As in the Wiener-based case, the expectation and the variance are decreasing with respect to ρ.*

The expectation and the variance of the gamma-based maintained processes $Y_1(t)$ and $Y_2(t)$ versus t are shown in Figure 8. The same parameters as above are considered.



(a) Expectation. (b) Variance.

Figure 8: Expectation and variance of $Y_1(t)$ (red) and $Y_2(t)$ (blue) versus t.

The covariance between $Y_1(t)$ and $Y_2(t)$ is given by

$$Cov(Y_1(t), Y_2(t)) = b^2 \left(\sum_{j=1}^{n} (1-\rho)^{2(n-j+1)} \Delta\alpha_0(jT) + \alpha_0(t) - \alpha_0(nT) \right)$$

hence the Pearson correlation coefficient between $Y_1(t)$ and $Y_2(t)$ varies with time and it is given by

$$\theta(t) = \frac{e_0(t)}{\sqrt{e_0(t)+e_1(t)}\sqrt{e_0(t)+e_2(t)}}, \qquad nT \leq t < (n+1)T, \qquad (12)$$

where

$$\begin{aligned} e_0(t) &= \sum_{j=1}^{n}(1-\rho)^{2(n-j+1)}\Delta\alpha_0(jT) + \alpha_0(t) - \alpha_0(nT) \\ e_i(t) &= \sum_{j=1}^{n}(1-\rho)^{2(n-j+1)}\Delta\alpha_i(jT) + \alpha_i(t) - \alpha_i(nT). \end{aligned}$$

The Pearson correlation coefficient between two gamma-based maintained processes $Y_1(t)$ and $Y_2(t)$ with parameters given by Example 2 is represented in Figure 9. Notice that it differs from the Wiener case, in which the Pearson correlation coefficient is constant (Eq.(10)).

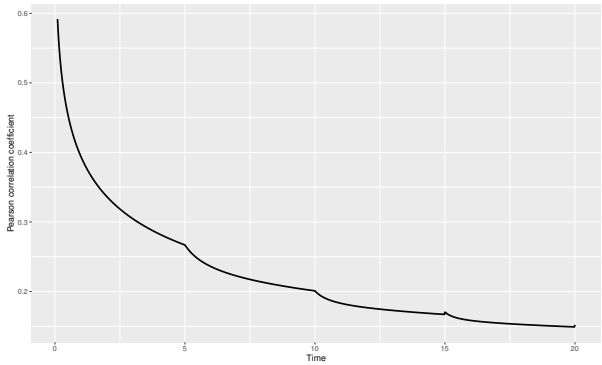


Figure 9: Pearson correlation coefficient between $Y_1(t)$ and $Y_2(t)$.

By analogy with the Wiener-based model, by choosing

$$\alpha_0(t) = \mu_0\Lambda(t), \quad \alpha_1(t) = \mu_1\Lambda(t), \quad \alpha_2(t) = \mu_2\Lambda(t),$$

we get that

$$\theta(t) = \frac{\mu_0}{\sqrt{\mu_0+\mu_1}\sqrt{\mu_0+\mu_2}},$$

and the correlation coefficient does not depend on t.

Lemmas 3 and 4 analyze the monotonicity of the expectation and variance of the degradation of the maintained system in the gamma-based model.

Lemma 3. *In the gamma-based model, for fixed T, if $\alpha_i(\cdot)$ and $\alpha_0(\cdot)$ have increasing increments, then*

$$\mathbb{E}[Y_i(nT)] \leq \mathbb{E}[Y_i((n+1)T)], \quad Var[Y_i(nT)] \leq Var[Y_i((n+1)T)].$$

Lemma 4. *Let t_1 and t_2 with $t_1 < t_2$. If $\alpha(\cdot)$ and $\alpha_0(\cdot)$ have increasing increments and t_1 and t_2 fulfil*

$$t_1 - \lfloor t_1/T \rfloor T = t_2 - \lfloor t_2/T \rfloor T,$$

then

$$\mathbb{E}[Y_i(t_1)] \leq \mathbb{E}[Y_i(t_2)], \quad Var(Y_i(t_1)) \leq Var(Y_i(t_2)).$$

Proof. Proofs of Lemmas 3 and 4 are analogous to those of Lemmas 1 and 2. □

As can be seen, the same results are obtained for the bivariate gamma-based model as in the Wiener-based case. Next section describes the time to system failure and, subsequently, the system reliability.

3 Failure time distribution

As we explain in Section 2, a series scheme is considered in this paper. It assumes that the system fails when the degradation level of a component first reaches a failure threshold. Let L_i be the failure thresholds for component i. The system is inspected each T time units and, in these inspections, imperfect PM actions are performed if the system is working. If the system is failed, the system is correctively replaced (CM) in the inspection time.

Next, we shall obtain the time to the system failure under this maintenance scheme considering the two models exposed above: the Wiener-based model and the gamma-based model.

3.1 Wiener-based model

In order to analyze the failure time distribution of the system, we shall use analytic results obtained in [23] in the case of a bivariate Wiener process. We extend the analysis developed by [23] incorporating imperfect PM actions.

Before, a realization of the maintained system is shown in the Example .

Example 3. *3.1 A realization of the evolution of the maintained system is represented in Figure 10. The parameters used in the Wiener-based model are similar to Example 1. Inspections are performed each $T = 5$ time units. The efficiency of the PM actions is equalto $\rho = 0.5$ and $L_1 = L_2 = 20$. As we can check, PM actions are performed at times T and $2T$. At time $3T$, the degradation level of one of the components exceeds the failure threshold, hence the system failure T_f occurs in the time interval $(2T, 3T)$ with downtime equals to $3T - T_f$. Successive maintenance actions are performed at times $4T$ and $5T$ in a similar way.*

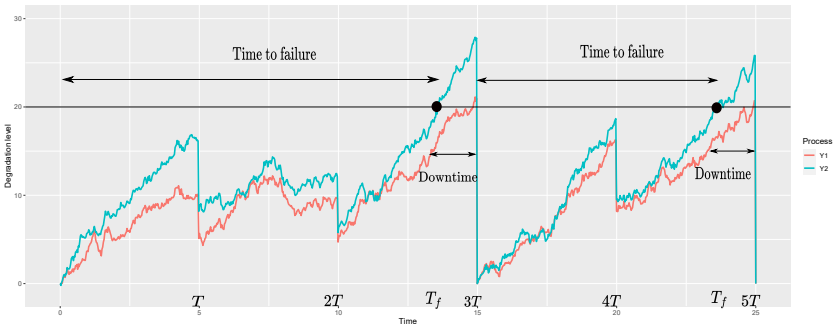


Figure 10: Degradation of the maintained components for the Wiener-based model.

Let $T_f = \min(T_1, T_2)$ be the time to the system failure, where

$$T_i = \inf\{t \geq 0,\ Y_i(t) \geq L_i\}, \quad i = 1, 2. \tag{13}$$

On the other hand, $\tilde{T}_f$ is defined as the first hitting time of the unmaintained process $\mathbf{X}(t) = (X_1(t), X_2(t))$. That is, $\tilde{T}_f = \min(\tilde{T}_1, \tilde{T}_2)$, with

$$\tilde{T}_i = \inf\{t \geq 0,\ X_i(t) > L_i\}, \quad i = 1, 2. \tag{14}$$

Let $g(\cdot)$ be the joint probability density function of the maintained process $\mathbf{Y}(t)$ up to time T_f (Eq.(13)). It is given by

$$g(t, w_1, w_2) = \frac{\partial^2}{\partial w_1 \partial w_2} P(T_f > t, Y_1(t) < w_1, Y_2(t) < w_2), \tag{15}$$

for $w_1 < L_1$ and $w_2 < L_2$. To develop Eq.(15), we shall use the analytic expression of the joint conditional pdf and the first hitting time of the bivariate process $\mathbf{X}(t)$ in absence of PM actions. The conditional pdf of $\mathbf{X}(t)$ up to the time $\tilde{T}_f$ is given by

$$f_{s,x_1,x_2}(t, y_1, y_2) = \frac{\partial^2}{\partial y_1 \partial y_2} P(\tilde{T}_f > t, X_1(t) < y_1, X_2(t) < y_2 | \mathbf{X}(s) = (x_1, x_2)). \tag{16}$$

These results are valid for any $\Lambda(t)$ (linear or not), but Eq.(16) has an analytical form only for the linear case, which is given by Lemma 2 in [23]. With this density, the distribution of the time to system failure is obtained as consequence of the following lemma.

Lemma 5. *Given* $\mathbf{X}(0) = (0, 0)$*, the function* $g^{(n)}(t, w_1, w_2)$ *given by*

$$g^{(n)}(t, w_1, w_2) = \frac{\partial^2}{\partial w_1 \partial w_2} P(T_f > t, Y_1(t) < w_1, Y_2(t) < w_2),$$

with $nT < t < (n+1)T$, $w_1 < L_1$ and $w_2 < L_2$ is recursively obtained as follows. For $0 < t < T$,

$$g^{(0)}(t, w_1, w_2) = f_{(0,0,0)}(t, w_1, w_2), \quad w_1 < L_1,\ w_2 < L_2,$$

and for $nT \leq t < (n+1)T$ we get that

$$g^{(n)}(t, w_1, w_2) = \int_0^{L_1} \int_0^{L_2} g^{(n-1)}(nT, y_1, y_2) f_{nT,(1-\rho)y_1,(1-\rho)y_2}(t, w_1, w_2) dy_1 dy_2,$$

where $f(\cdot)$ is given by Eq.(16).

Proof. At time $t = 0$, both components are new.

- For $0 < t \leq T$, since imperfect PM actions have not yet been performed, we get that $T_f = \tilde{T}_f$ and

$$\begin{aligned} g^{(0)}(t, w_1, w_2) &= \frac{\partial^2}{\partial w_1 \partial w_2} P\left(T_f > t, Y_1(t) < w_1, Y_2(t) < w_2\right) \\ &= \frac{\partial^2}{\partial w_1 \partial w_2} P\left(\tilde{T}_f > t, X_1(t) < w_1, X_2(t) < w_2\right) \\ &= f_{(0,0,0)}(t, w_1, w_2), \end{aligned} \tag{17}$$

  where $f_{(0,0,0)}(t, w_1, w_2)$ is given by Eq.(16).

- For $T < t \leq 2T$, the degradation of each component is reduced in a $\rho\%$ at time T. Let $g^{(1)}(t, w_1, w_2)$ be the following probability

$$g^{(1)}(t, w_1, w_2) = \frac{\partial^2}{\partial w_1 \partial w_2} P(T_f > t, Y_1(t) < w_1, Y_2(t) < w_2).$$

  Conditioning to the degradation level of each component after the first PM action, we get that

$$\begin{aligned} &g^{(1)}(t, w_1, w_2) = \frac{\partial^2}{\partial w_1 \partial w_2} P(T_f > t, Y_1(t) < w_1, Y_2(t) < w_2) \\ &= \int_0^{L_1} \int_0^{L_2} g^{(0)}(T, y_1, y_2) \left(\frac{\partial^2}{\partial w_1 \partial w_2} P\left(T_f > t, Y_1(t) < w_1,\right.\right. \\ &\left.\left. Y_2(t) < w_2 | Y_1(T) = (1-\rho)y_1, Y_2(T) = (1-\rho)y_2\right) \right) dy_1 dy_2, \end{aligned}$$

  where $g^{(0)}$ is given by Eq.(17). On the other hand,

$$\begin{aligned} &\frac{\partial^2}{\partial w_1 \partial w_2} P\left(T_f > t, \mathbf{Y}(t) < (w_1, w_2) | \mathbf{Y}(T) = ((1-\rho)y_1, (1-\rho)y_2)\right) \\ &= \frac{\partial^2}{\partial w_1 \partial w_2} P\left(\tilde{T}_f > t, \mathbf{X}(t) < (w_1, w_2) | \mathbf{X}(T) = ((1-\rho)y_1, (1-\rho)y_2)\right) \\ &= f_{T,(1-\rho)y_1,(1-\rho)y_2}(t, w_1, w_2), \end{aligned}$$

  where f is given by Eq.(16).

- In a general setting, for $nT < t \leq (n+1)T$, let $g^{(n)}(t, w_1, w_2)$ be the following probability:

$$g^{(n)}(t, w_1, w_2) = \frac{\partial^2}{\partial w_1 \partial w_2} P\left(T_f > t, Y_1(t) < w_1, Y_2(t) < w_2\right).$$

For $w_1 < L_1$ and $w_2 < L_2$, we get that

$$\begin{aligned} g^{(n)}(t, w_1, w_2) &= \frac{\partial^2}{\partial w_1 \partial w_2} P\left(T_f > t, Y_1(t) < w_1, Y_2(t) < w_2\right) \\ &= \int_0^{L_1} \int_0^{L_2} g^{(n-1)}(nT, y_1, y_2) \left(\frac{\partial^2}{\partial w_1 \partial w_2} P\left(\tilde{T}_f > t, X_1(t) < w_1, X_2(t) < w_2 | \right.\right. \\ & \left.\left. X_1((n-1)T) = (1-\rho)y_1, X_2((n-1)T) = (1-\rho)y_2\right) \right) dy_1 dy_2 \\ &= \int_0^{L_1} \int_0^{L_2} g^{(n-1)}(nT, y_1, y_2) f_{(nT,(1-\rho)y_1,(1-\rho)y_2)}(t, w_1, w_2) dy_1 dy_2, \end{aligned}$$

for $nT < t \leq (n+1)T$.

□

We can get the distribution of the system lifetime as follows.

Corollary 1. *The survival function of T_f at time t with $nT < t \leq (n+1)T$ is given by*

$$\bar{F}_{T_f}(t) = P(T_f > t) = \int_0^{L_1} \int_0^{L_2} g^{(n)}(t, w_1, w_2)\ dw_1\ dw_2$$

where $g^{(n)}(\cdot)$ is given by Lemma 5.

Proof. It is directly obtained by using Lemma 5. □

The distribution of T_f is simpler to obtain in the gamma-based case than in the Wiener-based case since the trajectories of the former are non-decreasing. The results are given in the next section.

3.2 Gamma-based model

In order to obtain the distribution of T_f, it is necessary to identify clearly the dependency between $Y_1(t)$ and $Y_2(t)$ as follows. By using the gamma-based model, $Y_i(t)$ is expressed as

$$Y_i(t) = \tilde{W}_i(t) + \tilde{W}_0(t), \quad nT \leq t < (n+1)T,$$

where

$$\begin{aligned} \tilde{W}_i(t) &= \sum_{j=1}^{n} (1-\rho)^{n-j+1} \Delta W_i(jT) + (W_i(t) - W_i(nT)) \\ \tilde{W}_0(t) &= \sum_{j=1}^{n} (1-\rho)^{n-j+1} \Delta W_0(jT) + (W_0(t) - W_0(nT)) \end{aligned}$$

are given by Eq.(11).

The distribution of $\tilde{W}_i(t)$ and $\tilde{W}_0(t)$ corresponds to the sum of $n+1$ independent gamma variables with different scale parameters. They can be seen as a linear combination of increments $Z_{j,i}(t)$,

$$\tilde{W}_i(t) = \sum_{j=1}^{n+1} z_j Z_{j,i}(t), \quad t \geq 0,$$

with

$$\begin{aligned} Z_{j,i}(t) &= \Delta W_i(jT) = W_i(jT) - W_i((j-1)T), \quad j = 1, 2, \ldots, n \\ Z_{n+1,i}(t) &= W_i(t) - W_i(nT), \end{aligned}$$

and weights

$$z_j = (1-\rho)^{n-j+1}$$

for $j = 0, 1, \ldots, n$ and $z_{n+1} = 1$.

The distribution of a sum of independent gamma variables with different scale parameters was obtained by Moschopoulos in [16]. According to it (see [27] for more details), the density of $\tilde{W}_i(t)$ is given by

$$f_{\tilde{W}_i(t)}(y) = D_i(t) \sum_{k=0}^{\infty} \frac{\xi_{k,i}(t) \tilde{\beta}^{-s_i(t)-k}}{\gamma(s_i(t)+k)} y^{s_i(t)+k-1} e^{-y/\tilde{\beta}}, \quad y \geq 0, \tag{18}$$

where

$$\tilde{\beta} = \min_{1 \leq j \leq n} ((1-\rho)^{n-j+1} b, b) = (1-\rho)^n b,$$

being $0 \leq \rho < 1$. Functions $D_i(t)$ and $s_i(t)$ are given by

$$\begin{aligned} D_i(t) &= \prod_{k=1}^{n} \left(\frac{\tilde{\beta}}{(1-\rho)^{n-k+1} b} \right)^{\alpha_i(kT) - \alpha_i((k-1)T)} \left(\frac{\tilde{\beta}}{b} \right)^{\alpha_i(t) - \alpha_i(nT)}, \\ &= \prod_{k=1}^{n} \left((1-\rho)^{k-1} \right)^{\alpha_i(kT) - \alpha_i((k-1)T)} \left((1-\rho)^n \right)^{\alpha_i(t) - \alpha_i(nT)} \end{aligned}$$

and

$$s_i(t) = \sum_{k=1}^{n} \alpha_i(kT) - \alpha_i((k-1)T) + (\alpha_i(t) - \alpha_i(nT)) = \alpha_i(t), \quad t \geq 0,$$

respectively, and $\xi_{k+1,i}(t)$ is obtained in a recursive way as

$$\xi_{k+1,i}(t) = \frac{1}{k+1} \sum_{j=1}^{k+1} j \nu_{j,i}(t) \xi_{k+1-j,i}(t),$$

with $\xi_{0,i}(t) = 1$ and $\nu_{k,i}(t)$ is given by

$$\begin{aligned}\nu_{k,i}(t) &= \sum_{j=1}^{n} \frac{\alpha_i(jT) - \alpha_i((j-1)T)}{k} \left(1 - (1-\rho)^{j-1}\right)^k \\ &+ \frac{\alpha_i(t) - \alpha_i(nT)}{k} \left(1 - (1-\rho)^n\right)^k.\end{aligned}$$

Let $\bar{F}_{\tilde{W}_1(t)}$, $\bar{F}_{\tilde{W}_2(t)}$ and $\bar{F}_{\tilde{W}_0(t)}$ be the survival distributions of $\tilde{W}_1(t)$, $\tilde{W}_2(t)$ and $\tilde{W}_0(t)$ respectively. Using Eq.(18), the following result is obtained.

Lemma 6. *By using the gamma-based model, the survival distribution of the FHT is given by*

$$\bar{F}_{T_f}(t) = \int_0^{\min(L_1,L_2)} f_{\tilde{W}_0(t)}(y) F_{\tilde{W}_1(t)}(L_1 - y) F_{\tilde{W}_2(t)}(L_2 - y)\, dy, \tag{19}$$

where $F_{\tilde{W}_1(t)}$, $F_{\tilde{W}_2(t)}$ are obtained from their corresponding p.d.f. and $f_{\tilde{W}_0(t)}$ is given directly by Eq.(18).

Proof. In the gamma-based model, the processes $Y_1(t)$ and $Y_2(t)$ are strictly increasing. We get that

$$P(T_f \leq t) = P(Y_1(t) \leq L_1,\, Y_2(t) \leq L_2),$$

hence

$$\begin{aligned}P(T_f \geq t) &= P(Y_1(t) \leq L_1, Y_2(t) \leq L_2) \\ &= P(\tilde{W}_1(t) + \tilde{W}_0(t) \leq L_1, \tilde{W}_2(t) + \tilde{W}_0(t) \leq L_2) \\ &= \int_0^{\min(L_1,L_2)} f_{\tilde{W}_0(t)}(y) F_{\tilde{W}_1(t)}(L_1 - y) F_{\tilde{W}_2(t)}(L_2 - y)\, dy,\end{aligned}$$

and the result holds.

□

Example 4. *Similarly to the Wiener-based model, a realization of the processes $Y_1(t)$ and $Y_2(t)$ is represented in Figure 11. Imperfect PM actions are performed each $T = 5$ time units considering $\rho = 0.5$.*

Next section describes the maintenance strategy for this system. Given the complexity of the analytical expressions obtained for $\bar{F}_{T_f}(t)$, which involve multiple integrals and derivatives, they have not been evaluated numerically directly. Instead of, Monte Carlos simulation have been performed to obtain the results presented in Section 4.

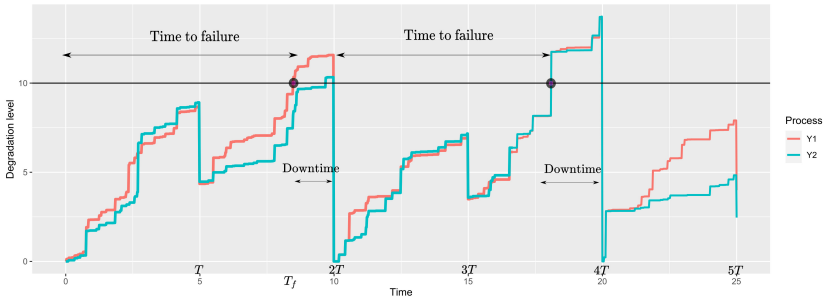


Figure 11: Degradation of the maintained components for the gamma-based model.

4 Cost model and optimization

Maintenance optimization approaches based on the average cost rate are widely studied in reliability. The aim is to optimise decision variables and parameters influencing the model in order to execute an appropriate decision-making for the given maintenance strategy. The search for this optimal maintenance strategy has been traditionally based on an asymptotic approach, considering an infinite life cycle in which the system can be maintained an infinite number of times. However, this situation rarely occurs in practice and infinite-horizon assumption may provide a suboptimal solution [20]. For this reason, the optimization problem in this paper is formulated in the finite horizon setting assuming in this paper that the system is working up to a time T_{max}.

As we explained in the model assumptions in Section 2, the system is inspected each T time units. If the system is not failed in an inspection time, an imperfect PM action ARD(∞) is performed and the accumulated degradation of each component is reduced in a $\rho\%$ $(0 < \rho < 1)$. If the system is failed in an inspection time, the system is replaced by a new one. $C_I(\rho)$ and C_c denote the cost associated to the preventive (corrective) maintenance cost. Furthermore, $r_i(x)$ is the reward obtained by component i when it is working and the degradation level of this component is x.

Let $C(t_1, t_2)$ be the maintenance cost in the interval $(t_1, t_2]$ and $C(t)$ the total maintenance cost of the system up to time t. The objective cost function in the finite time horizon is equal to

$$\mathbb{C}_{T_{max}}(T, \rho) = \frac{\mathbb{E}[C(T_{max})]}{T_{max}}. \tag{20}$$

The decision variables considered for the optimization of this maintenance policy are the time between inspections T and the maintenance efficiency ρ. Hence, the optimal policy (T_{opt}, ρ_{opt}) is the one that minimize $\mathbb{C}_{T_{max}}(T, \rho)$ and its cost is given by

$$\mathbb{C}_{T_{max}}(T_{opt}, \rho_{opt}) = \inf \left\{ \mathbb{C}_{T_{max}}(T, \rho),\, 0 < T < T_{max},\, 0 \leq \rho \leq 1 \right\}. \tag{21}$$

4.1 Expected cost per time unit

By replacement cycle we mean the time between successive replacements of the system. Let R be the replacement time of the system, with $R = (\lfloor T_f/T \rfloor + 1)T$ and where T_f is the time to the system failure. Hence, the expected cost in a replacement cycle is given by

$$C_I(\rho)\mathbb{E}(\lfloor T_f/T \rfloor) - \sum_{i=1}^{2} \mathbb{E}\left(\int_0^{T_f} r_i(Y_i(t))dt\right) + C_d\mathbb{E}(R - T_f) + C_c, \qquad (22)$$

where $C_I(\rho)\mathbb{E}(\lfloor T_f/T \rfloor)$ represents the cost due to imperfect PM actions, C_c stands for the replacement cost, $C_d\mathbb{E}(R - T_f)$ corresponds to the downtime cost and finally the expected reward in a replacement cycle is given by

$$\sum_{i=1}^{2} \mathbb{E}\left(\int_0^{T_f} r_i(Y_i(t))dt\right).$$

This reward function is inspired by utility functions of insurance literature [22] and has been used by several authors in degradation models, such as [17] or [14].

We focus on the expectation $\mathbb{E}[C(t)]$. The following result is obtained.

Lemma 7. *For $t \geq T$, the expected cost at time t fulfils the following renewal equation*

$$\mathbb{E}[C(t)] = h(t) + \int_0^t f_{T_f}(u)\mathbb{E}[C(t - (\lfloor u/T \rfloor + 1)T)]du,$$

given the initial condition $\mathbb{E}[C(0)] = 0$ with $n = \lfloor t/T \rfloor$ where

$$\begin{aligned} h(t) &= nC_I(\rho)\bar{F}_{T_f}(nT) + C_d\int_{nT}^{t} f_{T_f}(u)(t-u)du \\ &- \mathbb{E}\left[\mathbf{1}_{\{nT<T_f\}}\sum_{i=1}^{2}\int_0^{\min(T_f,t)} r_i(Y_i(s))ds\right] \\ &+ \int_0^{nT} f_{T_f}(u)\left(C_I(\rho)\lfloor u/T \rfloor + C_d((\lfloor u/T \rfloor + 1)T - u) + C_c\right) \\ &- \mathbb{E}\left(\mathbf{1}_{\{(\lfloor T_f/T+1 \rfloor)T \leq t\}}\sum_{i=1}^{2}\int_0^{T_f} r_i(Y_i(s))ds\right). \end{aligned}$$

For $t < T$, we get that the expected cost is equal to

$$\mathbb{E}[C(t)] = C_d\int_0^t (t-u)f_{T_f}(u)\ du - \mathbb{E}\left(\sum_{i=1}^{2}\int_0^{\min(T_f,t)} r_i(Y_i(s))\ ds\right).$$

Proof. • For $t < T$, two cases are distinguished. If $t < T$ and $t < T_f$, the only cost is given by the reward. On the other hand, for $t < T$ and $t \geq T_f$, the cost is the reward between 0 and T_f and the downtime cost between T_f and t. Then, the expected cost at time t is given by

$$\mathbb{E}[C(t)] = \mathbb{E}\left[C(t)\mathbf{1}_{\{T_f>t\}}\right] + \mathbb{E}\left[C(t)\mathbf{1}_{\{T_f\leq t\}}\right].$$

Furthermore,

$$\begin{aligned}
\mathbb{E}\left[C(t)\mathbf{1}_{\{T_f>t\}}\right] &= -\mathbb{E}\left(\mathbf{1}_{\{T_f>t\}}\sum_{i=1}^{2}\int_0^t r_i(Y_i(s))ds\right)\\
\mathbb{E}\left[C(t)\mathbf{1}_{\{T_f\leq t\}}\right] &= -\mathbb{E}\left(\mathbf{1}_{\{T_f\leq t\}}\sum_{i=1}^{2}\int_0^{T_f} r_i(Y_i(s))ds\right)\\
&+ \mathbb{E}\left[C_d(t-T_f)\mathbf{1}_{\{T_f\leq t\}}\right]\\
&= -\mathbb{E}\left(\mathbf{1}_{\{T_f\leq t\}}\sum_{i=1}^{2}\int_0^{T_f} r_i(Y_i(s))ds\right)\\
&+ C_d\int_0^t (t-u)f_{T_f}(u)du.
\end{aligned}$$

Hence, for $t < T$, the expected cost at time t is given by

$$\mathbb{E}[C(t)] = C_d\int_0^t F_{T_f}(u)\,du - \mathbb{E}\left(\sum_{i=1}^{2}\int_0^{\min(T_f,t)} r_i(Y_i(s))\,ds\right).$$

• For $t \geq T$, denoting $n = \lfloor t/T \rfloor$, the expected cost is conditioned to the time of the first replacement.

$$\begin{aligned}
\mathbb{E}[C(t)] &= \mathbb{E}\left[C(t)\mathbf{1}_{\{R\leq t\}}\right] + \mathbb{E}\left[C(t)\mathbf{1}_{\{R\geq t\}}\right] \qquad (23)\\
&= \mathbb{E}\left[(C(0,R)+C(R,t))\,\mathbf{1}_{\{R\leq t\}}\right] + \mathbb{E}\left[C(t)\mathbf{1}_{\{R\geq t\}}\right]\\
&= \mathbb{E}\left[C(t)\mathbf{1}_{\{R\geq t\}}\right] + \mathbb{E}\left[C(0,R)\mathbf{1}_{\{R\leq t\}}\right] + \mathbb{E}\left[C(R,t)\mathbf{1}_{\{R\leq t\}}\right].
\end{aligned}$$

Next, the development of the three terms in Eq.(23) is shown. If $T_f > t$, then the total cost is given by the cost due to PM minus the reward. If $T_f < t$, the downtime cost is added in addition to the above. For the first summation,

we get that

$$\begin{aligned}
\mathbb{E}\left[C(t)\mathbf{1}_{\{R\geq t\}}\right] &= \mathbb{E}\left[C(t)\mathbf{1}_{\{T_f\geq t\}} + C(t)\mathbf{1}_{\{nT<T_f<t<(n+1)T\}}\right] \\
&= \mathbb{E}\left[\mathbf{1}_{\{T_f\geq t\}}\left(nC_I(\rho) - \sum_{i=1}^{2}\int_0^t r_i(Y_i(s))ds\right)\right] \\
&+ \mathbb{E}\left[\mathbf{1}_{\{nT\leq T_f<t<(n+1)T\}}\left(nC_I(\rho) - \sum_{i=1}^{2}\int_0^{T_f} r_i(Y_i(s))ds\right)\right] \\
&+ \mathbb{E}\left[\mathbf{1}_{\{nT\leq T_f<t<(n+1)T\}}C_d(t-T_f)\right] \\
&= nC_I(\rho)\bar{F}_{T_f}(nT) + C_d\int_{nT}^{t} f_{T_f}(u)(t-u)du \\
&- \mathbb{E}\left[\mathbf{1}_{\{nT\leq T_f\}}\sum_{i=1}^{2}\int_0^{\min(t,T_f)} r_i(Y_i(s))ds\right]. \qquad (24)
\end{aligned}$$

The second summation corresponds to the expected cost in a replacement cycle given by Eq.(22) if $R \leq t$. In the case $nT \leq T_f < (n+1)T$, then the replacement time $R = (n+1)T$. Hence,

$$\begin{aligned}
\mathbb{E}[C(0,R)\mathbf{1}_{\{R\leq t\}}] &= \mathbb{E}\left[\mathbf{1}_{\{R\leq t\}}(C_I(\rho)\lfloor T_f/T\rfloor + C_d(R-T_f) + C_c)\right] \\
&- \mathbb{E}\left[\mathbf{1}_{\{R\leq t\}}\sum_{i=1}^{2}\int_0^{T_f} r_i(Y_i(s))ds\right] \\
&= \int_0^{nT} f_{T_f}(u)\left(\lfloor u/T\rfloor C_I(\rho) + C_d((\lfloor u/T\rfloor+1)T-u) + C_c\right)du \\
&- \mathbb{E}\left[\mathbf{1}_{\{R\leq t\}}\sum_{i=1}^{2}\int_0^{T_f} r_i(Y_i(s))ds\right]. \qquad (25)
\end{aligned}$$

Then, as $\mathbb{E}[C(R,t)\mathbf{1}_{\{R\leq t\}}] = \mathbb{E}[C(t-R)\mathbf{1}_{\{R\leq t\}}]$, we get that

$$\mathbb{E}\left[C(R,t)\mathbf{1}_{\{R\leq t\}}\right] = \int_0^t f_{T_f}(u)C(t-(\lfloor u/T\rfloor+1)T)du. \qquad (26)$$

Finally, given Eqs.(24), (25) and (26), the result holds. □

Next section shows some numerical examples to illustrate the results obtained.

5 Numerical examples

Firstly, numerical examples are shown to describe the first hitting time distribution developed in the previous section. Secondly, numerical examples of the maintenance strategy are developed.

5.1 First hitting time

Figure 12 shows the estimation of the failure time density versus time t with $\rho = 0.25$ (green), $\rho = 0.5$ (red), $\rho = 0.75$ (blue) and $\rho = 1$ (purple). This plot has been performed considering the following underlying degrading processes from Eqs.(8) and (9) and $\Lambda(t) = t$, that is,

$$\begin{aligned} X_1(t) &= 3.5t + 4.5B_1(t) + 4B_0(t) \\ X_2(t) &= 4t + 3.5B_2(t) + 4B_0(t) \end{aligned}$$

and assuming that the maintenance has been performed each $T = 5$ time units with failure thresholds $L_1 = L_2 = 20$. The exact expression of the density could theoretically be obtained by deriving the *cdf* given in Corollary 1. But this computation is very complex, so Figure 12 has been obtained by simulations for a Wiener process-based model.

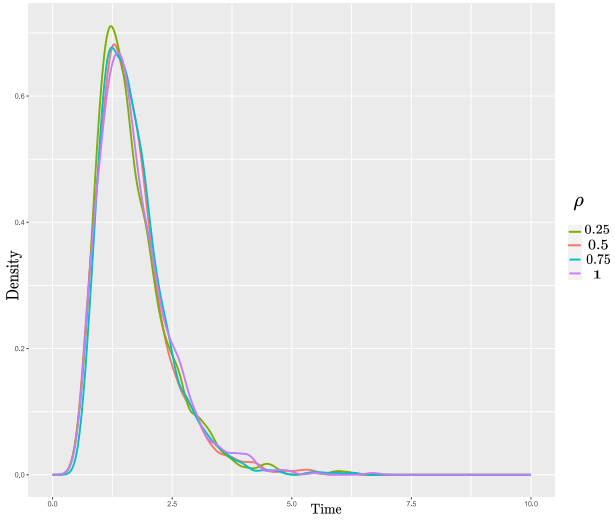


Figure 12: Estimation of the failure time density for a Wiener process-based model obtained by simulations.

Considering the processes and parameters used in Example 1, Figure 13 shows the density estimation of the failure time for a Wiener process-based model in absence of maintenance actions for the same failure threshold value as in Figure 12. Notice that, in this case, the failure usually occurs earlier than in the case with maintenance actions. Both densities described in Figures 12 and 13 have been obtained performing simulations and using non parametric estimation of the respective kernels.

In Figure 14, the expected failure time is represented for the degradation Wiener-based processes given by Eqs.(8) and (9) considering different values for the time between imperfect maintenance actions and the efficiency parameter ρ.

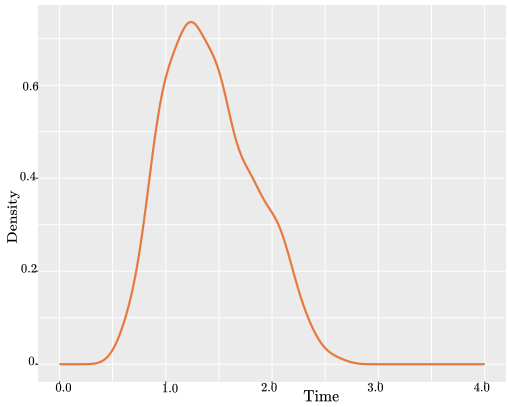


Figure 13: Estimation of the failure time density for a Wiener process-based model in absence of maintenance ($\rho = 0$).

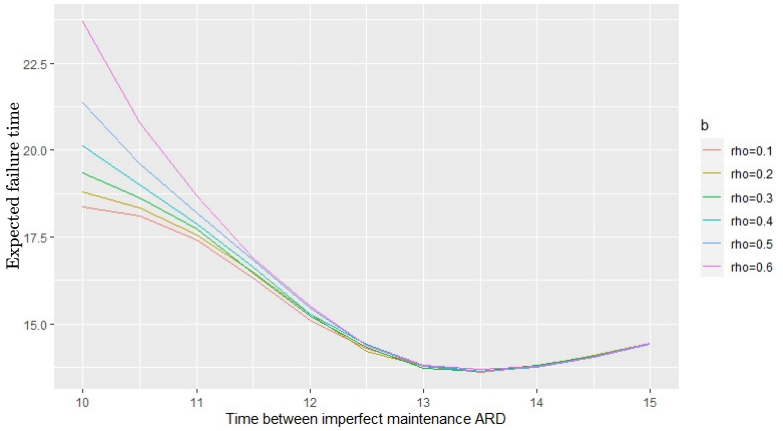


Figure 14: Expected failure time for different values of ρ and varying the time between imperfect maintenance.

When the time between maintenance actions decreases (around 10 time units), greatest disparities are found in the expected failure times for different values of *rho*. But, as the time between maintenance actions increases, the behaviour of the expected failure time is similar for different values of parameter ρ. This will of course largely depend on the parameters chosen for the model.

Similarly to the Wiener-based case, Figures 15 and 16 represent the density estimation of the failure time versus time for a gamma process-based for different values of ρ and in absence of maintenance, respectively. The gamma processes used to obtain these plots are the following:

$$X_1(t) = W_0(t) + W_1(t)$$
$$X_2(t) = W_0(t) + W_2(t),$$

where $W_0(t), W_1(t)$ and $W_2(t)$ are gamma processes with shape parameters $\alpha_0(t) = t, \alpha_1(t) = 1.3t^{1.2}$ and $\alpha_2(t) = 1.8t^{1.5}$, respectively, and scale parameter $b = 1$. Maintenance actions are performed each $T = 5$ time units and the failure thresholds are $L_1 = L_2 = 20$.

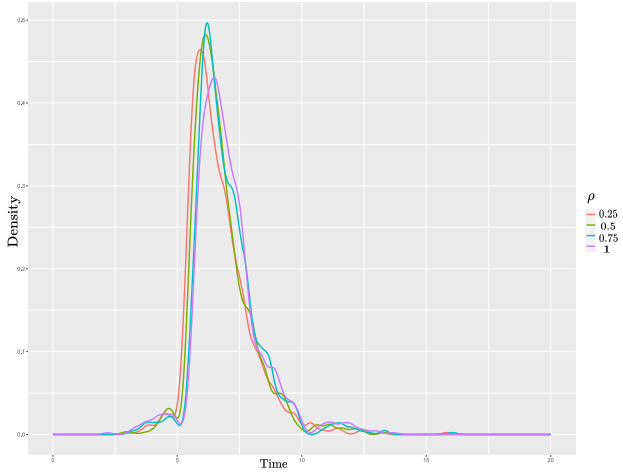


Figure 15: Estimation of the failure time density for a gamma process-based model obtained by simulations.

Finally, in the next numerical example, the expected cost per time unit versus time is compared in the Wiener and gamma cases.

Example 5. *The expected cost per time unit* $\mathbb{E}[C(t)]/t$ *versus* t *is shown in Figure 17 for the Wiener and gamma-based models. The values* $\mu_1 = \mu_2 = 3$, $\sigma_0 = 0.5$, $\sigma_1 = 0.3$ *and* $\sigma_2 = 0.25$. *The failure thresholds are* $L_1 = L_2 = 20$. *Function* $\Lambda(t) = t$ *is considered for the Wiener-based model and* $\alpha_0(t) = 1.5t^{1.5}$, $\alpha_1(t) = 3t^{1.5}$, $\alpha_2(t) = 2.5t^{1.5}$ *for the gamma-based model. PM actions are performed each* $T = 5$ *time units with an efficiency equals to* $\rho = 0.5$. *The*

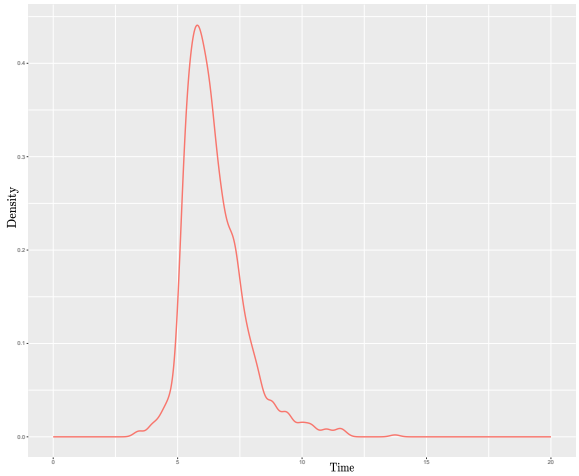


Figure 16: Estimation of the failure time density for a gamma process-based model in absence of maintenance ($\rho = 0$).

sequence of cost is equal to $C_I(\rho) = 100\rho$ *m.u.,* $C_c = 120$ *m.u. and* $C_d = 60$ *m.u. per t.u. No reward is considered for this numerical example, that is* $r_i(Y_i(t)) = 0$ *for* $i = 1, 2$ *and* $t \geq 0$. *The plot is drawn from numerical simulations. Notice that, with this parameter values, which were chosen in order that both models give close trajectories for new systems, the expected cost rate of both models is very similar at the beginning of system lifes (but they may differ before the first periodic inspection has taken place).*

5.2 Maintenance cost

The objective function for the maintenance strategy (the expected cost rate) is assessed by using Monte-Carlo simulation. Afterwards, the optimization of the objective cost function with respect to maintenance decision variables ρ and T as in Eq.(21) under the proposed policy is explored.

5.2.1 Wiener-based model

We assume that the cost of a corrective replacement is equal to $C_c = 120$ monetary units (m.u.). When the system is down, it incurs a cost equal to $C_d = 60$ monetary units per time unit. We also assume that the preventive maintenance cost depends on the efficiency degree and it is equal to $C_I(\rho) = 100\rho$ monetary units.
Furthermore, when the components are working, a reward is obtained. We assume that the reward function is equal to

$$r_i(x) = k_i - a_i e^{b_i x}, \quad x > 0, \quad i = 1, 2, \tag{27}$$

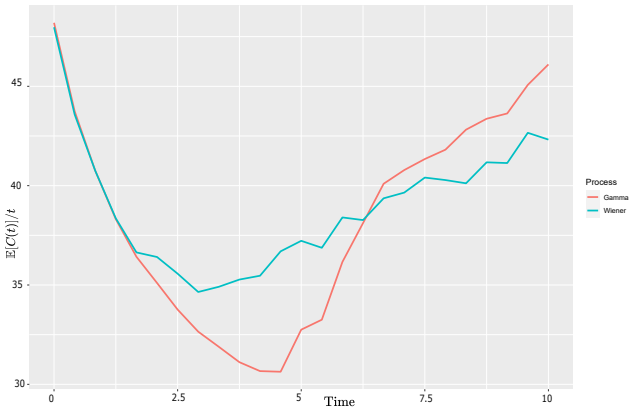


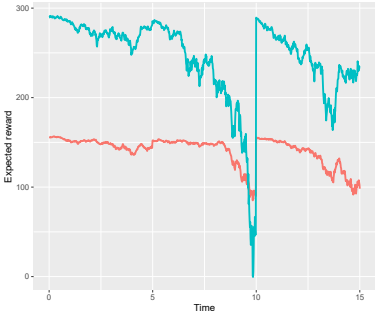
Figure 17: Expected cost per time unit in the Wiener-based (blue) and gamma-based (red) models obtained by simulations.

where x denotes the degradation level of the corresponding component. In particular, the following functions are considered for each component:

$$r_1(x) = 8 - 0.2e^{0.1x}, \qquad r_2(x) = 15 - 0.5e^{0.1x}. \tag{28}$$

The realization of a Wiener-based degradation model given by Eqs.(8) with $T = 5$ t.u. and its corresponding reward functions given by Eq.(28) are represented in Figure 18. The more degraded the component is, the smaller the reward is.

(a) $Y_1(t)$ (red) and $Y_2(t)$ (blue).

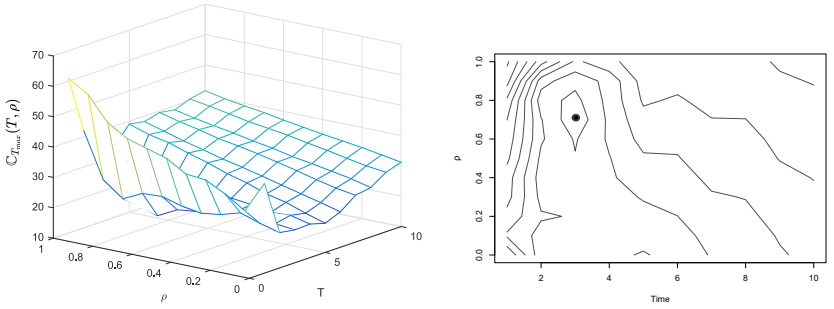


(b) Reward.

Figure 18: Reward with respect to time for a Wiener-based realization.

The Wiener-based process is simulated up to the finite horizon $T_{max} = 100$ with parameters $\sigma_0 = 0.5$, $\sigma_1 = 0.3$, $\sigma_2 = 0.25$, $\mu_1 = \mu_2 = 3$ and $\Lambda(t) = t$. The failure thresholds are $L_1 = L_2 = 20$. For the optimization process, a grid in (T, ρ) is used, with 11 points $(0, 1)$ for ρ and 10 points in $(0, 10)$ for T. All the

results are obtained by Monte-Carlo simulation method with 10000 simulations in each pair. As we can see in Figure 19, the expected cost per time unit in a finite horizon $\mathbb{E}[\mathbb{C}(T_{max}, \rho)]/T_{max}$ is a convex function. The optimal values for the objective cost function are obtained for $T_{opt} = 3.33$ t.u. and $\rho = 0.7$, with an expected cost per time unit equals to 16.26 m.u. per t.u.



(a) Expected cost rate

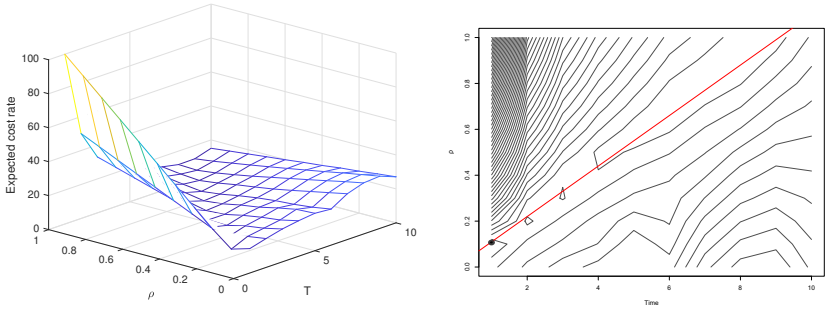
(b) Contour plot

Figure 19: Expected cost per time unit and contour plot for a Wiener-based model considering a reward.

Next, the expected cost per time unit is analyzed when $b_i = k_i = a_i = 0$, for $i = 1, 2$ in Eq.(28). That is, the system reward is null. Figures 20, 21 and 22 show the cost function per unit time in the finite horizon. It is deduced that there are several local minima. The straight line formed by them shows the relationship between the time between imperfect PM actions, T, and the maintenance efficiency ρ. If the maintenance actions are close one to each other, the system has not enough time to deteriorate, so a not very efficient maintenance is enough to maintain the system in a stable state. If the maintenance actions become more spaced, then a more efficient imperfect maintenance action is required to reach the same level of overall efficiency for the maintenance policy. The location of these minima is largely influenced by the drift parameters μ_1 and μ_2 (see Figures 20, 21 and 22) since they represent the expected degradation rate of each marginal Wiener process $X_1(t)$ and $X_2(t)$, respectively. The set of parameters is the same for these figures, the only difference is the values of the drift parameters μ_i, for $i = 1, 2$.

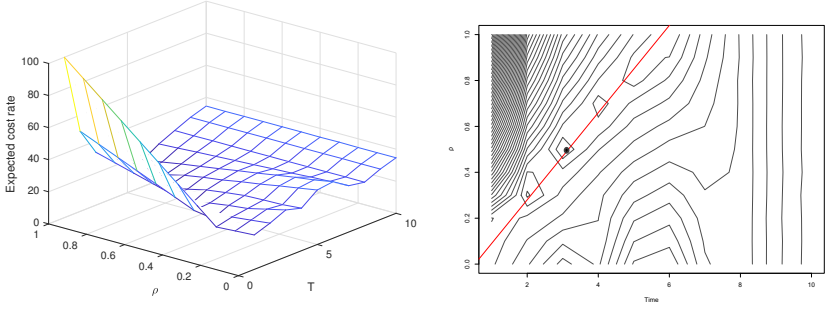
5.2.2 Gamma-based model

The proposed maintenance policy is then evaluated for a gamma-based degradation model. The values $\alpha_1(t) = 3t^{1.5}$, $\alpha_2(t) = 2.5t^{1.5}$, $\alpha_0(t) = 1.5t^{1.5}$ for the shape parameters and $b = 1$ for the scale parameter are used. A component fails when its degradation level exceeds the thresholds $L_1 = L_2 = 20$. The finite horizon is equal to $T_{max} = 100$. Considering the sequence of costs $C_c = 120$ m.u., $C_I = 100\rho$ m.u., $C_d = 60$ m.u. per t.u. and the reward functions given



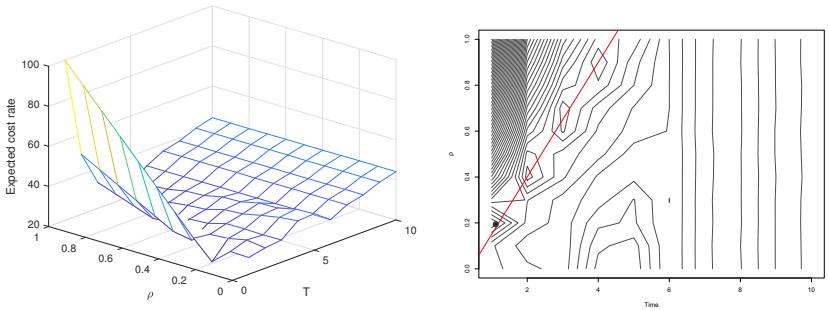
(a) Expected cost rate (b) Contour plot

Figure 20: Optimization for $\mu_1 = \mu_2 = 2$ and no reward. The absolute minimum is 11.2 m.u. per t.u., obtained at $T = 1$ and $\rho = 0.1$.



(a) Expected cost rate (b) Contour plot

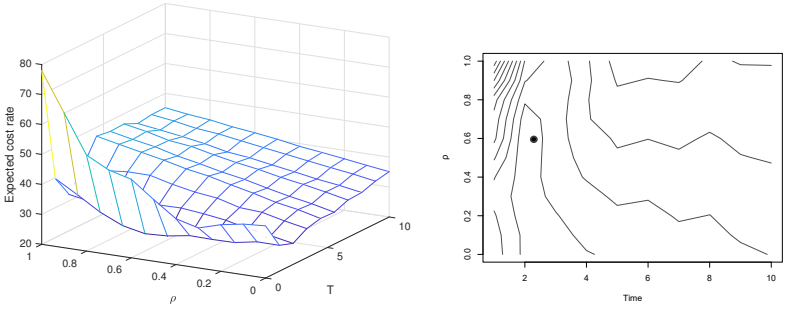
Figure 21: Optimization for $\mu_1 = \mu_2 = 3$ and no reward. The absolute minimum is 16.7 m.u. per t.u., obtained at $T = 3$ and $\rho = 0.5$.



(a) Expected cost rate (b) Contour plot

Figure 22: Optimization for $\mu_1 = \mu_2 = 4$ and no reward. The absolute minimum is 20.9 m.u. per t.u., obtained at $T = 1$ and $\rho = 0.2$.

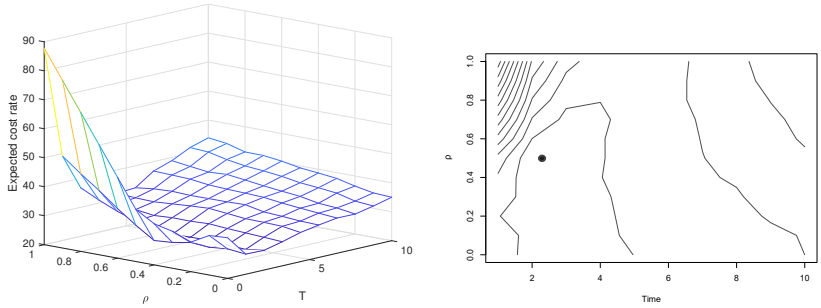
by Eq.(28), the expected cost per time unit in the finite horizon is shown in Figure 23. The optimal expected cost rate is equal to 25.78 m.u. per t.u., with optimal maintenance efficiency $\rho_{opt} = 0.6$ and optimal time between inspections $T_{opt} = 2.22$ time units. Model parameters have been chosen so that the average degradation is as close as possible of that obtained in the Wiener-based case. This is why the optimization results are similar with both models.



(a) Expected cost rate (b) Contour plot

Figure 23: Expected cost per time unit and contour plot for a gamma-based model considering a reward.

With respect to the analysis of the gamma-based model in the case with no reward, that is, considering $b_i = k_i = a_i = 0$, for $i = 1, 2$ in the reward function given by Eq.(27), the expected cost rate represented in Figure 24 is obtained. The optimal value for the PM efficiency is equal to $\rho_{opt} = 0.5$. The optimal time between inspections is equal to $T_{opt} = 2.22$ t.u. The optimal expected cost rate is equal to 25.81 m.u. per t.u., which is slightly higher than in the



(a) Expected cost rate (b) Contour plot

Figure 24: Expected cost per time unit and contour plot in a gamma-based model with no reward.

case with reward.

6 Conclusions and future work

In this paper, a degrading system with two dependent components is analysed. The model used to describe the correlation is the trivariate reduction method. This method is used in this paper considering Wiener processes and gamma processes. This system is periodically maintained and imperfect PM actions are performed. Under the imperfect PM scheme, if Wiener processes are used to describe the degradation of the components, the evolution of the maintained system can be described as a bivariate Wiener process. By contrast, if gamma processes are used to describe the components degradation, the evolution of the maintained system is no longer bivariate gamma distributed.

This paper focuses on the analysis of the first hitting time distribution, that is, the distribution of the time until the system reaches its failure threshold. Theoretical results have been obtained in relation to the first hitting time distribution for two models: a Wiener-based and a gamma-based model. It is complicated to deal with this FHT distribution analytically or numerically because of its complexity, but it is easy to obtain numerical results using simulation. Furthermore, the expected cost in the finite time horizon fulfils a renewal-type equation. Finally, an ARD(∞) model is used to describe the imperfect PM actions. The reduction of the degradation is more realistic than the age reduction ARA(∞) where the component age is reduced. On the contrary, from a mathematical point of view, the use of ARA models allows to keep the gamma distribution after successive imperfect PM actions.

Notwithstanding wide use of Wiener process and gamma process, as future works, it is possible to consider other stochastic processes to describe the degradation of the components in the trivariate reduction method. An Inverse

Gaussian process is an appropriate candidate to model the degradation of the components when this degradation is monotone. As in the Wiener and gamma cases studied, random effects can be flexibly incorporated when Inverse Gaussian processes are used. The estimation of the parameters of the model is a very important issue indeed. By considering only the univariate case, the development of the parameter's estimation is shown in [11], where statistical inference for a Wiener-based degradation model with imperfect maintenance actions is analysed. In our model, the estimation procedure is more complicated than the univariate case since it involves, not only the dependence between jumps along time, but also the dependence of the components deterioration. This will be addressed as future work.

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