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Monotone coupling of Feller Markov processes on the real line

Jean Bérard and Briec Frénais

Abstract

We prove that, for any stochastically non-decreasing Feller Markov process on the real line, two versions of the process starting from distinct initial values always admit an order-preserving coupling, defined through a Feller Markov process on the set of ordered pairs.

1 Introduction

1.1 Statement of the main result

Consider a continuous-time homogeneous Markov process on the real line, characterized by a homogeneous family of Markov kernels¹ $(p_t)_{t \geq 0}$ on \mathbb{R} , and the corresponding Markov semi-group $(P_t)_{t \geq 0}$, which is defined on the space \mathcal{B}_b of bounded real-valued Borel functions on \mathbb{R} by²

$$\forall f \in \mathcal{B}_b \forall t \geq 0 \forall x \in \mathbb{R} \quad P_t f(x) = \int_{\mathbb{R}} f(y) p_t(x, dy).$$

If $(X_t)_{t \geq 0}$ is a family of random variables defined on a probability space $(\Omega, \mathcal{F}, (\mathbb{P}_x)_{x \in \mathbb{R}})$ providing a version of the process starting from $X_0 = x \in \mathbb{R}$, we thus have $p_t(x, \cdot) = \mathbb{P}_x(X_t \in \cdot)$, and $P_t f(x) = \mathbb{E}_x(f(X_t))$.

In addition to \mathcal{B}_b , let us denote by \mathcal{C}_b (resp. \mathcal{C}_0) the set of continuous bounded (resp. continuous vanishing at $\pm\infty$) real-valued functions on \mathbb{R} . On \mathcal{B}_b (and thus also \mathcal{C}_b and \mathcal{C}_0), we consider the supremum norm defined by $\|f\|_{\infty} = \sup_{x \in \mathbb{R}} |f(x)|$, with respect to which \mathcal{B}_b , \mathcal{C}_b and \mathcal{C}_0 are Banach spaces.

Recall that the semi-group $(P_t)_{t \geq 0}$ associated with $(p_t)_{t \geq 0}$ is said to enjoy the Feller property if:

- (a) $\forall t \geq 0, \forall f \in \mathcal{C}_0, P_t f \in \mathcal{C}_0$;
- (b) $\forall f \in \mathcal{C}_0, \lim_{t \rightarrow 0^+} \|P_t f - f\|_{\infty} = 0$.

Note that property (a) implies, but is not equivalent to

- (a') $\forall t \geq 0, \forall f \in \mathcal{C}_b, P_t f \in \mathcal{C}_b$,

with (a') often being termed "strong Feller property" in the literature (see e.g. [3]). Moreover, (b) may be replaced by the apparently weaker assumption of pointwise (instead of uniform) convergence

- (b') $\forall f \in \mathcal{C}_0, \forall x \in \mathbb{R}, \lim_{t \rightarrow 0^+} P_t f(x) = f(x)$,

¹Given a measurable space (E, \mathcal{E}) , a Markov kernel on E is a map $k : E \times \mathcal{E} \rightarrow \mathbb{R}$ such that (i) for all $x \in E$, $k(x, \cdot)$ is a probability measure on (E, \mathcal{E}) , and (ii) for all $B \in \mathcal{E}$, $k(\cdot, B)$ is a measurable real-valued function on E . Given two Markov kernels k, ℓ on E , the composition of the two kernels is yet another Markov kernel $k\ell$ defined by $(k\ell)(x, B) = \int_E k(x, dy)\ell(y, B)$. The composition is an associative (but in general non-commutative) operation on Markov kernels. We say that $(k_t)_{t \geq 0}$ is a homogeneous family of Markov kernels on E if (I) for all $x \in E$ and $B \in \mathcal{E}$, $k_0(x, B) = \delta_x(B)$, and (II) the Chapman-Kolmogorov equation holds: for all $s, t \geq 0$, $k_{s+t} = k_s k_t$.

²For all $f \in \mathcal{B}_b$, we denote by $P_t f$ the function $x \mapsto P_t f(x)$, and we have that $P_t f \in \mathcal{B}_b$. Moreover, P_t defines a linear operator from \mathcal{B}_b into itself, and satisfies (α) $\|P_t f\|_{\infty} \leq \|f\|_{\infty}$ (β) $P_t f \geq 0$ when $f \geq 0$ (γ) $P_t 1 = 1$. As a consequence of the Chapman-Kolmogorov equation for $(p_t)_{t \geq 0}$, the semi-group property holds: for all $s, t \geq 0$, $P_{s+t} = P_s P_t$.

but it turns out that a Markov semi-group satisfying (a)(b') also satisfies (b).

The Feller property has various useful consequences, among which the possibility of building versions of the corresponding Markov process enjoying "nice" properties: càdlàg trajectories, strong Markov property, quasi-left-continuity, etc. We refer e.g. to [4] (especially Chapter 19) for the key definitions and properties of Feller processes.

Then recall the definition of the stochastic dominance ordering between probability measures on \mathbb{R} . Denote by \mathcal{B}_b^\nearrow the set of non-decreasing bounded Borel functions on \mathbb{R} . Among several equivalent characterizations (see e.g. [7]), we say that $\mu \preceq \nu$ if and only if, for every $f \in \mathcal{B}_b^\nearrow$, one has that $\int f(x)\mu(dx) \leq \int f(x)\nu(dx)$. A celebrated result due to Strassen [9] states that the condition $\mu \preceq \nu$ is equivalent to the existence of a pair of random variables (Y, Z) such that $Y \sim \mu$, $Z \sim \nu$, and $Y \leq Z$ a.s.

Now we say that $(p_t)_{t \geq 0}$ is stochastically non-decreasing when, for all $t \geq 0$, and all $x, y \in \mathbb{R}$ such that $x \leq y$, one has $p_t(x, \cdot) \preceq p_t(y, \cdot)$. In terms of the semi-group, this condition rewrites as

$$(c) \quad \forall t \geq 0, \forall f \in \mathcal{B}_b^\nearrow, P_t f \in \mathcal{B}_b^\nearrow.$$

Thanks to Strassen's result, condition (c) also means that, starting from two initial positions $x^1 \leq x^2$, and for all $t \geq 0$, there exists a couple of random variables (X_t^1, X_t^2) such that X_t^1 and X_t^2 respectively follow the distribution of the Markov process at time t starting respectively from x^1 and from x^2 , and such that $X_t^1 \leq X_t^2$ a.s. The question we investigate in this paper is whether this statement can be strengthened to the effect that there exists a process of pairs $(X_t^1, X_t^2)_{t \geq 0}$ such that $(X_t^1)_{t \geq 0}$ and $(X_t^2)_{t \geq 0}$ respectively follow the distribution of the Markov process starting respectively from x^1 and from x^2 , and such that a.s. $\forall t \geq 0, X_t^1 \leq X_t^2$, while also enjoying "nice" properties. We shall consider the space of ordered pairs

$$S = \{(x^1, x^2) \in \mathbb{R}^2, x^1 \leq x^2\},$$

and show that it is indeed possible to build a Feller process of pairs on S , as we now explain.

Note that the definition and key properties of Feller processes that we have quoted above, are valid not just for \mathbb{R} -valued Markov processes, but in the general context of a locally compact separable metric space, which is clearly the case of S (note that S is also a complete metric space, as a closed subset of \mathbb{R}^2). One just has to replace the spaces $\mathcal{B}_b, \mathcal{C}_b, \mathcal{C}_0$ of real-valued functions defined on \mathbb{R} , by the corresponding spaces of real-valued functions defined on S , hereafter denoted by $\mathcal{B}_b(S), \mathcal{C}_b(S), \mathcal{C}_0(S)$.

Theorem 1 *If $(P_t)_{t \geq 0}$ enjoys the Feller property (a)(b) and the stochastic monotonicity property (c), there exists a homogeneous family of Markov kernels $(r_t)_{t \geq 0}$ on S , with a corresponding Markov semi-group $(R_t)_{t \geq 0}$, such that:*

- for all $t \geq 0$, all $x^1 \leq x^2$, and every Borel subset B of \mathbb{R} ,

$$\begin{cases} r_t((x_1, x_2), (B \times \mathbb{R}) \cap S) = p_t(x_1, B) \\ r_t((x_1, x_2), (\mathbb{R} \times B) \cap S) = p_t(x_2, B) \end{cases} ;$$

- the semi-group $(R_t)_{t \geq 0}$ enjoys the Feller property.

In terms of trajectories, Theorem 1 shows that, starting from any two initial positions $x^1 \leq x^2$, we can define a Markov process of pairs $(X_t^1, X_t^2)_{t \geq 0}$ on \mathbb{R}^2 such that $(X_t^1)_{t \geq 0}$ and $(X_t^2)_{t \geq 0}$ respectively follow the distribution of the Markov process starting respectively from x^1 and from x^2 , and such that a.s. $\forall t \geq 0, X_t^1 \leq X_t^2$. Moreover, the Feller property of $(R_t)_{t \geq 0}$ entails that the process of pairs can be constructed so as to have càdlàg paths, the strong Markov property, and quasi-left-continuity, these properties being inherited by the coordinate processes $(X_t^1)_{t \geq 0}$ and $(X_t^2)_{t \geq 0}$.

1.2 Discussion

Note the following two classical settings in which Assumption (c) is satisfied, and for which a process of pairs can be defined in a rather direct and explicit way:

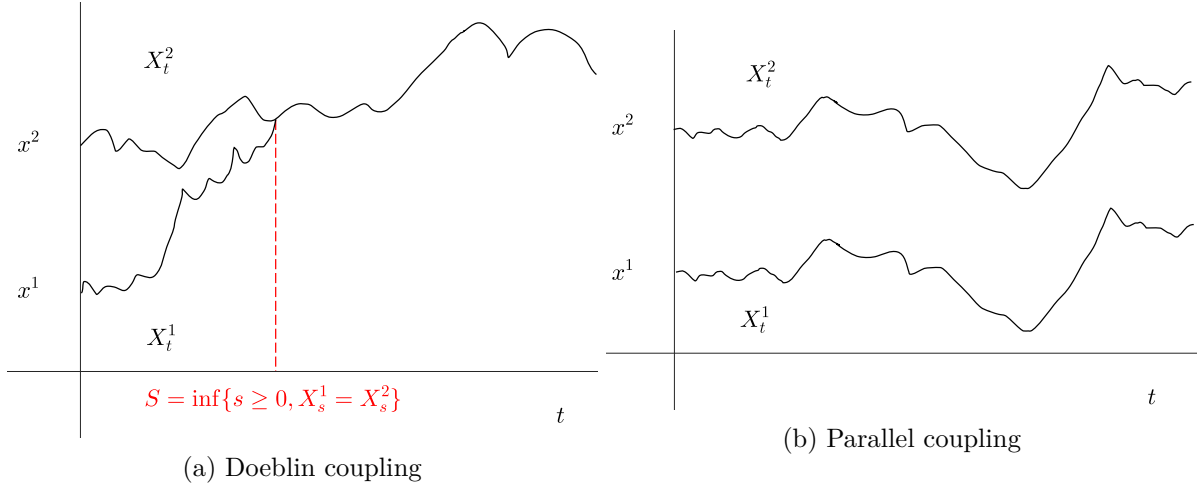


Figure 1: Special cases of Theorem 1 with easier constructions

- strong Markov processes with continuous paths; one can use the so-called *Doebelin* or *classical coupling*, in which the two processes evolve independently until they meet, and then stick together (see Figure 1a);
- space-homogeneous Markov processes (e.g. Lévy processes); one can use the *parallel coupling*, *i.e.* always keeping the same distance between both versions of the process (see Figure 1b).

In our view, the interest of Theorem 1 is that it holds in a completely general setting: the existence of a Feller process of ordered pairs on \mathbb{R}^2 is established under optimal assumptions. One may wonder whether weaker assumptions on the transition function $(p_t)_{t \geq 0}$ may lead to a (weaker) version of the conclusion of Theorem 1, but we do not pursue such a question here.

Note that a similar question has been considered by Kamae et al. ([5]), in the broader context of partially ordered Polish spaces, and with two possibly distinct Markov processes $(X_t)_{t \geq 0}$ and $(Y_t)_{t \geq 0}$ being compared (not just two versions of the same process started at two distinct initial values) through the appropriate extension of condition (c). Specifically, Theorem 5 in [5] states that, provided that individual càdlàg versions of the two processes exist, it is possible to define càdlàg versions of both processes on the same probability space, in such a way that a.s. $\forall t \geq 0 X_t \leq Y_t$ (Feller-type properties are not considered in [5]). Unfortunately, we believe that a part of the proof of Theorem 5 in [5] is problematic, as explained in Section 3.

Let us also mention [10], which deals with the case of jump processes on partially ordered Polish spaces (relying on [1]), and [6], which provides a rather general sufficient criterion for a (suitably regular) one-dimensional Feller process to enjoy property (c).

Finally, note that, although we focused on a coupling between two versions of a Markov process, the arguments used to prove Theorem 1 can be immediately extended to deal with an ordered n -tuple of processes, with $n \geq 2$, instead of just a pair.

2 Proof of Theorem 1

Step 1. For any fixed dyadic time t , a sequence of Markov kernels $(q_t^{(n)})_{n \geq n_0}$.

Fix $t > 0$ and, for every $x \in \mathbb{R}$, denote by $F_{x,t}^{[-1]}$ the quantile function of X_t starting from x , *i.e.* for $u \in (0, 1)$

$$F_{x,t}^{[-1]}(u) = \inf\{y \in \mathbb{R} \text{ such that } u \leq \mathbb{P}_x(X_t \leq y)\}.$$

Then consider a random variable U with uniform distribution on $(0, 1)$. For every $x \in \mathbb{R}$, the distribution of the random variable $F_{x,t}^{[-1]}(U)$ is the distribution of X_t starting from x , i.e. $p_t(x, \cdot)$. Moreover, the stochastic monotonicity property (c) implies that, for every $u \in (0, 1)$, and every $x^1 \leq x^2$, one has $F_{x^1,t}^{[-1]}(u) \leq F_{x^2,t}^{[-1]}(u)$, so that $(F_{x^1,t}^{[-1]}(U), F_{x^2,t}^{[-1]}(U))$ is an S -valued random pair. Now, for every $(x^1, x^2) \in S$, we define $q_t((x^1, x^2), \cdot)$ as the probability distribution of $(F_{x^1,t}^{[-1]}(U), F_{x^2,t}^{[-1]}(U))$ on S , and we observe that its marginals coincide respectively with $p_t(x^1, \cdot)$ and $p_t(x^2, \cdot)$. We also define $Q_t f(x^1, x^2) = \int_S f(x'^1, x'^2) q_t((x^1, x^2), (dx'^1, dx'^2))$, for every $(x^1, x^2) \in S$ and every bounded real-valued Borel function f on S .

Our goal is now to check that for any Borel set B of S , the map $(x^1, x^2) \mapsto q_t((x^1, x^2), B)$ is Borel, so that q_t defines a Markov kernel on S . We start by checking that, if $f \in \mathcal{C}_b(S)$, then $Q_t f$ is a continuous function.

By the Feller property (a), if we take a sequence of real numbers $(x_k)_{k \geq 1}$ converging to $x \in \mathbb{R}$, $p_t(x_k, \cdot)$ converges weakly³ to $p_t(x, \cdot)$. Hence $F_{x_k,t}^{[-1]}(U)$ converges a.s. to $F_{x,t}^{[-1]}(U)$ (see for instance [2], Chapter 14, Proposition 5). As a consequence, if we take a sequence $(x_k^1, x_k^2)_{k \geq 1}$ converging to (x^1, x^2) in S , we have the a.s. convergence of $f(F_{x_k^1,t}^{[-1]}(U), F_{x_k^2,t}^{[-1]}(U))$ to $f(F_{x^1,t}^{[-1]}(U), F_{x^2,t}^{[-1]}(U))$, for every $f \in \mathcal{C}_b(S)$. By the dominated convergence theorem, this implies that $Q_t f(x_k^1, x_k^2)$ converges to $Q_t f(x^1, x^2)$, so we have proved that $Q_t f$ is a continuous function.

Now we apply Theorem 0.2.2 in [8], which is a functional version of the monotone class theorem: the set \mathcal{H} of bounded real-valued Borel functions on S such that $Q_t f$ is Borel contains $\mathcal{C}_b(S)$ – which plays the role of \mathcal{C} in [8] – and the assumptions of the theorem are met⁴, so that \mathcal{H} contains all $\sigma(\mathcal{C}_b(S))$ -measurable functions, and thus all bounded real-valued Borel functions on S . In particular $Q_t \mathbf{1}_B$ is a Borel function for any Borel set B in S , so q_t is indeed a Markov kernel on S .

Now we call D_+ the set of positive dyadic rational numbers. Given $t \in D_+$, we write $t = k2^{-n_0} \in D_+$ where $k \geq 1$ and $n_0 \geq 0$ are integers, and n_0 has the minimum possible value in such an expression. Then, for every integer $n \geq n_0$, we let

$$q_t^{(n)} = (q_{2^{-n}})^{k2^{n-n_0}},$$

i.e. $q_t^{(n)}$ is the repeated composition of kernels $q_{2^{-n}} \cdots q_{2^{-n}}$ with a total of $k2^{n-n_0}$ kernels in the composition. As a result, $(q_t^{(n)})_{n \geq n_0}$ is a sequence of Markov kernels on S . Moreover, thanks to the fact that, for every $(x^1, x^2) \in S$, the marginals of $q_{2^{-n}}((x^1, x^2), \cdot)$ coincide respectively with $p_{2^{-n}}(x^1, \cdot)$ and $p_{2^{-n}}(x^2, \cdot)$, and to the fact that, thanks to the Chapman-Kolmogorov equation, $p_t = (p_{2^{-n}})^{k2^{n-n_0}}$, we have that, for every $(x^1, x^2) \in S$ and $n \geq n_0$, the marginals of $q_t^{(n)}((x^1, x^2), \cdot)$ coincide respectively with $p_t(x^1, \cdot)$ and $p_t(x^2, \cdot)$.

In the sequel, we denote by $Q_t^{(n)} f$ the action of $q_t^{(n)}$ on the bounded real-valued Borel function f on S .

Step 2. Equicontinuity of $(Q_t^{(n)} f)_{n \geq 1}$.

For every $K > 0$, we define a function $\phi_K : \mathbb{R} \rightarrow \mathbb{R}$ by setting, for every $x \in \mathbb{R}$,

$$\phi_K(x) = -K \mathbf{1}_{\{x < -K\}} + x \mathbf{1}_{\{-K \leq x \leq K\}} + K \mathbf{1}_{\{x > K\}},$$

and we note that ϕ_K is non-decreasing, bounded (by K) and continuous. Now, for every bounded real-valued Lipschitz function f on S , we define a function f_K on S by setting, for every $(x^1, x^2) \in S$,

$$f_K(x^1, x^2) = f(\phi_K(x^1), \phi_K(x^2)).$$

³We call weak convergence the convergence in distribution for probability measures.

⁴The set \mathcal{H} is a vector space that contains the constant functions and the supremum of any bounded non-decreasing sequence of its non-negative elements (thanks to the monotone convergence theorem); the set \mathcal{C} is stable under pointwise multiplication.

We claim that, for fixed t , K and f , the sequence $(Q_t^{(n)} f_K)_{n \geq n_0}$ is equicontinuous: given $(x^1, x^2) \in S$, for any $\varepsilon > 0$, there exists $\eta > 0$ such that for all $n \geq n_0$ and $(x'^1, x'^2) \in S$ such that $|x'^1 - x^1| + |x'^2 - x^2| \leq \eta$, we have

$$|Q_t^{(n)} f_K(x'^1, x'^2) - Q_t^{(n)} f_K(x^1, x^2)| \leq \varepsilon. \quad (1)$$

To begin with, since f is a Lipschitz function, there exists a real number $L > 0$ such that for all $(x^1, x^2), (x'^1, x'^2) \in S$ we have

$$|f(x'^1, x'^2) - f(x^1, x^2)| \leq L(|x'^1 - x^1| + |x'^2 - x^2|).$$

Now, using a sequence $(U_1, \dots, U_{k2^{n-n_0}})$ of i.i.d. random variables with uniform distribution on $(0, 1)$, we build two pairs of random variables (X^1, X^2) and (X'^1, X'^2) whose distributions are respectively $q_t^{(n)}((x^1, x^2), \cdot)$ and $q_t^{(n)}((x'^1, x'^2), \cdot)$, and such that the ordering between x^1, x'^1, x^2, x'^2 is the same as the ordering between X^1, X'^1, X^2, X'^2 . To do so, define inductively

$$\begin{cases} (X_0^1, X_0^2) = (x^1, x^2) \\ (X_i^1, X_i^2) = (F_{X_{i-1}^1, 2^{-n}}^{[-1]}(U_i), F_{X_{i-1}^2, 2^{-n}}^{[-1]}(U_i)) \quad 1 \leq i \leq k2^{n-n_0} \end{cases}$$

and set $(X^1, X^2) = (X_{k2^{n-n_0}}^1, X_{k2^{n-n_0}}^2)$. Then make the same construction starting from $(X_0'^1, X_0'^2) = (x'^1, x'^2)$ to build $(X'^1, X'^2) = (X_{k2^{n-n_0}}'^1, X_{k2^{n-n_0}}'^2)$, using the same random variables U_i . It is apparent from the definition that both (X^1, X^2) and (X'^1, X'^2) have the desired distributions, since each step in the inductive construction above amounts to one Markov transition step according to the kernel $q_{2^{-n}}$. Moreover, we have already observed in Step 1 that, thanks to the stochastic monotonicity property (c), the map $x \mapsto F_{x,t}^{[-1]}(u)$ is non-decreasing for every given $u \in (0, 1)$. As a consequence, since we used the same U_i 's for both (X^1, X^2) and (X'^1, X'^2) , the initial ordering between $(X_0^1, X_0'^1, X_0^2, X_0'^2) = (x^1, x'^1, x^2, x'^2)$ is preserved at each step of the construction, and is thus identical to the ordering between (X^1, X'^1, X^2, X'^2) .

We now write

$$|Q_t^{(n)} f_K(x'^1, x'^2) - Q_t^{(n)} f_K(x^1, x^2)| = |\mathbb{E}(f_K(X'^1, X'^2) - f_K(X^1, X^2))|.$$

Using the triangle inequality and the Lipschitz property of f , we obtain

$$|Q_t^{(n)} f_K(x'^1, x'^2) - Q_t^{(n)} f_K(x^1, x^2)| \leq L\mathbb{E}(|\phi_K(X'^1) - \phi_K(X^1)| + |\phi_K(X'^2) - \phi_K(X^2)|). \quad (2)$$

Now by construction the sign of $X'^1 - X^1$, and hence of $\phi_K(X'^1) - \phi_K(X^1)$ since ϕ_K is non-decreasing, is the same as that of $x'^1 - x^1$, regardless of the randomness, so we can get the absolute value out of the expectation:

$$\mathbb{E}(|\phi_K(X'^1) - \phi_K(X^1)|) = |\mathbb{E}(\phi_K(X'^1) - \phi_K(X^1))| = |P_t \phi_K(x'^1) - P_t \phi_K(x^1)|,$$

and the same goes for X^2 and X'^2 . Plugging these identities into (2), we obtain that

$$|Q_t^{(n)} f_K(x'^1, x'^2) - Q_t^{(n)} f_K(x^1, x^2)| \leq L(|P_t \phi_K(x'^1) - P_t \phi_K(x^1)| + |P_t \phi_K(x'^2) - P_t \phi_K(x^2)|). \quad (3)$$

By the Feller property (a') for $(P_t)_{t \geq 0}$, since ϕ_K is in $\mathcal{C}_b(\mathbb{R})$, $P_t \phi_K$ is in $\mathcal{C}_b(\mathbb{R})$ too, so that, invoking continuity,

$$|P_t \phi_K(x'^1) - P_t \phi_K(x^1)| \leq \frac{\varepsilon}{2L}$$

as soon as x'^1 is close enough to x^1 . Doing the same with x'^2 and x^2 , we can bound the the r.h.s. in (3) independently from n by $\varepsilon > 0$, as soon as x'^1 and x'^2 are close enough to x^1 and x^2 (*i.e.* when the distance between them is smaller than a certain $\eta > 0$ given by the continuity of $P_t \phi_K$ at x^1 and x^2), which gives (1).

Step 3. Construction of a limiting transition kernel by diagonal extraction.

We have already observed that, given $t \in D_+$ and $(x^1, x^2) \in S$, the marginals of $q_t^{(n)}((x^1, x^2), \cdot)$ are respectively $p_t(x^1, \cdot)$ and $p_t(x^2, \cdot)$ for every $n \geq n_0$. From this observation, we deduce that the sequence $(q_t^{(n)}((x^1, x^2), \cdot))_{n \geq n_0}$ is tight⁵, allowing us to extract a weakly converging subsequence.

Now fix a countable dense subset D_S of S . By diagonal extraction, there exists an increasing sequence of integers $\varphi(n)$ such that for all $t \in D_+$ and $(x^1, x^2) \in D_S$, as $n \rightarrow +\infty$, the sequence $q_t^{(\varphi(n))}((x^1, x^2), \cdot)$ converges weakly to some probability measure $r_t((x^1, x^2), \cdot)$ on S . We shall prove that r_t can be extended to a limiting Markov kernel on the whole of S .

For any bounded real-valued Borel function f on S , and every $(x^1, x^2) \in D_S$ let

$$R_t f(x^1, x^2) = \int_S f(x'^1, x'^2) r_t((x^1, x^2), (dx'^1, dx'^2)).$$

Now take $t \in D_+$ and $(x^1, x^2) \in S$, fix $K > 0$ and a certain bounded Lipschitz function f as in Step 2, and let $\varepsilon > 0$. Since D_S is dense in S , we can find $(x'^1, x'^2) \in D_S$ such that $|x'^1 - x^1| + |x'^2 - x^2| \leq \eta$, where η is the modulus of equicontinuity at (x^1, x^2) given in Step 2 for the sequence $(Q_t^{(n)} f_K)_{n \geq n_0}$. Thus we have, for all $n \geq n_0$,

$$|Q_t^{(n)} f_K(x'^1, x'^2) - Q_t^{(n)} f_K(x^1, x^2)| \leq \varepsilon.$$

Now since (x'^1, x'^2) is in D_S , $Q_t^{(\varphi(n))} f_K(x'^1, x'^2)$ converges to $R_t f_K(x'^1, x'^2)$ as n goes to infinity, so for all large enough n we have that

$$|Q_t^{(\varphi(n))} f_K(x'^1, x'^2) - R_t f_K(x'^1, x'^2)| \leq \varepsilon,$$

so that, combining the two inequalities,

$$|Q_t^{(\varphi(n))} f_K(x^1, x^2) - R_t f_K(x^1, x^2)| \leq 2\varepsilon. \quad (4)$$

Now if we also take $m \geq n$, we have, using (4) twice and the triangle inequality

$$|Q_t^{(\varphi(n))} f_K(x^1, x^2) - Q_t^{(\varphi(m))} f_K(x^1, x^2)| \leq 4\varepsilon.$$

We have thus proved that the sequence $(Q_t^{(\varphi(n))} f_K(x^1, x^2))_{n \geq n_0}$ is Cauchy, and thus converges to a certain limit as n goes to infinity. We denote this limit by $R_t f_K(x^1, x^2)$, thus extending the definition of $R_t f_K$ to every $(x^1, x^2) \in S$ (when $(x^1, x^2) \in D_S$, we already know that the limit exists and is equal to $R_t f_K(x^1, x^2)$). As a consequence, any weak limit ν along a subsequence of $(q_t^{(\varphi(n))}((x^1, x^2), \cdot))_{n \geq n_0}$ has to satisfy $\int_S f_K(y^1, y^2) \nu(dy^1, dy^2) = R_t f_K(x^1, x^2)$. Since this is true for every $K > 0$ and bounded Lipschitz f , we deduce⁶ that there can be at most one such weak limit. Moreover, as we have already noted, the sequence $(q_t^{(\varphi(n))}((x^1, x^2), \cdot))_{n \geq n_0}$ is tight, so it indeed has a weak limit, which we denote by $r_t((x^1, x^2), \cdot)$, thus extending the definition of r_t to every $(x^1, x^2) \in S$. We extend the definition of R_t accordingly, so that $R_t f(x^1, x^2)$ is defined for every $(x^1, x^2) \in S$ and every bounded Borel real-valued function f on S .

To conclude that r_t defines a Markov kernel on S , we prove that the class of functions f such that $R_t f$ is Borel contains $\mathcal{C}_b(S)$. Given $f \in \mathcal{C}_b(S)$, weak convergence implies that $R_t f$ is the pointwise limit of the sequence of functions $Q_t^{(\varphi(n))} f$ as $n \rightarrow +\infty$. Each function in the sequence is Borel since f is bounded and Borel and $Q_t^{(\varphi(n))}$ is a Markov kernel, so the pointwise limit of the sequence is Borel. We then argue exactly as in Step 1.

⁵Let H_1, H_2 be compact subsets of \mathbb{R} such that $p_t(x^1, H_1^c) \leq \frac{\varepsilon}{2}$ and $p_t(x^2, H_2^c) \leq \frac{\varepsilon}{2}$ for a certain $\varepsilon > 0$. Then $H = (H_1 \times H_2) \cap S$ is compact and, by the union bound, $q_t^{(n)}((x^1, x^2), H^c) \leq p_t(x^1, H_1^c) + p_t(x^2, H_2^c) \leq \varepsilon$ for all $n \geq n_0$.

⁶Any bounded Lipschitz function with compact support can be written $f = f_K$ for a sufficiently large K . And the class of bounded Lipschitz functions with compact support separates Borel probability measures on S .

Step 4. $(R_t)_{t \in D_+}$ has the Feller property.

We have built for all $t \in D_+$ a Markov kernel r_t , and now we prove that $(R_t)_{t \in D_+}$ enjoys the characteristic properties of a Feller semi-group, restricted to indices $t \in D_+$.

- First we prove that, if $f \in \mathcal{C}_b(S)$, then $R_t f \in \mathcal{C}_b(S)$. Fix $\varepsilon > 0$ and $(x^1, x^2) \in S$. As in Steps 2 and 3, fix a bounded Lipschitz function f and $K > 0$, and take $\eta > 0$ a modulus of equicontinuity as in Step 2, meaning that for every $(x'^1, x'^2) \in S$ such that $|x'^1 - x^1| + |x'^2 - x^2| \leq \eta$, we have for all n such that $\varphi(n) \geq n_0$

$$|Q_t^{(\varphi(n))} f_K(x'^1, x'^2) - Q_t^{(\varphi(n))} f_K(x^1, x^2)| \leq \varepsilon.$$

Now, thanks to the weak convergence, we also have for n large enough (depending on x^1, x^2, t, f and K but not on x'^1, x'^2)

$$|Q_t^{(\varphi(n))} f_K(x^1, x^2) - R_t f_K(x^1, x^2)| \leq \varepsilon,$$

so that

$$|Q_t^{(\varphi(n))} f_K(x'^1, x'^2) - R_t f_K(x^1, x^2)| \leq 2\varepsilon, \quad (5)$$

and taking the limit $n \rightarrow +\infty$, we obtain

$$|R_t f_K(x'^1, x'^2) - R_t f_K(x^1, x^2)| \leq 2\varepsilon. \quad (6)$$

Thus, we have proved that $R_t f_K(x'^1, x'^2)$ converges to $R_t f_K(x^1, x^2)$ as (x'^1, x'^2) goes to (x^1, x^2) . Moreover, the class of functions of the form f_K , where $K > 0$ and f is a bounded Lipschitz function, characterizes weak convergence towards a probability measure⁷, so that $r_t((x'^1, x'^2), \cdot)$ converges weakly to $r_t((x^1, x^2), \cdot)$ as (x'^1, x'^2) goes to (x^1, x^2) . In particular, for $g \in \mathcal{C}_b(S)$, $R_t g(x'^1, x'^2)$ converges to $R_t g(x^1, x^2)$ as (x'^1, x'^2) goes to (x^1, x^2) . We have proved that $R_t g$ is continuous. It is also bounded by $\|g\|_\infty$, so it is in $\mathcal{C}_b(S)$.

- Now let us prove that if g is in $\mathcal{C}_0(S)$, then $R_t g$ is also in $\mathcal{C}_0(S)$. We already know it is continuous, so we just have to prove that it vanishes at infinity. First, fix $\varepsilon > 0$ and let $M > 0$ such that $|g|$ is bounded by ε on $S \setminus [-M, M]^2$ (M exists since g vanishes at $\pm\infty$). Then we have

$$|R_t g(x^1, x^2)| \leq \|g\|_\infty r_t((x^1, x^2), S \cap [-M, M]^2) + \varepsilon.$$

Now, since $(P_t)_t$ maps \mathcal{C}_0 into \mathcal{C}_0 , and r_t 's marginals are respectively $p_t(x^1, \cdot)$ and $p_t(x^2, \cdot)$, $r_t((x^1, x^2), S \cap [-M, M]^2)$ goes to zero as x^1 and x^2 go to infinity⁸, so we can make $|R_t g(x^1, x^2)|$ arbitrarily small by taking (x^1, x^2) far enough from the origin, and thus $R_t g$ is in $\mathcal{C}_0(S)$.

- Let us prove the semi-group property. First, recall $t = k_0 2^{-n_0}$ and also take $s = l_0 2^{-m_0}$ another dyadic time. If n is such that $\varphi(n) \geq \max(n_0, m_0)$, we have by construction

$$Q_s^{(\varphi(n))} Q_t^{(\varphi(n))} = Q_{s+t}^{(\varphi(n))}.$$

To prove that $R_s R_t = R_{s+t}$ we will prove that the corresponding operators coincide on the functions f_K , with f Lipschitz and $K > 0$. Fix $M > 0$. We have for every $g, h \in \mathcal{C}_b(S)$ bounded by a same constant $C > 0$

$$\begin{aligned} & |Q_s^{(\varphi(n))} g(x^1, x^2) - Q_s^{(\varphi(n))} h(x^1, x^2)| \\ & \leq \|g - h\|_{\infty, S \cap [-M, M]^2} + 2C q_s^{(\varphi(n))}((x^1, x^2), S \setminus [-M, M]^2), \end{aligned} \quad (7)$$

⁷Any bounded Lipschitz function with compact support can be written $f = f_K$ for a sufficiently large K . And the class of bounded Lipschitz functions with compact support characterizes weak convergence towards a Borel probability measure on S .

⁸We have $r_t((x^1, x^2), S \cap [-M, M]^2) \leq p_t(x^1, [-M, M])$, and, given a function $f \in \mathcal{C}_0(\mathbb{R})$ such that $f \geq \mathbf{1}_{[-M, M]}$, $P_t f$ vanishes at infinity.

where $\|\cdot\|_{\infty, B}$ is the supremum norm on functions restricted to a domain B . Using the fact that the sequence of probability measures $(q_s^{(\varphi(n))}((x^1, x^2), \cdot))_{n \in \mathbb{N}}$ is tight since it converges weakly, we can make the second term in the r.h.s. of (7) arbitrarily small by taking M large enough. Now if we take $g = Q_t^{(\varphi(n))} f_K$ and $h = R_t f_K$, both bounded by $\|f\|_{\infty}$, we will see that we can also make the first term as small as we want. Indeed, looking back at (5) and (6), we have that, for all $(y^1, y^2) \in S$, and all $\varepsilon > 0$, there exists an $\eta > 0$ such that, for all large enough n and (y^1, y^2) such that $|y^1 - y^1| + |y^2 - y^2| \leq \eta$,

$$|Q_t^{(\varphi(n))} f_K(y^1, y^2) - R_t f_K(y^1, y^2)| \leq 4\varepsilon,$$

which means that the convergence of $Q_t^{(\varphi(n))} f_K$ to $R_t f_K$ is locally uniform around (y^1, y^2) , and thus uniform on any compact subset of S . Taking $S \cap [-M, M]^2$ for this compact, we can make (7) arbitrarily small, namely

$$|Q_s^{(\varphi(n))} Q_t^{(\varphi(n))} f_K(x^1, x^2) - Q_s^{(\varphi(n))} R_t f_K(x^1, x^2)| \leq \varepsilon \quad (8)$$

for n large enough.

Now since $R_t f_K$ is in $\mathcal{C}_b(S)$ (because f_K is) and $r_s((x^1, x^2), \cdot)$ is the weak limit of $q_s^{(\varphi(n))}$, $Q_s^{(\varphi(n))} R_t f_K(x^1, x^2)$ converges to $R_s R_t f_K(x^1, x^2)$, so that $Q_s^{(\varphi(n))} Q_t^{(\varphi(n))} f_K(x^1, x^2)$ also converges to $R_s R_t f_K(x^1, x^2)$ thanks to (8). Furthermore, since $Q_s^{(\varphi(n))} Q_t^{(\varphi(n))} = Q_{s+t}^{(\varphi(n))}$, and since $s+t$ is also dyadic, the weak convergence of $q_{s+t}^{(\varphi(n))}((x^1, x^2), \cdot)$ ensures that $Q_s^{(\varphi(n))} Q_t^{(\varphi(n))} f_K(x^1, x^2)$ converges to $R_{s+t} f_K(x^1, x^2)$, hence $R_s R_t f_K(x^1, x^2) = R_{s+t} f_K(x^1, x^2)$. Using again the fact that the functions f_K separate probability measures on S , we deduce that $r_s r_t = r_{s+t}$ and that $R_s R_t = R_{s+t}$.

- Finally we prove the uniform convergence of $R_t f$ to f , for any $f \in \mathcal{C}_0(S)$, as t goes to 0 (for $t \in D_+$). Fix $\varepsilon > 0$ and η a modulus of uniform continuity for f , *i.e.* for all $(x^1, x^2), (x^1, x^2)$ in S such that $|x^1 - x^1| + |x^2 - x^2| \leq \eta$, we have $|f(x^1, x^2) - f(x^1, x^2)| \leq \varepsilon$. Also fix $M > 0$ such that $|f| < \varepsilon$ on $S \setminus [-M, M]^2$. We want to dominate $|R_t f(x^1, x^2) - f(x^1, x^2)|$ uniformly in (x^1, x^2) .
 - Assume that (x^1, x^2) is outside $[-(M+1), M+1]^2$. Since $|f| \leq \varepsilon$ outside $[-M, M]^2$,

$$\begin{aligned} |R_t f(x^1, x^2) - f(x^1, x^2)| &\leq \varepsilon + |R_t f(x^1, x^2)| \\ &\leq 2\varepsilon + R_t(|f| \mathbf{1}_{[-M, M]^2})(x^1, x^2) \\ &\leq 2\varepsilon + \|f\|_{\infty} p_t(x^1, [-M, M]). \end{aligned}$$

Now observe that the Feller property for P_t used on the function h depicted in Figure 2a proves that $p_t(x, [-M, M])$ converges to zero as $t \rightarrow 0$, uniformly for x outside $[-(M+1), M+1]$: indeed, $p_t(x, [-M, M])$ is smaller than $P_t h(x)$, which converges uniformly to $h(x) = 0$. We deduce that $|R_t f(x^1, x^2) - f(x^1, x^2)|$ is smaller than 3ε for t small enough, uniformly for $(x^1, x^2) \notin [-(M+1), M+1]^2$.

- Assume that (x^1, x^2) is in $[-(M+1), M+1]^2$. Denote (X_t^1, X_t^2) a couple of random variables whose distribution is $r_t((x^1, x^2), \cdot)$, and, for $x \in \mathbb{R}$ and $\delta > 0$, denote by $I_\delta(x)$ the interval $[x - \delta, x + \delta]$. We have

$$\begin{aligned} |R_t f(x^1, x^2) - f(x^1, x^2)| &\leq \mathbb{E}(|f(X_t^1, X_t^2) - f(x^1, x^2)|) \\ &\leq \varepsilon + 2\|f\|_{\infty} \mathbb{P}(|X_t^1 - x^1| + |X_t^2 - x^2| > \eta) \\ &\leq \varepsilon + 2\|f\|_{\infty} \left(\mathbb{P}\left(|X_t^1 - x^1| > \frac{\eta}{2}\right) + \mathbb{P}\left(|X_t^2 - x^2| > \frac{\eta}{2}\right) \right) \\ &= \varepsilon + 2\|f\|_{\infty} \left(p_t(x^1, I_{\eta/2}^c(x^1)) + p_t(x^2, I_{\eta/2}^c(x^2)) \right). \end{aligned}$$

Now, we only need to prove that $p_t(x, I_{\eta/2}^c(x))$ converges to zero uniformly on the interval $[-(M+1), M+1]$. To do so, we prove that this convergence is locally

uniform around every x in \mathbb{R} , so that it is uniform on every compact subset of \mathbb{R} . To this end, we apply the Feller property for P_t to the function h depicted in Figure 2b. Since $h \leq \mathbf{1}_{I_{2\delta}(x)}$, we have for $x' \in I_\delta(x)$, $P_t h(x') \leq p_t(x', I_{2\delta}(x)) \leq p_t(x', I_{3\delta}(x'))$. Since $P_t h$ converges uniformly to h as t goes to zero, $P_t h(x')$ converges to 1 uniformly for $x' \in I_\delta(x)$, hence so does $p_t(x', I_{3\delta}(x'))$. Taking $\delta = \frac{\eta}{6}$ allows us to complete the proof of uniform convergence of $R_t f$ to f as t goes to zero.

Step 5. Extension of $(R_t)_{t \in D_+}$ to all times.

Let $t > 0$ be a non-dyadic time, and $s, s' \in D_+$ such that $t < s < s'$. Also fix $f \in \mathcal{C}_0(S)$. Thanks to the semi-group property for dyadic times, we have

$$\|R_{s'}f - R_s f\|_\infty \leq \|R_s(R_{s'-s}f - f)\|_\infty \leq \|R_{s'-s}f - f\|_\infty,$$

so $(R_s f)_{s>t}$ satisfies the Cauchy condition with respect to the $\|\cdot\|_\infty$ norm as s goes to t with $s > t$, thanks to the uniform convergence of $R_{s'-s}f$ to f as $s' - s$ goes to 0. Then we can define $R_t f = \lim_{s \rightarrow t^+} R_s f \in \mathcal{C}_0(S)$. Consider a sequence $(s_n)_{n \geq 1}$ converging to t such that, for all $n \geq 1$, $s_n \in D_+$, and $s_n > t$. For all $x^1 \leq x^2$, the marginals of $r_{s_n}((x^1, x^2), \cdot)$ are given by $p_{s_n}(x^1, \cdot)$ and $p_{s_n}(x^2, \cdot)$, which, by the Feller property (b), converge weakly to $p_t(x^1, \cdot)$ and $p_t(x^2, \cdot)$. As a consequence, the sequence $(r_{s_n}((x^1, x^2), \cdot))_{n \geq 1}$ is tight, and the convergence of $R_{s_n} f$ to $R_t f$ for every $f \in \mathcal{C}_0(S)$ shows that there indeed is a probability distribution $r_t((x^1, x^2), \cdot)$ which is the weak limit of $r_{s_n}((x^1, x^2), \cdot)$ as $n \rightarrow +\infty$, with $R_t f(x^1, x^2) = \int_S f(x^1, x^2) r_t((x^1, x^2), (dx^1, dx^2))$ for every $f \in \mathcal{C}_0(S)$, and whose marginals are $p_t(x^1, \cdot)$ and $p_t(x^2, \cdot)$. As in Step 3, we check that r_t is a Markov kernel as a (pointwise) weak limit of Markov kernels.

Note that, by construction, R_t maps $\mathcal{C}_0(S)$ into $\mathcal{C}_0(S)$, and, composing uniform limits, we also obtain that $R_t f$ goes uniformly to f as t goes to zero, so properties (a) and (b) are satisfied. Finally, the semi-group property comes from the uniform convergence on the dyadics: let $s, t > 0$ and consider two sequences of elements of D_+ denoted by $(s_n)_{n \geq 1}$, $(t_n)_{n \geq 1}$, with $s_n \geq s$ and $t_n \geq t$ for all n , and $s_n \rightarrow s, t_n \rightarrow t$. Then we have, for all $f \in \mathcal{C}_0(S)$,

$$\begin{aligned} \|R_{s+t}f - R_s R_t f\|_\infty &\leq \|R_{s+t}f - R_{s_n} R_{t_n} f\|_\infty + \|R_{s_n} R_{t_n} f - R_{s_n} R_t f\|_\infty \\ &\quad + \|R_{s_n} R_t f - R_s R_t f\|_\infty. \end{aligned}$$

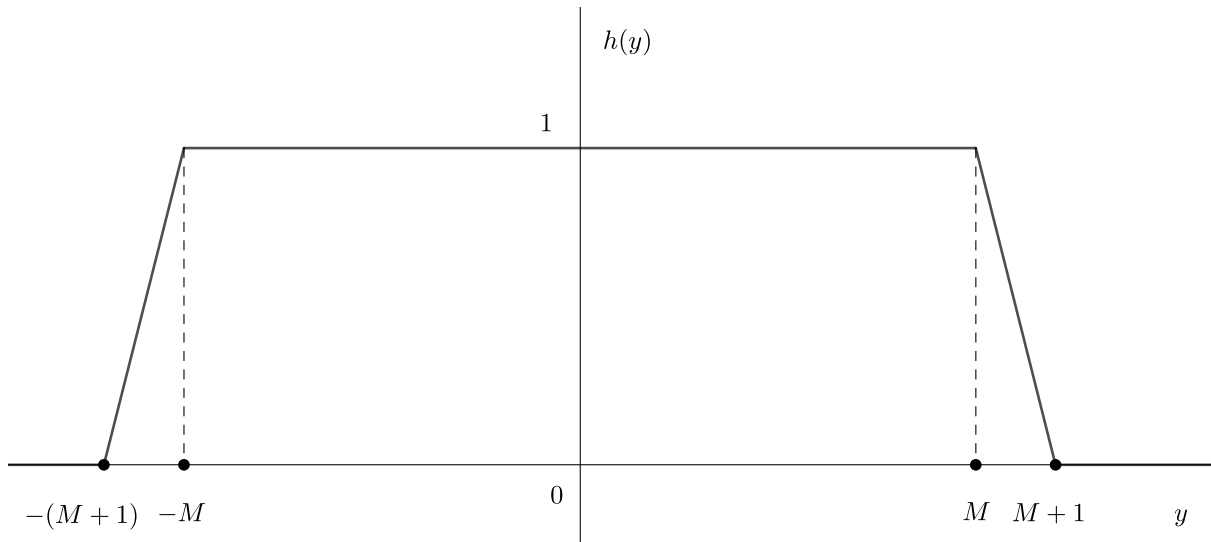
Now, thanks to the semi-group property on D_+ , the first term is $\|R_{s+t}f - R_{s_n+t_n}f\|_\infty$ and can be made arbitrarily small since $s_n + t_n$ converges to $s + t$. The third term is also small, because $R_{s_n} g$ converges to $R_s g$ uniformly for all $g \in \mathcal{C}_0(S)$. Finally, the second term is smaller than $\|R_{t_n} f - R_t f\|_\infty$ and thus also converges to zero. Hence we obtain the semi-group property of $(R_t)_{t \geq 0}$ on $\mathcal{C}_0(S)$, and the corresponding Chapman-Kolmogorov equation for $(r_t)_{t \geq 0}$.

3 About the proof of Kamae et al.

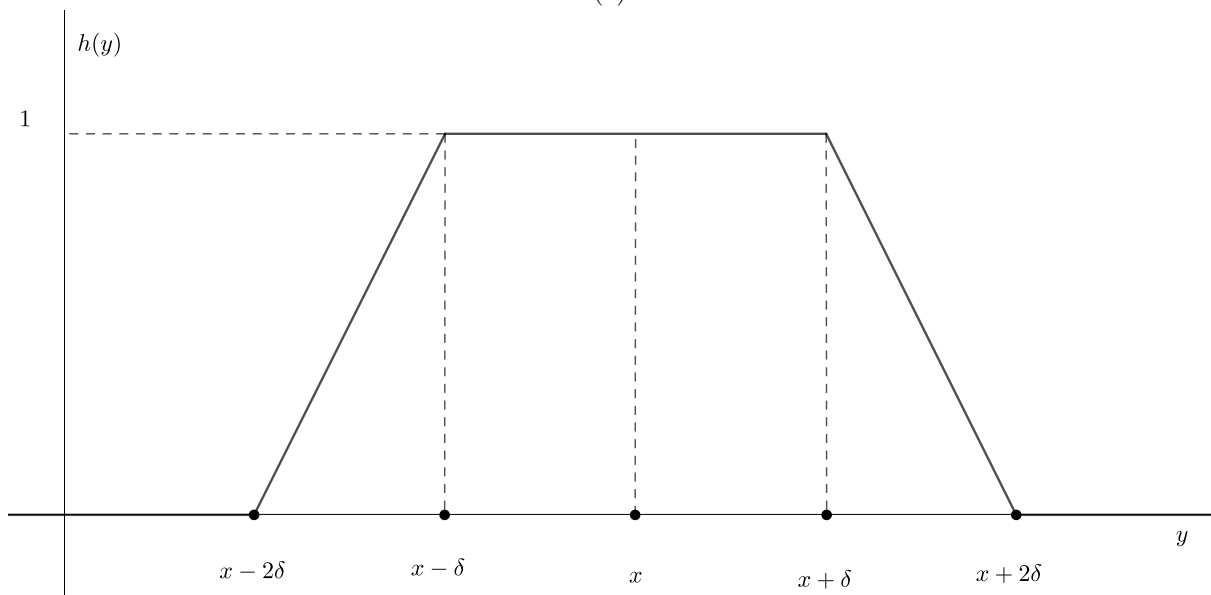
The argument of Kamae et al. in [5] to deal with the case of continuous-time Markov processes with càdlàg paths (Theorem 5) is by reduction to the discrete-time case (Theorem 2 via Theorem 4), invoking a step-by-step construction over rational time indices which is then extended to time-indices in \mathbb{R}_+ .

We now explain the issue with the step-by-step construction, in the context of two versions of the same process Markov X starting from distinct initial values being compared. The construction relies on an enumeration $(t_n)_{n \geq 1}$ of non-negative rational numbers, and the Markov kernels

$$\tilde{p}_{t_1, \dots, t_n}((x_1, \dots, x_{n-1}), \cdot) = \mathbb{P}(X_{t_n} \in \cdot | X_{t_1} = x_1, \dots, X_{t_{n-1}} = x_{n-1}).$$



(a)



(b)

Figure 2: Auxiliary functions used to obtain Feller property

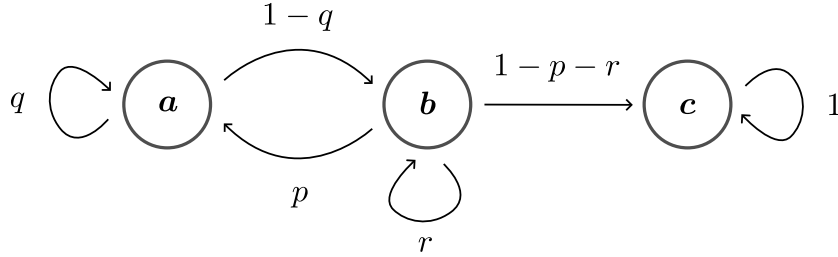


Figure 3: A non-decreasing Markov chain whose post-conditioned kernels are not non-decreasing

with the key assumption that, for all $n \geq 1$ and all $x^1, x^2 \in \mathbb{R}^{n-1}$ such that $x_i^1 \leq x_i^2$ for $1 \leq i \leq n-1$

$$\tilde{p}_{t_1, \dots, t_n}((x_1^1, \dots, x_{n-1}^1), \cdot) \preceq \tilde{p}_{t_1, \dots, t_n}((x_1^2, \dots, x_{n-1}^2), \cdot), \quad (9)$$

where \preceq denotes the stochastic dominance ordering between probability measures.

From property (c) and the Markov property, (9) is certainly true when $t_n \geq t_i$ for all $1 \leq i \leq n-1$. However, since it is certainly not possible to enumerate a dense subset of \mathbb{R}_+ using an increasing sequence, the sequence $(t_n)_{n \geq 1}$ cannot be increasing, and the success of the step-by-step construction relies on (9) being true also when there exist $1 \leq i, j \leq n-1$ such that $t_i < t_n < t_j$. Unfortunately, property (c) is not sufficient to ensure that (9) is true in such situations, as we now show using a counter-example.

Consider a Markov chain X on a set of three states $a < b < c$, whose transition kernel involves parameters p, q, r , as depicted on Figure 3. For $x_0, x_1, x_2 \in \{a, b, c\}$, denote by F_{x_0} the c.d.f. of X_1 under the condition $X_0 = x_0$, $F_{x_0 x_2}$ that of X_1 under the condition $X_0 = x_0, X_2 = x_2$, and $p_{x_0 x_1}$ the probability of transition from x_0 to x_1 (in one jump). One checks that, as soon as $p \leq q$, we have $F_a \geq F_b \geq F_c$, so that the Markov chain is stochastically non-decreasing. Now let us compute $F_{aa}(a)$ and $F_{bb}(a)$. We have

$$\begin{cases} F_{aa}(a) = \frac{p_{aa} p_{aa}}{p_{aa} p_{aa} + p_{ab} p_{ba} + p_{ac} p_{ca}} = \frac{q^2}{q^2 + (1-q)p}, \\ F_{bb}(a) = \frac{p_{ba} p_{ab}}{p_{ba} p_{ab} + p_{bb} p_{bb} + p_{bc} p_{cb}} = \frac{p(1-q)}{p(1-q) + r^2}, \end{cases}$$

As a consequence, if we find $p, q, r \in [0, 1]$ such that

$$\frac{p(1-q)}{p(1-q) + r^2} - \frac{q^2}{q^2 + (1-q)p} > 0$$

with conditions $p \leq q$ and $p+r \leq 1$, we have $F_{bb}(a) > F_{aa}(a)$, which shows that we do not have the ordering $\mathbb{P}(X_1 \in \cdot | X_0 = a, X_2 = a) \preceq \mathbb{P}(X_1 \in \cdot | X_0 = b, X_2 = b)$, despite the fact that $a < b$.

For instance⁹, taking $p = \frac{1}{2}, q = \frac{3}{5}$ and $r = \frac{1}{4}$ works.

⁹Notice that if we had $r = 1-p$, the third state c would be useless. However it is not, precisely because the inequality has no solution in that case.

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