



## Discrete Voss surfaces: Designing geodesic gridshells with planar cladding panels

Nicolas Montagne, Cyril Douthe, Xavier Tellier, Corentin Fivet, Olivier Baverel

### ► To cite this version:

Nicolas Montagne, Cyril Douthe, Xavier Tellier, Corentin Fivet, Olivier Baverel. Discrete Voss surfaces: Designing geodesic gridshells with planar cladding panels. Automation in Construction, 2022, 140, pp.104200. 10.1016/j.autcon.2022.104200 . hal-04309288

HAL Id: hal-04309288

<https://hal.science/hal-04309288>

Submitted on 27 Nov 2023

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

1                   Discrete Voss Surfaces: Toward the design of  
2                   Geodesic Gridshells

4                   Nicolas MONTAGNE<sup>\*a,b</sup>, Cyril DOUTHE<sup>a</sup>, Xavier Tellier<sup>a</sup>, Corentin FIVET<sup>b</sup>, Olivier BAVEREL<sup>a</sup>

5                   <sup>\* a</sup> Laboratoire Navier, UMR 8205, Ecole des Ponts, IFSTTAR, CNRS, UPE  
6                   77455 Champs-sur-Marne – MLV Cedex 2 – France  
7                   [nicolas.montagne@enpc.fr](mailto:nicolas.montagne@enpc.fr)

8                   <sup>b</sup> EPFL, ENAC, Structural Xploration Lab, Passage du Cardinal 13b CH-1700 Fribourg

10                  ABSTRACT

---

11                  The design and construction of doubly-curved structures often reveals to be challenging and can result  
12                  in complex manufacturing and assembly. A recent strategy to tackle this difficulty consists in exploiting  
13                  the connection between discrete differential geometry and constructive properties to identify curve  
14                  networks with good fabrication or mechanical properties. Following this approach, a family of surfaces,  
15                  called Voss surfaces, is here presented. Among other features, they can be built from flat panels,  
16                  initially flat straight strips, and hinged connections. These properties arise from the existence of a  
17                  conjugate network of geodesic curves. Two generation methods are presented to shape discrete Voss  
18                  surfaces: the first allows their exploration through linear spaces; the second provides a unique solution  
19                  by means of a direct computation.

20                  Keywords: Voss surface; Geodesic gridshell; Architectural geometry; Space exploration; Chebyshev net

21                  1 INTRODUCTION

---

22                  Building envelopes are a key aspect of the materialization of architectural intents. The recent advent  
23                  of computer graphics opened the door to more complex, free-form shapes in architecture, and thus  
24                  challenged conventional strategies of structural design and construction. Irregularly-curved  
25                  geometries introduce new design issues that are not encountered with traditional spatial structures,  
26                  often leading to fabrication and construction aberrations. For example, the use of triangulated  
27                  networks to adapt target shapes often results in unique beam members and intricate nodal  
28                  arrangements. On the other side, alternative non-triangular patterns do not ensure the possibility of a  
29                  covering with flat panels. In this respect, bending-active structures are particularly appealing for their  
30                  capacity to smoothly render freeform shapes and ease their fabrication [1]. Their members are  
31                  elastically bent into a target position, and, together, form a network of curved elements fitting a doubly

32 curved surface. Furthermore, the erection process may contribute to the good mechanical behaviour  
33 of the system when the pre-stressing effect of bent members stiffens the structure.

34 A remarkable family of bending-active structures are elastic gridshells [2,3]. These structures are  
35 assembled flat on the ground from initially straight beams connected with simple rotational joints, and  
36 then erected into a target curved shape. The erection of the structure as a whole is permitted by a  
37 special arrangement of the beams into a regular grid, known as Chebyshev net. Elastic gridshells  
38 illustrate both the advantages and the difficulties faced when dealing with active-bending systems.  
39 Although, they create lightweight and large spanning support structures from elements that are easy  
40 to manufacture and assemble, their design also presents many issues. The actual geometry and the  
41 mechanical behaviour of elastic gridshells is hard to predict since it is given by the static equilibrium of  
42 the beam network and that members are largely-deformed and interconnected in dense layouts. In  
43 addition, members, generally wooden laths with rectangular cross sections, are subject to severe local  
44 stresses since the layout is imposed by the use of a Chebyshev net [4,5]. Additionally, the covering of  
45 the support structure usually requires custom curved panels, which reveals to be expensive and  
46 difficult to construct, or tailor-made fabric elements which assume a perfect control of the geometry  
47 and an increased maintenance.

48 **1.1 RELATED WORKS**

49 Research responding to the challenges raised for the design of elastic gridshells also proved useful for  
50 the more general field of active-bending structures. Past and current developments address various  
51 design aspects from geometric fitting, to numerical simulation of mechanical behaviour, envelope  
52 design and structural optimization.

53 From a geometrical point of view, the problem of mapping a given layout, namely a Chebyshev net, on  
54 an input target surface is a main focus. The IL team in Stuttgart pioneered the so-called compass  
55 method and hanging chain models to form-find the geometry of grids on given surfaces [6]. Bouhaya  
56 et al [7] addressed the same problem by projection and later optimized the positioning using genetic  
57 algorithms [8]. Alternatively, Lafuente Hernández et al. [9] proposed a methodology to design with an  
58 approximate Chebyshev net, while Masson [10] proposed solutions to introduce and manage  
59 singularities in the net. Sageman-Furnas et al. [11] gathered these different approaches in a single  
60 framework.

61 On a numerical modelling level, the capacity to simulate and evaluate the mechanical behaviour of  
62 bending-active structures has been greatly improved by the development of special numerical  
63 methods like dynamic relaxation [12]. Combined with the use of discrete element models, dynamic  
64 relaxation allows the quick and reliable assessment of equilibrium states and stresses in structures.

65 Since the initial development of the discrete element with three degrees of freedom accounting for  
66 axial stresses and bending [13], more complete formulations have been proposed to improve the  
67 precision of the model [14–18]. In contrast, Douthe et al. [19] evaded the problem of excessive stresses  
68 in members imposed by the Chebyshev net layout by using circular hollow cross section beams, hence  
69 avoiding torsion and improving bending compliancy. This approach proved to be conclusive in other  
70 works [5,20]. Other types of bending-active systems were also investigated mechanically [1].

71 Although bending-active structures and more specifically elastic gridshells have regained interest at  
72 the beginning of this century, their construction remains rare. This can be explained by the difficulty  
73 to cover these systems with envelopes, which has not much been addressed in research. In this regard,  
74 Schober and Schlaich [22] pioneered the use of translational net to ensure the covering of the structure  
75 with flat quadrangular panels. Recently, Douthe et al. [23] studied the special case of isoradial surfaces.  
76 The particularity of these manifolds is that they can be mapped by a Chebyshev grid and covered with  
77 planar faces. In geometry, it was found that conjugate nets of curves on smooth surfaces have a  
78 discrete equivalent: meshes with planar quadrilateral faces [24]. Since grid support structures are  
79 modelled as nets on surfaces, this property can be applied to ensure a covering of the structure with  
80 flat quadrilateral panels. In the case of isoradial surfaces, the interesting cladding property is due to  
81 the duality of the Chebyshev net with the conjugate net of principal curvature lines.

82 On another aspect, the problem of designing active bending with rectangular laths was not solved in  
83 the case of elastic gridshells but only wisely avoided by using an alternative cross section. Inspired by  
84 the precursory work of J. Natterer [25], Pirazzi and Weinand [26] propose an interesting solution for  
85 the design of bending-active shells with rectangular laths by using geodesic lines. Geodesic lines on  
86 surfaces have a vanishing geodesic curvature, therefore, a lath following such a line will not bend  
87 sideways. Since the slender element is not bent along its strong axis, severe stresses are avoided, as  
88 well as local instabilities. A geodesic line corresponds to the natural path of such elements on surfaces.  
89 Geodesic patterns on surfaces were further studied by Pottmann et al. [27]. The mapping of curved  
90 surfaces is addressed using several families of geodesic lines. Rabinovich et al. [26,27] used a special  
91 case of geodesic net, namely orthogonal geodesic nets, to model, deform, and explore the shape space  
92 of developable surfaces. In relation with the erection process of Chebyshev nets, Soriano et al. [30] as  
93 well as Pillwein et al. [31] explored the design of geodesic shells that can be constructed from an  
94 initially flat grid of laths. Wang et al. [32] used a similar notion, referred to as geodesic parallels to map  
95 surfaces with piecewise initially-flat strips.

96 In conclusion, various strategies for designing complex shaped envelopes can be found within the  
97 scope of bending-active systems. In particular, geodesic shells allow to build structures with

98 rectangular cross-sections tangent to the target surface. However, much like elastic gridshells, the lack  
99 of covering solutions with flat panels is a major limitation and contribute to a low uptake of such shells.

## 100 1.2 OVERVIEW

101 This paper studies the generation and transformation of discrete Voss surfaces, a special family of  
102 surfaces, whose geometrical properties ease fabrication and construction processes of freeform shells  
103 and gridshells. Voss surfaces were introduced and defined by Aurel E. Voss in 1888 [33] in their smooth  
104 version as surfaces that can be mapped by a conjugate network of geodesics. In their discrete version,  
105 Voss surfaces are meshes with flat quadrilateral faces whose opposite angles between edges at each  
106 vertex are equal [34]. Architectural envelopes based on discrete Voss surfaces can be built from regular  
107 elements despite their freeform shape. In particular, discrete Voss surfaces support the construction  
108 of geodesic gridshells, but can also be applied to flexible shell frameworks [35] made of quadrangular  
109 flat panels, and assembled with simply-hinged connections. In the case of geodesic gridshells, they  
110 allow the use of initially-straight rectangular-cross-section beams without sideways bending, and  
111 covered with flat panels.

112 In section 2, the representation of discrete *Voss nets*, i.e. conjugate geodesic nets carried by Voss  
113 surfaces, through normal vectors is studied. This approach, inspired by existing studies on smooth  
114 cases, provides a dual shape encompassing all necessary constraints characterizing Voss nets. Then, in  
115 section 3, transformations are introduced to characterize the domain of possible shapes. Particular  
116 families of transformations enable the geometric alteration of Voss nets while maintaining their good  
117 constructive properties. In section 4, the above-mentioned characteristics are used for the generation  
118 of meshes from a relevant family of surfaces, namely Chebyshev nets on the unit sphere  $S^2$ . More  
119 accurately, the shape space of Chebyshev nets on  $S^2$  creates a linear space of discrete Voss surfaces.  
120 The linear space is then used in section 5 to investigate and deform Voss surfaces using well-chosen  
121 transformation modes. This methodology addresses the design of surfaces from an exploratory point  
122 of view. A more straightforward method is then described, in which Voss surfaces are built from two  
123 boundary curves. Finally, applications of these techniques are exposed in section 6. Resulting doubly-  
124 curved shapes illustrate the large potential of discrete Voss surfaces for architectural applications.

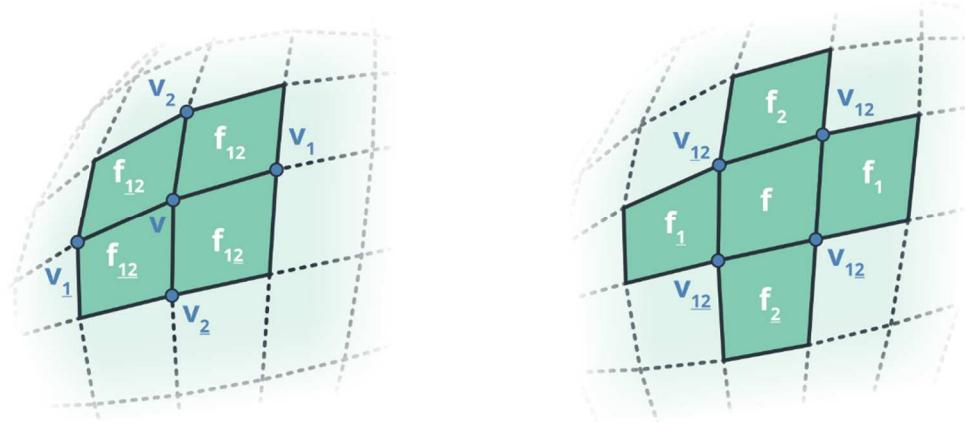
## 125 1.3 NOTATIONS

126 The remainder of this paper focuses on meshes with planar quadrilateral faces. In this context, a  
127 general mesh, a mesh with planar faces and a planar quad mesh (so-called PQ mesh) will be denoted  
128 respectively by the script letters  $\mathcal{M}$ ,  $\mathcal{P}$  and  $\mathcal{Q}$ . Similarly,  $\mathcal{V}$  will refer to a Voss net. Meshes are defined

129 by their list of vertices  $(v_i)_{i \leq I}$ , of edges  $(e_j)_{j \leq J}$  and of faces  $(f_k)_{k \leq K}$  where the size of these lists are  
 130 respectively  $I, J, K$ . The vertices of the mesh are stored in a column vector  $\mathbf{V}$  of size  $3I$ , where matrixes  
 131 will be distinguished by the use of bold capital letters. Therefore,  $\mathbf{V}$  is given by:

132 
$$\mathbf{V}^T = [v_{1,x}, v_{1,y}, v_{1,z}, v_{2,x}, v_{2,y}, \dots, v_{I,y}, v_{I,z}]$$

133 Specific notions are introduced to qualify planar quad meshes. The vertex-star refers to the  
 134 configuration depicted on Figure 1 - left and the face-star to the configuration on Figure 1 - right.  
 135 Notations on these figures will be used in the following to depict the neighbourhood of vertices and  
 136 faces. Vertices (Figure 1 - left), respectively faces (Figure 1 - right), on opposite sides of a vertex,  
 137 respectively a face, have the same index. One is underlined, the other is not. Indices of a face in the  
 138 neighbourhood of a vertex  $v$  (Figure 1 - left) are equivalent to the indices of the two other vertices  
 139 bounding that face. The same is true when swapping vertices and faces (Figure 1 - right). Additionally,  
 140 the affiliation to a mesh will be denoted in exponent when needed.



141  
 142 *Figure 1: Illustration and naming convention for a vertex-star (left) and a face-star (right)*

## 143 2 GAUSS MAP OF DISCRETE VOSS NETS

---

144 In the domain of classical differential geometry, a prolific strategy to extract properties of a surface is  
 145 to study the variation its normal vectors. Its importance is highlighted by the central notion of Gauss  
 146 Map. With  $S$  an orientable smooth surface, the Gauss map is the application that associates to each  
 147 unit normal vector of  $S$  a point on the unit sphere  $S^2$ , producing a new surface on  $S^2$ . The study of

148 this application has proven useful to characterize and represent smooth varieties like developable  
149 surfaces.

150 The discrete counterpart of the Gauss Map is equivocal and subject to discussion. Indeed, when  
151 addressing polygonal meshes, the set of normal vectors can be defined upon several entities, i.e. the  
152 faces, the edges, or the vertices. Thus, different notions of Gauss maps can be defined, as long as they  
153 are consistent with one another. The relevance of a Gauss map is evaluated from the image produced  
154 on the unit sphere  $S^2$  and its capacity to characterize the underlying mesh. For example, Pottmann et  
155 al. [36] established a definition of Gauss map in the context of parallel meshes in which normals vectors  
156 are determined by comparing a pair of parallel meshes  $\mathcal{P}$  and  $\mathcal{P}'$ . The normal vectors are defined by  
157 unitizing the distance between the corresponding vertices of  $\mathcal{P}$  and  $\mathcal{P}'$ . This formulation proves to be  
158 relevant to analyse offset properties of support structures.

159 When it comes to planar quadrilateral meshes, a variety of options exists to determine normal vectors.  
160 The following sub-sections detail two of them that prove to be particularly relevant and  
161 complementary when dealing with discrete Voss surfaces.

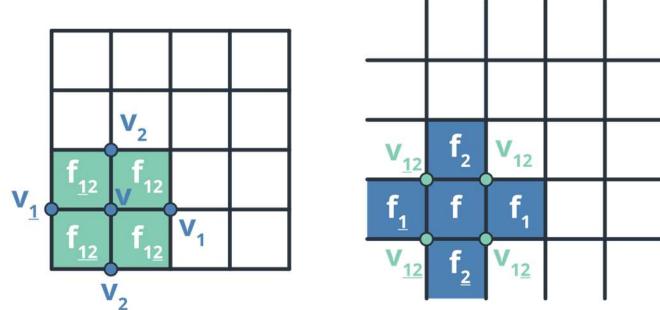
## 162 2.1 DEFINITION FROM FACE NORMALS

163 Since discrete Voss nets are meshes with planar faces, a first way to determine normal vectors on the  
164 mesh builds on the faces' normals.

165 *Definition 1: Let  $\mathcal{Q}$  be a planar quad net, the Gauss map, or spherical projection of  $\mathcal{Q}$  is the application  
166 which maps each normal vector of the faces of  $\mathcal{Q}$  to the corresponding point on the unit sphere  $S^2$ .*

167 In the remainder, the Gauss map  $\mathcal{G}$  designates either this application or the resulting mesh on  $S^2$   
168 without ambiguity. As a consequence of the definition, the image of the Gauss map  $\mathcal{G}$  is topologically  
169 dual to the initial mesh  $\mathcal{Q}$  (Figure 2): every face, edge, or vertex of  $\mathcal{Q}$  is transformed respectively to a  
170 vertex, edge or face of  $\mathcal{G}$ . This relation is reciprocal: every face, internal edge, internal vertex of  $\mathcal{G}$  is  
171 respectively transformed to a vertex, edge, face of  $\mathcal{Q}$ .

172 The above definition of the Gauss map is actually more general than just the context of PQ meshes,  
173 and can be used to represent any mesh  $\mathcal{P}$  with planar faces.



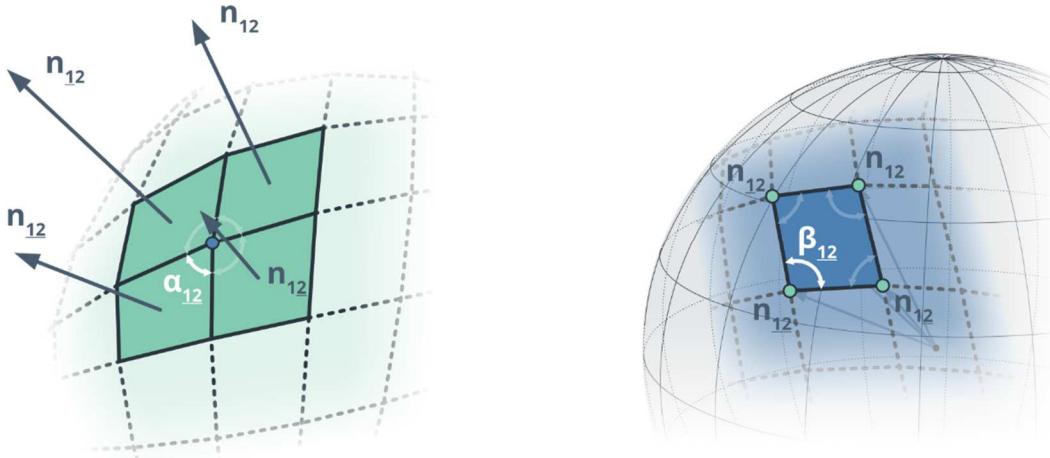
174

175 *Figure 2: Schematic representation of the dual relation between a PQ net  $\mathcal{Q}$  (left) and its Gauss map  $\mathcal{G}$  (right).*

176 It follows a first property on the projection of the angles made by edges around a vertex-star [37]:

177 **Property 1:** Let  $\mathcal{Q}$  be a planar quad net,  $\mathcal{G}$  its Gauss map, and the angles  $\alpha_{12}, \alpha_{\underline{12}}, \alpha_{1\underline{2}}, \alpha_{\underline{1}\underline{2}}$  around the  
 178 vertex  $v$  of  $\mathcal{Q}$ , at a vertex-star. Let also  $\beta_{12}, \beta_{\underline{1}2}, \beta_{1\underline{2}}, \beta_{\underline{1}\underline{2}}$  be the angles of the face  $f$  of  $\mathcal{G}$ , dual to  $v$ , at  
 179 the vertices  $n_{12}, n_{\underline{1}2}, n_{1\underline{2}}, n_{\underline{1}\underline{2}}$ , dual to the faces  $f_{12}, f_{\underline{1}2}, f_{1\underline{2}}, f_{\underline{1}\underline{2}}$  in  $\mathcal{Q}$  (Figure 3). By construction, the  
 180 spherical projection implies that the angles  $\alpha_{12}, \alpha_{\underline{12}}, \alpha_{1\underline{2}}, \alpha_{\underline{1}\underline{2}}$  are transformed to their supplementary  
 181 angles  $\beta_{12}, \beta_{\underline{1}2}, \beta_{1\underline{2}}, \beta_{\underline{1}\underline{2}}$  respectively:

182 
$$\beta_{12} = \pi - \alpha_{12}, \quad \beta_{\underline{1}2} = \pi - \alpha_{\underline{1}2}, \quad \beta_{1\underline{2}} = \pi - \alpha_{1\underline{2}}, \quad \beta_{\underline{1}\underline{2}} = \pi - \alpha_{\underline{1}\underline{2}}.$$

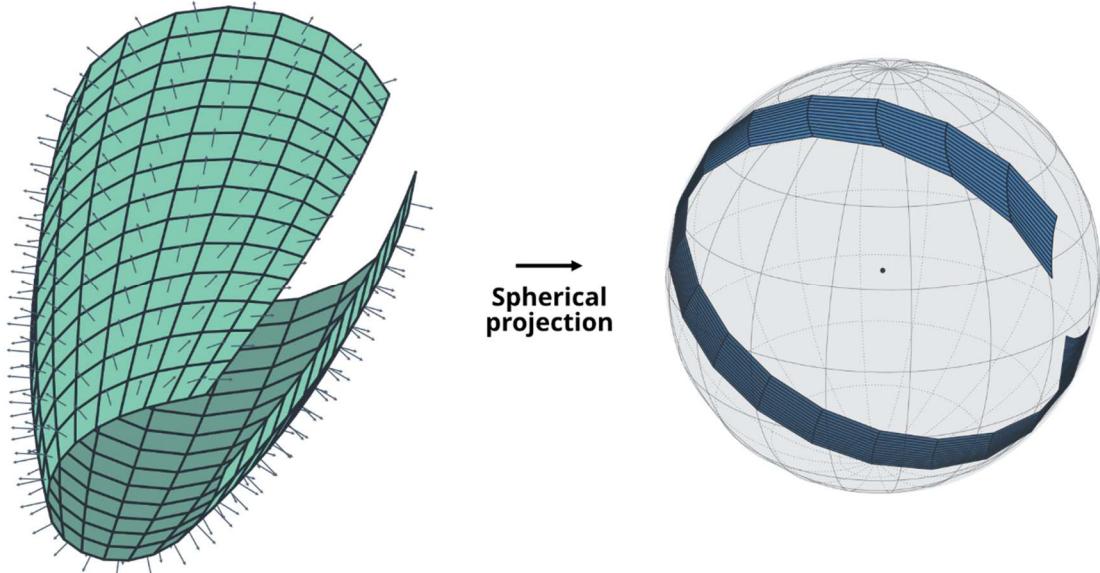


183

184 *Figure 3: Spherical projection of a vertex-star. Angles around a vertex (left) are transformed to supplementary angles in the  
 185 dual face of the Gauss map (right).*

186 As an immediate consequence of the spherical projection of angles around a vertex, the faces of the  
 187 Gauss map  $\mathcal{G}$  of  $\mathcal{V}$ , a Voss surface, have equal opposite angles too. Thus, since  $\mathcal{G}$  is a net on the unit  
 188 sphere, the faces of  $\mathcal{G}$  are spherical parallelograms (Figure 5). This is the defining property of discrete  
 189 Chebyshev nets, which implies the essential characteristic of Voss nets [34]:

190 **Property 2:** Let  $\mathcal{V}$  be a discrete Voss surface. The Gauss map of  $\mathcal{V}$  is a Chebyshev net on the unit  
 191 sphere  $S^2$ , and will be denoted by  $\mathcal{C}$ .



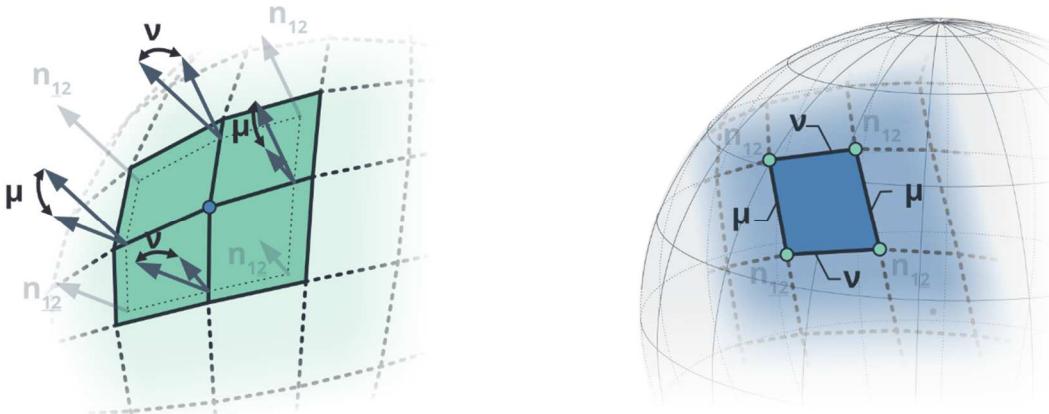
192

193

*Figure 4: An example of a Voss net (left) and its Gauss Map (right)*

194 As a consequence, any discrete Voss surfaces  $\mathcal{V}$  is associated with a unique Chebyshev net  $\mathcal{C}$  on the  
 195 sphere  $S^2$  (Figure 4). The exact smooth counterpart of this property was stated in the founding article  
 196 by A. Voss [33].

197 In this paper, a *discrete Chebyshev net* refers to a quad mesh whose faces are parallelograms on the  
 198 surface. Therefore, the faces of a discrete Chebyshev net have equal opposite angles but also equal  
 199 opposite edge lengths. On  $S^2$ , since the curvature is constant, it is possible to rightfully replace the  
 200 smooth notion of arc length with the discrete notion of length. Some papers make a distinction  
 201 between regular Chebyshev and weak (or generalized) Chebyshev nets [38]. In the first case all edges  
 202 of the quad mesh have equal length, while it is not necessarily true in the second case for which the  
 203 length equality only stands for opposite edges in each faces. In 2D, the distinction segregates nets  
 204 whose faces are all sheared squares in the first case or sheared rectangles in the second. The remainder  
 205 of this paper considers the more general case of weak Chebyshev nets.



206

207 *Figure 5: Spherical projection of dihedral angles of a discrete Voss net (left) onto its Gauss map (right), on which dihedral*  
208 *angles corresponds to the arc length of edges.*

209 Since the Gauss map of a Voss net  $\mathcal{V}$  is a Chebyshev net, the dihedral angles correspond to the angles  
210 made by the normal vectors of two adjacent faces. Because the length of opposite edges of a face of  
211  $\mathcal{C}$  are equal, the duality between  $\mathcal{C}$  and  $\mathcal{V}$  suggests that in both direction of the PQ mesh  $\mathcal{V}$ , dihedral  
212 angles are constant along each parameter line (Figure 5). According to Schief et al. [39], this property  
213 on dihedral angles can be taken as an alternative definition for discrete Voss surfaces. Thus:

214 Property 3: *Let  $\mathcal{P}$  be a planar quadrilateral net and  $\mathcal{G}$  its Gauss map on the unit sphere. If  $\mathcal{G}$  is a*  
215 *Chebyshev net on  $S^2$ , then  $\mathcal{P}$  is Voss net.*

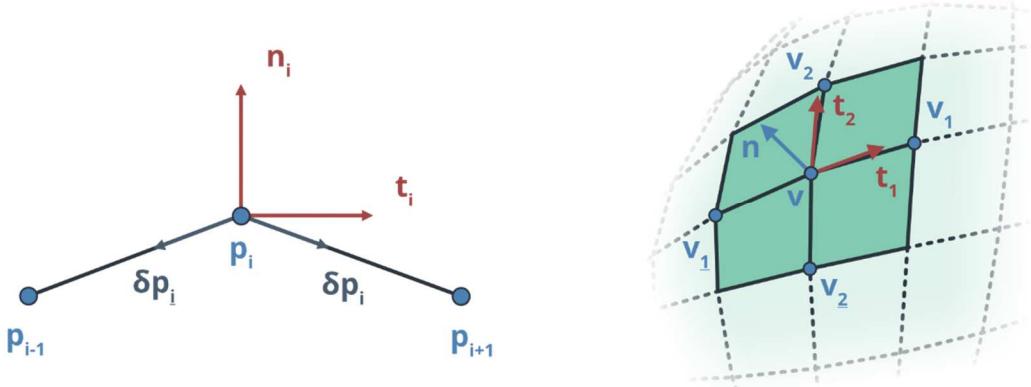
## 216 2.2 DEFINITION FROM VERTEX NORMALS

217 An alternative choice of normal vectors, described upon the vertices, can be found interpreting PQ  
218 meshes as discrete parametrizations of smooth surfaces. The notions of tangent and normal vectors  
219 to a curve need to be presented first. If  $C$  is a regular, non-degenerate curve at arc length  $s_1$ , i.e.  $C'(s_1)$   
220 and  $C''(s_1)$  are non-null, then it is possible to compute the Frenet frame  $(t_1, n_1, b_1)$  at  $s_1$ , in which  $t_1$   
221 is the tangent vector,  $n_1$  the normal vector and  $b_1$  the binormal vector [40]. A discrete analogous  
222 definition exists for non-degenerate discrete curves, i.e. polylines for which any three consecutive  
223 vertices are not aligned:

224 Definition 2: *Let  $P$  be a non-degenerate polyline and  $p_{i-1}, p_i, p_{i+1}$  three consecutive vertices of  $P$*   
225 *(Figure 6 - left). The Frenet frame of  $P$  at  $p_i$ , is designated by  $(t_i, n_i, b_i)$ , where:*

$$226 \quad \delta p_i = p_{i-1} - p_i, \quad \delta p_{\underline{i}} = p_{i+1} - p_i,$$

$$227 \quad t_i = \frac{\delta p_i - \delta p_{\underline{i}}}{\|\delta p_i - \delta p_{\underline{i}}\|}, \quad n_i = \frac{\delta p_i + \delta p_{\underline{i}}}{\|\delta p_i + \delta p_{\underline{i}}\|}, \quad b_i = t_i \wedge n_i.$$



228

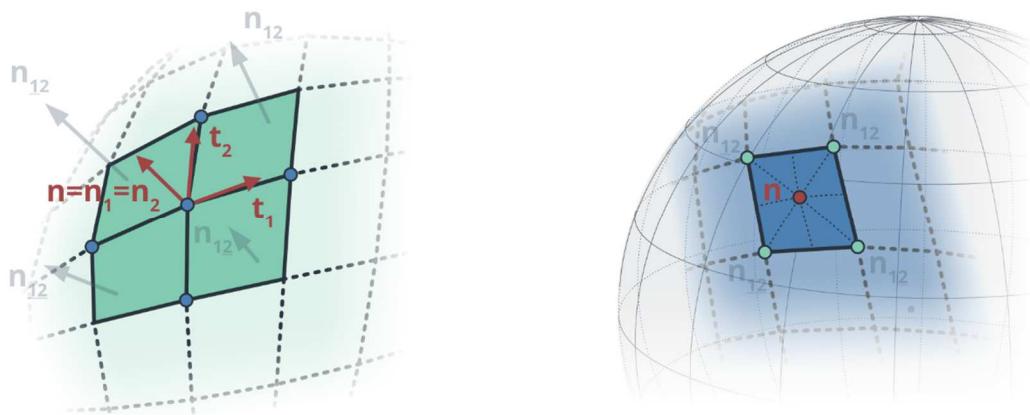
229 *Figure 6: Frenet frame of a non-degenerate polyline (left) and mesh normal vector at a vertex computed from the tangent*  
 230 *values of the coordinate-lines (right).*

231 As mentioned above, PQ meshes define a two-way grid in space. It is thus possible to compute two  
 232 Frenet frames  $(t_1, n_1, b_1)$  and  $(t_2, n_2, b_2)$  in both directions at a vertex-star  $v$  (Figure 6 - right). With  $t_1$   
 233 and  $t_2$  the tangent vectors at vertex  $v$ , the normal vector at the vertex is defined by:

$$n = \frac{t_1 \times t_2}{\|t_1 \times t_2\|}. \quad (1)$$

234 This definition imitates the smooth case: with  $C_1$  and  $C_2$  two curves on a surface, intersecting at a point  
 235  $v$ , and  $t_1, t_2$  the tangent vectors of these curves at  $v$ , the normal of the surface at  $v$  is aligned with the  
 236 cross product of the tangent vectors  $t_1$  and  $t_2$ .

237 For discrete geodesic nets, and therefore for Voss nets, Rabinovich et al. [28] showed that, just like in  
 238 the smooth case, the normal vectors of two intersecting discrete geodesic curves agree at their  
 239 intersection, meaning at their common vertex. As a consequence, the common normal vector of the  
 240 curves coincides with the normal  $n$  in equation (1). Rabinovich et al. [28] used this result to define an  
 241 adapted set of normals upon the vertices for the Gauss map of discrete orthogonal geodesic nets.



242

243 *Figure 7: Projection of a normal at a node in a Voss net (left) onto its Gauss Map (right)*

244 As a matter of fact, it is possible to identify this set of normal vectors defined upon the vertices in the  
245 Gauss Map of discrete Voss nets (Figure 7).

246 *Property 4: Let  $\mathcal{V}$  be a discrete Voss net and  $\mathcal{C}$  its Gauss Map. Let's consider a vertex-star configuration*  
247 *and its spherical projection on  $S^2$ . Thus, the face's normal  $n_{12}^{\mathcal{V}}, n_{\underline{12}}^{\mathcal{V}}, n_{\underline{12}}^{\mathcal{V}}, n_{12}^{\mathcal{V}}$  around the vertex v of  $\mathcal{V}$*   
248 *are dual to  $n_{12}^f, n_{\underline{12}}^f, n_{\underline{12}}^f, n_{12}^f$  the vertices of  $\mathcal{C}$ , belonging to the face f, dual to v. The normal vector n*  
249 *at the vertex v of  $\mathcal{V}$  corresponds on  $S^2$  to the centre of the spherical parallelogram f.*

250 The centre of the spherical parallelogram can be defined alternatively as the centre or intersection of  
251 its diagonals, or as the centre or the intersection of its medians [41] (id non vidi). This connection  
252 between the two notions of Gauss map, upon faces and vertices, enriches significantly the  
253 representation of discrete Voss surfaces on the unit sphere. The Chebyshev net on the unit sphere not  
254 only provides information on the face normals but also on the vertex normals of the Voss net.

### 255 3 TRANSFORMATIONS OF VOSS NETS

---

256 The study of mesh transformation is a very active field in geometry processing. Editing a mesh by  
257 prescribing displacements [42], or by using adapted subdivision patterns has been explored in many  
258 different situations and different paradigms compete.

259 A common method for mesh editing operates handle-based transformations: the user controls a small  
260 set of vertices and drags them around. Following these modifications, a new shape is computed after  
261 the resolution of an optimization problem [40,41]. Another method for editing polyhedral meshes  
262 operates constraint-based transformations: a given smooth surface is approximated by a mesh on  
263 which geometric constraints are defined [45]. The constraints, prescribed upon the vertices or the  
264 faces, express the geometrical behaviour expected for the tangential mesh. In practice this method  
265 tends to be time consuming and the resulting mesh only loosely respects the constraints [43,44].

266 Recently, an alternative approach has been proposed by Vaxman et al. [48] for polyhedral meshes and  
267 meshes with planar faces and has later been generalized by Poranne et al. [49]: polyhedral meshes and  
268 their transformation are described as linear spaces. The advantage of this method is that only exact  
269 solutions are explored in the process, which, on the other hand means that the design space is smaller  
270 than the previous methods. In [48], the shape space allows affine transformations of faces. Inspired by  
271 this approach, family of meshes, addressed as linear space, have been used in different contexts such  
272 as the design of planar quadrilateral meshes [50], and the form-finding of a shell-nexorade hybrid [51].  
273 Pottmann et al. [36] address the linear space of parallel meshes for other reasons. This approach will  
274 be of particular interest in our context as well.

275 In this section, transformations that do not alter the defining attributes of discrete Voss nets are  
276 identified. The aim is to simplify the exploration of the shape space of Voss nets by using the point of  
277 view of meshes as linear spaces.

278 **3.1 ISOMETRIC TRANSFORMATIONS**

279 Isometric transformations have been a primary concern in discrete differential geometry during the  
280 past decade. In simple terms, they are defined as distance-preserving transformations. Their study is  
281 highly related to research on developable surfaces and on the exploration of deployable systems.  
282 Discrete isometric transformations are extensively used in origami for it preserves the length of the  
283 edge and only “folds” the mesh along these edges, and not at all faces.

284 Schief et al. [39] provide a thorough mathematical study on infinitesimal isometric transformation of  
285 discrete nets which reveals that discrete Voss surfaces admit a one-parameter family of isometric  
286 transformation that preserve their defining properties. Therefore, given a discrete Voss surface  $\mathcal{V}$ , it is  
287 possible to append the overall shape of the net without changing the edge length. In fact, only the  
288 dihedral angles of the mesh are modified. The transformation is ruled by the following relation (Figure  
289 8):

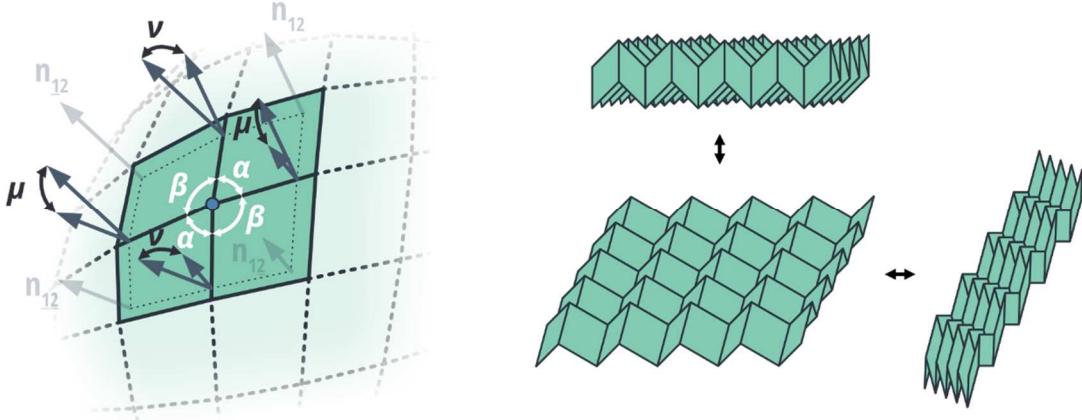
$$290 \tan\left(\frac{\mu}{2}\right) \tan\left(\frac{\nu}{2}\right) = \frac{\sin(\alpha + \beta)}{\sin(\alpha) + \sin(\beta)}.$$

291 Keeping  $\alpha$  and  $\beta$  constant, the isometric transformation of a discrete Voss surface is ruled by one  
292 parameter  $r \in \mathbb{R}$ :

$$293 \begin{cases} \tan\left(\frac{\mu}{2}\right) \\ \tan\left(\frac{\nu}{2}\right) \end{cases} \xrightarrow{r} \begin{cases} r \cdot \tan\left(\frac{\mu}{2}\right) \\ \frac{1}{r} \cdot \tan\left(\frac{\nu}{2}\right) \end{cases}$$

294 In essence, the isometric transformation of Voss surfaces is determined by the choice of a single  
295 dihedral angle between two consecutive faces.

296 Tachi et al. [52] identified that generalized eggbox patterns in origami are discrete Voss surfaces.  
297 Indeed, opposite angles made by edges around each vertex are equal. For generalized eggbox patterns,  
298 the mesh corresponds to an alternation of valleys and mountain folds. It is this alternation that gives  
299 the bidirectional flat rigid-foldability property to the eggbox pattern. Mitchell et al. [53] made a  
300 connection between this rigid-foldability property of Voss surfaces and graphic statics, and used it to  
301 build a kinematic pavilion.



302

303

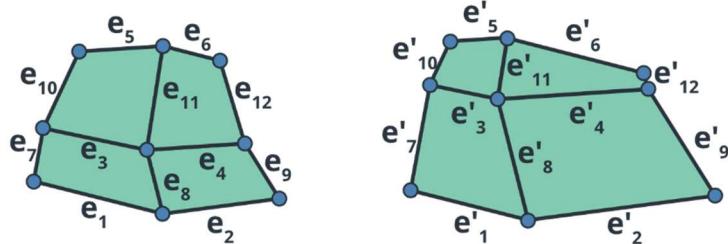
Figure 8: Angle properties of a Voss vertex-star (left); Folding motion of the eggbox pattern (right).

### 304 3.2 COMBESCURE TRANSFORMATIONS

305 Further essential examples of transformations are Combescure transformations. They are  
 306 encountered for parallel meshes [36]. Two meshes are termed parallel if they have the same  
 307 connectivity and if their corresponding edges are parallel. The transformation between two parallel  
 308 meshes is called a Combescure transformation. Thus, given two parallel meshes  $\mathcal{P}$  and  $\mathcal{P}'$ , with  
 309  $(e_j)_{j \leq J}$ ,  $(e'_j)_{j \leq J}$  their edges respectively, they respect the following linear relation:

310

$$\forall j \in J, e_j \times e'_j = 0$$



311

312

Figure 9: Two meshes related by Combescure transform.

313 Any planar quadrilateral net  $\mathcal{Q}$  admits an infinite number of Combescure transforms. However, the  
 314 dimension of the space of Combescure transforms  $\mathfrak{C}_{\mathcal{Q}}$  of  $\mathcal{Q}$  is finite and is combinatorically given by  
 315 the following equation [36]:

316

$$\dim(\mathfrak{C}_{\mathcal{Q}}) = N_e - 2N_f + 3$$

317 where  $N_e$  is the number of edges in  $\mathcal{Q}$ , and  $N_f$  the number of faces in  $\mathcal{Q}$ . The dimension of the space of  
 318 Combescure transforms must be understood as the number of degrees of freedom of the  
 319 transformation. It represents the number of edge lengths that can be chosen within the mesh  $\mathcal{Q}$ , under  
 320 the condition that all the constraints are compatible with one another. The number 3 at the end of the

321 equation stands for the global translations in the 3D Euclidean space, which leaves the edge length of  
322 the net unchanged.

323 If  $\mathcal{P}$  and  $\mathcal{P}'$  are two planar quadrilateral meshes, related by a Combescure transform, then their Gauss  
324 maps are identical. This results from the fact that by keeping edges parallel, the normal vectors of the  
325 faces remain unchanged. Therefore, Combescure transformations change Voss nets into other Voss  
326 nets, while preserving the Gauss map [39].

## 327 4 LINEAR SPACES OF VOSS NETS

---

328 The previous sections reviewed discrete surfaces whose geometric features are relevant for modelling  
329 shells from geodesic lines, with useful constructive and architectural properties – e.g. absence of  
330 sideways bending in rectangular cross-sections and covering with flat quadrilateral panels. The study  
331 highlighted remarkable properties, already known to geometers, notably on the Gauss map. Those  
332 properties are now used to set a methodology to identify appropriate design spaces of Voss surfaces.

### 333 4.1 REQUIREMENTS

334 The linear space of Voss nets is obtained after characterizing an appropriate shape space of  
335 quadrilateral meshes. This family of PQ meshes serves as the initialization for the creation of the linear  
336 space, in a similar fashion as proposed by Poranne et al. [49]. This ensures that the mesh, called a  
337 realization of the topology, always complies with the constraints used to create the linear space.

338 When identifying and computing a proper design space for Voss nets, the difficulty is to find the  
339 appropriate set of properties that corresponds to Voss nets. On the one hand, the number of  
340 constraints should be minimal: since explored shape spaces are already small in comparison with other  
341 non-linear methods, it is important to ensure that no potential member is missing out. On the other  
342 hand, constraints should not be too weak, in order to ensure that the linear space does not contain  
343 any mesh that is not a discrete Voss net. In addition to this, the design space exploration should be  
344 convenient.

### 345 4.2 ROLE OF THE GAUSS MAP

346 Following section 2, a relevant design space for discrete Voss nets is the Gauss map of this variety,  
347 namely the space of discrete Chebyshev nets on the unit sphere. In other words, the design of Voss  
348 nets is facilitated and guaranteed by starting from a Chebyshev net on  $S^2$ .

349 Mimicking the behaviour of smooth Voss surfaces, all defining properties of discrete Voss nets are  
350 enclosed in their Gauss map and not more. Indeed, as stated, with  $\mathcal{P}$  a planar quad net, and  $\mathcal{G}$  its Gauss

351 map, if  $\mathcal{G}$  is a Chebyshev net then  $\mathcal{P}$  is a Voss net since the dihedral angles are constant along a  
352 coordinate-line. In addition, with the definition adopted for the spherical projection, it is always  
353 possible to define a planar quad net from a given mesh on the unit sphere (section 4.3). Hence, the  
354 Gauss map defines a surjection between the space of discrete Voss surfaces and the space of  
355 Chebyshev nets on the unit sphere. This demonstrates that the space of Chebyshev net on  $S^2$ ,  
356 representing the Gauss map of an underlying Voss net, is maximal.

357 However, for this space to be relevant for the design, it should allow a surjection from the space of  
358 Voss surfaces but also enable the exploration of all feasible discrete Voss surfaces. Further study into  
359 the inverse mapping is required.

### 360 4.3 COMPUTATION

361 In line with the previous section, the set of Chebyshev nets on  $S^2$ , to be understood as the space of  
362 the Gauss maps of Voss nets, allows to narrow down the degrees of freedom for design exploration.  
363 For this manifold, the space of Gauss maps encloses just enough information, ensuring that the  
364 realization of the mesh will indeed be a Voss net. A construction of the linear space of Voss surfaces is  
365 provided in this section. More precisely, given an input Chebyshev net  $\mathcal{C}$  on the unit sphere  $S^2$ , the  
366 design space of Voss surfaces whose Gauss map is that input is characterized. Dealing with planar  
367 meshes, the following property holds:

368 *Property 6: Let  $\mathcal{P}$  be a mesh with planar faces, and e an edge of  $\mathcal{P}$  that does not lie on the boundary.  
369 Let  $n_l$  and  $n_r$  be the normal vectors of the faces adjacent to the edge e. Provided that  $n_l$  and  $n_r$  are  
370 not parallel,  $n_l \times n_r$  is parallel to edge e, i.e.:*

$$371 \quad (n_l \times n_r) \times e = 0.$$

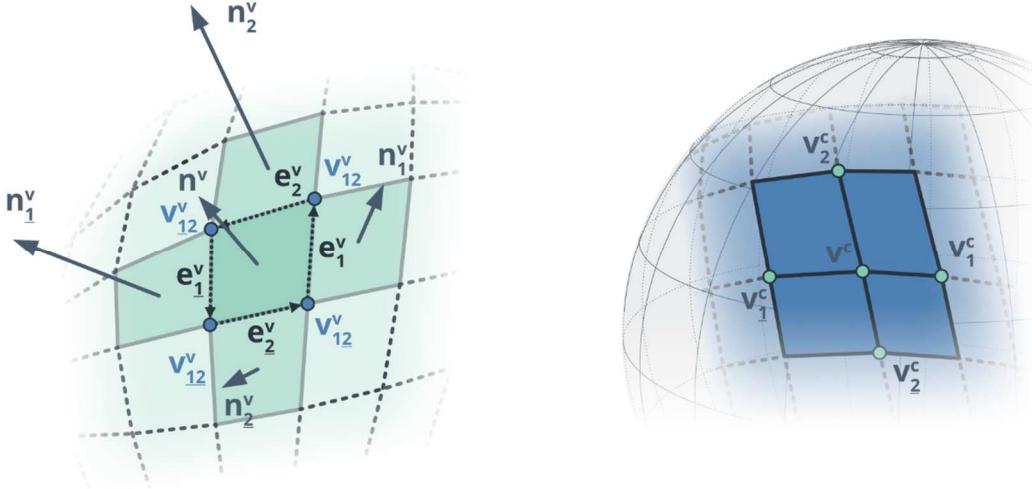
372 Considering the Gauss map of a net with planar faces, this property can be reformulated as follows:

373 *Corollary 1: Let  $\mathcal{P}$  be a mesh with planar faces and  $\mathcal{G}$  its Gauss map. Let also  $e_i^{\mathcal{P}}$  be an internal edge of  
374  $\mathcal{P}$  and  $e_i^{\mathcal{G}}$  the corresponding edge in  $\mathcal{G}$  by the spherical projection. The normal vectors of the faces  
375 adjacent to  $e_i^{\mathcal{P}}$  correspond to the start vertex  $v_{start}^{\mathcal{G}}$  and the end vertex  $v_{end}^{\mathcal{G}}$  of the edge  $e_i^{\mathcal{G}}$ . And thus  
376  $(e_{start}^{\mathcal{G}} \times e_{end}^{\mathcal{G}})$  is parallel to the direction of  $e_i^{\mathcal{P}}$ .*

377 As a consequence, the Gauss map encloses the direction of the edges of the underlying meshes. To  
378 simplify the discussion, the reconstruction of a single-face Voss net from the Gauss map is examined  
379 (Figure 10). Considering a Chebyshev net  $\mathcal{C}$  reduced to only a vertex-star, the direction of the edges of  
380 the underlying face  $f^{\mathcal{V}}$  of  $\mathcal{V}$ , dual to the vertex  $v^{\mathcal{C}}$  of  $\mathcal{C}$ , are given by:

$$\begin{aligned}
e_1^{\mathcal{V}} &\parallel (v_1^c \times v^c) \\
e_2^{\mathcal{V}} &\parallel (v_2^c \times v^c) \\
e_{\underline{1}}^{\mathcal{V}} &\parallel (v_{\underline{1}}^c \times v^c) \\
e_{\underline{2}}^{\mathcal{V}} &\parallel (v_{\underline{2}}^c \times v^c)
\end{aligned} \tag{2}$$

381



382

383 *Figure 10: A Chebyshev net on the unit sphere (right) as the Gauss map  $\mathcal{C}$  of a Voss surface  $\mathcal{V}$  (left) to be determined. Edges*  
384  *$e_j^{\mathcal{V}}$  of  $\mathcal{V}$ , common to faces of normal  $n^{\mathcal{V}}$  and  $n_j^{\mathcal{V}}$  (left), are dual to edges  $e_j^{\mathcal{C}}$  of  $\mathcal{C}$ , linking vertices  $v_j^{\mathcal{C}}$  and  $v_j^{\mathcal{V}}$  (right).*

385 The directions of the edges are computed linearly upon the vertices of the Chebyshev net. However,  
386 recalling the notion of mesh parallelism, it is clear that having the information of the edge directions  
387 does not uniquely determine the face  $f$ . As illustrated in Figure 9, two distinct faces, related by a  
388 Combescure transform, can have parallel edges. With  $\mathbf{V}_f$  the column vector containing the coordinates  
389 of vertices  $v_{12}^{\mathcal{V}}, v_{12}^{\mathcal{V}}, v_{12}^{\mathcal{V}}, v_{12}^{\mathcal{V}}$  of face  $f$ , above relations (2) are rewritten:

$$\begin{aligned}
390 \quad & \begin{cases} (v_1^c \times v^c) \times (v_{12}^{\mathcal{V}} - v_{12}^{\mathcal{V}}) = 0 \\ (v_2^c \times v^c) \times (v_{12}^{\mathcal{V}} - v_{12}^{\mathcal{V}}) = 0 \\ (v_{\underline{1}}^c \times v^c) \times (v_{12}^{\mathcal{V}} - v_{12}^{\mathcal{V}}) = 0 \\ (v_{\underline{2}}^c \times v^c) \times (v_{12}^{\mathcal{V}} - v_{12}^{\mathcal{V}}) = 0 \end{cases} \Rightarrow \mathbf{B}_f(\mathcal{C}) \cdot \mathbf{V}_f = 0
\end{aligned}$$

391 The relation of edge parallelism is expressed linearly using the vertices of the face to rebuild, and a  
392 matrix  $\mathbf{B}_f$  which only depends on the vertices of  $\mathcal{C}$ . In this equation, the unknown is the column  $\mathbf{V}_f$ ,  
393 which contains the vertices of face  $f$ . Consequently, the null space  $\text{null}(\mathbf{B}_f)$  can be interpreted as the  
394 linear space containing all quadrilateral faces whose edges are parallel to the direction prescribed by  
395 the Gauss map  $\mathcal{C}$ .

396 Now, with  $\mathcal{C}$ , a more complex Chebyshev net, with several faces, the characterization of the underlying  
397 meshes  $\mathcal{V}$ , with several faces, relies on the generalization of matrix  $\mathbf{B}_f$ . Computing all matrixes  $\mathbf{B}_f$

398 independently, for each face  $f$  of  $\mathcal{V}$ , using each vertex-stars of  $\mathcal{C}$ , the matrixes  $\mathbf{B}_f$  are combined in a  
399 single large matrix  $\mathbf{B}$  of size  $[3J^{\mathcal{V}} \times 3I^{\mathcal{V}}]$ , which gives:

400 
$$\mathbf{B}(\mathcal{C}) \cdot \mathbf{V} = 0$$

401 Where  $\mathbf{V}$  is the column vector of the vertices of  $\mathcal{V}$ , interpreted as the combination of all  $\mathbf{V}_f$ , and where  
402  $\mathbf{B}$  only depends on the vertices of  $\mathcal{C}$ . Consequently,  $\text{null}(\mathbf{B})$ , defined only from  $\mathcal{C}$ , is the linear space  
403 of dimension  $N$  composed of meshes whose Gauss map is  $\mathcal{C}$ .

404 **4.4 GEOMETRIC INTERPRETATION**

405 The study of the reconstruction of meshes from a given Gauss map led to the definition of a linear  
406 space, described as the null space of a matrix. Therefore, the design of a Chebyshev net  $\mathcal{C}$  on  $S^2$  leads  
407 to the definition of a linear space of Voss surfaces, only dependent on  $\mathcal{C}$ . Actually, the null space  
408  $\text{null}(\mathbf{B})$ , contains Voss nets that are all related by Combescure transforms. The elementary matrixes  
409  $\mathbf{B}_f$  defined on the each face, help describing the space of quadrilateral face whose edges are parallel  
410 to directions given by the Gauss Map. Each  $\text{null}(\mathbf{B}_f)$  is the linear space of quad faces related by  
411 Combescure transforms respecting the constraints on the edges. Combining the face matrixes in a  
412 single matrix  $\mathbf{B}$  guarantees that the constraints on the faces are geometrically compatible with one  
413 another, making sure that  $\text{null}(\mathbf{B})$  is not always reduced to zero. Therefore,  $\text{null}(\mathbf{B})$  is the linear space  
414 of meshes whose edge directions are prescribed by the initial Chebyshev net  $\mathcal{C}$  on the unit sphere.  
415 They are related by Combescure transforms. From a mathematical perspective, the space of  
416 Chebyshev nets on  $S^2$  is in bijection with the quotient set of Voss surfaces under the equivalence  
417 relation of Combescure transformations.

418 Importantly,  $\text{null}(\mathbf{B})$  is never reduced to zero when the Chebyshev net is larger than a  $2 \times 2$  complex  
419 of faces. A planar quad mesh, realization of the given Gauss Map, always exists by definition of the  
420 Gauss map.

421 Remarkably, the reconstruction of the linear space from a Gauss map presented above is very general.  
422 It can actually be applied for the characterization of general meshes with planar faces, since it mainly  
423 depends on the definition of the Gauss map as the projection on  $S^2$  of the normal of the face.  
424 Therefore, polygonal meshes with different types of faces can be studied in this way, as long as faces  
425 are planar. In our context, this approach proved to be particularly relevant since all and only defining  
426 constraints of discrete Voss surfaces are enclosed in their Gauss map.

---

## 427 5 GENERATION OF VOSS SURFACES

---

428 The design space for discrete Voss surfaces identified in section 4 is used for the generation of discrete  
429 Voss surfaces. Two different methods are presented. The first consists in exploring the linear space of  
430 discrete Voss surfaces (section 5.2). The second consists in a direct computation of the Voss net from  
431 two target boundary curves (section 5.3). Beforehand, section 5.1 exposes existing approaches to  
432 generate Chebyshev nets on surfaces.

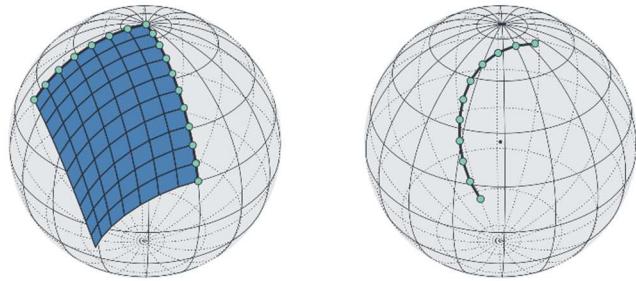
### 433 5.1 GENERATION OF CHEBYSHEV NET ON THE SPHERE

434 The study of the space of Chebyshev nets on  $S^2$  is a preliminary to the generation of discrete Voss  
435 surfaces and has already been addressed in the literature.

436 Historically, Chebyshev introduced this variety of nets on smooth surfaces in 1878 for their capacity to  
437 approximate the behaviour of fabric around the human body. Such a net can be assembled initially as  
438 a flat two-way grid, and then its transformation approximates the behaviour of flexible but inextensible  
439 rods. Thus, Chebyshev nets can be found in various contexts, such as stylish furniture or medical stents.  
440 Its use for architectural venues, initiated by Frei Otto, resulted in the association of Chebyshev nets  
441 with the structural notion of elastic gridshells.

442 Mathematically, Chebyshev nets on a surface are defined by hyperbolic equation and their generation  
443 involves the solution of the Sine-Gordon equation [54]. The determination of such nets is therefore a  
444 problem of propagation. On this kind of problems, unique solutions are determined from the  
445 prescription of boundary conditions or from the propagation of an element. As exposed in section 1.1,  
446 the problem of mapping Chebyshev nets on given smooth surfaces has been addressed many times,  
447 geometrically and computationally. Unlike the case of elastic gridshells, where these nets directly  
448 model the support structures, Chebyshev nets are used here as a representation of Voss nets on the  
449 unit sphere. Thus, requirements on the net differ from usual applications, since smoothness is here  
450 not necessary. Still, similar generation methods can still be applied in our case, but the scope is reduced  
451 to the generation of Chebyshev nets on the unit sphere.

452 A first choice of generation method consists two primal boundary curves as initial conditions [10]. This  
453 process corresponds to the so-called “compass method” introduced by Otto et al. [6]. Let  $P_a = (a_i)_{i \leq A}$   
454 and  $P_b = (b_j)_{j \leq B}$  be two polylines, whose starting vertices coincide. Recalling that a Chebyshev net is  
455 defined by equal opposite lengths in each face, it is possible to rebuild iteratively the parallelograms  
456 by starting from the intersection of  $P_a$  and  $P_b$ , and by using the subdivisions of both polylines.  
457 Parallelograms are computed line by line, or diagonally. The process stops when the  $(A - 1)(B - 1)$   
458 quadrilateral faces are mapped.

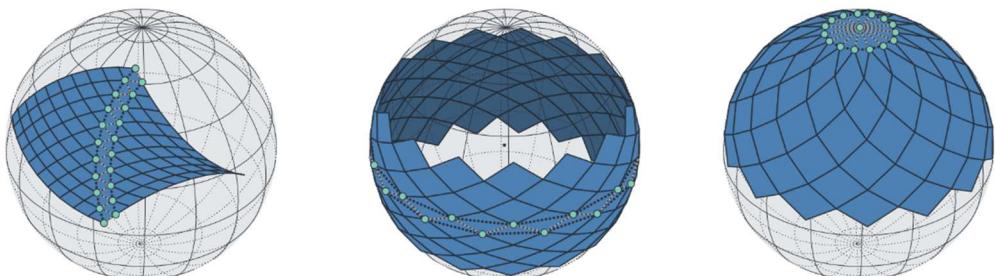


459

460 *Figure 11: Chebyshev nets generated from two primal boundary conditions.*

461 Figure 11 illustrates two examples of Chebyshev nets on the unit sphere generated from two primal  
 462 boundary conditions. The primal conditions are the discretized lines, depicted by thicker lines. The  
 463 parallelograms are successively computed from the intersection of the two lines to the opposite corner  
 464 of the mesh. The situation on the right sphere is a degenerate case where one of the two lines is  
 465 reduced to a point. Thus, the resulting Chebyshev net is curve. The underlying Voss net is in fact a  
 466 developable net, mapped by a conjugate geodesic network. This is a well-known fact of differential  
 467 geometry: all developable surfaces are indeed Voss surfaces.

468 A second choice consists in generating Chebyshev nets from secondary conditions denoted as “initial  
 469 zigzag” [36,51]. This method relies on the initial prescription of the diagonal parallelograms of the  
 470 Chebyshev net. Secondary conditions correspond to the vertices on a diagonal and to the next or  
 471 previous diagonal line, which is equivalent to a zigzag (Figure 12). Similarly, the parallelograms are then  
 472 computed diagonally, on one side and the other.

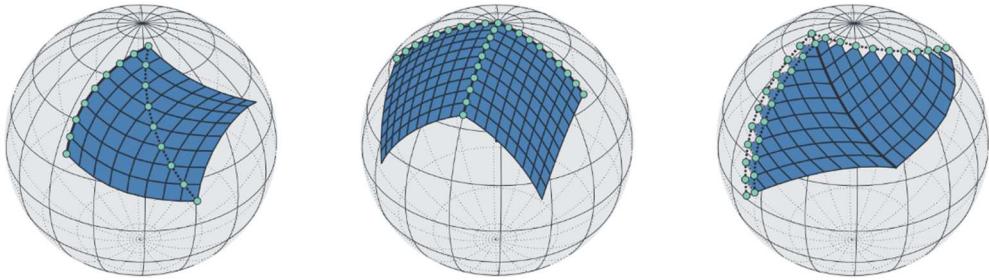


473

474 *Figure 12: Chebyshev nets on the unit sphere generated from secondary conditions. Open (left) and closed (middle) polylines  
 475 on  $S^2$  are used. The rosette condition (left) is a degenerate case.*

476 Figure 12 shows three applications of this second generation method where the secondary conditions  
 477 are represented with dotted lines. Open (left) and closed (middle) polylines are used as support for  
 478 the secondary conditions, resulting respectively in an open and closed looping Chebyshev net on  $S^2$ .  
 479 The degenerate situations where one of the secondary curves is reduced to a point is also illustrated  
 480 (right).

481 A third choice consists in mixing the two aforementioned boundary conditions (Figure 13 left).



482

483 *Figure 13: Examples of a mixed condition (left) and of patches of Chebyshev nets generated from primal (center) or dual*  
484 *(right) curves.*

485 Masson studied the notion of patches or junction between Chebyshev nets [10]. The bounding  
486 polylines of a given Chebyshev net are seen as potential input curves for the generation of attached  
487 Chebyshev nets. This allows the mapping of surfaces that could not be approximated by a single  
488 Chebyshev net by defining several nets compatible with one another, in the sense that they match at  
489 their common polyline.

490 Figure 13 illustrates the patch of primal (center) and secondary (right) conditions. In the first case, the  
491 compatibility of the generated Chebyshev nets results from the use of a common boundary polyline.  
492 Regarding the patch of secondary conditions, only one secondary polyline is required to create the  
493 second Chebyshev net (on the right). The generation of this second net can also be seen as a mixed  
494 generation which uses the new boundary of the first Chebyshev net as a primal polyline.

495 The generation methods presented above always lead to a Chebyshev net. Depending on the situation  
496 addressed, one condition may be more relevant than another, in a similar way as for partial differential  
497 equations where the resolution is eased by the use of adapted initial conditions. The methods differ  
498 by their initial inputs for resolution. Depending on the problem addressed, one boundary condition  
499 may prove to be more relevant than another. For the generation of Voss nets, an arrangement of the  
500 grid on the boundary can turn out to be technically or architecturally more interesting. A methodology  
501 to compute the spherical parallelograms of Chebyshev nets on  $S^2$  by means of unit quaternions is  
502 presented in the Appendix: Computation of a Chebyshev net  $S^2$ .

## 503 5.2 EXPLORATION OF DISCRETE VOSS SURFACES

504 As exposed in the previous section, a number of options exists to generate Chebyshev nets, supporting  
505 the choice made for a reduced design space for Voss surfaces. Following results from sections 4.3 and  
506 4.4, once a Chebyshev net  $\mathcal{C}$  is computed on  $S^2$ , the matrix  $\mathbf{B}$  is calculated upon the vertices of  $\mathcal{C}$ , and  
507 the linear space of  $\text{null}(\mathbf{B})$  is to explore.

508 Contrary to usual approaches, the starting point is not an initial mesh but its Gauss map, which is  
509 topologically dual. Therefore, best-fit algorithms comparing trial shapes with a target surface cannot

510 be directly be implemented in this context. Instead, the objective is to identify a basis of the linear  
 511 space  $\text{null}(\mathbf{B})$  which allows its smooth and informed description.

512 **EIGENSHAPES**

513 Poranne et al. [49] suggested the use of eigenshapes to generate initial smooth meshes and transform  
 514 them. Eigenshapes are defined as the eigenvectors of the Laplacian  $\mathbf{L}$  of the mesh, a quadratic function.  
 515 For each vertex  $v$ , the Laplacian gives a direction for the smoothing of the mesh, based on the  
 516 neighbour vertices  $v_i$  of  $v$ :

$$517 \quad \mathbf{L}(v) = v - \frac{1}{n} \sum_{i=1}^n v_i$$

518 Eigenshapes, or eigenvectors of the Laplacian  $\mathbf{L}$  constrained on  $\text{null}(\mathbf{B})$ , form a special basis of the  
 519 space  $\text{null}(\mathbf{B})$ . Unlike a random basis of this linear space, each eigenshape corresponds to a smooth  
 520 Voss surface. Alternatively, eigenshapes can also be seen as transformation modes if they are added  
 521 to a realization of this space. Therefore, in the present case, eigenshapes serve for the generation of  
 522 an initial Voss surface of  $\text{null}(\mathbf{B})$  and then for the transformation of this element. The best candidates  
 523 for the initial realization are found by minimizing the Rayleigh quotient:

$$524 \quad \min_{V \in \text{null}(\mathbf{B})} \frac{\mathbf{V}^T \cdot \mathbf{L} \cdot \mathbf{V}}{\mathbf{V}^T \cdot \mathbf{V}}$$

525 Since we are interested in applying transformations and thus in exploring the whole linear space  
 526 described by  $\text{null}(\mathbf{B})$ , the minimization problem is indirectly solved by finding the eigenvectors of  $\mathbf{L}$ .  
 527 To account for the constraints of  $\mathbf{B} \cdot \mathbf{X} = \mathbf{0}$ , an orthonormal basis  $\mathbf{N}$  of  $\text{null}(\mathbf{B})$  is computed by  
 528 running a Singular Value Decomposition (SVD) on  $\mathbf{B}$ . Thus, every vector of  $\text{null}(\mathbf{B})$  can be written as  
 529  $\mathbf{N} \cdot \mathbf{W}$  for some column vector  $\mathbf{W}$ . This also has the effect of reducing drastically the number of  
 530 unknowns to  $N$ , the exact dimension of  $\text{null}(\mathbf{B})$  given earlier as the dimension of the space of  
 531 Combescure transforms:

$$532 \quad \min_{V \in \text{null}(\mathbf{B})} \frac{\mathbf{V}^T \cdot \mathbf{L} \cdot \mathbf{V}}{\mathbf{V}^T \cdot \mathbf{V}} \Leftrightarrow \min_{\mathbf{W}} \frac{\mathbf{W}^T \cdot \mathbf{N}^T \cdot \mathbf{L} \cdot \mathbf{N} \cdot \mathbf{W}}{\mathbf{W}^T \cdot \mathbf{W}}$$

533 The eigenshapes are then given by computing an eigenvalue decomposition of the matrix  $\mathbf{N}^T \cdot \mathbf{L} \cdot \mathbf{N}$   
 534 and taking the eigenvectors of the resulting matrix. Consequently, the smoothest meshes of the linear  
 535 space  $\text{null}(\mathbf{B})$  are the eigenshapes of  $\mathbf{N}^T \cdot \mathbf{L} \cdot \mathbf{N}$  with the lowest eigenvalues. In the present case, this  
 536 allows to define a smooth initial discrete Voss surface  $\mathcal{V}$ , whose Gauss map is the Chebyshev net  $\mathcal{C}$   
 537 which was used to compute  $\mathbf{B}$ .

538 **MINIMAL INFLUENCE MODES**

539 Interpreting  $\text{null}(\mathbf{B})$  as a space of transformation modes for Voss nets, an alternative solution for the  
540 exploration of the linear space is to create a basis of modes with minimal influence on the overall  
541 geometry.

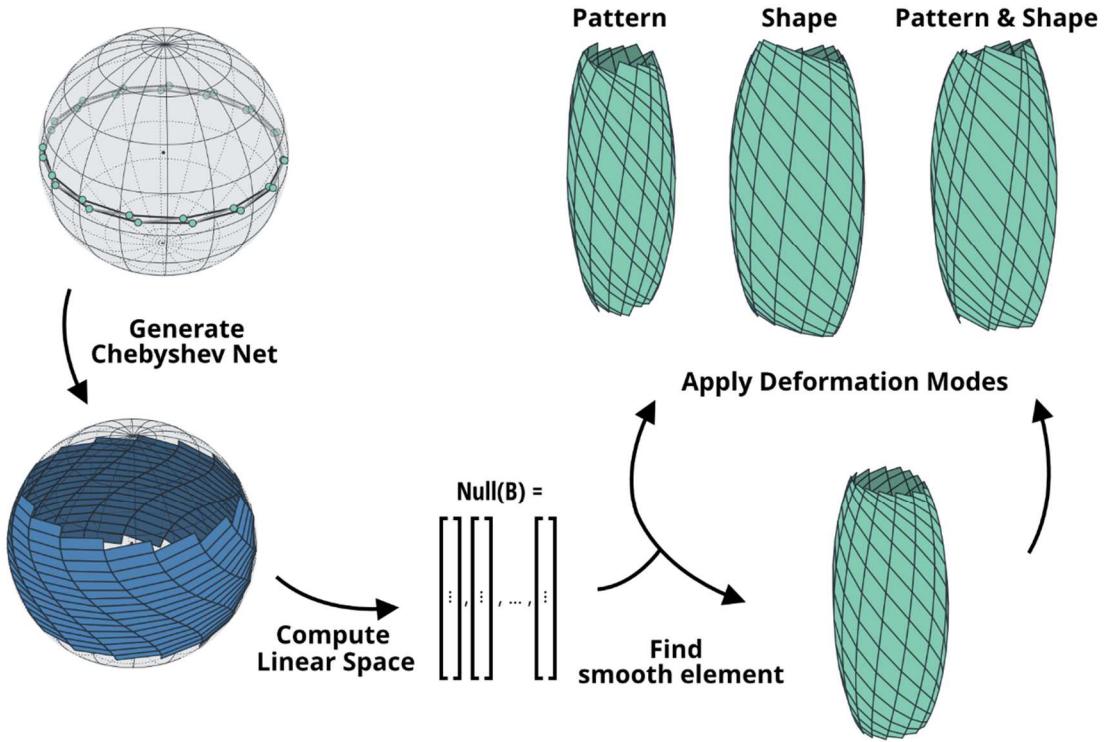
542 The formulation of such basis derives from the observation that given a planar mesh  $\mathcal{P}$ , the  
543 Combescure transformations of  $\mathcal{P}$  sorts the edges of the mesh into groups of co-dependent edges.  
544 Thus, if an edge is stretched or shortened under a Combescure transformation, then all the other edges  
545 from its group are also modified. Combescure transformation form an equivalence relation on the edge  
546 of  $\mathcal{P}$  and the number of equivalence class is  $N - 3$ , living out the three global translations in space.  
547 For example, considering the simple case depicted on Figure 9, the equivalence class would be  $[e_1] =$   
548  $\{e_1, e_3, e_5\}$ ,  $[e_2] = \{e_2, e_4, e_6\}$ ,  $[e_7] = \{e_7, e_8, e_9\}$ ,  $[e_{10}] = \{e_{10}, e_{11}, e_{12}\}$ . Taking one representative  
549 edge by equivalence class, the aforementioned basis of  $\text{null}(\mathbf{B})$  is composed of transformation modes  
550 which alter one of the representative while leaving the length of all the other selected edges  
551 unchanged.

552 The research of the modes is achieved on a reduced space  $\mathbf{N}_{\text{reduced}}$  of size  $[3I_{\text{reduced}}^{\mathcal{V}} \times N - 3]$   
553 obtained from  $\mathbf{N} = \text{null}(\mathbf{B})$ , of size  $[3I^{\mathcal{V}} \times N]$ , by filtering the three global translations and then by  
554 keeping the line of the matrix correspond to the vertices involved at the start or the end of the  
555 representative edges. With  $\mathbf{R}$  the matrix computing the edge vectors of the representatives from their  
556 start and end vertex coordinate in  $\mathbf{N}_{\text{reduced}}$ , the problem of finding the basis is equivalent to solving:

$$557 \quad \forall i, 1 \leq i \leq N - 3, \quad \mathbf{R} \cdot \mathbf{N}_{\text{reduced}} \cdot \mathbf{X} = \mathbf{E}_i, \quad \text{where } \mathbf{E}_i = [0 \cdots 1 \cdots 0] \text{ canonical} \\ 558 \quad \text{vectors of } E^{N-3}.$$

559 Finally, the  $N - 3$  solution vectors  $\mathbf{X}$  are each multiplied by  $\mathbf{N}_{\text{filtered}}$  in which the three global  
560 translations were previously filtered. This produces basis vectors on the whole space and not on the  
561 reduced one.

562 Compared with the eigenshapes, this alternative method to explore  $\text{null}(\mathbf{B})$  proved more intuitive for  
563 the transformation of already computed Voss net. The ability to only have local modifications revealed  
564 useful for the design of Voss nets as well as for patching them together.



565

566

*Figure 14: Process for the exploration of linear spaces of discrete Voss surfaces.*

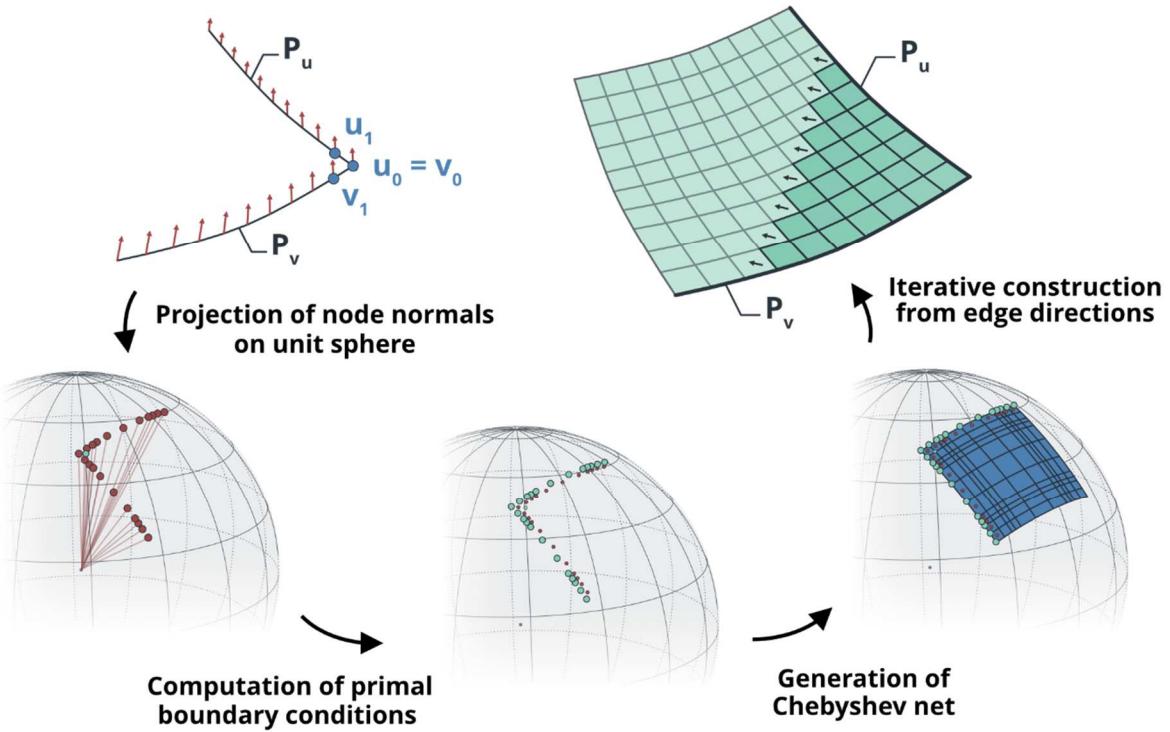
567 Thus, the entirety of  $\text{null}(\mathbf{B})$  can be explored by adding either eigenshapes or minimal influence  
 568 modes to  $\mathcal{V}$ . The effect is the transformation of the mesh while keeping the Gauss map of  $\mathcal{V}$   
 569 unchanged. These approaches give very satisfying solutions for closed and open meshes and allows to  
 570 have relevant transformation modes (Figure 14), which is not the case when simply using an  
 571 orthonormal basis of  $\text{null}(\mathbf{B})$ .

### 572 5.3 DIRECT GENERATION OF DISCRETE VOSS SURFACES

573 The methodology exposed in the previous section explores linear spaces of Voss surfaces, i.e. produces  
 574 a variety of shapes. However, in some cases, one can be interested in a more explicit approach, i.e. in  
 575 directly computing a discrete Voss surface from initial constraints expressed in the Euclidean space  
 576 and not on the Gauss map. Therefore, a direct generation method is presented, which uses two  
 577 boundary curves as input.

578 Let  $P_a$  and  $P_b$  be two polylines whose first vertices are coincident. These curves correspond to the  
 579 boundary coordinate-lines of the Voss net to generate. Given that  $P_a$  and  $P_b$  are not degenerated, the  
 580 Frenet frame at each internal vertex is computable. This is equivalent to interpreting the curves  $P_a$  and  
 581  $P_b$  as discrete geodesic lines on a surface. The Frenet frames give the normal vector at each internal  
 582 vertex, which corresponds to the centre of each spherical parallelogram of the Gauss map. With this  
 583 information only, there are some degrees of freedom left for the design of the Chebyshev net on  $S^2$ .  
 584 The missing information is to be found at the intersection of the input curves. Since  $P_a$  and  $P_b$  are the

585 boundary curves, their vertices  $a_0 = b_0$ ,  $a_1$  and  $b_1$  are contained in the same planar facets at the  
 586 starting corner. Therefore, the first vertices of the polylines give the normal vector of the initial facet.  
 587 The Chebyshev net is then generated from the first face normal and the set of vertex normals (Figure  
 588 15).



589

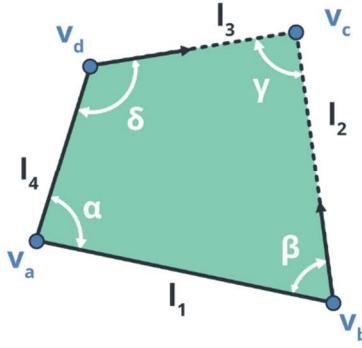
590

*Figure 15: Process for the direct computation of discrete Voss surfaces*

591 The Chebyshev net computed on the unit sphere gives access to the direction of each edge of the  
 592 discrete Voss net to be constructed from the boundary curves. In relation with the previous generation  
 593 method, a possible technique to construct the Voss net from there would be to compute the matrix  $B$   
 594 and the eigenshapes of the Laplacian, to describe  $\text{null}(B)$ . Then, to find the unique solution that fits  
 595 with the vertices of the boundary curves  $P_a$  and  $P_b$ , an additional linear system would have to be  
 596 solved. Instead, a more straightforward approach is adopted. In order to reduce significantly  
 597 computational time, constraints are solved locally, at the scale of the face and not globally on the mesh  
 598 [56]. Considering  $v_a, v_b, v_c$  and  $v_d$  the vertices of a planar quad face of  $\mathcal{V}$ , and  $d_1, d_2, d_3$  and  $d_4$  the  
 599 direction of the edge given by the Gauss map (Figure 16), if the length of two edges is known then it is  
 600 possible to know the length of the two others by solving:

601

$$\begin{pmatrix} l_1 - l_4 \cdot \cos(\alpha) \\ l_4 \cdot \sin(\alpha) \end{pmatrix} = \begin{pmatrix} \cos(\beta) & -\cos(\alpha + \delta) \\ \sin(\beta) & \sin(\alpha + \delta) \end{pmatrix} \begin{pmatrix} l_2 \\ l_3 \end{pmatrix}$$



602

603 *Figure 16: Computation of the last vertex of a planar face from two edge length.*

604 Thus, the Voss net is given by iteratively computing the faces, starting at the intersection of the  
 605 boundary discrete geodesic coordinate-line, in a similar way than for the Chebyshev net. In conclusion,  
 606 the direct generation of the discrete Voss net is permitted by the extensive use of the different  
 607 interpretations and analyses of the Gauss map presented in the preceding sections.

## 6 APPLICATIONS

---

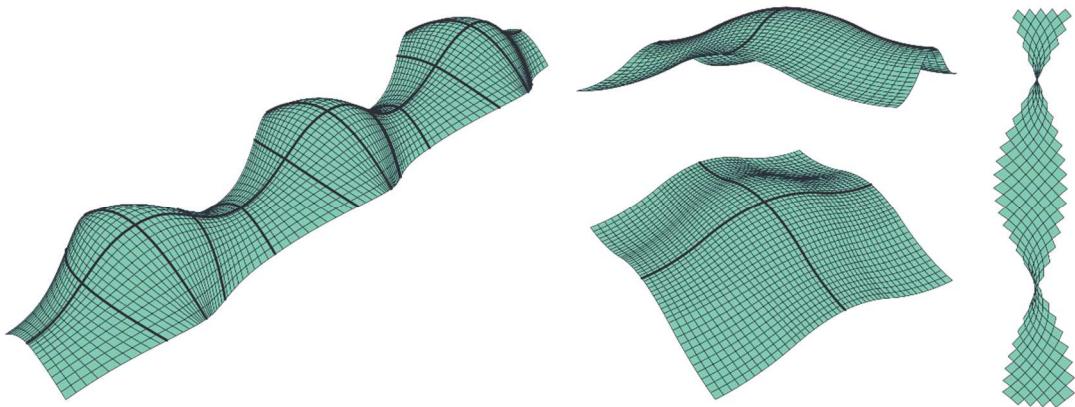
609 The methodologies presented in the previous section are applied for the computation of doubly-  
 610 curved shapes, all based on discrete Voss surfaces.

### 6.1 FORMAL EXPLORATION

612 The framework proposed gives access to a large variety of complex geometries. The design of discrete  
 613 Voss surfaces with positive, negative or sign-changing Gaussian curvature is possible, although the  
 614 design space, the unit sphere, has a constant positive Gaussian curvature.

615 The first methodology, based on the computation of a linear space, provides enough freedom to  
 616 investigate diverse shapes and to alter them quickly. Interestingly, new shapes can be found from  
 617 linear combinations of realizations, since the Voss nets are parts of the same linear space. Because the  
 618 generation of Voss nets is done through an abstract representation, it requires a certain intuition to  
 619 manipulate the Gauss Map. On the other hand, the determination of discrete Voss surfaces from two  
 620 boundary curves reveals more intuitive. With this second method, the designer has an explicit and  
 621 prompt way to build Voss nets.

622 Resorting to Chebyshev patches permits the concatenation of discrete Voss surfaces while ensuring  
 623 continuity between them. In a similar fashion as for Chebyshev nets, using patches provides a mean to  
 624 insert and manage singularities in the discrete Voss nets. Examples of constructions are compiled on  
 625 Figure 17, where the separation between patches is denoted by thick black lines.



626

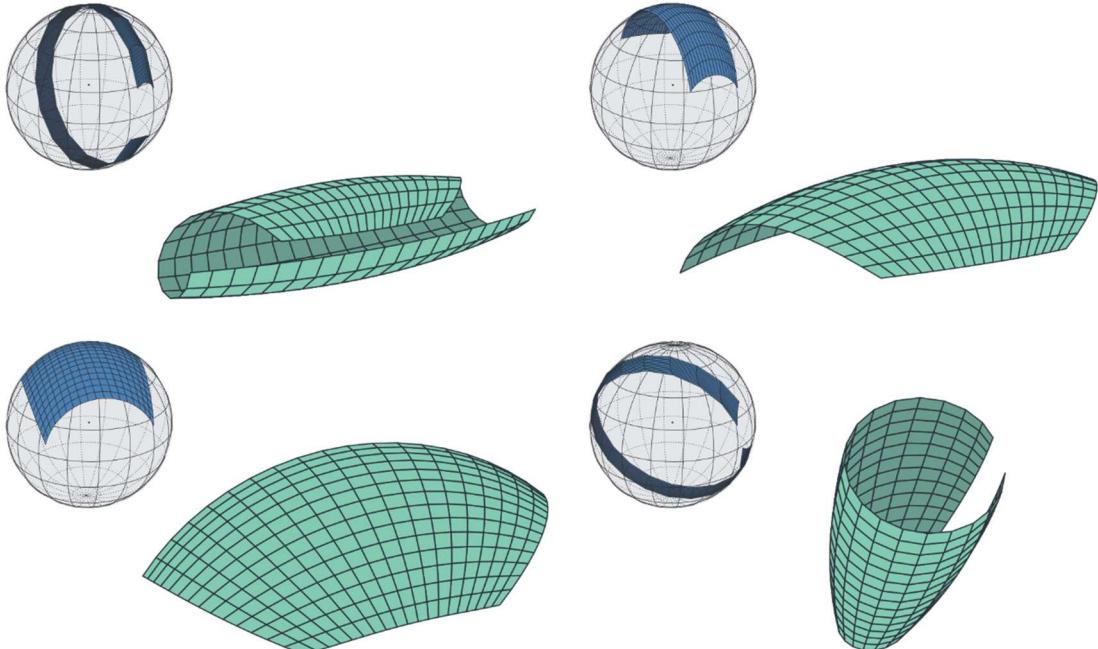
627 *Figure 17: Discrete Voss surfaces generated from Chebyshev patches. Left: Geodesic gridshell inspired by the gridshell of*  
628 *Downland [3]. Center: Geodesic gridshell canopy which can be repeated in both directions. Right: Twisting periodic Voss net.*

## 629 6.2 TRANSFORMATIONS

### 630 **ISOMETRIC TRANSFROMATIONS**

631 Isometric transformations are distance preserving transformations. As an extension of developable  
632 surfaces, discrete Voss nets admit a one-parameter family of isometric transformations. During such  
633 transformations, the shape of each face is kept identical, while their orientation varies in space.  
634 Consequently, given an initial Voss net, a family of meshes with same faces is deduced by prescribing  
635 one parameter for the transformation.

636 Figure 18 shows examples of discrete Voss surfaces related by isometric transformations. The different  
637 cases have been obtained by choosing the dihedral angle between two consecutives faces. The Gauss  
638 Map is then deformed accordingly.



639

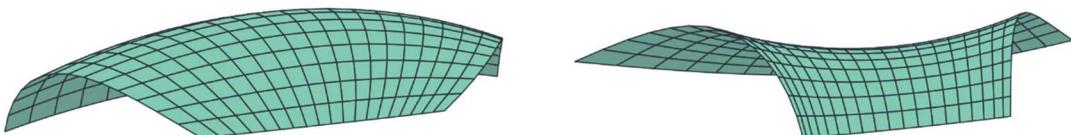
640

*Figure 18: Discrete Voss surfaces related by isometric transformations.*

#### 641 **COMBESCURE TRANSFORMATIONS**

642 Furthermore, Combescure transformations also preserve the defining properties of discrete Voss  
 643 surfaces. Here, they are applied to alter the geometry of Voss and explore the shape space. In Figure  
 644 19, the initial discrete surface with a positive Gaussian curvature (left) is transformed into a net with  
 645 negative Gaussian curvature (right).

646



647

648

*Figure 19: Voss surfaces with positive Gaussian curvature (left) and negative Gaussian curvature (right) both related by a Combescure transform.*

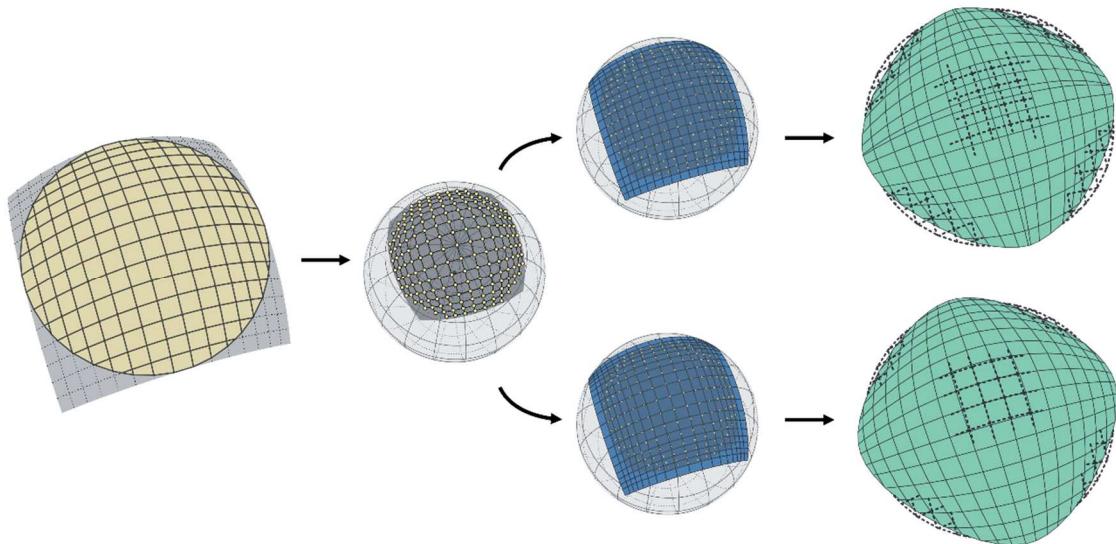
#### 649 **6.3 SHAPE FITTING WITH VOSS NETS**

650 The framework developed in this paper ensure that the surfaces drawn do not have approximate but  
 651 the exact properties. This guideline led to the formulation of Voss nets as linear space but has evaded  
 652 options such as ruled-based optimization, heavily used for fitting target shapes. However, the goal of  
 653 fitting a target surface is not inherently incompatible with the work presented. In this subsection,  
 654 illustrating the potential of our methodology, the fitting of a translational surface is studied (Figure 20  
 655 – left). The surface considered originates from an example given by Schober and Schlaich [22] and is  
 656 obtained by translating a parabola along another orthogonal parabola. By cutting this surface with a  
 657 circle, Schober and Schlaich were able to obtain a doubly curved dome that can be covered with flat

658 panels only, and which could have been implemented in projects such as the Neckarsulm swimming pool  
659 cover.

660 For the shape-fitting optimization on this target surface  $S$ , an adapted linear space of Voss nets is first  
661 defined. Knowing the surface  $S$ , its Gauss map  $\mathcal{G}$  is retrieved by projecting the normal vectors on the  
662 unit sphere. To ensure a complete covering of the Voss net  $\mathcal{V}$  on the target  $S$ , the Gauss map  $\mathcal{C}$  of  $\mathcal{V}$   
663 has to be greater than  $\mathcal{G}$ . In the case of the translational surface, a simple bi-symmetrical Chebyshev  
664 net  $\mathcal{C}$  has been designed on  $S^2$  (Figure 20 – top). From this initial guess  $\mathcal{C}$ , the linear space null( $\mathcal{B}$ )  
665 and then the minimal influence modes were computed. For a mesh  $\mathcal{V}$  of  $17 \times 17$  faces, it resulted in  
666 34 transformation modes of  $\mathcal{V}$ , to combine with the three global translation.

667 For practical reasons, the optimization method used is an implementation of the genetic algorithm  
668 called Galapagos [57]. The genotype is composed of the 37 modes of transformation. The fitness function  
669 is composed of three parts: the first one sums the distance of each vertices of  $\mathcal{V}$  to the target surface  
670  $S$  to ensure closeness; the second enforces symmetry on  $\mathcal{V}$  by summing the length discrepancies  
671 among symmetrical edges, and the last energy aligns the meshes by weighting the distance between  
672 the in-plane projection of their barycentre. The respective weights of the parts are 1.0, 20 and 20.



673

674 *Figure 20: Step of the fitting optimization of a translational surface with a Voss net. On the right, the Voss net resulting from*  
675 *the optimization with the exceeding portions of target surface in dotted lines*

676 After optimization, a first convincing result was obtained. The maximal and mean distances of the  
677 vertices of  $\mathcal{V}$  to the target geometry are respectively about 2.5 % and 0.7% of the surface diameter.  
678 However the pattern of the Voss net looks uneven due to the fact that the Gauss map  $\mathcal{C}$  was a regular  
679 Chebyshev net. To smoothen the division pattern of the net  $\mathcal{V}$ , edge length of the Chebyshev net could  
680 have been added in the genotype. This option assumes that the linear space has to be computed again

681 after each generation which is time consuming. Therefore, the Gauss map of  $\mathcal{V}$  was altered manually  
682 to a weak Chebyshev and after few iterations, another conclusive Voss net was found (Figure 20 –  
683 bottom). On this result, the mean and max distances are similar with respectively 2.8% and 0.78% of  
684 the target surface diameter.

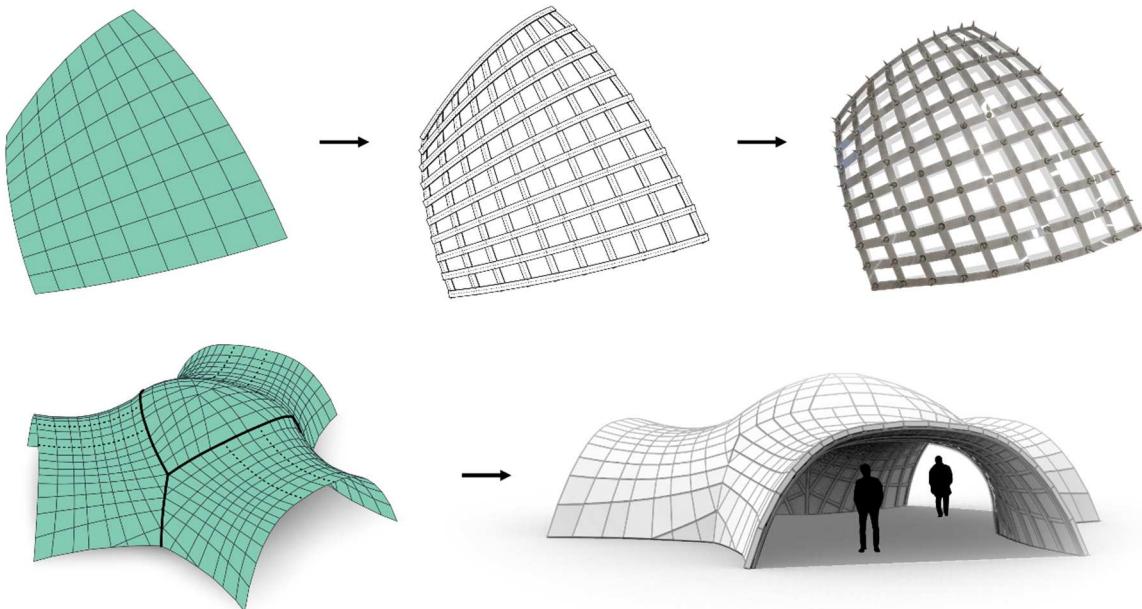
685 The advantage of using a linear space is that the optimization ran on only 37 parameters, while it would  
686 have been done on  $3 \times I = 972$  values corresponding to the coordinate in space of each vertices of  
687  $\mathcal{V}$ . On the other hand, the Voss nets resulting from the optimization fit the target geometry only  
688 approximately. The fact that the Gauss map has to be a Chebyshev net limits the shapes accessible to  
689 Voss nets.

#### 690 6.4 CONSTRUCTIVE INTERPRETATION

691 In this section, the relation between Voss nets and their constructability as geodesic gridshells is  
692 illustrated through two examples. For such interpretation, the edges of the meshes represent discrete  
693 geodesic lines and are thus approximations of the path taken by rectangular cross-section beams.  
694 Similarly, the faces of the mesh correspond exactly to flat panels used for the covering of the support  
695 structure.

696 The first example is based on a Voss net with a positive Gaussian curvature (Figure 21 – top right). The  
697 simplicity of the geometry (Figure 21 – top right) allows to grasp the transition leading to the simulated  
698 layout of the rectangular cross section beams (Figure 21 – top middle) and then to a real mock-up  
699 (Figure 21 – top right). The real mock-up is made of organic glass beams with 1.5mm by 8mm  
700 rectangular cross sections, assembled together with earing connectors to enforce tangency of  
701 intersecting members. The overall geometry of the grid took form progressively while assembling the  
702 beams on by one.

703 The second case study comprises an arrangement of several Voss nets (Figure 21 – bottom left). The  
704 central net itself is a composition of Voss nets which generate a singularity at the top. Then, three  
705 identical branches are patched on the sides. To display the feasibility of the geometry, a model of the  
706 pavilion with a beam layout and panel thickness is exposed (Figure 21 – bottom right). The panel  
707 thickness is set to 100 mm to emphasise on the capacity of Voss net to adapt large thickness. After a  
708 preliminary sizing of the bent members based on their curvature, 8mm by 80mm rectangular cross-  
709 section laths are used in the gridshells. The beams at free borders of the branches are double in place  
710 to fix their geometry while the supports are better fixed than hinged. This guarantees the proximity of  
711 the surface model with the relaxed configuration of the gridshell, thus reducing the tolerances and  
712 improving the capacity to fix the rigid panels over the support structure.



713

714 *Figure 21: A composition of Voss surfaces with a singularity at the centre (top - left) along with its constructive*  
 715 *interpretation which includes the flexible beam layout and the thickness of the covering panels (top - left); A simple Voss*  
 716 *surface (bottom - left), with the expected beam layout (bottom - middle) and the built mock-up (bottom - right).*

717

## CONCLUSIONS

---

718 This paper addresses the design of geodesic gridshells, a family of shells that can be built from initially-  
 719 straight beams arranged in a layout which avoids sideways bending, and that can be covered with flat  
 720 quadrilateral panels only. These unique beneficial construction features are made possible by using  
 721 discrete Voss surfaces and their unique geometrical properties.

722 A generation process is presented, which starts with the creation of a Chebyshev net  $\mathcal{C}$  on the unit  
 723 sphere, and interprets it as the Gauss map of Voss surfaces. The vertices of the Gauss map are used to  
 724 compute edge directions and create a matrix, whose null space is a linear space containing all  
 725 corresponding Voss surfaces. Smooth realizations of this space are obtained after computing a special  
 726 basis of the null space, called eigenshapes, the best candidate being the eigenshapes associated with  
 727 the lowest eigenvalue. The exploration of other realizations associated with the same Gauss map  $\mathcal{C}$  is  
 728 achieved at no computational cost by employing other eigenvectors or with minimal influence modes.

729 In addition, a process to directly compute Voss surfaces from two boundary curves is presented. It uses  
 730 the normal at the vertices of the mesh and generates the unique underlying Voss surface in linear time.

731 Finally, applications comprise the generation of the best-fit discrete surface to a target mesh. The  
 732 fitting is tackled from the Gauss map since it reduces the number of degrees of freedom for the  
 733 optimisation process. Additional geometries are presented, including the model of a pavilion.

734 Further developments should address the mapping of a Chebyshev net on complex Gauss maps.  
735 Although several methods already deal with the approximation of surfaces with Chebyshev nets  
736 (section 1.1), Gauss maps often presents singularities and cusps, a problem that is not systematically  
737 addressed in the past proposed methodologies. In addition, the described generation strategies do not  
738 apply to the exploration of the developable surfaces subfamily, since the gauss map is reduced to a  
739 curve. Alternative methods can be formulated with more information than solely the Gauss map.

740 Following the research presented in this article, focused on the theoretical framework, a pavilion has  
741 been designed and built. Thus, the constructive details, the mechanical study of Voss nets, and the  
742 built structure will be presented in an upcoming article.

## 743 AUTHOR'S CONTRIBUTIONS

---

744 Nicolas Montagne developed the theoretical formalism in the paper with the help of Xavier Tellier  
745 regarding differential geometry and under the supervision of Cyril Douthe. Nicolas Montagne devised  
746 the mathematical interpretations, generation methodologies and their computational implementation  
747 with advices from Cyril Douthe. Nicolas Montagne wrote the manuscript in consultation with Cyril  
748 Douthe for the development. All the authors provided critical feedback all along the development of  
749 the research, and supervised the writing of the paper.

## 750 ACKNOWLEDGMENT

---

751 The authors warmly thank Romain Mesnil for the fruitful discussions on topics ranging from  
752 architectural geometry, to structural morphogenesis to implementation details, as well as Laurent  
753 Hauswirth for the enlightening advices in differential geometry. The authors also thank Arthur Leb  e  
754 for valuable enlightenments in the domain of origami structures.

755

## BIBLIOGRAPHY

---

- 757 [1] J. Lienhard, H. Alpermann, C. Gengnagel, and J. Knippers, 'Active Bending, a Review on  
758 Structures where Bending is Used as a Self-Formation Process', *International Journal of Space  
759 Structures*, vol. 28, no. 3–4, pp. 187–196, 2013, doi: 10.1260/0266-3511.28.3-4.187.
- 760 [2] E. Happold and W. I. Liddell, 'Timber lattice roof for the Mannheim Bundesgartenschau',  
761 *Structural Engineering*, vol. 53, no. 3, pp. 99–135, 1975.
- 762 [3] R. Harris, J. Romer, O. Kelly, and S. Johnson, 'Design and construction of the Downland  
763 Gridshell', *Building Research & Information*, vol. 31, no. 6, pp. 427–454, 2003, doi:  
764 10.1080/0961321032000088007.
- 765 [4] G. C. Quinn and C. Gengnagel, 'A review of elastic grid shells, their erection methods and the  
766 potential use of pneumatic formwork', in *WIT Transactions on the Built Environment*, Belgium,  
767 2014, vol. 136, pp. 129–144. doi: 10.2495/MAR140111.
- 768 [5] L. Du Peloux, F. Tayeb, O. Baverel, and J.-F. Caron, 'Construction of a Large Composite Gridshell  
769 Structure: A Lightweight Structure Made with Pultruded Glass Fibre Reinforced Polymer Tubes',  
770 *Structural Engineering International*, vol. 26, no. 2, pp. 160–167, 2016, doi:  
771 10.2749/101686616X14555428758885.
- 772 [6] F. Otto, J. Hennicke, and K. Matsushita, *IL10 Gitterschalen*. 1974.
- 773 [7] L. Bouhaya, O. Baverel, and J.-F. Caron, 'Mapping two-way continuous elastic grid on an  
774 imposed surface: application to grid shells', in *IASS 2009: Evolution and Trends in Design,  
775 Analysis and Construction of Shell and Spatial Structures*, Valencia, 2009, pp. 989–998.  
776 Accessed: May 30, 2020. [Online]. Available: <https://hal.archives-ouvertes.fr/hal-00531430>
- 777 [8] L. Bouhaya, O. Baverel, and J.-F. Caron, 'Optimization of gridshell bar orientation using a  
778 simplified genetic approach', *Structural and Multidisciplinary Optimization*, vol. 50, no. 5, pp.  
779 839–848, 2014, doi: 10.1007/s00158-014-1088-9.
- 780 [9] E. Lafuente Hernández, O. Baverel, and C. Gengnagel, 'On the Design and Construction of  
781 Elastic Gridshells with Irregular Meshes', *International Journal of Space Structures*, vol. 28, no.  
782 4, pp. 161–174, 2013, doi: 10.1260/0266-3511.28.3-4.161.
- 783 [10] Y. Masson, 'Existence and construction of Chebyshev nets and application to gridshells',  
784 Université Paris-Est, Champs-sur-Marne, 2017. Accessed: May 30, 2020. [Online]. Available:  
785 <https://tel.archives-ouvertes.fr/tel-01676984v2>
- 786 [11] A. O. Sagedan-Furnas, A. Chern, M. Ben-Chen, and A. Vaxman, 'Chebyshev Nets from  
787 Commuting PolyVector Fields', *ACM Trans. Graph.*, vol. 38, no. 6, 2019, doi:  
788 10.1145/3355089.3356564.
- 789 [12] A. S. Day, 'An Introduction to dynamic relaxation', *The Engineer*, vol. 219, pp. 218–221, 1965.
- 790 [13] S. M. L. Adriaenssens, M. R. Barnes, and C. Williams, 'A new analytic and numerical basis for the  
791 form-finding and analysis of spline and gridshell structures', in *Computing Developments in Civil  
792 and Structural Engineering*, Civil-Comp Press, 1999, pp. 83–91. [Online]. Available:  
793 <https://researchportal.bath.ac.uk/en/publications/a-new-analytic-and-numerical-basis-for-the-form-finding-and-analy>
- 795 [14] M. R. Barnes, S. M. L. Adriaenssens, and M. Krupka, 'A novel torsion/bending element for  
796 dynamic relaxation modeling', *Computers & Structures*, vol. 119, pp. 60–67, 2013, doi:  
797 10.1016/j.compstruc.2012.12.027.
- 798 [15] B. D'Amico, A. Kermani, and H. Zhang, 'A form finding method for post formed timber grid shell  
799 structures', Canada, 2014, p. 6. doi: 10.13140/2.1.3949.8888.
- 800 [16] L. Du Peloux, F. Tayeb, B. Lefevre, O. Baverel, and J.-F. Caron, 'Formulation of a 4-DoF  
801 torsion/bending element for the formfinding of elastic gridshells', in *IASS 2015: Future Visions*,  
802 Amsterdam, 2015, p. 16. Accessed: May 30, 2020. [Online]. Available: <https://hal.archives-ouvertes.fr/hal-01199049>

- 804 [17] J. Bessini, C. Lázaro, and S. Monleón, ‘A form-finding method based on the geometrically exact  
805 rod model for bending-active structures’, *Engineering Structures*, vol. 152, pp. 549–558, 2017,  
806 doi: 10.1016/j.engstruct.2017.09.045.
- 807 [18] M. Sakai, Y. Mori, X. Sun, and K. Takabatake, ‘Recent Progress on Mesh-free Particle Methods  
808 for Simulations of Multi-phase Flows: A Review’, *KONA Powder and Particle Journal*, vol. 37, pp.  
809 132–144, 2020, doi: 10.14356/kona.2020017.
- 810 [19] C. Douthe, O. Baverel, and J.-F. Caron, ‘Form-finding of a grid shell in composite materials’, *J  
811 IASS*, vol. 47, p. 10, 2006.
- 812 [20] L. Du Peloux, F. Tayeb, J.-F. Caron, and O. Baverel, ‘The Ephemeral Cathedral of Créteil : a  
813 350m<sup>2</sup> lightweight gridshell structure made of 2 kilometers of GFRP tubes’, in *CIGOS 2015:  
814 Innovations in Constructions*, Cachan, 2015, p. 10. Accessed: May 30, 2020. [Online]. Available:  
815 <https://hal.archives-ouvertes.fr/hal-01199044>
- 816 [21] O. Baverel, J.-F. Caron, F. Tayeb, and L. Du Peloux, ‘Gridshells in Composite Materials:  
817 Construction of a 300 m<sup>2</sup> Forum for the Solidays’ Festival in Paris’, *Structural Engineering  
818 International*, vol. 22, no. 3, pp. 408–414, 2012, doi: 10.2749/101686612X13363869853572.
- 819 [22] J. Schlaich and H. Schober, ‘Glass Roof for the Hippo House at the Berlin Zoo’, *Structural  
820 Engineering International*, vol. 7, no. 4, pp. 252–254, 1997, doi: 10.2749/101686697780494581.
- 821 [23] C. Douthe, R. Mesnil, H. Orts, and O. Baverel, ‘Isoradial meshes: Covering elastic gridshells with  
822 planar facets’, *Automation in Construction*, vol. 83, pp. 222–236, 2017, doi:  
823 10.1016/j.autcon.2017.08.015.
- 824 [24] A. I. Bobenko and Y. B. Suris, *Discrete Differential Geometry : Integrable Structure*. American  
825 Mathematical Society, 2008.
- 826 [25] J. Natterer, N. Burger, and A. Müller, ‘The roof structure “Expodach” at the World Exhibition  
827 Hanover 2000’, in *Space Structures 5*, Guildford, 2002, pp. 185–193. doi:  
828 10.1680/ss5v1.31739.0020.
- 829 [26] C. Pirazzi and Y. Weinand, ‘Geodesic Lines on Free-Form Surfaces - Optimized Grids for Timber  
830 Rib Shells’, 2006, pp. 595–601. Accessed: May 30, 2020. [Online]. Available:  
831 <https://core.ac.uk/display/147936818?recSetID=>
- 832 [27] H. Pottmann *et al.*, ‘Geodesic Patterns’, *ACM Trans. Graph.*, vol. 29, no. 4, p. 10, 2010, doi:  
833 10.1145/1833349.1778780.
- 834 [28] M. Rabinovich, T. Hoffmann, and O. Sorkine-Hornung, ‘Discrete Geodesic Nets for Modeling  
835 Developable Surfaces’, *ACM Trans. Graph.*, vol. 37, no. 2, 2018, doi: 10.1145/3180494.
- 836 [29] M. Rabinovich, T. Hoffmann, and O. Sorkine-Hornung, ‘The Shape Space of Discrete Orthogonal  
837 Geodesic Nets’, *ACM Trans. Graph.*, vol. 37, no. 6, p. 17, 2018, doi: 10.1145/3272127.3275088.
- 838 [30] E. Soriano, R. Sastre, and D. Boixader, ‘G-shells: Flat collapsible geodesic mechanisms for  
839 gridshells’, in *IASS 2019: Form and Force*, Barcelona, 2019, pp. 1894–1901. Accessed: May 30,  
840 2020. [Online]. Available:  
841 <https://www.ingentaconnect.com/content/iass/piass/2019/00002019/00000011/art00006>
- 842 [31] S. Pillwein, K. Leimer, M. Birsak, and P. Musialski, ‘On Elastic Geodesic Grids and Their Planar to  
843 Spatial Deployment’, *ACM Trans. Graph.*, vol. 39, no. 4, 2020, doi: 10.1145/3386569.3392490.
- 844 [32] H. Wang, D. Pellis, F. Rist, H. Pottmann, K. Matsushita, and C. Müller, ‘Discrete Geodesic Parallel  
845 Coordinates’, *ACM Trans. Graph.*, vol. 38, no. 6, p. 13, 2019, doi: 10.1145/3355089.3356541.
- 846 [33] A. E. Voss, ‘Ueber diejenigen Flächen, auf denen zwei Scharen geodätischer Linien ein  
847 conjugirtes System bilden.’, *Math.-Naturwiss.*, 1888.
- 848 [34] W. Wunderlich, ‘Zur Differenzengeometrie der Flächen konstanter negativer Krümmung’, Wien,  
849 1951, pp. 39–77.
- 850 [35] S. E. Taylor, A. Gupta, J. Kirkpatrick, and A. E. Long, ‘Testing a novel flexible concrete arch  
851 system’, presented at the Advanced Composites in Construction, Bath, 2007. Accessed: Sep. 15,  
852 2020. [Online]. Available:  
853 <https://citeseerx.ist.psu.edu/viewdoc/summary?doi=10.1.1.1075.5870>

- 854 [36] H. Pottmann, Y. Liu, J. Wallner, A. I. Bobenko, and W. Wang, ‘Geometry of Multi-Layer Freeform  
855 Structures for Architecture’, *ACM Trans. Graph.*, vol. 26, no. 3, 2007, doi:  
856 10.1145/1276377.1276458.  
857 [37] P. Romon, *Introduction à la géométrie différentielle discrète*. Ellipses, 2013. [Online]. Available:  
858 <https://hal.archives-ouvertes.fr/hal-00926137>  
859 [38] U. Pinkall, ‘Selected topics in Discrete Differential Geometry and Visualization’, Beijing, 2009.  
860 [Online]. Available: <https://www3.math.tu-berlin.de/geometrie/gpsdde/2009DDGandVis/iddg.pdf>  
861 [39] W. K. Schief, A. I. Bobenko, and T. Hoffmann, ‘On the Integrability of Infinitesimal and Finite  
862 Deformations of Polyhedral Surfaces’, in *Discrete Differential Geometry*, A. I. Bobenko, J. M.  
863 Sullivan, P. Schröder, and G. M. Ziegler, Eds. Basel: Birkhäuser Basel, 2008, pp. 67–93. [Online].  
864 Available: [https://doi.org/10.1007/978-3-7643-8621-4\\_4](https://doi.org/10.1007/978-3-7643-8621-4_4)  
865 [40] J.-F. Frenet, ‘Sur les courbes à double courbure’, *Journal de Mathématiques Pures et  
866 Appliquées*, vol. 17, no. 1, pp. 437–447, 1852.  
867 [41] R. Sauer, *Differenzengeometrie*. Springer Verlag, 1970. [Online]. Available:  
868 <https://doi.org/10.1007/978-3-642-86411-7>  
869 [42] O. Sorkine-Hornung, D. Cohen-Or, Y. Lipman, M. Alexa, C. Rössl, and H. P. Seidel, ‘Laplacian  
870 Surface Editing’, in *Eurographics Symposium on Geometry Processing 2004*, 2004, pp. 175–184.  
871 doi: 10.1145/1057432.1057456.  
872 [43] M. Kilian, N. J. Mitra, and H. Pottmann, ‘Geometric Modeling in Shape Space’, *ACM Trans.  
873 Graph.*, vol. 26, no. 3, pp. 64–71, 2007, doi: 10.1145/1276377.1276457.  
874 [44] M. Botsch and O. Sorkine-Hornung, ‘On Linear Variational Surface Deformation Methods’, *IEEE  
875 Trans. Vis Comput. Graph.*, vol. 14, no. 1, pp. 213–230, 2008, doi: 10.1109/TVCG.2007.1054.  
876 [45] Y.-L. Yang, Y.-J. Yang, H. Pottmann, and N. J. Mitra, ‘Shape Space Exploration of Constrained  
877 Meshes’, *ACM Trans. Graph.*, vol. 30, no. 6, p. 12, 2011, doi: 10.1145/2070781.2024158.  
878 [46] Y. Liu, H. Pottmann, J. Wallner, Y.-L. Yang, and W. Wang, ‘Geometric Modeling with Conical  
879 Meshes and Developable Surfaces’, *ACM Trans. Graph.*, vol. 25, no. 3, pp. 681–689, 2006, doi:  
880 10.1145/1141911.1141941.  
881 [47] S. Bouaziz, M. M. Deuss, Y. Schwartzburg, T. Weise, and M. Pauly, ‘Shape-Up: Shaping Discrete  
882 Geometry with Projections’, in *Eurographics Symposium on Geometry Processing 2012*, Tallinn,  
883 2012, vol. 31, pp. 1657–1667. doi: 10.1111/j.1467-8659.2012.03171.x.  
884 [48] A. Vaxman, ‘A Projective Framework for Polyhedral Mesh Modelling’, *Computer Graphics  
885 Forum*, vol. 33, no. 8, pp. 121–131, 2014, doi: 10.1111/cgf.12405.  
886 [49] R. Poranne, R. Chen, and C. Gotsman, ‘On Linear Spaces of Polyhedral Meshes’, *IEEE Trans. Vis  
887 Comput. Graph.*, vol. 21, no. 5, pp. 652–662, 2015, doi: 10.1109/TVCG.2014.2388205.  
888 [50] R. Mesnil, C. Douthe, O. Baverel, and B. Léger, ‘Marionette Meshes: Modelling free-form  
889 architecture with planar facets’, *International Journal of Space Structures*, vol. 32, no. 3–4, pp.  
890 184–198, 2017, doi: 10.1177/0266351117738379.  
891 [51] C. Douthe *et al.*, ‘Design and construction of a shell-nexorade hybrid timber structure’,  
892 presented at the International Association for Shell and Spatial Structures Symposium, Boston,  
893 2018. [Online]. Available: <https://hal-enpc.archives-ouvertes.fr/hal-01852280>  
894 [52] T. Tachi, ‘Freeform Rigid-Foldable Structure using Bidirectionally Flat-Foldable Planar  
895 Quadrilateral Mesh’, 2010, pp. 87–102. doi: 10.1007/978-3-7091-0309-8\_6.  
896 [53] T. Mitchell, A. Masurek, C. Hartz, M. Miki, and W. Baker, ‘Structural Applications of the Graphic  
897 Statics and StaticKinematic Dualities: Rigid Origami, Self-Centering Cable Nets, and Linkage  
898 Meshes’, presented at the International Association for Shell and Spatial Structures Symposium,  
899 Boston, 2018. Accessed: Jun. 15, 2020. [Online]. Available:  
900 <https://www.ingentaconnect.com/content/iass/piass/2018/00002018/00000016/art00010>  
901 [54] E. Ghys, ‘Sur la coupe des vêtements: variation autour d’un thème de Tchebychev’,  
902 *L’Enseignement Mathématique*, vol. 57, no. 2, pp. 165–208, 2011.

- 904 [55] U. Pinkall, ‘Designing Cylinders with Constant Negative Curvature’, in *Discrete Differential*  
905 *Geometry*, A. I. Bobenko, J. M. Sullivan, P. Schröder, and G. M. Ziegler, Eds. Birkhäuser Basel,  
906 2008, pp. 55–66. [Online]. Available: [https://doi.org/10.1007/978-3-7643-8621-4\\_3](https://doi.org/10.1007/978-3-7643-8621-4_3)
- 907 [56] R. Mesnil, C. Douthe, O. Baverel, and B. Léger, ‘Morphogenesis of surfaces with planar lines of  
908 curvature and application to architectural design’, *Automation in Construction*, vol. 95, pp. 129–  
909 141, 2018, doi: 10.1016/j.autcon.2018.08.007.
- 910 [57] D. Rutten, ‘Galapagos: On the Logic and Limitations of Generic Solvers’, *Architectural Design*,  
911 vol. 83, no. 2, pp. 132–135, 2013, doi: 10.1002/ad.1568.
- 912

913

## APPENDIX: COMPUTATION OF A CHEBYSHEV NET $S^2$

---

914 Literature is filled with strategies for generating Chebyshev nets on general surfaces. They are based  
 915 on geometrical considerations or optimisation principles, which are computationally heavy. In [38] a  
 916 specific method is explained to generate Chebyshev nets on the unit sphere using unit quaternions.  
 917 The embedding in the 4D space of quaternions allows to build the Chebyshev net from simple addition  
 918 and multiplication.

919 Considering  $v_a, v_b, v_c$  and  $v_d$  the vertices of a spherical parallelogram contained in a hemisphere of  
 920 the unit sphere  $S^2$  (Figure 22), the following relation holds:

921 
$$(v_a + v_c) \times (v_b + v_d) = 0,$$

922 which is equivalent to stating that the rotation about the axis  $(v_a + v_c)$  or  $(v_b + v_d)$  of an angle of  $\pi$ ,  
 923 transforms the spherical parallelogram to itself. The position of the vertices  $v_a$  and  $v_c$ , and  $v_b$  and  $v_d$   
 924 are inverted. Thus, if the vertices  $v_a, v_b$  and  $v_d$  are known,  $v_c$  can be computed from  $v_a$  by rotation of  
 925 an angle  $\pi$  around the axis  $(v_b + v_d)$ .

926 The space of unit quaternions is then used to compute this rotation in a straightforward way. Let's  
 927 consider the rotation of a point  $p = (p_x, p_y, p_z)$  around the unitized axis  $d = (d_x, d_y, d_z)$  of an angle  
 928  $\alpha$ . The unit rotation quaternion is defined by:

929 
$$q = \left( \cos\left(\frac{\alpha}{2}\right), \sin\left(\frac{\alpha}{2}\right) \cdot d_x, \sin\left(\frac{\alpha}{2}\right) \cdot d_y, \sin\left(\frac{\alpha}{2}\right) \cdot d_z \right).$$

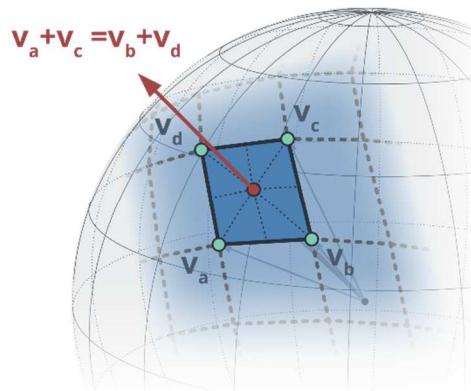
930 Using the identification between the Euclidean space  $\mathbb{R}^3 = \{(x, y, z); x, y, z \in \mathbb{R}\}$  and the space of  
 931 imaginary quaternions  $\mathbb{H} = \{(0, x, y, z); x, y, z \in \mathbb{R}\}$ , the resulting point of the rotation is computed  
 932 as:

933 
$$r = q \cdot p \cdot q^{-1}.$$

934 Using this formula with:

935 
$$p = v_a, \quad d = \frac{v_b + v_d}{\|v_b + v_d\|}, \quad \alpha = \pi,$$

936 the resulting  $r$ , or more precisely its imaginary, is the position of  $v_c$ .



937

938 *Figure 22: Spherical parallelogram and the rotation axis mapping the quad to itself through a rotation of an angle  $\pi$*