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► **To cite this version:**

Silvia M. Bianchi, Dipayan Chakraborty, Yanina Lucarini, Annegret K. Wagler. Location-Domination Type Problems Under the Mycielski Construction. 2023. hal-04309171

HAL Id: hal-04309171

<https://hal.science/hal-04309171v1>

Preprint submitted on 27 Nov 2023

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Location-Domination Type Problems Under the Mycielski Construction*

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Abstract. We consider the following variants of the classical minimum dominating set problem in graphs: locating-dominating set, locating total-dominating set and open locating-dominating set. All these problems are known to be hard for general graphs. A typical line of attack, therefore, is to either determine the minimum cardinalities of such sets in general or to establish bounds on these minimum cardinalities in special graph classes. In this paper, we study the minimum cardinalities of these variants of the dominating set under a graph operation defined by Mycielski in [22] and is called the Mycielski construction. We provide some general lower and upper bounds on the minimum sizes of the studied sets under the Mycielski construction. We apply the Mycielski construction to stars, paths and cycles in particular, and provide lower and upper bounds on the minimum cardinalities of such sets in these graph classes. Our results either improve or attain the general known upper bounds.

Keywords: Locating-dominating set · Open locating-dominating set · Locating total-dominating set · Mycielski construction.

1 Introduction

For a graph modeling a facility, the placement of monitoring devices, for example, fire detectors or surveillance cameras, motivates the study of various location-domination type problems in graphs. The problem of placing monitoring devices so that every site of a facility is visible from a monitor leads to a domination problem. In addition, the position of a fire, a thief, or a saboteur

* This work was sponsored by a public grant overseen by the French National Research Agency as part of the “Investissements d’Avenir” through the IMobS3 Laboratory of Excellence (ANR-10-LABX-0016) and the IDEX-ISITE initiative CAP 20-25 (ANR-16-IDEX-0001).

in the facility can be uniquely located by a specific subset of the monitoring devices which leads to location problems. During the last decades, several combined location-domination problems of this type have been actively studied, see, for example, the bibliography maintained by Lobstein and Jean [20]. In this work, we study three different location-domination type problems under a graph operation known as the Mycielski construction and defined by Mycielski himself in [22].

All graphs in this paper are finite, simple and connected. Given a graph $G = (V, E)$, the (open) neighborhood of a vertex $u \in V$ is the set $N(u) = N_G(u)$ of all vertices of G adjacent to u , and $N[u] = N_G[u] = \{u\} \cup N(u)$ is the closed neighborhood of u . A subset $C \subseteq V$ is *dominating* (respectively, *total-dominating*) if the set $N[u] \cap C$ (respectively, $N(u) \cap C$) is non-empty for all $u \in V$. In addition, a subset $C \subseteq V$ *separates* (respectively, *total-separates*) a pair $u, v \in V$ if $N[u] \cap C \neq N[v] \cap C$ (respectively, $N(u) \cap C \neq N(v) \cap C$). In such a case, we also say that $u, v \in V$ are *separated by C* (respectively, *total-separated by C*). A subset $C \subseteq V$ is called

- a *locating-dominating set* [26] (or *LD-set* for short) of G if it is a dominating set of G that separates all pairs of distinct vertices outside of C , that is, $N(u) \cap C \neq N(v) \cap C$, for all distinct $u, v \in V - C$;
- a *locating total-dominating set* [16] (or *LTD-set* for short) of G if it is a total-dominating set of G that separates all pairs of distinct vertices outside of C , that is, $N(u) \cap C \neq N(v) \cap C$, for all distinct $u, v \in V - C$;
- an *open locating-dominating set* [24] (or *OLD-set* for short) of G if it is a total-dominating set of G that total-separates all pairs of distinct vertices of the graph, that is, $N(u) \cap C \neq N(v) \cap C$, for all distinct $u, v \in V$.

Two distinct vertices u, v of a graph $G = (V, E)$ are called *false twins* if $N(u) = N(v)$, see [24]. Similarly, any two vertices $u, v \in V$ with $N[u] = N[v]$ are called *true twins*. Now, for $X \in \{LD, LTD, OLD\}$, the X -problem on G is the problem of finding an X -set of minimum size in G . The size of such a set is called the X -number of G and is denoted by $\gamma_X(G)$. Note that a graph G without isolated vertices admits an *OLD-set* if there are no false twins in G . On the other hand, *LD-sets* and *LTD-sets* are admitted by all graphs.

From the definitions themselves, the following relations hold for any graph G admitting any two X -sets for $X \in \{LD, LTD, OLD\}$:

$$\gamma_{LD}(G) \leq \gamma_{LTD}(G) \leq \gamma_{OLD}(G). \quad (1)$$

It has been shown that determining $\gamma_X(G)$ is in general NP-hard for all $X \in \{LD, LTD, OLD\}$. Apart from determining $\gamma_{LD}(G)$ being NP-hard in general [24], it remains so for bipartite graphs [9] and some subclasses of chordal graphs like split graphs and interval graphs [14]. This result is also extended to planar bipartite unit disk graphs in [21] and intersection graphs in [13]. Closed formulas for the exact values of $\gamma_{LD}(G)$ have so far been found for restricted graph families, for example, for paths [26], cycles [6], stars, complete multipartite graphs, some subclasses of split graphs and thin suns [1,4]. Bounds for the

LD -number of trees were provided in [7]. A linear-time algorithm to determine $\gamma_{LD}(G)$ for G being a tree was provided by Slater in [27] and has been extended to block graphs (graphs which generalize the concept of trees in that any 2-connected subgraph in a block graph is complete) in [2]. Moreover, in connection to block graphs and hence, trees, tight upper and lower bounds for LD -numbers of block graphs and twin-free block graphs have been established in [8].

Determining $\gamma_{OLD}(G)$ is NP-hard not only in general [24] but also on other graph classes like perfect elimination bipartite graphs [23], interval graphs [14] and is APX-hard on chordal graphs of maximum degree 4 [23]. Closed formulas for the exact value of $\gamma_{OLD}(G)$ have so far been found only for restricted graph families such as cliques and paths [24], some subclasses of split graphs and thin suns [1]. Tight lower and upper bounds for OLD -numbers certain classes of graphs like trees [24], block graphs [8], lower bounds for interval graphs, permutation graphs and cographs [13] and upper bounds for cubic graphs [18] have been established. Lastly, some algorithmic aspects of the problem have been discussed in [2,23].

Concerning LTD -sets, it can be checked that it is as hard as the OLD -problem by using the same arguments as in [24]. Bounds for the LTD -number of trees are given in [16,17]. In addition, the LTD -number in special families of graphs, including cubic graphs, grid graphs, complete multipartite graphs, some subclasses of split graphs and thin suns is investigated in [1,17].

In fact, giving bounds for the X -numbers in special graphs is a popular way to tackle the problems. In this work, we study the behavior of the three X -sets of graphs under the following graph operation defined by Mycielski in [22]. Given a graph $G = (V, E)$ with $V = \{v_1, \dots, v_n\}$, a new graph $M(G)$ is constructed as follows: for every vertex v_i of G , add a new vertex u_i and make u_i adjacent to all vertices in $N_G(v_i)$. Finally add a vertex u which is adjacent to all u_i . Let the set containing all the vertices u_i 's be called U , that is, $U = \{u_1, u_2, \dots, u_n\}$.

Originally, Mycielski introduced this construction in the context of graph coloring and used it to generate graphs $M(G)$ whose chromatic number increases by one compared to the chromatic number of G . In [12], it is proved that the application of the Mycielski construction also increases the dominating number by one. In this paper, we show that the same holds for total domination and study the X -numbers of the graphs $M(G)$, where G is a star $K_{1,n}$, a path P_n and a cycle C_n (see Fig. 1, Fig. 2 and Fig. 3, respectively, for examples of their illustrations). As far as previous works on such variants of the dominating sets of Mycielski constructions is concerned, we know of only one such, namely, in [25] where the authors find tight upper bounds of ID -numbers of $M(G)$ for G being an identifiable graph (that is, a graph without true twins). The ID -number of an identifiable graph G is the minimum cardinality of a dominating set C of G such that $N[u] \cap C \neq N[v] \cap C$ for all distinct pairs u, v of vertices of G (see [19]).

In Section 2, we show that the application of the Mycielski construction increases the total-dominating number by at least one and give a general lower bound on the studied X -numbers of the graphs $M(G)$ in terms of $\gamma_X(G)$ when

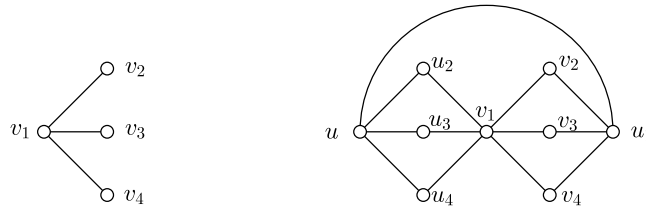


Fig. 1. The star $K_{1,3}$ and the resulting graph $M(K_{1,3})$

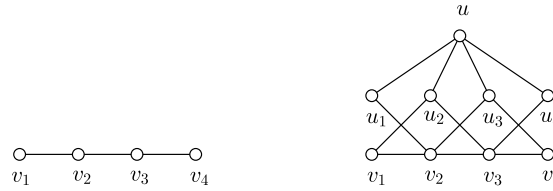


Fig. 2. The path P_4 and the resulting graph $M(P_4)$

G is either a path or a cycle. We then combine this bound with previously known results on $\gamma_X(P_n)$ (respectively, on $\gamma_X(C_n)$) to obtain lower bounds on the X -numbers of $M(P_n)$ (respectively, of $M(C_n)$).

In Section 3, we give a general upper bound on the X -numbers of the graphs $M(G)$. We also show that this bound is attained when G is a star and improve the bound for the cases when G is a path or cycle.

We note that there are some particularities in applying the Mycielski construction to paths and cycles with a small number of vertices. In fact, we have $M(P_2) = C_5$. While we have $\gamma_{LD}(P_2) = 1$ and $\gamma_{LTD}(P_2) = \gamma_{OLD}(P_2) = 2$, it is easy to see that $\gamma_{LD}(C_5) = 2$, $\gamma_{LTD}(C_5) = 3$, and $\gamma_{OLD}(C_5) = 4$ hold. Moreover, P_3 and C_4 have false twins, and so do $M(P_3)$ and $M(C_4)$. Hence, there exist no OLD -sets of these graphs. However, we have $\gamma_X(P_3) = \gamma_X(C_4) = 2$ and $\gamma_X(M(P_3)) = \gamma_X(M(C_4)) = 4$ for $X \in \{LD, LTD\}$. Hence, in the rest of what follows, we study paths P_n and cycles C_n with larger values of n .

We close with some concluding remarks and open problems for future research.

2 Lower bounds on X -numbers of graphs $M(G)$

To start with, observe the following fact that, for every graph G ,

1. two vertices v_i and v_j are false twins in G if and only if the vertices v_i, v_j and u_i, u_j are pairs of false twins in $M(G)$; and
2. $M(G)$ has no true twins.

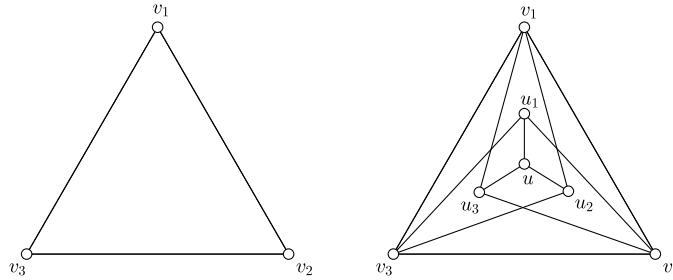


Fig. 3. The cycle C_3 and the resulting graph $M(C_3)$

In [12], it is proved that, for every graph G , the equality $\gamma(M(G)) = \gamma(G) + 1$ holds, where $\gamma(G)$ is the dominating number of G . Analogously, for the total-dominating number $\gamma_t(M(G))$, we can prove:

Lemma 1. *For every graph G without isolated vertices, we have $\gamma_t(M(G)) = \gamma_t(G) + 1$.*

Proof (sketch). Let $C \subseteq V$ be a total-dominating set of G and let $u_i \in U$. We define $C_i = C \cup \{u_i\}$. As G has no isolated vertices, every vertex in $V(M(G))$ is adjacent to a vertex in C_i and so, $\gamma_t(M(G)) \leq \gamma_t(G) + 1$.

Now, let C_M be a total-dominating set of $M(G)$ of cardinality $\gamma_t(M(G))$. Every vertex in V is adjacent to a vertex in C_M . Let us define the sets $C_V = C_M \cap V$ and $C_U = C_M \cap U$. Then it can be verified that the set $C_V \cup \{v_i : u_i \in C_U\}$ is a total-dominating set of G and $|C_V| \leq |C_M - \{u\}|$. Thus, if $u \in C_M$, we are done. Therefore, let us assume that $u \notin C_M$. Then, there exists $u_j \in C_M$ in order for C_M to total-dominate u . Now, for every vertex $u_i \in C_U$, any neighbor v_k ($\in V$) of u_i also has a neighbor in C_V (the same vertex in C_V that is a neighbor of u_k). This implies that $C_M - C_U$ is a total-dominating set of V . Since, $u_j \in C_U$, we have $|C_U| \geq 1$ and hence, the result follows. \square

This motivates us to study the parameter $\gamma_X(M(G))$ in terms of $\gamma_X(G)$. In doing so, we now establish a general lower bound on the X -numbers of the graphs $M(G)$, where G is either a path P_n or a cycle C_n and $X \in \{LD, LTD, OLD\}$.

Theorem 1. *Let $X \in \{LD, LTD, OLD\}$. For a graph G that is either a path P_n or a cycle C_n admitting an X -set, we have*

$$\gamma_X(M(G)) \geq \gamma_X(G) + 1.$$

Proof (sketch). As a proof sketch, we provide here the proof of the theorem only for the case that $X = LD$. The proof in the other cases when $X \in \{LTD, OLD\}$ follows with similar proof techniques. To begin with, let us assume that G is any graph (not necessarily a path or a cycle) and that C_M is a minimum LD -set of $M(G)$. Let $C_V = C_M \cap V$ and $C_U = C_M \cap U$. Then define the set $C = C_V \cup \{v_i : u_i \in C_U\}$.

Claim. C is an LD -set of G .

Proof of Claim. Firstly, since C_M is a dominating set of $M(G)$, one can verify that the set C is also a dominating set of G . We now show that C is also a separating set of G . Let $v_i, v_j \in V$ be any pair of arbitrary vertices such that $v_i, v_j \notin C$. Then we show that v_i, v_j are separated by C in G . Let $w_k \in C_M$ with $k \notin \{i, j\}$ separate v_i and v_j in $M(G)$, where $w_k \in \{u_k, v_k\}$. If $w_k = v_k$, then C clearly separates v_i, v_j . So, let $w_k = u_k$ and that u_k is a neighbour of v_i and not of v_j in $M(G)$. Now, if $k = j$, then $v_j \in C$ and is trivially separated from every other vertex of G by the definitions of LD -sets. So, let $k \neq j$. Then again, v_k is a neighbour of v_i and not of v_j in G and thus C separates v_i, v_j . This establishes the claim. \square

Thus we have,

$$\gamma_{LD}(M(G)) \geq |C_M - \{u\}| \geq |C| \geq \gamma_{LD}(G). \quad (2)$$

So, if $u \in C_M$, then $\gamma_{LD}(M(G)) > |C_M|$ and hence, the statement of the theorem holds. So, let us assume that $u \notin C_M$, in which case, we have $\gamma_{LD}(M(G)) \geq \gamma_{LD}(G)$. Toward contradiction, let us assume that $\gamma_{LD}(M(G)) = \gamma_{LD}(G)$. Then, by the assumed equalities in (2), C must be a minimum LD -set of G . This in turn implies that for each i , we have $|\{u_i, v_i\} \cap C_M| \leq 1$. In other words, if $u_i \in C_M$, then $v_i \notin C_M$ and vice-versa.

For the rest of the proof sketch, let us assume that G is either a path P_n or a cycle C_n . First of all, we observe that if any three consecutive vertices $v_i, v_{i+1}, v_{i+2} \in C$, then C cannot be a minimum LD -set of G , as one can discard v_{i+1} from C and the latter still remains an LD -set. Similarly, for some i , if $v_i, v_{i+1}, v_{i+3}, v_{i+4} \in C$, then again C cannot be a minimum LD -set of G , as one can discard v_{i+1}, v_{i+3} and include v_{i+2} in C and the latter still remains an LD -set of G . With those observations, let us first assume that some vertex $u_i \in C_M$ in order to dominate u . If one of its neighbours in G , say v_{i+1} , without loss of generality, belongs to C_M , then we must also have $v_{i+2} \in C_M$ in order for C_M to dominate u_{i+1} (note that $v_i \notin C_M$). Thus, $v_i, v_{i+1}, v_{i+2} \in C$, a contradiction to the minimality of C by our earlier observation. Hence, let us assume that for all $u_i \in C_M$, none of its neighbours in G , that is, v_{i-1} and v_{i+1} , belong to C_M . So, fix one such $u_i \in C_M$. Then $v_i \notin C_M$. Therefore, without loss of generality, let $u_{i+1} \in C_M$ in order for the latter to dominate v_i . If any of $u_{i-1}, v_{i-1} \in C_M$, then again we would have three consecutive vertices of G in C , a contradiction. So, let us assume that $u_{i-1}, v_{i-1} \notin C_M$. In order for v_{i-1}, v_{i+1} to be separated, we must have either $w_{i-2} \in C_M$ or $w_{i+2} \in C_M$, where $w_{i-2} \in \{u_{i-2}, v_{i-2}\}$ and $w_{i+2} \in \{u_{i+2}, v_{i+2}\}$. However, we cannot have $w_{i+2} \in C_M$, as otherwise, we would have $v_i, v_{i+1}, v_{i+2} \in C$, the same contradiction as before. Hence, $w_{i-2} \in C_M$. If $w_{i-2} = v_{i-2}$, then $u_{i-2} \notin C_M$. This implies that $v_{i-3} \in C_M$ for C_M to dominate u_{i-2} . This implies that $v_{i-3}, v_{i-2}, v_i, v_{i+1} \in C$, a contradiction by our earlier observation. Moreover, if $w_{i-2} = u_{i-2}$, then $v_{i-2} \notin C$ and hence, $w_{i-3} \in C_M$ for v_{i-2} to be dominated by C_M , where $w_{i-3} \in \{u_{i-3}, v_{i-3}\}$. Here again, we have $v_{i-3}, v_{i-2}, v_i, v_{i+1} \in C$, the same contradiction.

This proves that our assumption of $\gamma_{LD}(M(G)) = \gamma_{LD}(G)$ is wrong which, in turn, proves the theorem for the case that $X = LD$. The other cases when $X \in \{LTD, OLD\}$ follow with similar proof techniques. \square

Note that this lower bound in Theorem 1 is tight:

- for $X \in \{LD, LTD\}$, we have $\gamma_X(C_3) = 2$ as $\{v_1, v_2\}$ is a minimum X -set, and $\gamma_X(M(C_3)) = 3$ as $\{v_1, v_2, u\}$ is a minimum X -set (see Figure 3 for C_3 and $M(C_3)$),
- for $X = OLD$, no tight examples are yet known in this case.

We deduce lower bounds for $\gamma_X(M(P_n))$ and $\gamma_X(M(C_n))$ from the respective values of $\gamma_X(P_n)$ and $\gamma_X(C_n)$. Theorem 1 together with the results from [6] on $\gamma_{LD}(P_n)$ and $\gamma_{LD}(C_n)$ yield:

Corollary 1. *If G equals P_n or C_n for $n \geq 3$, we have as lower bound:*

$$\gamma_{LD}(M(G)) \geq \left\lceil \frac{2n}{5} \right\rceil + 1.$$

The exact *OLD*-numbers of path and cycles are studied in [24] and [3], respectively. However, the latter result for cycles of even order needed to be corrected and as such, we state and prove the result in its entirety as follows.

Theorem 2. *For any cycle C_n on n vertices such that $n \geq 3$ and $n \neq 4$, we have*

$$\gamma_{OLD}(C_n) = \begin{cases} \left\lceil \frac{2n}{3} \right\rceil, & \text{for odd } n, \\ 2 \left\lceil \frac{n}{3} \right\rceil, & \text{for even } n. \end{cases}$$

Proof. We prove the theorem by first showing that $\left\lceil \frac{2n}{3} \right\rceil$ for odd n and $2 \left\lceil \frac{n}{3} \right\rceil$ for even n is a lower bound on $\gamma_{OLD}(C_n)$ and then providing an *OLD*-set of C_n of exactly the same cardinality as the lower bound. We start with establishing the lower bound first.

Seo and Slater showed in [24] that if G is a regular graph on n vertices, of regular-degree r and with no open twins, then we have $\gamma_{OLD}(G) \geq \frac{2}{1+r}n$. Using this result in [24] for the cycle C_n , therefore, we have $\gamma_{OLD}(C_n) \geq \frac{2}{3}n$, that is, $\gamma_{OLD}(C_n) \geq \left\lceil \frac{2}{3}n \right\rceil$. Now, for $n \neq 6k + 4$ for any non-negative integer k , we have

$$\left\lceil \frac{2n}{3} \right\rceil = \begin{cases} \left\lceil \frac{2n}{3} \right\rceil, & \text{for odd } n, \\ 2 \left\lceil \frac{n}{3} \right\rceil, & \text{for even } n. \end{cases}$$

Thus, the only case left to prove is the following claim.

Claim. For $n = 6k + 4$ with $k \geq 1$, we have $\gamma_{OLD}(C_n) \geq 2 \left\lceil \frac{n}{3} \right\rceil = 4k + 4$.

Proof (of Claim). The proof of the last claim is by induction on k with the base case being for $k = 1$, that is, when C_n is a cycle on $n = 10$ vertices. We first show the result for $n = 10$.

Subclaim. $\gamma_{OLD}(C_{10}) \geq 8$.

Proof (of Subclaim). Let $V(C_{10}) = \{v_1, v_2, \dots, v_{10}\}$ and S be a minimum open *OLD*-set of C_{10} . Then, $|S| < 10$ by a characterization result by Foucaud et al. [11] on the extremal graphs G for which $\gamma_{OLD}(G) = |V(G)|$. Hence, there exists a vertex v_1 (without loss of generality) such that $v_1 \notin S$. We consider the induced 5-paths $P_1 : v_1v_2v_3v_4v_5$ and $P_2 : v_1v_{10}v_9v_8v_7$. Then, from the path P_1 , the vertex $v_3 \in S$ for the latter to total-dominate v_2 and the vertex $v_5 \in S$ for the latter to separate the pair v_2, v_4 . By the same argument, from path P_2 , the vertices $v_9, v_7 \in S$. Moreover, at least one vertex from each of the pairs $(v_2, v_4), (v_4, v_6), (v_6, v_8), (v_8, v_{10}), (v_{10}, v_2)$ must belong to S for the latter to total-dominate v_3, v_5, v_7, v_9, v_1 , respectively. Hence, the result follows from counting. \square

Thus, the result holds for the base case of the induction hypothesis. We, therefore, assume $k \geq 2$ and that $\gamma_{OLD}(C_m) \geq 4q + 4$ for all cycles C_m with $|V(C_m)| = 6q + 4$ and $q \in \{1, 2, \dots, k-1\}$. Toward contradiction, let us assume that $\gamma_{OLD}(C_n) < 4k + 4$. Moreover, let $V(C_n) = \{v_1, v_2, \dots, v_n\}$. Then again, by the characterization result in [11], we have $\gamma_{OLD}(C_n) < n$. This implies that, for any minimum *OLD*-set S of C_n , there exists a pair (v_{n-6}, v_{n-5}) (by a possible renaming of vertices) such that $v_{n-6} \in S$ and $v_{n-5} \notin S$. Let C'_{n-6} be the cycle on $n-6$ vertices formed by adding the edge v_1v_{n-6} in the graph $C_n - \{v_{n-5}, v_{n-4}, \dots, v_n\}$. Note that $|V(C'_{n-6})| = 6(k-1) + 4$ and hence, the induction hypothesis applies to it to give

$$\gamma_{OLD}(C'_{n-6}) \geq 4(k-1) + 4 = 4k. \quad (3)$$

Now, let $S' = S - \{v_{n-5}, v_{n-4}, \dots, v_n\}$.

Subclaim. S' is an *OLD*-set of C'_{n-6} .

Proof (of Subclaim). To show that S' , first of all, is a total-dominating set of C'_{n-6} , we notice that the vertices $v_1, v_5 \notin S$. Therefore, all vertices in the set $\{v_2, v_3, \dots, v_{n-6}\}$ remain total-dominated by S' . Moreover, S' total-dominates v_1 by virtue of $v_{n-6} \in S'$. This proves that S' is a total-dominating set of C'_{n-6} .

We now show that S' is also a total-separating set of C'_{n-6} . To that end, since $v_{n-5} \notin S$, if now the vertex $v_n \notin S$ as well, then S' clearly total-separates every pair of vertices in C'_{n-6} and hence, is an *OLD*-set. If however, $v_n \in S$ and total-separates a pair of vertices in C'_{n-6} , the pair can either be (v_1, v_2) or (v_1, v_3) . Since $n \geq 16$, we have $3 < n-6$ and hence, $v_{n-6} \in S'$ total-separates the pairs (v_1, v_2) and (v_1, v_3) in C'_{n-6} . Therefore, S' is an *OLD*-set of C'_{n-6} . \square

Subclaim. $|S \cap \{v_{n-5}, v_{n-4}, \dots, v_n\}| \geq 4$.

Proof (of Subclaim). Since $v_{n-6} \notin S$, it implies that $v_{n-3} \in S$ in order for the latter to total-dominate the vertex v_{n-4} . Moreover, we also have $v_{n-1} \in S$ in order for S to total-separate the pair (v_{n-2}, v_{n-4}) . Furthermore, we must have at least one vertex each from the pairs (v_{n-4}, v_{n-2}) and (v_{n-2}, v_n) belonging to S in

order for the latter to total-dominate the vertices v_{n-3} and v_{n-1} , respectively. Finally, at least one vertex from the pair (v_{n-4}, v_n) must also belong to S in order for S to total-separate the pair (v_{n-3}, v_{n-1}) . This proves that the result holds. \square

Recall that $|S| = \gamma_{OLD}(C_n) < 4k + 4$, by assumption. Thus, we have

$$\gamma_{OLD}(C'_{n-6}) \leq |S'| = |S| - |S \cap \{v_{n-5}, v_{n-4}, \dots, v_n\}| < 4k + 4 - 4 = 4k,$$

a contradiction to the Inequality (3). This proves the claim and establishes the lower bound on $\gamma_{OLD}(C_n)$. \square

The theorem is, therefore, proved by providing an *OLD*-set S of C_n of the exact same cardinality as the lower bound, that is,

$$|S| = \begin{cases} \lceil \frac{2n}{3} \rceil, & \text{for odd } n, \\ 2 \lfloor \frac{n}{3} \rfloor, & \text{for even } n. \end{cases} \quad (4)$$

Let $= V(C_n) = \{v_1, v_2, \dots, v_n\}$ and that $n = 6k + r$, where $r \in \{0, 1, 2, 3, 4, 5\}$. For $k = 0$, that is, C_n being either a 3-cycle or a 5-cycle, it can be checked that the sets $\{v_1, v_2\}$ and $\{v_1, v_2, v_3, v_4\}$ are the respective *OLD*-sets. Thus, the result holds in this case. For the rest of this proof, therefore, we assume that $n \geq 6$, that is, $k \geq 1$. We now construct a vertex subset S of C_n by including in S the vertices

1. $v_{6i-4}, v_{6i-3}, v_{6i-1}, v_{6i}$ for all $i \in \{1, 2, \dots, k\}$ if $r = 0, 3$. In this case, we have $|S| = 4k$ for $r = 0$ and $|S| = 4k + 2$ for $r = 3$.
2. $v_{6i-4}, v_{6i-3}, v_{6i-2}, v_{6i-1}$ for all $i \in \{1, 2, \dots, k\}$ if $r \neq 0$; with
 - (a) the vertices $v_{6k}, v_{6k+1}, \dots, v_{6k+r-1}$ if $r = 1, 2, 4$. In this case, we have $|S| = 4k + r$; and
 - (b) the vertices $v_{6k+1}, v_{6k+2}, v_{6k+3}, v_{6k+4}$ if $r = 5$. In this case, we have $|S| = 4k + 4$.

It can be checked that the constructed set S is, indeed, an *OLD*-set of C_n and of the cardinality as in Equation (4). This proves the result. \square

Combining Theorem 1 with results on $\gamma_{OLD}(P_n)$ in [24] and on $\gamma_{OLD}(C_n)$ in Theorem 2, we deduce:

Corollary 2. *Consider P_n with $n = 6k + r$ for $k \geq 1$ and $r \in \{0, \dots, 5\}$, then we have:*

$$\gamma_{OLD}(M(P_n)) \geq \begin{cases} 4k + r + 1 & \text{if } r \in \{0, \dots, 4\}, \\ 4k + 5 & \text{if } r = 5; \end{cases}$$

and for $n \geq 3$ and $n \neq 4$, we have

$$\gamma_{OLD}(M(C_n)) \geq \begin{cases} \lceil \frac{2n}{3} \rceil + 1, & \text{for odd } n, \\ 2 \lfloor \frac{n}{3} \rfloor + 1, & \text{for even } n. \end{cases}$$

Theorem 1 together with the results from [16] on $\gamma_{LTD}(P_n)$ and from [17] on $\gamma_{LTD}(C_n)$ imply:

Corollary 3. *If G equals P_n or C_n for $n \geq 3$, we have as lower bound:*

$$\gamma_{LTD}(M(G)) \geq \lfloor \frac{n}{2} \rfloor - \lfloor \frac{n}{4} \rfloor + \lceil \frac{n}{4} \rceil + 1.$$

3 Upper bounds on X -numbers of graphs $M(G)$

We first establish a general upper bound on X -numbers of graphs $M(G)$ in terms of $\gamma_X(G)$.

Theorem 3. *Let $X \in \{LD, LTD, OLD\}$. For a graph G admitting an X -set, we have*

$$\gamma_X(M(G)) \leq 2\gamma_X(G).$$

Proof (sketch). Let C be a minimum X -set of G . Then, we construct a new set $C' = C \cup \{u_i : v_i \in C\}$. It can be checked that if C is a dominating (respectively, total-dominating) set of G , then so is it of $M(G)$. For any $x \in V(M(G))$, let $N_M(x)$ (respectively, $N_M[x]$) denote the neighborhood (respectively, closed neighborhood) of x in $M(G)$. If C is a total-separating set of G , then for any $x \in V(M(G))$, we have

$$\begin{aligned} - C' \cap N_M(u) &= \{u_i : v_i \in C\} \\ - C' \cap N_M(u_j) &= (C \cup \{u_i : v_i \in C\}) \cap N_M(u_j) = C \cap N(v_j) \\ - C' \cap N_M(v_j) &= (C \cup \{u_i : v_i \in C\}) \cap N_M(v_j) = \{v_k, u_k : v_k \in N(v_j) \cap C\} \end{aligned}$$

As is evident, the set $C' \cap N_M(x)$ is unique for each $x \in V(M(G))$. Thus, C' is also a total-separating set of $M(G)$. Moreover, $|C'| = 2|C|$. This proves the result. \square

Based on results from [4,15,17] on X -numbers of stars and the relation (1), we can show that the bound given in Theorem 3 is tight for stars (see Fig. 4 for illustration):

Theorem 4. *For stars $K_{1,n}$ with $n \geq 3$, we have $\gamma_X(K_{1,n}) = n$ and*

$$\gamma_X(M(K_{1,n})) = 2n$$

whenever $X \in \{LD, LTD\}$.

Note that stars $K_{1,n}$ have false twins and, therefore, so does $M(K_{1,n})$. Hence, $M(K_{1,n})$ does not admit an OLD -set. Fig. 5 provides an example for $\gamma_X(M(G)) = 2\gamma_X(G)$ when $X = OLD$. For OLD -sets, we can further prove the following.

Theorem 5. *Let G be a graph without isolated vertices and false twins. Then $\gamma_{OLD}(M(G)) \leq \gamma_{OLD}(G) + 2$.*

Proof (sketch). Let $C \subset V$ be an OLD -set of G and let $u_i \in U$. We define $C_i = C \cup \{u, u_i\}$. As G has no isolated vertices, every vertex in $V(M(G))$ is adjacent to a vertex in C_i . This implies that C_i is a total-dominating set of $M(G)$. Moreover, by the fact that C is a total-separating set of G , it can be checked that each of the following sets is unique.

$$\begin{aligned} C_i \cap N_M(u) &= \{u_i\}; \\ C_i \cap N_M(v_j) &= (C \cap N(v_j)) \cup \{u_i\} \quad \text{for } v_j \in N(u_i); \\ C_i \cap N_M(v_j) &= C \cap N(v_j) \quad \text{for } v_j \notin N(u_i); \text{ and} \\ C_i \cap N_M(u_j) &= (C \cap N(v_j)) \cup \{u\} \quad \text{for } u_j \in U. \end{aligned}$$

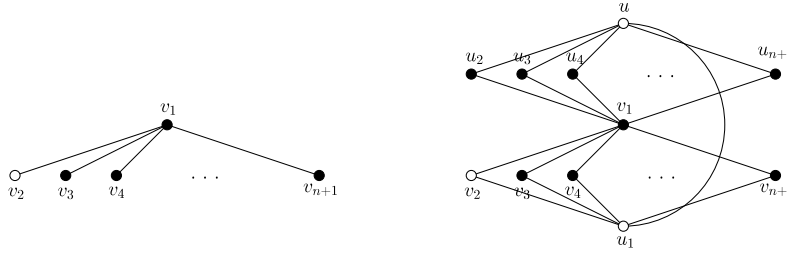


Fig. 4. $K_{1,n}$ and $M(K_{1,n})$ (black vertices form a minimum X -set when $X \in \{LD, LTD\}$)

This proves that C_i is an OLD -set of $M(G)$ and since, $|C_i| = |C| + 2$, the result follows. \square

The bound given in Theorem 5 is tight, as $M(P_2)$ and $M(C_3)$ (in Figure 5) show. In addition, it enables us to prove the following for $\gamma_{OLD}(M(P_n))$ and $\gamma_{OLD}(M(C_n))$.

Theorem 6. For all $n \geq 2$ and $n \neq 3$, we have

$$\gamma_{OLD}(M(P_n)) = \gamma_{OLD}(P_n) + 2$$

and for all $n \geq 3$, we have

$$\gamma_{OLD}(C_n) + 1 \leq \gamma_{OLD}(M(C_n)) \leq \gamma_{OLD}(C_n) + 2.$$

Proof (sketch). The result for cycles follows directly from Theorems 1 and 5. For paths, again using Theorem 5, we only need to show that $\gamma_{OLD}(P_n) \geq \gamma_{OLD}(P_n) + 2$ for all $n \geq 2$ and $n \neq 3$. As far as small paths a concerned, it can be checked that $\gamma_{OLD}(P_2) = 2$, $\gamma_{OLD}(P_4) = \gamma_{OLD}(P_5) = 4$; and $\gamma_{OLD}(M(P_2)) = 4$,

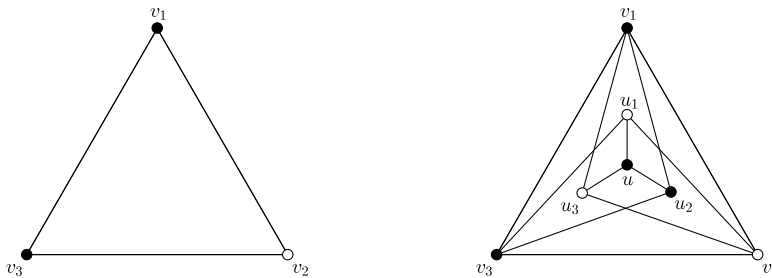


Fig. 5. C_3 and $M(C_3)$ (black vertices form a minimum OLD -set)

$\gamma_{OLD}(M(P_4)) = \gamma_{OLD}(M(P_5)) = 6$. Thus, the result holds for these small paths. Therefore, we assume that $n = 6k + r$ with $k \geq 1$, where $r \in \{0, 1, \dots, 5\}$. If $V(P) = \{v_1, v_2, \dots, v_n\}$, the proof follows by partitioning the vertex set of $M(P_n)$ into $\lceil \frac{n}{6} \rceil$ parts, the first $\lfloor \frac{n}{6} \rfloor$ of which are given by $B_i = \{v_j, u_j : 6i - 5 \leq j \leq 6i\}$ for all $1 \leq i \leq \lfloor \frac{n}{6} \rfloor$; and the last (if exists, that is, if $r \neq 0$) part $B_i = \{v_j, u_j : 6k + 1 \leq j \leq r\}$. Further analysis of any block B_i for $1 \leq i \leq \lfloor \frac{n}{6} \rfloor$ shows that any *OLD*-set C of $M(P_n)$ must contain at least 4 vertices from B_i . Moreover, we would have $|C \cap U| \geq 2$. This gives the total count for the cardinality of C to be $4k + 2$ and thus proves the theorem for $r = 0$. Moreover, each other case for $r \in \{1, 2, 3, 4, 5\}$ is dealt with separately where it can be shown that, for $r \in \{1, 2, 3, 4\}$, exactly r vertices and, for $r = 5$, exactly 4 vertices need to be included in C . This proves the theorem by comparison to the results for $\gamma_{OLD}(P_n)$ in [24]. \square

Concerning *LD*-numbers, we note that $\gamma_{LD}(M(P_2)) = 2$ and $\gamma_{LD}(M(P_3)) = \gamma_{LD}(M(P_4)) = \gamma_{LD}(M(P_5)) = 4$ holds. We can improve the general upper bounds for $\gamma_{LD}(M(P_n))$ and $\gamma_{LD}(M(C_n))$ as follows:

Theorem 7. *Consider P_n with $n = 3k + r$ for $k \geq 2$, $r \in \{0, 1, 2\}$ and C_n with $n \geq 3$, then we have:*

$$\gamma_{LD}(M(P_n)) \leq \begin{cases} 2k + 1 & \text{if } r = 0 \\ 2k + 2 & \text{if } r \in \{1, 2\} \end{cases}$$

and

$$\gamma_{LD}(M(C_n)) \leq \begin{cases} n - \lfloor \frac{n}{3} \rfloor + 1 & \text{if } n \text{ is odd} \\ n - 2 \lfloor \frac{n}{6} \rfloor + 1 & \text{if } n \text{ is even} \end{cases}$$

Proof (sketch). We provide the proof sktech for paths to illustrate the proof technique. The proof for cycles follows with similar techniques. Let $n \geq 6$, $n = 3k + r$ with $k \geq 2$ and $r \in \{0, 1, 2\}$. Then, according to three possible values of r , we define the following sets.

- If $r = 0$, we define $C = \{v_2\} \cup \{v_i, v_{i+1} : i = 3\ell + 1, \ell \in \{1, \dots, k - 1\}\} \cup \{u_{3k}, u\}$. In this case, we have $|C| = 2k + 1$.
- If $r = 1$, we define $C_1 = (C - \{u_{3k}\}) \cup \{v_{3k+1}, u_{3k+1}\}$. Here we have, $|C_1| = 2k + 2$
- If $r = 2$, we define $C_2 = (C - \{u_{6k}\}) \cup \{v_{6k+1}, v_{6k+2}\}$. In this case, we have $|C_2| = 2k + 2$

Further analysis of the above sets C , C_1 and C_2 shows that in each of the above three cases, the sets are *LD*-sets of $M(P_n)$. The result then follows by the cardinalities of the sets in the above three cases. \square

We observe that the upper bounds are tight for $M(P_n)$ with $6 \leq n \leq 8$ and for $M(C_n)$ with $n \in \{3, 6, 7\}$, but are not tight for $M(C_4)$ and $M(C_5)$, for example. There are no examples yet known where the upper bounds are not tight for $M(P_n)$.

The next theorem provides an upper bound for the LTD -numbers of $M(P_n)$ and $M(C_n)$. However, before coming to it, as far as small graphs of these graph classes are concerned, we note that $\gamma_{LTD}(M(P_2)) = 3$, $\gamma_{LTD}(M(P_3)) = 4$ and $\gamma_{LTD}(M(P_4)) = \gamma_{LTD}(M(P_5)) = 5$. The next result improves the general upper bounds for $\gamma_{LTD}(M(P_n))$ and $\gamma_{LTD}(M(C_n))$ as follows.

Theorem 8. *Consider P_n with $n = 6k + r$ for $k \geq 1$, $r \in \{0, \dots, 5\}$ and C_n with $n \geq 3$, then we have:*

$$\gamma_{LTD}(M(P_n)) \leq \begin{cases} 4k + 2 & \text{if } r = 0 \\ 4k + r + 1 & \text{if } r \in \{1, 2, 3\} \\ 4k + r & \text{if } r \in \{4, 5\} \end{cases}$$

and

$$\gamma_{LTD}(M(C_n)) \leq \begin{cases} n - \lfloor \frac{n}{3} \rfloor + 2 & \text{if } n \text{ is odd} \\ n - 2 \lfloor \frac{n}{6} \rfloor + 2 & \text{if } n \text{ is even} \end{cases}$$

Proof (sketch). The upper bound on the LTD -number of $M(P_n)$ follows by the fact that $\gamma_{LTD}(M(P_n)) \leq \gamma_{OLD}(M(P_n)) = \gamma_{OLD}(P_n) + 2$ (by Theorem 6) and by the known exact values of $\gamma_{OLD}(P_n)$ from [24].

For the upper bound on the LTD -number of $M(C_n)$, we consider the following two graphs G and G' . Let $G = (V, E)$ be the graph such that $V = \{v_1, v_2, \dots, v_n\}$ and $E = \{\{v_i, v_{i+2}\}, \{v_i, v_{i+4}\}, \{v_i, v_{i+3}\} : i \in \{1, \dots, n\}\}$ (where the sum of the indices is taken modulo n). Renaming the vertices of V in such a way that $w_i = v_{1+2i}$ for $i \in \{0, \dots, n-1\}$, we consider the second graph G' with vertex set $\{w_i : i \in \{0, \dots, n-1\}\}$ and edge set $\{\{w_i, w_{i+1}\}, \{w_i, w_{i+2}\} : i \in \{0, \dots, n-1\}\}$. We then look at the graph G' with vertex set $V(C_n)$, and denote by $C_{G'}$ its minimum vertex cover. Then, it is easy to check that $C_{G'} \cup \{u_i, u\}$ for some $i \in \{1, \dots, n\}$ is an LTD -set of $M(C_n)$. The theorem for C_n therefore follows by the use of another result proven separately that the size of a minimum vertex cover of G is

- $n - \lfloor \frac{n}{3} \rfloor$ when n odd and not a multiple of 3,
- $n - \lfloor \frac{n}{4} \rfloor$ when n is odd and multiple of 3,
- $n - 2 \lfloor \frac{n}{6} \rfloor$ when n is even.

□

We observe that for $\gamma_{LTD}(M(C_n))$, there are values of n where the upper bound is attained (for example, $n \in \{6, 9\}$), but also where this is not the case (for example, $n \in \{3, 4, 5, 7, 8\}$). There are no examples yet known where the upper bounds are not tight for $M(P_n)$.

4 Concluding remarks

To summarize, we studied three location-domination type problems under the Mycielski construction. In Section 2, we showed that $\gamma_X(G) + 1$ is a general lower

bound of $\gamma_X(M(G))$ for all paths and cycles and all $X \in \{LD, LTD, OLD\}$. Using results on $\gamma_X(P_n)$ (respectively, on $\gamma_X(C_n)$) from [3,6,10,16,24], this allowed us to deduce the appropriate lower bounds on the X -numbers of $M(P_n)$ (respectively, of $M(C_n)$) for $X \in \{LD, LTD, OLD\}$. As a related extension of one of the main focuses of this paper, namely, the OLD -numbers of $M(C_n)$, we also establish the exact OLD -numbers for cycles.

In Section 3, we firstly provided two general upper bounds on X -numbers of the graphs $M(G)$. We showed that the upper bound of $2\gamma_X(G)$ is attained when G is a star for $X \in \{LD, LTD\}$. For OLD -numbers of $M(G)$, we could further establish the general upper bound of $\gamma_{OLD}(G) + 2$. We showed that this bound is attained for $\gamma_{OLD}(M(P_n))$ and, combining our results on the lower and upper bound of the OLD -numbers, we obtained a Vizing-type result for $M(C_n)$, namely, $\gamma_{OLD}(C_n) + 1 \leq \gamma_{OLD}(M(C_n)) \leq \gamma_{OLD}(C_n) + 2$. For the other X -problems with $X \in \{LD, LTD\}$, we could improve the general upper bounds for the X -numbers of both $M(P_n)$ and $M(C_n)$.

For the studied X -numbers, there are examples where the upper bounds are attained (and, therefore, cannot be improved any further). On the other hand, there are also examples where the upper bounds are not tight. Therefore, our future research includes finding these exact values. In view of the fact that lower bounds were obtained by considering the domination aspect only, we expect that the true values are closer to the upper bounds. This applies particularly to $\gamma_{LD}(M(P_n))$ and to $\gamma_{LTD}(M(P_n))$ where no examples are yet known where the upper bound is not tight.

Moreover, it would be interesting to study similar questions for other locating-dominating type problems, for example, differentiating total-dominating sets (defined as total-dominating sets that separate all vertices of the graph).

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