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# Skew Cyclic and Skew Constacyclic Codes over a mixed alphabet

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**Abstract** In this note, we study skew cyclic and skew constacyclic codes over the mixed alphabet $\mathcal{R} = \mathbb{F}_q\mathcal{R}_1\mathcal{R}_2$ where $\mathcal{R}_1 = \mathbb{F}_q + u\mathbb{F}_q$ with $u^2 = u$ and $\mathcal{R}_2 = \mathbb{F}_q + u\mathbb{F}_q + v\mathbb{F}_q$ with $u^2 = u, v^2 = v, uv = vu = 0$. Such codes consist of the juxtaposition of three codes of the same size over respectively $\mathbb{F}_q$, $\mathcal{R}_1$, and $\mathcal{R}_2$.

We investigate the generator polynomial for skew cyclic codes over $\mathcal{R}$. Furthermore, we discuss the structural properties of the skew cyclic and skew constacyclic codes over $\mathcal{R}$. We also study their $q$-ary images under suitable Gray maps.

**Keywords** Linear codes · Skew-cyclic codes · Gray map · Skew Constacylic codes

**Mathematics Subject Classification (2000)** 94B05 · 94B15 · 94B35 · 94B60

## 1 Introduction

Constacyclic codes constitute a remarkable generalization of cyclic codes and form an important class of linear codes in the coding theory. They were introduced by Berlekamp to construct the analogue of BCH codes in the Lee metric [9].

In 2007, Boucher et al. [14] was the first one to study cyclic codes using a non commutative ring over the skew polynomials ring $\mathbb{F}[x;\theta]$ where $\theta$ denotes the automorphism of the finite field $\mathbb{F}$ and they produced many numerical examples which improved the tables of best known codes. The advantage of skew polynomials ring is that the polynomial $x^n - 1$ has more factors in skew polynomials ring than in commutative rings. Later, in [15] Boucher et al. generalised this idea to skew constacyclic codes. In 2011, Siap et al. [34] studied skew cyclic codes of arbitrary length, and established a strong connection

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with classical cyclic and quasi-cyclic codes. In 2012, Abualrub et al. [1] studied $\theta$-cyclic codes over the non-chain ring $\mathbb{F}_2 + u\mathbb{F}_2, v^2 = v$ with respect to Euclidean and Hermitian inner products, and Jitman et al. [27] studied skew constacyclic codes over finite chain rings, and gave the generators of Euclidean and Hermitian dual codes. Later, these codes over non-chain rings are extensively studied. For instance, the rings $\mathbb{F}_3 + v\mathbb{F}_3$ in [3], $\mathbb{F}_q + v\mathbb{F}_q, v^2 = v$ in [21], $\mathbb{F}_q + u\mathbb{F}_q + v\mathbb{F}_q, u^2 = u, v^2 = v, uv = vu = 0$ in [4] are considered as alphabet for skew cyclic codes. Also, Yao et al. [33] and Dertli & Cengellenmis [17] studied these codes over $\mathbb{F}_q + u\mathbb{F}_q + v\mathbb{F}_q + uv\mathbb{F}_q, u^2 = u, v^2 = v, uv = vu$. In 2017, Gao et al. [20] obtained the structure of skew constacyclic codes over the non-chain ring $\mathbb{F}_q + v\mathbb{F}_q, v^2 = v$ and they obtained skew $(-1 + 2v)$- constacyclic codes. Islam and Prakash have determined the structural properties of skew constacyclic codes over $\mathbb{F}_q + u\mathbb{F}_q + v\mathbb{F}_q + uv\mathbb{F}_q, u^2 = u, v^2 = v, uv = vu$ in [24], and $\mathbb{F}_q + u\mathbb{F}_q + v\mathbb{F}_q, u^2 = u, v^2 = v, uv = vu = 0$ in [25]. In 2019, Bhardwaj and Raka [12] studied the skew constacyclic codes over the ring $\mathbb{F}_q[u, v]\langle f(u), g(v), uv - vu\rangle$ by using two non trivial automorphisms.

In another tack, Borges et al. [10] introduced $\mathbb{Z}_2\mathbb{Z}_4$-linear codes that is to say codes over the **mixed alphabet** $\mathbb{Z}_2\mathbb{Z}_4$, where some coordinates are binary and the rest is quaternary. They have obtained their dual codes as well as their generator matrices. In continuation, Fernandez-Cordoba et al. [19] determined the rank and kernel of $\mathbb{Z}_2\mathbb{Z}_4$-linear codes. These codes found an engineering application in the area of steganography [29]. Additive codes over the mixed alphabet $\mathbb{Z}_2\mathbb{Z}_{2^s}$ were considered in [5]. Then, the mixed alphabet $\mathbb{Z}_p\mathbb{Z}_{p^s}$, and, more generally $\mathbb{Z}_{p^r}\mathbb{Z}_{p^s}$ was studied in [7,31,35,36]. On the other hand, to the progress of cyclic codes on mixed alphabets, in 2014, Abualrub et al. [2] defined $\mathbb{Z}_2\mathbb{Z}_4$-additive cyclic codes as $\mathbb{Z}_4[x]$-submodule of $\mathbb{Z}_2[x]/\langle x^r - 1\rangle \times \mathbb{Z}_4[x]/\langle x^s - 1\rangle$ and derived the unique set of generators, and minimal spanning set for these codes where $s$ is an odd integer. Also, Borges et al. [11] found generator polynomials and duals for $\mathbb{Z}_2\mathbb{Z}_4$-additive cyclic codes. After introducing the new mixed alphabets $\mathbb{Z}_2\mathbb{Z}_2[u]$-additive codes, where $u^2 = 0$ in [6], Aydogdu et al. [8] were also investigated constacyclic codes over mixed alphabets by defining them as $\mathbb{Z}_2[u][x]$-submodules of $\mathbb{Z}_2[x]/\langle x^\alpha - 1\rangle \times \mathbb{Z}_2[u][x]/\langle x^\beta - (1 + u)\rangle$. They obtained some optimal binary linear codes as the Gray images of $\mathbb{Z}_2\mathbb{Z}_2[u]$-cyclic codes. Meanwhile, [32] studied the algebraic properties of $\mathbb{Z}_2\mathbb{Z}_2[u]$-additive cyclic and constacyclic codes with the unit $1 + u$, respectively. Therefore, in continuation of these studies, the expected generalization should be $\mathbb{Z}_{2^r}\mathbb{Z}_{2^s}[u]$-additive cyclic and constacyclic codes, where $u^2 = 0$.

Motivated by the two kinds of above studies, here we consider a mixed alphabet $\mathcal{R} = \mathbb{F}_q\mathcal{R}_1\mathcal{R}_2$ where $\mathcal{R}_1 = \mathbb{F}_q + u\mathbb{F}_q$ with $u^2 = u$ and $\mathcal{R}_2 = \mathbb{F}_q + u\mathbb{F}_q + v\mathbb{F}_q$ with $u^2 = u, v^2 = v, uv = vu = 0$ and study $\theta_t$-cyclic and $(\theta_t, \alpha)$-cyclic codes over $\mathcal{R}$. We completely determine the algebraic structure of these codes. We study their $q$-ary images under Gray maps, and give some numerical examples in short lengths.

The material is organized as follows. The next section collects some background material. Section 3 studies Gray maps. Section 4 is dedicated to skew cyclic codes, and Section 5 to skew constacyclic codes. Section 6 concludes the article.

## 2 Preliminaries

Let $\mathbb{F}_q$ denote the finite field of characteristic $p$ with $q$ elements, where $q = p^m$ for odd prime $p$ and positive integer $m$. The set $\mathbb{F}_q^n$ of all ordered $n$-tuples over $\mathbb{F}_q$ forms a vector space with the usual component-wise addition and scalar multiplication of vectors. A non-empty subset $\mathcal{C}$ of $\mathbb{F}_q^n$ is called a code of length $n$ over $\mathbb{F}_q$ and it is called a linear code if $\mathcal{C}$ is a subspace of $\mathbb{F}_q^n$. From now onward, we denote $\mathcal{R}_1 = \mathbb{F}_q + u\mathbb{F}_q$, with $u^2 = u$ and $\mathcal{R}_2 = \mathbb{F}_q + u\mathbb{F}_q + v\mathbb{F}_q$, with $u^2 = u, v^2 = v, uv = vu = 0$, where $q = p^m$ for odd prime $p$ and positive integer $m$.

Note that $\mathcal{R}_1$ and $\mathcal{R}_2$ are finite non-chain rings. Let $a + ub + vc$ be an element of $\mathcal{R}_2$. Then we define two

maps $\eta$ and $\delta$ as follows:

$$\eta : \mathcal{R}_2 \to \mathbb{F}_q, \ \ \delta : \mathcal{R}_2 \to \mathcal{R}_1,$$
$$\eta(a + ub + vc) = a, \ \ \delta(a + ub + vc) = a + ub,$$

It is clear that $\eta$ and $\delta$ are ring homomorphisms. We consider the ring $\mathcal{R}$,

$$\mathcal{R} = \mathbb{F}_q\mathcal{R}_1\mathcal{R}_2 = \{(x, y, z) \mid x \in \mathbb{F}_q, y \in \mathcal{R}_1 \text{ and } z \in \mathcal{R}_2\}$$

We define a $\mathcal{R}_2$-multiplication in this ring as follows,

$$\star : \mathcal{R}_2 \times \mathcal{R} \to \mathcal{R}$$
$$r \star (x, y, z) = (\eta(r)x, \delta(r)y, rz)$$

This is a well defined multiplication and it can be extended componentwise to $\mathcal{R}_\gamma = \mathbb{F}_q^{n_1} \times \mathcal{R}_1^{n_2} \times \mathcal{R}_2^{n_3}$ by

$$\star : \mathcal{R}_2 \times \mathcal{R}_\gamma \to \mathcal{R}_\gamma$$
$$r \star (x_1, \cdots, x_{n_1}, y_1, \cdots, y_{n_2}, z_1, \cdots, z_{n_3}) = (\eta(r)x_1, \cdots, \eta(r)x_{n_1}, \delta(r)y_1, \cdots, \delta(r)y_{n_2}, rz_1, \cdots, rz_{n_3})$$

where $(x_1, \cdots, x_{n_1}, y_1, \cdots, y_{n_2}, z_1, \cdots, z_{n_3}) \in \mathcal{R}_\gamma$. By this multiplication, $\mathcal{R}_\gamma$ is a $\mathcal{R}_2$ module. A non-empty subset $\mathcal{C}$ of $\mathcal{R}_\gamma$ is said to be a $\mathcal{R}$-linear code of length $(n_1, n_2, n_3)$ if $\mathcal{C}$ is an $\mathcal{R}_2$-submodule of $\mathcal{R}_\gamma$. Now we define the inner product as follows,

$$\langle c, c' \rangle = \sum_1^{n_1} x_i x_i' + \sum_1^{n_2} y_j y_j' + \sum_1^{n_3} z_k z_k'$$

where $c = (x_1, \cdots, x_{n_1}, y_1, \cdots, y_{n_2}, z_1, \cdots, z_{n_3})$, $c' = (x_1', \cdots, x_{n_1}', y_1', \cdots, y_{n_2}', z_1', \cdots, z_{n_3}')$ are in $\mathcal{R}_\gamma$. Let $\mathcal{C}$ be a $\mathcal{R}$-linear code of length $(n_1, n_2, n_3)$. Then the dual code of $\mathcal{C}$ is defined as,

$$\mathcal{C}^\perp = \{c' \in \mathcal{R}_\gamma \mid \langle c, c' \rangle = 0 \ \forall \ c \in \mathcal{C}\}$$

## 3 Decomposition and Properties of Gray Maps

Recall that, $\mathcal{R}_1 = \mathbb{F}_q + u\mathbb{F}_q$, with $u^2 = u$. Consider the idempotent orthogonal elements $e_1 = u$ and $e_2 = 1 - u$, then

$$\mathcal{R}_1 = e_1\mathcal{R}_1 \oplus e_2\mathcal{R}_1 \cong e_1\mathbb{F}_q \oplus e_2\mathbb{F}_q$$

$\because e_1e_2 = 0, e_1^2 = e_1, e_1 + e_2 = 1$. Hence, $\mathcal{R}_1 = \{ae_1 + be_2 \mid a, b \text{ in } \mathbb{F}_q\}$. We now define the Gray map,

$$\varphi_1 : \mathcal{R}_1 \to \mathbb{F}_q^2$$
$$\varphi_1(ae_1 + be_2) = (a, b)$$

It can be extended to the length $n$ by

$$\varphi_1 : \mathcal{R}_1^n \to \mathbb{F}_q^{2n}$$
$$\varphi_1((a_1, \cdots, a_n)e_1 + (b_1, \cdots, b_n)e_2) = (a_1, \cdots, a_n, b_1, \cdots, b_n)$$

Note that it is a linear map. We define the Gray weight of a codeword in $\mathcal{R}_1$ as

$$wt_G(ae_1 + be_2) = wt_H(a, b)$$

where $wt_H$ denotes the Hamming weight. If $x, y$ lie in $\mathcal{R}_1^n$, then their mutual distance is given by

$$d_G(x,y) = \sum_1^n wt_G(x_i - y_i) = \sum_1^{2n} wt_H(\varphi_1(x) - \varphi_1(y)) = d_H(\varphi_1(x), \varphi_1(y)).$$

Hence $\varphi_1$ is a weight preserving map. A non-empty subset $\mathcal{C}$ of $\mathcal{R}_i^n$ is said to be a linear code of length $n$ if $\mathcal{C}$ is $\mathcal{R}_i$ submodule of $\mathcal{R}_i^n$.

For $i \in \{1,2\}, A_i \subseteq \mathcal{R}_1$

$$A_1 \oplus A_2 = \{a_1 + a_2 \mid a_i \in A_i\} \text{ and } A_1 \otimes A_2 = \{(a_1, a_2) \mid a_i \in A_i\}.$$

Let $\mathcal{C}_e$ be a linear code of length $n$ over $\mathcal{R}_1$. Then we define,

$$\mathcal{C}_{e_1} = \{y_1 \in \mathbb{F}_q^n \mid e_1 y_1 + e_2 y_2 \in \mathcal{C}_e, \text{ for some } y_2 \in \mathbb{F}_q^n\}$$

$$\mathcal{C}_{e_2} = \{y_2 \in \mathbb{F}_q^n \mid e_1 y_1 + e_2 y_2 \in \mathcal{C}_e, \text{ for some } y_1 \in \mathbb{F}_q^n\}$$

Therefore any linear code $\mathcal{C}_e$ over $\mathcal{R}_1$ can be represented as $\mathcal{C}_e = e_1\mathcal{C}_{e_1} \oplus e_2\mathcal{C}_{e_2}$ and $\varphi_1(\mathcal{C}_e) = \mathcal{C}_{e_1} \otimes \mathcal{C}_{e_2}$. Hence $\mathcal{C}_{e_1}$ and $\mathcal{C}_{e_2}$ are $\mathbb{F}_q$-linear codes. Also note that $\varphi_1(\mathcal{C}_e^\perp) = \varphi_1(\mathcal{C}_e)^\perp$.

Recall that, $\mathcal{R}_2 = \mathbb{F}_q + u\mathbb{F}_q + v\mathbb{F}_q$, with $u^2 = u, v^2 = v, uv = vu = 0$. Let $o_1 = (1 - u - v), o_2 = u, o_3 = v$ be idempotent orthogonal elements in $\mathcal{R}_2$, then

$$\mathcal{R}_2 = o_1\mathcal{R}_2 \oplus o_2\mathcal{R}_2 \oplus o_3\mathcal{R}_2 \cong o_1\mathbb{F}_q \oplus o_2\mathbb{F}_q \oplus o_2\mathbb{F}_q$$

$\because o_i o_j = 0\ (i \neq j), o_i^2 = o_i,\ o_1 + o_2 + o_3 = 1$. Hence any element in $\mathcal{R}_2$ can be written as $ao_1 + bo_2 + o_3c_3$. We now define a weight preserving linear Gray map $\varphi_2$,

$$\varphi_2 : \mathcal{R}_2 \to \mathbb{F}_q^3$$
$$\varphi_2(ao_1 + bo_2 + co_3) = (a, b, c)$$

It can be extended to length $n$ by the formula

$$\varphi_2((a_1, \cdots, a_n)o_1 + (b_1, \cdots, b_n)o_2 + (c_1, \cdots, c_n)o_2) = (a_1, \cdots, a_n, b_1, \cdots, b_n, c_1, \cdots, c_n).$$

We define the Gray weight of a codeword in $\mathcal{R}_2$ as

$$wt_G(ao_1 + bo_2 + co_3) = wt_H(a, b, c)$$

where $wt_H$ denotes the Hamming weight. If $x, y$ are in $\mathcal{R}_2^n$, then their Gray distance is given by

$$d_G(x,y) = \sum_1^n wt_G(x_i - y_i) = \sum_1^{3n} wt_H(\varphi_2(x) - \varphi_2(y)) = d_H(\varphi_2(x), \varphi_2(y)).$$

For $i \in \{1,2,3\}, A_i \subseteq \mathcal{R}_2$

$$A_1 \oplus A_2 \oplus A_3 = \{a_1 + a_2 + a_3 \mid a_i \in A_i\} \text{ and } A_1 \otimes A_2 \otimes A_3 = \{(a_1, a_2, a_3) \mid a_i \in A_i\}.$$

Let $\mathcal{C}_o$ be a linear code of length $n$ over $\mathcal{R}_2$. We define,

$$\mathcal{C}_{o_1} = \{z_1 \in \mathbb{F}_q^n \mid o_1 z_1 + o_2 z_2 + o_3 z_3 \in \mathcal{C}_o, \text{ for some } z_2, z_3 \in \mathbb{F}_q^n\}$$
$$\mathcal{C}_{o_2} = \{z_2 \in \mathbb{F}_q^n \mid o_1 z_1 + o_2 z_2 + o_3 z_3 \in \mathcal{C}_o, \text{ for some } z_1, z_3 \in \mathbb{F}_q^n\}$$
$$\mathcal{C}_{o_3} = \{z_3 \in \mathbb{F}_q^n \mid o_1 z_1 + o_2 z_2 + o_3 z_3 \in \mathcal{C}_o, \text{ for some } z_1, z_2 \in \mathbb{F}_q^n\}$$

Then any linear code $\mathcal{C}_o$ over $\mathcal{R}_2$ can be represented as $\mathcal{C}_o = o_1\mathcal{C}_{o_1} \oplus o_2\mathcal{C}_{o_2} \oplus o_3\mathcal{C}_{o_3}$ and $\varphi_2(\mathcal{C}_o) = \mathcal{C}_{o_1} \otimes \mathcal{C}_{o_2} \otimes \mathcal{C}_{o_3}$, where $\mathcal{C}_{o_1}, \mathcal{C}_{o_2}$ and $\mathcal{C}_{o_3}$ are $\mathbb{F}_q$-linear codes. Also note that $\varphi_2(\mathcal{C}_o^{\perp}) = \varphi_2(\mathcal{C}_o)^{\perp}$.
Henceforth, we define the Gray map $\varphi$ on $\mathcal{R}$ using the maps defined previously,

$$\begin{aligned} \varphi : \mathcal{R} &\to \mathbb{F}_q^6 \\ \varphi(x, y, z) &= (x, \varphi_1(y), \varphi_2(z)) \end{aligned}$$

now we can extend this map to $\mathcal{R}_\gamma$,

$$\varphi(x_1, \cdots, x_{n_1}, y_1, \cdots, y_{n_2}, z_1, \cdots, z_{n_3}) = (x_1, \cdots, x_{n_1}, \varphi_1(y_1), \cdots, \varphi_1(y_{n_2}), \varphi_2(z_1), \cdots, \varphi_2(z_{n_3}))$$

then the Gray weight of an element in $\mathcal{R}_\gamma$ can be denoted by $wt_G(\alpha) = wt_H(\varphi(\alpha))$ Any linear code $\mathcal{C}$ of $\mathcal{R}_\gamma$ can be represented by $\mathcal{C} = \mathcal{C}_1 \otimes \mathcal{C}_e \otimes \mathcal{C}_o$, where $\mathcal{C}_1, \mathcal{C}_e$ and $\mathcal{C}_o$ are linear code over $\mathbb{F}_q, \mathcal{R}_1$ and $\mathcal{R}_2$. Let $G_{\mathbb{F}_q}$ be the generator matrix for linear code over $\mathbb{F}_q$. The generator matrix $G_{\mathcal{R}_1}$ for a linear code over $\mathcal{R}_1$ is denoted by

$$G_{\mathcal{R}_1} = \begin{bmatrix} e_1 G_{e_1} \\ e_2 G_{e_2} \end{bmatrix}$$

where $G_{e_i}$ is the generator matrix for the linear code $\mathcal{C}_{ei}$, for $i = \{1, 2\}$. The generator matrix $G_{\mathcal{R}_2}$ for the linear code over $\mathcal{R}_2$ is

$$G_{\mathcal{R}_2} = \begin{bmatrix} o_1 G_{o_1} \\ o_2 G_{o_2} \\ o_3 G_{o_3} \end{bmatrix}$$

where $G_{o_i}$ is the generator matrix for the linear code $\mathcal{C}_{oi}$, for $i = \{1, 2, 3\}$. Using the generator matrices above, we can say that the generator matrix $G$ for the linear code over $\mathcal{R}$ is,

$$G = \begin{bmatrix} G_{\mathbb{F}_q} & 0 & 0 \\ 0 & G_{\mathcal{R}_1} & 0 \\ 0 & 0 & G_{\mathcal{R}_2} \end{bmatrix}$$

Note that the minimum distance of $\mathcal{C}$ is $\min\{d_H(\mathcal{C}_1), d_H(\varphi_1(\mathcal{C}_e)), d_H(\varphi_2(\mathcal{C}_o))\}$. The following Theorem provides the weight preserving nature of the Gray map,

**Theorem 1** *The Gray map $\varphi$ defined above is linear and weight preserving.*

*Proof* Let $x = (x_1, x_2, x_3), x' = (x_1', x_2', x_3')$ be in $\mathcal{R}_\gamma$ where $x_1, x_1' \in \mathbb{F}_q^{n_1}, x_2, x_2' \in \mathcal{R}_1^{n_2}, x_3, x_3' \in \mathcal{R}_2^{n_3}$

$$\begin{aligned} \varphi(x + x') &= \varphi(x_1 + x_1', x_2 + x_2', x_3 + x_3') \\ &= (x_1 + x_1', \varphi_1(x_2 + x_2'), \varphi_2(x_3 + x_3')) \\ &= (x_1, \varphi_1(x_2), \varphi_2(x_3)) + (x_1', \varphi_1(x_2'), \varphi_2(x_3'))(\because \varphi_1 \text{ and } \varphi_2 \text{ are linear}) \\ &= \varphi(x) + \varphi(x') \end{aligned}$$

Using the linear map $\varphi$,

$$d_G(x, x') = wt_G(x - x') = wt_H(\varphi(x) - \varphi(x')) = d_H(\varphi(x), \varphi(x'))$$

Hence $\varphi$ is a weight preserving linear map.

The following theorem gives the parameters of the Gray image of a linear code.

**Theorem 2** *Let $\mathcal{C} \subseteq \mathcal{R}_\gamma$ be $(n_1+n_2+n_3, d_G)$ linear code then $\varphi(\mathcal{C})$ is $(n_1+2(n_2)+3(n_3), d_H)$ linear code over $\mathbb{F}_q$, where $d_G = d_H$.*

*Proof* The proof can extended from the proof of Theorem 1.

The following Theorem characterizes $\varphi(\mathcal{C})$,

**Theorem 3** *Let $\mathcal{C} \subseteq \mathcal{R}_\gamma$ be linear, then $\varphi(\mathcal{C}) = \mathcal{C}_1 \otimes_{i=1}^{i=2} \mathcal{C}_{e_i} \otimes_{j=1}^{j=3} \mathcal{C}_{o_j}, \mid \mathcal{C} \mid = \mid \mathcal{C}_1 \mid \prod_{i=1}^{i=2} \mid \mathcal{C}_{e_i} \mid \prod_{j=1}^{j=3} \mid \mathcal{C}_{o_j} \mid$.*

*Proof* Let

$$\varphi(x,y,z) = (x, \varphi_1(y), \varphi_3(z)) = (a_1, \ldots, a_6) \in \varphi(\mathcal{C}) \subseteq \mathbb{F}_q^6.$$

Note that $\varphi$ is bijective and $\mathcal{C} = \mathcal{C}_1 \otimes \mathcal{C}_e \otimes \mathcal{C}_o$ is linear. So $a_1 = x \in \mathcal{C}_1$. Also note that $\varphi_1(y) = (a_2, a_3), \varphi_2(z) = (a_4, a_5, a_6)$. Since $\varphi_i's$ are bijective, $a_2e_1 + a_3e_2 \in e_1\mathcal{C}_{e_1} \oplus e_2\mathcal{C}_{e_2} = \mathcal{C}_e$. Hence $(a_2, a_3) \in \mathcal{C}_{e_1} \otimes \mathcal{C}_{e_2}$ and similarly $(a_4, a_5, a_6) \in \mathcal{C}_{o_1} \otimes \mathcal{C}_{o_2} \otimes \mathcal{C}_{o_2}$. The converse holds in a similar way. The second part of the statement follows from the fact that $\varphi$ is bijective.

The following Theorem furnishes the decompostion of the dual of the linear code $\mathcal{C}$.

**Theorem 4** *Let $\mathcal{C} = \mathcal{C}_1 \otimes \mathcal{C}_e \otimes \mathcal{C}_o$ be a linear code over $\mathcal{R}$ then $\mathcal{C}^\perp = \mathcal{C}_1^\perp \otimes \mathcal{C}_e^\perp \otimes \mathcal{C}_o^\perp$ where $\mathcal{C}_1^\perp, \mathcal{C}_e^\perp$ and $\mathcal{C}_o^\perp$ are duals for the respective linear codes.*

*Proof* Let $\mathcal{C}^\perp = \{c' \in \mathcal{R}_\gamma | \langle c, c' \rangle = 0 \text{ for all } c \in \mathcal{C}\} = \{(x', y', z') \in \mathcal{R}_\gamma | x' \in \mathbb{F}_q^{n_1}, y' \in \mathcal{R}_1^{n_2}, z' \in \mathcal{R}_2^{n_3}\}$. Let $c = (x, y, z) \in \mathcal{C} = \mathcal{C}_1 \otimes \mathcal{C}_e \otimes \mathcal{C}_o$. Then

$$\langle c, c' \rangle = xx' + yy' + zz' = 0.$$

So $x' \in \mathcal{C}_1^\perp, y' \in \mathcal{C}_e^\perp, z' \in \mathcal{C}_o^\perp$ and so $\mathcal{C}^\perp \subseteq \mathcal{C}_1^\perp \otimes \mathcal{C}_e^\perp \otimes \mathcal{C}_o^\perp$. Since $|\mathcal{C}^\perp| = |\mathcal{C}_1^\perp||\mathcal{C}_e^\perp||\mathcal{C}_o^\perp|$, the statement holds.

The next result shows that the Gray maps is compatible with duality.

**Theorem 5** *If $\mathcal{C} \subseteq \mathcal{R}_\gamma$ is linear, then $\varphi(\mathcal{C}^\perp) = \varphi(\mathcal{C})^\perp$.*

*Proof* Let $(x, y, z) \in \mathcal{C}$ and $(x', y', z') \in \mathcal{C}^\perp$ where $x \in \mathcal{C}_1, y \in \mathcal{C}_e, z \in \mathcal{C}_o$ and $x' \in \mathcal{C}_1^\perp, y' \in \mathcal{C}_e^\perp, z' \in \mathcal{C}_o^\perp$, then $\langle (x|y|z, x'|y'|z') \rangle = 0$. Using Theorem 4, $\mathcal{C}_1^\perp, \mathcal{C}_e^\perp$ and $\mathcal{C}_o^\perp$ are duals for $\mathcal{C}_1, \mathcal{C}_e$ and $\mathcal{C}_o$. Now, we have $\varphi(x, y, z) = (x, \varphi_1(y), \varphi_2(z)), \varphi(x', y', z') = (x', \varphi_1(y'), \varphi_2(z'))$, then the inner product is given by

$$\begin{aligned}
\langle \varphi(x,y,z), \varphi(x',y',z') \rangle &= \langle (x, \varphi_1(y), \varphi_2(z)), (x', \varphi_1(y'), \varphi_2(z')) \rangle \\
&= \langle (x,0,0), (x',0,0) \rangle + \langle (0, \varphi_1(y), 0), (0, \varphi_1(y'), 0) \rangle \\
&\quad + \langle (0,0,\varphi_2(z)), (0,0,\varphi_2(z')) \rangle \\
&= 0 \quad (\because \varphi_1(C^\perp) = \varphi_1(C)^\perp, \varphi_2(C^\perp) = \varphi_2(C)^\perp)
\end{aligned}$$

So, $\varphi(\mathcal{C}^\perp) \subseteq \varphi(\mathcal{C})^\perp$. Since the cardinality is same, the statement holds.

The following result provides the self duality nature of the linear code and its Gray image,

**Corollary 1** *Let $\mathcal{C} \subseteq \mathcal{R}_\gamma$ be linear, then $\mathcal{C}$ is self-dual iff $\varphi(C)$ is self-dual. Further $\varphi(C)$ is a self-orthogonal over $\mathbb{F}_q$ iff $C$ is self-orthogonal.*

*Proof* Let $\mathcal{C}$ be a self-dual linear code of length $n$ over $\mathcal{R}$. That is $\mathcal{C} = \mathcal{C}^\perp$. Then $\varphi(C) = \varphi(C^\perp)$, and hence by Theorem 5, we have $\varphi(\mathcal{C}) = (\varphi(\mathcal{C}))^\perp$. Thus $\varphi(\mathcal{C})$ is a self-dual linear code of length $n_1+2n_2+3n_3$ over $\mathbb{F}_q$. Conversely, let $\varphi(\mathcal{C})$ be a self-dual linear code of length $n_1+2n_2+3n_3$ over $\mathbb{F}_q$. Then $\varphi(C) = (\varphi(\mathcal{C}))^\perp$, and hence by Theorem 5, we have $\varphi(\mathcal{C}) = \varphi(\mathcal{C}^\perp)$. Since $\varphi$ is bijection, $\mathcal{C} = \mathcal{C}^\perp$. Therefore, $\mathcal{C}$ is a self-dual linear code over $\mathcal{R}_\gamma$. Similarly the self orthogonal case holds.

## 4 Skew Cyclic $\mathcal{R}$-codes

Let $\theta_t$ be a non-trivial Frobenius automorphism defined by

$$\theta_t : \mathbb{F}_q \to \mathbb{F}_q, \theta_t(a) = a^{p^t},$$

where $t$ divides $m$. It can be extended to $\mathcal{R}_1$ and $\mathcal{R}_2$ by

$$\theta_t(a+ub) = \theta_t(a) + u\theta_t(b), \theta_t(a+ub+vc) = \theta_t(a) + u\theta_t(b) + v\theta_t(b).$$

Since $t|m$, the order of automorphism $\theta_t$ is $\frac{m}{t}$. We define a polynomial ring $\mathcal{R}_i[x,\theta_t]$ $(1 \leq i \leq 2)$ as follows,

$$\mathcal{R}_i[x,\theta_t] = \{a_1 + \cdots + a_n x^n | a_j \in \mathcal{R}_i, 1 \leq j \leq n\}$$

Clearly $\mathcal{R}_i[x,\theta_t]$ is a ring with respect to usual addition and the multiplication defined by

$$ax^m bx^n = a\theta_t^m(a)x^{m+n}$$

Note that it is a non commutative ring unless $\theta_t$ is an identity map. A non empty set $\mathcal{C}$ is said to be linear code of length $n_i$ over $\mathcal{R}_i$ if it is a $\mathcal{R}_i$ submodule of $\mathcal{R}_i^{n_i}$. Using the above polynomial rings above, we extend the polynomial ring to $\mathcal{R}$ by

$$\mathcal{R}[x,\theta_t] = \{(a(x), b(x), c(x)) : a(x) \in \mathbb{F}_q[x], b(x) \in \mathcal{R}_1[x], c(x) \in \mathcal{R}_2[x]\}.$$

It can be seen that $\mathcal{R}[x,\theta_t]$ is a $\mathcal{R}_2[x;\theta_t]$ submodule with respect to usual addition and multiplication defined by

$$\begin{aligned} \star : \mathcal{R}_2[x] \times \mathcal{R}[x,\theta_t] &\to \mathcal{R}[x,\theta_t] \\ (ax^s) \star (b_1x^i, b_2x^j, b_3x^k) &= (\eta(a)x^s b_1 x^i, \delta(a)x^s b_2 x^j, ax^s b_3 x^k) \\ &= (\eta(a)\theta_t^s(b_1)x^{s+i}, \delta(a)\theta_t^s(b_2)x^{s+j}, a\theta_t^s(b_3)x^{s+k}) \end{aligned}$$

However under associative and distributive laws, the multiplication can be extended to $\mathcal{R}_\gamma[x;\theta_t] = \frac{F_q[x;\theta_t]}{\langle x^{n_1}-1\rangle} \times \frac{R_1[x;\theta_t]}{\langle x^{n_2}-1\rangle} \times \frac{R_2[x;\theta_t]}{\langle x^{n_3}-1\rangle}$ as follows,

$$\star : \mathcal{R}_2[x;\theta_t] \times R_\gamma[x;\theta_t] \to R_\gamma[x;\theta_t]$$

$$r(x)\star(f_1(x)+\langle x^{n_1}-1\rangle, f_2(x)+\langle x^{n_2}-1\rangle, f_3(x)+\langle x^{n_3}-1\rangle) = (\eta(r(x))f_1(x)+\langle x^{n_1}-1\rangle, \delta(r(x))f_2(x)+\langle x^{n_2}-1\rangle, r(x)f_3(x) + \langle x^{n_3}-1\rangle)$$

**Definition 1** [14] A non-trivial $\mathcal{R}$-submodule $\mathcal{C}$ of $\mathcal{R}^n$ is called a $\theta_t$-cyclic code if for any $c = (c_0, c_1, \ldots, c_{n-1}) \in C$, $\sigma_1(c) = (\theta_t(c_{n-1}), \theta_t(c_0), \ldots, \theta_t(c_{n-2})) \in \mathcal{C}$. The operator $\sigma_1$ is called a $\theta_t$-cyclic shift operator on $\mathcal{R}^n$.

**Definition 2** A non-trivial $\mathcal{R}_2$-submodule $\mathcal{C}$ of $\mathcal{R}_\gamma$ is called a $\theta_t$-cyclic code if for any $c = (c_{0,1}, c_{1,1}, \ldots, c_{n_1-1,1}, c_{0,e}, c_{1,e}, \ldots, c_{n_2-1,e}, c_{0,o}, c_{1,o}, \ldots, c_{n_3-1,o}) \in \mathcal{C}$, $\sigma(c) = (\theta(c_{n_1-1,1}), \theta(c_{0,1})$ $, \ldots, \theta(c_{n_1-2,1}), \theta(c_{n_2-1,e}), \theta(c_{0,e}), \ldots, \theta(c_{n_2-2,e}), \theta(c_{n_3-1,o}), \theta(c_{0,o}), \ldots, \theta(c_{n_3-2,o})) \in \mathcal{C}$. The operator $\sigma$ is called as $\theta_t$-cyclic shift operator on $\mathcal{R}^n$.

The following result yields the relationship between the $\theta_t$-cyclic codes over $\mathcal{R}$ and $\mathbb{F}_q$,

**Theorem 6** *Let $\mathcal{C} = \mathcal{C}_1 \otimes \mathcal{C}_e \otimes \mathcal{C}_o \subseteq \mathcal{R}_\gamma$ be linear. Then $\mathcal{C}$ is a $\theta_t$-cyclic code if and only if $\mathcal{C}_1, \mathcal{C}_e$ and $\mathcal{C}_o$ are $\theta_t$-cyclic codes of length $n_1, n_2$ and $n_3$ over $\mathbb{F}_q, \mathcal{R}_1$ and $\mathcal{R}_2$ respectively.*

*Proof* Let $C = \mathcal{C}_1 \otimes \mathcal{C}_e \otimes \mathcal{C}_o$ be a $\theta_t$-cyclic code over $\mathcal{R}$. Let $z = (z_1, z_e, z_o) \in \mathcal{C}$ that is

$$z = (z_{0,1}, z_{1,1}, \ldots, z_{n_1-1,1}, z_{0,e}, z_{1,e}, \ldots, z_{n_2-1,e}, z_{0,o}, z_{1,o}, \ldots, z_{n_3-1,o}) \in \mathcal{C}.$$

Then $\sigma(z) = (\theta(z_{n_1-1,1}), \theta(z_{0,1}), \ldots, \theta(z_{n_1-2,1}), \theta(z_{n_2-1,e}), \theta(z_{0,e}), \ldots, \theta(z_{n_2-2,e}), \theta(z_{n_3-1,o})$ $, \theta(z_{0,o}), \ldots, \theta(z_{n_3-2,o})) = (\sigma(z_1), \sigma(z_e), \sigma(z_o)) \in \mathcal{C}$. From this we can conclude that,

$$\sigma(z_1) \in \mathcal{C}_1, \sigma(z_e) \in \mathcal{C}_e \text{ and } \sigma(z_o) \in \mathcal{C}_o$$

Hence $\mathcal{C}_1, \mathcal{C}_e$ and $\mathcal{C}_o$ are $\theta_t$-cyclic code of length $n_i$. The converse holds in a similar way.

We recall the following Theorem from [4].

**Theorem 7** *[4] Let $\mathcal{C}_o = o_1\mathcal{C}_{o_1} \oplus o_2\mathcal{C}_{o_2} \oplus o_3\mathcal{C}_{o_3}$ be a linear code over $\mathcal{R}_2$ of length $n_3$ then $\mathcal{C}_o$ is $\theta_t$-cyclic code iff $\mathcal{C}_{o_i} (1 \leq i \leq 3)$ is a $\theta_t$-cyclic code of length $n_3$ over $\mathbb{F}_q$.*

The analogue of this result in our setting is as follows.

**Theorem 8** *[21] Let $\mathcal{C}_e = e_1\mathcal{C}_{e_1} \oplus e_2\mathcal{C}_{e_2}$ be a linear code over $\mathcal{R}_1$ of length $n_2$ then $\mathcal{C}_e$ is $\theta_t$-cyclic code iff $\mathcal{C}_{e_i} (1 \leq i \leq 2)$ is a $\theta_t$-cyclic code of length $n_2$ over $\mathbb{F}_q$.*

**Theorem 9** *$\mathcal{C} = \mathcal{C}_1 \otimes \mathcal{C}_e \otimes \mathcal{C}_o$ be a linear code of length $\gamma = n_1 + n_2 + n_3$, then $\mathcal{C}$ is $\theta_t$-cyclic iff $\mathcal{C}_1, \mathcal{C}_{e_i}, \mathcal{C}_{o_j} (1 \leq i \leq 2, 1 \leq j \leq 3)$ are $\theta_t$-cyclic code of length $n_1, n_2, n_3$ over $\mathbb{F}_q$ respecively.*

*Proof* We obtain the proof on combining proofs of Theorem 6,7 and 8.

These notions are well-behaved with respect to duality as the next result shows.

**Theorem 10** *If $\mathcal{C}$ is a $\theta_t$-cyclic code of length $n$, then its dual $\mathcal{C}^\perp$ is also a $\theta_t$-cyclic code.*

*Proof* From Theorem 9, $\mathcal{C}_1, \mathcal{C}_{e_i}, \mathcal{C}_{o_j} (1 \leq i \leq 2, 1 \leq j \leq 3)$ are $\theta_t$-cyclic codes over $\mathbb{F}_q$. Then $\mathcal{C}_1^\perp, \mathcal{C}_{e_i}^\perp, \mathcal{C}_{o_j}^\perp (1 \leq i \leq 2, 1 \leq j \leq 3)$ are $\theta_t$-cyclic codes over $\mathbb{F}_q$ from [16] and once again by using Theorem 9, $\mathcal{C}^\perp$ becomes a $\theta_t$-cylic code.

Recall the following result from [34].

**Lemma 1** *[34] Let $C$ be a $\theta_t$-cyclic code of length $n$ over $\mathbb{F}_q$. Then there exists a polynomial $f(x) \in \mathbb{F}_q[x; \theta_t]$ such that $C = \langle f(x) \rangle$ and $x^n - 1 = g(x)f(x)$ in $\mathbb{F}_q[x; \theta_t]$.*

By assuming $o(\theta_t)|n$, the counterpart follows;

**Theorem 11** *Let $\mathcal{C} = \mathcal{C}_1 \otimes \mathcal{C}_e \otimes \mathcal{C}_o$ be a $\theta_t$-cyclic code of length $n$ over $\mathcal{R}$ and order of $\theta_t$ divides $n$. Then $\mathcal{C} = \langle B_1, B_e, B_o \rangle$ where $B_1 = \langle (f_1(x), 0, 0) \rangle, B_e = \langle (0, f_e(x), 0) \rangle$, and $B_o = \langle (b_1(x), b_e(x), f_o(x)) \rangle$, such that $\mathcal{C}_1 = \langle f_1(x) \rangle$, $\mathcal{C}_e = \langle f_e(x) \rangle$, $\mathcal{C}_o = \langle f_o(x) \rangle$, $b_1(x) \in \mathcal{C}_1$ and $b_2(x) \in \mathcal{C}_e$.*

*Proof* Let $\mathcal{C} = \mathcal{C}_1 \otimes \mathcal{C}_e \otimes \mathcal{C}_o$ be a $\theta_t$-cyclic code of length $\gamma = n_1 + n_2 + n_3$ over $\mathcal{R}$. Then by Thereom 6, $\mathcal{C}_1, \mathcal{C}_e, \mathcal{C}_o$ are $\theta_t$-cyclic codes of length $n_i$ over $\mathbb{F}_q, \mathcal{R}_1$ and $\mathcal{R}_2$. Define a homomorphism from $\mathcal{C}$ to $\mathcal{R}$ as follows,

$$\psi : \mathcal{C} \mapsto \mathcal{R}$$
$$\psi(c_1(x), c_e(x), c_o(x)) = (0, 0, c_o(x))$$

Define

$$ker(\psi) = \{(c_1(x), c_e(x), 0) : c_1(x) \in \mathcal{C}_1, c_e(x) \in \mathcal{C}_e\}$$

$$\mathcal{I} = \{(c_1(x), c_e(x)) \in \mathbb{F}_q[x;\theta_t] \times \mathcal{R}_1[x;\theta_t] : (c_1(x), c_e(x), 0) \in \ \ker(\psi)\}.$$

Clearly $\mathcal{I} = \mathcal{I}_1 \times \mathcal{I}_e$ forms a submodule of $\mathbb{F}_q[x;\theta_t] \times \mathcal{R}_1[x;\theta_t]$. Therefore there exist a polynomial $f_1(x)$ and $f_e(x)$ in $\mathbb{F}_q[x;\theta_t]$ and $\mathcal{R}_1[x;\theta_t]$ respectively, generating $\mathcal{I}_1$ and $\mathcal{I}_e$ with $f_1(x)|x^{n_1} - 1$ and $f_e(x)|x^{n_e} - 1$. Thus $\mathcal{I} = \langle (f_1(x), 0), (0, f_e(x))\rangle$, then for any $(c_1(x), c_e(x), 0) \in \ \ker(\psi)$, $(c_1(x), c_e(x)) = v(x)\star(f_1(x), 0), (0, f_e(x))$ for some $v(x) \in \mathcal{R}_1[x;\theta_t]$. Finally it leads to $\ker(\psi) = \langle (f_1(x), 0, 0), (0, f_e(x), 0)\rangle$. The fact that $\mathcal{C}$ is a submodule implies that $\psi(\mathcal{C})$ is a submodule. By using the first isomorphism theorem,

$$\mathcal{C}/ker(\psi) \cong \psi(\mathcal{C}).$$

Let $(b_1(x), b_e(x), f_o(x)) \in \mathcal{C}$ then $\psi(b_1(x), b_e(x), f_o(x)) = (0, 0, f_o(x))$. From this any $\theta_t$-cyclic code of length $n$ can be represented by $\mathcal{C} = \langle (f_1(x), 0, 0)(0, f_e(x), 0), (b_1(x), b_e(x), f_o(x))\rangle$ where $f_1(x)|(x^{n_1} - 1), f_e(x)|(x^{n_2} - 1)$ and $f_o(x)|(x^{n_3} - 1)$.

Further, we have $\mathcal{C}$ is $\theta_t$-cyclic then $\mathcal{C}_k$ where $k \in \{1, e_1, e_2, o_1, o_2, o_3\}$ is skew $\theta_t$-cyclic code over $\mathbb{F}_q$ with respective lengths. From Theorem 3, $|\mathcal{C}| =\mid \mathcal{C}_1 \mid \prod_{i=1}^{i=2} \mid C_{e_i} \mid \prod_{j=1}^{j=3} \mid C_{o_j} \mid$ since each $\mathcal{C}_k$ is $\theta_t$-cyclic it is generated by a polynomial $f_k(x)$ thus $\mid \mathcal{C} \mid= q^{\gamma - \sum_{i=1}^{6} \epsilon_k}$ where $\gamma = n_1 + 2(n_2) + 3(n_3)$. The following Theorem provides the generator polynomials for $\theta_t$-cylic codes over $\mathbb{F}_q$.

**Theorem 12** *Let $\mathcal{C} = \mathcal{C}_1 \otimes \mathcal{C}_e \otimes \mathcal{C}_o$ be a skew cyclic code over $\mathcal{R}$ of length $\gamma = n_1 + n_2 + n_3$. Then there exists a polynomial*

*(i)* $f_1(x) \in \mathbb{F}_q[x;\theta_t]$ *such that* $\mathcal{C}_1 = \langle f_1(x)\rangle$ *and* $x^{n_1} - 1 = g_1(x)f_1(x)$.
*(ii)* $f_e(x) \in \mathcal{R}_1[x;\theta_t]$ *such that* $\mathcal{C}_e = \langle f_e(x)\rangle$ *and* $x^{n_2} - 1 = g_e(x)f_e(x)$ *where* $f_e(x) = \sum_{i=1}^{2} e_i f_{e_i}(x)$.
*(iii)* $f_o(x) \in \mathcal{R}_2[x;\theta_t]$ *such that* $\mathcal{C}_o = \langle f_o(x)\rangle$ *and* $x^{n_3} - 1 = g_o(x)f_o(x)$ *where* $f_o(x) = \sum_{i=1}^{3} o_i f_{o_i}(x)$.

*Proof* Let $\mathcal{C}$ be a $\theta_t$-cyclic code of length $\gamma = n_1 + n_2 + n_3$. From Theorem 6, we have that $\mathcal{C}_1, \mathcal{C}_e$ and $\mathcal{C}_o$ are $\theta_t$-cyclic codes. Using Lemma 1, $(i)$ follows.

The the proof of $(ii)$ is as follows, let $\mathcal{C}_e = e_1\mathcal{C}_{e_1} \oplus e_2\mathcal{C}_{e_2}$ be a $\theta_t$-cyclic code of length $n_2$ over $\mathcal{R}_1$. Thereom 7 says that, $\mathcal{C}_{e_1}$ and $\mathcal{C}_{e_2}$ are $\theta_t$-cyclic codes of length $n_2$ over $\mathbb{F}_q$. Lemma 1 says that we have $\mathcal{C}_i = \langle f_{e_i}(x)\rangle$ and $x^{n_2} - 1 = g_{e_i}(x)f_{e_i}(x)$ in $\mathbb{F}_q[x;\theta_t]$ for $i \in \{1, 2\}$. Then $e_i f_{e_i}(x) \in \mathcal{C}$ for $i \in \{1, 2\}$. Also for any $f_e(x) \in \mathcal{C}$, we have $f_e(x) = \sum_{i=1}^{2} e_i h_{e_i}(x) f_{e_i}(x)$ where $h_{e_i}(x) \in \mathbb{F}_q[x;\theta_t]$ for $i \in \{1, 2\}$. Thus $f_e(x) \in \langle e_1 f_{e_1}(x), e_2 f_{e_2}(x)\rangle$. Therefore, $\mathcal{C} = \langle e_1 f_{e_1}(x), e_2 f_{e_2}(x)\rangle$. As $x^{n_2} - 1 = g_{e_i}(x)f_{e_i}(x)$ in $\mathbb{F}_q[x;\theta_t]$ for $i \in \{1, 2\}$. Let $f_e(x) = e_1 f_{e_1}(x) + e_2 f_{e_2}(x) \in \mathcal{R}_1[x;\theta_t]$. Then $f_e(x) \in C$. On the other hand $e_i f_{e_i}(x) = e_i f_e(x) \in \langle f_e(x)\rangle$ for $i = 1, 2$. Consequently, $\mathcal{C} = \langle f_e(x)\rangle$. Further, $[\sum_{i=1}^{2} e_i g_{e_i}(x)]f_e(x) = \sum_{i=1}^{2} e_i g_{e_i}(x) f_{e_i}(x) = \sum_{i=1}^{2} e_i(x^{n_2} - 1) = x^{n_2} - 1$. Then $x^{n_2} - 1 = g_e(x)f_e(x)$ in $\mathcal{R}_1[x;\theta_t]$, where $g_e(x) = \sum_{i=1}^{2} e_i g_{e_i}(x)$. Thus $(ii)$ is follows. $(iii)$ is similar to the proof of $(ii)$.

## 5 Skew Constacyclic Code over $\mathcal{R}$

In this section, we study skew $\theta_t$-constacyclic codes over $\mathcal{R}$. We choose a unit element $\alpha \in \mathcal{R}_2^*$ such that $\alpha$ satisfying the condition $\alpha^2 = 1, (\alpha = 1, -1, \cdots)$.

**Definition 3** Let $\alpha_i \in \mathbb{F}_{p^t}/\{0\}$, a linear code $\mathcal{C} \subseteq \mathcal{R}_\gamma[x,\theta]$ is called skew $\alpha = \alpha_1 + u\alpha_2 + v\alpha_3$-constacyclic code if it is invariant under the cyclic shift operator $\lambda_\alpha$ that is whenever

$$\begin{aligned} c &= (x_0, x_1, \cdots, x_{n_1-1}, y_0, y_1, \cdots, y_{n_2-1}, z_0, z_1, \cdots, z_{n_3-1}) \in \mathcal{C} \\ \lambda_\alpha(c) &= (\alpha_1\theta_t(x_{n_1-1}), \theta_t(x_0), \cdots, \theta_t(x_{n_1}), (\alpha_1 + u\alpha_2)\theta_t(y_{n_1-1}), \theta_t(y_1), \cdots, \theta_t(y_{n_2-2}), \\ &\quad (\alpha_1 + u\alpha_2 + v\alpha_3)\theta_t(z_{n_3-1}), \theta_t(z_0), \cdots, \theta_t(z_{n_3-2})) \in \mathcal{C} \end{aligned}$$

The following two results translate symmetry conditions into algebraic constraints. The proof of the first result is immediate and omitted.

**Theorem 13** *Let $R_{n,\lambda} = R[x;\theta_t]/\langle x^n - \lambda\rangle$. A linear code $C$ of length $n$ over $R$ is $(\theta_t, \lambda)$-cyclic code if and only if $C$ is a left $R[x;\theta_t]$-submodule of $R_{n,\lambda}$.*

**Theorem 14** *A code $\mathcal{C}$ is skew $\alpha$-cyclic code over $\mathcal{R}_\gamma = \frac{\mathbb{F}_q[x,\theta_t]}{x^{n_1}-\alpha} \times \frac{\mathcal{R}_1[x,\theta_t]}{x^{n_2}-\alpha} \times \frac{\mathcal{R}_2[x,\theta_t]}{x^{n_3}-\alpha}$ iff $\mathcal{C}$ is a left $\mathcal{R}_2[x,\theta_t]$ module over $\mathcal{R}_\gamma$.*

*Proof* Let $\mathcal{C}$ be a skew $\alpha$-cyclic code. Then by definition $x \star (f(x)|g(x)|h(x)) \in \mathcal{C}$

$$\begin{aligned} x \star (f(x)|g(x)|h(x)) = (\theta_t(f_0)x + \theta_t(f_1)x^2 + \cdots + \alpha_1\theta_t(f_{n_1-1}), \theta_t(g_0)x + \theta_t(g_1)x^2 + \cdots + (\alpha_1 + u\alpha_2)\theta_t(g_{n_2-1}), \\ \theta_t(h_0)x + \theta_t(h_1)x^2 + \cdots + (\alpha_1 + u\alpha_2 + v\alpha_3)\theta_t(h_{n_3-1})) \in \mathcal{C} \end{aligned}$$

And by using linearity of $\mathcal{C}$,

$$r(x) \star (g_1(x)|g_2(x)|g_3(x)) \in \mathcal{C}$$

for some $r(x) \in \mathcal{R}_2[x,\theta_t]$. Hence $\mathcal{C}$ is an left $\mathcal{R}_2[x,\theta_t]$ submodule over $\mathcal{R}_\gamma$. Conversely, assume that $\mathcal{C}$ is is an left $\mathcal{R}_2[x,\theta_t]$ submodule over $\mathcal{R}_\gamma$, then we have $x \star (f(x)|g(x)|h(x)) \in \mathcal{C}$ implies $\mathcal{C}$ is skew $\alpha$-cyclic code.

**Theorem 15** *Let $\mathcal{C}_o \subseteq \mathcal{R}_2^n$ be a skew $\alpha = \alpha_1 + u\alpha_2 + v\alpha_3$-cyclic code of length $n$ iff $\mathcal{C}_{o_1}, \mathcal{C}_{o_2}$ and $\mathcal{C}_{o_3}$ are skew $\alpha_1, \alpha_1 + \alpha_2, \alpha_1 + \alpha_3$-cyclic codes over $\mathbb{F}_q$ of length $n$.*

*Proof* Let $\mathcal{C}_o$ be a skew $\alpha$-cyclic code, Let $a = xo_1 + yo_2 + zo_3 \in \mathcal{C}_o$ where $x = (x_0, x_1, \cdots, x_{n-1}) \in \mathcal{C}_{o_1}, y = (y_0, y_1, \cdots, y_{n-1}) \in \mathcal{C}_{o_2}$ and $x = (z_0, z_1, \cdots, z_{n-1}) \in \mathcal{C}_{o_3}$. Then we have by definiton, $\lambda_\alpha(x) \in \mathcal{C}_o$,

$$\begin{aligned} &\lambda_\alpha(o_1(x_0, x_1, \cdots, x_{n-1}) + o_2(y_0, y_1, \cdots + y_{n-1}) \\ &\qquad + o_3(z_0, z_1, \cdots, z_{n-1})) \quad = \quad ((\alpha_1 + u\alpha_2 + v\alpha_3) \star o_1(\theta_t(x_{n-1}), \theta_t(x_0), \cdots, \theta_t(x_{n-2})) + \\ &\qquad (\alpha_1 + u\alpha_2 + v\alpha_3) \star o_2(\theta_t(y_{n-1}), \theta_t(y_0), \cdots + \theta_t(y_{n-2})) + \\ &\qquad (\alpha_1 + u\alpha_2 + v\alpha_3) \star o_3(\theta_t(z_{n-1}), \theta_t(z_0), \cdots, \theta_t(z_{n-1}))) \\ &\qquad \implies \lambda_{\alpha_1}(x) + \lambda_{\alpha_1+\alpha_2}(y) + \lambda_{\alpha_1+\alpha_3}(z) \in \mathcal{C}_o \\ &\qquad \implies \lambda_{\alpha_1}(x) \in \mathcal{C}_{o_1}, \lambda_{\alpha_1+\alpha_2}(y) \in \mathcal{C}_{o_2}, \lambda_{\alpha_1+\alpha_3}(z) \in \mathcal{C}_{o_3} \end{aligned}$$

Hence, $\mathcal{C}_{o_1}, \mathcal{C}_{o_2}$ and $\mathcal{C}_{o_3}$ are skew $\alpha_1, \alpha_1 + \alpha_2, \alpha_1 + \alpha_3$-cyclic codes over $\mathbb{F}_q$ of length $n$.

Conversely, assume that $\mathcal{C}_{o_1}, \mathcal{C}_{o_2}$ and $\mathcal{C}_{o_3}$ are skew $\alpha_1, \alpha_1 + \alpha_2, \alpha_1 + \alpha_3$-cyclic codes over $\mathbb{F}_q$ of length $n$. Let $m_0, m_1, \cdots, m_{n-1}$ be an element in $\mathcal{C}_o$, where $m_i = o_1x_i + o_2y_i + o_3z_i$ such that $x = (x_0, x_2, \cdots, x_{n-1}) \in \mathcal{C}_{o_1}, y = (y_0, y_2, \cdots, y_{n-1}) \in \mathcal{C}_{o_2}$ and $z = (z_0, z_2, \cdots, z_{n-1}) \in \mathcal{C}_{o_3}$. Then we have $\lambda_{\alpha_1}(x) \in \mathcal{C}_{o_1}, \lambda_{\alpha_1+\alpha_2}(y) \in \mathcal{C}_{o_2}$ and $\lambda_{\alpha_1+\alpha_3}(z) \in \mathcal{C}_{o_3}$. So we get,

$$\begin{aligned} o_1\lambda_{\alpha_1}(x) + o_2\lambda_{\alpha_1+\alpha_2}(y) + o_3\lambda_{\alpha_1+\alpha_3}(z) &= o_1\lambda_{\alpha_1}(x_0, x_1, \cdots, x_{n-1}) + o_2\lambda_{\alpha_1+\alpha_2}(y_0, y_1, \cdots + y_{n-1}) \\ &\quad + o_3\lambda_{\alpha_1+\alpha_3}(z_0, z_1, \cdots, z_{n-1}) \in \mathcal{C} \\ &= \lambda_\alpha(m_0, m_1, \cdots, m_{n-1}) \in \mathcal{C} \end{aligned}$$

Hence, $\mathcal{C}$ is skew $\alpha-$cyclic code over $\mathcal{R}_2^n$

**Theorem 16** *Let $\mathcal{C}_e$ be a a skew $\alpha = \alpha_1 + u\alpha_2$-cyclic code over $\mathcal{R}_1$ iff $\mathcal{C}_{e_1}$ and $\mathcal{C}_{e_2}$ are skew $\alpha_1 + \alpha_2$ and $\alpha_1$-cyclic codes over $\mathbb{F}_{11}$.*

*Proof* The proof is similar to Theorem 15 taking mod $v$ to the above condition.

**Theorem 17** *Let $\mathcal{C}$ be a skew $\alpha = \alpha_1 + u\alpha_2 + v\alpha_3$-cyclic code over $\mathcal{R}$ of length $\gamma = n_1 + n_2 + n_3$ iff $\mathcal{C}_1, \mathcal{C}_e$ and $\mathcal{C}_o$ are $\alpha_1, \alpha_1 + u\alpha_2$ and $\alpha_1 + u\alpha_2 + v\alpha_3$-cyclic codes over $\mathbb{F}_q, \mathcal{R}_1$ and $\mathcal{R}_2$ respectively.*

*Proof* Let $\mathcal{C}_1, \mathcal{C}_e$ and $\mathcal{C}_o$ be $\alpha_1, \alpha_1 + u\alpha_2$ and $\alpha_1 + u\alpha_2 + v\alpha_3$-cyclic. Consider $x = (x_0, x_1, \cdots, x_{n_1-1}), y = (y_0, y_1, \cdots, y_{n_2-1})$ and $z = (z_0, z_1, \cdots, z_{n_3-1})$. Consider $\alpha_1 + u\alpha_2 = \beta$. Then we have,

$$(x, y, z) \in \mathcal{C} \implies (\lambda_{\alpha_1}(x), \lambda_\beta(y), \lambda_\alpha(y)) \in \mathcal{C}$$

Hence $\mathcal{C}$ is skew $\alpha$-cyclic. The converse part holds similarly.

**Theorem 18** *Let $\mathcal{C}$ be a skew $\alpha$-cyclic code of length $\gamma = n_1+n_2+n_3$ iff $\mathcal{C}_1$ is skew $\alpha_1$-cyclic code of length $n_1$, $\mathcal{C}_{e_1}$& $\mathcal{C}_{e_2}$ are $\alpha_1 + \alpha_2, \alpha_1$-cyclic codes of length $n_2$ and $\mathcal{C}_{o_1}, \mathcal{C}_{o_2}$& $\mathcal{C}_{o_3}$ are skew $\alpha_1, \alpha_1 + \alpha_2, \alpha_1 + \alpha_3$-cyclic codes over $\mathbb{F}_q$ of length $n_3$.*

*Proof* Using Theorem 15,16,17 the result follows.

**Theorem 19** *Let $\mathcal{C}$ be a skew $\alpha = \alpha_1 + u\alpha_2 + v\alpha_3$-cyclic code over $\mathcal{R}$ of length $\gamma = n_1 + n_2 + n_3$ iff $\mathcal{C}_1^\perp, \mathcal{C}_e^\perp$ and $\mathcal{C}_o^\perp$ are $(\alpha_1)^{-1}, (\alpha_1 + u\alpha_2)^{-1}$ and $(\alpha_1 + u\alpha_2 + v\alpha_3)^{-1}$-cyclic.*

*Proof* Let $\mathcal{C}$ be a skew $\alpha$-cyclic code, [ Lemma 3.1,[27]] says that $\mathcal{C}^\perp$ is skew $(\alpha_1+u\alpha_2+v\alpha_3)^{-1}$-cyclic code. From Theorem 17, we have $\mathcal{C}_1^\perp, \mathcal{C}_e^\perp$ and $\mathcal{C}_o^\perp$ are skew $(\alpha_1)^{-1}, (\alpha_1 + u\alpha_2)^{-1}$ and $(\alpha_1 + u\alpha_2 + v\alpha_3)^{-1}$-cyclic.

**Corollary 2** *Let $\mathcal{C} = \mathcal{C}_1 \otimes \mathcal{C}_e \otimes \mathcal{C}_o$. be a skew $\alpha = \alpha_1 + u\alpha_2 + v\alpha_3$-cyclic code over $\mathcal{R}$ of length $\gamma = n_1 + n_2 + n_3$. Then there exists a polynomial.*

*(i)* $f_1(x) \in \mathbb{F}_q[x; \theta_t]$ *such that* $\mathcal{C}_1 = \langle f_1(x) \rangle$ *and* $x^{n_1} - \alpha_1 = g_1(x) f_1(x)$ .
*(ii)* $f_e(x) \in \mathcal{R}_1[x; \theta_t]$ *such that* $\mathcal{C}_e = \langle f_e(x) \rangle$ *and* $x^{n_2} - (\alpha_1 + u\alpha_2) = g_e(x) f_e(x)$.
*(iii)* $f_o(x) \in \mathcal{R}_2[x; \theta_t]$ *such that* $\mathcal{C}_o = \langle f_o(x) \rangle$ *and* $x^{n_3} - (\alpha_1 + u\alpha_2 + v\alpha_3) = g_o(x) f_o(x)$.

*Proof* The proof is similar to the proof of Theorem 12.

**Theorem 20** *Let $\mathcal{C} = \mathcal{C}_1 \otimes \mathcal{C}_e \otimes \mathcal{C}_o$ be a $\theta_t$-constacyclic code of length $\gamma$ over $\mathcal{R}$. Then $\mathcal{C} = \langle \mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_o \rangle$ where $\mathcal{B}_1 = \langle (f_1(x), 0, 0) \rangle, B_1 = \langle (0, f_e(x), 0) \rangle$, and $B_1 = \langle (b_1(x), b_e(x), f_o(x)) \rangle$.*

*Proof* The proof is similar to proof of Theorem 11.

*Example 1* Let $q = 9$ and $\mathbb{F}_9 = \mathbb{F}_3[z]$ with $z^2+1 = 0$ then consider the ring $R_\gamma = \frac{\mathbb{F}_9[x,\theta_t]}{x^4-1} \times \frac{\mathcal{R}_1[x,\theta_t]}{x^5-1} \times \frac{\mathcal{R}_2[x,\theta_t]}{x^5-1}$ and $\theta_3$ be the Frobenius automorphism that is $\theta_3(a) = a^3$ where $a \in \mathbb{F}_9$.

$x^4 - 1 = (x + 1)(x + 2)(x + z)(x + 2z) \in \mathbb{F}_9[x, \theta_3]$

$x^5 - 1 = (x + 2)(x^4 + x^3 + x^2 + x + 1) \in \mathbb{F}_9[x, \theta_3]$

$f_1(x) = \langle (x + 1) \rangle, f_e(x) = \langle e_1(x + 2) + e_2(x + 2) \rangle, f_o(x) = \langle o_1(x + 2) + o_2(x + 2) + o_3(x + 2) \rangle$ from Theorem 12 we have $f_i$ divides $x^{n_i} - 1$ for $(i = 1, e, o)$ thus we have a code with parameter $[29, 18, 2]$ over $\mathbb{F}_9$.

*Example 2* Let $q = 25$ and $\mathbb{F}_9 = \mathbb{F}_3[z]$ with $z^2+z+1 = 0$ then consider the ring $R_\gamma = \frac{\mathbb{F}_{25}[x,\theta_t]}{x^4-1} \times \frac{\mathcal{R}_1[x,\theta_t]}{x^6-1} \times \frac{\mathcal{R}_2[x,\theta_t]}{x^4-1}$ and $\theta_5$ be the frobemious automorphism that is $\theta_5(a) = a^5$ where $a \in \mathbb{F}_{25}$.

$x^4 - 1 = (x + 2)(x + 3)(x + z)(x + z + 1) \in \mathbb{F}_{25}[x, \theta_5]$

$x^6 - 1 = (x^2 - 1)(x^2 + x + 1)(x^2 - x + 1) \in \mathbb{F}_{25}[x, \theta_5]$

$f_1(x) = \langle (x+2) \rangle, f_e(x) = \langle e_1(x^2-1)+e_2(x^2-1) \rangle, f_o(x) = \langle o_1(x+z+1)+o_2(x+z+1)+o_3(x+z+1) \rangle$ from Theorem 12 we have $f_i$ divides $x^{n_i} - 1$ for $(i = 1, e, o)$ thus we have a code with parameter $[28, 20, 2]$ over $\mathbb{F}_{25}$.

## 6 Conclusion and open problems

In this note, we have studied the algebraic and metric structure of skew cyclic and skew constacyclic codes over a special mixed alphabet. Thus, our codes have a structure of module over the largest of the three alphabets $\mathcal{R}_2$. Codes over the product ring $\mathbb{F}_q \times \mathcal{R}_1 \times \mathcal{R}_2$ would be modules over that larger ring. The two algebraic structures are different and should not be confused.

The present work leads itself to two paths of generalization: consider different mixed alphabets or replace the concepts of cyclicity by that of quasi-cyclicity. The former path seems easier than the latter, in view of the many examples of rings that have been used as alphabets of cyclic codes in recent years. On the other hand, the structure of quasi-cyclic codes is always more subtle than that of cyclic codes.

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