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# SIMPLE ANOSOV REPRESENTATIONS OF CLOSED SURFACE GROUPS

NICOLAS THOLOZAN AND TIANQI WANG

ABSTRACT. We introduce and study *simple Anosov representations* of closed hyperbolic surface groups, analogous to Minsky's *primitive stable representations* of free groups. We prove that the set of simple Anosov representations into  $\mathrm{SL}(d, \mathbb{C})$  with  $d \geq 4$  strictly contains the set of Anosov representations. As a consequence, we construct domains of discontinuity for the mapping class group action on character varieties which contain non-discrete representations.

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## 1. INTRODUCTION

Given a finitely generated group  $\Gamma$  and a complex linear group  $G$ , the group of outer automorphisms  $\mathrm{Out}(\Gamma)$  of  $\Gamma$  acts on the *character variety*  $\chi(\Gamma, G)$ , the GIT quotient of  $\mathrm{Hom}(\Gamma, G)$  under the conjugation action of  $G$ , by precomposition. This action is of primordial interest in various topics, such as the study of locally homogeneous geometric structures on manifolds, or isomonodromic deformations of complex differential equations.

When  $\Gamma$  has a large automorphism group (e.g. when  $\Gamma$  is a free group or a surface group), the action of  $\mathrm{Out}(\Gamma)$  on character varieties can be very chaotic, and a first interesting question is whether one can construct large domains of discontinuity for this action. A broad family of examples have been produced by the theory of *Anosov representations* of hyperbolic groups, see [Lab06]. Anosov representations are quasi-isometrically embedded (they are equivalent for  $\mathrm{SL}(2, \mathbb{C})$ ) and stable under small deformations. They thus form open domains of character varieties on which  $\mathrm{Out}(\Gamma)$  acts properly discontinuously (see [Lab08]).

However, these domains of discontinuity are not necessarily maximal. In [Min13], Minsky constructed examples of so-called *primitive stable representations* of a free group into  $\mathrm{SL}(2, \mathbb{C})$ , and proved that they form an open domain of discontinuity

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which contains strictly the set of quasi-isometric embeddings. Minsky’s construction has been generalized to higher rank by Kim–Kim [KK21] and by the second author in [Wan23] who developed the notion of *primitive Anosov representation*.

Roughly speaking, primitive Anosov representations of a non-abelian free group are representations with an Anosov behaviour “in restriction to” primitive elements. More precisely, the second author proved in [Wan23] that primitive Anosov representations are representations such that the associated local system over the geodesic flow of the free group admits a dominated splitting *in restriction to* the closure of the union of all closed orbits corresponding to primitive elements. This motivates the more general study of *restricted Anosov representations*, initiated in [Wan23], which we carry on in the present paper.

With this point of view, primitive Anosov representations have a natural analog for closed surface groups, which we call *simple Anosov*:

**Definition 1.1.** A representation  $\rho$  of the fundamental group  $\Gamma$  of a closed connected hyperbolic surface  $S$  into  $\mathrm{SL}(d, \mathbb{C})$  is called *simple  $k$ -Anosov* if the local system associated to  $\rho$  over the geodesic flow of  $S$  admits a dominated splitting of rank  $k$  in restriction to the closure of the union of simple closed geodesics.

We will call the closure of simple closed geodesics the *Birman–Series set*, in reference to Birman and Series [BS85], who proved that this set is “sparse” in the unit tangent bundle of the hyperbolic surface (it has Hausdorff dimension 1). We refer to Section 3 for precisions about the above definition (in particular, the definition of dominated splitting of rank  $k$ ).

We prove in Section 5 the following expected property:

**Proposition 1.2.** *Let  $\Gamma$  be the fundamental group of a closed connected hyperbolic surface  $S$ . Then the set of simple Anosov representations  $\Gamma$  into  $\mathrm{SL}(d, \mathbb{C})$  modulo conjugation is an open domain of discontinuity for the action of  $\mathrm{Out}(\Gamma) = \mathrm{Mod}(S)$ .*

For non-oriented surfaces, Lee proved in [Lee15] that the domain of simple Anosov representations into  $\mathrm{SL}(2, \mathbb{C})$  contains strictly the domain of Anosov (i.e. quasifuchsian) representations. In contrast, for oriented surfaces, the domain of quasifuchsian representations is known to be a maximal domain of discontinuity by results of Lee [Lee15] and Souto–Storm [SS06]. The main goal of the present paper is to construct new examples of simple Anosov representations in higher rank:

**Theorem 1.3.** *Let  $\Gamma$  be the fundamental group of a closed connected hyperbolic surface. Then, for every  $d \geq 2$ , there exists a simple  $d$ -Anosov representation of  $\Gamma$  into  $\mathrm{SL}(2d, \mathbb{C})$  in the boundary of the domain of  $d$ -Anosov representations.*

Taking a “generic deformation”, we obtain the following:

**Corollary 1.4.** *Let  $\Gamma$  be the fundamental group of a closed connected hyperbolic surface. Then, for every  $d \geq 2$ , there exist simple  $d$ -Anosov representation of  $\Gamma$  into  $\mathrm{SL}(2d, \mathbb{C})$  with analytically dense image.*

As a consequence, we obtain domains of discontinuity of the mapping class group action that properly contain the domain of Anosov representations.

**Corollary 1.5.** *Let  $\Gamma$  be the fundamental group of a closed connected hyperbolic surface. Then, for every  $d \geq 2$ , there exists an open domain of discontinuity for the action of  $\mathrm{Out}(\Gamma)$  on  $\mathrm{Hom}(\Gamma, \mathrm{SL}(2d, \mathbb{C})) / \mathrm{SL}(2d, \mathbb{C})$  which contains points of the boundary of domain of Anosov representations.*

In particular, the domain of Anosov representations modulo conjugation is not a maximal domain of discontinuity.

### 1.1. Further results and open questions.

1.1.1. *Simple  $P$ -Anosov representations into  $G$ .* More generally, there is a notion of simple  $P$ -Anosov representation into  $G$ , for any pair of a semisimple (real or complex) linear group  $G$  and a parabolic subgroup  $P$ . Here we will focus on simple  $P_d$ -Anosov representations into  $\mathrm{SL}(2d, \mathbb{C})$ , where  $P_d$  is the stabilizer of a  $d$ -dimensional subspace of  $\mathbb{C}^d$ , but one could easily elaborate on these to build more examples. For instance, the direct sum of one our representations with a trivial representation will give simple  $d$ -Anosov representations into  $\mathrm{SL}(d', \mathbb{C})$  for any  $d' \geq 2d$ . Another example is the following: Among the exotic simple 2-Anosov representations into  $\mathrm{SL}(4, \mathbb{C})$  that we construct, some take values in the complex symplectic group  $\mathrm{Sp}(4, \mathbb{C})$ . Through the isomorphism  $\mathrm{Sp}(4, \mathbb{C}) \simeq \mathrm{SO}(5, \mathbb{C})$ , one obtains simple 1-Anosov representations into  $\mathrm{SO}(5, \mathbb{C}) \subset \mathrm{SL}(5, \mathbb{C})$ .

These constructions, however, do not seem to exhaust all the possibilities. In particular, it seems that our construction cannot provide *simple Borel Anosov* representations, i.e. simple Anosov with respect to a minimal parabolic subgroup. This raises the following question:

*Question 1.6.* Does there exist a representation of a closed oriented surface group into  $\mathrm{SL}(d, \mathbb{C})$ ,  $d \geq 2$  that is simple Borel Anosov but not Borel Anosov? In particular, does there exist a representation of a closed oriented surface group into  $\mathrm{SL}(3, \mathbb{C})$  which is simple Anosov but not Anosov ?

On a related topic, Maloni, Martone, Mazzoli and Zhang have initiated in [MMMZ23] the study of representations which are Borel Anosov in restriction to a fixed lamination.

1.1.2. *Other mapping class group invariant subflows.* We develop more generally the basic properties of Anosov representations in restriction to a subflow of the geodesic flow. In particular, any subflow which is globally preserved by the mapping class group gives rise to a domain of discontinuity for the mapping class group on the character variety. We mention various examples in Section 5. There, we also prove that any closed subflow of the geodesic flow that is preserved by a finite index subgroup of the mapping class group contains the Birman–Series set. As a consequence, all these other domains of discontinuity associated to subflows are contained in the domain of simple Anosov representations. This provides evidence for a positive answer to the following question:

*Question 1.7.* Is the domain of simple Anosov representations a maximal domain of discontinuity for the mapping class group action on a character variety of a surface group?

1.2. **Structure of the paper.** In Section 2, we introduce a general notion of  $\Gamma$ -flow of a finitely generated group, of which the main examples are subflows of geodesic flows of hyperbolic or relatively hyperbolic groups. In Section 3, we develop the general notion of Anosov representation *in restriction to a  $\Gamma$ -flow*, which specially emphasis on relatively Anosov and simple Anosov representations. This section contains several general results of independent interest which will make the main construction rather natural.

Section 4 presents our main construction of exotic simple Anosov representations. In a word, these are obtained as the induced representations of a geometrically finite but not quasifuchsian representation into  $\mathrm{SL}(2, \mathbb{C})$  of the fundamental group of a covering of degree  $d$ . Finally, Section 5 develops the applications to mapping class group actions on character varieties.

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## 2. GEODESIC FLOWS OF HYPERBOLIC AND RELATIVELY HYPERBOLIC GROUPS

**2.1. Hyperbolic and relatively hyperbolic groups.** Let  $(X, d_X)$  be a metric space. Recall that a map  $\ell$  from  $\mathbb{R}$  (resp.  $\mathbb{R}_{\geq 0}$ ,  $[a, b] \subset \mathbb{R}$ ) to  $X$  is a *geodesic* (resp. *geodesic ray*, *geodesic segment*) if

$$d_X(\ell(t), \ell(s)) = |t - s|$$

for all  $s, t$  in the interval of definition. The space  $X$  is called *geodesic* if any two points are joined by a geodesic segment, and called *proper* if closed balls are compact. We say that  $X$  is *taut* if every point is at uniformly bounded distance from a bi-infinite geodesic. When  $(X, d_X)$  is proper and geodesic, we say that it is  $\delta$ -*hyperbolic*, if for any geodesic triangle in  $X$ , there exists a point at distance at most  $\delta$  from all three sides of the triangle, which is called a center of the triangle. The (*Gromov*) *boundary* of a  $\delta$ -hyperbolic space  $X$ , denoted by  $\partial_\infty X$ , is defined to be the asymptotic classes of geodesic rays.

When  $(X, d_X)$  is a  $\delta$ -hyperbolic space, a *horofunction* about a boundary point  $p \in \partial_\infty X$  is a function  $h : X \rightarrow \mathbb{R}$  such that for any  $x, x' \in X$ ,

$$|(h(x) - d_X(x, x_0)) - (h(x') - d_X(x', x_0))|$$

is uniformly bounded, where  $x_0$  is a center of the ideal triangle with vertices  $x, x'$  and  $p$ . A subset  $B \subset X$  is called a *horoball* centered at  $p$  if there exists a horofunction  $h$  about  $p$ , such that  $B = \{x \in X | h(x) < 0\}$ .

Finally, a finitely generated group  $\Gamma$  is (*Gromov*) *hyperbolic* if it admits a properly discontinuous and cocompact isometric action on a  $\delta$ -hyperbolic space  $(X, d_X)$  (for some constant  $\delta$ ). The space  $X$  is called a *Gromov model* of  $\Gamma$ .

The notion of Gromov hyperbolic groups is a far-reaching generalization of convex-cocompact Kleinian groups. Similarly, the notion of *relatively hyperbolic groups* generalizes that of geometrically finite Kleinian groups. We follow here the definition given by Gromov [Gro87].

Let  $\Gamma$  be a finitely generated group and let  $\Pi = \{\Pi_i\}_{i \in I}$  be a finite collection of finitely generated, infinite subgroups of  $\Gamma$ . Let

$$\Pi^\Gamma = \{\gamma \Pi_i \gamma^{-1} \mid \gamma \in \Gamma, \Pi_i \in \Pi\}$$

be the collection of conjugates of the subgroups in  $\Pi$ .

**Definition 2.1.** The pair  $(\Gamma, \Pi)$  is called a *relatively hyperbolic pair*, and  $\Gamma$  is called *hyperbolic relative to*  $\Pi$ , if there exists a  $\delta$ -hyperbolic space  $(X, d_X)$  with a properly discontinuous isometric action of  $\Gamma$  such that

- (1).  $X$  is either taut or a horoball;

- (2). There exists  $\mathcal{B} = \{B_i\}_{i \in I}$ , a collection of horoballs of  $X$ , such that  $\mathcal{B}^\Gamma = \{\gamma B_i \mid \gamma \in \Gamma, i \in I\}$  is a collection of disjoint open horoballs with  $\gamma \Pi_i \gamma^{-1}$  the stabilizer of  $\gamma B_i$  for each  $\gamma \in \Gamma$  and  $i \in I$ ;
- (3).  $\Gamma$  acts on  $X^{th} = X \setminus \bigcup_{i \in I, \gamma \in \Gamma} \gamma B_i$ , the thick part of  $X$ , cocompactly.

The space  $X$  is called a *Gromov model* of  $(\Gamma, \Pi)$ . The Gromov boundary of  $X$  is called the Bowditch boundary of  $(\Gamma, \Pi)$ , denoted by  $\partial_\infty(\Gamma, \Pi) = \partial_\infty X$ . The subgroups in  $\Pi^\Gamma$  are called *peripheral subgroups* of  $\Gamma$  and the centers of horoballs in  $\mathcal{B}^\Gamma$  are called *parabolic points*.

While two Gromov models  $X$  and  $X'$  of a relatively hyperbolic pair  $(\Gamma, \Pi)$  are not necessarily quasi-isometric, there always exists a  $\Gamma$ -equivariant homeomorphism between their Gromov boundaries (see [Bow12]). The Bowditch boundary  $\partial_\infty(\Gamma, \Pi)$  is thus well-defined independently of choice of the Gromov models.

Recall the definition of a convergence group action.

**Definition 2.2.** Let  $Z$  be a metrizable compact set. An action of a discrete group  $\Gamma$  by homeomorphisms on  $Z$  is a *convergence group action* if for every unbounded sequence  $(\gamma_n)_{n \in \mathbb{N}}$  in  $\Gamma$ , there exists a subsequence  $(\gamma_{n_k})_{k \in \mathbb{N}}$  and a pair of points  $(x_-, x_+) \in Z^2$  such that  $\gamma_{n_k} z \rightarrow x_+$  as  $k \rightarrow +\infty$  for all  $z \in Z \setminus \{x_-\}$ . The points  $x_-$  and  $x_+$  are called respectively the *repelling* and *attracting* points of the subsequence  $(\gamma_{n_k})_{k \in \mathbb{N}}$ . In this case, we say a point  $p \in Z$  is

- (a) a *conical limit point* if there exists a sequence  $(\gamma_n)_{n \in \mathbb{N}}$  with attracting point  $p$  and repelling point distinct from  $p$ ;
- (b) a *bounded parabolic point* if the stabilizer of  $p$  in  $\Gamma$  is infinite and acts properly discontinuously and cocompactly on  $Z \setminus \{p\}$ .

We say that the  $\Gamma$ -action on  $Z$  is *geometrically finite* if  $Z$  consists of only conical limit points and bounded parabolic points.

If  $\Gamma$  is a discrete group of isometries of a  $\delta$ -hyperbolic space  $(X, d_X)$ , then the action of  $\Gamma$  on  $\partial_\infty X$  is a convergence group action. Moreover, a point  $p \in \partial_\infty X$  is a conical limit point if and only if there exists a sequence  $(\gamma_n)_{n \in \mathbb{N}}$  such that, for some (hence any) point  $o \in X$  and some (hence any) geodesic ray  $\ell$  converging to  $p$ , the sequence  $(\gamma_n o)_{n \in \mathbb{N}}$  converges to  $p$  and remains at bounded distance from  $\ell$ .

When  $(\Gamma, \Pi)$  is a relatively hyperbolic pair,  $\Gamma$  acts on  $\partial_\infty(\Gamma, \Pi)$  geometrically finitely (see [Bow12]). Conversely, Yaman [Yam04] proved the following theorem.

**Theorem 2.3** (Yaman [Yam04]). *Let  $\Gamma$  be a finitely generated group with a convergence group action on a perfect metrizable compact space  $Z$ . If the  $\Gamma$ -action is geometrically finite, then there are finitely many orbits  $\{\Gamma p_i\}_{i \in I}$  of bounded parabolic points, and  $\Gamma$  is hyperbolic relative to  $\Pi = \{\text{Stab}_\Gamma(p_i)\}_{i \in I}$ . Moreover,  $Z$  is  $\Gamma$ -equivariantly homeomorphic to the Bowditch boundary of  $(\Gamma, \Pi)$ .*

A relatively hyperbolic pair  $(\Gamma, \Pi)$  is called *elementary* if  $\partial_\infty(\Gamma, \Pi)$  contains at most two points. This happens when the  $\Pi_i$  are finite, and  $\Gamma$  is finite or virtually isomorphic to  $\mathbb{Z}$ , and when  $\Pi$  consists of only one subgroup of  $\Gamma$  of finite index.

Otherwise,  $\partial_\infty(\Gamma, \Pi)$  is a perfect metrizable compact set and the pair  $(\Gamma, \Pi)$  is called *non-elementary*. From now on, we always assume that the hyperbolic groups and hyperbolic pairs we consider are non-elementary.

We will be interested in the situation where a Gromov hyperbolic group also admits a relatively hyperbolic structure. This was studied by Osin [Osi06], Bowditch [Bow12], Tran [Tra13], Manning [Man15] etc.

Let  $\Gamma$  be a group, a finite collection  $\Pi = \{\Pi_i\}_{i \in I}$  of subgroups of  $\Gamma$  is called *almost malnormal* if for any  $i, j \in I$ ,  $\Pi_i \cap \gamma \Pi_j \gamma^{-1}$  is infinite only when  $i = j$  and  $\gamma \in \Pi_i$ .

**Theorem 2.4** ([Osi06] Proposition 2.36, Lemma 5.4, [Bow12] Theorem 7.11). *Let  $\Gamma$  be a nonelementary hyperbolic group and  $\Pi$  a finite, almost malnormal collection of quasiconvex subgroups. Then  $(\Gamma, \Pi)$  is relatively hyperbolic. Conversely, if  $\Gamma$  is both hyperbolic and hyperbolic relative to  $\Pi$ , then  $\Pi$  is an almost malnormal collection of quasiconvex subgroups.*

In the situation of Theorem 2.4, the relation between the Gromov and Bowditch boundaries is described in the following way.

**Theorem 2.5** ([Tra13] Main Theorem, [Man15] Theorem 1.3). *Let  $\Gamma$  be a nonelementary hyperbolic group and let  $\Pi$  be a finite almost malnormal collection of quasiconvex subgroups. Then there is a  $\Gamma$ -equivariant continuous surjective map*

$$\eta : \partial_\infty(\Gamma) \rightarrow \partial_\infty(\Gamma, \Pi) ,$$

such that,

- (a) *the preimage of a conical limit point by  $\eta$  is a singleton;*
- (b) *the preimage of the bounded parabolic point  $\gamma p_i$  fixed by  $\gamma \Pi_i \gamma^{-1}$  is  $\gamma \partial_\infty \Pi_i \subset \partial_\infty \Gamma$  for any  $i \in I$  and  $\gamma \in \Gamma$ .*

Informally, this theorem states that the Bowditch boundary of  $(\Gamma, \Pi)$  is obtained from the Gromov boundary of  $\Gamma$  by contracting the Gromov boundary of each conjugate of some  $\Pi_i, i \in I$  to a point.

**2.2. Geodesic flows.** We now introduce a very general notion of flow space for a finitely generated group  $\Gamma$ , of which an important source of examples will be given by geodesic flows of Gromov models of relatively hyperbolic groups. Let  $\Gamma$  be a discrete group.

**Definition 2.6.** A  $\Gamma$ -flow is the data of a Hausdorff topological space  $Y$  with a continuous flow  $\phi$  and a properly discontinuous action of  $\Gamma$  on  $Y$  that commutes with  $\phi$ . This  $\Gamma$ -flow is called *cocompact* if the quotient  $\Gamma \backslash Y$  is compact. A  $\Gamma$ -subflow of a  $\Gamma$ -flow  $(Y, \phi)$  is a  $\Gamma$ -invariant subflow of  $(Y, \phi)$ .

*Remark 2.7.* If  $\Gamma'$  is a subgroup of  $\Gamma$ , then every  $\Gamma$ -flow is automatically a  $\Gamma'$ -flow by restriction of the action.

We now introduce a weak notion of *morphism* between  $\Gamma$ -flows. Importantly, while such morphisms map orbits of one flow to orbits of the other, we do not require that they preserve the time of the flow.

**Definition 2.8.** A *morphism*  $\sigma$  between  $\Gamma$ -flows  $(Y_1, \phi_1)$  and  $(Y_2, \phi_2)$  is a  $\Gamma$ -equivariant continuous map  $\sigma : Y_1 \rightarrow Y_2$  for which there exists a map  $c_\sigma : Y_1 \times \mathbb{R} \rightarrow \mathbb{R}$  such that

$$\sigma(\phi_1^t(y)) = \phi_2^{c_\sigma(y,t)}(\sigma(y))$$

for all  $(y, t) \in Y_1 \times \mathbb{R}$ . Such a morphism  $\sigma$  is *quasi-isometric* if there exist constants  $\lambda \geq 1$  and  $\epsilon \geq 0$ , such that for any  $y \in Y_1$ , the map  $c_\sigma(y, \cdot) : \mathbb{R} \rightarrow \mathbb{R}$  is  $(\lambda, \epsilon)$ -quasi-isometric and  $c_\sigma(y, t) \rightarrow +\infty$  as  $t \rightarrow +\infty$ . A morphism  $\sigma$  is an isomorphism if it is a homeomorphism. In that case,  $\sigma^{-1}$  is automatically a morphism from  $(Y_2, \phi_2)$  to  $(Y_1, \phi_1)$ .

*Remark 2.9.* The map  $c_\sigma : Y_1 \times \mathbb{R} \rightarrow \mathbb{R}$  in the above definition is  $\Gamma$ -invariant and satisfies the cocycle rule

$$c_\sigma(y, t + s) = c_\sigma(y, t) + c_\sigma(\phi_1^t(y), s)$$

for all  $y \in Y_1$  and all  $s, t \in \mathbb{R}$ .

If  $\Gamma$  acts properly discontinuously by isometries on a  $\delta$ -hyperbolic space  $(X, d_X)$ , a direct way to construct a  $\Gamma$ -flow is to consider the collection of parametrized geodesics of  $X$ . Define

$$\mathcal{G}(X) = \{\ell \mid \ell : \mathbb{R} \rightarrow X \text{ geodesic}\},$$

equipped with the flow  $\psi$  defined by

$$\psi^t(\ell) : s \mapsto \ell(s + t).$$

Then  $(\mathcal{G}(X), \psi)$  is a  $\Gamma$ -flow. One can define the metric  $d'$  on  $\mathcal{G}(X)$  by

$$d'(\ell, \ell') = \int_{\mathbb{R}} \frac{e^{-|t|}}{2} d_X(\ell(s), \ell'(s)) ds,$$

for any  $\ell, \ell' \in \mathcal{G}(X)$  and verify that the projection  $\pi' : \mathcal{G}(X) \rightarrow X$  defined by  $\pi'(\ell) = \ell(0)$  is a  $\text{Isom}(X)$ -equivariant quasi-isometry.

The drawback of this construction is that quasi-isometries between hyperbolic spaces do not induce morphisms of flows. For our purposes, we will need a more “canonical” notion of geodesic flow, typically one for which pairs of distinct points in the boundary at infinity define a unique orbit of the flow. Such a flow is provided to us by a general theorem of Mineyev. We denote by  $A^{(2)}$  the set of ordered pairs of distinct points in a set  $A$ .

**Theorem 2.10** ([Min05] Theorem 60). *Let  $(X, d_X)$  be a taut hyperbolic metric space and  $\Gamma$  a discrete subgroup of  $\text{Isom}(X)$ . Then there exists a metric  $d$  on the topological space  $\mathcal{F}(X) = \partial_\infty^{(2)} X \times \mathbb{R}$  and a continuous cocycle  $c : \Gamma \times \partial_\infty X^{(2)} \rightarrow \mathbb{R}$  with the following properties:*

- (1). *The action of  $\Gamma$  on  $\mathcal{F}(X)$  given by*

$$\gamma(z_-, z_+, t) = (\gamma z_-, \gamma z_+, t + c(z_-, z_+, \gamma))$$

*is properly discontinuous and isometric;*

- (2). *There is a  $\Gamma$ -equivariant quasi-isometry  $\pi : (\mathcal{F}(X), d) \rightarrow (X, d_X)$ ;*  
(3). *There exist constants  $\lambda \geq 1$  and  $\epsilon \geq 0$  such that for any  $(z_-, z_+) \in \partial_\infty X^{(2)}$ , the curve  $\{\pi(z_-, z_+, t), t \in \mathbb{R}\}$  is a  $(\lambda, \epsilon)$ -quasi-geodesic in  $X$  with backward endpoint  $z_-$  and forward endpoint  $z_+$ .*

Note that the definition of the action of  $\Gamma$  on  $\mathcal{F}(X)$  implies that it commutes with the flow  $\phi$  defined by

$$\phi^t(z_-, z_+, s) = (z_-, z_+, t + s).$$

Hence  $(\mathcal{F}(X), \phi)$  is a  $\Gamma$ -flow.

When  $\Gamma$  is a hyperbolic group and  $X$  is a Gromov model of  $\Gamma$  (e.g. its Cayley graph with respect to a finite generating set), we define *geodesic flow of  $\Gamma$*  to be the space  $\mathcal{F}(\Gamma) \stackrel{\text{def}}{=} \mathcal{F}(X)$  equipped with the flow  $\phi$  and the action of  $\Gamma$ . This  $\Gamma$ -flow is well-defined up to quasi-isometric isomorphisms.



When  $X$  is a Gromov model of a relatively hyperbolic pair  $(\Gamma, \Pi)$  (e.g. its *Groves–Manning cusp space* as defined in [GM08]), we call *geodesic flow of  $(\Gamma, \Pi)$*  the space  $\mathcal{F}(\Gamma, \Pi) \stackrel{\text{def}}{=} \mathcal{F}(X)$  equipped with the flow  $\phi$  and the action of  $\Gamma$ . Now, it is only well-defined up to isomorphisms of  $\Gamma$ -flow, since different Gromov models of  $(\Gamma, \Pi)$  may not be quasi-isometric to each other. The next proposition is independent to the choice of the Gromov models for defining  $\mathcal{F}(\Gamma, \Pi)$ .

**Proposition 2.11.** *Let  $\Gamma$  be a hyperbolic group and  $\Pi$  a finite almost malnormal collection of quasi-convex subgroups. Then there is a morphism of  $\Gamma$ -flows*

$$\sigma : \mathcal{F}(\Gamma) \setminus \bigcup_{i \in I, \gamma \in \Gamma} \gamma \mathcal{F}(\Pi_i) \longrightarrow \mathcal{F}(\Gamma, \Pi)$$

that extends the quotient map  $\eta : \partial_\infty \Gamma \rightarrow \partial_\infty(\Gamma, \Pi)$  from Theorem 2.5. Moreover, the restriction of  $\sigma$  to any cocompact  $\Gamma$ -subflow is a quasi-isometric morphism.

*Proof.* Let  $X$  be any Gromov model of the relatively hyperbolic pair  $(\Gamma, \Pi)$ . We fix  $\mathcal{F}(\Gamma, \Pi)$  to be the  $\Gamma$ -flow  $\mathcal{F}(X)$ . Consider the space

$$U = \left\{ \left( (z_-, z_+, s_1), (\eta z_-, \eta z_+, s_2) \right) : z_-, z_+ \in \partial_\infty \Gamma, s_1, s_2 \in \mathbb{R}, \eta z_- \neq \eta z_+ \right\} \subset \mathcal{F}(\Gamma) \times \mathcal{F}(\Gamma, \Pi)$$

There is a natural  $\Gamma$ -action on  $U$  by coordinates since  $\eta$  is  $\Gamma$ -equivariant. Note that the projection to the first coordinate maps  $U$  to  $\mathcal{F}(\Gamma) \setminus \bigcup_{i \in I, \gamma \in \Gamma} \gamma \mathcal{F}(\Pi_i)$  by Theorem 2.5. This projection factors to a fiber bundle

$$p_1 : \Gamma \backslash U \rightarrow \Gamma \backslash \left( \mathcal{F}(\Gamma) \setminus \bigcup_{i \in I, \gamma \in \Gamma} \gamma \mathcal{F}(\Pi_i) \right)$$

with fibers homeomorphic to  $\mathbb{R}$ . Let  $\hat{\sigma}$  be a continuous section of  $p_1$ . Lifting  $\hat{\sigma}$  to  $\mathcal{F}(\Gamma) \setminus \bigcup_{i \in I, \gamma \in \Gamma} \gamma \mathcal{F}(\Pi_i)$  and composing with the projection to the second coordinate, we obtain a morphism of  $\Gamma$ -flows

$$\sigma : \mathcal{F}(\Gamma) \setminus \bigcup_{i \in I, \gamma \in \Gamma} \gamma \mathcal{F}(\Pi_i) \longrightarrow \mathcal{F}(X(\Gamma, \Pi))$$

which extends the boundary map  $\eta$ .

It remains to prove that  $\sigma$  is quasi-isometric in restriction to any cocompact  $\Gamma$ -subflow. Let us thus introduce the cocycle

$$c : \left( \mathcal{F}(\Gamma) \setminus \bigcup_{i \in I, \gamma \in \Gamma} \gamma \mathcal{F}(\Pi_i) \right) \times \mathbb{R} \longrightarrow \mathbb{R}$$

such that

$$\sigma(\phi^t(z)) = \psi^{c(z,t)}(\sigma(z))$$

for any  $z \in \mathcal{F}(\Gamma) \setminus \bigcup_{i \in I, \gamma \in \Gamma} \gamma \mathcal{F}(\Pi_i)$  and  $t \in \mathbb{R}$ .

Let  $\tilde{K}$  be a cocompact  $\Gamma$ -subflow of  $\mathcal{F}(\Gamma) \setminus \bigcup_{i \in I, \gamma \in \Gamma} \gamma \mathcal{F}(\Pi_i)$ . Since  $c$  satisfies the cycle rule

$$c(x, t + s) = c(x, t) + c(\phi^t(x), s) ,$$

in order to prove that it is quasi-isometric, it suffices to verify the following statement.

*Claim.* There exist constants  $T, m, M > 0$ , such that  $m \leq c(z, T) \leq M$  for any  $z \in \tilde{K}$ .

Let  $K$  be a compact subset of  $\mathcal{F}(\Gamma) \setminus \bigcup_{i \in I, \gamma \in \Gamma} \gamma \mathcal{F}(\Pi_i)$  such that  $\Gamma K = \widetilde{K}$ . Since  $c$  is continuous and  $\Gamma$ -invariant, we only need to show that there exists a constant  $T > 0$  such that  $c(z, T) > 0$  for any  $z \in K$ .

We argue by contradiction. Suppose that there is a sequence  $(z^n)_{n \in \mathbb{N}} \subset K$  and a sequence  $(T_n)_{n \in \mathbb{N}} \subset \mathbb{R}_+$  with  $T_n \rightarrow +\infty$  as  $n \rightarrow +\infty$ , such that  $c(z^n, T_n) \leq 0$ .

Up to a subsequence, we may assume  $z^n = (z_-^n, z_+^n, s_n)$  converges to  $z = (z_-, z_+, s) \in K$  as  $n \rightarrow +\infty$ . For each  $n$ , there exists  $\gamma_n \in \Gamma$  such that  $\gamma_n^{-1} \phi^{T_n}(z^n) \in K$  by  $\Gamma K = \widetilde{K}$ . Then for a base point  $z^0 \in K$ , we have  $\gamma_n z^0 \rightarrow z_+$  as  $n \rightarrow +\infty$  since  $d(\gamma_n z^0, \phi^{T_n}(z^n)) = d(z^0, \gamma_n^{-1} \phi^{T_n}(z^n))$  is bounded by the diameter of  $K$  and  $\phi^{T_n}(z^n) \rightarrow z_+$  as  $n \rightarrow +\infty$ . Therefore, the sequence  $(\gamma_n)_{n \in \mathbb{N}}$  has attracting point  $z_+$  for the convergence group action of  $\Gamma$  on  $\partial_\infty \Gamma$ .

We apply the argument above by replacing  $z^n$  by  $\sigma(z^n)$  in the compact set  $\sigma(K)$ , and replacing  $T_n$  by  $c(z^n, T_n)$ . Since  $c(z^n, T_n) \leq 0$ , after further extraction, either  $\gamma_n \sigma(z^0)$  is bounded in  $\mathcal{F}(X(\Gamma, \Pi))$  or  $\gamma_n \sigma(z^0) \rightarrow \eta(z_-)$  as  $n \rightarrow +\infty$ , depending on whether  $c(z^n, T_n)$  is bounded or diverges to  $-\infty$ . Since  $\gamma_n$  is unbounded and  $\Gamma$  acts properly on  $\mathcal{F}(X)$ , we conclude that  $\gamma_n \sigma(z^0) \rightarrow \eta(z_-)$  as  $n \rightarrow +\infty$  and  $\eta(z_-)$  is the attracting point of  $(\gamma_n)$  in  $\partial_\infty(\Gamma, \Pi)$ .

On the other hand, since  $\eta$  is  $\Gamma$ -equivariant, it must send the attracting point of  $\gamma_n$  in  $\partial_\infty \Gamma$  to its attracting point in  $\partial_\infty(\Gamma, \Pi)$ . We deduce that  $\eta(z_+) = \eta(z_-)$ . By Proposition 2.5, this implies that  $z_-$  and  $z_+$  both belong to  $\gamma \partial_\infty P_i$  for some  $i \in I$  and some  $\gamma \in \Gamma$ , contradicting the assumption that

$$z \in K \subset \mathcal{F}(\Gamma) \setminus \bigcup_{i \in I, \gamma \in \Gamma} \gamma \mathcal{F}(\Pi_i).$$

□

### 3. RESTRICTED ANOSOV AND RELATIVELY ANOSOV REPRESENTATIONS

We always fix the field  $\mathbb{K}$  to be  $\mathbb{R}$  or  $\mathbb{C}$ .

The notion of *Anosov representation* of a hyperbolic group  $\Gamma$  admits many equivalent definitions. One of them, which is close to Labourie's original definition and was developed by Bochi–Potrie–Sambarino [BPS19], states that a linear representation  $\rho : \Gamma \rightarrow \mathrm{SL}(d, \mathbb{K})$  is  $k$ -Anosov if the associated flat bundle admits a dominated splitting over the geodesic flow of  $\Gamma$ . The purpose of this section is to investigate the generalization of this definition when the geodesic flow is replaced by any  $\Gamma$ -flow.

**3.1. Linear representations and dominated splittings.** Let  $\Gamma$  be a countable group and let  $(Y, \phi)$  be a  $\Gamma$ -flow. For a representation  $\rho : \Gamma \rightarrow \mathrm{SL}(d, \mathbb{K})$ , we consider the flat bundle  $E_\rho(Y) = \Gamma \backslash (Y \times \mathbb{K}^d)$ , where the  $\Gamma$ -action is given by  $\gamma(y, v) = (\gamma y, \rho(\gamma)v)$  for all  $\gamma \in \Gamma$ ,  $y \in Y$  and  $v \in \mathbb{K}^d$ . The flow  $\phi$  on  $Y$  lifts to a flow on  $Y \times \mathbb{K}^d$  by parallel transformations, which we still denote by  $\phi$ , namely,

$$\phi^t(y, v) = (\phi^t(y), \phi_y^t(v)) = (\phi^t(y), v)$$

for all  $y \in Y$ ,  $v \in \mathbb{K}^d$  and  $t \in \mathbb{R}$ . This flow commutes with the  $\Gamma$ -action and thus factors to a flow on  $E_\rho(Y)$ , which we again denote by  $\phi$ .

**Definition 3.1.** A representation  $\rho$  of  $\Gamma$  into  $\mathrm{SL}(d, \mathbb{K})$  is  $k$ -Anosov in restriction to the  $\Gamma$ -flow  $(Y, \phi)$  if there exists a metric  $\|\cdot\|$  on the vector bundle  $E_\rho(Y)$  such that  $E_\rho(Y)$  admits a *dominated splitting of rank  $k$* , that is, a continuous  $\phi$ -invariant

decomposition  $E_\rho(Y) = E_\rho^s \oplus E_\rho^u$  with  $E_\rho^s$  of rank  $k$ , for which there exist constants  $C, \lambda > 0$ , such that

$$\frac{\|\phi_y^t(v)\|}{\|\phi_y^t(w)\|} \leq C e^{-\lambda t} \frac{\|v\|}{\|w\|},$$

for all  $y \in \Gamma \backslash Y$ ,  $t \in \mathbb{R}_+$  and all non zero vectors  $v \in (E_\rho^s)_y$  and  $w \in (E_\rho^u)_y$ . We respectively call  $E_\rho^s$  and  $E_\rho^u$  the *stable direction* and *unstable direction* of  $E_\rho(Y)$  with respect to the metric  $\|\cdot\|$ .

*Remark 3.2.* When the  $\Gamma$ -flow  $(Y, \phi)$  is cocompact, the dominated splitting is unique and does not depend on the choice of the metric, since any two metrics on  $E_\rho(Y)$  are uniformly bi-Lipschitz.

*Remark 3.3.* By abuse of notations, we will write  $Y \times \mathbb{K}^d = E_\rho^s \oplus E_\rho^u$  to represent the lift of the dominated splitting over  $Y$ , as a decomposition of  $Y \times \mathbb{K}^d$  into  $\Gamma$ -invariant,  $\phi$ -invariant subbundles. The metric  $\|\cdot\|$  will lift to a  $\Gamma$ -invariant one, still denoted by  $\|\cdot\|$ . We call it a dominated splitting of  $Y \times \mathbb{K}^d$  of rank  $k$  associated to  $\rho$  and  $\|\cdot\|$ .

*Remark 3.4.* Let  $\|\cdot\|_0$  denote the standard metric on  $\mathbb{K}^d$ . Since  $Y \times \mathbb{K}^d$  is a trivial bundle, for any metric  $\|\cdot\|$ , there exists a continuous map  $A : Y \rightarrow \text{GL}(d, \mathbb{K})$ , such that at any point  $y \in Y$ , the metric  $\|\cdot\|$  is expressed by  $\|A(y) \cdot\|_0$ . We will say that  $\|\cdot\|$  is of unit volume if there exists such a map  $A$  takes values in  $\text{SL}(d, \mathbb{K})$ . Since a rescaling of the metric preserves the ratio of the norms of two vectors, we may assume without loss of generality that the metric in Definition 3.1 is always of unit volume.

One of the good properties of the restricted Anosov definition is that it is preserved under pull-back by quasi-isometric morphisms of  $\Gamma$ -flows.

**Proposition 3.5.** *Let  $\sigma : (Y_2, \phi_2) \rightarrow (Y_1, \phi_1)$  be a quasi-isometric morphism of  $\Gamma$ -flows, and let  $\rho : \Gamma \rightarrow \text{SL}(d, \mathbb{K})$  be a  $k$ -Anosov representation in restriction to  $(Y_1, \phi_1)$ . Then  $\rho$  is  $k$ -Anosov in restriction to  $(Y_2, \phi_2)$ .*

*Proof.* Let  $c : Y \times \mathbb{R} \rightarrow \mathbb{R}$  be the cocycle such that

$$\sigma(\phi_2^t(y)) = \phi_1^{c(y,t)}(\sigma(y)).$$

By definition of a quasi-isometric morphism, there exist constants  $a > 0, b > 0$  such that

$$c(y, t) \geq at - b$$

for all  $y \in Y$  and all  $t \geq 0$ .

The morphism  $\sigma$  naturally lifts to a continuous bundle morphism  $\sigma : E_\rho(Y_1) \rightarrow E_\rho(Y_2)$ . Let  $\|\cdot\|$  be a continuous metric for which  $E_\rho(Y_1)$  has a dominated splitting of rank  $k$

$$E_\rho(Y_1) = E_\rho^s(Y_1) \oplus E_\rho^u(Y_1).$$

Pulling back this splitting by  $\sigma$ , we get a continuous  $\phi_1$ -invariant splitting

$$E_\rho(Y_2) = E_\rho^s(Y_2) \oplus E_\rho^u(Y_2).$$

Let us still denote by  $\|\cdot\|$  the pull-back by  $\sigma$  of the metric on  $E_\rho(Y_1)$ . With these choices, we have for all  $y \in \Gamma \setminus Y_2$  and all  $v \in E_\rho^s(Y_2)$ ,  $w \in E_\rho^u(Y_2)$ :

$$\begin{aligned} \frac{\|\phi_2^t(v)\|}{\|\phi_2^t(w)\|} &= \frac{\|\sigma(\phi_2^t(v))\|}{\|\sigma(\phi_2^t(w))\|} = \frac{\|\phi_1^{c(y,t)}(\sigma(v))\|}{\|\phi_1^{c(y,t)}(\sigma(w))\|} \\ &\leq C e^{-\lambda c(y,t)} \frac{\|\sigma(v)\|}{\|\sigma(w)\|} \\ &\leq C e^b e^{-a\lambda t} \frac{\|v\|}{\|w\|}, \end{aligned}$$

showing that the splitting  $E_\rho^s(Y_2) \oplus E_\rho^u(Y_2)$  is dominated.  $\square$

Another good property of the notion of restricted Anosov representation is its stability under passing to a subgroup. Let  $\Gamma$  be a countable group,  $(Y, \phi)$  a  $\Gamma$ -flow and  $\Gamma'$  a subgroup of  $\Gamma$ . Then  $(Y, \phi)$  can be seen as a  $\Gamma'$ -flow by restricting the  $\Gamma$ -action to  $\Gamma'$ .

**Proposition 3.6.** *Let  $\rho : \Gamma \rightarrow \mathrm{SL}(n, \mathbb{K})$  be a linear representation.*

- (1) *If  $\rho$  is  $k$ -Anosov in restriction to  $(Y, \phi)$ . Then  $\rho|_{\Gamma'}$  is a  $k$ -Anosov in restriction to  $(Y, \phi)$ .*
- (2) *Conversely, assume that  $(Y, \phi)$  is cocompact and  $\Gamma'$  has finite index in  $\Gamma$ . If  $\rho|_{\Gamma'}$  is  $k$ -Anosov in restriction to  $(Y, \phi)$ , then  $\rho$  is  $k$ -Anosov in restriction to  $(Y, \phi)$ .*

*Proof.* For part (1), let  $\|\cdot\|$  be a  $\Gamma$ -invariant norm on  $E_\rho(Y)$  and let  $E_\rho(Y) = E_\rho^s(Y) \oplus E_\rho^u(Y)$  be a  $\Gamma$ -invariant and  $\phi$ -invariant splitting of rank  $k$  that is dominated for  $\|\cdot\|$ . Then the splitting and the norm are in particular  $\Gamma'$ -invariant and define a dominated splitting over  $\Gamma' \setminus Y$ .

Now we prove part (2). By (1) we can restrict to a smaller subgroup of finite index and assume that  $\Gamma'$  is normal in  $\Gamma$ . let  $E_\rho(Y) = E_\rho^s(Y) \oplus E_\rho^u(Y)$  be a  $\phi$ -invariant and  $\Gamma'$ -invariant splitting of rank  $k$  over  $Y$ , which is dominated for a  $\Gamma'$ -invariant norm  $\|\cdot\|$ . Since  $\Gamma'$  acts cocompactly on  $Y$ , the domination condition does not depend on the choice of the norm and we can assume without loss of generality that  $\|\cdot\|$  is in fact  $\Gamma$ -invariant.

For  $\gamma \in \Gamma$ , consider the push-forward of the splitting, defined by

$$\gamma_* E_\rho^s(Y)_x \oplus \gamma_* E_\rho^u(Y)_x = \rho(\gamma) E_\rho^s(Y)_{\gamma^{-1}x} \oplus \rho(\gamma) E_\rho^u(Y)_{\gamma^{-1}x}$$

Since the action of  $\Gamma$  on  $Y \times \mathbb{R}^d$  commutes with the flow  $\phi$ , the push-forward splitting is again  $\phi$ -invariant. Moreover, for  $\eta \in \Gamma'$ , we have

$$\begin{aligned} \gamma_* E_\rho^s(Y)_{\eta x} &= \rho(\gamma) E_\rho^s(Y)_{\gamma^{-1}\eta x} \\ &= \rho(\gamma) E_\rho^s(Y)_{\gamma^{-1}\eta\gamma(\gamma^{-1}x)} \\ &= \rho(\gamma)\rho(\gamma^{-1}\eta\gamma) E_\rho^s(Y)_{\gamma^{-1}x} \\ &\quad \text{since } \Gamma' \text{ is normal in } \Gamma \text{ and } E_\rho^s(Y) \text{ is } \Gamma' \text{ invariant} \\ &= \rho(\eta)\gamma_* E_\rho^s(Y)_x. \end{aligned}$$

The same holds for  $\gamma_* E_\rho^u(Y)$ , showing that the push-forward splitting is again  $\Gamma'$  invariant. Finally, since  $\|\cdot\|$  is  $\Gamma$ -invariant, we get that the push-forward splitting

is a dominated splitting over  $\Gamma' \setminus Y$ . By uniqueness of such a splitting we conclude that

$$\gamma_* E_\rho^s(Y) \oplus \gamma_* E_\rho^u(Y) = E_\rho^s(Y) \oplus E_\rho^u(Y) .$$

Hence  $E_\rho^s(Y) \oplus E_\rho^u(Y)$  is a dominated splitting over  $\Gamma \setminus Y$  and  $\rho$  is  $k$ -Anosov in restriction to  $Y$ .  $\square$

Finally, one of the main properties of restricted Anosov representations over cocompact  $\Gamma$ -flows is their stability under small deformation.

**Proposition 3.7.** *Let  $(Y, \phi)$  be a cocompact  $\Gamma$ -flow. Then the space*

$$A_Y^k(\Gamma, \mathbf{SL}(d, \mathbb{K})) = \{\rho : \Gamma \rightarrow \mathbf{SL}(d, \mathbb{K}) \text{ } k\text{-Anosov in restriction to } Y\}$$

*is open in  $\text{Hom}(\Gamma, \mathbf{SL}(d, \mathbb{K}))$ .*

Proposition 3.7 essentially follows from a general stability theorem for dominated splittings (see for instance [Shu13] Corollary 5.19. [CZZ22] Theorem 8.1 or [Wan23] Theorem 7.1). A key point is that one can see the linear flows associated to representations in the neighbourhood of a representation  $\rho_0$  as perturbations of the flow  $\phi$  on the fixed vector bundle  $E_\rho(Y)$ . This is ensured by the following Lemma.

**Lemma 3.8.** *Let  $\rho_0 : \Gamma \rightarrow \mathbf{SL}(d, \mathbb{K})$  be a representation. Then there exists a neighborhood  $O$  of  $\rho_0$  in  $\text{Hom}(\Gamma, \mathbf{SL}(d, \mathbb{K}))$  and a continuous map*

$$g : O \times Y \rightarrow \mathbf{GL}(d, \mathbb{K}),$$

*such that*

$$\rho(\gamma)g(\rho, y) = g(\rho, \gamma y)\rho_0(\gamma)$$

*and  $g|_{\rho_0 \times Y} \equiv \text{Id}$ .*

*Proof.* Let  $K$  be a compact subset of  $Y$  such that  $\Gamma K = Y$ . Let  $U$  be an open, relatively compact subset of  $Y$  that contains  $K$ . Then there exists a continuous function  $f : Y \rightarrow \mathbb{R}_{\geq 0}$  such that  $f = 1$  on  $K$  and  $f = 0$  on  $Y \setminus U$ . Define

$$g : \text{Hom}(\Gamma, \mathbf{SL}(d, \mathbb{K})) \times Y \rightarrow \text{Mat}_{d \times d}(\mathbb{K})$$

$$(\rho, y) \mapsto \frac{1}{\sum_{\gamma \in \Gamma} f(\gamma^{-1}y)} \sum_{\gamma \in \Gamma} f(\gamma^{-1}y) \rho(\gamma) \circ \rho_0(\gamma)^{-1} .$$

Note that  $f(\gamma^{-1}y) = 0$  for all but finitely many  $\gamma$  (by the properness of the  $\Gamma$  action and relative compactness of  $U$ ),  $\sum_{\gamma \in \Gamma} f(\gamma^{-1}y) \geq 1$  since  $\Gamma y \cap K \neq \emptyset$  and  $f \equiv 1$  on  $K$ . Hence  $g$  is well-defined and continuous. One easily verifies that

$$(1) \quad \rho(\gamma)g(\rho, y) = g(\rho, \gamma y)\rho_0(\gamma)$$

and that  $g|_{\rho_0 \times Y} \equiv \text{Id}$ . By the continuity of  $g$ , there is a neighborhood  $O$  of  $\rho_0$  such that  $g|_{O \times K}$  takes values in  $\mathbf{GL}(d, \mathbb{K})$ . Finally the equivariance property (1) implies that  $g|_{O \times Y}$  takes values in  $\mathbf{GL}(d, \mathbb{K})$ .  $\square$

The equivariance property of  $g$  precisely means that  $g(\rho, \cdot)$  factors to a bundle isomorphism from  $E_{\rho_0}(Y)$  to  $E_\rho(Y)$  which depends continuously on  $\rho$ . The rest of the proof of Proposition 3.7 follows classical stability arguments for dominated splittings. We sketch here for completeness.

*Proof of Proposition 3.7.* Let  $\iota_0$  denote the  $\Gamma$ -action on  $O \times Y \times \mathbb{K}^d$  defined by  $\gamma(\rho, y, v) = (\rho, \gamma y, \rho_0(\gamma)y)$  for any  $\rho \in O$ ,  $\gamma \in \Gamma$  and  $v \in \mathbb{K}^d$ , and let  $\iota$  denote the  $\Gamma$ -action on  $O \times Y \times \mathbb{K}^d$  defined by  $\gamma(\rho, y) = (\rho, y, \rho(\gamma)v)$   $\rho \in O$ ,  $\gamma \in \Gamma$  and  $v \in \mathbb{K}^d$ . It is natural to extend  $\phi$  on  $O \times Y$  by  $\phi^t(\rho, y) = (\rho, \phi^t(y))$ .

The map  $g$  given by the lemma above induces a  $\iota_0$ - $\iota$ -equivariant map

$$O \times Y \times \mathbb{K}^d \simeq O \times Y \times \mathbb{K}^d$$

by  $(\rho, y, v) \rightarrow (\rho, y, g(\rho, y)v)$  for any  $\rho \in O$ ,  $y \in Y$  and  $v \in \mathbb{K}^d$ , which is an isomorphism of vector bundles fibring over  $id_{O \times Y}$ , and hence induces a continuous isomorphism

$$\hat{g} : \iota_0(\Gamma) \backslash (O \times Y \times \mathbb{K}^d) \simeq \iota(\Gamma) \backslash (O \times Y \times \mathbb{K}^d) .$$

A dominated splitting  $E_{\rho_0}(Y) = E_{\rho_0}^s \oplus E_{\rho_0}^u$  induces a dominated splitting

$$\iota_0(\Gamma) \backslash (O \times Y \times \mathbb{K}^d) = (O \times E_{\rho_0}^s) \oplus (O \times E_{\rho_0}^u)$$

with respect to a  $\Gamma$ -invariant metric. Then there is a decomposition

$$\iota(\Gamma) \backslash (O \times Y \times \mathbb{K}^d) = \hat{g}(O \times E_{\rho_0}^s) \oplus \hat{g}(O \times E_{\rho_0}^u) .$$

We denote  $E_O^s = \hat{g}(O \times E_{\rho_0}^s)$  and  $E_O^u = \hat{g}(O \times E_{\rho_0}^u)$  in brief. The decomposition is  $\phi$ -invariant over  $\{\rho_0\} \times \Gamma \backslash Y$ , but may not be  $\phi$ -invariant over the whole  $O \times \Gamma \backslash Y$ . We wish to make it  $\phi$ -invariant by small deformation. More concretely, we define a flow  $\Phi$  (respectively,  $\Psi$ ) on the space of continuous sections of  $\text{Hom}(E_O^u, E_O^s)$  (respectively,  $\text{Hom}(E_O^s, E_O^u)$ ) with norm at most 1, such that  $\phi^t$  maps the graph of  $f_{(\rho, y)}$  to the graph of  $\Phi^t(f_{(\rho, y)})$  (respectively,  $\Psi^t(f_{(\rho, y)})$ ) for any section  $f$  with norm at most 1 and  $(\rho, y) \in O \times \Gamma \backslash Y$ . Following the argument of Lemma 7.4 in [Wan23],  $\Phi$  and  $\Psi$  are well-defined contraction maps. The images of the unique fixed point of  $\Phi$  and the unique fixed point of  $\Psi$ , which are independent to  $t$ , give a new decomposition of  $\iota(\Gamma) \backslash (O \times Y \times \mathbb{K}^d)$ , which is  $\phi$ -invariant. Up to replacing  $O$  by a smaller neighborhood of  $\rho_0$ , this decomposition of  $\iota(\Gamma) \backslash (O \times Y \times \mathbb{K}^d)$  gives a dominated splitting for each piece  $\iota(\Gamma) \backslash (\{\rho\} \times Y \times \mathbb{K}^d) \simeq E_\rho(Y)$ , which completes the proof.  $\square$

**3.2. Relatively Anosov representations.** While Anosov representations are meant to generalize convex-cocompact representations to higher rank Lie groups, the notion of *relatively Anosov representation* introduced by Zhu [Zhu21] and Zhu–Zimmer [ZZ22a], which is equivalent to that of *asymptotically embedded representation* introduced previously by Kapovich–Leeb [KL23], is meant to extend to higher rank the geometrically finite representations of relatively hyperbolic groups.

Let  $(\Gamma, \Pi)$  be a relatively hyperbolic pair.

**Definition 3.9.** A representation  $\rho : \Gamma \rightarrow \text{SL}(d, \mathbb{K})$  is  $k$ -Anosov relative to  $\Pi$  if there exists a pair of continuous maps

$$\xi = (\xi^k, \xi^{d-k}) : \partial_\infty(\Gamma, \Pi) \rightarrow \text{Gr}_K(\mathbb{K}^d) \times \text{Gr}_{d-k}(\mathbb{K}^d)$$

which is

- $\rho$ -equivariant, that is,  $\rho(\gamma)\xi(\cdot) = \xi(\gamma(\cdot))$  for any  $\gamma \in \Gamma$ ;
- transverse, that is,  $\xi^k(x) \oplus \xi^{d-k}(y) = \mathbb{K}^d$  for any  $x \neq y \in \partial_\infty(\Gamma, \Pi)$ ;

- strongly dynamics preserving, that is, for any sequence  $(\gamma_n)_{n \in \mathbb{N}} \subset \Gamma$  such that  $\gamma_n \rightarrow x \in \partial_\infty(\Gamma, \Pi)$  and  $\gamma_n^{-1} \rightarrow y \in \partial_\infty(\Gamma, \Pi)$  as  $n \rightarrow +\infty$ , one has  $\rho(\gamma_n)V \rightarrow \xi^k(x)$  as  $n \rightarrow +\infty$  for any  $V \in \text{Gr}_k(\mathbb{K}^d)$  that transverse to  $\xi^{d-k}(y)$ .

Let  $X$  be a Gromov model of  $(\Gamma, \Pi)$ . Let  $\rho : \Gamma \rightarrow \text{SL}(d, \mathbb{K})$  be a  $k$ -Anosov representation relative to  $\Pi$ . The pair of transverse boundary maps associated to  $\rho$  defines a  $\Gamma$ -invariant and  $\phi$ -invariant splitting of  $E_\rho(\mathcal{G}(X))$ , which Zhu and Zimmer prove to be dominated:

**Theorem 3.10** ([ZZ22a] Theorem 1.3). *Let  $\rho : \Gamma \rightarrow \text{SL}(d, \mathbb{K})$  be a  $k$ -Anosov representation relative to  $\Pi$ . Then  $\rho$  is  $k$ -Anosov in restriction to the  $\Gamma$ -flow  $\mathcal{G}(X)$ .*

*Remark 3.11.* The notion of being “Anosov in restriction  $\mathcal{G}(X)$ ” is called “Anosov relative to  $X$ ” in [ZZ22a].

The main example of a relatively representation is the inclusion of a *geometrically finite subgroup* of  $\text{SL}(2, \mathbb{C})$ . Recall that a subgroup  $\Gamma$  of  $\text{SL}(2, \mathbb{C})$  is called a *geometrically finite subgroup* if the action of  $\Gamma$  on its limit set  $\Lambda(\Gamma) \subset \mathbb{C}\mathbb{P}^1$  is geometrically finite in the sense of Definition 2.2 (see [Bow12]). In particular, such a group  $\Gamma$  is hyperbolic relatively to the collection  $\Pi$  of stabilizers of its parabolic points.

**Proposition 3.12** ([ZZ22b] Proposition 1.7). *Let  $\Gamma$  be a geometrically finite subgroup of  $\text{SL}(2, \mathbb{C})$  and  $\Pi$  the collection of its parabolic stabilizers. Then the inclusion  $\Gamma \hookrightarrow \text{SL}(2, \mathbb{C})$  is 1-Anosov relative to  $\Pi$ .*

When  $\Gamma \subset \text{SL}(2, \mathbb{C})$  is geometrically finite with  $\Pi$  a set of representatives of the conjugacy classes of maximal parabolic subgroups of  $\Gamma$ , the convex hull of  $\Lambda(\Gamma)$  in  $\mathbb{H}^3$ , denoted by  $\mathcal{C}(\Gamma)$ , is a Gromov model of the relatively hyperbolic pair  $(\Gamma, \Pi)$ . Since  $\mathbb{H}^3$  is uniquely geodesic,  $\mathcal{C}(\Gamma)$  contains a unique bi-infinite geodesic between two given points of  $\Lambda(\Gamma) \simeq \partial_\infty(\Gamma, \Pi)$ , and one can thus identify  $\mathcal{G}(\mathcal{C}(\Gamma))$  with  $\mathcal{F}(\Gamma, \Pi)$  by an isomorphism of  $\Gamma$ -flows. In particular, the inclusion  $\Gamma \hookrightarrow \text{SL}(2, \mathbb{C})$  is also 1-Anosov in restriction to  $\mathcal{F}(\Gamma, \Pi)$ .

*Remark 3.13.* While the relatively property is independent of the Gromov model, Zhu and Zimmer do not state that it implies the Anosov property in restriction to  $\mathcal{F}(\Gamma, \Pi)$ . This will be proven by the second author in a forthcoming paper. Here, we do not care about this subtlety because we will ultimately consider geometrically finite subgroups of  $\text{PSL}(2, \mathbb{C})$ , for  $\mathcal{F}(\Gamma, \Pi)$  is isomorphic to the geodesic flow of the convex core in  $\mathbb{H}^3$ .

**3.3. Simple Anosov representations.** An example that motivates the study of restricted Anosov representations is the notion of primitive-stable representations introduced by Minsky [Min13]. Let  $F_n$  be a free group of order  $n$ . Let  $\mathcal{F}_{\text{Prim}} \subset \mathcal{F}(F_n)$  denote the *primitive geodesic flow*, which is the closure of the collection of all geodesic axes of primitive elements of  $F_n$ . The second author proved in [Wan23] Section 8.1 that a representation  $\rho : F_n \rightarrow \text{SL}(d, \mathbb{K})$  is  $k$ -primitive-stable in the sense of Minsky [Min13], Guéritaud–Guichard–Kassel–Wienhard [GGKW17] and Kim–Kim [KK21], if and only if  $\rho$  is  $k$ -Anosov in restriction to  $\mathcal{F}_{\text{Prim}}$ .

Here we introduce the notion of *simple Anosov representations* which can be thought of as an analogue of primitive-stable representations for closed surface groups.

Let  $\pi_1(S)$  be the fundamental group of a closed connected oriented surface  $S$  of genus  $g \geq 2$ . Such a surface carries hyperbolic metrics, and for each hyperbolic metric  $h$ , there is a discrete and faithful representation  $j : \pi_1(S) \rightarrow \mathrm{PSL}(2, \mathbb{R})$  (a *Fuchsian representation*) such that  $(S, h)$  is isomorphic to  $j(\pi_1(S)) \backslash \mathbb{H}^2$ .

Since  $\mathbb{H}^2$  is uniquely geodesic with the  $\pi_1(S)$ -action cocompact, the  $\pi_1(S)$ -flow  $\mathcal{F}(\pi_1(S))$  is isomorphic to the geodesic flow  $\psi$  on the unit tangent bundle  $T^1(\mathbb{H}^2)$  equipped with the  $\pi_1(S)$ -action given by  $\rho$ .<sup>1</sup>

Therefore, by Proposition 3.5 and [BPS19] Section 4, a representation  $\rho : \pi_1(S) \rightarrow \mathrm{SL}(d, \mathbb{K})$  is  $k$ -Anosov (in the classical sense) if and only if  $\rho$  is  $k$ -Anosov in restriction to  $(T^1(\mathbb{H}^2), \psi)$ .

Now, let  $\mathcal{F}_p(S) \subset T^1(S)$  be the closure of the union of all closed geodesics with at most  $p$  self-intersections, and denote by  $\mathcal{F}_p(\pi_1(S))$  its preimage in  $T^1(\mathbb{H}^2) \simeq \mathcal{F}(\pi_1(S))$ . Then  $\mathcal{F}_p(\pi_1(S))$  is again a cocompact  $\pi_1(S)$ -subflow of  $\mathcal{F}(\pi_1(S))$ .

**Proposition 3.14.** *Let  $c \subset T^1(S)$  be a closed geodesic with more than  $p$  self-intersections. Then  $c \subset T^1(S) \setminus \mathcal{F}_p(S)$ .*

*Proof.* Let  $v$  be a point in  $c$  and let  $T$  be the first positive time such that  $\psi_T(v) = v$ . By assumption, there exist  $p + 1$  pairs of times  $(t_i, t'_i) \in [0, T]^2$  with  $t_i \neq t'_i$  such that  $\psi_{t_i}(v)$  and  $\psi_{t'_i}(v)$  have the same projection to  $S$ . Up to replacing the initial vector  $v$  by another point on  $c$ , we can assume that none of the  $t_i$  is 0. The corresponding self-intersection in  $S$  is transverse and thus stable by small perturbation. In particular, if  $v_n$  converges to  $v$ , then for  $n$  large enough, one can find times  $(t_{i,n}, t'_{i,n}) \in (0, T)^2$ ,  $1 \leq i \leq p+1$  such that  $\psi_{t_{i,n}}(v_n)$  and  $\psi_{t'_{i,n}}(v_n)$  have the same projection to  $S$ . In other words, for  $n$  large enough, the geodesic arc  $\psi_{(0,T)}(v_n)$  has at least  $p + 1$  self-intersections. In particular,  $v$  cannot be approximated by points belonging to a closed geodesic with at most  $p$  self-intersections, showing that  $c \subset T^1(S) \setminus \mathcal{F}_p(S)$ .  $\square$

Note that  $\mathcal{F}_p(S) \subset \mathcal{F}_q(S)$  for  $p \leq q$ . In particular, all these subflows contain  $\mathcal{F}_0(S)$ , the closure of the union of all simple closed geodesics in  $S$ . This set has been studied by Birman and Series [BS85], who proved in particular that it has Hausdorff dimension 1. It is thus a very “small” subset of the geodesic flow of  $S$ . We call it the *Birman–Series set* of  $S$ .

**Definition 3.15.** A representation  $\rho : \pi_1(S) \rightarrow \mathrm{SL}(d, \mathbb{K})$  is called *simple  $k$ -Anosov* if  $\rho$  is  $k$ -Anosov in restriction to  $\mathcal{F}_0(\pi_1(S))$ .

#### 4. CONSTRUCTION OF SIMPLE ANOSOV REPRESENTATIONS

We can now state more precisely the main result of the paper:

**Theorem 4.1.** *For every  $p \geq 0$  and every  $d \geq 2$ , there exists a representation  $\rho : \Gamma \rightarrow \mathrm{SL}(2d, \mathbb{C})$  that is  $d$ -Anosov in restriction to  $\mathcal{F}_p(\Gamma)$  but not in restriction to  $\mathcal{F}_{p+1}(\Gamma)$ .*

By stability of the restricted Anosov property, we can deform such a representation in order to guarantee further generic properties. In particular, we have the following:

<sup>1</sup>Concretely, one can fix a base point  $x_0 \in \mathbb{H}^2$  and parametrize each oriented geodesic in  $\mathbb{H}^2$  by length, with the projection of the base point at 0. This gives flow-equivariant homomorphism from  $\partial_\infty \pi_1(S)^{(2)}$  to  $T^1(\mathbb{H}^2)$ .



**Corollary 4.2.** *For every  $d \geq 2$ , there exists a non-empty open set of  $\text{Hom}(\Gamma, \text{SL}(2d, \mathbb{C}))$  consisting of Zariski dense representations that are either non-discrete or unfaithful.*

The rest of the section is devoted to the proof of these results.

**4.1. Geometrically finite representations of surface groups.** Let us start by recalling that there exist geometrically finite representations of closed surface groups with parabolic subgroups given by any prescribed simple closed curve. More precisely, let  $S$  be a closed connected oriented surface of genus  $g \geq 2$ ,  $c$  a simple closed curve on  $S$  and  $\Pi = \langle \gamma \rangle$  the cyclic subgroup of  $\Gamma = \pi_1(S)$  generated by an element of  $\gamma$  representing  $c$ .

The following proposition is well-known to Kleinian group experts:

**Proposition 4.3.** *There exists  $\rho : \Gamma \rightarrow \text{SL}(2, \mathbb{C})$  discrete and faithful with geometrically finite image and whose stabilizers of parabolic points are exactly the conjugates of  $\Pi$ .*

*In other terms, there exists a relative 1-Anosov representation of the relatively hyperbolic pair  $(\Gamma, \Pi)$  into  $\text{SL}(2, \mathbb{C})$ .*

Such representations can be constructed using the Maskit combination theorem for amalgamated products (when  $\gamma$  is separating) and HNN extensions (see Theorem 4.104, Theorem 4.105 and Example 4.106 of Kapovich's book [Kap01]). A priori, the representations obtained take values in  $\text{Isom}_+(\mathbb{H}^3) \simeq \text{PSL}(2, \mathbb{C})$ . However, discrete and faithful representations of surface groups into  $\text{PSL}(2, \mathbb{C})$  have vanishing second Stiefel–Whitney class and can thus be lifted to  $\text{SL}(2, \mathbb{C})$ .

**4.2. The induced representation.** Let us now recall classical construction of an *induced representation* from a finite index subgroup to the whole group.

Let  $\Gamma$  be a countable group and  $\Gamma'$  be a subgroup of  $\Gamma$  of finite index. Let  $V$  be a finite dimensional complex vector space and let  $\rho : \Gamma' \rightarrow \text{GL}(V)$  be a linear representation. Recall that in representation theory of groups, giving such representation  $\rho$  is equivalent to equipping  $V$  with a structure of  $\mathbb{C}[\Gamma']$ -module. The induced representation  $\text{Ind}_{\Gamma'}^{\Gamma}(\rho)$  of  $\Gamma$  is the representation associated to the  $\mathbb{C}[\Gamma]$ -module structure of  $\mathbb{C}[\Gamma] \otimes_{\mathbb{C}[\Gamma']} V$ . If  $\Gamma'$  has index  $d$  in  $\Gamma$ , then  $\mathbb{C}[\Gamma]$  is a free  $\mathbb{C}[\Gamma']$ -module of rank  $d$ , hence  $\text{Ind}_{\Gamma'}^{\Gamma}(\rho)$  is a representation of rank  $d \cdot \dim_{\mathbb{C}}(V)$ .

In more concrete terms, pick a collection  $\{\gamma_1 = id, \gamma_2, \dots, \gamma_d\} \subset \Gamma$  of representatives of left cosets of  $\Gamma'$ , so that

$$\Gamma = \bigsqcup_{i=1}^d \gamma_i \Gamma' .$$

The  $\mathbb{C}[\Gamma]$ -module  $\mathbb{C}[\Gamma] \otimes_{\mathbb{C}[\Gamma']} V$  can be identified with  $\bigoplus_{i=1}^d \gamma_i V$ , where each  $\gamma_i V$  is a copy of  $V$ . For any  $1 \leq i \leq d$ , we denote the copy of  $v \in V$  in  $\gamma_i V$  by  $(\gamma_i v)$ . Then the induced  $\Gamma$ -action is defined by  $\gamma(\gamma_i v) = (\gamma_j(\gamma' v))$  for any  $\gamma \in \Gamma$ , where  $\gamma_j$  and  $\gamma' \in \Gamma'$  are such that  $\gamma \gamma_i = \gamma_j \gamma'$ .

In particular, when  $\Gamma'$  is a normal subgroup in  $\Gamma$ , the restriction of  $\text{Ind}_{\Gamma'}^{\Gamma}(\rho)$  to  $\Gamma'$  is precisely  $\bigoplus_{i=1}^d \rho_i$ , where  $\rho_i$  is the representation of  $\Gamma'$  defined by  $\rho_i(\gamma) = \rho(\gamma_i^{-1} \gamma \gamma_i)$  for all  $\gamma \in \Gamma'$ .

**Lemma 4.4.** *Let  $Y$  be a cocompact  $\Gamma$ -flow. If  $\Gamma' \subset \Gamma$  is a normal subgroup of index  $d$  and  $\rho : \Gamma' \rightarrow \text{SL}(2, \mathbb{C})$  is 1-Anosov in restriction to  $Y$ , then  $\text{Ind}_{\Gamma'}^{\Gamma}(\rho) : \Gamma \rightarrow \text{SL}(2d, \mathbb{C})$  is  $d$ -Anosov in restriction to  $Y$ .*

*Proof.* Since  $\rho : \Gamma' \rightarrow \mathrm{SL}(2, \mathbb{C})$  is 1-Anosov in restriction to  $Y$ , there exists a dominated splitting of rank 1, denoted by  $Y \times \mathbb{C}^2 = E_\rho^s \oplus E_\rho^u$ , where  $E_\rho^s$  is the stable direction and  $E_\rho^u$  is the unstable direction, with respect to a  $\rho$ -invariant metric  $\|\cdot\|$  on  $Y \times \mathbb{C}^2$ . Following Remark 3.4, we may assume that  $\|\cdot\|$  is of unit volume and that  $E_\rho^s$  and  $E_\rho^u$  are orthogonal with respect to  $\|\cdot\|$ .

Fix  $y \in Y$ ,  $v \in (E_\rho^s)_y$  and  $w \in (E_\rho^u)_y$ . Since both  $E_\rho^s$  and  $E_\rho^u$  have rank 1, the previous conditions on  $\|\cdot\|$  imply that the product  $\|\phi^t(v)\| \cdot \|\phi^t(w)\|$  is constant, and the dominated splitting condition then gives us constants  $C, \lambda > 0$  (independent of  $y, v, w$ ), such that

$$(2) \quad \|\phi^t(v)\| \leq C e^{-\lambda t} \|v\| \text{ and } \|\phi^t(w)\| \geq C e^{\lambda t} \|w\| ,$$

for all  $t \in \mathbb{R}_+$ . This will imply that the direct sum of several such dominated splittings is again a dominated splitting.

Let  $\{\gamma_1 = \mathrm{id}, \gamma_2, \dots, \gamma_d\}$  be a collection of representatives of the left cosets of  $\Gamma' \subset \Gamma$ . For each  $i \in \{1, \dots, d\}$ , we define the subbundles  $E_i^s$  and  $E_i^u$  of  $Y \times \mathbb{C}^2$  by

$$(E_i^s)_y = (E_\rho^s)_{\gamma_i^{-1}y} , \quad (E_i^u)_y = (E_\rho^u)_{\gamma_i^{-1}y} .$$

The splitting  $E_i^s \oplus E_i^u$  is the pull-back of the splitting  $E_\rho^s \oplus E_\rho^u$  by  $\gamma_i^{-1}$  acting on  $Y$ . One can easily show that  $Y \times \mathbb{C}^2 = E_i^s \oplus E_i^u$  is a  $\phi$ -invariant and  $\rho_i$ -equivariant splitting, where  $\rho_i(\gamma) = \rho(\gamma_i^{-1}\gamma\gamma_i)$  for any  $\gamma \in \Gamma'$ . Moreover, this splitting is dominated with respect to the pull-back metric  $\gamma_i^{-1*}\|\cdot\|$ .

Consider  $Y \times \mathbb{C}^{2d} = \bigoplus_{i=1}^d Y \times \mathbb{C}^2$ , the direct sum of  $d$  copies of a the trivial rank 2 bundle, where the  $i$ . Setting  $F^s = \bigoplus_{i=1}^d E_i^s$  and  $F^u = \bigoplus_{i=1}^d E_i^u$ , one obtains a rank  $d$  splitting which is equivariant for the representation  $\bigoplus_{i=1}^d \rho_i$ . By (2), this splitting is dominated (for the metric  $\bigoplus_{i=1}^d \gamma_i^* \|\cdot\|$ , for instance). Then we conclude that  $\bigoplus_{i=1}^d \rho_i : \Gamma' \rightarrow \mathrm{SL}(2d, \mathbb{C})$  is  $d$ -Anosov in restriction to  $Y$ . Since  $\bigoplus_{i=1}^d \rho_i$  is the restriction to  $\Gamma'$  of  $\mathrm{Ind}_{\Gamma'}^\Gamma(\rho)$ , we conclude that  $\mathrm{Ind}_{\Gamma'}^\Gamma(\rho)$  is  $d$ -Anosov in restriction to  $Y$  by Proposition 3.6.  $\square$

**4.3. Construction of simple Anosov representations.** We now have all the tools to prove Theorem 4.1.

Let  $S = \Gamma \backslash \mathbb{H}^2$  be a closed connected oriented hyperbolic surface. Fix  $p \geq 0$  and  $d \geq 2$ .

**Proposition 4.5.** *There exists a Galois covering  $\pi : \tilde{S} \rightarrow S$  of degree  $d$  and a simple closed geodesic  $c \in \tilde{S}$  such that  $\pi(\gamma)$  has  $p + 1$  self-intersections.*

An example of such a pair  $(\tilde{S}, c)$  is shown in Figure 1.

Let  $\Gamma' \subset \Gamma$  be the fundamental group of  $\tilde{S}$  and let  $\Pi$  be the cyclic subgroup generated by a representative  $\gamma$  of  $c$  in  $\Gamma'$ . By Proposition 4.3, there exists a representation  $\rho' : \Gamma' \rightarrow \mathrm{SL}(2, \mathbb{C})$  which is 1-Anosov relative to  $\Pi$ . We set

$$\rho = \mathrm{Ind}_{\Gamma'}^\Gamma(\rho') : \Gamma \rightarrow \mathrm{SL}(2d, \mathbb{C}) .$$

**Theorem 4.6.** *The representation  $\rho$  is  $d$ -Anosov in restriction to  $\mathcal{F}_p$  and not  $d$ -Anosov in restriction to  $\mathcal{F}_{p+1}$ .*

*Proof.* Denote by  $Y \subset T^1(\mathbb{H}^2)$  the preimage of  $T^1(S) \setminus \pi(c)$ . It is an open  $\Gamma$ -subflow of  $T^1(\mathbb{H}^2)$ .

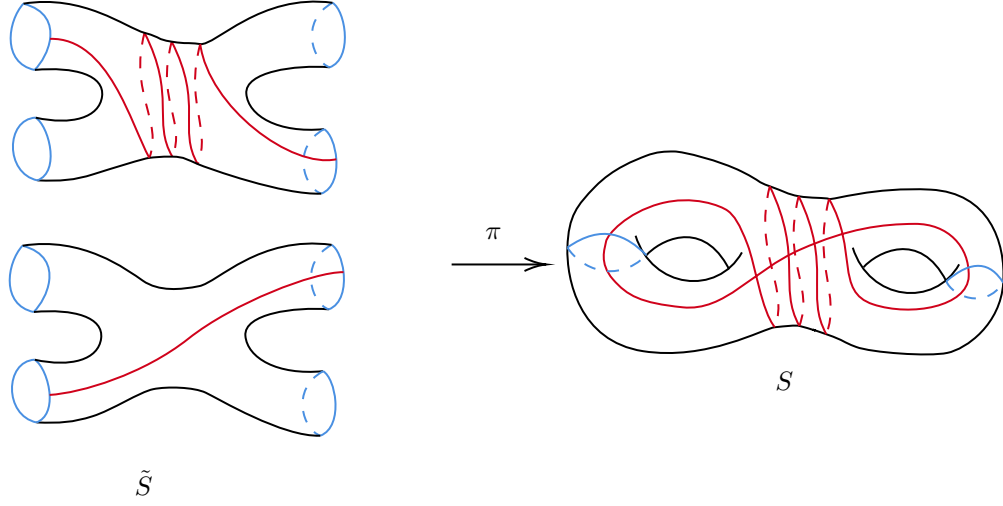


FIGURE 1.

By Proposition 2.11 and Corollary 3.10, the representation  $\rho'$  is 1-Anosov in restriction to any cocompact subflow of  $Y$ . By Lemma 4.4, the induced representation  $\rho$  is  $d$ -Anosov in restriction to any cocompact subflow of  $Y$ . Finally, by Proposition 3.14, the curve  $c$  is disjoint from  $\mathcal{F}_p(S)$ . Hence  $\mathcal{F}_p(\Gamma)$  is a cocompact subflow of  $Y$ , and  $\rho$  is  $d$ -Anosov in restriction to  $\mathcal{F}_p$ .

On the other hand,  $\rho(\gamma)$  is a direct sum of  $d$  matrices in  $\mathrm{SL}(2, \mathbb{C})$ , hence its eigenvalues have the form  $\lambda_1, \dots, \lambda_d, \lambda_d^{-1}, \dots, \lambda_1^{-1}$ , with  $|\lambda_1| \geq \dots \geq |\lambda_d| \geq 1$ . One of these matrices is  $\rho'(\gamma)$  which has eigenvalues  $\pm 1$ . We deduce that  $|\lambda_d| = 1 = |\lambda_d^{-1}|$  and  $\rho(\gamma)$  is not  $d$ -proximal. Since  $\pi(c) \subset \mathcal{F}_{p+1}(S)$ , this implies that  $\rho$  is not  $d$ -Anosov in restriction to  $\mathcal{F}_{p+1}(S)$ .

This concludes the proof of Theorem 4.6, hence that of Theorem 4.1.  $\square$

**4.4. Deformations and generic simple Anosov representations.** In this section, we consider small deformations of the above constructed representation to deduce Corollary 4.2.

Let  $\mathrm{Hom}_{df}(\Gamma, \mathrm{SL}(n, \mathbb{C})) \subset \mathrm{Hom}(\Gamma, \mathrm{SL}(n, \mathbb{C}))$  denote the set of discrete and faithful representations and  $\mathrm{Hom}_{ndf}(\Gamma, \mathrm{SL}(n, \mathbb{C}))$  its complement. As a consequence of the Zassenhaus lemma,  $\mathrm{Hom}_{df}(\Gamma, \mathrm{SL}(n, \mathbb{C}))$  is closed, hence  $\mathrm{Hom}_{ndf}(\Gamma, \mathrm{SL}(n, \mathbb{C}))$  is open in  $\mathrm{Hom}(\Gamma, \mathrm{SL}(n, \mathbb{C}))$ .

Let  $\mathrm{Hom}^Z(\Gamma, \mathrm{SL}(n, \mathbb{C})) \subset \mathrm{Hom}(\Gamma, \mathrm{SL}(n, \mathbb{C}))$  denote the set of representations with Zariski dense image. It is a Zariski open subset of the representation variety which intersects every irreducible component (see for instance [Lab17]).

Finally, denote by  $\mathrm{An}_{\mathcal{F}_p}^d(\Gamma, \mathrm{SL}(2d, \mathbb{C}))$  the set of representations that are  $d$ -Anosov in restriction to  $\mathcal{F}_p$ , which is open by Proposition 3.7. Corollary 4.2 admits the following reformulation:

**Proposition 4.7.** *The intersection*

$$\mathrm{Hom}_{ndf}(\Gamma, \mathrm{SL}(2d, \mathbb{C})) \cap \mathrm{Hom}^Z(\Gamma, \mathrm{SL}(2d, \mathbb{C})) \cap \mathrm{An}_{\mathcal{F}_p}^d(\Gamma, \mathrm{SL}(2d, \mathbb{C}))$$

is non-empty.

*Proof.* In the previous section, we constructed a representation  $\rho \in \text{An}_{\mathcal{F}_p}^d(\Gamma, \text{SL}(2d, \mathbb{C}))$  of the form  $\text{Ind}_{\Gamma'}^{\Gamma}(\rho')$  where  $\Gamma'$  is a subgroup of  $\Gamma$  of index  $d$  and  $\rho' : \Gamma' \rightarrow \text{SL}(2, \mathbb{C})$  is discrete and faithful but not convex-cocompact.

By Sullivan's stability theorem for Kleinian groups [Sul86], there exists a sequence  $\rho'_n \in \text{Hom}_{\text{ndf}}(\Gamma', \text{SL}(2, \mathbb{C}))$  converging to  $\rho'$ .<sup>2</sup>

Then  $\rho_n \stackrel{\text{def}}{=} \text{Ind}_{\Gamma'}^{\Gamma}(\rho'_n)$  belongs to  $\text{Hom}_{\text{ndf}}(\Gamma, \text{SL}(2d, \mathbb{C}))$  and converges to  $\rho$ . For  $n$  large enough,  $\rho_n$  is  $d$ -Anosov in restriction to  $\mathcal{F}_p$  and we conclude that

$$\text{Hom}_{\text{ndf}}(\Gamma, \text{SL}(2d, \mathbb{C})) \cap \text{An}_{\mathcal{F}_p}^d(\Gamma, \text{SL}(2d, \mathbb{C})) \neq \emptyset .$$

Finally, since  $\text{Hom}^Z(\Gamma, \text{SL}(2d, \mathbb{C}))$  is the complement of a subvariety of  $\text{Hom}(\Gamma, \text{SL}(2d, \mathbb{C}))$  that does not contain an irreducible component,  $\text{Hom}^Z(\Gamma, \text{SL}(2d, \mathbb{C}))$  intersects every non-empty open subset, hence

$$\text{Hom}^Z(\Gamma, \text{SL}(2d, \mathbb{C})) \cap \text{Hom}_{\text{ndf}}(\Gamma, \text{SL}(2d, \mathbb{C})) \cap \text{An}_{\mathcal{F}_p}^d(\Gamma, \text{SL}(2d, \mathbb{C})) \neq \emptyset .$$

□

## 5. MAPPING CLASS GROUP DYNAMICS

In the final section of this paper, we introduce the action of the mapping class group  $\text{Mod}(S)$  on geodesic flows and character varieties. We remark that the subflows  $\mathcal{F}_p(S)$  are “invariant under the Mapping Class Group”, and that  $\mathcal{F}_0(S)$  is the unique minimal subflow with this property. We then deduce that the domains of Anosov representations in restriction to  $\mathcal{F}_p$  form domains of discontinuity for the mapping class group action on character varieties, among which the domains of Anosov representations in restriction to the Birman–Series flow are maximal.

**5.1. Mapping class group invariant closed subflows.** Let  $\Gamma$  be a finitely generated hyperbolic group. Recall that every automorphism of  $\Gamma$  is a quasi-isometry, and thus extends to a homeomorphism of  $\partial_{\infty}\Gamma$ . This defines an action of  $\text{Aut}(\Gamma)$  on  $\partial_{\infty}\Gamma$ . The restriction of this action to the inner automorphism group  $\text{Inn}(\Gamma)$  on  $\partial_{\infty}\Gamma$  is the standard action of  $\Gamma$  (i.e. it coincides with the action on the boundary extending left translations on the Cayley graph).

The action of  $\text{Aut}(\Gamma)$  on the boundary at infinity naturally induces an action on  $\partial_{\infty}^{(2)}\Gamma$ . Now, every cocompact  $\Gamma$ -subflow of  $\mathcal{F}(\Gamma)$ , has the form  $Y_P = P \times \mathbb{R}$ , where  $P$  is a closed,  $\Gamma$ -invariant subset of  $\partial_{\infty}^{(2)}\Gamma$ . Given a subgroup  $H$  of  $\text{Out}(\Gamma)$  with  $\hat{H}$  its preimage in  $\text{Aut}(\Gamma)$ , we say that a cocompact  $\Gamma$ -subflow  $Y_P$  is  $H$ -invariant if  $P$  is  $\hat{H}$ -invariant. This is well-defined since  $P$  is always  $\text{Inn}(\Gamma)$ -invariant by the definition.

**Example 5.1.** Recall the notations of Section 3.3,

- for  $F_n$ , the free group of order  $n$ , the primitive geodesic flow  $\mathcal{F}_{\text{Prim}}$  is an  $\text{Out}(F_n)$ -invariant  $F_n$ -subflow, since the set of primitive elements is  $\text{Out}(F_n)$ -invariant;
- for a compact hyperbolic surface  $S$  (possibly with boundary), the flow of geodesics with at most  $p$  self-intersections  $\mathcal{F}_p(S)$  is a  $\text{Mod}(S)$ -invariant  $\pi_1(S)$ -subflow, where  $\text{Mod}(S)$  denotes the mapping class group of  $S$ .

<sup>2</sup>In this precise case, there are more explicit ways to construct the sequence  $\rho_n$  than to invoke Sullivan's stability.

In particular, the Birman–Series flow is  $\text{Mod}(S)$ -invariant. Here we prove that it is the unique minimal  $\text{Mod}(S)$ -invariant subflow.

**Theorem 5.2.** *Let  $S$  be a closed connected hyperbolic surface,  $H$  a finite index subgroup of  $\text{Mod}(S)$  and  $Y$  a (non-empty)  $H$ -invariant cocompact subflow of  $\mathcal{F}(\pi_1(S))$ . Then  $Y$  contains the Birman–Series set  $\mathcal{F}_0(\pi_1(S))$ .*

*Proof.* Let  $\ell$  be a geodesic contained in  $Y$  and let  $c$  be a simple closed geodesic that intersects with  $\ell$ . Let  $T_c$  denote the Dehn twist along  $c$ . Then there exists  $k > 0$  such that  $T_c^k \in H$ . Since  $Y$  is closed and  $H$ -invariant, it contains

$$\bigcup_{n \in \mathbb{Z}} T_c^{nk}(\ell) \supset c .$$

We conclude that  $Y$  contains any simple closed geodesic that intersects it. In particular it contains a simple closed curve  $c_0$ . Let  $c$  be any simple closed geodesic. There exists a simple closed geodesic  $c'$  that intersects both  $c$  and  $c_0$ . By the preceding argument,  $Y$  contains  $c'$ , hence it contains  $c$ . We conclude that  $Y$  contains every simple closed curve, and thus contains the Birman–Series set  $\mathcal{F}_0(S)$ .  $\square$

**5.2. Action on character varieties.** Recall that for a  $\Gamma$ -flow  $(Y, \phi)$ ,  $\text{An}_Y^k(\Gamma, \text{SL}(d, \mathbb{K})) \subset \text{Hom}(\Gamma, \text{SL}(d, \mathbb{K}))$  denotes the collection of representations of  $\Gamma$  into  $\text{SL}(d, \mathbb{K})$  that are  $k$ -Anosov in restriction to  $Y$ . We denote the collection of such representations modulo conjugations by  $\mathcal{A}_Y^k(\Gamma, \text{SL}(d, \mathbb{K})) = \text{An}_Y^k(\Gamma, \text{SL}(d, \mathbb{K})) / \text{SL}(d, \mathbb{K}) \subset \chi(\Gamma, \text{SL}(d, \mathbb{K}))$ .

Now we are ready to finish the proof of Proposition 1.2.

**Theorem 5.3.** *Let  $S$  be a closed connected hyperbolic surface. Then the set  $\mathcal{A}_{\mathcal{F}_p(S)}^d(\pi_1(S), \text{SL}(2d, \mathbb{C}))$  is open in  $\chi(\pi_1(S), \text{SL}(2d, \mathbb{C}))$ , and  $\text{Mod}(S)$  acts properly discontinuously on it.*

We will need the following theorem for the proof.

**Theorem 5.4** ([Wan23] Theorem 1.1 and Remark 7.5). *Let  $\mathcal{S}$  be a finite, symmetric generating set of  $\Gamma$  and  $\lambda \geq 1$ ,  $\epsilon, b \geq 0$  are any given constants. Let*

$Q_{P,b} = \{\ell : \mathbb{R} \rightarrow \text{Cay}(\Gamma, \mathcal{S}) \mid \ell \text{ is a geodesic with } d(\ell(0), \text{id}) \leq b \text{ and } (\ell(-\infty), \ell(+\infty)) \in P\}$   
and

$$\Gamma_{P,b}^+ = \{\ell(t) \mid \ell \in Q_{P,b}, t \in \mathbb{R}_+ \text{ with } \ell(t) \in \Gamma \text{ a vertex of } \text{Cay}(\Gamma, \mathcal{S})\} \subset \Gamma .$$

Let  $O$  be an open, relatively compact subset in  $A_{\mathcal{S}_P}^k(\Gamma, \text{SL}(d, \mathbb{K}))$ . Then there exist constants  $A \geq 1$  and  $B \geq 0$ , such that

$$\log \frac{\sigma_k(\rho(\gamma))}{\sigma_{k+1}(\rho(\gamma))} \geq A^{-1}|\gamma| - B ,$$

for any  $\rho \in O$  and  $\gamma \in \Gamma_{P,b}^+$ , where  $|\gamma|$  is the word length of  $\gamma$  with respect to  $\mathcal{S}$ , and  $\sigma_k(\rho(\gamma))$  is the  $k^{\text{th}}$  singular value of  $\rho(\gamma)$ .

*Proof of Theorem 5.3.* Let  $\mathcal{S} = \{s_1, s_2, \dots, s_n\}$  be a generating set of  $\pi_1(S)$  such that

- Each  $s_i$  represents a simple closed geodesic;
- For any  $i \neq j$ , at least one of  $s_i s_j$  and  $s_i s_j^{-1}$  represents a simple closed geodesic.

Let  $D$  denote the collection of all  $s_i$  and all  $s_i s_j^{\pm 1}$  that represent simple closed geodesics.

Let  $P \subset \partial_\infty^{(2)} \pi_1(S)$  be the closed subset such that  $S_P = \mathcal{F}_P(S)$ . Since  $\mathcal{A}_{\mathcal{F}_P(S)}^d(\Gamma, \mathrm{SL}(2d, \mathbb{C}))$  is open in  $\mathrm{Hom}(\Gamma, \mathrm{SL}(2d, \mathbb{C}))$  (by Proposition 3.7) and conjugation invariant, we have  $\mathcal{A}_{\mathcal{F}_P(S)}^d(\Gamma, \mathrm{SL}(2d, \mathbb{C}))$  is open in  $\chi(\Gamma, \mathrm{SL}(2d, \mathbb{C}))$ .

Let  $\ell$  be a compact subset of  $\mathcal{A}_{\mathcal{F}_P(S)}^d(\pi_1(S), \mathrm{SL}(2d, \mathbb{C}))$ . We pick  $b \geq 0$  large enough such that all powers of elements in  $D$  contains in  $\Gamma_{P,b}^+$ . By Theorem 5.4, for any  $\rho_0 \in \mathcal{A}_{\mathcal{F}_P(S)}^d(\Gamma, \mathrm{SL}(2d, \mathbb{C}))$  with  $[\rho_0] \in L$ , there exists a relatively compact neighborhood  $O$  and constants  $A_O \geq 1$  and  $B_O \geq 0$ , such that

$$A_O^{-1} |\gamma| - B_O \leq \log \frac{\sigma_d(\rho(\gamma))}{\sigma_{d+1}(\rho(\gamma))} \leq \log \frac{\sigma_1(\rho(\gamma))}{\sigma_{2d}(\rho(\gamma))} \leq A_O |\gamma| ,$$

for any  $\rho \in O$  and  $\gamma \in \Gamma_{P,b}^+$ . Here the third inequality follows from that  $\pi_1(S)$  is finitely generated. Then for any  $\gamma \in D$ , we have

$$A_O^{-1} \|\gamma\| \leq \log \frac{|\lambda_d(\rho(\gamma))|}{|\lambda_{d+1}(\rho(\gamma))|} = \lim_{n \rightarrow \infty} \frac{1}{n} \log \frac{\sigma_d(\rho(\gamma^n))}{\sigma_{d+1}(\rho(\gamma^n))} \leq A_O \|\gamma\| ,$$

where  $\|\gamma\| = \lim_{n \rightarrow \infty} \frac{|\gamma|^n}{n}$  is the stable length of  $\gamma$  in  $\mathrm{Cay}(\pi_1(S), \mathcal{S})$  and  $\lambda_k$  denote the  $k^{\mathrm{th}}$  eigenvalue (ranking by absolute values). Since both stable lengths in  $\mathrm{Cay}(\pi_1(S), \mathcal{S})$  and eigenvalues are invariant by conjugation, the inequality holds for all  $\rho \in \mathrm{SL}(d, \mathbb{C}) \cdot O$ . Let  $[O]$  denote the conjugation classes of  $\mathrm{SL}(d, \mathbb{C}) \cdot O$ , then  $\ell$  is covered by finitely many such open set  $[O]$ . Therefore, there exists a constant  $A$ , such that

$$A^{-1} \|\gamma\| \leq \log \frac{|\lambda_d(\rho(\gamma))|}{|\lambda_{d+1}(\rho(\gamma))|} \leq A \|\gamma\| ,$$

for all  $\gamma \in D$ .

Suppose there exists  $[f] \in \mathrm{Out}(\pi_1(S))$  with a representative  $f \in \mathrm{Aut}(\pi_1(S))$ , such that there is a representation  $[\rho] \in L$  with  $[\rho \circ f] \in L$ , where  $\rho$  is a representative of  $[\rho]$ . Then we have

$$\|f(\gamma)\| \leq A^2 \|\gamma\| ,$$

for any  $\gamma \in D$ .

Therefore, to show the  $\mathrm{Mod}(S)$ -action is properly discontinuous, it suffices to check

$$\{[f] \in \mathrm{Out}(\pi_1(S)) \mid \|f(\gamma)\| \leq A^2 \|\gamma\| \text{ for all } \gamma \in D\}$$

is finite. This immediately follows from the proof of Lemma 12 in [Lee15].  $\square$

*Remark 5.5.* It provides a sequence of domains of discontinuous

$$\mathcal{A}_{\mathcal{F}_{P+1}(S)}^d(\pi_1(S), \mathrm{SL}(2d, \mathbb{C})) \subset \mathcal{A}_{\mathcal{F}_P(S)}^d(\pi_1(S), \mathrm{SL}(2d, \mathbb{C})) ,$$

all of which are contained in  $\mathcal{A}_{\mathcal{F}_0(S)}^d(\pi_1(S), \mathrm{SL}(2d, \mathbb{C}))$ .

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