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Equivalent definitions of Arthur packets for real unitary groups

Nicolas Arancibia Robert, Paul Mezo

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Abstract

Mok and Mœglin-Renard have defined Arthur packets for unitary groups. Their definitions follow Arthur's work on classical groups and rely on harmonic analysis. For real groups there is an alternative definition of Arthur packets, due to Adams-Barbasch-Vogan. It relies on sheaf-theoretic techniques instead of harmonic analysis. We prove that these two definitions of Arthur packets are equivalent in the case of real unitary groups.

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1 Introduction

In an effort to characterize the automorphic spectrum of a connected reductive group, Arthur introduced a set of parameters together with a collection of conjectures concerning them ([A1], [A2]). The parameters are commonly called Arthur-parameters or simply A-parameters. These automorphic A-parameters are global objects. Conjecturally, each global A-parameter gives rise to a local A-parameter for every valuation of the underlying field.

In the present work, we study local A-parameters only for a real valuation and only for unitary groups. Moreover, our first main theorem concerns real quasisplit unitary groups. A real quasisplit unitary group is a real form of a general linear group, which we denote here as $G(\mathbb{R})$. An *A-parameter* for G is a group homomorphism

$$\psi_G : W_{\mathbb{R}} \times \mathrm{SL}_2 \rightarrow {}^{\vee}G^{\Gamma} \quad (1)$$

in which, $\Gamma = \mathrm{Gal}(\mathbb{C}/\mathbb{R})$, ${}^{\vee}G^{\Gamma} = {}^{\vee}G \rtimes \Gamma$ is the Galois form of the L-group of G , and $W_{\mathbb{R}}$ is the real Weil group. In addition, $\psi_G|_{W_{\mathbb{R}}}$ is a tempered L-parameter and $\psi_G|_{\mathrm{SL}_2}$ is algebraic (see [B] for the definitions). Arthur conjectured the existence of a stable virtual character η_{ψ_G} of $G(\mathbb{R})$ with several properties. The set of irreducible characters appearing in η_{ψ_G} with non-zero multiplicity are, by definition, the Arthur-packet (or A-packet) Π_{ψ_G} . Arthur conjectured the irreducible characters in Π_{ψ_G} to be characters of irreducible unitary representations.

In this limited scope, Arthur's conjectures have been proven to a great extent. However, there have been two disparate methods employed, and it has been unknown whether the two methods lead to the same conclusions. The first method relies on global harmonic analysis ([R], [A3], [KMSW], [M3]). We denote the stable virtual character defined by Mok in [M3] by $\eta_{\psi_G}^{\mathrm{Mok}}$. The second method is local and relies on sheaf theory ([ABV]). We denote the stable virtual character defined by Adams, Barbasch and Vogan in [ABV] by $\eta_{\psi_G}^{\mathrm{ABV}}$. Our main theorem for quasisplit unitary groups is

$$\eta_{\psi_G}^{\mathrm{Mok}} = \eta_{\psi_G}^{\mathrm{ABV}} \quad (2)$$

(Theorem 9.3). An immediate consequence of this theorem is that the sets of irreducible characters appearing in each of the virtual characters are the same, that is

$$\Pi_{\psi_G}^{\mathrm{Mok}} = \Pi_{\psi_G}^{\mathrm{ABV}}.$$

The irreducible characters in $\Pi_{\psi_G}^{\mathrm{Mok}}$ are all unitary ([M3, Theorem 3.2.1 (b)]), whereas this was not known for $\Pi_{\psi_G}^{\mathrm{ABV}}$. These identities have consequences for all real forms of unitary groups, not just the quasisplit forms. To explain why this is so requires some background which is also present in the proof of (2). For this reason we first provide an overview of the proof of (2) for quasisplit forms, and return to the remaining real forms thereafter.

The principal difficulty in proving (2) lies in the disparate manner in which the two virtual characters are defined. Let us examine the definitions beginning with the virtual character $\eta_{\psi_G}^{\mathrm{Mok}}$. The main idea here is to express the unitary group G as a *twisted endoscopic group* of a pair $(\mathrm{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N, \vartheta)$. In this pair $\mathrm{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N$ is the algebraic group obtained from the general linear group GL_N by restriction of scalars from \mathbb{C} to \mathbb{R} ([S7, Proposition 11.4.22], [B, I.5]). It may be regarded as $\mathrm{GL}_N \times \mathrm{GL}_N$ together with a real structure whose real points determine the real form $\mathrm{GL}_N(\mathbb{C})$. The other member of the pair is an automorphism ϑ of $\mathrm{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N$. It is defined by

$$\vartheta(g_1, g_2) = (\tilde{J}(g_2^{-1})^{\top} \tilde{J}^{-1}, \tilde{J}(g_1^{-1})^{\top} \tilde{J}^{-1}), \quad g_1, g_2 \in \mathrm{GL}_N, \quad (3)$$

where \tilde{J} is the anti-diagonal matrix

$$\tilde{J} = \begin{bmatrix} 0 & & & & 1 \\ & & & -1 & \\ & & \ddots & & \\ & & & & 0 \\ (-1)^{N-1} & & & & \end{bmatrix}.$$

Clearly, ϑ is an automorphism of order two, and as a real form, the semidirect product $\mathrm{GL}_N(\mathbb{C}) \rtimes \langle \vartheta \rangle$ is a disconnected algebraic group. The group G is attached to the pair $(\mathrm{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N, \vartheta)$ in that ${}^{\vee}G$

is isomorphic to the identity component of the fixed-point set $({}^\vee\mathrm{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N)^\vartheta$. This furnishes an inclusion

$$\epsilon : {}^\vee G^\Gamma \hookrightarrow {}^\vee\mathrm{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N^\Gamma$$

which allows us to define the A-parameter

$$\psi = \epsilon \circ \psi_G \tag{4}$$

for $\mathrm{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N$ using (1). Since the real form $\mathrm{GL}_N(\mathbb{C})$ is particularly well-understood, there is an obvious choice of stable virtual character η_ψ^{Mok} . It is in fact a single irreducible character

$$\eta_\psi^{\mathrm{Mok}} = \pi_\psi$$

of $\mathrm{GL}_N(\mathbb{C})$. Furthermore, as a representation, π_ψ is stable under composition with ϑ . This allows one to extend π_ψ to a representation π_ψ^\sim of the disconnected group $\mathrm{GL}_N(\mathbb{C}) \rtimes \langle \vartheta \rangle$. At this stage some care must be taken, as the extension is only unique up to a sign. If we ignore this wrinkle for the time being, then we obtain the *twisted character* of π_ψ^\sim as the restriction of the character of π_ψ^\sim to the non-identity component $\mathrm{GL}_N(\mathbb{C}) \rtimes \vartheta$. If we identify π_ψ^\sim with its twisted character then the stable virtual character $\eta_{\psi_G}^{\mathrm{Mok}}$ is uniquely determined by the twisted endoscopic transfer identity

$$\pi_\psi^\sim = \mathrm{Trans}_{G(\mathbb{R})}^{\mathrm{GL}_N(\mathbb{C}) \rtimes \vartheta}(\eta_{\psi_G}^{\mathrm{Mok}}) \tag{5}$$

([M3, Proposition 8.2.1]). The twisted endoscopic transfer map $\mathrm{Trans}_{G(\mathbb{R})}^{\mathrm{GL}_N(\mathbb{C}) \rtimes \vartheta}$ is defined on the space of stable virtual characters of $G(\mathbb{R})$. It is defined for real reductive groups in [S5], [M1] and [M2].

Turning now to the definition of $\eta_{\psi_G}^{\mathrm{ABV}}$, we come upon completely different methods. A remarkable innovation of Adams, Barbasch and Vogan is their introduction of a complex variety $X({}^\vee G^\Gamma)$, together with a ${}^\vee G$ -action, such that the ${}^\vee G$ -orbits are in bijection with the equivalence classes of L-parameters ([ABV, Section 6]). The orbits stratify the variety. Thus, one may consider ${}^\vee G$ -equivariant local systems on the orbits, and ${}^\vee G$ -equivariant *constructible sheaves* or *perverse sheaves* on $X({}^\vee G)$. We define a *complete geometric parameter* to be a pair

$$\xi = (S, \mathcal{V}) \tag{6}$$

consisting of an orbit $S \subset X({}^\vee G^\Gamma)$, together with a ${}^\vee G$ -equivariant local system \mathcal{V} on S ([ABV, Definition 7.6]). The set of complete geometric parameters is denoted by $\Xi({}^\vee G^\Gamma)$. This definition ignores the more general local systems in [ABV], which are equivariant for an algebraic cover of ${}^\vee G$. By [ABV, Theorem 10.11] there is a canonical bijection

$$\Xi({}^\vee G^\Gamma) \longleftrightarrow \Pi(G/\mathbb{R}) \tag{7}$$

which may be regarded as a more precise version of the local Langlands correspondence. The set on the right is the set of (equivalence classes of) irreducible admissible representations of *pure forms* of G . The pure forms include the quasisplit form $G(\mathbb{R})$. We write bijection (7) as

$$\xi \mapsto \pi(\xi). \tag{8}$$

Let $K\Pi(G/\mathbb{R})$ be the Grothendieck group of the finite-length admissible representations of pure forms of G . This Grothendieck group has $\{\pi(\xi)\}$ as a \mathbb{Z} -basis. It contains the virtual characters of $G(\mathbb{R})$ as a subgroup.

There is a similar picture for sheaves when considering the dual group ${}^\vee G$. Suppose $\xi = (S, \mathcal{V}) \in \Xi({}^\vee G^\Gamma)$. The local system \mathcal{V} is a ${}^\vee G$ -equivariant sheaf on $S \subset X({}^\vee G^\Gamma)$. Applying the functor of intermediate extension to the closure of S , and then taking the direct image to $X({}^\vee G)$ produces an irreducible ${}^\vee G$ -equivariant perverse sheaf $P(\xi)$, and a bijection

$$\xi \mapsto P(\xi)$$

([ABV, Section 7]). Let $KX({}^\vee G^\Gamma)$ be the Grothendieck group of the category of ${}^\vee G$ -equivariant perverse sheaves on $X({}^\vee G^\Gamma)$. It has $\{P(\xi)\}$ as a \mathbb{Z} -basis.

There is a perfect pairing

$$\langle \cdot, \cdot \rangle_G : K\Pi(G/\mathbb{R}) \times KX({}^\vee G^\Gamma) \rightarrow \mathbb{Z} \quad (9)$$

which satisfies

$$\langle \pi(\xi), P(\xi') \rangle_G = e(\xi) (-1)^{d(\xi)} \delta_{\xi, \xi'}, \quad \xi, \xi' \in \Xi({}^\vee G^\Gamma) \quad (10)$$

([ABV, Theorem 1.24]). Here, $d(\xi)$ is the dimension of the orbit S in $\xi = (S, \mathcal{V})$, $e(\xi)$ is the Kottwitz sign ([ABV, Definition 15.8]), and $\delta_{\xi, \xi'}$ is the Kronecker delta.

Using pairing (9), we identify virtual characters as \mathbb{Z} -valued linear functionals on $KX({}^\vee G^\Gamma)$. The theory of microlocal geometry provides a family of linear functionals

$$\chi_S^{\text{mic}} : KX({}^\vee G^\Gamma) \rightarrow \mathbb{Z} \quad (11)$$

parameterized by the ${}^\vee G$ -orbits $S \subset X({}^\vee G^\Gamma)$. These microlocal multiplicity maps appear in the theory of *characteristic cycles* ([ABV, Chapter 19], [BGK⁺]), and are associated with ${}^\vee G$ -equivariant local systems on a conormal bundle over $X({}^\vee G^\Gamma)$ ([ABV, Section 24], [GM]). The virtual characters associated by the pairing to these linear functionals are stable ([ABV, Theorems 1.29 and 1.31]).

The stable virtual character $\eta_{\psi_G}^{\text{ABV}}$ is defined from (11) as follows. There is an L-parameter ϕ_{ψ_G} associated to ψ_G ([ABV, Definition 22.4]). Let $S_{\psi_G} \subset X({}^\vee G^\Gamma)$ be the unique ${}^\vee G$ -orbit associated to ϕ_{ψ_G} , and $\eta_{\psi_G}^{\text{mic}}$ be the unique virtual character satisfying

$$\langle \eta_{\psi_G}^{\text{mic}}, \mu \rangle_G = \chi_{S_{\psi_G}}^{\text{mic}}(\mu), \quad \mu \in KX({}^\vee G^\Gamma).$$

As a distribution, the stable virtual character $\eta_{\psi_G}^{\text{mic}}$ is supported on real forms of G which include the quasisplit form $G(\mathbb{R})$. For the purpose of proving (2), it suffices to consider the restriction to the quasisplit form. We therefore define

$$\eta_{\psi_G}^{\text{ABV}} = \eta_{\psi_G}^{\text{mic}}|_{G(\mathbb{R})}. \quad (12)$$

Recall that the definition of $\eta_{\psi_G}^{\text{Mok}}$ in (5) relies on the theory of twisted endoscopy. One might hope to find a bridge between $\eta_{\psi_G}^{\text{Mok}}$ and $\eta_{\psi_G}^{\text{ABV}}$ by working in a theory of twisted endoscopy for $\eta_{\psi_G}^{\text{ABV}}$. Fortunately, the theory of standard endoscopy already appears in [ABV] and is extended to the twisted setting in [CM]. There are two main tasks in this extension.

The first task is the definition of a meaningful pairing between the \mathbb{Z} -modules of *twisted characters* $K\Pi(\text{GL}_N(\mathbb{C}), \vartheta)$ and *twisted sheaves* $KX({}^\vee \text{R}_{\mathbb{C}/\mathbb{R}} \text{GL}_N^\Gamma, \vartheta)$ ([LV, Section 2.3])

$$\langle \cdot, \cdot \rangle : K\Pi(\text{GL}_N(\mathbb{C}), \vartheta) \times KX({}^\vee \text{GL}_N^\Gamma, \vartheta) \rightarrow \mathbb{Z}. \quad (13)$$

This is a serious task to which we shall return later in the introduction.

The second task is to define a twisted endoscopic lifting map

$$\text{Lift}_0 : K\Pi(G(\mathbb{R}))^{\text{st}} \rightarrow K(\text{GL}_N(\mathbb{C}), \vartheta) \quad (14)$$

from stable virtual characters to twisted characters, *i.e.* the counterpart of $\text{Trans}_{G(\mathbb{R})}^{\text{GL}_N(\mathbb{C}) \rtimes \vartheta}$ in (5). This is not that serious, for there is an inverse image functor on sheaves

$$\epsilon^* : KX({}^\vee \text{R}_{\mathbb{C}/\mathbb{R}} \text{GL}_N^\Gamma, \vartheta) \rightarrow KX({}^\vee G^\Gamma) \quad (15)$$

which allows us to define Lift_0 by the identity

$$\langle \text{Lift}_0(\eta), \mu \rangle = \langle \eta, \epsilon^*(\mu) \rangle_G, \quad \mu \in KX({}^\vee \text{R}_{\mathbb{C}/\mathbb{R}} \text{GL}_N^\Gamma, \vartheta). \quad (16)$$

In this identity both pairings (13) and (9) are used.

The equality between $\eta_{\psi_G}^{\text{Mok}}$ and $\eta_{\psi_G}^{\text{ABV}}$ may then be established by returning to (5), proving

$$\text{Lift}_0 = \text{Trans}_{G(\mathbb{R})}^{\text{GL}_N(\mathbb{C}) \rtimes \vartheta}, \quad (17)$$

and

$$\text{Lift}_0(\eta_{\psi_G}^{\text{ABV}}) = \pi_{\tilde{\psi}}. \quad (18)$$

Equation (17) ends up being a simple consequence of the definition of Lift_0 and [AMR1, (1.0.3)]. Equation (18) is to a large extent proven in [ABV, Theorem 26.25]. From these two equations it follows that

$$\text{Lift}_0(\eta_{\psi_G}^{\text{Mok}}) = \text{Lift}_0(\eta_{\psi_G}^{\text{ABV}})$$

and then an injectivity result yields the main theorem (2).

The proof we have sketched follows [AAM] entirely. The classical groups in [AAM] are twisted endoscopic groups of GL_N . In the present work the classical groups are replaced by unitary groups, and GL_N is replaced by $\text{R}_{\mathbb{C}/\mathbb{R}}\text{GL}_N$. The good properties of GL_N which were harnessed in [AAM] (e.g. connected centralizers) are also properties of $\text{R}_{\mathbb{C}/\mathbb{R}}\text{GL}_N$. In truth, both the structure and the representation theory of $\text{GL}_N(\mathbb{C}) = \text{R}_{\mathbb{C}/\mathbb{R}}\text{GL}_N(\mathbb{R})$ are simpler than for $\text{GL}_N(\mathbb{R})$. We have made efforts to highlight the simplifications. Where we have not been able to improve on preliminary material, we have copied passages from [AAM].

Thus far, we have discussed identity (2) which pertains only to quasisplit unitary groups. We have indicated how the theory of twisted endoscopy plays a crucial role in the proof of the identity. In the final two sections of this paper, we explore two variants on the proof of (2). The first variant still concerns a quasisplit unitary group $G(\mathbb{R})$, but now for *standard* (non-twisted) endoscopy. In this setting, $G'(\mathbb{R})$ is a quasisplit endoscopic group of $G(\mathbb{R})$. The relationship between $\text{R}_{\mathbb{C}/\mathbb{R}}\text{GL}_N$ and $G(\mathbb{R})$ in the twisted setting is replaced by the relationship between $G(\mathbb{R})$ and $G'(\mathbb{R})$ in the standard setting. One may choose $G'(\mathbb{R})$ to be a product of smaller unitary groups so that identity (2) holds for $G'(\mathbb{R})$. One may then examine the standard endoscopic lifts of the stable virtual characters appearing in (2) for $G'(\mathbb{R})$. Explicit formulae for these lifts are given in both [M3] and [ABV]. A detailed comparison of these formulae is presented in Section 10.

In the second variant, we again consider standard endoscopy. However, in this variant the quasisplit unitary group $G(\mathbb{R})$ is the endoscopic group of a (pure) form $G(\mathbb{R}, \delta) = \text{U}(p, q)$. Mœglin and Renard define a stable virtual character $\eta_{\psi_G}^{\text{MR}}$ on $G(\mathbb{R}, \delta)$ as the standard endoscopic lift of $\eta_{\psi_G}^{\text{Mok}}$ ([MR1], [MR2]). The irreducible characters appearing in $\eta_{\psi_G}^{\text{MR}}$ form a packet $\Pi_{\psi_G}^{\text{MR}}(G(\mathbb{R}, \delta))$ of unitary representations. Analogues for these objects, $\eta_{\psi_G}^{\text{ABV}}(\delta)$ and $\Pi_{\psi_G}^{\text{ABV}}(G(\mathbb{R}, \delta))$, appear in [ABV]. In Section 11, we prove our second main theorem, namely that

$$e(\delta) \eta_{\psi_G}^{\text{MR}} = \eta_{\psi_G}^{\text{ABV}}(\delta)$$

where $e(\delta) = \pm 1$ is a Kottwitz invariant. It is then immediate that

$$\Pi_{\psi_G}^{\text{MR}}(G(\mathbb{R}, \delta)) = \Pi_{\psi_G}^{\text{ABV}}(G(\mathbb{R}, \delta)).$$

In fact, these identities are a special case of Theorem 11.1, which expresses identities for all standard endoscopic groups of $G(\mathbb{R}, \delta)$. An immediate consequence of the identity of packets is that $\Pi_{\psi_G}^{\text{ABV}}(G(\mathbb{R}, \delta))$ consists of unitary representations. This extends the unitarity results for special unipotent representations of unitary groups in [BMSZ]. We expect that the methods used in the proof of Theorem 11.1 should carry over to pure inner forms of special orthogonal groups—a work in progress.

We have just given a synopsis of the last two sections. We now give synopses of the remaining sections. Section 2.1 is a review of the preliminary material of [ABV]. Two important objects appearing here are the extended group G^Γ which mirrors the L-group ${}^\vee G^\Gamma$ and the complex variety $X({}^\vee \mathcal{O}, {}^\vee G^\Gamma) \subset X({}^\vee G^\Gamma)$. The term ${}^\vee \mathcal{O}$ is a semisimple ${}^\vee G$ -orbit in the complex Lie algebra ${}^\vee \mathfrak{g}$, and is to be thought of as an infinitesimal character. This infinitesimal character accompanies all of the arguments in the sequel and for the most part is assumed to be regular. The notions of *pure inner form* and *pure strong involution* are also introduced and compared. The section culminates with a description of the revised local Langlands correspondence (7).

In Section 2.2 the specifics of some of this preliminary material are given for quasisplit unitary groups and $\text{R}_{\mathbb{C}/\mathbb{R}}\text{GL}_N$. In particular it is proven that $\text{R}_{\mathbb{C}/\mathbb{R}}\text{GL}_N$ has only one pure strong involution which corresponds to the sole inner form, namely $\text{GL}_N(\mathbb{C})$.

Section 2.3 presents an alternative set of parameters to the complete geometric parameters (6). They are the parameters introduced in [AdC] and [AV], and so we call them *Atlas parameters*. The

Atlas parameters are more convenient for the computations appearing in later sections. The Atlas parameters also have an easily discernible involution which pertains to Vogan duality, a tool used later as well.

A third advantage to the Atlas parameters is that they may be extended to provide a parameterization for the irreducible representations of $\mathrm{GL}_N(\mathbb{C}) \rtimes \langle \vartheta \rangle$. This is the subject of Section 2.4. One of the favourable features of the Atlas parameters in this special context is the existence of a preferred extension $\pi(\xi)^+$ to $\mathrm{GL}_N(\mathbb{C}) \rtimes \langle \vartheta \rangle$ of any ϑ -stable irreducible representation $\pi(\xi)$ of $\mathrm{GL}_N(\mathbb{C})$. We refer to this preferred extension as the *Atlas extension* of $\pi(\xi)$.

Section 2.5 lays out the notation for the Grothendieck group of admissible representations and provides the construction for the related concept of the \mathbb{Z} -module of twisted characters of $\mathrm{GL}_N(\mathbb{C}) \rtimes \langle \vartheta \rangle$.

Section 3 is devoted to the pairings, (9) and (13), and the definitions of the virtual characters, $\eta_{\psi_G}^{\mathrm{ABV}}$ and $\eta_{\psi}^{\mathrm{ABV}}$, which are defined through them. The values of the ordinary pairing (9) were given in (10) for irreducible representations and perverse sheaves. It is equally important to understand the values of this pairing on *standard representations* and irreducible *constructible sheaves*. Before saying why, we recall that any irreducible representation $\pi(\xi)$ in (8) is the unique Langlands quotient of a standard representation, which we denote by $M(\xi)$. If the parameter ξ is ϑ -stable then there is an Atlas extension $M(\xi)^+$ which contains $\pi(\xi)^+$ as a quotient. On the other hand, if one replaces the intermediate extension with extension by zero in the construction of $P(\xi)$ above, then one arrives at an irreducible ${}^\vee G$ -equivariant constructible sheaf $\mu(\xi)$. The Grothendieck group of the ${}^\vee G$ -equivariant constructible sheaves on $X({}^\vee \mathcal{O}, {}^\vee G^\Gamma)$ is isomorphic to the Grothendieck group $KX({}^\vee \mathcal{O}, {}^\vee G^\Gamma)$ for the perverse sheaves ([ABV, Lemma 7.8]). It therefore makes sense to evaluate the pairing on these two objects. As a matter of fact pairing (9) is *defined* by

$$\langle M(\xi), \mu(\xi') \rangle = e(\xi) \delta_{\xi, \xi'}, \quad \xi, \xi' \in \Xi({}^\vee \mathcal{O}, {}^\vee G^\Gamma)$$

and the content of [ABV, Theorem 1.24] is (10). It is important to know the values of the pairing on these objects, since there is a well-known basis for the stable virtual characters (*cf.* (14)) given in terms of standard representations ([S3]). In addition, the inverse image functor (15) is computed relative to constructible sheaves.

Much of the technical work in this paper is spent on defining the twisted pairing (13) and proving that its values on standard representations and constructible sheaves are related to its values on irreducible representations and perverse sheaves as in the ordinary case. The first undertaking is to define preferred $({}^\vee \mathrm{R}_{\mathbb{C}/\mathbb{R}} \mathrm{GL}_N \rtimes \langle \vartheta \rangle)$ -equivariant sheaves $\mu(\xi)^+$ and $P(\xi)^+$ which restrict to $\mu(\xi)$ and $P(\xi)$ respectively as ${}^\vee \mathrm{R}_{\mathbb{C}/\mathbb{R}} \mathrm{GL}_N$ -equivariant sheaves. These are the twisted sheaves mentioned above. We define a \mathbb{Z} -module $K(X({}^\vee \mathcal{O}, {}^\vee \mathrm{R}_{\mathbb{C}/\mathbb{R}} \mathrm{GL}_N), \vartheta)$ of twisted sheaves akin to the module of twisted characters $K\Pi({}^\vee \mathcal{O}, \mathrm{GL}_N(\mathbb{C}), \vartheta)$. We then define the twisted pairing (13) by setting

$$\langle M(\xi)^+, \mu(\xi')^+ \rangle = (-1)^{l^\vartheta(\xi) - l_\vartheta^\xi(\xi)} \delta_{\xi, \xi'} \quad (19)$$

The definition of the signs on the right appears in (60) and (61). As we shall see, these signs are crucial in making comparisons with other extensions $M(\xi)^\sim$ and $\pi(\xi)^\sim$. The principal result pertaining to the twisted pairing is

$$\langle \pi(\xi)^+, P(\xi')^+ \rangle = (-1)^{d(\xi)} (-1)^{l^\vartheta(\xi) - l_\vartheta^\xi(\xi)} \delta_{\xi, \xi'} \quad (20)$$

(Theorem 3.5).

The sole objective of Section 4 is to prove (20). Our proof in this twisted setting is an adaptation of the proof in the ordinary setting ([ABV, Sections 16-17]) using the tools of [AV]. Hecke operators are among these tools. There is a difference between [ABV] and [AV] in the objects upon which the Hecke operators act. In [ABV] Hecke operators are defined on both characters and sheaves. By contrast, the Hecke operators of [AV, Section 7] are defined only on (twisted) characters. The links between characters and sheaves in the Hecke actions are the Riemann-Hilbert and Beilinson-Bernstein correspondences ([ABV, Theorems 7.9 and 8.3]). In Section 4.1 we describe these correspondences as a bijection

$$P(\xi) \longleftrightarrow \pi({}^\vee \xi), \quad \xi \in \Xi({}^\vee \mathcal{O}, {}^\vee G^\Gamma),$$

where $\pi({}^\vee\xi)$ is the *Vogan dual* of $\pi(\xi)$ (as the equivalence class of a Harish-Chandra module) (6.1 [AV]). For $\mathbf{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N$ the correspondence is extended to

$$P(\xi)^+ \longleftrightarrow \pi({}^\vee\xi)^+$$

for ϑ -fixed complete geometric parameters ξ . Once sheaves are aligned with characters in this manner, the rest of the proof of (20) follows [ABV] and [AAM, Section 4].

In Section 5 we describe the theory of endoscopy, both standard and twisted, for $\mathbf{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N$. The standard theory of endoscopy is included to motivate the twisted theory and is also used in Section 6. The twisted theory of endoscopy in Section 5.2 is a specialization of [CM, Section 5.4]. We compute the values of the twisted endoscopic lifting map Lift_0 on a basis of the stable virtual characters. This is instrumental in proving (17) and in proving that Lift_0 is injective. The value $\mathrm{Lift}_0(\eta_{\psi_G}^{\mathrm{ABV}})$ is described as an element $\eta_\psi^{\mathrm{ABV}^+} \in K\Pi({}^\vee\mathcal{O}, \mathrm{GL}_N(\mathbb{C}), \vartheta)$, which may be regarded as an extension of η_ψ^{ABV} .

In Section 6 we prove that for *any* A-parameter ψ of $\mathbf{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N$ (not necessarily of the form (4)) $\eta_\psi^{\mathrm{ABV}} = \pi_\psi$. The proof begins under the assumption that ψ is an A-parameter studied by Adams and Johnson ([AJ]). Adams and Johnson defined A-packets for these parameters, and it is easily shown that their packets are singletons for $\mathbf{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N$. The equality of the Adams-Johnson packets with the ABV-packets is proven in [AR]. The proof that ABV-packets are singletons follows from a decomposition of ψ in terms of Adams-Johnson A-parameters of smaller general linear groups, and an application of standard endoscopic lifting from the direct product of these smaller general linear groups (Proposition 6.3).

In section 7 we describe another extension $\pi(\xi)^\sim$ of the ϑ -stable irreducible representations $\pi(\xi)$ of $\mathrm{GL}_N(\mathbb{C})$ —the so-called *Whittaker extension*. Regarded as a twisted character $\pi(\xi)^\sim$ differs from the Atlas extension $\pi(\xi)^+$ by at most a sign. For special complete geometric parameters ξ it is shown that the Whittaker extension agrees with the Atlas extension. The key to determining the sign for other complete geometric parameters is to compute the sign when $\pi(\xi)$ is *generic*, *i.e.* has a Whittaker model. Indeed, the Whittaker extensions are built from extensions of generic representations and irreducible generic representations occur as subrepresentations of any standard representation. If one knows the (signed) multiplicity with which an irreducible twisted generic representation $\pi(\xi_0)^+$ appears in the decomposition of a twisted standard representation $M(\xi)^+$, then one can use this knowledge together with the agreement at the special complete geometric parameters to prove that

$$\pi(\xi)^+ = (-1)^{l^t(\xi) - l_\vartheta^t(\xi)} \pi(\xi)^\sim.$$

Observe that the sign on the right appears on the right of (19) and (20). Replacing $\pi(\xi)^+$ with $(-1)^{l^t(\xi) - l_\vartheta^t(\xi)} \pi(\xi)^\sim$ yields a cosmetic simplification to the pairing. More importantly, it is the Whittaker extension which appears in (5). With the substitution of the Whittaker extensions into the computed values of Lift_0 , the identities (17) and (18) are established.

These last observations are spelled out in Section 8. This short section assembles the essential ingredients already outlined in the introduction in proving the main theorem (2). However, it works under the assumption that the infinitesimal character is regular in ${}^\vee\mathbf{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N$. This assumption is removed in Section 9 by applying *Jantzen-Zuckerman translation*. There is nothing novel in this approach and the ideas are all present in [ABV, Section 16].

In closing, let us briefly mention a different approach to obtaining the equality between the A-packets (2). Mœglin and Renard give an explicit description of the representations in $\Pi_{\psi_G}^{\mathrm{Mok}}$. More precisely, by [MR2, Equation (5.1)], ψ in (4) decomposes as $\psi = \psi_0 \oplus \psi_1$, where

$$\psi_i : W_{\mathbb{R}} \times \mathrm{SL}_2 \longrightarrow {}^\vee\mathbf{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_{N_i}^\Gamma, \quad i = 0, 1, \quad N_0 + N_1 = N.$$

The A-parameter ψ_0 corresponds to an irreducible unitary representation π_0 of $\mathrm{GL}_{N_0}(\mathbb{C})$. In addition, the A-parameter ψ_1 factors to an A-parameter ψ_{G_1} of a smaller rank unitary group G_1

$$\psi_1 : W_{\mathbb{R}} \times \mathrm{SL}_2 \xrightarrow{\psi_{G_1}} {}^\vee G_1^\Gamma \hookrightarrow {}^\vee\mathbf{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_{N_1}^\Gamma.$$

According to ([MR2, Proposition 5.2]), every irreducible representation in $\Pi_{\psi_G}^{\mathrm{Mok}}$ is parabolically induced from a representation in

$$\left\{ \pi_0 \otimes \pi_1 : \pi_1 \in \Pi_{\psi_{G_1}}^{\mathrm{Mok}} \right\}.$$

Furthermore, each $\pi_1 \in \Pi_{\psi_{G_1}}^{\text{Mok}}$ is cohomologically induced from a character of an inner form of G_1 determined by ψ_{G_1} ([MR2, Theorem 4.1]). One should also be able to prove that each representation in $\Pi_{\psi_G}^{\text{ABV}}$ is parabolically induced from a representation in $\{\pi_0 \otimes \pi_1 : \pi_1 \in \Pi_{\psi_{G_1}}^{\text{ABV}}\}$ by imitating Proposition 5.1. This would reduce the proof of the equality of A-packets (2) to showing that $\Pi_{\psi_{G_1}}^{\text{ABV}} = \Pi_{\psi_{G_1}}^{\text{Mok}}$. The proof of this last equality should follow the proof of [AR, Theorem 4.16], which asserts the equality of ABV-packets and packets defined by Adams-Johnson ([AJ]). Although to do this, one must first extend the framework in [AR] to include unitary groups and A-parameters with singular infinitesimal character.

2 The local Langlands correspondence

Unless otherwise stated, the group G in this section may be taken to be an arbitrary connected complex reductive algebraic group. Our goal is to review the local Langlands correspondence as developed in [ABV]. Our review differs in two ways from [ABV]. First, we replace the notion of *strong real form* with the equivalent notion of *strong involution* ([AdC]). Second, we limit the theory to the set of *pure strong involutions* (or equivalently *pure strong real forms*). This limitation simplifies the review, while retaining the necessary information for quasisplit forms of G .

2.1 Extended groups and complete geometric parameters

The L-group of G is an essential feature of the local Langlands correspondence. The dual group ${}^\vee G$ is an index two subgroup of (the Galois form of) the L-group of G . One of the innovations in the local Langlands correspondence of [ABV] is the introduction of an *extended group* for G , which mirrors the L-group in that G , not ${}^\vee G$, appears as an index two subgroup.

We begin the definition of an extended group for G by fixing a pinning

$$(B, H, \{X_\alpha\})$$

in which $B \subset G$ is a Borel subgroup, $H \subset B$ is a maximal torus and $\{X_\alpha\}$ is a set of simple root vectors relative to the positive root system $R^+(G, H) = R(B, H)$ of $R(G, H)$. We fix an inner class of real forms for G , or equivalently, an algebraic involution δ_0 of G fixing the pinning ([ABV, Proposition 2.12], [AV, (5)]). The (*weak*) *extended group* defined by the inner class is

$$G^\Gamma = G \rtimes \langle \delta_0 \rangle$$

([AdC, Definition 5.1], *cf.* [ABV, Definition 2.13]).

A *strong involution* is an element $\delta \in G^\Gamma - G$ such that $\delta^2 \in Z(G)$ is central and has finite order ([AdC, Definition 5.5]). Two strong involutions are *equivalent* if they are G -conjugate. There is a surjective map

$$\delta \mapsto G(\mathbb{R}, \delta) \tag{21}$$

from (equivalence classes of) strong involutions to (isomorphism classes of) real inner forms of G ([AdC, Lemma 5.7]). There is a well-known bijection between the real inner forms of G and $H^1(\mathbb{R}, G/Z(G))$ ([S7, 12.3.7]). Using this bijection one may also think of (21) as a surjection onto $H^1(\mathbb{R}, G/Z(G))$. It is natural to juxtapose this surjection with the the quotient map

$$H^1(\mathbb{R}, G) \rightarrow H^1(\mathbb{R}, G/Z(G)). \tag{22}$$

The cohomology set $H^1(\mathbb{R}, G)$ is known as the set of *pure inner forms* ([V5, Section 2]). The pure inner forms may be realized as strong involutions in the following fashion. Let $\sigma \in \Gamma$ be the nontrivial element of the Galois group. Set

$${}^\vee \rho = \frac{1}{2} \sum_{\alpha \in R^+(G, H)} {}^\vee \alpha.$$

Any 1-cocycle $z \in Z^1(\mathbb{R}, G)$ is equivalent to a 1-cocycle taking values in H under the δ_0 -action ([AT, Proposition 7.4]). After replacing z with such an equivalent cocycle, $z(\sigma) \in H$ and

$$z(\sigma) \exp(\pi i {}^\vee \rho) \delta_0 \in G^\Gamma \tag{23}$$

is seen to be a strong involution. ($\exp(\pi i \vee \rho) \delta_0$ is the *large* involution in [AV, (11f)-(11h)]). This assignment sends classes in $H^1(\mathbb{R}, G)$ to equivalence classes of strong involutions. The (equivalence classes of) *pure strong involutions* are defined as the image of this map.

The quasisplit real form corresponds to the trivial cocycle of $H^1(\mathbb{R}, G/Z(G))$. It lies in the image of (22) as the image of the trivial cocycle in $H^1(\mathbb{R}, G)$. Equivalently, the quasisplit real form is the image under (21) of the pure strong involution

$$\delta_q = \exp(\pi i \vee \rho) \delta_0. \quad (24)$$

Given a strong involution δ we set

$$K = K_\delta \subset G \quad (25)$$

to be the fixed-point subgroup of $\text{Int}(\delta)$. The real form $G(\mathbb{R}, \delta)$ contains

$$K(\mathbb{R}) = G(\mathbb{R}, \delta) \cap K$$

as a maximal compact subgroup and is determined by K ([AV, (5f)-(5g)]). By a representation of $G(\mathbb{R}, \delta)$ we usually mean an admissible (\mathfrak{g}, K) -module, although we will need veritable admissible group representations ([V3, Definition 1.1.5]) in Section 7. A *representation of a strong involution* is a pair (π, δ) in which δ is a strong involution and π is an admissible (\mathfrak{g}, K) -module. We let $\Pi(G(\mathbb{R}, \delta))$ be the set of equivalence classes of irreducible representations (π', δ') of strong involutions in which δ' is equivalent to δ . Let

$$\Pi(G/\mathbb{R}) = \coprod_{\delta} \Pi(G(\mathbb{R}, \delta)) \quad (26)$$

be the disjoint union over the (equivalence classes of) pure strong involutions δ .¹

Returning to the more familiar territory of L-groups, we fix a pinning

$$(\vee B, \vee H, \{X_{\vee \alpha}\})$$

of $\vee G$. The previous two pinnings and the involution δ_0 fix an involution $\vee \delta_0$ of $\vee G$ as prescribed in [AV, (12)]. The group

$$\vee G^\Gamma = \vee G \rtimes \langle \vee \delta_0 \rangle$$

is the L-group of our inner class.

Suppose λ is a semisimple element of the complex Lie algebra $\vee \mathfrak{g}$. After conjugating by $\vee G$ we may assume $\lambda \in \vee \mathfrak{h}$. Using the canonical isomorphism $\vee \mathfrak{h} \simeq \mathfrak{h}^*$ we identify λ with an element of \mathfrak{h}^* , and hence via the Harish-Chandra homomorphism, with an infinitesimal character for G . This construction depends only on the $\vee G$ -orbit of λ . We refer to a semisimple element $\lambda \in \vee \mathfrak{g}$, or a $\vee G$ -orbit $\vee \mathcal{O} \subset \vee \mathfrak{g}$ of semisimple elements, as an *infinitesimal character* for G . Let

$$\Pi(\vee \mathcal{O}, G/\mathbb{R}) \subset \Pi(G/\mathbb{R})$$

be the representations (of pure strong involutions) with infinitesimal character $\vee \mathcal{O}$.

Let $P(\vee G^\Gamma)$ be the set of *quasiadmissible* homomorphisms $\phi : W_{\mathbb{R}} \rightarrow \vee G^\Gamma$ ([ABV, Definition 5.2]). There is an infinitesimal character associated to $\phi \in P(\vee G^\Gamma)$ ([ABV, Proposition 5.6]). Let

$$P(\vee \mathcal{O}, \vee G^\Gamma) \quad (27)$$

be the set of quasiadmissible homomorphisms with infinitesimal character $\vee \mathcal{O}$. The group $\vee G$ acts on $P(\vee \mathcal{O}, \vee G^\Gamma)$ by conjugation. It is to the set of $\vee G$ -orbits

$$P(\vee \mathcal{O}, \vee G^\Gamma) / \vee G \quad (28)$$

that the Langlands correspondence, in its original form, assigns L-packets of representations.

Another great innovation of [ABV] is the introduction of the complex variety $X(\vee \mathcal{O}, \vee G^\Gamma)$ of *geometric parameters*, which lies between (27) and (28) ([ABV, Definition 6.9]). It may be regarded

¹Warning! Identical notation is used in [ABV] in which δ runs over all, not necessarily pure, involutions.

as a set of equivalence classes in $P({}^\vee\mathcal{O}, {}^\vee G^\Gamma)$ upon which ${}^\vee G$ still acts by conjugation with finitely many orbits ([AAM, Section 2.2], [ABV, Proposition 6.16]). Furthermore, the quotient map

$$P({}^\vee\mathcal{O}, {}^\vee G^\Gamma) \rightarrow X({}^\vee\mathcal{O}, {}^\vee G^\Gamma) \quad (29)$$

passes to a bijection at the level of ${}^\vee G$ -orbits ([ABV, Proposition 6.17]).

Let $S \subset X({}^\vee\mathcal{O}, {}^\vee G^\Gamma)$ be a ${}^\vee G$ -orbit, and for $p \in S$ let ${}^\vee G_p = \text{Stab}_{{}^\vee G}(p)$. A *pure complete geometric parameter* for $X({}^\vee\mathcal{O}, {}^\vee G^\Gamma)$ is a pair (S, τ_S) where τ_S is (an equivalence class of) an irreducible representation of the component group ${}^\vee G_p/({}^\vee G_p)^0$ ([ABV, Definitions 7.1 and 7.6]). We denote the set of pure complete geometric parameters for $X({}^\vee\mathcal{O}, {}^\vee G^\Gamma)$ by $\Xi({}^\vee\mathcal{O}, {}^\vee G^\Gamma)$.

A special case of the local Langlands correspondence as stated in [ABV, Theorem 10.11] is a bijection

$$\Pi({}^\vee\mathcal{O}, G/\mathbb{R}) \longleftrightarrow \Xi({}^\vee\mathcal{O}, {}^\vee G^\Gamma) \quad (30)$$

between representations of pure strong involutions and pure complete geometric parameters. It is important to bear in mind that the left-hand side of (30) contains the subset $\Pi({}^\vee\mathcal{O}, G(\mathbb{R}, \delta_q))$ of representations of the quasisplit form of G .

2.2 Extended groups for unitary groups and the complex general linear group

We specialize the discussion of the previous section to two groups, each with a fixed inner class. The first group is GL_N . When N is even we fix the inner class whose quasisplit form is the indefinite unitary group $\text{U}(N/2, N/2)$. When N is odd we fix the inner class to contain the quasisplit form $\text{U}((N-1)/2, (N+1)/2)$. The second group is

$$\text{R}_{\mathbb{C}/\mathbb{R}}\text{GL}_N = \text{GL}_N \times \text{GL}_N$$

together with the inner class whose quasisplit form is $\text{GL}_N(\mathbb{C})$.

Let us begin with the unitary groups. Fix the usual pinning for GL_N in which B is the upper-triangular Borel subgroup, H is the diagonal subgroup, and X_α is the matrix with 1 in the entry corresponding to the simple root α and zeroes elsewhere. Regardless of whether N is even or odd the specified inner classes contain the compact real form $\text{U}(N) = \text{U}(N, 0)$. This implies that δ_0 acts as the trivial automorphism on GL_N ([AV, (5)]). Consequently, the extended group for the inner class of unitary groups is

$$\text{GL}_N^\Gamma = \text{GL}_N \times \langle \delta_0 \rangle \cong \text{GL}_N \times \mathbb{Z}/2\mathbb{Z}.$$

It follows from [AV, (12c)] that the involution ${}^\vee\delta_0$ acts on ${}^\vee\text{GL}_N$ as

$${}^\vee\delta_0(g) = \tilde{J}(g^{-1})^\top \tilde{J}^{-1}, \quad g \in {}^\vee\text{GL}_N.$$

The corresponding L-group is

$${}^\vee\text{GL}_N^\Gamma = {}^\vee\text{GL}_N \rtimes \langle {}^\vee\delta_0 \rangle. \quad (31)$$

The strong involutions and pure strong involutions of real unitary groups are presented in [A, Section 9].

For the group $\text{R}_{\mathbb{C}/\mathbb{R}}\text{GL}_N$ and the inner class containing $\text{GL}_N(\mathbb{C})$, we shall follow [M3, Section 2.1] and begin with the L-group first. The L-group is defined as

$${}^\vee\text{R}_{\mathbb{C}/\mathbb{R}}\text{GL}_N^\Gamma = {}^\vee(\text{GL}_N \times \text{GL}_N)^\Gamma = ({}^\vee\text{GL}_N \times {}^\vee\text{GL}_N) \rtimes \langle {}^\vee\delta_0 \rangle, \quad (32)$$

where the involution ${}^\vee\delta_0$ is now defined by

$${}^\vee\delta_0(g_1, g_2) = (g_2, g_1), \quad g_1, g_2 \in {}^\vee\text{GL}_N.$$

Let us now fix the pinning $(B, H, \{X_\alpha\})$ for $\text{GL}_N \times \text{GL}_N$ (and its dual) by taking B to be the direct product of the upper-triangular subgroups, H to be the direct product of the diagonal subgroups and $\{X_\alpha\}$ to be the union of simple root vectors for each of the two factors in the direct

product. Then, by the prescription [AV, (12c)], the involution δ_0 on $\mathbb{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N = \mathrm{GL}_N \times \mathrm{GL}_N$ is defined by

$$\delta_0(g_1, g_2) = (\tilde{J}(g_2^{-1})^\top \tilde{J}^{-1}, \tilde{J}(g_1^{-1})^\top \tilde{J}^{-1}), \quad g_1, g_2 \in \mathrm{GL}_N, \quad (33)$$

and the extended group of $\mathbb{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N$ is

$$\mathbb{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N^\Gamma = (\mathrm{GL}_N \times \mathrm{GL}_N)^\Gamma = (\mathrm{GL}_N \times \mathrm{GL}_N) \rtimes \langle \delta_0 \rangle. \quad (34)$$

It is coincidental that δ_0 defines the same automorphism as ϑ (3). The real form σ associated to δ_0 is given by composing δ_0 with the compact real form ([AV, (5)]). This turns out to be

$$\sigma(g_1, g_2) = (\tilde{J}\tilde{g}_2\tilde{J}^{-1}, \tilde{J}\tilde{g}_1\tilde{J}^{-1}), \quad g_1, g_2 \in \mathrm{GL}_N. \quad (35)$$

The group $\mathbb{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N(\mathbb{R}, \delta_0)$ from (21) is by definition the fixed-point subgroup of σ ,

$$\mathbb{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N(\mathbb{R}, \delta_0) = \{(g, \tilde{J}\tilde{g}\tilde{J}^{-1}) : g \in \mathrm{GL}_N\} \cong \mathrm{GL}_N(\mathbb{C}).$$

In what follows we will reserve the notation $\mathrm{GL}_N(\mathbb{C})$ for this particular *real form*, and reserve the notation GL_N for the absolute theory of reductive groups.

Lemma 2.1. *The Galois cohomology sets*

$$H^1(\mathbb{R}, \mathbb{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N) \text{ and } H^1(\mathbb{R}, \mathbb{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N/Z(\mathbb{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N))$$

are both trivial. In particular, $\mathrm{GL}_N(\mathbb{C})$ is the only real form in its inner class, there is only one equivalence class of pure strong involutions, and this equivalence class corresponds to $\mathrm{GL}_N(\mathbb{C})$ via (22).

Proof. We begin with a cocycle $z \in Z^1(\mathbb{R}, \mathbb{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N)$. Here, the implicit action of the non-trivial element $\sigma \in \Gamma$ on $\mathbb{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N$ is given by (35). Suppose $z(\sigma) = (g_1, g_2)$. Then by definition,

$$(1, 1) = z(\sigma^2) = z(\sigma)\sigma(z(\sigma)) = (g_1\tilde{J}\tilde{g}_2\tilde{J}^{-1}, g_2\tilde{J}\tilde{g}_1\tilde{J}^{-1}),$$

which implies

$$z(\sigma) = (g_1, \tilde{J}\tilde{g}_1^{-1}\tilde{J}^{-1}).$$

The cocycle z is trivial in $H^1(\mathbb{R}, \mathbb{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N)$ since

$$z(\sigma) = (g_1, 1)\sigma((g_1, 1)^{-1}).$$

This proves the triviality of $H^1(\mathbb{R}, \mathbb{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N)$. The proof of the triviality of the second cohomology set follows in the same manner. \square

Lemma 2.1 reduces the set of representations in (26) to the unique real form $\mathrm{GL}_N(\mathbb{C})$. Thus, we write

$$\Pi(\vee \mathcal{O}, (\mathbb{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N)/\mathbb{R}) = \Pi(\vee \mathcal{O}, \mathrm{GL}_N(\mathbb{C})). \quad (36)$$

This is an opportune moment to bring up two peculiarities of $\mathbb{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N$ that will be of importance later. The first is that we could equally well have reversed the roles of the L-group and extended group by defining the L-group as (34) and the extended group as (32). Indeed, with this reversal the (unique) real form corresponding to the extended group is the fixed-point set of

$$(g_1, g_2) \mapsto ((\tilde{g}_2^\top)^{-1}, (\tilde{g}_1^\top)^{-1})$$

([AV, (5)]), which is easily seen to be

$$\{(g, (\tilde{g}^\top)^{-1}) : g \in \mathrm{GL}_N\} \cong \mathrm{GL}_N(\mathbb{C}).$$

The recovery of the same inner class, namely $\mathrm{GL}_N(\mathbb{C})$, under this reversal shall be useful when we explore Vogan duality in Section 4.1.

The second peculiarity has to do with the connectedness of centralizers in the dual group ${}^\vee\mathbb{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N = {}^\vee\mathrm{GL}_N \times {}^\vee\mathrm{GL}_N$. It is well-known that the centralizer of the image of any L-parameter

for the general linear group $\mathrm{GL}_N(\mathbb{R})$ is connected. This peculiarity is shared by $\mathrm{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N(\mathbb{R}) = \mathrm{GL}_N(\mathbb{C})$ and may be seen as follows. To lighten the notation take $N = 2$. It is a simple exercise to show that after conjugation any L-parameter $\phi \in P({}^\vee\mathrm{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_2^\Gamma)$ may be taken to have the form

$$\phi(z) = \left(\left[\begin{array}{cc} z^{\lambda_1} \bar{z}^{\lambda'_1} & 0 \\ 0 & z^{\lambda_2} \bar{z}^{\lambda'_2} \end{array} \right], \left[\begin{array}{cc} z^{\lambda'_1} \bar{z}^{\lambda_1} & 0 \\ 0 & z^{\lambda'_2} \bar{z}^{\lambda_2} \end{array} \right], 1 \right), \quad z \in \mathbb{C}^\times$$

$$\phi(j) = \left(\left[\begin{array}{cc} (-1)^{(\lambda_1 - \lambda'_1)/2} & 0 \\ 0 & (-1)^{(\lambda_2 - \lambda'_2)/2} \end{array} \right], \left[\begin{array}{cc} (-1)^{(\lambda_1 - \lambda'_1)/2} & 0 \\ 0 & (-1)^{(\lambda_2 - \lambda'_2)/2} \end{array} \right], {}^\vee\delta_0 \right)$$

for some $\lambda_1, \lambda'_1, \lambda_2, \lambda'_2 \in \mathbb{C}$. It is not difficult to see that the centralizer of $\phi(j)$ in ${}^\vee\mathrm{GL}_2 \times {}^\vee\mathrm{GL}_2$ reduces the further computation of the centralizer of $\phi(\mathbb{C}^\times)$ to the well-known case of ${}^\vee\mathrm{GL}_2$, where it is known to be connected. The connectedness of the centralizer of ϕ is the same as the triviality of the component group, which we write as

$$({}^\vee\mathrm{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N)_\phi / ({}^\vee\mathrm{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N)_\phi^0 = \{1\}. \quad (37)$$

2.3 Atlas parameters for $\mathrm{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N$

We introduce another set of parameters for representations of $\mathrm{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N$ which may be used in place of the complete geometric parameters of Section 2.1. These *Atlas parameters* are convenient for Hecke algebra computations and are well-suited to a description of Vogan duality (see Section 4.1). The main references for this section are [AdC] and [AV, Section 3].

We start by working in the context of the extended group (34). We take $H \subset \mathrm{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N$ to be the direct product of the diagonal subgroups as in the pinning of the previous section. Following [AV, Section 3] we set

$$\mathcal{X}_{\vee\rho} = \{ \delta \in \mathrm{Norm}_{(\mathrm{GL}_N \times \mathrm{GL}_N)_{\delta_0}}(H) : \delta^2 = \exp(2\pi i {}^\vee\rho) \} / H$$

where the quotient is by the conjugation action of H . This is a set of H -conjugacy classes of strong involutions with *infinitesimal cocharacter* ${}^\vee\rho$ ([AV, (16e)], cf. (24)). By Lemma 2.1, these strong involutions are all pure and correspond to the real form $\mathrm{GL}_N(\mathbb{C})$.

We fix a ϑ -fixed, regular, integrally dominant element $\lambda \in {}^\vee\mathfrak{h}$. This means

$$\begin{aligned} \vartheta(\lambda) &= \lambda \\ \langle \lambda, {}^\vee\alpha \rangle &\neq 0, \quad \alpha \in R(\mathrm{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N, H) \\ \langle \lambda, {}^\vee\alpha \rangle &\notin \{-1, -2, -3, \dots\}, \quad \alpha \in R^+(\mathrm{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N, H). \end{aligned} \quad (38)$$

The ${}^\vee\mathrm{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N$ -orbit ${}^\vee\mathcal{O}$ of λ will be the infinitesimal character of our representations of $\mathrm{GL}_N(\mathbb{C})$. The assumption of integral dominance is harmless ([AV, Lemma 4.1]). We shall remove the regularity assumption at the beginning of Section 9.

The action of δ_0 in the extended group (34) induces an action on the Weyl group $W(\mathrm{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N, H)$. Consider the set

$$\{w \in W(\mathrm{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N, H) : w \delta_0(w) = 1\}. \quad (39)$$

If $x \in \mathcal{X}_{\vee\rho}$ then the action by conjugation of x on H is equal to $w\delta_0$ for some w in the set (39). The map $x \mapsto w$ is surjective ([AV, Proposition 3.2]). Let $\mathcal{X}_{\vee\rho}^w$ be the fibre of this map over w , i.e.

$$\mathcal{X}_{\vee\rho}^w = \{x \in \mathcal{X}_{\vee\rho} : xhx^{-1} = w\delta_0 \cdot h, \text{ for all } h \in H\}. \quad (40)$$

Turning to the L-group (32), we have an analogous set in which the infinitesimal cocharacter ${}^\vee\rho$ is replaced by the infinitesimal character λ , and δ_0 is replaced by ${}^\vee\delta_0$, namely

$${}^\vee\mathcal{X}_\lambda = \{{}^\vee\delta \in \mathrm{Norm}_{{}^\vee\mathrm{GL}_N \vee \delta_0}({}^\vee H) : {}^\vee\delta^2 = \exp(2\pi i \lambda)\} / {}^\vee H.$$

The analogue of the set (39) is

$$\{w \in W({}^\vee\mathrm{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N, {}^\vee H) : w {}^\vee\delta_0(w) = 1\} \quad (41)$$

and there is an obvious analogue of (40), which we denote by ${}^\vee\mathcal{X}_\lambda^w$.

If we identify ${}^\vee\mathrm{GL}_N$ with GL_N then the actions of δ_0 and ${}^\vee\delta_0$ on H are related by

$${}^\vee\delta_0(h) = w_0\delta_0(h^{-1}), \quad h \in H,$$

where w_0 is the long Weyl group element ([AV, (12c)]). From this it is easily verified that for all $h \in H$ and $w \in W(\mathrm{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N, H)$

$$w\delta_0(w) \cdot h = w\delta_0w\delta_0 \cdot h = ww_0{}^\vee\delta_0ww_0{}^\vee\delta_0 \cdot h = ww_0{}^\vee\delta_0(ww_0) \cdot h.$$

It follows that

$$w \mapsto ww_0$$

defines a bijection from (39) onto (41). This map allows us to pair any set $\mathcal{X}_{\check{\rho}}^w$ with the set ${}^\vee\mathcal{X}_\lambda^{ww_0}$.

The next result follows from [AdC], [ABV] and [AV, Theorem 3.11]. Our proof follows that of [AAM, Lemma 2.2].

Lemma 2.2. *There is a canonical bijection*

$$\coprod_{\{w:w\delta_0(w)=1\}} \mathcal{X}_{\check{\rho}}^w \times {}^\vee\mathcal{X}_\lambda^{ww_0} \longleftrightarrow \Xi({}^\vee\mathcal{O}, {}^\vee\mathrm{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N^\Gamma).$$

Proof. First of all $|\mathcal{X}_{\check{\rho}}^w| = 1$ for all w . This follows from [AdC, Proposition 12.19(5)] which equates the cardinality with that of the component group of a dual Cartan subgroup. As indicated in the first peculiarity near the end of Section 2.2, the dual inner class consists of (products of) complex general linear groups. The dual Cartan subgroup is therefore isomorphic to N copies of \mathbb{C}^\times and is evidently connected, so its component group is trivial.

The lemma is now reduced to defining a canonical bijection

$$\coprod_{w\delta_0(w)=1} {}^\vee\mathcal{X}_\lambda^{ww_0} \longleftrightarrow \Xi({}^\vee\mathcal{O}, {}^\vee\mathrm{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N^\Gamma).$$

As explained in [AAM, Lemma 2.2], the triviality of the component groups (37) further reduces the task to defining a canonical bijection with ${}^\vee\mathrm{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N$ -orbits in $X({}^\vee\mathcal{O}, {}^\vee\mathrm{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N^\Gamma)$, or equivalently, with ${}^\vee\mathrm{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N$ -orbits of quasiamissible homomorphisms. The definition of the latter bijection is almost identical to the one in [AAM, Lemma 2.2] and is left to the reader. \square

Together with (30) this gives

Theorem 2.3. *Let ${}^\vee\mathcal{O}$ be the ${}^\vee\mathrm{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N$ -orbit of λ . There are canonical bijections*

$$\coprod_{\{w:w\delta_0(w)=1\}} \mathcal{X}_{\check{\rho}}^w \times {}^\vee\mathcal{X}_\lambda^{ww_0} \longleftrightarrow \Xi({}^\vee\mathcal{O}, {}^\vee\mathrm{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N^\Gamma) \longleftrightarrow \Pi({}^\vee\mathcal{O}, \mathrm{GL}_N(\mathbb{C})).$$

As in [AV, Theorem 3.11] the bijection of Theorem 2.3 is written as

$$\mathcal{X}_{\check{\rho}}^w \times {}^\vee\mathcal{X}_\lambda^{ww_0} \ni (x, y) \mapsto J(x, y, \lambda) \tag{42}$$

We call the pair (x, y) on the left the *Atlas parameter* of the irreducible representation $J(x, y, \lambda)$. By Lemma 2.2, the Atlas parameter (x, y) is equivalent to a unique complete geometric parameter $\xi \in \Xi({}^\vee\mathcal{O}, {}^\vee\mathrm{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N^\Gamma)$, and accordingly we define

$$\pi(\xi) = J(x, y, \lambda).$$

The representation $\pi(\xi)$ is the Langlands quotient of a standard representation which we denote by $M(\xi)$ or $M(x, y)$.

2.4 Twisted Atlas parameters for $\mathbf{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N$

We next describe the generalization of Theorem 2.3 to the ϑ -twisted setting, which involves representations of the group $\mathrm{GL}_N(\mathbb{C}) \rtimes \langle \vartheta \rangle$. We specialize the results of [AV, Sections 3-5] to this case. Some of the more complicated issues that arise in [AV] do not occur for $\mathbf{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N$.

We continue with the hypotheses of (38). Recall that both $\vee\rho$ and λ are fixed by ϑ . By Clifford theory, an irreducible representation of $\mathrm{GL}_N(\mathbb{C}) \rtimes \langle \vartheta \rangle$ restricted to $\mathrm{GL}_N(\mathbb{C})$ is either an irreducible ϑ -fixed representation, or the direct sum of two irreducible representations which are exchanged by the action of ϑ . Since we shall be restricting our attention to twisted characters, we only need representations of the first type.

By [CM, Theorem 4.1] and Lemma 2.2, the map (42) is ϑ -equivariant. Therefore $J(x, y, \lambda)$ is ϑ -stable if and only if $(x, y) \in \mathcal{X}_{\vee\rho}^w \times \vee\mathcal{X}_{\lambda}^{ww_0}$ is fixed by ϑ . Let

$$\Pi(\vee\mathcal{O}, \mathrm{GL}_N(\mathbb{C}))^\vartheta \subset \Pi(\vee\mathcal{O}, \mathrm{GL}_N(\mathbb{C}))$$

be the subset of ϑ -stable irreducible representations and set

$$W(\delta_0, \vartheta) = \{w \in W(\mathbf{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N, H) : w\delta_0(w) = 1, w = \vartheta(w)\}$$

(cf. (39)). The ϑ -equivariance of (42) allows us to restrict Theorem 2.3 to these sets and we obtain

Corollary 2.4. *Suppose λ satisfies the hypotheses of (38) and let $\vee\mathcal{O}$ be its $\vee\mathbf{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N$ -orbit. Then there is a canonical bijection*

$$\coprod_{\{w \in W(\delta_0, \vartheta)\}} \mathcal{X}_{\vee\rho}^w \times \vee\mathcal{X}_{\lambda}^{ww_0} \longleftrightarrow \Pi(\vee\mathcal{O}, \mathrm{GL}_N(\mathbb{C}))^\vartheta$$

written $(x, y) \mapsto J(x, y, \lambda)$.

We now introduce the *extended parameters* of [AV, Sections 3-5], and summarize some facts. Fix $w \in W(\delta_0, \vartheta)$. An *extended parameter* for w is a set

$$E = (\lambda, \tau, \ell, t), \quad \lambda, \tau \in X^*(H), \quad \ell, t \in X_*(H)$$

satisfying certain conditions depending on w (see [AV, Definition 5.4]).² There is a surjective map

$$E \mapsto (x(E), y(E)) \tag{43}$$

carrying extended parameters for w to $\mathcal{X}_{\vee\rho}^w \times \vee\mathcal{X}_{\lambda}^{ww_0}$. This map only depends on λ and ℓ . In addition,

$$J(x(E), y(E), \lambda) \in \Pi(\vee\mathcal{O}, \mathrm{GL}_N(\mathbb{C}))^\vartheta,$$

and every ϑ -fixed irreducible representation arises this way. The remaining parameters τ and t in E define an irreducible representation $J(E, \lambda)$ of $\mathrm{GL}_N(\mathbb{R}) \rtimes \langle \vartheta \rangle$ satisfying

$$J(E, \lambda)|_{\mathrm{GL}_N(\mathbb{C})} = J(x(E), y(E), \lambda).$$

The representation $J(x(E), y(E), \lambda)$ is determined by a quasicharacter of a Cartan subgroup of $\mathrm{GL}_N(\mathbb{C})$. The representation $J(E, \lambda)$ is determined by the semidirect product of this Cartan subgroup with an element $h\vartheta \in \mathbf{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N \rtimes \vartheta$ ([AV, (24e)]) and a choice of extension of the quasicharacter to the semidirect product. The value of the extended quasicharacter on the element $h\vartheta$ depends on a choice of sign [AV, Definition 5.2], and the square root of this sign is given by

$$z(E) = i^{\langle \tau, (1+w)t \rangle} (-1)^{\langle \lambda, t \rangle}.$$

The preceding discussion is a specialization of a general framework to $\mathbf{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N \rtimes \langle \vartheta \rangle$. One of the special properties of $\mathbf{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N$ is that the preimage of any $(x, y) \in \mathcal{X}_{\vee\rho}^w \times \vee\mathcal{X}_{\lambda}^{ww_0}$ under (43) has a *preferred extended parameter* of the form

$$(\lambda, \tau, 0, 0). \tag{44}$$

²Warning! The symbols λ and τ here are not to be confused with symbols λ and τ appearing elsewhere. Note the slight difference in font. We have chosen to use λ and τ for ease of comparison with [AV].

This comes down to the fact that $X_{\nu_\rho}^w$ is a singleton (see the proof of Lemma 2.2). In taking $t = 0$ we see $z(\lambda, \tau, 0, 0) = 1$, and this in turn amounts to taking the aforementioned semidirect product of the Cartan subgroup with $h\vartheta = \vartheta$, and setting the value of the extended quasicharacter at ϑ equal to 1. In this way, the preferred extended parameter defines a canonical extension

$$J(x, y, \lambda)^+ = J((\lambda, \tau, 0, 0), \lambda) \quad (45)$$

of $J(x, y, \lambda)$ to $\mathrm{GL}_N(\mathbb{C}) \rtimes \langle \vartheta \rangle$. We call this extension the *Atlas extension* of $J(x, y, \lambda)$.

Going back to Theorem 2.3 and Corollary 2.4, we may formulate the result as follows.

Corollary 2.5. *There is a natural bijection of ϑ -fixed sets*

$$\coprod_{\{w \in W(\delta_0, \vartheta)\}} \mathcal{X}_{\nu_\rho}^w \times {}^\vee \mathcal{X}_\lambda^{w w_0} \longleftrightarrow \Xi({}^\vee \mathcal{O}, {}^\vee \mathrm{R}_{\mathbb{C}/\mathbb{R}} \mathrm{GL}_N^\Gamma)^\vartheta \longleftrightarrow \Pi({}^\vee \mathcal{O}, \mathrm{GL}_N(\mathbb{C}))^\vartheta.$$

Furthermore, if $\xi \in \Xi({}^\vee \mathcal{O}, {}^\vee \mathrm{R}_{\mathbb{C}/\mathbb{R}} \mathrm{GL}_N^\Gamma)^\vartheta$ is identified with (x, y) under the first bijection, then there is a canonical representation

$$\pi(\xi)^+ = J(x, y, \lambda)^+$$

extending $\pi(\xi)$ to $\mathrm{GL}_N(\mathbb{C}) \rtimes \langle \vartheta \rangle$.

The irreducible representation $\pi(\xi)^+$ is defined as the unique (Langlands) quotient of a representation $M(\xi)^+$ such that $M(\xi)^+|_{\mathrm{GL}_N(\mathbb{C})} = M(\xi)$. We call $\pi(\xi)^+$ and $M(\xi)^+$ the *Atlas extensions* of $\pi(\xi)$ and $M(\xi)$ respectively.

2.5 Grothendieck groups of characters and twisted characters

The setting for studying characters of reductive groups is the Grothendieck group of admissible representations. There is a corresponding notion in the twisted setting. In this section we establish the notation for these objects.

For the moment let G be an arbitrary complex connected reductive group. Fix a semisimple orbit ${}^\vee \mathcal{O} \subset {}^\vee \mathfrak{g}$. Recall from Section 2.1 that $\Pi({}^\vee \mathcal{O}, G/\mathbb{R})$ is the set of equivalence classes of irreducible representations (π, δ) of pure strong involutions with infinitesimal character ${}^\vee \mathcal{O}$. We define $K\Pi({}^\vee \mathcal{O}, G/\mathbb{R})$ to be the Grothendieck group of admissible representations of pure strong involutions with infinitesimal character ${}^\vee \mathcal{O}$ (see [ABV, (15.5)-(15.6)]). We identify this with the \mathbb{Z} -span of distribution characters of the irreducible representations in $\Pi({}^\vee \mathcal{O}, G/\mathbb{R})$, and refer to elements of this \mathbb{Z} -module as virtual characters.

For the unitary groups we often refer to the submodule of stable characters for the quasisplit form. So we define

$$K\Pi({}^\vee \mathcal{O}, G(\mathbb{R}, \delta_q))^{\mathrm{st}} \subset K\Pi({}^\vee \mathcal{O}, G(\mathbb{R}, \delta_q))$$

to be the subspace spanned by the (strongly) stable virtual characters. If we identify virtual characters with functions on $G(\mathbb{R}, \delta_q)$ these are the virtual characters η which satisfy $\eta(g) = \eta(g')$ whenever strongly regular semisimple elements $g, g' \in G(\mathbb{R}, \delta_q)$ are G -conjugate. See [S3, Section 5] or [ABV, Definition 18.2].

We now turn the twisted setting of the inner class $\mathrm{GL}_N(\mathbb{C})$ of $\mathrm{R}_{\mathbb{C}/\mathbb{R}} \mathrm{GL}_N$, equipped with the involution ϑ . An immediate consequence of (36) is the expression of the Grothendieck group

$$K\Pi(\mathrm{R}_{\mathbb{C}/\mathbb{R}} \mathrm{GL}_N/\mathbb{R}) = K\Pi(\mathrm{GL}_N(\mathbb{C}))$$

in terms of the unique real form. We define

$$K\Pi({}^\vee \mathcal{O}, \mathrm{GL}_N(\mathbb{C}))^\vartheta \subset K\Pi({}^\vee \mathcal{O}, \mathrm{GL}_N(\mathbb{C}))$$

to be the submodule spanned by $\Pi({}^\vee \mathcal{O}, \mathrm{GL}_N(\mathbb{C}))^\vartheta$. This is not the Grothendieck group of ϑ -stable representations of $\mathrm{GL}_N(\mathbb{C})$, but we retain the “ K ” to help align the object with its ambient Grothendieck group. On the other hand we let

$$K\Pi({}^\vee \mathcal{O}, \mathrm{GL}_N(\mathbb{C}) \rtimes \langle \vartheta \rangle) \quad (46)$$

be the veritable Grothendieck group of admissible representations of $\mathrm{GL}_N(\mathbb{C}) \rtimes \langle \vartheta \rangle$ with infinitesimal character ${}^\vee \mathcal{O}$.

We are ready to construct the \mathbb{Z} -module of twisted characters of $\mathrm{GL}_N(\mathbb{C})$. An irreducible character in $K\Pi({}^\vee \mathcal{O}, \mathrm{GL}_N(\mathbb{C}) \rtimes \langle \vartheta \rangle)$ is the usual distribution on $\mathrm{GL}_N(\mathbb{C}) \rtimes \langle \vartheta \rangle$ given by $\mathrm{Tr} \pi$, where π is an irreducible representation of $\mathrm{GL}_N(\mathbb{C}) \rtimes \langle \vartheta \rangle$. The restriction of such a distribution character to the non-identity component $\mathrm{GL}_N(\mathbb{C}) \rtimes \vartheta$ (when non-zero) is what we mean by an irreducible *twisted character* of $\mathrm{GL}_N(\mathbb{C})$ (cf. [AAM, (42)]). We define

$$K\Pi({}^\vee \mathcal{O}, \mathrm{GL}_N(\mathbb{C}), \vartheta)$$

to be the \mathbb{Z} -module generated by the irreducible twisted characters of $\mathrm{GL}_N(\mathbb{C})$ of infinitesimal character ${}^\vee \mathcal{O}$.

As noted in Section 2.4, an irreducible representation of $\mathrm{GL}_N(\mathbb{C}) \rtimes \langle \vartheta \rangle$ restricts either to an irreducible ϑ -fixed representation of $\mathrm{GL}_N(\mathbb{C})$, or to a direct sum $\pi \oplus (\pi \circ \vartheta)$ of inequivalent irreducible representations. In the second case the twisted character is 0, so we only need to consider the first case. The first case describes the irreducible representations in $K\Pi({}^\vee \mathcal{O}, \mathrm{GL}_N(\mathbb{C}))^\vartheta$. If $\pi \in \Pi({}^\vee \mathcal{O}, \mathrm{GL}_N(\mathbb{C}))^\vartheta$ then it has two extensions π^\pm to $\mathrm{GL}_N(\mathbb{C}) \rtimes \langle \vartheta \rangle$, satisfying

$$\pi^-(\vartheta) = -\pi^+(\vartheta). \quad (47)$$

Consequently the twisted characters of π^\pm agree up to sign. If we set $U_2 = \{\pm 1\}$ then it follows that the homomorphism

$$K\Pi({}^\vee \mathcal{O}, \mathrm{GL}_N(\mathbb{C}))^\vartheta \otimes_{\mathbb{Z}} \mathbb{Z}[U_2] \rightarrow K\Pi({}^\vee \mathcal{O}, \mathrm{GL}_N(\mathbb{C}), \vartheta),$$

which restricts the distribution character of $\pi(\xi)^+$ to the non-identity component, is surjective. By (47), the homomorphism passes to an isomorphism

$$K\Pi({}^\vee \mathcal{O}, \mathrm{GL}_N(\mathbb{C}), \vartheta) \cong K\Pi({}^\vee \mathcal{O}, \mathrm{GL}_N(\mathbb{C}))^\vartheta \otimes_{\mathbb{Z}} \mathbb{Z}[U_2] / \langle (\pi \otimes 1) + (\pi \otimes -1) \rangle, \quad (48)(a)$$

where the quotient runs over $\pi \in \Pi({}^\vee \mathcal{O}, \mathrm{GL}_N(\mathbb{C}))^\vartheta$. The map carrying $\pi(\xi) \in \Pi({}^\vee \mathcal{O}, \mathrm{GL}_N(\mathbb{C}))^\vartheta$ to the twisted character of the Atlas extension $\pi(\xi)^+$ extends to an isomorphism

$$K\Pi({}^\vee \mathcal{O}, \mathrm{GL}_N(\mathbb{C}))^\vartheta \cong K\Pi({}^\vee \mathcal{O}, \mathrm{GL}_N(\mathbb{C}), \vartheta).$$

We again remind the reader that the \mathbb{Z} -modules appearing in (48) are not Grothendieck groups in any natural fashion, notwithstanding the appearance of the “ K ”.

3 Sheaves, Pairings and Characteristic Cycles

Suppose ψ_G is an Arthur parameter for G as in (1). In this section we give more details on the definition of the ABV-packet $\Pi_{\psi_G}^{\mathrm{ABV}}$ and its stable virtual character $\eta_{\psi_G}^{\mathrm{ABV}}$ (2). The results apply in the more general context of complex connected reductive groups G ([ABV, Sections 19, 22]). However, for this section G will be GL_N or $\mathrm{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N$, with the setup of Section 2.2.

The definitions depend on a pairing between characters and sheaves. We also define a pairing between *twisted* characters and *twisted* sheaves for $\mathrm{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N$ ([CM, Sections 5-6], [AAM, Section 3]). The key properties of this twisted pairing are listed in this section and shall be proved in Section 4.

3.1 The pairing and the ABV-packets in the non-twisted case

Let ϕ_{ψ_G} be the Langlands parameter associated to ψ_G ([ABV, Definition 22.4]), ${}^\vee \mathcal{O}$ be the infinitesimal character of ϕ_{ψ_G} , and $S_{\psi_G} \subset X({}^\vee \mathcal{O}, {}^\vee G^\Gamma)$ be the corresponding orbit (29). Recall that $\Xi({}^\vee \mathcal{O}, {}^\vee G^\Gamma)$ is the set of pure complete geometric parameters.

Let $\mathcal{C}(X({}^\vee \mathcal{O}, {}^\vee G^\Gamma))$ be the category of ${}^\vee G$ -equivariant constructible sheaves of complex vector spaces on $X({}^\vee \mathcal{O}, {}^\vee G^\Gamma)$. This is an abelian category and its simple objects are parameterized by the set of complete geometric parameters $\xi = (S, \tau_S) \in \Xi({}^\vee \mathcal{O}, {}^\vee G^\Gamma)$ as follows. Choose $p \in S$, let

${}^\vee G_p = \text{Stab}_G(p)$, and choose a character τ_ξ of the component group of ${}^\vee G_p$ so that (p, τ_ξ) is a representative of ξ . Then τ_ξ pulled back to ${}^\vee G_p$ defines an algebraic vector bundle

$${}^\vee G \times_{{}^\vee G_p} V \rightarrow S. \quad (49)$$

The sheaf of sections of this vector bundle is, by definition, a ${}^\vee G$ -equivariant local system on S ([ABV, Section 7, Lemma 7.3]). Extend this local system to the closure \bar{S} by zero and then take the direct image into $X({}^\vee \mathcal{O}, {}^\vee G^\Gamma)$ to obtain an irreducible (*i.e.* simple) ${}^\vee G$ -equivariant constructible sheaf denoted by $\mu(\xi)$ ([ABV, (7.10)(c)]).

Now let $\mathcal{P}(X({}^\vee \mathcal{O}, {}^\vee G^\Gamma))$ be the abelian category of ${}^\vee G$ -equivariant perverse sheaves of complex vector spaces on $X({}^\vee \mathcal{O}, {}^\vee G^\Gamma)$ ([BL, Section 5]). The simple objects of $\mathcal{P}(X({}^\vee \mathcal{O}, {}^\vee G^\Gamma))$ are defined from $\xi = (S, \tau_S) \in \Xi({}^\vee G^\Gamma, {}^\vee \mathcal{O})$ and the algebraic vector bundle (49) by taking the *intermediate* extension ([BBD, Section 2]) to the closure \bar{S} instead of the extension by zero. This is denoted $P(\xi)$ ([ABV, (7.10)(d)]). It is an irreducible ${}^\vee G$ -equivariant perverse sheaf on $X({}^\vee \mathcal{O}, {}^\vee G^\Gamma)$.

The Grothendieck groups of the two categories $\mathcal{C}(X({}^\vee \mathcal{O}, {}^\vee G^\Gamma))$ and $\mathcal{P}(X({}^\vee \mathcal{O}, {}^\vee G^\Gamma))$ are canonically isomorphic ([BBD], [ABV, Lemma 7.8]). We identify the two Grothendieck groups via this isomorphism and denote them by $KX({}^\vee \mathcal{O}, {}^\vee G^\Gamma)$. This Grothendieck group has two natural bases

$$\{\mu(\xi) : \xi \in \Xi({}^\vee \mathcal{O}, {}^\vee G^\Gamma)\} \quad \text{and} \quad \{P(\xi) : \xi \in \Xi({}^\vee \mathcal{O}, {}^\vee G^\Gamma)\}.$$

Suppose $\xi = (S, \tau) \in \Xi({}^\vee \mathcal{O}, {}^\vee G^\Gamma)$. We define two invariants associated to ξ . First, let $d(\xi)$ be the dimension of S_ξ . Second, associated to ξ is the representation $\pi(\xi)$ of a pure strong involution of G (30). Let $e(\xi) = \pm 1$ be the Kottwitz invariant of the underlying real form of this strong involution ([ABV, Definition 15.8]).

As stated in the introduction, we define a perfect pairing

$$\langle \cdot, \cdot \rangle : K\Pi({}^\vee \mathcal{O}, G/\mathbb{R}) \times KX({}^\vee \mathcal{O}, {}^\vee G^\Gamma) \rightarrow \mathbb{Z} \quad (50)$$

by

$$\langle M(\xi), \mu(\xi') \rangle = e(\xi) \delta_{\xi, \xi'}.$$

The pairing takes a deceptively simple form relative to the bases given by $\pi(\xi)$ and $P(\xi')$ ([ABV, Theorem 1.24, Sections 15-17]). We state it as a theorem.

Theorem 3.1. *The pairing (50) satisfies*

$$\langle \pi(\xi), P(\xi') \rangle = (-1)^{d(\xi)} e(\xi) \delta_{\xi, \xi'}, \quad \xi, \xi' \in \Xi({}^\vee \mathcal{O}, {}^\vee G^\Gamma).$$

This pairing allows us to regard elements of $K\Pi({}^\vee \mathcal{O}, G/\mathbb{R})$ as \mathbb{Z} -linear functionals of $KX({}^\vee \mathcal{O}, {}^\vee G^\Gamma)$. The microlocal multiplicity maps χ_S^{mic} discussed in (11) are \mathbb{Z} -linear functionals on $KX({}^\vee \mathcal{O}, {}^\vee G^\Gamma)$. Before making the connection with the pairing (50), we review some facts needed to define χ_S^{mic} . To begin, we consider the category of ${}^\vee G$ -equivariant coherent \mathcal{D} -modules on $X({}^\vee \mathcal{O}, {}^\vee G^\Gamma)$. We denote this category by $\mathcal{D}(X({}^\vee \mathcal{O}, {}^\vee G^\Gamma))$. Here, \mathcal{D} is the sheaf of algebraic differential operators on $X({}^\vee \mathcal{O}, {}^\vee G^\Gamma)$ ([BGK⁺, VIII.14.4], [ABV, Section 7], [HTT, Part I]).

The equivariant Riemann-Hilbert correspondence ([BGK⁺, Theorem VIII.14.4]) induces an isomorphism

$$DR : K\mathcal{D}(X({}^\vee \mathcal{O}, {}^\vee G^\Gamma)) \rightarrow KX({}^\vee \mathcal{O}, {}^\vee G^\Gamma). \quad (51)$$

For simplicity we write $X = X({}^\vee \mathcal{O}, {}^\vee G^\Gamma)$, and $\mathcal{D}X = \mathcal{D}(X({}^\vee \mathcal{O}, {}^\vee G^\Gamma))$.

The sheaf \mathcal{D} is filtered by the order of the differential operators, and the associated graded ring is canonically isomorphic to $\mathcal{O}_{T^*(X)}$, the coordinate ring of the cotangent bundle of X ([HTT, Section 1.1]). Suppose $\mathcal{M} \in \mathcal{D}X$. Then \mathcal{M} has a filtration such that the resulting graded sheaf $\text{gr}\mathcal{M}$ is a coherent $\mathcal{O}_{T^*(X)}$ -module ([HTT, Section 2.1]).

The support of $\text{gr}\mathcal{M}$ is a closed subvariety of $T^*(X)$ ([ABV, Definition 19.7]). Each minimal ${}^\vee G$ -invariant component of this closed subvariety is the closure of a conormal bundle $T_S^*(X)$, where $S \subset X$ is a ${}^\vee G$ -orbit ([ABV, Proposition 19.12(c)]). Therefore to each conormal bundle $T_S^*(X)$ we may attach a non-negative integer, denoted by $\chi_S^{\text{mic}}(\mathcal{M})$, which (when nonzero) is the length of the module $\text{gr}\mathcal{M}$ localized at $T_S^*(X)$ ([HTT, Section 2.2]).

The characteristic cycle of \mathcal{M} is defined as

$$\mathrm{Ch}(\mathcal{M}) = \sum_{S \in X/\vee G} \chi_S^{\mathrm{mic}}(\mathcal{M}) \overline{T_S^*(X)}.$$

For a given $\vee G$ -orbit S we may regard χ_S^{mic} as a function on \mathcal{D} -modules which is additive for short exact sequences ([ABV, Proposition 19.12(e)]). It therefore defines a homomorphism $K\mathcal{D}(X(\vee\mathcal{O}, \vee G^\Gamma)) \rightarrow \mathbb{Z}$, called the *microlocal multiplicity* along S . Using the isomorphism (51), we interpret this as a homomorphism

$$\chi_S^{\mathrm{mic}} : KX(\vee\mathcal{O}, \vee G^\Gamma) \rightarrow \mathbb{Z}.$$

We now return to the pairing (50) and its relationship to χ_S^{mic} . This relationship defines $\eta_{\psi_G}^{\mathrm{ABV}}$. We first define $\eta_{\psi_G}^{\mathrm{mic}} \in K\Pi(\vee\mathcal{O}, G/\mathbb{R})$ to be the element of $K\Pi(\vee\mathcal{O}, G/\mathbb{R})$ corresponding via the pairing to the \mathbb{Z} -linear functional χ_S^{mic} on $KX(\vee\mathcal{O}, \vee G^\Gamma)$. Explicitly working through the identifications in the definition we see

$$\eta_{\psi_G}^{\mathrm{mic}} = \sum_{\xi \in \Xi(\vee\mathcal{O}, \vee G^\Gamma)} (-1)^{d(S_\xi) - d(S_{\psi_G})} \chi_{S_{\psi_G}}^{\mathrm{mic}}(P(\xi)) \pi(\xi). \quad (52)$$

An important result of Kashiwara and Adams-Barbasch-Vogan is

Proposition 3.2 ([ABV, Theorem 1.31, Corollary 19.16]). *The virtual character $\eta_{\psi_G}^{\mathrm{mic}}$ is stable.*

The *microlocal packet* $\Pi_{\psi_G}^{\mathrm{mic}}$ of ψ_G is defined to be the irreducible representations in the support of $\eta_{\psi_G}^{\mathrm{mic}}$. In other words

$$\Pi_{\psi_G}^{\mathrm{mic}} = \{\pi(\xi) : \xi \in \Xi(\vee\mathcal{O}, \vee G^\Gamma), \chi_{S_{\psi_G}}^{\mathrm{mic}}(P(\xi)) \neq 0\}.$$

This is a set of irreducible representations of pure strong involutions of G . We are primarily interested in the packet for the quasisplit strong involutions. We therefore define

$$\eta_{\psi_G}^{\mathrm{ABV}} = \eta_{\psi_G}^{\mathrm{mic}}(\delta_q) \quad (53)$$

to be the restriction of $\eta_{\psi_G}^{\mathrm{mic}}$ to the submodule of $K\Pi(\vee\mathcal{O}, G/\mathbb{R})$ generated by the representations in $\Pi(\vee\mathcal{O}, G(\mathbb{R}, \delta_q))$. The ABV-packet $\Pi_{\psi_G}^{\mathrm{ABV}}$ is defined as the support of $\eta_{\psi_G}^{\mathrm{ABV}}$, that is

$$\Pi_{\psi_G}^{\mathrm{ABV}} = \{\pi(\xi) : \xi \in \Xi(\vee\mathcal{O}, \vee G^\Gamma), \chi_{S_{\psi_G}}^{\mathrm{mic}}(P(\xi)) \neq 0, \pi(\xi) \in \Pi(G(\mathbb{R}, \delta_q))\}. \quad (54)$$

In definitions (53) and (54) we may easily replace δ_q with any other pure strong involution δ . Although we shall only use these further objects in Section 11, it seems appropriate to define them now. Let

$$\eta_{\psi_G}^{\mathrm{ABV}}(\delta) = \eta_{\psi_G}^{\mathrm{mic}}(\delta) \quad (55)$$

be the restriction of $\eta_{\psi_G}^{\mathrm{mic}}$ to the submodule of $K\Pi(\vee\mathcal{O}, G/\mathbb{R})$ generated by the representations in $\Pi(\vee\mathcal{O}, G(\mathbb{R}, \delta))$. In addition, let

$$\Pi_{\psi_G}^{\mathrm{ABV}}(G(\mathbb{R}, \delta)) = \{\pi(\xi) : \xi \in \Xi(\vee\mathcal{O}, \vee G^\Gamma), \chi_{S_{\psi_G}}^{\mathrm{mic}}(P(\xi)) \neq 0, \pi(\xi) \in \Pi(G(\mathbb{R}, \delta))\}. \quad (56)$$

We conclude this section with a restatement of Theorem 3.1 which will be valuable later on. Define the representation-theoretic transition matrix m_r by

$$M(\xi) = \sum_{\xi' \in \Xi(\vee\mathcal{O}, \vee G^\Gamma)} m_r(\xi', \xi) \pi(\xi').$$

Define the geometric “transition matrix” c_g by

$$P(\xi) = \sum_{\xi' \in \Xi(\vee\mathcal{O}, \vee G^\Gamma)} (-1)^{d(\xi)} c_g(\xi', \xi) \mu(\xi').$$

(see [ABV, (7.11)(c)]). Then [ABV, Corollary 15.13] says

Proposition 3.3. *Theorem 3.1 is equivalent to the identity*

$$m_r(\xi', \xi) = (-1)^{d(\xi) - d(\xi')} c_g(\xi, \xi').$$

This equation relates the decomposition of characters with the decomposition of sheaves.

3.2 The pairing in the twisted case

In the previous section the pairing (50) plays a fundamental role in the definition of ABV-packets. We now develop a twisted version of this pairing for $\mathrm{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N$.

We replace $K\Pi(\vee\mathcal{O}, \mathrm{GL}_N(\mathbb{C}))$ with the \mathbb{Z} -module $K\Pi(\vee\mathcal{O}, \mathrm{GL}_N(\mathbb{C}), \vartheta)$ of twisted characters (48). Associated to $\xi \in \Xi(\vee\mathcal{O}, \vee\mathrm{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N^\Gamma)^\vartheta$ are an irreducible representation $\pi(\xi) \in \Pi(\vee\mathcal{O}, \mathrm{GL}_N(\mathbb{C}))^\vartheta$ as well as the Atlas extension $\pi(\xi)^+$ to $\mathrm{GL}_N(\mathbb{C}) \rtimes \langle \vartheta \rangle$ (Corollary 2.5). The twisted character of $\pi(\xi)^+$ is an element of $K\Pi(\vee\mathcal{O}, \mathrm{GL}_N(\mathbb{C}), \vartheta)$, the \mathbb{Z} -module of twisted characters, and this gives a basis of $K\Pi(\vee\mathcal{O}, \mathrm{GL}_N(\mathbb{C}), \vartheta)$ parameterized by $\Xi(\vee\mathcal{O}, \vee\mathrm{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N^\Gamma)^\vartheta$. See (46) and the end of Section 2.4.

The twisted characters are to be paired with twisted sheaves which are elements in a \mathbb{Z} -module generalizing $KX(\vee\mathcal{O}, \vee G^\Gamma)$. The twisted objects for this pairing are given in [ABV, (25.7)] (see also [CM, Section 5.4]). We provide a short summary.

The automorphism ϑ acts on $X(\vee\mathcal{O}, \vee\mathrm{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N^\Gamma)$ in a manner which is compatible with its $\vee\mathrm{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N$ -action ([ABV, (25.1)]), and so also acts on $\vee\mathrm{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N$ -equivariant sheaves. Let

$$\mathcal{P}(X(\vee\mathcal{O}, \vee\mathrm{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N^\Gamma); \vartheta)$$

be the category of $\vee\mathrm{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N$ -equivariant perverse sheaves with a compatible ϑ -action. An object in this category is a pair (P, ϑ_P) in which P is an equivariant perverse sheaf and ϑ_P is an automorphism of P which is compatible with ϑ ([CM, Section 5.4]). Similarly, we define

$$\mathcal{C}(X(\vee\mathcal{O}, \vee\mathrm{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N^\Gamma); \vartheta)$$

to be the category of $\vee\mathrm{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N$ -equivariant constructible sheaves with a compatible ϑ -action. An object in this category is a pair (μ, ϑ_μ) in which μ is an equivariant constructible sheaf and ϑ_μ is an automorphism of μ which is compatible with ϑ .

The Grothendieck groups of these two categories are isomorphic ([CM, (35)]). We identify them and denote their Grothendieck groups by $K(X(\vee\mathcal{O}, \vee\mathrm{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N^\Gamma); \vartheta)$. This is the sheaf-theoretic analogue of $K\Pi(\mathrm{GL}_N(\mathbb{C}) \rtimes \langle \vartheta \rangle)$.

As with the representations (see (45)), we seek a canonical choice of extension of $P(\xi)$, *i.e.* a canonical automorphism $\vartheta_{P(\xi)}$ of $P(\xi)$. This is achieved by the following lemma, whose proof follows exactly as for [AAM, Lemma 3.4] by virtue of (37).

Lemma 3.4. *Let $\vee G = \vee\mathrm{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N$, $\xi = (S, \tau_S) \in \Xi(\vee\mathcal{O}, \vee\mathrm{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N^\Gamma)^\vartheta$, $p \in S$, and (49) be the equivariant vector bundle representing $\mu(\xi)$.*

(a) *Suppose $p' \in S$ and $p' = a \cdot p$ for some $a \in \vee\mathrm{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N$. Then the maps*

$$\begin{aligned} (g, v) &\mapsto (ga^{-1}, v) \\ g \cdot p &\mapsto (ga^{-1}) \cdot p' \end{aligned}$$

define an isomorphism of equivariant vector bundles

$$\vee G \times_{\vee G_p} V \rightarrow \vee G \times_{\vee G_{p'}} V.$$

which is independent of the choice of a .

(b) *There exist canonical choices of pairs*

$$\begin{aligned} \mu(\xi)^+ &= (\mu(\xi), \vartheta_{\mu(\xi)}^+) \in \mathcal{C}(X(\vee\mathcal{O}, \vee\mathrm{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N^\Gamma); \vartheta), \\ P(\xi)^+ &= (P(\xi), \vartheta_{P(\xi)}^+) \in \mathcal{P}(X(\vee\mathcal{O}, \vee\mathrm{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N^\Gamma); \vartheta) \end{aligned}$$

such that if $p \in S$ is fixed by ϑ then $\vartheta_{\mu(\xi)}^+$ (and $\vartheta_{P(\xi)}^+$) acts trivially on the stalk of $\mu(\xi)$ (and $P(\xi) \in KX(\vee\mathcal{O}, \vee\mathrm{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N^\Gamma)$) at p .

We now imitate the definition of $K\Pi(\vee\mathcal{O}, \mathrm{GL}_N(\mathbb{C}), \vartheta)$ (48) for the sheaves appearing in Lemma 3.4. Attached to $\xi \in \Xi(\vee\mathcal{O}, \vee\mathrm{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N^\Gamma)^\vartheta$ are perverse sheaves $P(\xi)^\pm$, where $P(\xi)^+$ is defined in Lemma 3.4, and $P(\xi)^-$ is the unique other choice of extension. Furthermore, the *microlocal traces* of $P(\xi)^\pm$ differ by sign ([ABV, (25.1)(j)]). Similar comments apply to $\mu(\xi)^\pm$. The microlocal traces are used in Proposition 5.5 and are the analogues of twisted characters.

We are interested only in irreducible sheaves with non-vanishing microlocal trace. We consequently follow the definition of (48) in defining the quotient

$$KX(\vee\mathcal{O}, \vee\mathrm{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N^\Gamma, \vartheta) = K(X(\vee\mathcal{O}, \vee\mathrm{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N^\Gamma)^\vartheta \otimes \mathbb{Z}[U_2]/\langle(P(\xi) \otimes 1) + (P(\xi) \otimes -1)\rangle \quad (57)$$

where the quotient runs over $\xi \in \Xi(\vee\mathcal{O}, \vee\mathrm{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N^\Gamma)^\vartheta$.

This is the \mathbb{Z} -module which we shall pair with

$$K\Pi(\vee\mathcal{O}, \mathrm{GL}_N(\mathbb{C}))^\vartheta \cong K\Pi(\vee\mathcal{O}, \mathrm{GL}_N(\mathbb{C}), \vartheta)$$

in Section 4. We call the elements of this module *twisted sheaves*, and remind the reader that these modules are not naturally Grothendieck groups, even though we have kept the “ K ” in the notation.

For reasons that will only become clear in Section 7, the definition of our twisted pairing involves some additional signs. The signs depend on the integral lengths of parameters, which may be described as follows.

From now on we assume $\lambda \in \vee\mathcal{O}$ satisfies the regularity condition (38). Let $\xi \in \Xi(\vee\mathcal{O}, \vee\mathrm{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N^\Gamma)^\vartheta$. Lemma 2.2 tells us that associated to ξ is an element $x \in \mathcal{X}_{\vee\rho}$. Set

$$\theta_x = \mathrm{Int}(x)|_H. \quad (58)$$

Let

$$R(\lambda) = \{\alpha \in R(\mathrm{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N, H) : \langle\lambda, \vee\alpha\rangle \in \mathbb{Z}\} \quad (59)(a)$$

be the λ -integral roots, with positive λ -integral roots

$$R^+(\lambda) = \{\alpha \in R(\lambda) : \langle\lambda, \vee\alpha\rangle > 0\}. \quad (59)(b)$$

Define the *integral length*, following [ABV, (16.16)], as

$$l^I(\xi) = -\frac{1}{2} (|\{\alpha \in R^+(\lambda) : \theta_x(\alpha) \in R^+(\lambda)\}| + \dim(H^{\theta_x})) + \frac{N}{2}. \quad (60)$$

The integral length takes values in the non-positive integers.

Furthermore define

$$R_\vartheta^+(\lambda) = \{\alpha \in R((\mathrm{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N^\vartheta)^0, (H^\vartheta)^0) : \langle\lambda, \vee\alpha\rangle \in \mathbb{Z}_{>0}\}.$$

We define the *ϑ -integral length* by

$$l_\vartheta^I(\xi) = -\frac{1}{2} (|\{\alpha \in R_\vartheta^+(\lambda) : \theta_x(\alpha) \in R_\vartheta^+(\lambda)\}| + \dim((H^\vartheta)^{\theta_x})) + \frac{[N/2]}{2}, \quad (61)$$

where $[N/2]$ is the least integer greater than or equal to $N/2$. This is the integral length for the fixed-point group $(\mathrm{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N)^\vartheta \cong \mathrm{GL}_N$, or more precisely, for a quasisplit unitary group of Section 2.2.

Now we define a perfect pairing (under the assumption (38))

$$\langle\cdot, \cdot\rangle : K\Pi(\vee\mathcal{O}, \mathrm{GL}_N(\mathbb{C}), \vartheta) \times KX(\vee\mathcal{O}, \vee\mathrm{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N^\Gamma, \vartheta) \rightarrow \mathbb{Z} \quad (62)$$

by setting

$$\langle M(\xi)^+, \mu(\xi')^+ \rangle = (-1)^{l^I(\xi) - l_\vartheta^I(\xi)} \delta_{\xi, \xi'} \quad (63)$$

for $\xi, \xi' \in \Xi(\vee\mathcal{O}, \vee\mathrm{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N^\Gamma)^\vartheta$. The analogue of Theorem 3.1 is

Theorem 3.5. *Suppose $\lambda \in {}^\vee\mathcal{O}$ satisfies (38). Define the pairing (62) by (63). Then*

$$\langle \pi(\xi)^+, P(\xi')^+ \rangle = (-1)^{d(\xi)} (-1)^{l^\dagger(\xi) - l^\dagger_\vartheta(\xi)} \delta_{\xi, \xi'}$$

where $\xi, \xi' \in \Xi({}^\vee\mathcal{O}, {}^\vee\mathbf{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N^\Gamma)^\vartheta$.

The proof of this theorem is the primary purpose of Section 4. Its proof is modelled on the proof of Theorem 3.1 in [ABV, Sections 15-17].

We conclude this section by giving a twisted analogue of Proposition 3.3. This analogue will only be needed in Sections 7 and 9, so the reader may wish to skip this discussion and return to it later.

For $\xi, \xi' \in \Xi({}^\vee\mathcal{O}, {}^\vee\mathbf{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N^\Gamma)^\vartheta$, define $m_r(\xi'_\pm, \xi_+)$ to be the multiplicity of the representation $\pi(\xi')^\pm$ in $M(\xi)^+$ as elements of the Grothendieck group $K\Pi({}^\vee\mathcal{O}, \mathrm{GL}_N(\mathbb{C}) \rtimes \langle \vartheta \rangle)$ (Section 2.5). In other words

$$M(\xi)^+ = \sum_{\xi' \in \Xi({}^\vee\mathcal{O}, {}^\vee G^\Gamma)^\vartheta} m_r(\xi'_+, \xi_+) \pi(\xi')^+ + m_r(\xi'_-, \xi_+) \pi(\xi')^- + \dots$$

where the omitted summands are irreducible representations of $\mathrm{GL}_N(\mathbb{C}) \rtimes \langle \vartheta \rangle$ which restrict to the sum of two irreducible representations of $\mathrm{GL}_N(\mathbb{C})$. Define the *twisted multiplicity* of $\pi(\xi')^+$ in $M(\xi)^+$ by

$$m_r^\vartheta(\xi', \xi) = m_r(\xi'_+, \xi_+) - m_r(\xi'_-, \xi_+), \quad \xi, \xi' \in \Xi({}^\vee\mathcal{O}, {}^\vee\mathbf{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N^\Gamma)^\vartheta \quad (64)$$

(cf. [AvLTV, (19.3d)]). By construction, the image of $M(\xi)^+$ in $K\Pi({}^\vee\mathcal{O}, \mathrm{GL}_N(\mathbb{C}), \vartheta)$ (48) decomposes as

$$M(\xi)^+ = \sum_{\xi' \in \Xi({}^\vee\mathcal{O}, {}^\vee\mathrm{GL}_N^\Gamma)^\vartheta} m_r^\vartheta(\xi', \xi) \pi(\xi')^+. \quad (65)$$

The matrix given by (64) is invertible ([AAM, Lemma 3.6]).

In a parallel fashion, we define $c_g(\xi'_\pm, \xi_+)$ for $\xi, \xi' \in \Xi({}^\vee\mathcal{O}, {}^\vee G^\Gamma)^\vartheta$ by

$$P(\xi)^+ = \sum_{\xi' \in \Xi({}^\vee\mathcal{O}, {}^\vee G^\Gamma)^\vartheta} (-1)^{d(\xi')} c_g(\xi'_+, \xi_+) \mu(\xi')^+ + (-1)^{d(\xi')} c_g(\xi'_-, \xi_+) \mu(\xi')^- + \dots$$

in the Grothendieck group $KX({}^\vee\mathcal{O}, {}^\vee\mathbf{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N; \vartheta)$ of Section 3.2. Setting

$$c_g^\vartheta(\xi', \xi) = c_g(\xi'_+, \xi_+) - c_g(\xi'_-, \xi_+). \quad (66)$$

we see that the image of $P(\xi)^+$ in $KX({}^\vee\mathcal{O}, {}^\vee\mathbf{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N; \vartheta)$ is

$$\sum_{\xi' \in \Xi({}^\vee\mathcal{O}, {}^\vee G^\Gamma)^\vartheta} (-1)^{d(\xi')} c_g^\vartheta(\xi', \xi) \mu(\xi')^+.$$

Just as Theorem 3.1 is equivalent to Proposition 3.3. We have the following equivalence.

Proposition 3.6 ([AAM, Proposition 3.7]). *Theorem 3.5 is equivalent to the identity*

$$m_r^\vartheta(\xi', \xi) = (-1)^{l^\dagger_\vartheta(\xi) - l^\dagger_\vartheta(\xi')} c_g^\vartheta(\xi, \xi')$$

for all $\xi, \xi' \in \Xi({}^\vee\mathcal{O}, {}^\vee\mathbf{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N^\Gamma)^\vartheta$.

4 The proof of Theorem 3.5

We continue working with an infinitesimal character $\lambda \in {}^\vee\mathcal{O}$ which satisfies the hypothesis of (38). The proof of Theorem 3.5 is nearly identical to the proof of [AAM, Theorem 3.5]. In fact, the proof of Theorem 3.5 is somewhat simpler. In order to convince the reader of these claims we review the proof of [AAM, Theorem 3.5] in our context, calling attention to the points which are simplified.

The general idea of the proof is already given in the non-twisted context of [ABV, Sections 15-17]. In the proof one first extends the \mathbb{Z} -modules appearing in Theorem 3.5 to *Hecke modules*

acted upon by a *Hecke algebra*. The extended pairing is then shown to furnish an isomorphism between one of the Hecke modules and the dual of the other. Special bases are chosen for the Hecke modules. The values of the special bases are explicitly computed under the pairing. Theorem 3.5 then follows by restricting these values to the setting of the original \mathbb{Z} -modules.

There are numerous Hecke module computations underlying this proof, and many of them have been completed in [LV] and [AV]. The computations of [AV] are given in representation-theoretic language and are therefore suitable when working with $K\Pi(\vee\mathcal{O}, \mathrm{GL}_N(\mathbb{C}), \vartheta)$. In order to adapt the computations of [AV] to $KX(\vee\mathcal{O}, \vee R_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N^{\Gamma}, \vartheta)$ we replace this module with an equivalent module of representations. In so doing, we touch on the notion of *Vogan duality* ([AV, Section 6.1]). We attend to this preliminary work in the next section.

4.1 Vogan duality and $KX(\vee\mathcal{O}, R_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N^{\Gamma}, \vartheta)$

We wish to replace the sheaf-theoretic module $KX(\vee\mathcal{O}, R_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N^{\Gamma}, \vartheta)$ with an equivalent module of representations. In the non-twisted setting this is achieved by [ABV, Theorem 8.5]. This theorem relies on two correspondences. The first correspondence is the Riemann-Hilbert correspondence, which furnishes an equivalence between the category of equivariant perverse sheaves on $X(\vee\mathcal{O}, R_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N^{\Gamma})$ and a category of equivariant D-modules on the same space ([ABV, Theorem 7.9], [BBD, Theorem VIII.14.4]). The second correspondence is the Beilinson-Bernstein localization theorem, which furnishes an equivalence between the category of equivariant D-modules and a category of Harish-Chandra modules ([ABV, Theorem 8.3], [V1, Proposition 1.2]).

Combining the two correspondences produces a bijection between a set of irreducible perverse sheaves and a set of irreducible representations (Harish-Chandra modules). The set of irreducible perverse sheaves is

$$\{P(\xi) : \xi \in \Xi(\vee\mathcal{O}, \vee R_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N^{\Gamma})\}$$

(Section 3.1). We denote the set of corresponding representations by $\vee\Pi(\vee\mathcal{O}, \mathrm{GL}_N(\mathbb{C}))$, so that the bijection may be written as

$$\{P(\xi) : \xi \in \Xi(\vee\mathcal{O}, \vee R_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N^{\Gamma})\} \longleftrightarrow \vee\Pi(\vee\mathcal{O}, \mathrm{GL}_N(\mathbb{C})). \quad (67)$$

The notation for the set of representations on the right hints at some manner of duality with $\Pi(\vee\mathcal{O}, \mathrm{GL}_N(\mathbb{C}))$. The particulars of this duality are given in [AAM, Section 4.2] for $\mathrm{GL}_N(\mathbb{R})$. The arguments there apply just as well to $\mathrm{GL}_N(\mathbb{C})$ and are summarized as follows. By Lemma 2.2 a complete geometric parameter $\xi \in \Xi(\vee\mathcal{O}, \vee R_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N^{\Gamma})$ corresponds to a unique Atlas parameters $(x, y) \in \mathcal{X}_{\vee\rho} \times \mathcal{X}_{\lambda}$. By reversing the order of the entries in the Atlas parameter, one obtains an Atlas parameter

$$(y, x) \in \mathcal{X}_{\lambda} \times \mathcal{X}_{\vee\rho}$$

for the group

$$\vee R_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N(\lambda) = \text{centralizer in } \vee R_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N \text{ of } \exp(2\pi i\lambda) \quad (68)$$

(*cf.* [AAM, Lemma 4.2]). The irreducible representation $J(y, x, \vee\rho)$ (*cf.* (42)) is a $(\vee\mathfrak{gl}_N \times \vee\mathfrak{gl}_N)(\lambda, \vee\tilde{K}_y)$ -module, where $\vee\tilde{K}_y$ is a two-fold cover of $\vee K_y$ (*cf.* (25)). It has infinitesimal character $\vee\rho$. This representation may be made plainer by computing that $\vee R_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N(\lambda)$ is a product of groups $\prod_i \vee R_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_{n_i}$. As noted in Section 2.2, this product has only one real form, namely $\prod_i \mathrm{GL}_{n_i}(\mathbb{C})$. Consequently, $J(y, x, \vee\rho)$ is a representation of a double-cover of $\prod_i \mathrm{GL}_{n_i}(\mathbb{C})$.

Bijection (67) is given by

$$P(\xi) \mapsto J(y, x, \vee\rho)$$

([AAM, Proposition 4.3]). We streamline the notation by the identification $\xi = (x, y)$ using Lemma 2.2, and defining the dual parameter $\vee\xi$ by

$$\vee\xi = (y, x) \iff \xi = (x, y). \quad (69)$$

We denote the dual representation by

$$\pi(\vee\xi) = J(y, x, \vee\rho).$$

In this way, bijection (67) takes the form

$$P(\xi) \mapsto \pi({}^\vee\xi), \quad \xi \in \Xi({}^\vee\mathcal{O}, {}^\vee\mathbf{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N^\Gamma).$$

Every representation $\pi({}^\vee\xi)$ is the unique irreducible (Langlands) quotient of a standard module, which we denote by $M({}^\vee\xi)$ ([AV, (20)]). Bijection (67) extends to an isomorphism of Grothendieck groups

$$KX({}^\vee\mathcal{O}, {}^\vee\mathbf{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N^\Gamma) \cong K^\vee\Pi({}^\vee\mathcal{O}, \mathrm{GL}_N(\mathbb{C})). \quad (70)$$

which satisfies

$$(-1)^{d(\xi)}\mu(\xi) \rightarrow M({}^\vee\xi), \quad \xi \in \Xi({}^\vee\mathcal{O}, {}^\vee\mathbf{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N^\Gamma)$$

([AAM, Proposition 4.3]).

The isomorphism (70) may be generalized to the twisted setting as well. Recall from (45) that the Atlas extension $\pi(\xi)^+$ is defined from a preferred extended parameter. The same extended parameter also determines a unique extension $\pi({}^\vee\xi)^+$ of $\pi({}^\vee\xi)$ ([AV, (39h)]). It is a $(({}^\vee\mathfrak{gl}_N \times {}^\vee\mathfrak{gl}_N)(\lambda), {}^\vee\tilde{K}_y \rtimes \langle \vartheta \rangle)$ -module, which is the unique irreducible quotient of an extension $M({}^\vee\xi)^+$ of $M({}^\vee\xi)$. Define a bijection

$$P(\xi)^+ \mapsto \pi({}^\vee\xi)^+, \quad \xi \in \Xi({}^\vee\mathcal{O}, {}^\vee\mathbf{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N^\Gamma)^\vartheta.$$

The bijection induces an isomorphism of \mathbb{Z} -modules

$$KX({}^\vee\mathcal{O}, {}^\vee\mathbf{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N^\Gamma, \vartheta) \cong K^\vee\Pi({}^\vee\mathcal{O}, \mathrm{GL}_N(\mathbb{C}), \vartheta)$$

which satisfies

$$(-1)^{d(\xi)}\mu(\xi)^+ \mapsto M({}^\vee\xi)^+, \quad \xi \in \Xi({}^\vee\mathcal{O}, {}^\vee\mathbf{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N^\Gamma)^\vartheta$$

([AAM, Proposition 4.5]).

The representation-theoretic replacement for $KX({}^\vee\mathcal{O}, {}^\vee\mathbf{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N^\Gamma, \vartheta)$ is the \mathbb{Z} -module $K^\vee\Pi({}^\vee\mathcal{O}, \mathrm{GL}_N(\mathbb{C}), \vartheta)$. Making this replacement in Theorem 3.5, and taking into account the fixed relationship between $d(\xi)$ and $l^I(\xi)$ ([AMR2, Proposition B.1]), we obtain

Lemma 4.1 ([AAM, Lemma 4.6]). *Theorem 3.5 is equivalent to the following assertion. The pairing*

$$\langle \cdot, \cdot \rangle : K\Pi({}^\vee\mathcal{O}, \mathrm{GL}_N(\mathbb{C}), \vartheta) \times K^\vee\Pi({}^\vee\mathcal{O}, \mathrm{GL}_N(\mathbb{C}), \vartheta) \rightarrow \mathbb{Z} \quad (71)$$

defined by

$$\langle M(\xi)^+, M({}^\vee\xi')^+ \rangle = (-1)^{l_\vartheta^I(\xi)} \delta_{\xi, \xi'}$$

satisfies

$$\langle \pi(\xi)^+, \pi({}^\vee\xi')^+ \rangle = (-1)^{l_\vartheta^I(\xi)} \delta_{\xi, \xi'}$$

where $\xi, \xi' \in \Xi({}^\vee\mathcal{O}, {}^\vee\mathbf{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N^\Gamma)^\vartheta$.

4.2 Twisted Hecke modules

The proof of Theorem 3.5 relies on a Hecke algebra and Hecke modules, which we introduce in the context of (71). In the twisted setting, Lusztig and Vogan define a Hecke algebra which we denote by $\mathcal{H}(\lambda)$ ([LV, Section 3.1]). This Hecke algebra acts on the Hecke modules

$$\mathcal{K}\Pi({}^\vee\mathcal{O}, \mathrm{GL}_N(\mathbb{C}), \vartheta) = K\Pi({}^\vee\mathcal{O}, \mathrm{GL}_N(\mathbb{C}), \vartheta) \otimes_{\mathbb{Z}} \mathbb{Z}[q^{1/2}, q^{-1/2}]$$

and

$$\mathcal{K}^\vee\Pi({}^\vee\mathcal{O}, \mathrm{GL}_N(\mathbb{C}), \vartheta) = K^\vee\Pi({}^\vee\mathcal{O}, \mathrm{GL}_N(\mathbb{C}), \vartheta) \otimes_{\mathbb{Z}} \mathbb{Z}[q^{1/2}, q^{-1/2}]$$

as in [LV, Section 7]. We extend the pairing (71) to these Hecke modules

$$\langle \cdot, \cdot \rangle : \mathcal{K}\Pi({}^\vee\mathcal{O}, \mathrm{GL}_N(\mathbb{C}), \vartheta) \times \mathcal{K}^\vee\Pi({}^\vee\mathcal{O}, \mathrm{GL}_N(\mathbb{C}), \vartheta) \rightarrow \mathbb{Z}[q^{1/2}, q^{-1/2}], \quad (72)$$

by setting

$$\langle M(\xi)^+, M({}^\vee\xi')^+ \rangle = (-1)^{l_\vartheta^I(\xi)} q^{(l^I(\xi) + l^I({}^\vee\xi'))/2} \delta_{\xi, \xi'}$$

for all $\xi, \xi' \in \Xi(\vee \mathcal{O}, \vee \mathbf{R}_{\mathbb{C}/\mathbb{R}} \mathbf{GL}_N^{\Gamma})^{\vartheta}$. In view of the Kronecker delta, the term $q^{1/2(l^I(\xi)+l^I(\vee \xi'))}$ in the pairing could be replaced by $q^{1/2(l^I(\xi)+l^I(\vee \xi))}$ or $q^{1/2(l^I(\xi')+l^I(\vee \xi'))}$. In fact, both of the latter terms are independent of ξ or ξ' ([AAM, Lemma 4.7]).

To say more about the Hecke algebra $\mathcal{H}(\lambda)$, we must examine the set of integral roots $R(\lambda)$ (59). Let κ be a ϑ -orbit on the set of simple roots of $R^+(\lambda)$. The orbit κ is equal to one of the following:

$$\begin{aligned} & \text{one root } \{\alpha = \vartheta(\alpha)\} && \text{(type 1)} \\ & \text{two roots } \{\alpha, \beta = \vartheta(\alpha)\}, \quad \langle \alpha, \vee \beta \rangle = 0 && \text{(type 2)} \\ & \text{two roots } \{\alpha, \beta = \vartheta(\alpha)\}, \quad \langle \alpha, \vee \beta \rangle = -1 && \text{(type 3)}. \end{aligned} \tag{73}$$

It is clear that our automorphism ϑ renders all orbits κ to be of type 2. This is a notable simplification in our setting.

Write $W(\lambda)$ for the Weyl group of the integral roots $R(\lambda)$, and let

$$W(\lambda)^{\vartheta} = \{w \in W(\lambda) : \vartheta(w) = w\}.$$

The group $W(\lambda)^{\vartheta}$ is a Coxeter group ([LV, Section 4.3]) with generators

$$w_{\kappa} = s_{\alpha} s_{\vartheta(\alpha)}, \quad \kappa = \{\alpha, \vartheta(\alpha)\}. \tag{74}$$

The Hecke algebra $\mathcal{H}(\lambda)$ ([AV, Section 10], [LV, Section 4.7]) is a free $\mathbb{Z}[q^{1/2}, q^{-1/2}]$ -algebra with basis

$$\{T_w : w \in W(\lambda)^{\vartheta}\}.$$

It is a consequence of [LV, Equation 4.7 (a)] that $\mathcal{H}(\lambda)$ is generated by the operators $T_{\kappa} := T_{w_{\kappa}}$, where κ is a ϑ -orbit as in (73).

The action of T_{κ} is defined in terms of the types listed in (73), which for us are only of type 2. The action also depends on the relationship of κ relative to a fixed parameter $\xi \in \Xi(\vee \mathcal{O}, \vee \mathbf{R}_{\mathbb{C}/\mathbb{R}} \mathbf{GL}_N^{\Gamma})^{\vartheta}$. To say a bit more about this dependence, recall that the parameter ξ is equivalent to an Atlas parameter (x, y) as in Lemma 2.2. The adjoint action of x acts as an involution on $R(\lambda)$ (see (40)). This action separates the ϑ -orbits of roots κ into various types, *e.g.* real, imaginary, etc. Lusztig and Vogan combine this information with the types of (73) and also with the types defined by Vogan in [V3, Section 8.3]. The list of combined types may be found in [LV, Section 7] or [AV, Table 1].

Very few of the types that appear in these lists are relevant for $\mathbf{GL}_N(\mathbb{C})$. We have already observed that only type 2 orbits appear in the sense of (73). Furthermore, $\mathbf{GL}_N(\mathbb{C})$ has only complex roots relative to x (33). The only relevant types for $\mathbf{GL}_N(\mathbb{C})$ in [AV, Table 1] are labelled as

$$2\mathbf{C}+, 2\mathbf{C}-, 2\mathbf{C}i, 2\mathbf{C}r. \tag{75}$$

That only these four types are relevant to our setting is another notable simplification. Any ϑ -orbit κ also has a type relative to the dual parameter $\vee \xi$ (69). The dual parameter is equivalent to the Atlas parameter (y, x) and the adjoint action of y is essentially the negative of the adjoint action of x ([AV, Definition 3.10]). In consequence, it is easy to compute the types and see that we again recover exactly those listed in (75).

In [LV, Section 4] and [AV, Section 7] the Hecke algebra action on $\mathcal{K}\Pi(\vee \mathcal{O}, \mathbf{GL}_N(\mathbb{C}), \vartheta)$ is given by defining the action of the operators T_{κ} on the generating set $\{M(\xi)^+ : \xi \in \Xi(\vee \mathcal{O}, \vee \mathbf{R}_{\mathbb{C}/\mathbb{R}} \mathbf{GL}_N^{\Gamma})^{\vartheta}\}$. The actions are presented in terms of extended Atlas parameters in [AV, Proposition 10.4]. A case-by-case summary of the actions is given in [AV, Table 5].

There are obvious parallel constructions for (68) which define a Hecke algebra $\vee \mathcal{H}(\lambda)$ and a Hecke module structure for $\mathcal{K}^{\vee} \Pi(\vee \mathcal{O}, \mathbf{GL}_N(\mathbb{C}), \vartheta)$. The Hecke algebra $\vee \mathcal{H}(\lambda)$ for (68) is generated by Hecke operators $T_{\vee \kappa}$, where $\vee \kappa$ runs over the simple coroots corresponding to κ . The bijection between the two sets of operators

$$\{T_{\kappa} : \kappa \in R^+(\lambda) \text{ simple}\} \longleftrightarrow \{T_{\vee \kappa} : \kappa \in R^+(\lambda) \text{ simple}\}$$

extends to an isomorphism $\mathcal{H}(\lambda) \cong \vee \mathcal{H}(\lambda)$. For this reason, we also regard $\mathcal{K}^{\vee} \Pi(\vee \mathcal{O}, \vee \mathbf{GL}_N(\mathbb{C}), \vartheta)$ as an $\mathcal{H}(\lambda)$ -module.

4.3 A Hecke module isomorphism

Let

$$\mathcal{K}^\vee \Pi(\vee \mathcal{O}, \mathrm{GL}_N(\mathbb{C}), \vartheta)^* = \mathrm{Hom}_{\mathbb{Z}[q^{1/2}, q^{-1/2}]} \left(\mathcal{K}^\vee \Pi(\vee \mathcal{O}, \mathrm{GL}_N(\mathbb{C}), \vartheta), \mathbb{Z}[q^{1/2}, q^{-1/2}] \right).$$

The extended pairing (72) induces a $\mathbb{Z}[q^{1/2}, q^{-1/2}]$ -module isomorphism

$$\begin{aligned} \mathcal{K} \Pi(\vee \mathcal{O}, \mathrm{GL}_N(\mathbb{C}), \vartheta) &\rightarrow \mathcal{K}^\vee \Pi(\vee \mathcal{O}, \mathrm{GL}_N(\mathbb{C}), \vartheta)^* \\ M(\xi)^+ &\mapsto \langle M(\xi)^+, \cdot \rangle. \end{aligned} \quad (76)$$

We endow $\mathcal{K}^\vee \Pi(\vee \mathcal{O}, \mathrm{GL}_N(\mathbb{C}), \vartheta)^*$ with the Hecke module structure given in [AV, Section 11] (cf. [AAM, Section 4.5]). Specifically, for any $\mu \in \mathcal{K}^\vee \Pi(\vee \mathcal{O}, \mathrm{GL}_N(\mathbb{C}), \vartheta)^*$ and ϑ -orbit κ as in (73)

$$T_{w_\kappa}^* \cdot \mu = -(T_{w_\kappa})^t \cdot \mu + (q^{l(w_\kappa)} - 1)\mu, \quad (77)$$

where $l(w)$ is the usual length of w with respect to the simple reflections, and $(T_w)^t$ is the transpose of T_w . Since both the domain and codomain of (76) are $\mathcal{H}(\lambda)$ -modules it is natural to ask whether (76) extends to a $\mathcal{H}(\lambda)$ -module isomorphism.

Proposition 4.2. *The map (76) is an isomorphism of $\mathcal{H}(\lambda)$ -modules.*

The proof of this proposition is a simplified version of the proof of [AAM, Proposition 4.8]. We provide a sketch. In view of (77), Proposition 4.2 is equivalent to

$$\langle T_{w_\kappa} M(\xi_1)^+, M(\vee \xi_2)^+ \rangle = \langle M(\xi_1)^+, -T_{w_\kappa} M(\vee \xi_2)^+ + (q^{l(w_\kappa)} - 1)M(\vee \xi_2)^+ \rangle \quad (78)$$

for all $\xi_1, \xi_2 \in \Xi(\vee \mathcal{O}, \vee \mathrm{R}_{\mathbb{C}/\mathbb{R}} \mathrm{GL}_N^\Gamma)^\vartheta$ and w_κ as in (74). Looking back to the definition of (72), the left-hand side of (78) may be expressed as

$$\langle T_{w_\kappa} M(\xi_1)^+, M(\vee \xi_2)^+ \rangle = (-1)^{l_\vartheta(\xi_2)} q^{(l^I(\xi_2) + l^I(\vee \xi_2))/2} \cdot (\text{the coefficient of } M(\xi_2)^+ \text{ in } T_{w_\kappa} M(\xi_1)^+).$$

Similarly, the right-hand side of (78) may be expressed as the product of $(-1)^{l_\vartheta(\xi_1)} q^{(l^I(\xi_1) + l^I(\vee \xi_1))/2}$ with

$$\text{the coefficient of } M(\vee \xi_1)^+ \text{ in } -T_{w_\kappa} M(\vee \xi_2)^+ + (q^{l(w_\kappa)} - 1)M(\vee \xi_2)^+.$$

As in [AAM, Lemma 4.7],

$$l^I(\xi_1) + l^I(\vee \xi_1) = l^I(\xi_2) + l^I(\vee \xi_2),$$

so Equation (78) is equivalent to

$$\begin{aligned} &(-1)^{l_\vartheta(\xi_2) - l_\vartheta(\xi_1)} \cdot (\text{the coefficient of } M(\xi_2)^+ \text{ in } T_{w_\kappa} M(\xi_1)^+) \\ &= \text{the coefficient of } M(\vee \xi_1)^+ \text{ in } -T_{w_\kappa} M(\vee \xi_2)^+ + (q^{l(w_\kappa)} - 1)M(\vee \xi_2)^+. \end{aligned} \quad (79)$$

The values of $T_{w_\kappa} M(\xi_1)^+$ and $T_{w_\kappa} M(\vee \xi_2)^+$ are known ([AV, Table 5], [AAM, Proposition 4.9]). The proof of (79) may therefore be achieved by a case-by-case analysis of the types of κ relative to $\xi_1, \xi_2, \vee \xi_1$ and $\vee \xi_2$ (75).

We provide some more detail in the case that $\kappa = \{\alpha, \beta\}$ is of type 2Ci relative to ξ_1 , leaving the remaining cases to the reader. We identify ξ_1 with its corresponding Atlas parameter (x_1, y_1) (Lemma 2.2) and do likewise for all other parameters to come. Type 2Ci implies $\beta = x_1 \cdot \alpha$.

Let us start by computing the left-hand side of (79). According to [AV, Proposition 9.1 and Proposition 10.4], we have

$$T_{w_\kappa} M(\xi_1)^+ = qM(\xi_1)^+ + (q+1)M(\xi)^+,$$

where $\xi = (x, y) \in \Xi(\vee \mathcal{O}, \vee \mathrm{R}_{\mathbb{C}/\mathbb{R}} \mathrm{GL}_N^\Gamma)$ is a parameter satisfying

$$x = s_\alpha x_1 s_\alpha^{-1} = s_\alpha s_\beta x_1 = x_1 s_\alpha s_\beta \quad (80)$$

([AV, (46o)]). (We note that the signs appearing in [AV, Proposition 9.1] are always one for the Atlas extensions.) Consequently,

$$\text{the coefficient of } M(\xi_2)^+ \text{ in } T_{w_\kappa} M(\xi_1)^+ \text{ is } \begin{cases} q, & \text{if } \xi_2 = \xi_1 \\ (q+1), & \text{if } \xi_2 = \xi \\ 0, & \text{otherwise.} \end{cases}$$

To compute the ϑ -integral lengths (61), observe that

$$x \cdot \{\alpha, \beta\} = s_\alpha s_\beta x_1 \cdot \{\alpha, \beta\} = \{-\alpha, -\beta\},$$

and for any positive root $\gamma \in R^+(\lambda)$ with $\gamma \neq \alpha, \beta$, we have

$$x \cdot \{\gamma, \vartheta\gamma\} = x_1 s_\alpha s_\beta \{\gamma, \vartheta\gamma\} = x_1 \cdot \{\gamma', \vartheta\gamma'\},$$

where γ' is a positive root in $R^+(\lambda)$. Hence, after identifying the roots in $R_\vartheta^+(\lambda)$ with ϑ -orbits in $R^+(\lambda)$, we deduce

$$|\{\gamma \in R_\vartheta^+(\lambda) : x \cdot \gamma \in R_\vartheta^+(\lambda)\}| = |\{\gamma \in R_\vartheta^+(\lambda) : x_1 \cdot \gamma \in R_\vartheta^+(\lambda)\}| - 1,$$

and

$$\dim((H^\vartheta)^x) = \dim((H^\vartheta)^{x_1}) - 1.$$

It follows that $l_\vartheta^I(\xi) = l_\vartheta^I(\xi_1) - 1$ and $(-1)^{l_\vartheta^I(\xi) - l_\vartheta^I(\xi_1)} = -1$. The left-hand side of (79) is therefore equal to

$$\begin{cases} q, & \text{if } \xi_2 = \xi_1 \\ -(q+1), & \text{if } \xi_2 = \xi \\ 0, & \text{otherwise.} \end{cases} \quad (81)$$

Let us consider the right-hand side of (79), in which ${}^\vee\kappa$ is of type 2Cr relative to ${}^\vee\xi_1$. According to [AV, Table 5], $M({}^\vee\xi_1)^+$ occurs in $T_{w_\kappa} M({}^\vee\xi_2)^+$ only if one of the following holds

1. $M({}^\vee\xi_1)^+ = M({}^\vee\xi_2)^+$,
2. $M({}^\vee\xi_1)^+ = s_{\vee\alpha} \times M({}^\vee\xi_2)^+$,
3. $M({}^\vee\xi_1)^+ = w_{\vee\kappa} \times M({}^\vee\xi_2)^+$.

The third equation holds if and only if $M({}^\vee\xi_2)^+ = w_{\vee\kappa} \times M({}^\vee\xi_1)^+$, and for ${}^\vee\kappa$ of type 2Cr relative to ${}^\vee\xi_1$ one obtains

$$w_{\vee\kappa} \times M({}^\vee\xi_1)^+ = s_{\vee\alpha} \times (s_{\vee\beta} \times M({}^\vee\xi_1)^+) = M({}^\vee\xi_1)^+.$$

Therefore the third equation is equivalent to the first one.

The second equation is equivalent to ${}^\vee\xi_2$ being equal to ${}^\vee\xi$ ((80), [AAM, Proposition 4.9]). Moreover, ${}^\vee\kappa$ is of type 2Ci relative to ${}^\vee\xi$. The results [AV, Proposition 9.1 and Proposition 10.4] indicate that

$$\begin{aligned} T_{w_{\vee\kappa}} M({}^\vee\xi_1)^+ &= (q^2 - q - 1)M({}^\vee\xi_1)^+ + (q^2 - q)M({}^\vee\xi)^+, \\ T_{w_{\vee\kappa}} M({}^\vee\xi)^+ &= qM({}^\vee\xi)^+ + (q+1)M({}^\vee\xi_1)^+. \end{aligned}$$

Therefore, the right-hand side of (79) equals

$$\begin{cases} q, & \text{if } {}^\vee\xi_2 = {}^\vee\xi_1 \\ -(q+1), & \text{if } {}^\vee\xi_2 = {}^\vee\xi \\ 0, & \text{otherwise.} \end{cases}$$

In all cases we have equality with (81).

4.4 Special bases and the proof of Theorem 3.5

In addition to the $\mathcal{H}(\lambda)$ -action on $\mathcal{K}\Pi(\vee\mathcal{O}, \mathrm{GL}_N(\mathbb{C}), \vartheta)$, there is a \mathbb{Z} -linear involution D on $\mathcal{K}\Pi(\vee\mathcal{O}, \mathrm{GL}_N(\mathbb{C}), \vartheta)$ satisfying

$$\begin{aligned} D(q^{1/2}M(\xi)^+) &= q^{-1/2}D((\xi)^+) \\ D((T_\kappa + 1)M(\xi)^+) &= q^{-l(w_\kappa)}(T_\kappa + 1)D(M(\xi)^+) \end{aligned} \quad (82)$$

for all $\xi \in \Xi(\vee\mathcal{O}, \vee\mathrm{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N^\Gamma)^\vartheta$. This is the *Verdier duality map* ([LV, 4.8 (e)-(f)]). The Verdier duality map is uniquely determined by the additional conditions given in [LV, 8.1 (a)].

The special basis we seek for $\mathcal{K}\Pi(\vee\mathcal{O}, \mathrm{GL}_N(\mathbb{C}), \vartheta)$ is a basis of eigenvectors for D . It is defined in terms of the twisted KLV-polynomials $P^\vartheta(\xi', \xi) \in \mathbb{Z}[q^{1/2}, q^{-1/2}]$ defined in [LV, Section 0.1]. Its definition and characteristics are summarized in following theorem.

Theorem 4.3 ([LV, Theorem 5.2]). *For every $\xi \in \Xi(\vee\mathcal{O}, \vee\mathrm{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N^\Gamma)^\vartheta$, define*

$$C^\vartheta(\xi) = \sum_{\xi' \in \Xi(\vee\mathcal{O}, \vee\mathrm{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N^\Gamma)^\vartheta} (-1)^{l^\vartheta(\xi) - l^\vartheta(\xi')} P^\vartheta(\xi', \xi) M(\xi')^+, \quad (83)$$

an element in $\mathcal{K}\Pi(\vee\mathcal{O}, \mathrm{GL}_N(\mathbb{C}), \vartheta)$. Then

1. $D(C^\vartheta(\xi)) = q^{-l^\vartheta(\xi)} C^\vartheta(\xi)$
2. $P^\vartheta(\xi, \xi) = 1$
3. $P^\vartheta(\xi', \xi) = 0$ if $\xi' \not\leq \xi$
4. $\deg P^\vartheta(\xi', \xi) \leq (l^\vartheta(\xi) - l^\vartheta(\xi') - 1)/2$ if $\xi' \leq \xi$.

Conversely suppose $\{\underline{C}(\xi', \xi)\}$ and $\{\underline{P}(\xi', \xi)\}$ satisfy (83) and (1)-(4) above. Then $\underline{P}(\xi', \xi) = P^\vartheta(\xi', \xi)$ and $\underline{C}(\xi', \xi) = C^\vartheta(\xi', \xi)$ for all $\xi', \xi \in \Xi(\vee\mathcal{O}, \vee\mathrm{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N^\Gamma)^\vartheta$.

The third property of this theorem uses a partial order on $\Xi(\vee\mathcal{O}, \vee\mathrm{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N^\Gamma)^\vartheta$, the *Bruhat order*, which is defined in [LV, Section 5.1] (cf. [ABV, (7.11)(f)]).

There is also a Verdier duality map ${}^\vee D$ for the module $\mathcal{K}{}^\vee\Pi(\vee\mathcal{O}, \mathrm{GL}_N(\mathbb{C}), \vartheta)$ which satisfies obvious analogues of (82) and the additional conditions of [LV, 8.1 (a)]. Furthermore, there is an obvious analogue Theorem 4.3 for the dual basis elements

$$C^\vartheta(\vee\xi) = \sum_{\xi' \in \Xi(\vee\mathcal{O}, \vee\mathrm{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N^\Gamma)^\vartheta} (-1)^{l^\vartheta(\vee\xi) - l^\vartheta(\vee\xi')} {}^\vee P^\vartheta(\vee\xi', \vee\xi) M(\vee\xi')^+.$$

According to [AAM, Proposition 4.14], by setting $q = 1$ in the polynomials of the basis elements, one obtains

$$C^\vartheta(\xi)(1) = \pi(\xi)^+ \quad \text{and} \quad C^\vartheta(\vee\xi)(1) = \pi(\vee\xi)^+ \quad (84)$$

for all $\xi \in \Xi(\vee\mathcal{O}, \vee\mathrm{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N^\Gamma)^\vartheta$. It is immediate from Equation (84) and Lemma 4.1 that Theorem 3.5 is proved by the following theorem.

Theorem 4.4. *Pairing (72) satisfies*

$$\langle C^\vartheta(\xi), C^\vartheta(\vee\xi') \rangle = (-1)^{l_\vartheta^\vartheta(\xi)} q^{(l^\vartheta(\xi) + l^\vartheta(\vee\xi'))/2} \delta_{\xi, \xi'}.$$

for all $\xi, \xi' \in \Xi(\vee\mathcal{O}, \vee\mathrm{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N^\Gamma)^\vartheta$.

Proof. The isomorphism (76) allows us to define unique elements $\underline{C}^\vartheta(\xi) \in \mathcal{K}\Pi(\vee\mathcal{O}, \mathrm{GL}_N(\mathbb{C}), \vartheta)$, $\xi \in \Xi(\vee\mathcal{O}, \vee\mathrm{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N^\Gamma)^\vartheta$, satisfying

$$\langle \underline{C}^\vartheta(\xi), C^\vartheta(\vee\xi') \rangle = (-1)^{l_\vartheta^\vartheta(\xi')} q^{(l^\vartheta(\xi) + l^\vartheta(\vee\xi'))/2} \delta_{\xi, \xi'}.$$

The proof amounts to showing that

$$\underline{C}^\vartheta(\xi) = C^\vartheta(\vee\xi), \quad \xi \in \Xi(\vee\mathcal{O}, \vee\mathrm{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N^\Gamma)^\vartheta$$

and we may do so by showing that $\underline{C}^\vartheta(\xi)$ satisfies each of the four properties in Theorem 4.3. We outline the proof of the the first property of Theorem 4.3, as it is the most involved. We leave the reader to verify the remaining three properties by consulting [AAM, Section 4.7].

The first step is to prove that the two Verdier dualities are related by

$$\overline{\langle D(M(\xi)), M({}^\vee\xi') \rangle} = \langle M(\xi), {}^\vee D(M({}^\vee\xi')) \rangle, \quad \xi, \xi' \in \Xi({}^\vee\mathcal{O}, {}^\vee R_{\mathbb{C}/\mathbb{R}} \mathrm{GL}_N^\Gamma)^\vartheta \quad (85)$$

where

$$- : \mathbb{Z}[q^{1/2}, q^{-1/2}] \rightarrow \mathbb{Z}[q^{1/2}, q^{-1/2}]$$

is the unique automorphism sending $q^{1/2}$ to $q^{-1/2}$. To do this, one may define ${}^\vee D'$ by

$$\langle M(\xi), {}^\vee D'(M({}^\vee\xi')) \rangle = \overline{\langle D(M(\xi)), M({}^\vee\xi') \rangle}, \quad \xi, \xi' \in \Xi({}^\vee\mathcal{O}, {}^\vee R_{\mathbb{C}/\mathbb{R}} \mathrm{GL}_N^\Gamma)^\vartheta$$

and prove that ${}^\vee D' = {}^\vee D$ using the properties of (82) and [LV, 8.1 (a)] which characterize the Verdier duality (cf. [V4, Lemma 13.4]). Proposition 4.2 plays a key role in proving the property

$${}^\vee D'((T_\kappa + 1)M({}^\vee\xi')^+) = q^{-l(w_\kappa)}(T_\kappa + 1) {}^\vee D'(M({}^\vee\xi')^+) \quad (86)$$

of (82), so it seems fitting to supply the arguments for it. We are to prove that

$$\langle M(\xi), {}^\vee D'((T_\kappa + 1)M({}^\vee\xi')^+) \rangle = \langle M(\xi), q^{-l(w_\kappa)}(T_\kappa + 1) {}^\vee D'(M({}^\vee\xi')^+) \rangle$$

for all $\xi, \xi' \in \Xi({}^\vee\mathcal{O}, {}^\vee R_{\mathbb{C}/\mathbb{R}} \mathrm{GL}_N^\Gamma)^\vartheta$. According to the definition of ${}^\vee D'$, and Equation (78) of Proposition 4.2, this is equivalent to

$$(-T_\kappa + q^{l(w_\kappa)})D(M(\xi)) = D\left((-T_\kappa + q^{l(w_\kappa)})q^{-l(w_\kappa)}M(\xi)\right).$$

Beginning with the right-hand side and following ([V4, Lemma 13.4]) with (82), we compute

$$\begin{aligned} & -D\left((T_\kappa + 1)q^{-l(w_\kappa)}M(\xi)\right) + D\left((q^{l(w_\kappa)} + 1)q^{-l(w_\kappa)}M(\xi)\right) \\ &= -q^{-l(w_\kappa)}(T_\kappa + 1)D(q^{-l(w_\kappa)}M(\xi)) + (1 + q^{l(w_\kappa)})D(M(\xi)) \\ &= -(T_\kappa + 1)D(M(\xi)) + (1 + q^{l(w_\kappa)})D(M(\xi)) \\ &= (-T_\kappa + q^{l(w_\kappa)})D(M(\xi)). \end{aligned}$$

This proves (86). The remaining properties ensuring that ${}^\vee D' = {}^\vee D$ are also easily read from the proof of [V4, Lemma 13.4].

We now take for granted Equation (85), and may prove that $\underline{C}^\vartheta(\xi)$ satisfies the first property of Theorem 4.3 as follows.

$$\begin{aligned} \langle D\underline{C}^\vartheta(\xi), C^\vartheta({}^\vee\xi') \rangle &= \overline{\langle \underline{C}^\vartheta(\xi), {}^\vee DC^\vartheta({}^\vee\xi') \rangle} \\ &= q^{l({}^\vee\xi')} \overline{\langle \underline{C}^\vartheta(\xi), C^\vartheta({}^\vee\xi') \rangle} \\ &= (-1)^{l_\vartheta(\xi)} q^{l({}^\vee\xi')} q^{-(l^I(\xi) + l^I({}^\vee\xi'))/2} \delta_{\xi, \xi'} \\ &= \langle q^{-l^I(\xi)} \underline{C}^\vartheta(\xi), C^\vartheta({}^\vee\xi') \rangle. \end{aligned}$$

Since the elements $C^\vartheta({}^\vee\xi')$ form a basis we conclude that

$$D\underline{C}^\vartheta(\xi) = q^{-l^I(\xi)} \underline{C}^\vartheta(\xi)$$

and the first property of Theorem 4.3 is proved. As mentioned, the remaining properties of Theorem 4.3 follow easily from [AAM, Section 4.7]. \square

5 Endoscopic lifting for complex general linear groups following Adams-Barbasch-Vogan

We proceed with a review of standard endoscopy and twisted endoscopy from the perspective of [ABV], but restricted only to the particular case of the group $R_{\mathbb{C}/\mathbb{R}} \mathrm{GL}_N$. A similar review was made for GL_N in [AAM, Section 5]. The present review contains nothing new and so we shall refer to [AAM, Section 5] liberally. The background material for this section is found in [ABV, Section 26] and [CM, Section 5].

5.1 Standard endoscopy

In this section we do not place any restrictions on the infinitesimal characters. In addition the general framework applies to any connected real reductive group. For the purposes of motivation we specialize to the group ${}^\vee\mathrm{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N^\Gamma = ({}^\vee\mathrm{GL}_N \times {}^\vee\mathrm{GL}_N) \rtimes \langle {}^\vee\delta_0 \rangle$ as in Section 2.2. In the last two sections, we shall apply the theory of endoscopy to unitary groups.

An *endoscopic datum* for ${}^\vee\mathrm{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N^\Gamma$ is a pair

$$(s, {}^\vee G^\Gamma)$$

which satisfies

1. $s \in {}^\vee\mathrm{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N$ is semisimple
2. ${}^\vee G^\Gamma \subset {}^\vee\mathrm{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N^\Gamma$ is open in the centralizer of s in ${}^\vee\mathrm{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N^\Gamma$
3. ${}^\vee G^\Gamma$ is an E-group for a group G ([ABV, Definition 4.6]).

This is a specialization of [ABV, Definition 26.15] to ${}^\vee\mathrm{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N^\Gamma$. The groups ${}^\vee G$ and G here are isomorphic Levi subgroups. They are products of smaller general linear groups. Consequently, ${}^\vee G$ and ${}^\vee\mathrm{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N$ share the maximal torus ${}^\vee H$, which is two copies of the diagonal subgroup. Similarly, G and $\mathrm{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N$ share the maximal torus H . We shall abusively denote by δ_q the strong involution on both G and $\mathrm{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N$ which correspond to the quasisplit real forms. The group G in this definition is called the *endoscopic group*. We do not require the concept of an E-group in this section. From now on we assume that ${}^\vee G^\Gamma = {}^\vee G \rtimes \langle {}^\vee\delta_0 \rangle$. In other words, ${}^\vee G^\Gamma$ is an L-group for G .

There is a notion of equivalence for endoscopic data, and using this equivalence we may assume without loss of generality that $s \in {}^\vee H$. We fix $\lambda \in {}^\vee\mathfrak{h}$. Let ${}^\vee\mathcal{O}_G$ be the ${}^\vee G$ -orbit of λ and ${}^\vee\mathcal{O}$ be the ${}^\vee\mathrm{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N$ -orbit of λ . The second property of the endoscopic datum above allows us to define the inclusion

$$\epsilon : {}^\vee G^\Gamma \hookrightarrow {}^\vee\mathrm{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N^\Gamma. \quad (87)$$

This inclusion induces another map ([ABV, Corollary 6.21]), which we abusively also denote as

$$\epsilon : X({}^\vee\mathcal{O}_G, {}^\vee G^\Gamma) \rightarrow X({}^\vee\mathcal{O}, {}^\vee\mathrm{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N^\Gamma). \quad (88)$$

It is easily verified that the ${}^\vee G$ -action on $X({}^\vee\mathcal{O}_G, {}^\vee G^\Gamma)$ is compatibly carried under ϵ to the ${}^\vee\mathrm{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N$ -action on $X({}^\vee\mathcal{O}, {}^\vee\mathrm{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N^\Gamma)$ ([ABV, (7.17)]). As a result, the map ϵ induces a map from the orbits of the space $X({}^\vee\mathcal{O}_G, {}^\vee G^\Gamma)$ to the orbits of $X({}^\vee\mathcal{O}, {}^\vee\mathrm{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N^\Gamma)$.

The inverse image functor of ϵ on equivariant constructible sheaves induces a homomorphism

$$\epsilon^* : K({}^\vee\mathcal{O}, {}^\vee\mathrm{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N^\Gamma) \rightarrow K({}^\vee\mathcal{O}_G, {}^\vee G^\Gamma)$$

([ABV, Proposition 7.18]). When ϵ^* is combined with the pairings of Theorem 3.1, we obtain a map

$$\epsilon_* : K_{\mathbb{C}}\Pi({}^\vee\mathcal{O}_G, G/\mathbb{R}) \rightarrow K_{\mathbb{C}}\Pi({}^\vee\mathcal{O}, \mathrm{GL}_N(\mathbb{C}))$$

defined on $\eta_G \in K_{\mathbb{C}}\Pi({}^\vee\mathcal{O}_G, G/\mathbb{R})$ by

$$\langle \epsilon_*\eta_G, \mu(\xi) \rangle = \langle \eta_G, \epsilon^*\mu(\xi) \rangle_G, \quad \xi \in \Xi({}^\vee\mathcal{O}, {}^\vee\mathrm{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N^\Gamma). \quad (89)$$

Here, $K_{\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{Z}} K$ and we have placed a subscript G beside the pairing on the right to distinguish it from the pairing for $\mathrm{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N$ on the left.

The endoscopic lifting map Lift_0 is defined to be the restriction of ϵ_* to the submodule

$$K_{\mathbb{C}}\Pi({}^\vee\mathcal{O}_G, G(\mathbb{R}, \delta_q))^{\mathrm{st}} \subset K_{\mathbb{C}}\Pi({}^\vee\mathcal{O}_G, G/\mathbb{R})$$

of stable virtual characters of the quasisplit form $G(\mathbb{R}, \delta_q)$. Since $G(\mathbb{R}, \delta_q)$ is a product of complex general linear groups, stability is not an issue and we have

$$K_{\mathbb{C}}\Pi({}^\vee\mathcal{O}_G, G(\mathbb{R}, \delta_q))^{\mathrm{st}} = K_{\mathbb{C}}\Pi({}^\vee\mathcal{O}_G, G/\mathbb{R}).$$

This equality will not hold for twisted endoscopic groups in Section 5.2, and so it is better to write the endoscopic lifting map as

$$\text{Lift}_0 : K_{\mathbb{C}}\Pi(\mathcal{O}_G, G(\mathbb{R}, \delta_q))^{\text{st}} \rightarrow K_{\mathbb{C}}\Pi(\mathcal{O}, \text{GL}_N(\mathbb{C})). \quad (90)$$

According to [ABV, Lemma 18.11] and [MW, Corollary IV.2.8]), a basis for $K_{\mathbb{C}}\Pi(\mathcal{O}_G, G(\mathbb{R}, \delta_q))^{\text{st}}$ is provided by the virtual characters

$$\eta_{S_1}^{\text{loc}}(\delta_q) = \sum_{\tau_{S_1}} M(S_1, \tau_{S_1}), \quad (91)$$

where $(S_1, \tau_{S_1}) \in \Xi(\mathcal{O}_G, \mathcal{O}_G^{\Gamma})$ runs over those complete geometric parameters which correspond to the strong involution δ_q under the local Langlands correspondence (30). As indicated at the end of Section 2.2, the relevant component groups for complex general linear groups are trivial. Therefore the representations τ_{S_1} are all trivial for G and (91) reduces to

$$\eta_{S_1}^{\text{loc}}(\delta_q) = M(S_1, 1),$$

a single standard representation. The following proposition describes the image of $\eta_{S_1}^{\text{loc}}(\delta_q)$ under endoscopic lifting. Its proof follows from [AAM, Proposition 5.1] by replacing GL_N with $\text{R}_{\mathbb{C}/\mathbb{R}}\text{GL}_N$.

Proposition 5.1 ([AAM, Proposition 5.1]). *(a) Suppose $S_1 \subset X(\mathcal{O}_G, \mathcal{O}_G^{\Gamma})$ is a \mathcal{O}_G -orbit which is carried to the $\text{R}_{\mathbb{C}/\mathbb{R}}\text{GL}_N$ -orbit S under ϵ . Then*

$$\text{Lift}_0(\eta_{S_1}^{\text{loc}}(\delta_q)) = \eta_S^{\text{loc}},$$

or equivalently,

$$\text{Lift}_0(M(S_1, 1)) = M(S, 1).$$

(b) The endoscopic lifting map Lift_0 is equal to the parabolic induction functor $\text{ind}_{G(\mathbb{R}, \delta_q)}^{\text{GL}_N(\mathbb{C})}$ on $K_{\mathbb{C}}\Pi(\mathcal{O}_G, G(\mathbb{R}, \delta_q))^{\text{st}}$.

It is much more difficult to compute the value of Lift_0 on the stable virtual character $\eta_{\psi_G}^{\text{mic}}$ given in (52). Let $\psi = \epsilon \circ \psi_G$. According to [ABV, Theorem 26.25]

$$\text{Lift}_0(\eta_{\psi_G}^{\text{mic}}) = \sum_{\xi \in \Xi(\mathcal{O}_G^{\Gamma}, \mathcal{O}_G)} (-1)^{d(S_{\xi}) - d(S_{\psi})} \chi_{S_{\psi}}^{\text{mic}}(P(\xi)) \pi(\xi) = \eta_{\psi}^{\text{mic}}. \quad (92)$$

Recall from (54) that the ABV-packets $\Pi_{\psi_G}^{\text{ABV}}$ and Π_{ψ}^{ABV} are defined from $\eta_{\psi_G}^{\text{mic}}$ and η_{ψ}^{mic} respectively. We shall see in Section 6 that these ABV-packets are singletons.

5.2 Twisted endoscopy

Our aim in this section is to lay out the twisted versions of the concepts presented in the previous section. We define twisted endoscopic data relevant to unitary groups, the twisted endoscopic version of Lift_0 (90), compute twisted variants of $\text{Lift}_0(\eta_S^{\text{loc}})$ for $S \in X(\mathcal{O}_G, \mathcal{O}_G^{\Gamma})$, and compute twisted variants of $\text{Lift}_0(\eta_{\psi_G}^{\text{mic}})$. We shall work under the assumption of (38) on the infinitesimal characters. This assumption is made only to accommodate the definition of Atlas extensions.

An *endoscopic datum* for $(\text{R}_{\mathbb{C}/\mathbb{R}}\text{GL}_N^{\Gamma}, \vartheta)$ is a pair

$$(s, \mathcal{O}_G^{\Gamma})$$

which satisfies

1. $s \in \text{R}_{\mathbb{C}/\mathbb{R}}\text{GL}_N$ is ϑ -semisimple (see [KS, (2.1.3)])
2. $\mathcal{O}_G^{\Gamma} \subset \text{R}_{\mathbb{C}/\mathbb{R}}\text{GL}_N^{\Gamma}$ is open in the fixed-point set of $\text{Int}(s) \circ \vartheta$ in $\text{R}_{\mathbb{C}/\mathbb{R}}\text{GL}_N^{\Gamma} \rtimes \langle \vartheta \rangle$
3. \mathcal{O}_G^{Γ} is an E-group for a group G ([ABV, Definition 4.6]).

This is a special case of [CM, Definition 5.1] to ${}^\vee\mathrm{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N^\Gamma$. There is a notion of equivalence for these endoscopic data ([CM, Definition 5.6], [KS, (2.1.5)-(2.1.6)]).

Let us take $s = 1$ in the endoscopic pair above. Then the fixed-point subgroup

$$({}^\vee\mathrm{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N)^\vartheta = \{(g, \tilde{J}(g^\top)^{-1}\tilde{J}^{-1}) : g \in {}^\vee\mathrm{GL}_N\} \cong {}^\vee\mathrm{GL}_N$$

is a legitimate dual group for an endoscopic datum. Furthermore, by setting

$${}^\vee G^\Gamma = ({}^\vee\mathrm{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N)^\vartheta \rtimes \langle \delta_0 \rangle$$

with δ_0 as in (33) we see that ${}^\vee G^\Gamma$ is isomorphic to the L-group of a quasisplit unitary group as in (31). We have just shown that

$$(1, {}^\vee G^\Gamma) = (1, {}^\vee\mathrm{GL}_N^\Gamma)$$

is an endoscopic datum for $({}^\vee\mathrm{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N^\Gamma, \vartheta)$ and that the corresponding endoscopic group is the rank N quasisplit unitary group. This is the *only* endoscopic datum of interest to us here.

Unlike the previous section, we must distinguish between maximal tori in ${}^\vee\mathrm{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N$ and ${}^\vee G^\Gamma$. We let ${}^\vee H$ be the diagonal maximal torus in ${}^\vee\mathrm{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N$, and ${}^\vee H_G$ be a maximal torus in ${}^\vee G \cong {}^\vee\mathrm{GL}_N$. The two tori are related by

$${}^\vee H_G = ({}^\vee H)^\vartheta.$$

We fix $\lambda \in {}^\vee\mathfrak{h}^\vartheta$. Let ${}^\vee\mathcal{O}_G$ be the ${}^\vee G$ -orbit of λ and ${}^\vee\mathcal{O}$ be the ${}^\vee\mathrm{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N$ -orbit of λ .

The ϵ maps of (87)-(88) have obvious analogues and are equally valid in the twisted setting. The analogue of (87) is quite transparent as it takes the form

$$\epsilon(g) = (g, \tilde{J}^{-1}(g^\top)^{-1}\tilde{J}^{-1}), \quad g \in {}^\vee G$$

and carries the element ${}^\vee\delta_0$ in (31) to the element δ_0 in (32).

The crucial point in the twisted setting is to include the action of ϑ into the objects pertinent to endoscopy. In particular we must extend the sheaf theory of [ABV] for ${}^\vee\mathrm{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N$ to the disconnected group ${}^\vee\mathrm{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N \rtimes \langle \vartheta \rangle$. This mimics the extension of the representation theory of $\mathrm{GL}_N(\mathbb{C})$ to the disconnected group $\mathrm{GL}_N(\mathbb{C}) \rtimes \langle \vartheta \rangle$ in Section 2.4. Rather than viewing the sheaves in $\mathcal{C}({}^\vee\mathcal{O}, {}^\vee\mathrm{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N^\Gamma; \vartheta)$ as ${}^\vee\mathrm{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N$ -equivariant with compatible ϑ -action (Section 3.2), we view them simply as $({}^\vee\mathrm{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N \rtimes \langle \vartheta \rangle)$ -equivariant sheaves and apply the theory of [ABV] which is valid in this generality ([CM, Section 5.4]).

Let $\xi = (S, 1) \in \Xi({}^\vee\mathcal{O}, {}^\vee\mathrm{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N^\Gamma)^\vartheta$ and $p \in S$. Here, 1 is the trivial representation of the trivial group $({}^\vee\mathrm{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N)_p / (({}^\vee\mathrm{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N)_p)^0$ with representation space $V \cong \mathbb{C}$ as in (49). We define 1^+ on

$$({}^\vee\mathrm{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N)_p / (({}^\vee\mathrm{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N)_p)^0 \times \langle \vartheta \rangle$$

by

$$1^+(\vartheta) = \vartheta_{\mu(\xi)^+} = 1 \tag{93}$$

In this way, 1^+ defines the local system underlying the irreducible $({}^\vee\mathrm{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N \rtimes \langle \vartheta \rangle)$ -equivariant constructible sheaf $\mu(\xi)^+$ (Lemma 3.4, [ABV, p. 83]).

In a similar, but completely vacuous, fashion we may include the trivial action of ϑ on $\mu(\xi_1) \in \mathcal{C}({}^\vee\mathcal{O}_G, {}^\vee G)$ with $\xi_1 = (S_1, \tau_1)$ and $p_1 \in S_1$. In other words, we may regard $\mu(\xi_1)$ as a $({}^\vee G \times \langle \vartheta \rangle)$ -equivariant sheaf whose underlying local system is defined by a quasicharacter τ_1^+ on

$${}^\vee G_{p_1} / ({}^\vee G_{p_1})^0 \times \langle \vartheta \rangle \tag{94}$$

by $\tau_1^+(\vartheta) = 1$.

The inverse image functor

$$\epsilon^* : KX({}^\vee\mathcal{O}, {}^\vee\mathrm{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N^\Gamma, \vartheta) \rightarrow KX({}^\vee\mathcal{O}_G, {}^\vee G^\Gamma)$$

in the present twisted setting is defined on $({}^\vee\mathrm{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N \rtimes \langle \vartheta \rangle)$ -equivariant sheaves (Section 3.2). As in standard endoscopy, we combine ϵ^* with a pairing, namely the pairing of Theorem 3.5, to define

$$\epsilon_* : K_{\mathbb{C}}\Pi({}^\vee\mathcal{O}_G, G/\mathbb{R}) \rightarrow K_{\mathbb{C}}\Pi({}^\vee\mathcal{O}, \mathrm{GL}_N(\mathbb{C}), \vartheta).$$

To be precise, the image of any $\eta \in K_{\mathbb{C}}\Pi(\vee\mathcal{O}_G, G/\mathbb{R})$ under ϵ_* is determined by

$$\langle \epsilon_*\eta, \mu(\xi)^+ \rangle = \langle \eta, \epsilon^*\mu(\xi)^+ \rangle_G, \quad \xi \in \Xi(\vee\mathcal{O}, \vee\mathbb{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N^{\Gamma})^{\vartheta}. \quad (95)$$

(cf. (89)). The twisted endoscopic lifting map

$$\mathrm{Lift}_0 : K_{\mathbb{C}}\Pi(\vee\mathcal{O}_G, G(\mathbb{R}, \delta_q))^{\mathrm{st}} \rightarrow K_{\mathbb{C}}\Pi(\vee\mathcal{O}, \mathrm{GL}_N(\mathbb{C}), \vartheta) \quad (96)$$

is the restriction of ϵ_* to the stable submodule $K_{\mathbb{C}}\Pi(\vee\mathcal{O}_G, G(\mathbb{R}, \delta_q))^{\mathrm{st}}$.

Now, we wish to evaluate Lift_0 on the basis elements (91) of $K_{\mathbb{C}}\Pi(\vee\mathcal{O}_G, G(\mathbb{R}, \delta_q))^{\mathrm{st}}$. To maintain ease of comparison with [ABV] we evaluate Lift_0 on the virtual representations $\eta_{S_1}^{\mathrm{loc}}(\vartheta)(\delta_q)$ ([ABV, p. 279]). These virtual characters are defined by

$$\eta_{S_1}^{\mathrm{loc}}(\vartheta)(\delta_q) = \sum_{\tau_1} \mathrm{Tr}(\tau_1^+(\vartheta)) M(S_1, \tau_1) = \sum_{\tau_1} M(S_1, \tau_1),$$

where τ_1 runs over all quasicharacters of $\vee G_{p_1}/(\vee G_{p_1})^0$ as in (94) which correspond to the strong involution δ_q ([ABV, Definition 18.9]) under (30). It is immediate from the definition of τ_1^+ following (94) that

$$\eta_{S_1}^{\mathrm{loc}}(\vartheta)(\delta_q) = \eta_{S_1}^{\mathrm{loc}}(\delta_q)$$

and so this virtual character is stable ([ABV, Lemma 18.10]). The proof of the following proposition is the same as the proof of Proposition 5.3 [AAM] once GL_N is replaced by $\mathbb{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N$.

Proposition 5.2 (Proposition 5.3 [AAM]). *Suppose $S_1 \subset X(\vee\mathcal{O}_G, \vee G^{\Gamma})$ is a $\vee G$ -orbit which is carried to a $\vee\mathbb{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N$ -orbit S under ϵ . Then*

$$\mathrm{Lift}_0(\eta_{S_1}^{\mathrm{loc}}(\vartheta)(\delta_q)) = (-1)^{l(S,1) - l_{\vartheta}^l(S,1)} M(S, 1)^+$$

Proposition 5.3. *The twisted endoscopic lifting map Lift_0 is injective.*

Proof. Suppose $S_1, S_2 \subset X(\vee\mathcal{O}_G, \vee G^{\Gamma})$ are $\vee G$ -orbits which are carried to the same $\vee\mathbb{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N$ -orbit under ϵ . Then, after identifying these orbits with L-parameters ([ABV, Proposition 6.17]), [GGP, Theorem 8.1] implies $S_1 = S_2$ (cf. [M3, Lemma 2.2.1]). It now follows from Proposition 5.2 that Lift_0 sends the basis

$$\{\eta_{S_G}^{\mathrm{loc}}(\delta_q) : S_G \text{ a } \vee G\text{-orbit of } X(\vee\mathcal{O}_G, \vee G^{\Gamma})\}$$

of $K_{\mathbb{C}}\Pi(\vee\mathcal{O}_G, G(\mathbb{R}, \delta_q))^{\mathrm{st}}$ bijectively onto the linearly independent subset

$$\left\{ (-1)^{l(\epsilon(S_G),1) - l_{\vartheta}^l(\epsilon(S_G),1)} M(\epsilon(S_G), 1)^+ : S_G \text{ a } \vee G\text{-orbit of } X(\vee\mathcal{O}_G, \vee G^{\Gamma}) \right\}$$

of $K_{\mathbb{C}}\Pi(\vee\mathcal{O}, \mathrm{GL}_N(\mathbb{C}), \vartheta)$. □

The next and final goal of this section is to provide the twisted analogue of the endoscopic lifting of the virtual characters attached to A-parameters as in (92). As a guiding principle, it helps to remember that in moving from η_S^{loc} to $\eta_S^{\mathrm{loc}}(\vartheta)(\delta_q)$ we extended the component groups by $\langle \vartheta \rangle$ to obtain (94), and then extended the quasicharacters τ_1 defined on the original component groups. We shall follow the same process with $\eta_{\psi_G}^{\mathrm{mic}}$, doing our best to avoid the theory of microlocal geometry.

The stable virtual character (52) for the endoscopic group G is

$$\eta_{\psi_G}^{\mathrm{mic}} = \sum_{\xi \in \Xi(\vee\mathcal{O}_G, \vee G^{\Gamma})} (-1)^{d(S_{\xi}) - d(S_{\psi_G})} \chi_{S_{\psi_G}}^{\mathrm{mic}}(P(\xi)) \pi(\xi) \in K\Pi(\vee\mathcal{O}_G, G/\mathbb{R})^{\mathrm{st}}.$$

Here, $S_{\psi_G} \subset X(\vee\mathcal{O}_G, G^{\Gamma})$ is the $\vee G$ -orbit determined by the L-parameter ϕ_{ψ_G} , and $\xi = (S_{\xi}, \tau_{S_{\xi}})$. We may rewrite $\eta_{\psi_G}^{\mathrm{mic}}$ using the following deep theorem in microlocal analysis. It is a summary of [ABV, Theorem 24.8, Corollary 24.9, Definition 24.15].

Theorem 5.4. For each $\xi \in \Xi(\vee \mathcal{O}_G, \vee G^\Gamma)$ there is a representation $\tau_{S_{\psi_G}}^{\text{mic}}(P(\xi))$ of $\vee G_{\psi_G}/(\vee G_{\psi_G})^0$, the component group of the centralizer in $\vee G$ of the image of ψ_G , which satisfies the following properties

- (a) $\tau_{S_{\psi_G}}^{\text{mic}}(P(\xi))$ represents a (possibly zero) $\vee G$ -equivariant local system $Q^{\text{mic}}(P(\xi))$ of complex vector spaces.
- (b) The degree of $\tau_{S_{\psi_G}}^{\text{mic}}(P(\xi))$ is equal to $\chi_{S_{\psi_G}}^{\text{mic}}(P(\xi))$.
- (c) If $\xi = (S_{\psi_G}, \tau_{S_{\psi_G}})$ then $\tau_{S_{\psi_G}}^{\text{mic}}(P(\xi)) = \tau_{S_{\psi_G}} \circ i_{S_{\psi_G}}$, where

$$i_{S_{\psi_G}} : \vee G_{\psi_G}/(\vee G_{\psi_G})^0 \rightarrow \vee G_p/(\vee G_p)^0$$

is a surjective homomorphism for $p \in S_{\psi_G}$.

By Theorem 5.4 (b), we may rewrite $\eta_{\psi_G}^{\text{mic}}$ as

$$\eta_{\psi_G}^{\text{mic}} = \sum_{\xi \in \Xi(\vee \mathcal{O}_G, \vee G^\Gamma)} (-1)^{d(S_\xi) - d(S_{\psi_G})} \text{Tr} \left(\tau_{S_{\psi_G}}^{\text{mic}}(P(\xi))(1) \right) \pi(\xi). \quad (97)$$

Next, we extend $\vee G_{\psi_G}/(\vee G_{\psi_G})^0$ trivially to

$$\vee G_{\psi_G}/(\vee G_{\psi_G})^0 \times \langle \vartheta \rangle, \quad (98)$$

and extend $\tau_{S_{\psi_G}}^{\text{mic}}(P(\xi))$ trivially to (98) by defining $\tau_{S_{\psi_G}}^{\text{mic}}(P(\xi))(\vartheta)$ to be the identity map. We define

$$\begin{aligned} \eta_{\psi_G}^{\text{mic}}(\vartheta) &= \sum_{\xi \in \Xi(\vee \mathcal{O}_G, \vee G^\Gamma)} (-1)^{d(S_\xi) - d(S_{\psi_G})} \text{Tr} \left(\tau_{S_{\psi_G}}^{\text{mic}}(P(\xi))(\vartheta) \right) \pi(\xi) \\ &= \sum_{\xi \in \Xi(\vee \mathcal{O}_G, \vee G^\Gamma)} (-1)^{d(S_\xi) - d(S_{\psi_G})} \dim \left(\tau_{S_{\psi_G}}^{\text{mic}}(P(\xi)) \right) \pi(\xi). \end{aligned}$$

Clearly

$$\eta_{\psi_G}^{\text{mic}}(\vartheta) = \eta_{\psi_G}^{\text{mic}}. \quad (99)$$

Finally, define

$$\begin{aligned} \eta_{\psi_G}^{\text{mic}}(\vartheta)(\delta_q) &= \sum_{(S_\xi, \tau_{S_\xi})} (-1)^{d(S_\xi) - d(S_{\psi_G})} \text{Tr} \left(\tau_{S_{\psi_G}}^{\text{mic}}(P(\xi))(\vartheta) \right) \pi(\xi) \\ &= \sum_{(S_\xi, \tau_{S_\xi})} (-1)^{d(S_\xi) - d(S_{\psi_G})} \dim \left(\tau_{S_{\psi_G}}^{\text{mic}}(P(\xi)) \right) \pi(\xi) \end{aligned}$$

in which the sum runs over only those $\xi = (S_\xi, \tau_{S_\xi}) \in \Xi(\vee \mathcal{O}_G, \vee G^\Gamma)$ in which τ_{S_ξ} corresponds to the strong involution δ_q under (30). Therefore, by (53)

$$\eta_{\psi_G}^{\text{mic}}(\vartheta)(\delta_q) = \eta_{S_{\psi_G}}^{\text{mic}}(\delta_q) = \eta_{S_{\psi_G}}^{\text{ABV}}.$$

The virtual character $\eta_{\psi_G}^{\text{mic}}(\vartheta)(\delta_q)$ is a summand of the stable virtual character $\eta_{\psi_G}^{\text{mic}}$ and is therefore also stable ([ABV, Theorem 18.7]). Consequently, $\eta_{\psi_G}^{\text{mic}}(\vartheta)(\delta_q)$ lies in the domain of Lift_0 . In addition, the ABV-packet $\Pi_{\psi_G}^{\text{ABV}}$ consists of the irreducible characters in the support of $\eta_{\psi_G}^{\text{mic}}(\vartheta)(\delta_q)$ (54).

What we have done for $\eta_{\psi_G}^{\text{mic}}$ we begin to do for $\eta_{\psi}^{\text{mic}+}$, which we define as

$$\eta_{\psi}^{\text{mic}+} = \sum_{\xi \in \Xi(\vee \mathcal{O}, \vee \mathbb{R}_{\mathbb{C}/\mathbb{R}} \text{GL}_N^\Gamma)^\vartheta} (-1)^{d(S_\xi) - d(S_\psi)} \text{Tr}(\chi_{S_\psi}^{\text{mic}}(P(\xi))) (-1)^{l^I(\xi) - l_\vartheta^I(\xi)} \pi(\xi)^+ \quad (100)$$

for

$$\psi = \epsilon \circ \psi_G.$$

The main difference now is that ϑ does not act trivially on ${}^\vee\mathrm{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N$ and so the extensions require more attention. The properties of Theorem 5.4 hold for ψ and ${}^\vee\mathrm{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N$ as they do for ψ_G and G .

The first step is writing

$$\eta_\psi^{\mathrm{mic}+} = \sum_{\xi \in \Xi({}^\vee\mathcal{O}, {}^\vee\mathrm{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N^\Gamma)^\vartheta} (-1)^{d(S_\xi) - d(S_\psi)} \mathrm{Tr}(\tau_{S_\psi}^{\mathrm{mic}}(P(\xi))(1)) (-1)^{l(\xi) - l_\vartheta(\xi)} \pi(\xi)^+.$$

This holds from Theorem 5.4 (b) as (97) did for the endoscopic group G . What is simpler here is that the component group $({}^\vee\mathrm{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N)_\psi / (({}^\vee\mathrm{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N)_\psi)^0$ is trivial ([A1, Section 2.3]). It follows that $\tau_{S_\psi}^{\mathrm{mic}}(P(\xi))$ is either trivial or zero.

Let us digress briefly to examine Theorem 5.4 (c) for $\xi = (S_\psi, \tau_{S_\psi})$. Since the component group $({}^\vee\mathrm{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N)_p / (({}^\vee\mathrm{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N)_p)^0$ is trivial, the representation τ_{S_ψ} is trivial. It follows that

$$\tau_{S_\psi}^{\mathrm{mic}}(P(S_\psi, \tau_{S_\psi})) = \tau_{S_\psi} \circ i_{S_\psi} = 1 \circ i_{S_\psi} = 1 \neq 0.$$

In particular, $\pi(S_\psi, 1)$ is in the support of η_ψ^{mic} and belongs to Π_ψ^{ABV} . In the next section we will prove that this is the only representation in Π_ψ^{ABV} .

Returning to the matter of extensions, there is an obvious extension

$$({}^\vee\mathrm{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N)_\psi / (({}^\vee\mathrm{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N)_\psi)^0 \times \langle \vartheta \rangle$$

of the trivial component group, as ϑ fixes the image of ψ . We wish to extend the representation $\tau_{S_\psi}^{\mathrm{mic}}(P(\xi))$ to this group for $\xi \in \Xi({}^\vee\mathcal{O}, {}^\vee\mathrm{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N^\Gamma)^\vartheta$. The action of ϑ on $P(\xi) \in \mathcal{P}({}^\vee\mathcal{O}, {}^\vee\mathrm{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N^\Gamma; \vartheta)$ determines an action on the stalks of the local system $Q^{\mathrm{mic}}(P(\xi))$ as in (5.4) ([ABV, (25.1)]). [ABV, Proposition 26.23 (b)] allows us to choose a stalk over a ϑ -fixed point p' (related to S_ψ) in the topological space of $Q^{\mathrm{mic}}(P(\xi))$. This places us in the same setting as Lemma 3.4, with τ_S replaced by $\tau_{S_\psi}^{\mathrm{mic}}(P(\xi))$ and S replaced by the ${}^\vee\mathrm{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N$ -orbit of p' . As in that lemma, ϑ determines a canonical isomorphism of the stalk at p' equal to 1. In short, we define

$$\tau_{S_\psi}^{\mathrm{mic}}(P(\xi)^+)(\vartheta) = 1 \tag{101}$$

and extend $\tau_{S_\psi}^{\mathrm{mic}}(P(\xi))$ to a representation $\tau_{S_\psi}^{\mathrm{mic}}(P(\xi)^+)$. The representation $\tau_{S_\psi}^{\mathrm{mic}}(P(\xi)^+)$ represents the $({}^\vee\mathrm{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N \times \langle \vartheta \rangle)$ -equivariant local system of the restriction of $Q^{\mathrm{mic}}(P(\xi))$ to the orbit of p' . We may extend i_{S_ψ} in Theorem 5.4 (c) to include the products with $\langle \vartheta \rangle$. Definitions (93) and (101) are compatible in that

$$\tau_{S_\psi}^{\mathrm{mic}}(P(S_\psi, 1))^+ = 1^+ \circ i_{S_\psi}.$$

Finally, we define

$$\eta_\psi^{\mathrm{mic}+}(\vartheta) = \sum_{\xi \in \Xi({}^\vee\mathcal{O}, {}^\vee\mathrm{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N^\Gamma)^\vartheta} (-1)^{d(S_\xi) - d(S_\psi)} \mathrm{Tr}(\tau_{S_\psi}^{\mathrm{mic}}(P(\xi)^+)(\vartheta)) (-1)^{l(\xi) - l_\vartheta(\xi)} \pi(\xi)^+. \tag{102}$$

It is clear from definition (100) that $\eta_\psi^{\mathrm{mic}+}(\vartheta) = \eta_\psi^{\mathrm{mic}+}$.

The obvious definition of the representation $\tau_{S_\psi}^{\mathrm{mic}}(P(\xi)^-)$ is to take

$$\tau_{S_\psi}^{\mathrm{mic}}(P(\xi)^-)(\vartheta) = -1.$$

With this definition in place the following proposition is a consequence of [ABV, Corollary 24.9].

Proposition 5.5. *The functor $\tau_{S_\psi}^{\mathrm{mic}}(\cdot)$, from $({}^\vee\mathrm{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N \times \langle \vartheta \rangle)$ -equivariant perverse sheaves to representations of $({}^\vee G_\psi / ({}^\vee G_\psi)^0) \times \langle \vartheta \rangle$, induces a map from the Grothendieck group $K(X({}^\vee\mathcal{O}, {}^\vee\mathrm{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N^\Gamma); \vartheta)$ to the space of virtual representations. Furthermore the microlocal trace map*

$$\mathrm{Tr} \left(\tau_{S_\psi}^{\mathrm{mic}}(\cdot)(\vartheta) \right)$$

induces a homomorphism from $K(X({}^\vee\mathcal{O}, {}^\vee\mathrm{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N^\Gamma), \vartheta)$ (as in (57)) to \mathbb{C} .

A similar statement is true for $\tau_{S_{\psi_G}}^{\text{mic}}$ and the $({}^\vee G \times \langle \vartheta \rangle)$ -equivariant sheaves defined earlier. The proof of the next theorem is identical to the proof of [AAM, Theorem 5.6].

Theorem 5.6. [AAM, Theorem 5.6]

(a) As a function on $K(X({}^\vee \mathcal{O}, {}^\vee \mathbf{R}_{\mathbb{C}/\mathbb{R}} \text{GL}_N^\Gamma), \vartheta)$ we have

$$\langle \eta_\psi^{\text{mic}+}(\vartheta), \cdot \rangle = (-1)^{d(S_\psi)} \text{Tr} \left(\tau_{S_\psi}^{\text{mic}}(\cdot)(\vartheta) \right).$$

(b) The stable virtual character $\eta_\psi^{\text{mic}+}(\vartheta)$ is equal to

$$(-1)^{d(S_\psi)} \sum_{\xi \in \Xi({}^\vee \mathcal{O}, {}^\vee \mathbf{R}_{\mathbb{C}/\mathbb{R}} \text{GL}_N^\Gamma)^\vartheta} \text{Tr} \left(\tau_{S_\psi}^{\text{mic}}(\mu(\xi)^+)(\vartheta) \right) (-1)^{l^t(\xi) - l_\vartheta^t(\xi)} M(\xi)^+.$$

(c) $\text{Lift}_0 \left(\eta_{\psi_G}^{\text{mic}}(\vartheta)(\delta_q) \right) = \eta_\psi^{\text{mic}+}(\vartheta)$.

6 ABV-packets for complex general linear groups

In this section we prove that any ABV-packet for $\mathbf{R}_{\mathbb{C}/\mathbb{R}} \text{GL}(\mathbb{R}) = \text{GL}_N(\mathbb{C})$ consists of a single (equivalence class of an) irreducible representation. This implies that such an ABV-packet is equal to its corresponding L-packet ([ABV, Theorem 22.7 (a)]). From the classification of the unitary dual of $\text{GL}_N(\mathbb{C})$ we shall deduce that the single representation in the packet is unitary.

In this section we let

$$\psi : W_{\mathbb{R}} \times \text{SL}_2 \rightarrow {}^\vee \mathbf{R}_{\mathbb{C}/\mathbb{R}} \text{GL}_N^\Gamma$$

be an arbitrary A-parameter for $\mathbf{R}_{\mathbb{C}/\mathbb{R}} \text{GL}_N$. The description of the ABV-packet Π_ψ^{ABV} will be achieved in three steps. We follow the same proof as in [AAM, Section 6]. First, we treat the case of an *irreducible* A-parameter. Second, we compute the ABV-packet for a Levi subgroup of $\mathbf{R}_{\mathbb{C}/\mathbb{R}} \text{GL}_N$, whose dual group contains the image of ψ minimally. The final result is obtained from the second step by considering the Levi subgroup as an endoscopic group of $\mathbf{R}_{\mathbb{C}/\mathbb{R}} \text{GL}_N$ and applying the endoscopic lifting (92).

According to [M3, Section 2.3], any A-parameter ψ for $\mathbf{R}_{\mathbb{C}/\mathbb{R}} \text{GL}_N$ may be decomposed as a formal direct sum of A-parameters

$$\psi = \boxplus_{i=1}^r \ell_i \psi_i, \tag{103}$$

with $\ell_i \in \mathbb{N}$, ψ_i being an A-parameter of $\mathbf{R}_{\mathbb{C}/\mathbb{R}} \text{GL}_{N_i}$, and $N = \sum_{i=1}^r \ell_i N_i$. We may identify *real* L-parameters of $\mathbf{R}_{\mathbb{C}/\mathbb{R}} \text{GL}_{N_i}$ with their corresponding *complex* L-parameters of GL_{N_i} ([B, Section I.5]). This correspondence extends in an evident way to an analogous identification between A-parameters. The complex A-parameter of GL_{N_i} corresponding to each ψ_i is given by

$$\mu_i \boxtimes \nu_{N_i},$$

where ν_{N_i} is the unique irreducible representation of SL_2 of dimension N_i , and μ_i is an irreducible representation \mathbb{C}^\times .

The parameter ψ in (103) is said to be *irreducible* if $r = 1$ and $\ell_1 = 1$. For any irreducible A-parameter ψ of $\mathbf{R}_{\mathbb{C}/\mathbb{R}} \text{GL}_N$ the corresponding representation ν_N of SL_2 is irreducible and of dimension N . As a consequence the image of any unipotent subgroup of SL_2 under ν_N is *principally unipotent* (i.e. regular and unipotent) in ${}^\vee \text{GL}_N$. Equivalently, the image of any unipotent subgroup of SL_2 under the real A-parameter ψ is principally unipotent in ${}^\vee \text{GL}_N \times {}^\vee \text{GL}_N$. [AR, Theorem 4.11 (d)] therefore implies the following result (cf. [ABV, Theorem 27.18]).

Proposition 6.1. *Suppose ψ is an irreducible A-parameter of $\mathbf{R}_{\mathbb{C}/\mathbb{R}} \text{GL}_N$. Then Π_ψ^{ABV} consists of a single unitary character.*

Let us proceed to the case of a general A-parameter ψ as in Equation (103). Define

$${}^\vee G = \prod_{i=1}^r ({}^\vee G_i)^{\ell_i} \cong \prod_{i=1}^r ({}^\vee \mathrm{GL}_{N_i} \times {}^\vee \mathrm{GL}_{N_i})^{\ell_i} \quad (104)$$

to be the obvious Levi subgroup of ${}^\vee \mathrm{GL}_N \times {}^\vee \mathrm{GL}_N$ containing the image of ψ minimally. Set ${}^\vee G^\Gamma = {}^\vee G \rtimes \langle {}^\vee \delta_0 \rangle$, a subgroup of ${}^\vee \mathrm{R}_{\mathbb{C}/\mathbb{R}} \mathrm{GL}_N^\Gamma$. It is immediate that ψ factors through an A-parameter

$$\psi : W_{\mathbb{R}} \times \mathrm{SL}_2 \xrightarrow{\psi_G} {}^\vee G^\Gamma \hookrightarrow {}^\vee \mathrm{R}_{\mathbb{C}/\mathbb{R}} \mathrm{GL}_N^\Gamma,$$

where $\psi_G = \times_{i=1}^r \ell_i \psi_{G_i}$ and each ψ_{G_i} is an irreducible A-parameter of ${}^\vee G_i^\Gamma = {}^\vee \mathrm{R}_{\mathbb{C}/\mathbb{R}} \mathrm{GL}_{N_i}^\Gamma$. The description of the ABV-packet corresponding to ψ_G is a straightforward consequence of Proposition 6.1. We must only remind ourselves that the direct product of (104) translates into a tensor product of ABV-packets as it passes through the process defining the packets in Section 3.1. The proof follows exactly as for Corollary 6.2 [AAM].

Corollary 6.2. *The ABV-packet $\Pi_{\psi_G}^{\mathrm{ABV}}$ consists of a single irreducible unitary representation $\pi(S_{\psi_G}, 1)$.*

Finally, take ${}^\vee G$ as in (104), and take $s \in Z({}^\vee G) \subset {}^\vee \mathrm{GL}_N \times {}^\vee \mathrm{GL}_N$ to be as regular as possible so that its centralizer in ${}^\vee \mathrm{GL}_N \times {}^\vee \mathrm{GL}_N$ is equal to ${}^\vee G$. Then $(s, {}^\vee G^\Gamma)$ is an endoscopic datum for ${}^\vee \mathrm{R}_{\mathbb{C}/\mathbb{R}} \mathrm{GL}_N^\Gamma$ (Section 5.1). According to (92), Corollary 6.2, and Proposition 5.1 we have

$$\eta_{\psi}^{\mathrm{mic}} = \mathrm{Lift}_0(\eta_{\psi_G}^{\mathrm{mic}}) = \mathrm{Lift}_0(\pi(S_{\psi_G}, 1)) = \mathrm{ind}_{G(\mathbb{R}, \delta_q)}^{\mathrm{GL}_N(\mathbb{C})} \pi(S_{\psi_G}, 1).$$

Consequently, $\Pi_{\psi}^{\mathrm{ABV}}$ consists of a single representation, and this representation is parabolically induced from the single unitary representation of $\Pi_{\psi_G}^{\mathrm{ABV}}$. Since parabolic induction for general linear groups takes irreducible unitary representations to irreducible unitary representations ([T, Proposition 2.1]), we have just proved

Proposition 6.3. *Let ψ be an A-parameter for $\mathrm{R}_{\mathbb{C}/\mathbb{R}} \mathrm{GL}_N$ as in (103). Then the ABV-packet $\Pi_{\psi}^{\mathrm{ABV}}$ consists of a single irreducible unitary representation $\pi(S_{\psi}, 1)$.*

As a corollary, we have the next result. Its proof is the same as that of [AAM, Corollary 6.4].

Corollary 6.4. *The stable virtual character $\eta_{\psi}^{\mathrm{mic}+}(\vartheta)$ defined in (102) is equal to $(-1)^{l^1(\xi) - l^1_{\mathfrak{b}}(\xi)} \pi(\xi)^+$, where $\xi = (S_{\psi}, 1)$. In particular,*

$$\mathrm{Lift}_0(\eta_{\psi_G}^{\mathrm{mic}}(\vartheta)(\delta_q)) = (-1)^{l^1(\xi) - l^1_{\mathfrak{b}}(\xi)} \pi(\xi)^+.$$

7 Whittaker extensions and their relationship to Atlas extensions

Recall that (45) defines a preferred extension to $\mathrm{GL}_N(\mathbb{C}) \rtimes \langle \vartheta \rangle$ of any irreducible representation of $\mathrm{GL}_N(\mathbb{C})$. These are the extensions we have called ‘‘Atlas extensions’’. Since $\mathrm{GL}_N(\mathbb{C})$ is quasisplit as a real group, there is another choice of preferred extension, which depends on a Whittaker datum. This extension is introduced by Arthur ([A3, Section 2.2]) and followed by Mok ([M3, Section 3.2]). We call this alternative extension the *Whittaker extension*.

Here is a summary of the definition of Whittaker extensions. We fix a unitary character χ on the upper-triangular unipotent subgroup

$$U(\mathbb{R}) \subset \mathrm{R}_{\mathbb{C}/\mathbb{R}} \mathrm{GL}_N(\mathbb{R}) = \mathrm{GL}_N(\mathbb{C})$$

which satisfies $\chi \circ \vartheta = \chi$. In this manner (U, χ) is a ϑ -fixed Whittaker datum. We work under the hypothesis of (38) on an infinitesimal character $\lambda \in {}^\vee \mathfrak{h}$ and set ${}^\vee \mathcal{O}$ to be its ${}^\vee \mathrm{R}_{\mathbb{C}/\mathbb{R}} \mathrm{GL}_N$ -orbit. Let $\xi \in \Xi({}^\vee \mathcal{O}, {}^\vee \mathrm{R}_{\mathbb{C}/\mathbb{R}} \mathrm{GL}_N^\Gamma)^\vartheta$ so that $\pi(\xi)$ is (an infinitesimal equivalence class of) an irreducible

representation of $\mathrm{GL}_N(\mathbb{C})$. In defining Whittaker extensions we must work with an actual admissible group representation in this equivalence class which we also denote by $(\pi(\xi), V)$. If $\pi(\xi)$ is tempered then up to a scalar there is a unique Whittaker functional $\omega : V \rightarrow \mathbb{C}$ satisfying

$$\omega(\pi(\xi)(u)v) = \chi(u)\omega(v), \quad u \in U(\mathbb{R}), \quad (105)$$

for all smooth vectors $v \in V$. It follows that there is a unique operator \mathcal{I}^\sim which intertwines $\pi(\xi) \circ \vartheta$ with $\pi(\xi)$ and also satisfies $\omega \circ \mathcal{I}^\sim = \omega$. We extend $\pi(\xi)$ to a representation $\pi(\xi)^\sim$ of $\mathrm{GL}_N(\mathbb{C}) \rtimes \langle \vartheta \rangle$ by setting $\pi(\xi)^\sim(\vartheta) = \mathcal{I}^\sim$. We call this extension $\pi(\xi)^\sim$ the *Whittaker extension* of $\pi(\xi)$.

If $\pi(\xi)$ is not tempered then we express it as the Langlands quotient of a representation $M(\xi)$ induced from an essentially tempered representation of a Levi subgroup. The ϑ -stability of $\pi(\xi)$ and the uniqueness statement in the Langlands classification together imply the ϑ -stability of the essentially tempered representation. The earlier argument for tempered representations has an obvious analogue for the essentially tempered representation of the Levi subgroup. We may argue as above to extend the essentially tempered representation to the semi-direct product of the Levi subgroup with $\langle \vartheta \rangle$. One then induces this extended representation to $\mathrm{GL}_N(\mathbb{C}) \rtimes \langle \vartheta \rangle$. The unique irreducible quotient of this representation is the canonical extension of $\pi(\xi)$, namely the *Whittaker extension* $\pi(\xi)^\sim$ of $\pi(\xi)$. If one omits the Langlands quotient in this argument then we obtain, by definition, the Whittaker extension $M(\xi)^\sim$ of the standard representation $M(\xi)$. A Whittaker functional for the tempered representation of the Levi subgroup induces a Whittaker functional for $M(\xi)$ ([S2, Proposition 3.2]). It is a simple exercise to prove that $M(\xi)^\sim(\vartheta)$ is the unique intertwining operator that fixes an induced Whittaker functional as in (105).

How does the Whittaker extension $\pi(\xi)^\sim$ differ from the Atlas extension $\pi(\xi)^+$ of (45)? The operators $\pi(\xi)^\sim(\vartheta)$ and $\pi(\xi)^+(\vartheta)$ are involutive, and both intertwine $\pi(\xi) \circ \vartheta$ with $\pi(\xi)$. Therefore they are equal up to a sign, *i.e.*

$$\pi(\xi)^\sim(\vartheta) = \pm \pi(\xi)^+(\vartheta). \quad (106)$$

A direct comparison of the two extensions is problematic in that the Whittaker extension is essentially analytic in nature, and the Atlas extension is essentially algebraic.

Happily, there is a special type of parameter $\xi \in \Xi(\vee \mathcal{O}, \vee \mathrm{R}_{\mathbb{C}/\mathbb{R}} \mathrm{GL}_N^\Gamma)^\vartheta$ for which the construction of the two extensions is directly comparable. To describe this type, we must examine the construction of $\pi(\xi)$ in (45) in more detail. In this construction we identify ξ with its equivalent Atlas parameter (x, y) (Lemma 2.2), or its equivalent preferred extended parameter (44). Recall that θ_x is an automorphism of H (58). To the parameter (x, y) one associates the irreducible $(\mathfrak{h}, H^{\theta_x} \rtimes \langle \vartheta \rangle)$ -module π_0^+ of differential λ such that $\pi_0^+(\vartheta) = 1$ ([AV, (20b), Lemma 5.1, Definition 5.6]). Then to the module π_0^+ one applies the functor [AV, (20e)] to obtain $\pi(\xi)^+ = J(x, y, \lambda)^+$ (*cf.* [KV, (11.54b), (11.116b)]). The functor depends on θ_x . In the particular circumstance that θ_x sends all positive roots (determined by the Borel subgroup B in the pinning of $\mathrm{R}_{\mathbb{C}/\mathbb{R}} \mathrm{GL}_N$) to negative roots, the Borel subalgebra \mathfrak{b} is *real* relative to θ_x and B is a real parabolic subgroup ([AvLTV, Proposition 13.12 (2)]). Moreover, in this circumstance, the functor is equivalent to the (normalized) parabolic induction functor $\mathrm{ind}_{B(\mathbb{R}) \rtimes \langle \vartheta \rangle}^{\mathrm{GL}_N(\mathbb{C}) \rtimes \langle \vartheta \rangle}$ ([KV, Proposition 11.47]), and so

$$M(\xi)^+ = \mathrm{ind}_{B(\mathbb{R}) \rtimes \langle \vartheta \rangle}^{\mathrm{GL}_N(\mathbb{C}) \rtimes \langle \vartheta \rangle} \pi_0^+. \quad (107)$$

The following lemma shows that the assumption of the ϑ -stability of the parameter ξ is not necessary when θ_x sends all positive roots to negative roots, and asserts the equality of the Atlas and Whittaker extensions.

Lemma 7.1. *Suppose $\xi = (x, y) \in \Xi(\vee \mathcal{O}, \vee \mathrm{R}_{\mathbb{C}/\mathbb{R}} \mathrm{GL}_N^\Gamma)$ and θ_x sends all positive roots to negative roots. Then*

- (a) *$M(\xi)$ and $\pi(\xi)$ are ϑ -stable so that $\xi = (x, y) \in \Xi(\vee \mathcal{O}, \vee \mathrm{R}_{\mathbb{C}/\mathbb{R}} \mathrm{GL}_N^\Gamma)^\vartheta$.*
- (b) *The Whittaker and Atlas extensions of $M(\xi)$ and $\pi(\xi)$ are equal, *i.e.* $M(\xi)^\sim = M(\xi)^+$ and $\pi(\xi)^\sim = \pi(\xi)^+$.*

Proof. The assertions for $\pi(\xi)$ follow by definition from the assertions for $M(\xi)$. For $M(\xi)$ we imitate the process described before the lemma and thereby obtain an irreducible $(\mathfrak{h}, H^{\theta_x})$ -module π_0 of differential λ . In addition,

$$M(\xi) = \text{ind}_{B(\mathbb{R})}^{\text{GL}_N(\mathbb{C})} \pi_0.$$

As a representation of the connected Lie group $H(\mathbb{R}) \cong (\mathbb{C}^\times)^N$, π_0 is completely determined by its differential λ . Obviously, $\pi_0 \circ \vartheta$ has differential $\vartheta(\lambda) = \lambda$ (38). Consequently $\pi_0 \circ \vartheta = \pi_0$ and one may verify that

$$M(\xi) \circ \vartheta \cong \text{ind}_{\vartheta(B(\mathbb{R}))}^{\text{GL}_N(\mathbb{C})} (\pi_0 \circ \vartheta) = \text{ind}_{B(\mathbb{R})}^{\text{GL}_N(\mathbb{C})} \pi_0 = M(\xi).$$

This proves (a).

For (b) we may return to (107) and note that, vacuously, $\pi_0^+(\vartheta) = 1$ determines the Whittaker extension of π_0 , *i.e.* $\pi_0^+ = \pi_0^\sim$. Therefore, by definition, $M(\xi)^\sim$ is equal to (107). (The reader may wonder in what sense (107) is “standard”. An explanation may be found in [KV, Theorem 11.129 (a)]). \square

Lemma 7.1 provides a simple solution for the comparison of Atlas and Whittaker extensions of some special representations. As we shall soon see, an arbitrary irreducible representation appears as a subquotient of one of these special standard representations. Remarkably, the twisted multiplicity (64) with which this subquotient appears may be used to determine the sign in (106).

We begin this line of reasoning by recalling how irreducible generic representations appear in the characters of standard representations. Recall that a representation is *generic* if it admits a non-zero Whittaker functional as in (105).

Lemma 7.2. *Suppose $\xi \in \Xi(\vee \mathcal{O}, \vee \mathbb{R}_{\mathbb{C}/\mathbb{R}} \text{GL}_N^\Gamma)^\vartheta$. Then*

- (a) *(up to infinitesimal equivalence) there is a unique irreducible generic representation $\pi(\xi_0) = M(\xi_0)$ which occurs in $M(\xi)$ as a subquotient. Moreover $\pi(\xi_0)$ is ϑ -stable and occurs as a subquotient with multiplicity one;*
- (b) *(any representative in the class of) $\pi(\xi_0)$ embeds as a subrepresentation of (any representative in the class of) $M(\xi)$;*
- (c) *(any representative in the class of) $\pi(\xi_0)^\sim$ embeds as a subrepresentation of (any representative in the class of) $M(\xi)^\sim$.*

Proof. A result due to Vogan and Kostant states that every standard representation $M(\xi)$ contains a unique generic irreducible subquotient occurring with multiplicity one ([K2, Theorems E and L], [V2, Corollary 6.7]). In the rest of the proof we write $\pi(\xi_0)$ for the actual generic representation (not the equivalence class) for some $\xi_0 \in \Xi(\vee \mathcal{O}, \vee \mathbb{R}_{\mathbb{C}/\mathbb{R}} \text{GL}_N^\Gamma)$. It is straightforward to verify that $\pi(\xi_0) \circ \vartheta$ satisfies (105), just as $\pi(\xi_0)$ does. Therefore $\pi(\xi_0) \circ \vartheta$ is the unique irreducible generic subquotient of $M(\xi) \circ \vartheta \cong M(\xi)$. By uniqueness, $\pi(\xi_0) \circ \vartheta \cong \pi(\xi_0)$ and so $\xi_0 \in \Xi(\vee \mathcal{O}, \vee \mathbb{R}_{\mathbb{C}/\mathbb{R}} \text{GL}_N^\Gamma)^\vartheta$. The equality $\pi(\xi_0) = M(\xi_0)$ follows from [V2, Theorem 6.2 (f)].

For part (b), assume that $M(\xi)$ is an actual representation (not an equivalence class) with infinitesimal character $\lambda \in \mathfrak{h}$ satisfying the assumption of (38). As a standard representation, we may write it as

$$M(\xi) = \text{ind}_{P(\mathbb{R})}^{\text{GL}_N(\mathbb{C})} (\pi_M \otimes e^\nu).$$

Here, P is a standard real parabolic subgroup with Levi subgroup M , M has Langlands decomposition $M(\mathbb{R}) = M^1(\mathbb{R})A(\mathbb{R})$, π_M is an irreducible tempered representation of $M^1(\mathbb{R})$, and $\nu \in \mathfrak{a}^*$ has dominant real part. The Levi subgroup $M(\mathbb{R})$ is a product of smaller complex general linear groups ([B, Section 5.2]). Since $M(\mathbb{R})$ is a direct product of complex general linear groups, [T, Proposition 2.1] allows us to write the tempered representation as a parabolically induced representation from a discrete series representation on the unique standard cuspidal Levi subgroup $H(\mathbb{R})$. By induction in stages we may write

$$M(\xi) = \text{ind}_{B(\mathbb{R})}^{\text{GL}_N(\mathbb{C})} (\pi_0 \otimes e^\nu)$$

where we now regard ν as an element in the Lie algebra of the split component of $H(\mathbb{R})$. The differential of $\pi_0 \otimes e^\nu$ is λ . Since λ is integrally dominant and $\text{Re } \nu$ is dominant, $M(\xi)$ satisfies

the properties of [V2, Theorem 6.2 (e) (i)-(ii)]. From [V2, Theorem 6.2 (e)] we know that any irreducible subrepresentation of $M(\xi)$ is generic. By part (a), we conclude that $M(\xi)$ has a unique irreducible subrepresentation and this subrepresentation is equivalent to $\pi(\xi_0)$.

For part (c) we consider the standard representation $M(\xi)$, which has a Whittaker functional ω induced from π_M above ([S1, Proposition 3.2]). The functional ω restricts to a non-zero Whittaker functional on $\pi(\xi_0)$. It is simple to verify that $M(\xi)^\sim(\vartheta)$ is the intertwining operator which satisfies $\omega \circ M(\xi)^\sim(\vartheta) = \omega$. Restricting this equation to the subrepresentation $\pi(\xi_0)$ yields in turn that

$$\pi(\xi_0)^\sim(\vartheta) = M(\xi)^\sim(\vartheta)|_{\pi(\xi_0)} \quad \text{and} \quad \pi(\xi_0)^\sim \hookrightarrow M(\xi)^\sim. \quad (108)$$

□

Lemma 7.2 tells us that the multiplicity of $\pi(\xi_0)^\sim$ in $M(\xi)^\sim$ is one. On the other hand the twisted multiplicity $m_r^\vartheta(\xi_0, \xi)$ of (64) tells us about the “signed multiplicity” of $\pi(\xi_0)^\sim$ in $M(\xi)^\sim$. We investigate $m_r^\vartheta(\xi_0, \xi)$ further before comparing the two kinds of multiplicities.

Proposition 7.3. *Suppose $\xi \in \Xi(\vee \mathcal{O}, \vee \mathbb{R}_{\mathbb{C}/\mathbb{R}} \text{GL}_N^\Gamma)^\vartheta$ and $\pi(\xi_0)$ is the generic subrepresentation of $M(\xi)$ (Lemma 7.2). Then*

$$m_r^\vartheta(\xi_0, \xi) = (-1)^{l^I(\xi) - l_\vartheta^I(\xi) + l^I(\xi_0) - l_\vartheta^I(\xi_0)}. \quad (109)$$

Proof. It follows from the definition of the KLV-polynomials ([LV, Section 0.1]), (66) and [AMR2, Proposition B.1] that

$$\vee P^\vartheta(\vee \xi', \vee \xi)(1) = (-1)^{d(\xi) - d(\xi')} c_g^\vartheta(\xi', \xi) = (-1)^{l^I(\xi) - l^I(\xi')} c_g^\vartheta(\xi', \xi).$$

(Note that the definition of $P(\xi)$ in [LV] differs from ours by a shift in degree $d(\xi)$ cf. [ABV, (7.10)(d)].) According to Proposition 3.6 the above equation may be written as

$$\vee P^\vartheta(\vee \xi', \vee \xi)(1) = (-1)^{l^I(\xi) - l_\vartheta^I(\xi) + l^I(\xi_0) - l_\vartheta^I(\xi_0)} m_r^\vartheta(\xi_0, \xi)$$

Therefore the proposition is equivalent to proving

$$\vee P^\vartheta(\vee \xi, \vee \xi_0)(1) = 1.$$

This equation follows from [AAM, Proposition 4.17] once we establish that $\vee \xi_0$ satisfies the stated hypotheses. Indeed, the proof of [AAM, Proposition 4.17] depends on the group $\mathbb{R}_{\mathbb{C}/\mathbb{R}} \text{GL}_N$ only in terms of the types of roots (75). The proof relies crucially on the absence of some nuisance types, which fortunately do not appear in (75) either.

The hypotheses of [AAM, Proposition 4.17] refer to the *block* of ξ ([V4, Definition 1.14]). The hypotheses of [AAM, Proposition 4.17] are satisfied if we can prove that $\vee \xi_0$ is the unique maximal parameter in the block of $\pi(\vee \xi)$ with respect to the (dual) Bruhat order. This is equivalent to establishing that ξ_0 is the unique minimal parameter in the block of $\pi(\xi)$ ([V4, Theorem 1.15]). We use [ABV, Proposition 1.11] to convert the Bruhat order for the representations of $\text{GL}_N(\mathbb{C})$ into a closure relation between $\vee \mathbb{R}_{\mathbb{C}/\mathbb{R}} \text{GL}_N$ -orbits of $X(\vee \mathcal{O}, \vee \mathbb{R}_{\mathbb{C}/\mathbb{R}} \text{GL}_N^\Gamma)$. Moreover, this proposition implies that the minimality of $\xi_0 = (S_0, 1)$ is equivalent to the $\vee \mathbb{R}_{\mathbb{C}/\mathbb{R}} \text{GL}_N$ -orbit $S_0 \subset X(\vee \mathcal{O}, \vee \mathbb{R}_{\mathbb{C}/\mathbb{R}} \text{GL}_N^\Gamma)$ being maximal. The uniqueness and maximality of this orbit follows from [ABV, p. 19]. □

Equation (108) gives us information about the multiplicity of the Whittaker extension of a generic representation, and Equation (109) gives us information about the multiplicity of the Atlas extension of a generic representation. In the next theorem we combine this information with Lemma 7.1 to determine the sign between the two extensions of a generic representation. This may then be leveraged to determine the sign between the two extensions of an arbitrary irreducible representation.

Theorem 7.4. *Suppose $\xi \in \Xi(\vee \mathcal{O}, \vee \mathbb{R}_{\mathbb{C}/\mathbb{R}} \text{GL}_N^\Gamma)^\vartheta$. Then*

$$M(\xi)^\sim(\vartheta) = (-1)^{l^I(\xi) - l_\vartheta^I(\xi)} M(\xi)^\sim(\vartheta)$$

and

$$\pi(\xi)^\sim(\vartheta) = (-1)^{l^I(\xi) - l_\vartheta^I(\xi)} \pi(\xi)^\sim(\vartheta).$$

Proof. We first prove that there exists $\xi_p = (x_p, y_p) \in \Xi(\vee \mathcal{O}, \vee \mathbb{R}_{\mathbb{C}/\mathbb{R}} \mathrm{GL}_N^\Gamma)^\vartheta$ satisfying the hypothesis of Lemma 7.1 such that $\pi(\xi_p)$ lies in the block of $\pi(\xi)$. Let (x, y) be the Atlas parameter corresponding to ξ (Lemma 2.2) and θ_x be as in (58). A short exercise using (39) and (40) shows that

$$\theta_x = (w, w_0 w^{-1} w_0^{-1}) \delta_0$$

for some Weyl group element w for $\mathbb{R}_{\mathbb{C}/\mathbb{R}} \mathrm{GL}_N$, the long Weyl group element w_0 for $\mathbb{R}_{\mathbb{C}/\mathbb{R}} \mathrm{GL}_N$, and δ_0 as in (33). According to [V4, Theorem 8.8] there is a transitive action on the block of $\pi(\xi)$ given by cross actions (there are no Cayley transforms for $\mathbb{R}_{\mathbb{C}/\mathbb{R}} \mathrm{GL}_N$). Cross actions are recorded in the Atlas parameter (x, y) by conjugating both entries with an appropriate Weyl group element ([AdC, (9.11)(f)]). By taking the cross action of (x, y) with respect to $(w_0 w^{-1}, 1) \in W(\mathbb{R}_{\mathbb{C}/\mathbb{R}} \mathrm{GL}_N, H)$ we arrive at an Atlas parameter (x_p, y_p) such that

$$\theta_{x_p} = (w_0, w_0) \delta_0.$$

Evidently, θ_{x_p} sends all positive roots to negative roots. Setting $\xi_p = (x_p, y_p)$, we see that $\pi(\xi_p)$ lies in the block of $\pi(\xi)$ and satisfies the hypothesis of Lemma 7.1. Consequently, $M(\xi_p)^\sim = M(\xi_p)^\dagger$. It is straightforward to show $l^I(\xi_p) = l_\vartheta^I(\xi_p) = 0$. By (65) and (109)

$$\begin{aligned} M(\xi_p)^\sim &= m_r^\vartheta(\xi_0, \xi_p) \pi(\xi_0)^\dagger + \sum_{\xi' \neq \xi_0} m_r^\vartheta(\xi', \xi_p) \pi(\xi')^\dagger \\ &= (-1)^{l^I(\xi_0) - l_\vartheta^I(\xi_0)} \pi(\xi_0)^\dagger + \sum_{\xi' \neq \xi_0} m_r^\vartheta(\xi', \xi_p) \pi(\xi')^\dagger. \end{aligned}$$

According to (108), with $\xi = \xi_p$, this equation implies

$$(-1)^{l^I(\xi_0) - l_\vartheta^I(\xi_0)} \pi(\xi_0)^\dagger(\vartheta) = \pi(\xi_0)^\sim(\vartheta). \quad (110)$$

Thus, the theorem holds for $\xi = \xi_0$.

It remains to prove that the theorem holds when $\xi \neq \xi_0$. We compute, using (109) and (110), that

$$\begin{aligned} M(\xi)^\dagger &= \sum_{\xi' \neq \xi_0} m_r^\vartheta(\xi', \xi) \pi(\xi')^\dagger + m_r^\vartheta(\xi_0, \xi) \pi(\xi_0)^\dagger \\ &= \sum_{\xi' \neq \xi_0} m_r^\vartheta(\xi', \xi) \pi(\xi')^\dagger + (-1)^{l^I(\xi) - l_\vartheta^I(\xi) + l^I(\xi_0) - l_\vartheta^I(\xi_0)} \pi(\xi_0)^\dagger \\ &= \sum_{\xi' \neq \xi_0} m_r^\vartheta(\xi', \xi) \pi(\xi')^\dagger + (-1)^{l^I(\xi) - l_\vartheta^I(\xi)} \pi(\xi_0)^\sim. \end{aligned}$$

This equation and Lemma 7.2 imply

$$(-1)^{l^I(\xi) - l_\vartheta^I(\xi)} \pi(\xi_0)^\sim(\vartheta) = M(\xi)^\dagger(\vartheta) |_{\pi(\xi_0)}.$$

Combining this equation with (108), we see in turn that

$$(-1)^{l^I(\xi) - l_\vartheta^I(\xi)} M(\xi)^\dagger(\vartheta) |_{\pi(\xi_0)} = \pi(\xi_0)^\sim(\vartheta) = M(\xi)^\sim(\vartheta) |_{\pi(\xi_0)}$$

and $(-1)^{l^I(\xi) - l_\vartheta^I(\xi)} M(\xi)^\dagger(\vartheta) = M(\xi)^\sim(\vartheta)$. By taking Langlands quotients we obtain $(-1)^{l^I(\xi) - l_\vartheta^I(\xi)} \pi(\xi)^\dagger(\vartheta) = \pi(\xi)^\sim(\vartheta)$. \square

Theorem 7.4 allows us to replace the Atlas extensions with signed Whittaker extensions in the twisted pairing (62) and the endoscopic lifting (96). The results are recorded in the following two corollaries. The proofs are obtained from the analogous corollaries indicated in [AAM].

Corollary 7.5. [AAM, Corollary 7.9] *The pairing (62), defined by (63), satisfies*

$$\langle M(\xi)^\sim, \mu(\xi')^\dagger \rangle = \delta_{\xi, \xi'}$$

and

$$\langle \pi(\xi)^\sim, P(\xi')^+ \rangle = (-1)^{d(\xi)} \delta_{\xi, \xi'}$$

for $\xi, \xi' \in \Xi(\mathcal{O}, {}^\vee\mathrm{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N^\Gamma)^\vartheta$. Equivalently (cf. Proposition 3.6),

$$m_r^\sim(\xi', \xi) = (-1)^{d(\xi) - d(\xi')} c_g^\vartheta(\xi, \xi'), \quad (111)$$

where $m_r^\sim(\xi', \xi)$ is defined by the decomposition

$$M(\xi)^\sim = \sum_{\xi' \in \Xi(\mathcal{O}, {}^\vee\mathrm{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N^\Gamma)^\vartheta} m_r^\sim(\xi', \xi) \pi(\xi')^\sim.$$

in $K\Pi(\mathcal{O}, \mathrm{GL}_N(\mathbb{C}), \vartheta)$.

Corollary 7.6. [AAM, Corollary 7.10] *Suppose that $S_G \subset X(\mathcal{O}_G, {}^\vee G^\Gamma)$ is a ${}^\vee G$ -orbit and let $\epsilon(S_G) \subset X(\mathcal{O}, {}^\vee\mathrm{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N^\Gamma)$ be the ${}^\vee\mathrm{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N$ -orbit of the image of S_G under ϵ (88). Then*

(a)

$$\mathrm{Lift}_0(\eta_{S_G}^{\mathrm{loc}}(\sigma)(\delta_q)) = M(\epsilon(S_G), 1)^\sim,$$

(b)

$$\mathrm{Lift}_0 = \mathrm{Trans}_{G(\mathbb{R})}^{\mathrm{GL}_N(\mathbb{C}) \rtimes \vartheta}$$

on $K_{\mathbb{C}}\Pi(\mathcal{O}_G, G(\mathbb{R}, \delta_q))^{\mathrm{st}}$.

8 The equality of $\Pi_{\psi_G}^{\mathrm{Mok}}$ and $\Pi_{\psi_G}^{\mathrm{ABV}}$ for regular infinitesimal character

In this section we prove the equality of the stable virtual characters $\eta_{\psi_G}^{\mathrm{Mok}}$ and $\eta_{\psi_G}^{\mathrm{ABV}}$ under a regularity condition on the infinitesimal character. We shall work using the framework of Section 5.2. In particular, ψ_G and $\psi = \epsilon \circ \psi_G$ are A-parameters with respective infinitesimal characters ${}^\vee\mathcal{O}_G$ and ${}^\vee\mathcal{O}$. The assumption on the infinitesimal characters is that they are regular with respect to $\mathrm{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N$. This assumption shall be removed in the next section.

The definition of $\eta_{\psi_G}^{\mathrm{Mok}}$ was outlined in the introduction. Let us follow [M3] and provide a few more details. All we need is contained in the following lemma, which is a version of Lemma 8.1 [AAM] for unitary groups.

Lemma 8.1. *Let $S_\psi \subset X({}^\vee\mathcal{O}, {}^\vee\mathrm{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N^\Gamma)$ be the ${}^\vee\mathrm{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N$ -orbit corresponding to ϕ_ψ ([ABV, Proposition 6.17]).*

(a) *There exist integers n_S such that*

$$\pi(S_\psi, 1)^\sim = \sum_{(S, 1) \in \Xi({}^\vee\mathcal{O}, {}^\vee\mathrm{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N^\Gamma)^\vartheta} n_S M(S, 1)^\sim \quad (112)$$

in $K\Pi({}^\vee\mathcal{O}, \mathrm{GL}_N(\mathbb{C}), \vartheta)$.

(b) *For every S such that $n_S \neq 0$ in (112) there exists a unique ${}^\vee G$ -orbit $S_G \subset X({}^\vee\mathcal{O}_G, {}^\vee G^\Gamma)$ which is carried to S under ϵ .*

(c) *Writing*

$$S = \epsilon(S_G)$$

for the orbits in part (b), we have

$$\begin{aligned} \pi(S_\psi, 1)^\sim &= \mathrm{Trans}_{G(\mathbb{R})}^{\mathrm{GL}_N(\mathbb{C}) \rtimes \vartheta} \left(\sum_{S_G} n_{\epsilon(S_G)} \eta_{S_G}^{\mathrm{loc}}(\delta_q) \right) \\ &= \mathrm{Lift}_0 \left(\sum_{S_G} n_{\epsilon(S_G)} \eta_{S_G}^{\mathrm{loc}}(\delta_q) \right). \end{aligned} \quad (113)$$

Proof. The proof follows almost exactly as for [AAM, Lemma 8.1]. The only difference is that the existence of the orbit S_G in part (b) is established in the proof of [M3, Proposition 8.2.1]. \square

Let us recall that Mok defines $\eta_{\psi_G}^{\text{Mok}}$ through the identity

$$\pi(S_\psi, 1)^\sim = \text{Trans}_{G(\mathbb{R})}^{\text{GL}_N(\mathbb{C}) \rtimes \vartheta} (\eta_{\psi_G}^{\text{Mok}}).$$

Since $\text{Trans}_{G(\mathbb{R})}^{\text{GL}_N(\mathbb{C}) \rtimes \vartheta} = \text{Lift}_0$ (7.6) and Lift_0 is injective (Proposition 5.3), it follows by Equation (113) that

$$\eta_{\psi_G}^{\text{Mok}} = \sum_{S_G} n_{\epsilon(S_G)} \eta_{S_G}^{\text{loc}}(\delta_q) \in K_{\mathbb{C}} \Pi(\vee \mathcal{O}_G, G(\mathbb{R}, \delta_q))^{\text{st}} \quad (114)$$

(cf. [M3, Proposition 8.2.1]). By definition, the A-packet $\Pi_{\psi_G}^{\text{Mok}}$ consists of those irreducible representations in $\Pi(\vee \mathcal{O}_G, G(\mathbb{R}, \delta_q))$ which occur with non-zero multiplicity when (114) is expressed as a linear combination of irreducible representations.

Theorem 8.2. *Let ψ_G be an A-parameter for G with regular infinitesimal character. Then*

$$\eta_{\psi_G}^{\text{Mok}} = \eta_{\psi_G}^{\text{mic}}(\delta_q) = \eta_{\psi_G}^{\text{ABV}} \quad \text{and} \quad \Pi_{\psi_G}^{\text{Mok}} = \Pi_{\psi_G}^{\text{ABV}}.$$

Proof. The proof is the same as that of [AAM, Theorem 8.2 (a)]. We repeat it here in the case of unitary groups for the convenience of the reader. Let $\xi = (S_\psi, 1)$ as in Corollary 6.4.

$$\begin{aligned} \text{Lift}_0(\eta_{\psi_G}^{\text{mic}}(\delta_q)) &= \text{Lift}_0(\eta_{\psi_G}^{\text{mic}}(\vartheta)(\delta_q)) \quad (\text{by (99)}) \\ &= (-1)^{l^I(\xi) - l^I_\vartheta(\xi)} \pi(\xi)^+ \quad (\text{Corollary 6.4}) \\ &= \pi(\xi)^\sim \quad (\text{Theorem 7.4}) \\ &= \pi(S_\psi, 1)^\sim \\ &= \text{Trans}_{G(\mathbb{R})}^{\text{GL}_N(\mathbb{C}) \rtimes \vartheta} \left(\sum_{S_G} n_{\epsilon(S_G)} \eta_{S_G}^{\text{loc}}(\delta_q) \right) \quad (\text{Lemma 8.1, (114)}) \\ &= \text{Lift}_0(\eta_{\psi_G}^{\text{Mok}}) \quad (\text{Corollary 7.6(b)}). \end{aligned}$$

The equality of the stable virtual characters follows from the injectivity of Lift_0 (Proposition 5.3). The equality of packets follows immediately. \square

9 The equality of $\Pi_{\psi_G}^{\text{Mok}}$ and $\Pi_{\psi_G}^{\text{ABV}}$ for singular infinitesimal character

To conclude our comparison of stable virtual characters, we retain the setup of the previous section, but without the hypothesis of regularity on the infinitesimal character. In other words, the orbits $\vee \mathcal{O}_G$ and $\vee \mathcal{O}$ are now allowed to be orbits of singular infinitesimal characters and the reader should think of them as such. In order to prove $\eta_{\psi_G}^{\text{Mok}} = \eta_{\psi_G}^{\text{ABV}}$ for singular $\lambda \in \vee \mathcal{O}_G$, we must extend the pairing of Theorem 3.5 and extend the twisted endoscopic lifting (96) to include representations with singular infinitesimal character. This was done in Section 9 of [AAM] through the use of the *Jantzen-Zuckerman translation principle*, to which we refer from now on simply as *translation*. The same arguments can be used to extend the results of the previous sections to singular representations of $\text{GL}_N(\mathbb{C})$. For the convenience of the reader we are including it here again, adding the needed modifications to the context of the current paper.

In essence, translation allows one to transfer results for regular infinitesimal character to results for singular infinitesimal character. Applying this principle to the results of the previous section will allow us to compare Π_{ψ_G} with $\Pi_{\psi_G}^{\text{ABV}}$ with no restriction on the infinitesimal character.

The translation principle begins with the existence of a regular orbit $\vee \mathcal{O}' \subset \vee \mathfrak{gl}_N \times \vee \mathfrak{gl}_N$ and a *translation datum* \mathcal{T} from $\vee \mathcal{O}$ to $\vee \mathcal{O}'$ ([ABV, Definition 8.6, Lemma 8.7]). As $\vee \mathcal{O}$ is the $\vee \text{R}_{\mathbb{C}/\mathbb{R}} \text{GL}_N$ -orbit of $\lambda \in \vee \mathfrak{h}^\vartheta$ we may take $\vee \mathcal{O}'$ to be the $\vee \text{R}_{\mathbb{C}/\mathbb{R}} \text{GL}_N$ -orbit of

$$\lambda' = \lambda + \lambda_1 \in \vee \mathfrak{h} \quad (115)$$

where $\lambda_1 \in X_*(H)$ is regular and dominant with respect to the positive system of $R^+(\mathbb{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N, H)$. We may and shall take λ_1 to be the sum of the positive roots. In this way, each of λ , λ_1 and λ' are fixed by ϑ . The translation datum \mathcal{T} induces a ${}^\vee\mathbb{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N$ -equivariant morphism

$$f_{\mathcal{T}} : X({}^\vee\mathcal{O}', {}^\vee\mathbb{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N^\Gamma) \rightarrow X({}^\vee\mathcal{O}, {}^\vee\mathbb{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N^\Gamma) \quad (116)$$

of geometric parameters ([ABV, Proposition 8.8]). The morphism has connected fibres of fixed dimension, a fact we shall use when comparing orbit dimensions. The ${}^\vee\mathbb{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N$ -equivariance of (116) is tantamount to a coset map commuting with left-multiplication by ${}^\vee\mathbb{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N$ ([ABV, (6.10)(b)]). Since both λ and λ' are fixed by ϑ , it is just as easy to see that the action of ϑ commutes with the same coset map. We leave this exercise to the reader, taking for granted the resulting ${}^\vee\mathbb{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N$ -equivariance of (116).

According to [ABV, Proposition 7.15], the morphism $f_{\mathcal{T}}$ induces an inclusion

$$f_{\mathcal{T}}^* : \Xi({}^\vee\mathcal{O}, {}^\vee\mathbb{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N^\Gamma) \hookrightarrow \Xi({}^\vee\mathcal{O}', {}^\vee\mathbb{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N^\Gamma)$$

of complete geometric parameters. The ϑ -equivariance of (116) implies that this inclusion restricts to an inclusion (denoted by the same symbol)

$$f_{\mathcal{T}}^* : \Xi({}^\vee\mathcal{O}, {}^\vee\mathbb{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N^\Gamma)^\vartheta \hookrightarrow \Xi({}^\vee\mathcal{O}', {}^\vee\mathbb{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N^\Gamma)^\vartheta.$$

The (Jantzen-Zuckerman) translation functor ([AvLTV, (17.8j)])

$$T_{\lambda'}^\lambda = T_{\lambda+\lambda_1}^\lambda$$

is an exact functor on a category of Harish-Chandra modules, which we shall often regard as a homomorphism

$$T_{\lambda+\lambda_1}^\lambda : K\Pi({}^\vee\mathcal{O}', \mathrm{GL}_N(\mathbb{C}) \rtimes \langle \vartheta \rangle) \rightarrow K\Pi({}^\vee\mathcal{O}, \mathrm{GL}_N(\mathbb{C}) \rtimes \langle \vartheta \rangle) \quad (117)$$

of Grothendieck groups. It is surjective ([AvLTV, Corollary 17.9.8]). This translation functor is an extended version of the usual translation functor ([AvLTV, (16.8f)]), which we also denote by

$$T_{\lambda+\lambda_1}^\lambda : K\Pi({}^\vee\mathcal{O}', \mathrm{GL}_N(\mathbb{C})) \rightarrow K\Pi({}^\vee\mathcal{O}, \mathrm{GL}_N(\mathbb{C})). \quad (118)$$

Let us take a moment to make (117) more precise. The sum of the positive roots λ_1 is the infinitesimal character of a finite-dimensional representation of $\mathrm{GL}_N(\mathbb{C})$. Therefore, λ_1 is the differential of a ϑ -fixed quasicharacter Λ_1 of the real diagonal torus $H(\mathbb{R})$, which matches the weight of this finite-dimensional representation. The quasicharacter Λ_1 may be extended to a quasicharacter Λ_1^+ of the semi-direct product $H(\mathbb{R}) \rtimes \langle \vartheta \rangle$ by setting

$$\Lambda_1^+(\vartheta) = 1.$$

We define translation in the extended setting of (117) using this representation of the extended group. Since the extension is evident here we continue to write $T_{\lambda+\lambda_1}^\lambda$ instead of $T_{\lambda+\Lambda_1^+}^\lambda$.

In the ordinary setting of (118) we have

$$\begin{aligned} \pi(\xi) &= T_{\lambda+\lambda_1}^\lambda(\pi(f_{\mathcal{T}}^*(\xi))), \\ M(\xi) &= T_{\lambda+\lambda_1}^\lambda(M(f_{\mathcal{T}}^*(\xi))), \quad \xi \in \Xi({}^\vee\mathcal{O}, {}^\vee\mathbb{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N^\Gamma) \end{aligned}$$

([AvLTV, Corollary 16.9.4, 16.9.7 and 16.9.8], or [ABV, Theorem 16.4 and Proposition 16.6]). We define the Atlas extensions of $\pi(\xi)$ and $M(\xi)$, with $\xi \in \Xi({}^\vee\mathcal{O}, {}^\vee\mathbb{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N^\Gamma)^\vartheta$, by

$$\begin{aligned} \pi(\xi)^+ &= T_{\lambda+\lambda_1}^\lambda(\pi(f_{\mathcal{T}}^*(\xi))^+) \\ M(\xi)^+ &= T_{\lambda+\lambda_1}^\lambda(M(f_{\mathcal{T}}^*(\xi))^+). \end{aligned}$$

With the definition of Atlas extensions in place, the discussion of Section 2.5 is valid, and we see that $T_{\lambda+\lambda_1}^\lambda$ factors to a homomorphism of $K\Pi({}^\vee\mathcal{O}, \mathrm{GL}_N(\mathbb{C}), \vartheta)$ (see (48)). We use the same notation $T_{\lambda+\lambda_1}^\lambda$ to denote the functor of Harish-Chandra modules, and either of the earlier homomorphisms. The reader will be reminded of the context when it is important.

The definition of a Whittaker extension does not depend on the regularity of the infinitesimal character. The following proposition shows that translation sends Whittaker extensions to Whittaker extensions.

Proposition 9.1. *Suppose $\xi \in \Xi(\vee \mathcal{O}, \vee \mathbf{R}_{\mathbb{C}/\mathbb{R}} \mathrm{GL}_N^\Gamma)^\vartheta$. Then (as Harish-Chandra modules)*

$$T_{\lambda+\lambda_1}^\lambda (M(f_{\mathcal{T}}^*(\xi))^\sim) = M(\xi)^\sim,$$

and

$$T_{\lambda+\lambda_1}^\lambda (\pi(f_{\mathcal{T}}^*(\xi))^\sim) = \pi(\xi)^\sim.$$

Proof. The proof runs along the same lines as that of [AAM, Proposition 9.1]. The proof is even simpler in the current context, since every ϑ -invariant irreducible representation of $\mathrm{GL}_N(\mathbb{C})$ appears as a subquotient of a principal series representation as given in Lemma 7.1. \square

Our translation datum \mathcal{T} for $\mathbf{R}_{\mathbb{C}/\mathbb{R}} \mathrm{GL}_N$ is defined by (115), in which both λ and λ' are fixed by ϑ . For this reason (115) also determines a translation datum \mathcal{T}_G from $\vee G$ -orbits $\vee \mathcal{O}_G$ to $\vee \mathcal{O}'_G$ for the twisted endoscopic group G ([ABV, Definition 8.6 (e)]). Just as for $\mathbf{R}_{\mathbb{C}/\mathbb{R}} \mathrm{GL}_N$, we have maps

$$\begin{aligned} f_{\mathcal{T}_G} &: X(\vee \mathcal{O}', \vee G^\Gamma) \rightarrow X(\vee \mathcal{O}, \vee G^\Gamma) \\ f_{\mathcal{T}_G}^* &: \Xi(\vee \mathcal{O}, \vee G^\Gamma) \hookrightarrow \Xi(\vee \mathcal{O}', \vee G^\Gamma) \end{aligned}$$

and the translation functor $T_{\lambda+\lambda_1}^\lambda$ which satisfies

$$\pi(\xi) = T_{\lambda+\lambda_1}^\lambda (\pi(f_{\mathcal{T}_G}^*(\xi))), \quad \xi \in \Xi(\vee \mathcal{O}_G, \vee G^\Gamma)$$

([ABV, Proposition 16.6], [AvLTV, Section 16]).

The translation data \mathcal{T} and \mathcal{T}_G allow us to transport properties of our pairings at regular infinitesimal character to the same properties for pairings at singular infinitesimal character. More precisely, as explained at the end of [ABV, page 178], the translation datum \mathcal{T}_G applied to Proposition 3.3 allows us to extend Theorem 3.1 to any infinitesimal character. In a similar fashion, the translation datum \mathcal{T} applied this time to Equation (111), allows us to transport Corollary 7.5 from regular to singular infinitesimal character. These two procedures have the following result.

Proposition 9.2 ([AAM, Proposition 9.2]). *Define the pairing*

$$\langle \cdot, \cdot \rangle : K\Pi(\vee \mathcal{O}, \mathrm{GL}_N(\mathbb{C}), \vartheta) \times KX(\vee \mathcal{O}, \vee \mathbf{R}_{\mathbb{C}/\mathbb{R}} \mathrm{GL}_N^\Gamma, \vartheta) \rightarrow \mathbb{Z} \quad (119)$$

by

$$\langle M(\xi)^\sim, \mu(\xi')^+ \rangle = \delta_{\xi, \xi'}.$$

Then

$$\langle \pi(\xi)^\sim, P(\xi')^+ \rangle = (-1)^{d(\xi)} \delta_{\xi, \xi'}$$

where $\xi, \xi' \in \Xi(\vee \mathcal{O}, \vee \mathbf{R}_{\mathbb{C}/\mathbb{R}} \mathrm{GL}_N^\Gamma)^\vartheta$.

Proposition 9.2 is the final version of the twisted pairing, and we use it to extend the definition of endoscopic lifting Lift_0 to include singular infinitesimal characters ((95), (96)). In fact, all of the remaining results used in Section 8 easily carry over to the more general setting. In particular, using the pairing (119) in the proof of Proposition 5.2, we see that for any $\vee G$ -orbit $S_G \subset X(\vee \mathcal{O}_G, \vee G^\Gamma)$ we still have

$$\mathrm{Lift}_0(\eta_{S_G}^{\mathrm{loc}}(\vartheta)(\delta_q)) = M(\epsilon(S_G), 1)^\sim.$$

Finally, since the injectivity of Lift_0 still holds for singular infinitesimal character, the same argument used in the proof of Theorem 8.2 allows us to conclude

Theorem 9.3. *Let ψ_G be an A -parameter for G . Then*

$$\eta_{\psi_G}^{\mathrm{Mok}} = \eta_{\psi_G}^{\mathrm{mic}}(\delta_q) = \eta_{\psi_G}^{\mathrm{ABV}} \quad \text{and} \quad \Pi_{\psi_G}^{\mathrm{Mok}} = \Pi_{\psi_G}^{\mathrm{ABV}}.$$

10 Some consequences for standard endoscopy

In this section we explore standard endoscopic lifting from an endoscopic group G' of our quasisplit unitary group G . Both [ABV] and [M3] provide formulae for this lifting, and it is not obvious why the two types of formulae should be equal. Our goal is to show how Theorem 9.3 implies the equality of the formulae, arguing as in [AAM, Section 10].

A brief review of the definition of G' may be found in [ABV, Section 26]. The endoscopic group G' is defined to be a quasisplit form of a complex reductive group whose dual ${}^\vee G'$ is the identity component of the centralizer in ${}^\vee G$ of a semisimple element $s \in {}^\vee G$ (cf. Section 5). We further assume that the element s centralizes the image of a fixed A-parameter ψ_G as in (1). There is a natural embedding

$$\epsilon' : {}^\vee (G')^\Gamma \hookrightarrow {}^\vee G^\Gamma \quad (120)$$

(cf. [M3, (2.1.13)]), and an A-parameter $\psi_{G'}$ for G' such that

$$\psi_G = \epsilon' \circ \psi_{G'}$$

(cf. [A3, p. 36]). We write Π_{ψ_G} for $\Pi_{\psi_G}^{\text{Mok}} = \Pi_{\psi_G}^{\text{ABV}}$. For any $\pi \in \Pi_{\psi_G}$, we write $\tau_{\psi_G}^{\text{mic}}(\pi) = \tau_{S_{\psi_G}}^{\text{mic}}(\pi)$ for the representation of $A_{\psi_G} = {}^\vee G_{\psi_G} / ({}^\vee G_{\psi_G})^0$ introduced in (5.4).

Let us first look at endoscopic lifting from the perspective of [ABV]. Let $\text{Lift}_{G'(\mathbb{R})}^{G(\mathbb{R})}$ denote the endoscopic lifting map from stable virtual characters of the quasisplit group $G'(\mathbb{R})$ to the virtual characters of $G(\mathbb{R}) = G(\mathbb{R}, \delta_q)$ as given in [ABV, Definition 26.18]. According to [ABV, Theorem 22.7 and Theorem 26.25] the stable virtual character $\eta_{\psi_{G'}}^{\text{ABV}}$ of $G'(\mathbb{R})$ satisfies

$$\text{Lift}_{G'(\mathbb{R})}^{G(\mathbb{R})}(\eta_{\psi_{G'}}^{\text{ABV}}) = \sum_{\pi \in \Pi_{\psi_G}} (-1)^{d(\pi) - d(S_{\psi_G})} \text{Tr}(\tau_{\psi_G}^{\text{mic}}(\pi)(\bar{s})) \pi. \quad (121)$$

Here, $d(S_{\psi_G})$ is the dimension of the ${}^\vee G$ -orbit S_{ψ_G} , and \bar{s} is the coset of s in A_{ψ_G} .

The analogue of formula (121) from Mok's perspective is [M3, (8.2.4)]. It describes the endoscopic lifting map $\text{Trans}_{G'(\mathbb{R})}^{G(\mathbb{R})}$ defined by Shelstad ([S6]) on a stable virtual character $\eta_{\psi_{G'}}^{\text{Mok}}$ ([M3, Theorem 3.2.1]). With this notation Mok's formula is

$$\text{Trans}_{G'(\mathbb{R})}^{G(\mathbb{R})}(\eta_{\psi_{G'}}^{\text{Mok}}) = \sum_{\sigma \in \Sigma_{\psi_G}} \langle s_{\psi_G} \bar{s}, \sigma \rangle \sigma. \quad (122)$$

Here, Σ_{ψ_G} is a finite set of non-negative integral linear combinations

$$\sigma = \sum_{\pi \in \Pi_{\text{unit}}(G(\mathbb{R}))} m(\sigma, \pi) \pi$$

of irreducible unitary characters of $G(\mathbb{R})$. Furthermore, there is an injective map from Σ_{ψ_G} into the set of those quasicharacters of A_{ψ_G} which are trivial on the centre of ${}^\vee G$. The injection is denoted by

$$\sigma \mapsto \langle \cdot, \sigma \rangle.$$

The element s_{ψ_G} is the image of

$$\psi_G \left(1, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \right)$$

in A_{ψ_G} .

If one takes $s = 1$ in (122) then one recovers $G' = G$ and the equation reduces to

$$\begin{aligned} \eta_{\psi_G}^{\text{Mok}} &= \sum_{\sigma \in \Sigma_{\psi_G}} \langle s_{\psi_G}, \sigma \rangle \sigma \\ &= \sum_{\sigma \in \Sigma_{\psi_G}} \langle s_{\psi_G}, \sigma \rangle \sum_{\pi \in \Pi_{\text{unit}}(G(\mathbb{R}))} m(\sigma, \pi) \pi \\ &= \sum_{\pi \in \Pi_{\psi_G}} \left(\sum_{\sigma \in \Sigma_{\psi_G}} m(\sigma, \pi) \langle s_{\psi_G}, \sigma \rangle \right) \pi. \end{aligned} \quad (123)$$

For each $\pi \in \Pi_{\psi_G}$ define

$$\tau_{\psi_G}(\pi) = \sum_{\sigma \in \Sigma_{\psi_G}} m(\sigma, \pi) \langle \cdot, \sigma \rangle,$$

which is apparently a finite sum of quasicharacters of A_{ψ_G} . Mœglin and Renard have proven that $\tau_{\psi_G}(\pi)$ is actually irreducible ([MR2, Theorem 1.3]). Therefore (123) becomes

$$\eta_{\psi_G}^{\text{Mok}} = \sum_{\pi \in \Pi_{\psi_G}} \tau_{\psi_G}(\pi)(s_{\psi_G}) \pi,$$

and more generally (122) becomes

$$\text{Trans}_{G'(\mathbb{R})}^{G(\mathbb{R})}(\eta_{\psi_{G'}}^{\text{Mok}}) = \sum_{\pi \in \Pi_{\psi_G}} \tau_{\psi_G}(\pi)(s_{\psi_G} \bar{s}) \pi. \quad (124)$$

Our goal is to show that the two equations, (124) and (121), are identical. In order to compare the two equations, we must choose the element $s \in {}^\vee G$ defining the endoscopic group G' carefully. Recall that s lies in the centralizer of the image of ψ_G . We may suppose $\bar{s} \in A_{\psi_G}$ is not trivial. An explicit description of this centralizer is given in [M3, (2.4.13)] (following [GGP, Section 4]). In consideration of this description, it is not difficult to choose for each, $\bar{s} \in A_{\psi_G}$, a diagonal representative \dot{s} in the centralizer with eigenvalues ± 1 . The endoscopic group $G'(\dot{s})$ determined by \dot{s} is a direct product $G'_1(\dot{s}) \times G'_2(\dot{s})$ in which each of the two factors is a quasisplit unitary group whose rank is less than G (cf. [R, Proposition 4.6.1]). The A-parameter $\psi_{G'(\dot{s})}$ decomposes accordingly as a product $\psi_{G'_1(\dot{s})} \times \psi_{G'_2(\dot{s})}$ of A-parameters. Similarly, Mok's stable virtual character $\eta_{\psi_{G'(\dot{s})}}^{\text{Mok}}$ is defined as the tensor product $\eta_{\psi_{G'_1(\dot{s})}}^{\text{Mok}} \otimes \eta_{\psi_{G'_2(\dot{s})}}^{\text{Mok}}$. For these choices of endoscopic data, (124) reads as

$$\text{Trans}_{G'(\mathbb{R})}^{G(\mathbb{R})}(\eta_{\psi_{G'_1(\dot{s})}}^{\text{Mok}} \otimes \eta_{\psi_{G'_2(\dot{s})}}^{\text{Mok}}) = \sum_{\pi \in \Pi_{\psi_G}} \tau_{\psi_G}(\pi)(s_{\psi_G} \bar{s}) \pi. \quad (125)$$

We now turn to rewriting the left-hand side of (125) so as to match it with the left-hand side of (121). First, it is noted on [ABV, p. 289] that

$$\text{Trans}_{G'(\mathbb{R})}^{G(\mathbb{R})} = \text{Lift}_{G'(\mathbb{R})}^{G(\mathbb{R})}. \quad (126)$$

Second, using the arguments in the proof of Corollary 6.2, we see that

$$\eta_{\psi_{G'(\dot{s})}}^{\text{ABV}} = \eta_{\psi_{G'_1(\dot{s})}}^{\text{ABV}} \otimes \eta_{\psi_{G'_2(\dot{s})}}^{\text{ABV}}.$$

Third, since $G'_1(\dot{s})$ and $G'_2(\dot{s})$ are both quasisplit unitary groups, Theorem 9.3 tells us that

$$\eta_{\psi_{G'_j(\dot{s})}}^{\text{Mok}} = \eta_{\psi_{G'_j(\dot{s})}}^{\text{ABV}}, \quad j = 1, 2. \quad (127)$$

Taking these three observations together we conclude

$$\begin{aligned} \text{Trans}_{G'(\mathbb{R})}^{G(\mathbb{R})}(\eta_{\psi_{G'(\dot{s})}}^{\text{Mok}}) &= \text{Lift}_{G'(\mathbb{R})}^{G(\mathbb{R})}(\eta_{\psi_{G'_1(\dot{s})}}^{\text{Mok}} \otimes \eta_{\psi_{G'_2(\dot{s})}}^{\text{Mok}}) \\ &= \text{Lift}_{G'(\mathbb{R})}^{G(\mathbb{R})}(\eta_{\psi_{G'_1(\dot{s})}}^{\text{ABV}} \otimes \eta_{\psi_{G'_2(\dot{s})}}^{\text{ABV}}) \\ &= \text{Lift}_{G'(\mathbb{R})}^{G(\mathbb{R})}(\eta_{\psi_{G'(\dot{s})}}^{\text{ABV}}). \end{aligned}$$

It is now immediate from (125) and (121) that

$$\sum_{\pi \in \Pi_{\psi_G}} \tau_{\psi_G}(\pi)(s_{\psi_G} \bar{s}) \pi = \sum_{\pi \in \Pi_{\psi_G}} (-1)^{d(\pi) - d(S_{\psi_G})} \text{Tr}(\tau_{\psi_G}^{\text{ABV}}(\pi)(\bar{s})) \pi$$

for any $\bar{s} \in A_{\psi}$. By the linear independence of characters on $G(\mathbb{R})$

$$\tau_{\psi_G}(\pi)(s_{\psi_G} \bar{s}) = (-1)^{d(\pi) - d(S_{\psi_G})} \text{Tr}(\tau_{\psi_G}^{\text{ABV}}(\pi)(\bar{s}))$$

for any $\bar{s} \in A_\psi$. This may be regarded as an equality between virtual characters on A_ψ . By appealing to the linear independence of these characters we conclude that

$$\tau_{\psi_G}(\pi)(s_{\psi_G}) = (-1)^{d(\pi) - d(S_{\psi_G})}$$

and

$$\tau_{\psi_G}(\pi) = \tau_{\psi_G}^{\text{ABV}}(\pi).$$

In particular, $\tau_{\psi_G}^{\text{ABV}}(\pi)$ is irreducible. This completes our comparison of (121) and (124).

11 Pure inner forms

Fix a pure strong involution δ of G^Γ as given in (23). As δ runs over the equivalence classes of pure strong involutions, the real forms $G(\mathbb{R}, \delta)$ run over all isomorphism classes of indefinite unitary groups $U(p, q)$, $p + q = N$ ([A, Section 9]). Recall that δ_q is a pure strong involution. We shall extend the results of Section 10 by replacing $G(\mathbb{R}) = G(\mathbb{R}, \delta_q)$ with any $G(\mathbb{R}, \delta)$. The virtual characters $\eta_{\psi_G}^{\text{Mok}}$ for quasisplit unitary groups are replaced by virtual characters $\eta_{\psi_G}^{\text{MR}}$, defined and studied by Mœglin and Renard ([MR1], [MR2]). With these virtual characters in place the arguments are more or less the same as in Section 10.

There is one additional wrinkle in the non-quasisplit setting, which is to prove that the endoscopic maps of Shelstad and [ABV] are equal. We were able to manage this in Section 10 by using canonical transfer factors for quasisplit groups and citing an exercise on [ABV, p. 289] (*cf.* (126)). For non-quasisplit groups we refer to methods of Kaletha and [AM]. Let us settle this matter straightaway. We continue with the initial setup of Section 10, namely with an endoscopic group G' obtained from a semisimple element $s \in {}^\vee G$, but temporarily without the assumption of s being in the centralizer of ψ_G .

The first significant departure from the previous section arises in the nature of the spectral transfer map $\text{Trans}_{G'(\mathbb{R})}^{G(\mathbb{R}, \delta)}$ from stable virtual characters of the quasisplit group $G'(\mathbb{R})$ to the virtual characters of $G(\mathbb{R}, \delta)$. If $\delta = \delta_q$ so that $G(\mathbb{R}) = G(\mathbb{R}, \delta)$ is quasisplit, a Whittaker datum fixes a set of constants, the *transfer factors*, which specify the map $\text{Trans}_{G'(\mathbb{R})}^{G(\mathbb{R})}$ ([S4, Corollary 11.7]). When $G(\mathbb{R}, \delta)$ is not quasisplit, Shelstad does not provide canonical transfer factors, and as a result $\text{Trans}_{G'(\mathbb{R})}^{G(\mathbb{R}, \delta)}$ is defined only up to a non-zero constant. This ambiguity is rectified by Kaletha in [K1], where canonical transfer factors for $G(\mathbb{R}, \delta)$ are again defined relative to a fixed Whittaker datum. This specifies the transfer map $\text{Trans}_{G'(\mathbb{R})}^{G(\mathbb{R}, \delta)}$ on tempered L-packets and agrees with $\text{Trans}_{G'(\mathbb{R})}^{G(\mathbb{R})}$ when $G(\mathbb{R}, \delta)$ is quasisplit ([K1, (5.11), Proposition 5.10]). The extension of $\text{Trans}_{G'(\mathbb{R})}^{G(\mathbb{R}, \delta)}$ to the space of stable virtual characters follows from the characterization of this space ([MW, Corollary IV.2.8]), and an analytic continuation argument from the tempered setting (*cf.* Lemma 4.8 [AJ]). More concretely, the map $\text{Trans}_{G'(\mathbb{R})}^{G(\mathbb{R}, \delta)}$ is characterized by

$$\text{Trans}_{G'(\mathbb{R})}^{G(\mathbb{R}, \delta)} \left(\sum_{\tau_{S_1}} M(S_1, \tau_{S_1}) \right) = e(\delta) \sum_{\tau_S} \tau_S(s) M(S, \tau_S) \quad (128)$$

(*cf.* [K1, (5.9)]). On the left, $\text{Trans}_{G'(\mathbb{R})}^{G(\mathbb{R}, \delta)}$ is evaluated on a stable virtual character (91) for $(G')^\Gamma$. On the right, $e(\delta) = \pm 1$ is the Kottwitz invariant of $G(\mathbb{R}, \delta)$ ([ABV, Definition 15.8]), and the ${}^\vee G$ -orbit $S \subset X({}^\vee \mathcal{O}_G, {}^\vee G^\Gamma)$ is the one determined by the image of the ${}^\vee G'$ -orbit S_1 under (120). The sum on the right runs over certain characters τ_S of ${}^\vee G_p / ({}^\vee G_p)^0$, for fixed $p \in S$ (see the discussion following (29)). They are the characters which correspond to the pure strong involution δ under [K1, (5.13), Corollary 5.4, Theorem 5.2]. It follows from [AM, Theorem 1.1] that this correspondence between τ_S and δ coincides with the one given by (30) in [ABV]. (This is just another way of saying that we are justified in denoting the standard representations on the right-hand side of (128) by $M(S, \tau_S)$.)

Let $\text{Lift}_{G'(\mathbb{R})}^{G(\mathbb{R}, \delta)}$ denote the endoscopic lifting map written as $\text{Lift}_0(\delta)$ in [ABV, Definition 26.18]. By [ABV, Proposition 26.7],

$$\text{Lift}_{G'(\mathbb{R})}^{G(\mathbb{R}, \delta)} \left(\sum_{\tau_{S_1}} M(S_1, \tau_{S_1}) \right) = e(\delta) \sum_{\tau_S} \tau_S(s) M(S, \tau_S),$$

where the terms are identical to those of (128). In consequence,

$$\text{Trans}_{G'(\mathbb{R})}^{G(\mathbb{R}, \delta)} = \text{Lift}_{G'(\mathbb{R})}^{G(\mathbb{R}, \delta)}. \quad (129)$$

We now adopt all of the assumptions of Section 10, so that $s \in {}^\vee G$ is taken to centralize the image of a fixed A-parameter ψ_G and \bar{s} is the coset of s in A_{ψ_G} . In [MR2] Mœglin and Renard prove Arthur's local conjectures for all pure inner forms of unitary groups (*cf.* [KMSW, Section 1.6.1]). Let us give a brief description of their solution to the conjectures.

For each $\bar{s} \in A_{\psi_G}$ we choose a representative $s \in {}^\vee G$ and its endoscopic group G' as in the paragraph following Equation (124). Then there is an A-parameter $\psi_{G'}$ for G' such that

$$\psi_G = \epsilon' \circ \psi_{G'}.$$

For each $\bar{s} \in A_{\psi_G}$, Mœglin and Renard define the virtual character $\eta_{\psi_G}^{\text{MR}}(\bar{s})$ of $G(\mathbb{R}, \delta)$ by

$$e(\delta) \eta_{\psi_G}^{\text{MR}}(\bar{s}) = \text{Trans}_{G'(\mathbb{R})}^{G(\mathbb{R}, \delta)} \left(\eta_{\psi_{G'}}^{\text{Mok}} \right)$$

([MR1, Sections 1 and 2.1]). At first glance, the virtual character $\eta_{\psi_G}^{\text{MR}}(\bar{s})$ is a finite linear combination of irreducible representations with complex coefficients. In fact, the coefficients are integers and the irreducible representations are unitary. More precisely, in [MR2, Theorem 5.3] the authors define a finite set

$$\Pi_{\psi_G}^{\text{MR}}(G(\mathbb{R}, \delta))$$

of unitary representations, together with a mapping

$$\pi \longmapsto \tau_{\psi_G}(\pi),$$

from $\Pi_{\psi_G}^{\text{MR}}(G(\mathbb{R}, \delta))$ to the group of irreducible representations of A_{ψ_G} , such that

$$\eta_{\psi_G}^{\text{MR}}(\bar{s}) = \sum_{\pi \in \Pi_{\psi_G}^{\text{MR}}(G(\mathbb{R}, \delta))} \tau_{\psi_G}(\pi)(s_{\psi_G} \bar{s}) \pi, \quad \bar{s} \in A_{\psi_G}.$$

In particular, for $s = 1$, $G'(\mathbb{R}) = G(\mathbb{R}, \delta_q)$ and

$$e(\delta) \eta_{\psi_G}^{\text{MR}}(1) = \text{Trans}_{G(\mathbb{R}, \delta_q)}^{G(\mathbb{R}, \delta)} \left(\eta_{\psi_G}^{\text{Mok}} \right).$$

This virtual character is stable since $\text{Trans}_{G(\mathbb{R}, \delta_q)}^{G(\mathbb{R}, \delta)}$ carries stable virtual characters to stable virtual characters (*cf.* (128) with $s = 1$). The set $\Pi_{\psi_G}^{\text{MR}}(G(\mathbb{R}, \delta))$ is the Arthur packet attached to the stable virtual character $\eta_{\psi_G}^{\text{MR}} = \eta_{\psi_G}^{\text{MR}}(1)$.

We would like to compare these constructions of Mœglin and Renard with their analogues in [ABV, Theorem 22.7 and Definition 26.8]. For this we return to the discussion preceding (121) and specialize to the real form $G(\mathbb{R}, \delta)$. The analogous Arthur packet $\Pi_{\psi_G}^{\text{ABV}}(G(\mathbb{R}, \delta))$ was defined in (56). We use the representations $\tau_{\psi_G}^{\text{mic}}(\pi)$ given in (5.4) to define

$$\eta_{\psi_G}^{\text{ABV}}(\delta)(\bar{s}) = \sum_{\pi \in \Pi_{\psi_G}^{\text{ABV}}(G(\mathbb{R}, \delta))} e(\delta) (-1)^{d(\pi) - d(S_{\psi_G})} \text{Tr} \left(\tau_{\psi_G}^{\text{mic}}(\pi)(\bar{s}) \right) \pi \quad (130)$$

([ABV, p. 281]). In particular, $\eta_{\psi_G}^{\text{ABV}}(\delta)(1)$ is the stable virtual character $\eta_{\psi_G}^{\text{ABV}}(\delta)$ defined in (55).

Theorem 11.1. *Let ψ_G be an A -parameter for G . Then for any pure inner form $G(\mathbb{R}, \delta)$ we have*

$$e(\delta) \eta_{\psi_G}^{\text{MR}}(\bar{s}) = \eta_{\psi_G}^{\text{ABV}}(\delta)(\bar{s}), \quad \bar{s} \in A_{\psi_G}.$$

In particular,

$$\Pi_{\psi_G}^{\text{MR}}(G(\mathbb{R}, \delta)) = \Pi_{\psi_G}^{\text{ABV}}(G(\mathbb{R}, \delta)), \quad \tau_{\psi_G}(\pi) = \tau_{\psi_G}^{\text{ABV}}(\pi)$$

and

$$\tau_{\psi_G}(\pi)(s_{\psi_G}) = (-1)^{d(\pi) - d(S_{\psi_G})}.$$

Proof. For any $\bar{s} \in A_{\psi_G}$, we have

$$\begin{aligned} e(\delta) \eta_{\psi_G}^{\text{MR}}(\bar{s}) &= \text{Trans}_{G'}^{G(\mathbb{R}, \delta)}(\eta_{\psi_{G'}}^{\text{Mok}}) \\ &= \text{Lift}_{G'(\mathbb{R})}^{G(\mathbb{R}, \delta)}(\eta_{\psi_{G'}}^{\text{Mok}}) \quad (\text{Equation (129)}) \\ &= \text{Lift}_{G'(\mathbb{R})}^{G(\mathbb{R}, \delta)}(\eta_{\psi_{G'}}^{\text{ABV}}) \quad (\text{Theorem 9.3 and Equation (127)}) \\ &= \eta_{\psi_G}^{\text{ABV}}(\delta)(\bar{s}) \quad ([\text{ABV}, \text{Theorem 26.25}]). \end{aligned}$$

The equality of packets follows immediately after taking $s = 1$. The remaining equalities follow the argument used at the end of Section 10. \square

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