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L-packets over strong real forms

N. Arancibia Robert, P. Mezo

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Abstract

Langlands defined L-packets for real reductive groups. In order to refine the local Langlands correspondence, Adams-Barbasch-Vogan combined L-packets over all real forms belonging to an inner class ([ABV]). Using different methods, Kaletha also defines such combined L-packets with a refinement to the local Langlands correspondence ([K1]). We prove that the L-packets of Adams-Barbasch-Vogan and Kaletha are the same and are parameterized identically.

1 Introduction

Let G be a connected reductive algebraic group defined over \mathbb{R} . Langlands defined a partition of the (infinitesimal equivalence classes of) irreducible admissible representations of $G(\mathbb{R})$ into L-packets ([L]). The L-packets Π_ϕ are parameterized by (conjugacy classes of) L-homomorphisms

$$\phi : W_{\mathbb{R}} \rightarrow {}^\vee G^\Gamma \quad (1)$$

from the Weil group $W_{\mathbb{R}}$ of $\Gamma = \text{Gal}(\mathbb{C}/\mathbb{R})$ to the L-group ${}^\vee G^\Gamma$ ([B1, Section 8]).

Langlands proposed a further parameterization of the representations in each L-packet in terms of the centralizer ${}^\vee G_\phi$ of $\phi(W_{\mathbb{R}})$ in the dual group ${}^\vee G$. Specifically, the representations in Π_ϕ were to correspond to characters of the (abelian) component group ${}^\vee G_\phi / ({}^\vee G_\phi)^0$. Shelstad completed this proposal for a wide class of groups ([S]), but was obliged to introduce formal “ghost” representations in order to make the correspondence one-to-one.

Adams, Barbasch and Vogan proposed a different approach to achieve a one-to-one correspondence ([ABV, Theorem 10.11]). Rather than defining a packet for a single real form $G(\mathbb{R})$, they defined a packet of representations over several real forms appearing in the inner class of $G(\mathbb{R})$. We continue to call these compound packets “L-packets”, but denote them by

$$\Pi_\phi^{\text{ABV}} = \{ \pi_\tau^{\text{ABV}} : \tau \in ({}^\vee G_\phi / ({}^\vee G_\phi)^0)^\wedge \}. \quad (2)$$

In this preliminary version of the L-packet, each character τ determines a real form of G which is inner to a quasisplit form, and π_τ^{ABV} is (an infinitesimal equivalence class of) a representation of the inner form. Unfortunately, not

every real form of G appears in such an L-packet. (Only the *pure real forms* appear (Section 2).) In order to include all of the real forms of G one must introduce algebraic coverings

$$1 \rightarrow \hat{J} \rightarrow {}^\vee G^{\hat{J}} \rightarrow {}^\vee G \rightarrow 1.$$

in which we take \hat{J} to be a finite abelian group. The more general L-packets, which encompass all real forms in an inner class, take the form

$$\Pi_{\phi, J}^{\text{ABV}} = \left\{ \pi_{\tau}^{\text{ABV}} : \tau \in ({}^\vee G_{\phi}^{\hat{J}} / ({}^\vee G_{\phi}^{\hat{J}})^0)^{\wedge} \right\}.$$

Here, J is a (sufficiently large) finite subgroup of the centre $Z(G)$, defined over \mathbb{R} , and is related to \hat{J} through [ABV, Lemma 10.2]. The real forms parameterized by the characters τ are called *strong real forms of type J* .

In [V2] Vogan presented ideas towards defining compound L-packets for reductive groups over p-adic fields as well. A basic problem in this endeavour is to extend the notion of strong real form to groups over local fields. In characteristic zero, this problem was solved by Kaletha, who described the analogue as a *rigid inner twist* ([K1]). Kaletha combined his theory of rigid inner twists with the work of Langlands and Shelstad on L-packets for real groups. In doing so, he defined compound L-packets for *tempered* L-homomorphisms ([K1, Section 5.4]). Let us assume for the moment that ϕ is tempered, *i.e.* has bounded image in ${}^\vee G$. The compound L-packets of Kaletha, Langlands and Shelstad run over representations of strong real forms and are in one-to-one correspondence with the characters of ${}^\vee G_{\phi}^{\hat{J}} / ({}^\vee G_{\phi}^{\hat{J}})^0$. We denote these packets by

$$\Pi_{\phi, J}^{\text{KLS}} = \left\{ \pi_{\tau}^{\text{KLS}} : \tau \in ({}^\vee G_{\phi}^{\hat{J}} / ({}^\vee G_{\phi}^{\hat{J}})^0)^{\wedge} \right\}.$$

Our main theorem is

Theorem 1.1. *Suppose ϕ is tempered. Then for all $\tau \in ({}^\vee G_{\phi}^{\hat{J}} / ({}^\vee G_{\phi}^{\hat{J}})^0)^{\wedge}$*

$$\pi_{\tau}^{\text{ABV}} = \pi_{\tau}^{\text{KLS}}.$$

The equation in this theorem is to be interpreted as an equality of infinitesimal equivalence classes of irreducible representations. It should also be noted that τ determines a strong real form using the approach of Adams-Barbasch-Vogan, and determines another strong real form using the approach of Kaletha. Kaletha's approach relies heavily on Galois cohomology, and this is completely absent in the approach of Adams-Barbasch-Vogan. It is implicit in the theorem that the two strong real forms coincide.

An essential feature in the two approaches is the choice of a maximal torus related to ϕ . The choice of tori in the two approaches differs significantly when the representations of the L-packet have singular infinitesimal character. In this case we relate the two different tori through Cayley transforms. The representations are defined through the two tori and we identify them through Hecht-Schmid character identities ([SV, Section 5]).

There are two corollaries to Theorem 1.1 to which we wish to allude. The first pertains to the assumption of ϕ being tempered. The only reason for this assumption is the absence of the definition of the compound L-packets in [K1, Section 5.6] for general ϕ . The extension of this definition to arbitrary ϕ may be completed by imitating Langlands' definition of arbitrary L-packets from tempered ones ([B1, Section 11.3]). One starts with the compound essentially tempered L-packet of a Levi subgroup. The general L-packet is then obtained by taking Langlands quotients of representations induced from the Levi subgroup. Although Adams-Barbasch-Vogan do not define their L-packets using intermediate Levi subgroups, their definition is also consistent with this approach. In this way, the extension of Theorem 1.1 to arbitrary ϕ is an unremarkable exercise.

The second corollary pertains to the theory of endoscopy. Both Kaletha and Adams-Barbasch-Vogan define a notion of endoscopic datum for ${}^\vee G^J$ which extends the usual one for ${}^\vee G$ ([K1, Section 5.3], [ABV, Definition 26.15]), [LS]). In the absence of the technicality of z -extensions ([LS, Section 4.4]), both definitions are easily seen to coincide, and to boil down to an element $s \in {}^\vee G^J$ and a connected reductive group H . Let us restrict to this setting and further assume that ϕ is a tempered L-parameter for H . In this setting ϕ yields a compound L-packet for H and also for G . Suffice it to say that both Kaletha and Adams-Barbasch-Vogan prove character identities between the resulting L-packets ([K1, (5.11), Proposition 5.10], [ABV, Proposition 26.7, Definition 26.18]). The chief coefficients in both character identities are of the form $\tau(s)$, where τ is given in Theorem 1.1. From the equivalence of the representations attached to τ it becomes apparent that both endoscopic character identities are the same. This fact implies that the endoscopic lifting of tempered L-packets in the two perspectives is the same as well. (See [ABV, p. 289], where this is left as a "straightforward exercise".)

We anticipate that Theorem 1.1 will be valuable in similar endoscopic identities for compound Arthur packets of classical and unitary groups ([A, Chapter 9], [M], [KMSW], [MR], [ABV, Theorem 26.25], [AAM]).

This work is organized as follows. In Section 2 we review the equivalent definitions of strong real forms and rigid inner twists over the real numbers. We recall how representations are attached to these objects, and how pairings are attached to these objects. The relationships between the attached representations and pairings are discussed. Section 3 introduces the L-packets of [ABV] without any assumptions on ϕ . Section 4 introduces the L-packets of [K1] for tempered ϕ . The comparison of the two kinds of L-packets requires a comparison of certain component groups and the pairings between them and strong real forms. This is done in Sections 5.1-5.2 and is the core of the paper. The main point is that the component groups differ by Cayley transforms and that representations in the packets are seen to be equivalent through Hecht-Schmid character identities. This is briefly explained in Section 5.3.

2 Strong real forms and rigid inner twists

We first recall the notion of strong real form in [ABV]. We then review how Kaletha's notion of rigid inner twists, which are valid for groups over local fields, are equivalent to strong real forms when the local field is \mathbb{R} ([K1, Section 5.2]).

We mention in passing that there is yet another equivalent notion to strong real forms, namely that of *strong involutions* ([AdC, Definition 5.5], [AT, Remark 8.13]). Although we do not pursue strong involutions here, they are often, for good reasons, preferred over strong real forms. One could reformulate our results purely in terms of strong involutions.

Let G be a connected reductive complex algebraic group. Let G^Γ be a group containing G as an index two subgroup, with the additional condition that every element in $G^\Gamma - G$ acts by conjugation on G as an antiholomorphic automorphism. More precisely, given any $\delta \in G^\Gamma - G$ and an algebraic function $f \in \mathbb{C}[G]$, the function

$$g \mapsto \overline{f(\text{Int}(\delta)(g))}, \quad g \in G$$

is also in $\mathbb{C}[G]$. The group G^Γ is a *weak extended group containing G* ([ABV, Definition 2.13]). A *strong real form of G^Γ* is an element $\delta \in G^\Gamma - G$ such that $\delta^2 \in G$ belongs to the centre $Z(G)$ and has finite order. The strong real form δ determines a real form $\text{Int}(\delta)$ and a corresponding group of real points

$$G(\mathbb{R}, \delta) = \{g \in G : \text{Int}(\delta)(g) = g\}.$$

Two strong real forms of G^Γ are *equivalent* if they are conjugate under G .

These definitions are enriched by the addition of a G -conjugacy class \mathcal{W} of a triple

$$(\delta_q, N, \chi) \tag{3}$$

in which δ_q is a strong real form of G^Γ , $N \subset G$ is a maximal unipotent subgroup normalized by δ_q , and χ is a unitary character of $N(\mathbb{R}, \delta_q)$ which is non-trivial on each simple restricted root subgroup. We fix such a \mathcal{W} . The pair (G^Γ, \mathcal{W}) is called an *extended group for G* ([ABV, Definition 1.12]). In essence, \mathcal{W} specifies a Whittaker datum for the quasisplit real form $G(\mathbb{R}, \delta_q)$.

The extended groups for G are classified in [ABV, Proposition 3.6]. There it is shown that two extended groups (G^Γ, \mathcal{W}) and $((G^\Gamma)', \mathcal{W}')$ for G are equivalent only if $\delta_q^2 = (\delta'_q)^2$ for any choices $(\delta_q, N, \chi) \in \mathcal{W}$ and $(\delta'_q, N, \chi) \in \mathcal{W}'$. This allows for inequivalent extended groups when the centre of G is non-trivial. Nevertheless, as noted on [ABV, p. 46], there appears to be no reason to prefer one extended group over another. Thus, from now on, we assume that $\delta_q^2 = 1 \in Z(G)$. We fix $(\delta_q, N, \chi) \in \mathcal{W}$ so that as an internal semidirect product

$$G^\Gamma = G \rtimes \langle \delta_q \rangle. \tag{4}$$

This also fixes the unique Borel subgroup $B \supset N$. Let $Z(G)^{\text{tor}}$ be the torsion subgroup of $Z(G)$, *i.e.* the elements of finite order. Let J be a subgroup of

$Z(G)^{\text{tor}}$. A strong real form $\delta \in G^\Gamma - G$ is of *type J* if $\delta^2 \in J$ ([ABV, Definition 10.10]). We say that it is a *pure real form* if it is of type $\{1\}$, i.e. $\delta^2 = 1$. Obviously, δ_q is a pure real form.

Kaletha defines analogues of strong real forms uniformly for groups over any local field of characteristic zero ([K1, Section 5.1]). These analogues are called rigid inner twists. We summarize and paraphrase his construction of rigid inner twists over the real local field. A *rigid inner twist* of (G, δ_q) is a pair (ψ, z) . The first element $\psi : G \rightarrow G'$ is a \mathbb{C} -isomorphism of algebraic groups in which the group G has the quasisplit \mathbb{R} -structure defined by $\text{Int}(\delta_q)$. The group G' is taken to have an \mathbb{R} -structure defined by σ' such that $\psi^{-1} \circ \sigma' \circ \psi \circ \text{Int}(\delta_q) = \text{Int}(g_\psi)$ for some $g_\psi \in G$. The second element defining the rigid inner twist is a 1-cocycle $z \in Z^1(W, G)$. Here, W is an extension of $\Gamma = \text{Gal}(\mathbb{C}/\mathbb{R})$

$$1 \rightarrow u \rightarrow W \rightarrow \Gamma \rightarrow 1$$

in which u is a pro-algebraic group ([K1, Sections 3.1-3.2]). The 1-cocycle is defined to satisfy two conditions. The first is that the restriction $z|_u$ is an algebraic homomorphism into a finite subgroup J of $Z(G)$, defined over \mathbb{R} . The second is that $\text{Int}(z(\sigma)) = \text{Int}(g_\psi)$, that is

$$\psi^{-1} \circ \sigma' \circ \psi \circ \text{Int}(\delta_q) = \text{Int}(z(\sigma)), \quad (5)$$

where σ is a canonical element in W which maps to the non-trivial element in Γ . In this case, the rigid inner twist (ψ, z) is said to be *realized by J*. Observe that the finiteness condition on the subgroup $J \subset Z(G)$ is stronger than the condition on J for strong real forms stated above. A rigid inner twist is *pure* if it is realized by $\{1\}$.

Let (ψ, z) and (ψ_1, z_1) be rigid inner twists of (G, δ_q) with $\psi : G \rightarrow G'$ and $\psi_1 : G \rightarrow G'_1$. An *isomorphism* between them is a pair (f, g) in which $f : G' \rightarrow G'_1$ is an \mathbb{R} -isomorphism and $g \in G$. This pair is required to satisfy:

1. $\psi_1 \circ \text{Int}(g) = f \circ \psi$
2. $z_1(w) = gz(w)w \cdot (g^{-1})$ for all $w \in W$ (W acts on G through Γ alone.)

The first property in the definition is equivalent to the commutativity of the diagram

$$\begin{array}{ccc} G & \xrightarrow{\psi} & G' \\ \text{Int}(g) \downarrow & & \downarrow f \\ G & \xrightarrow{\psi_1} & G'_1 \end{array}$$

Let us make the \mathbb{R} -structures of the groups in this diagram more explicit by writing

$$\begin{array}{ccc} (G, \text{Int}(\delta_q)) & \xrightarrow{\psi} & (G', \sigma') \\ \text{Int}(g) \downarrow & & \downarrow f \\ (G, \text{Int}(\delta_q)) & \xrightarrow{\psi_1} & (G'_1, \sigma'_1) \end{array}$$

Here, the Γ -action of the group in each pair is given by the automorphism in the second entry, and $f \circ \sigma' \circ f^{-1} = \sigma'_1$.

Lemma 2.1. *Let $\text{id} : G \rightarrow G$ be the identity map and $\sigma \in W$ be the canonical element mapping to the non-trivial element in Γ (5). Suppose (ψ, z) is a rigid inner twist of (G, δ_q) . Then $(\psi^{-1}, 1)$ is an isomorphism between (ψ, z) and (id, z) , where the \mathbb{R} -structure of G in the codomain of id is defined by $\text{Int}(z(\sigma)\delta_q)$.*

Proof. The second property in the definition of isomorphism of rigid inner twists is trivially satisfied. We must therefore show the first property of the definition for $f = \psi^{-1}$, i.e. show that

$$\begin{array}{ccc} (G, \text{Int}(\delta_q)) & \xrightarrow{\psi} & (G', \sigma') \\ \text{Int}(1) \downarrow & & \downarrow \psi^{-1} \\ (G, \text{Int}(\delta_q)) & \xrightarrow{\text{id}} & (G, \text{Int}(z(\sigma)\delta_q)) \end{array}$$

is a commutative diagram. The commutativity follows from (5). \square

Suppose (ψ, z) is a rigid inner twist of (G, δ_q) . According to [K1, Theorem 5.2] the map

$$(\psi, z) \mapsto z(\sigma)\delta_q \quad (6)$$

takes values in the set of strong real forms of G^Γ . In addition, the map passes to a bijection from isomorphism classes of rigid inner twists to equivalence classes of strong real forms. It carries rigid inner twists realized by $J \subset Z(G)$ to strong real forms of type J . In particular, it carries pure rigid inner twists to pure real forms.

2.1 Representations

A *representation of a strong real form of G^Γ* is a pair (π, δ) in which δ is a strong real form of G^Γ and π is an admissible representation of $G(\mathbb{R}, \delta)$. Two such representations (π, δ) and (π_1, δ_1) are *equivalent* if there exists $g \in G$ such that $\delta_1 = g\delta g^{-1}$ and π_1 is infinitesimally equivalent to $\pi \circ \text{Int}(g^{-1})$ ([ABV, Definition 2.13]).

In the same vein, a *representation of a rigid inner twist of (G, δ_q)* is a triple (ψ, z, π) in which $\psi : G \rightarrow G'$ is the \mathbb{C} -isomorphism of a rigid inner twist (ψ, z) and π is an admissible representation of $G'(\mathbb{R})$. Suppose (ψ_1, z_1, π_1) is another representation of a rigid inner twist of (G, δ_q) with $\psi_1 : G \rightarrow G'_1$. An *isomorphism* between (ψ, z, π) and (ψ_1, z_1, π_1) is an isomorphism (f, g) between (ψ, z) and (ψ_1, z_1) such that π_1 is infinitesimally equivalent to $\pi \circ f^{-1}$.

Lemma 2.1 implies that any representation (ψ, z, π) of a rigid inner twist of (G, δ_q) is isomorphic to $(\text{id}, z, \pi \circ \psi)$. Furthermore, the map (6) makes it clear that $(z(\sigma)\delta_q, \pi \circ \psi)$ is a representation of a strong real form of G^Γ . This defines a map

$$(\psi, z, \pi) \mapsto (z(\sigma)\delta_q, \pi \circ \psi) \quad (7)$$

from representations of rigid inner twists of (G, δ_q) to representations of strong real forms of G^Γ .

Proposition 2.2. *The map (7) passes to a bijection of isomorphism classes of representations of rigid inner twists of (G, δ_q) to equivalence classes of representations of strong real forms of G^Γ .*

Proof. Suppose (f, g) is an isomorphism between (ψ, z, π) and (ψ_1, z_1, π_1) . We must prove that $(z(\sigma)\delta_q, \pi \circ \psi)$ is equivalent to $(z_1(\sigma)\delta_q, \pi_1 \circ \psi_1)$. By definition

$$\pi \circ \psi = \pi_1 \circ f \circ \psi = \pi_1 \circ \psi_1 \circ \text{Int}(g)$$

and so

$$\begin{aligned} (z_1(\sigma)\delta_q, \pi_1 \circ \psi_1) &= (z_1(\sigma)\delta_q, \pi \circ \psi \circ \text{Int}(g^{-1})) \\ &\sim (g^{-1}z_1(\sigma)\delta_q g, \pi \circ \psi) \\ &= (g^{-1}z_1(\sigma) \text{Int}(\delta_q)(g) \delta_q, \pi \circ \psi). \end{aligned}$$

The action of $\sigma \in W$ on G is by $\text{Int}(\delta_q)$. The second property in the definition of the isomorphism (f, g) therefore implies

$$(g^{-1}z_1(\sigma) \text{Int}(\delta_q)(g) \delta_q, \pi \circ \psi) = (z(\sigma)\delta_q, \pi \circ \psi),$$

completing the proof of the equivalence. This proves that (7) passes to a map from isomorphism classes to equivalence classes.

For surjectivity, suppose (δ, π) is a representation of a strong real form of G^Γ . Using the surjectivity of (6) and Lemma 2.1 we may take $\delta = z(\sigma)\delta_q$ for a rigid inner twist (id, z) in which the codomain of id has \mathbb{R} -structure defined by $\text{Int}(z(\sigma)\delta_q)$. This ensures that (id, z, π) is a representation of a rigid inner twist of (G, δ_q) . It is obvious that (id, z, π) maps to (δ, π) under (7).

For injectivity, we may assume without loss of generality that (id, z, π) and (id, z_1, π_1) map to (δ, π) and (δ_1, π_1) respectively. If there exists $g \in G$ such that

$$(\delta_1, \pi_1) = (g\delta g^{-1}, \pi \circ \text{Int}(g^{-1}))$$

then it is straightforward to show that (id, g) is an isomorphism between (id, z, π) and (id, z_1, π_1) . \square

2.2 Pairings for real algebraic tori

Pairings between equivalence classes of strong real forms of tori and certain component groups appear in both [ABV] and [K1]. We describe the pairings first in the context of [ABV] and then show that the pairings of [K1] agree with them when working over the real numbers.

Let T be a complex algebraic torus defined over \mathbb{R} , *i.e.* with a Γ -action. We define the extended group

$$T = T \rtimes \langle \delta_q \rangle, \quad \mathcal{W} = \{ \delta_q \},$$

where $\delta_q^2 = 1$ and δ_q acts on T by the non-trivial element $\sigma \in \Gamma$. Set $T(\mathbb{R}) = T(\mathbb{R}, \delta_q)$. The strong real forms of T are the elements of the form $t\delta_q \in T^\Gamma$ for which

$$(t\delta_q)^2 = t\delta_q(t)\delta_q^2 = t\delta_q(t)$$

is of finite order. Clearly, $\text{Int}(t\delta_q) = \text{Int}(\delta_q)$ and so

$$T(\mathbb{R}, t\delta_q) = T(\mathbb{R})$$

for any strong real form of T^Γ .

Let ${}^\vee T^\Gamma = {}^\vee T \rtimes \langle {}^\vee \delta_q \rangle$ be the L-group of T . Just as for the extended group, conjugation by any element ${}^\vee T^\Gamma - {}^\vee T$ defines the same automorphism ${}^\vee \theta = \text{Int}({}^\vee \delta_q)$ of ${}^\vee T$. The transpose ${}^\vee \theta^\tau$ of ${}^\vee \theta$ defines an automorphism of $X^*({}^\vee T) = X_*(T)$. The Lie algebra of T may be identified with $(X_*(T) \otimes \mathbb{C})/X_*(T)$ via the map $\exp(2\pi i \cdot)$. Under this identification, the differential of the action of $\sigma \in \Gamma$ is equal to ${}^\vee \theta^\tau$ ([ABV, Lemma 9.9 (d)]).

According to [ABV, Lemma 9.9, Proposition 9.10], the map

$$\lambda_1 \mapsto \exp(\pi i \lambda_1) (1 - \sigma)T, \quad \lambda_1 \in (X_*(T) \otimes \mathbb{Q})^{-\sigma} / (1 - \sigma)X_*(T) \quad (8)$$

defines an isomorphism onto the group

$$\{t \in T : t\delta_q(t) \text{ is of finite order}\} / (1 - \sigma)T.$$

In consequence, the map

$$\lambda_1 \mapsto \exp(\pi i \lambda_1) \delta_q, \quad \lambda_1 \in (X_*(T) \otimes \mathbb{Q})^{-\sigma} / (1 - \sigma)X_*(T) \quad (9)$$

passes to a bijection onto the set of equivalence classes of strong real forms of T^Γ .

The restriction of (8) to $X_*(T)^{-\sigma}$ produces an isomorphism from

$$X_*(T)^{-\sigma} / (1 - \sigma)X_*(T)$$

onto

$$\{t \in T : t\delta_q(t) = 1\} / (1 - \sigma)T = \{t \in T : t\sigma(t) = 1\} / (1 - \sigma)T \cong H^1(\Gamma, T)$$

which is the set of equivalence classes of pure real forms of T^Γ . Tate-Nakayama duality identifies the latter group with the characters of the component group

$${}^\vee T^{\vee \theta} / ({}^\vee T^{\vee \theta})^0$$

([K4, (3.3.1)]). In summary,

$$\begin{aligned} \text{Equivalence classes of pure real forms} &\cong X_*(T)^{-\sigma} / (1 - \sigma)X_*(T) & (10) \\ &\cong \text{Hom}\left({}^\vee T^{\vee \theta} / ({}^\vee T^{\vee \theta})^0, \mathbb{C}^\times\right). \end{aligned}$$

If one identifies the equivalence classes of pure real forms with the groups on the right, then one may equally well say that there is a perfect pairing between ${}^\vee T^{\vee\theta}/({}^\vee T^{\vee\theta})^0$ and the set of equivalence classes of pure real forms.

We would like a similar pairing for equivalence classes of strong real forms. Towards this end, we already have isomorphism (9). For the second isomorphism we need to introduce the *algebraic universal covering* ${}^\vee T^{\text{alg}}$ of ${}^\vee T$, which is the projective limit of all finite coverings of ${}^\vee T$. The algebraic universal covering is part of a short exact sequence

$$1 \rightarrow \pi_1({}^\vee T)^{\text{alg}} \rightarrow {}^\vee T^{\text{alg}} \rightarrow {}^\vee T \rightarrow 1 \quad (11)$$

in which $\pi_1({}^\vee T)^{\text{alg}}$ is a profinite group ([ABV, (5.10)]). According to [ABV, Proposition 9.8 (c)]

$$\begin{aligned} \text{Equivalence classes of strong real forms} &\cong (X_*(T) \otimes \mathbb{Q})^{-\sigma}/(1-\sigma)X_*(T) \\ &\cong \text{Hom}\left({}^\vee T^{\vee\theta}/({}^\vee T^{\vee\theta})^0, \mathbb{C}^\times\right). \end{aligned} \quad (12)$$

Here, the group $({}^\vee T^{\vee\theta})^{\text{alg}}$ is the preimage of ${}^\vee T^{\vee\theta} \subset {}^\vee T$ in (11). The isomorphisms of (12) amount to a perfect pairing between $({}^\vee T^{\vee\theta})^{\text{alg}}/({}^\vee T^{\vee\theta})^0$ and the set of equivalence classes of strong real forms.

Our next task is to determine the correct pairing for equivalence classes of strong real forms of type $J \subset T^{\text{tor}}$. In order to compare with rigid inner twists, we restrict to the case that J is finite and defined over \mathbb{R} . Under this assumption, equivalence classes of strong real forms of type J are a subset of (12) which is related to an isogeny of T . Set $\bar{T} = T/J$. As J is defined over \mathbb{R} , it is stable under δ_q , and so we may define $\bar{T}^\Gamma = \bar{T} \rtimes \langle \delta_q \rangle$. It is an algebraic torus which is related to T through the isogeny

$$\iota : T \rightarrow T/J.$$

The dual isogeny

$$\hat{\iota} : {}^\vee \bar{T} \rightarrow {}^\vee T$$

yields an algebraic cover

$$1 \rightarrow \ker \hat{\iota} \rightarrow {}^\vee \bar{T} \rightarrow {}^\vee T \rightarrow 1.$$

As $\ker \hat{\iota}$ is finite, there is a surjective map

$$\pi_1({}^\vee T)^{\text{alg}} \rightarrow \ker \hat{\iota}.$$

This surjection induces the commutative diagram

$$\begin{array}{ccccccc} 1 & \longrightarrow & \pi_1({}^\vee T)^{\text{alg}} & \longrightarrow & {}^\vee T^{\text{alg}} & \longrightarrow & {}^\vee T \longrightarrow 1 \\ & & \downarrow & & \downarrow & & \downarrow \cong \\ 1 & \longrightarrow & \ker \hat{\iota} & \longrightarrow & {}^\vee \bar{T} & \longrightarrow & {}^\vee T \longrightarrow 1 \end{array}$$

with surjective columns and exact rows ([ABV, 10.10]).

Lemma 2.3. *The kernel of $\hat{\iota}$ is isomorphic to $\hat{J} = \text{Hom}(J, \mathbb{C}^\times)$.*

Proof. The dual of the short exact sequence

$$1 \rightarrow J \rightarrow T \xrightarrow{\iota} \bar{T} \rightarrow 1$$

yields the short exact sequence

$$1 \rightarrow X^*(\bar{T}) \xrightarrow{\iota^*} X^*(T) \rightarrow \hat{J} \rightarrow 1$$

([B2, Section 8.12]). This implies in turn that $\hat{J} \cong X^*(T)/\iota^*X^*(\bar{T})$ and

$$J \cong \text{Hom}(X^*(T)/\iota^*X^*(\bar{T}), \mathbb{C}^\times). \quad (13)$$

Identical reasoning applied to the dual isogeny implies that

$$\ker \hat{\iota} \cong \text{Hom}(X_*(\bar{T})/\iota^*X_*(T), \mathbb{C}^\times),$$

where

$$\iota^* : X_*(T) \rightarrow X_*(\bar{T}) \quad (14)$$

is the transpose of ι^* . The perfect \mathbb{Z} -pairing between $X_*(T)$ and $X^*(T)$ allows us to rewrite this isomorphism as

$$\ker \hat{\iota} \cong \text{Hom}(\text{Hom}(X^*(\bar{T}), \mathbb{Z})/\iota^*\text{Hom}(X^*(T), \mathbb{Z}), \mathbb{C}^\times).$$

Looking back to (13) and applying duality, we are reduced to proving

$$X^*(T)/\iota^*X^*(\bar{T}) \cong \text{Hom}(\text{Hom}(X^*(\bar{T}), \mathbb{Z})/\iota^*\text{Hom}(X^*(T), \mathbb{Z}), \mathbb{C}^\times). \quad (15)$$

This is a statement about a quotient of $X^*(T) \cong \mathbb{Z}^n$ by a full-rank sublattice. By choosing \mathbb{Z} -bases for $X^*(T)$ and $X^*(\bar{T})$, we identify both groups with \mathbb{Z}^n and ι^* as an endomorphism of \mathbb{Z}^n . It is an elementary exercise to show that $\mathbb{Z} \cong \text{Hom}(\mathbb{Z}, \mathbb{Z})$, that ι^* has the same invariant factors, m_1, \dots, m_n , as the transpose of ι^* , and that the invariant factors do not depend on the choice of bases. Consequently, isomorphism (15) is equivalent to

$$\bigoplus_{j=1}^n \mathbb{Z}/m_j\mathbb{Z} \cong \text{Hom}(\bigoplus_{j=1}^n \mathbb{Z}/m_j\mathbb{Z}, \mathbb{C}^\times),$$

which follows from the well-known isomorphism

$$\mathbb{Z}/m_j\mathbb{Z} \cong \text{Hom}(\mathbb{Z}/m_j\mathbb{Z}, \mathbb{C}^\times).$$

□

This lemma allows us to denote ${}^\vee\bar{T}$ of [K1, Section 5.3] by ${}^\vee T^{\hat{J}}$. In this way

$$\begin{array}{ccccccc} 1 & \longrightarrow & \pi_1({}^\vee T)^{\text{alg}} & \longrightarrow & {}^\vee T^{\text{alg}} & \longrightarrow & {}^\vee T \longrightarrow 1 \\ & & \downarrow & & \downarrow & & \downarrow \cong \\ 1 & \longrightarrow & \hat{J} & \longrightarrow & {}^\vee T^{\hat{J}} & \longrightarrow & {}^\vee T \longrightarrow 1 \end{array}$$

which conforms with the notation of [ABV, (5.13)].

We shall see in a moment how the set of equivalence classes of strong real forms of T of type J is associated to $X_*(\bar{T})$. The precise statement depends on the injection (14), which is the bridge between the two relevant groups. It is possible to identify $X_*(T)$ with its image under (14) and this is done in [K1, Section 4.1]. Alternatively, one may extend \hat{i}^* to a vector space isomorphism

$$\hat{i}_{\mathbb{Q}}^* : X_*(T) \otimes \mathbb{Q} \rightarrow X_*(\bar{T}) \otimes \mathbb{Q},$$

and identify $X_*(\bar{T})$ with its preimage under $\hat{i}_{\mathbb{Q}}^*$ in $X_*(T) \otimes \mathbb{Q}$. We then set

$$\frac{X_*(\bar{T})^{-\sigma}}{(1-\sigma)X_*(\bar{T})} = (\hat{i}_{\mathbb{Q}}^*)^{-1} \left(\frac{X_*(\bar{T})^{-\sigma}}{(1-\sigma)X_*(\bar{T})} \right) \subset \frac{(X_*(T) \otimes \mathbb{Q})^{-\sigma}}{(1-\sigma)X_*(T)}. \quad (16)$$

We use this identification below.

Lemma 2.4. *The map (9) sends $X_*(\bar{T})^{-\sigma}/(1-\sigma)X_*(T)$ bijectively onto the set of equivalence classes of strong real forms of T^Γ of type J .*

Proof. Suppose $\bar{t}\delta_q$ is a strong real form of \bar{T}^Γ where $\bar{t} = tJ$ for some $t \in T$. Then $\bar{t}\delta_q(\bar{t})$ is of finite order. This in turn implies $(t\delta_q(t))^m \in J$ for some $m \geq 1$, $t\delta_q(t)$ has finite order, and $t\delta_q$ is a strong real form of T^Γ . Combining these observations with (9), we obtain the commutative diagram

$$\begin{array}{ccc} \frac{(X_*(T) \otimes \mathbb{Q})^{-\sigma}}{(1-\sigma)X_*(T)} & \longleftrightarrow & \text{equivalence classes strong real forms } T^\Gamma \\ \hat{i}_{\mathbb{Q}}^* \downarrow & & \downarrow /J \\ \frac{(X_*(\bar{T}) \otimes \mathbb{Q})^{-\sigma}}{(1-\sigma)X_*(T)} & \longleftrightarrow & \text{equivalence classes strong real forms } \bar{T}^\Gamma \end{array}$$

Arguing as above one sees that the strong real forms of T^Γ of type J map onto the pure real forms of \bar{T}^Γ upon taking the quotient by J . Therefore the previous diagram and (16) imply

$$\begin{array}{ccc} \frac{X_*(\bar{T})^{-\sigma}}{(1-\sigma)X_*(T)} & \longleftrightarrow & \text{equivalence classes strong real forms type } J \\ \hat{i}_{\mathbb{Q}}^* \downarrow & & \downarrow /J \\ \frac{X_*(\bar{T})^{-\sigma}}{(1-\sigma)X_*(T)} & \longleftrightarrow & \text{equivalence classes pure real forms } \bar{T}^\Gamma \end{array}$$

The top horizontal arrow of the latter diagram is the assertion of the lemma. \square

We now have a refinement of (12)

$$\begin{aligned} & \text{Equivalence classes of strong real forms of type } J \\ & \cong X_*(\bar{T})^{-\sigma}/(1-\sigma)X_*(T) \\ & \cong \text{Hom} \left(({}^\vee T^{\vee\theta})^J / (({}^\vee T^{\vee\theta})^J)^0, \mathbb{C}^\times \right) \end{aligned} \quad (17)$$

in which the bottom isomorphism is given by [ABV, Theorem 10.11] after writing ${}^\vee\bar{T}$ as ${}^\vee T^{\hat{J}}$. This is equivalent to a perfect pairing between $({}^\vee T^{\vee\theta})^{\hat{J}}/(({}^\vee T^{\vee\theta})^{\hat{J}})^0$ and the equivalence classes of real forms of type J .

An application of [K1, Corollary 5.4] to T produces a perfect pairing resembling the one we have just established. There are two apparent differences between the pairings. The first difference is the use of the torsion subgroup of $X_*(\bar{T})/(1-\sigma)X_*(T)$ in place of $X_*(\bar{T})^{-\sigma}/(1-\sigma)X_*(T)$ above. [K1, Fact 4.1] states that the two groups are actually equal, and this may be proven by decomposing a torsion element into $+1$ and -1 eigenvectors with respect to σ in $X_*(T) \otimes \mathbb{C}$.

The other apparent difference in the pairings is the isomorphism between $X_*(\bar{T})^{-\sigma}/(1-\sigma)X_*(T)$ and the equivalence classes of strong real forms of type J . The isomorphism is given in [K1, Theorem 4.8] and is rather intricate over p -adic fields. Fortunately, over the real numbers the setup simplifies considerably ([K1, Theorem 5.2]), and one may prove that the isomorphism is equal to (8). Without delving too far into the details, the proof runs as follows. In [K1, Section 4.6] the element $\lambda_1 \in X_*(\bar{T})^{-\sigma}$ (*cf.* (9)) is assigned to the the strong real form $t\delta_q$ in which

$$t = (l_k c_k \sqcup_{\mathbb{C}/\mathbb{R}} k! \lambda_1) (\sigma). \quad (18)$$

In this equation, $\sigma \in \Gamma$ is non-trivial, $k > 0$ is any integer divisible by $|J|$, l_k is the (usual) $k!$ -th root function on \mathbb{C}^\times , $c_k \in Z^2(\Gamma, \mathbb{C}^\times)$ is the 2-cocycle defining $W_{\mathbb{R}}$, and $\sqcup_{\mathbb{C}/\mathbb{R}}$ is a cup product defined in [K1, Section 4.3]. The formula for the computation of (18) appears at the end of [K1, Section 4.3] and one may compute that it is

$$k! \lambda_1(\exp(\pi i/k!)) = \lambda_1(\exp(\pi i)).$$

Since we are regarding $X_*(\bar{T})$ as a submodule of $X_*(T) \otimes \mathbb{Q}$, this element is regarded as an element in T . It is an elementary exercise to prove that $\lambda_1(\exp(\pi i))$ coincides with $\exp(\pi i \lambda_1)$ under the identification we have made for the Lie algebra of T . In consequence, the map $\lambda_1 \mapsto \lambda_1(\exp(\pi i))$ induces the same map as (8), and the first isomorphism of (17) coincides with Kaletha's. The second isomorphism of (17) also coincides with Kaletha's ([K1, Proposition 5.3]), so that the perfect pairings between $({}^\vee T^{\vee\theta})^{\hat{J}}/(({}^\vee T^{\vee\theta})^{\hat{J}})^0$ and the equivalence classes of real forms of type J in (17) and in [K1, Corollary 5.4] are the same.

3 L-packets in Adams-Barbasch-Vogan

In this section we review the definition of L-packets as given in [ABV]. In this framework the representations in an L-packet are representations of strong real forms, and usually run over different real forms of a given group. One may calibrate the size of an L-packet by specifying a subgroup $J \subset Z(G)^{\text{tor}}$ and considering only strong real forms of type J . We first complete the description of L-packets over *all* strong real forms, *i.e.* where $J = Z(G)^{\text{tor}}$. Towards the end of the section we shall discuss how the construction is affected in taking J to be a smaller subgroup.

An L-packet is determined by a ($\vee G$ -conjugacy class of an) L-homomorphism as in (1). The (equivalence classes of) irreducible representations which appear in the L-packet Π_ϕ^{ABV} are parameterized by characters of an abelian group. For this we need the algebraic universal covering ${}^\vee G^{\text{alg}}$ of ${}^\vee G$,

$$1 \rightarrow \pi_1({}^\vee G)^{\text{alg}} \rightarrow {}^\vee G^{\text{alg}} \rightarrow {}^\vee G \rightarrow 1 \quad (19)$$

in which $\pi_1({}^\vee G)^{\text{alg}}$ is a profinite group. Let ${}^\vee G_\phi$ be the isotropy subgroup of ${}^\vee G$ acting on ϕ under conjugation, *i.e.* the centralizer of $\phi(W_{\mathbb{R}})$ in ${}^\vee G$. Let ${}^\vee G_\phi^{\text{alg}}$ be the preimage of ${}^\vee G_\phi$ under (19), and let

$${}^\vee G_\phi^{\text{alg}} / ({}^\vee G_\phi^{\text{alg}})^0$$

be the component group of ${}^\vee G_\phi^{\text{alg}}$. The component group is abelian ([ABV, p. 61]). Let $({}^\vee G_\phi^{\text{alg}} / ({}^\vee G_\phi^{\text{alg}})^0)^\wedge$ be its group of characters. The elements of Π_ϕ^{ABV} are parameterized by the elements $\tau \in ({}^\vee G_\phi^{\text{alg}} / ({}^\vee G_\phi^{\text{alg}})^0)^\wedge$ and most of this section is devoted to the description of this parameterization.

For convenience, we fix Borel subgroups, $B \subset G$ and $\mathcal{B} \subset {}^\vee G$, and maximal tori $T \subset B$ and $\mathcal{T} \subset \mathcal{B}$. We may and do choose these subgroups to be stable under the Γ -actions of G^Γ and ${}^\vee G^\Gamma$ respectively. Write $W_{\mathbb{R}} = \mathbb{C}^\times \cup j\mathbb{C}^\times$. We may assume that $\phi : W_{\mathbb{R}} \rightarrow {}^\vee G^\Gamma$ is an L-homomorphism such that

$$\phi(z) = z^\lambda \bar{z}^{\text{Ad}(\phi(j))\lambda} \in \mathcal{T}, \quad z \in \mathbb{C}^\times \quad (20)$$

for some $\lambda \in X_*(\mathcal{T}) \otimes \mathbb{C}$. We identify the Lie algebra of \mathcal{T} with $(X_*(\mathcal{T}) \otimes \mathbb{C}) / X_*(\mathcal{T})$ via the map $\exp(2\pi i \cdot)$. Set $y = \exp(\pi i \lambda) \phi(j)$.

The pair (λ, y) attached to ϕ allows us to choose another maximal torus ${}^d T \subset {}^\vee G$ in the following manner. First, define the Levi subgroup

$${}^d L = {}^d L(\lambda, y \cdot \lambda) = \{g \in {}^\vee G : \text{Int}(g) \circ \lambda = \lambda, \text{Int}(g) \text{Int}(y) \circ \lambda = \text{Int}(y) \circ \lambda\} \subset {}^\vee G. \quad (21)$$

Evidently, \mathcal{T} is a maximal subtorus of ${}^d L$. Since $y^2 = \exp(2\pi i \lambda)$ ([ABV, Proposition 5.6]), it follows that y^2 is central in ${}^d L$. Therefore, in setting ${}^\vee \theta = \text{Int}(y)$ and restricting it to ${}^d L$, we obtain an involutive automorphism of ${}^d L$. By [ABV, Lemma 12.10] there exists a maximal torus ${}^d T \subset {}^d L$ with the property that $({}^d \mathfrak{t})^{-\vee \theta}$ is a maximal semisimple abelian subalgebra of $({}^d \mathfrak{t})^{-\vee \theta}$. The torus ${}^d T$ is unique up to conjugation by the identity component of the subgroup ${}^d K = ({}^d L)^{\vee \theta}$. One may interpret such a maximal torus as a maximally split torus of ${}^d L$.

As ${}^d L$ is a Levi subgroup of ${}^\vee G$, the torus ${}^d T$ is maximal in ${}^\vee G$ as well. Let ${}^d T^\Gamma$ be the group generated by y and ${}^d T$. It is not difficult to see that ϕ takes values in ${}^d T^\Gamma$ ([ABV, (12.11)]). Define

$$\phi_d : W_{\mathbb{R}} \rightarrow {}^d T^\Gamma$$

to be the resulting L-homomorphism for ${}^d T^\Gamma$.

This describes the passage from ϕ , a homomorphism associated to ${}^\vee G$, to ϕ_d , a homomorphism associated to ${}^d T$. We wish to accomplish the same thing for $\tau \in ({}^\vee G_\phi^{\text{alg}} / ({}^\vee G_\phi^{\text{alg}})^0)^\wedge$. The analogue of the component group for ${}^d T$ is defined by setting $({}^d T^{\vee\theta})^{\text{alg}, \vee G}$ to be the preimage of ${}^d T^{\vee\theta}$ in (19) and then considering its component group

$$({}^d T^{\vee\theta})^{\text{alg}, \vee G} / (({}^d T^{\vee\theta})^{\text{alg}, \vee G})^0. \quad (22)$$

The inclusion of $({}^d T^{\vee\theta})^{\text{alg}, \vee G}$ into ${}^\vee G_\phi^{\text{alg}}$ induces a surjection

$$({}^d T^{\vee\theta})^{\text{alg}, \vee G} / (({}^d T^{\vee\theta})^{\text{alg}, \vee G})^0 \rightarrow {}^\vee G_\phi^{\text{alg}} / ({}^\vee G_\phi^{\text{alg}})^0$$

([ABV, (12.11)(e)]). The kernel of this surjection is generated by the cosets of the elements

$${}^\vee \alpha(-1) \in ({}^d T^{\vee\theta})^{\text{alg}, \vee G} \quad (23)$$

([ABV, (12.6)(c), (12.11)(f)]), where $\alpha \in R({}^d L, {}^d T)$ satisfies $\alpha \circ {}^\vee \theta = -\alpha$ (i.e. α is ${}^\vee \theta$ -real). Note that the definition of ${}^d L$ forces these roots to satisfy $\langle \lambda, \alpha \rangle = 0$. In consequence, we call these the *real ϕ -singular* roots of $R({}^\vee G, {}^d T)$ and denote the set of them by $R_{\mathbb{R}}({}^d L, {}^d T)$. At this point, we abandon the ${}^\vee G$ in superscript and summarize by writing

$$({}^d T^{\vee\theta})^{\text{alg}} / (({}^d T^{\vee\theta})^{\text{alg}})^0 \langle {}^\vee \alpha(-1) : \alpha \in R_{\mathbb{R}}({}^d L, {}^d T) \rangle \cong {}^\vee G_\phi^{\text{alg}} / ({}^\vee G_\phi^{\text{alg}})^0. \quad (24)$$

This isomorphism induces an isomorphism of the character groups. We define τ_d to be the image of $\tau \in ({}^\vee G_\phi^{\text{alg}} / ({}^\vee G_\phi^{\text{alg}})^0)^\wedge$ under the isomorphism of character groups. We shall often identify τ_d with a character of $({}^d T^{\vee\theta})^{\text{alg}} / (({}^d T^{\vee\theta})^{\text{alg}})^0$ which is trivial on the cosets of ${}^\vee \alpha(-1)$, $\alpha \in R_{\mathbb{R}}({}^d L, {}^d T)$. So far we have described an assignment

$$(\phi, \tau) \mapsto (\phi_d, \tau_d). \quad (25)$$

The pair on the right is a *complete Langlands parameter of the Cartan subgroup* ${}^d T^\Gamma$ with respect to ${}^\vee G^\Gamma$ ([ABV, Definition 12.4]).

Using [ABV, Proposition 13.10 (b)] we obtain a Cartan subgroup $T^\Gamma \subset G^\Gamma$ ([ABV, Definition 12.1]) related to ${}^d T$. First off, T^Γ is a weak extended group of T , which is itself a maximal torus of G . The proposition also provides an element $\delta_d \in T^\Gamma$ such that $\delta_d = g\delta_q g^{-1}$ for some $g \in G$ (4). This implies that $(\delta_d, g^{-1}Ng, \chi \circ \text{Int}(g)) \in \mathcal{W}$ and that $G(\mathbb{R}, \delta_d)$ is a quasisplit real form of G . In addition, the proposition provides an isomorphism ${}^\vee T \cong {}^d T$ which identifies ${}^d T^\Gamma$ as an *E-group* of T ([ABV, Definition 4.6]). The classical Langlands correspondence matches ϕ_d with an irreducible representation π_d of $T(\mathbb{R}) = T(\mathbb{R}, \delta_d)$.

According to (12), there is a map from $({}^d T^{\vee\theta})^{\text{alg}} / (({}^d T^{\vee\theta})^{\text{alg}})^0$ to equivalence classes of strong real forms of T^Γ . According to [ABV, Proposition 13.12 (a)], the image of (22) under this map actually lies in the equivalence classes of strong real forms of G^Γ . The map is denoted by

$$\tau_d \mapsto \delta(\tau_d)$$

in [ABV, Proposition 13.12], and is chosen to send the trivial character to the equivalence class of δ_d , *i.e.* $\delta_d = \delta(1)$. We may progress from (25) by writing

$$(\phi, \tau) \mapsto (\phi_d, \tau_d) \mapsto (\pi_d, \delta(\tau_d)). \quad (26)$$

For the final step in this sequence of maps, observe that $T(\mathbb{R}, \delta(\tau_d)) = T(\mathbb{R})$. Hence, π_d is a representation of

$$T(\mathbb{R}, \delta(\tau_d)) \subset G(\mathbb{R}, \delta(\tau_d)). \quad (27)$$

This representation may be induced, cohomologically and parabolically, to a *standard* representation $M(\pi_d, \delta(\tau_d))$ of $G(\mathbb{R}, \delta(\tau_d))$ ([ABV, Section 11, Theorem 12.3, Proposition 13.2]). The standard representation has a unique Langlands quotient, which is an irreducible representation π_{τ_d} of $G(\mathbb{R}, \delta(\tau_d))$. The final step to be added to (26) is

$$(\phi, \tau) \mapsto (\phi_d, \tau_d) \mapsto (\pi_d, \delta(\tau_d)) \mapsto (\pi_{\tau_d}, \delta(\tau_d)). \quad (28)$$

The L-packet of ϕ is defined to consist of the distinct equivalence classes of the representations of strong real forms $(\pi_{\tau_d}, \delta(\tau_d))$, $\tau \in (\vee G_\phi^{\text{alg}} / (\vee G_\phi^{\text{alg}})^0)^\wedge$. Ignoring any indication of equivalences in our notation, we simply write

$$\Pi_\phi^{\text{ABV}} = \left\{ (\pi_{\tau_d}, \delta(\tau_d)) : \tau \in (\vee G_\phi^{\text{alg}} / (\vee G_\phi^{\text{alg}})^0)^\wedge \right\}.$$

We now consider a torsion subgroup $J \subset Z(G)^{\text{tor}}$ and the parameterization of the smaller L-packet defined by

$$\Pi_{\phi, J}^{\text{ABV}} = \left\{ (\pi_{\tau_d}, \delta(\tau_d)) : \tau \in (\vee G_\phi^{\text{alg}} / (\vee G_\phi^{\text{alg}})^0)^\wedge, \delta(\tau_d)^2 \in J \right\}.$$

There is a natural isomorphism between $Z(G)^{\text{tor}}$ and the group $(\pi_1(\vee G)^{\text{alg}})^\wedge$ of continuous characters of $\pi_1(\vee G)^{\text{alg}}$, the *Pontryagin dual* of $\pi_1(\vee G)^{\text{alg}}$ ([ABV, Lemma 10.9 (a)]). We may therefore identify J with a subgroup of $(\pi_1(\vee G)^{\text{alg}})^\wedge$ and write

$$J \hookrightarrow (\pi_1(\vee G)^{\text{alg}})^\wedge$$

Applying Pontryagin duality yields a surjection

$$\pi_1(\vee G)^{\text{alg}} \rightarrow \hat{J}$$

([F, Theorem 4.31, Corollary 4.41]). We form a profinite covering

$$1 \rightarrow \hat{J} \rightarrow \vee G^{\hat{J}} \rightarrow \vee G \rightarrow 1 \quad (29)$$

([ABV, (10.10)]), and repeat the earlier constructions replacing “alg” in superscript everywhere with \hat{J} . In particular, the component group becomes

$$\vee G_\phi^{\hat{J}} / (\vee G_\phi^{\hat{J}})^0.$$

According to [ABV, Theorem 10.11]

$$\Pi_{\phi, J}^{\text{ABV}} = \left\{ (\pi_{\tau_d}, \delta(\tau_d)) : \tau \in (\vee G_{\phi}^J / (\vee G_{\phi}^J)^0)^{\wedge} \right\}. \quad (30)$$

In anticipation of the comparison with the L-packets of the next section, let us rephrase inclusion (27) in the language of embeddings when the strong real forms $\delta(\tau_d)$ are of type J , where $J \subset Z(G)$ is a finite subgroup defined over \mathbb{R} . As in Lemma 2.1, the group $G(\mathbb{R}, \delta(\tau_d))$ is the group of real points of a rigid inner twist (id, z_{τ_d}) for which $z_{\tau_d}(\sigma) = \delta(\tau_d)\delta_q$. Inclusion (27) may be written as an \mathbb{R} -embedding

$$\eta_{\tau_d} : T \rightarrow G$$

in which the \mathbb{R} -structure of G is defined by $\delta(\tau_d)$ (or equivalently z_{τ_d}). It makes sense to call η_{τ_d} an *embedding of T into a strong real form of (or rigid inner twist of) G of type J* (cf. [K1, p. 594]).

Keeping Lemma 2.1 in mind, the discussion on [K1, pp. 593-594] reveals that the map

$$\delta(\tau_d) \rightarrow \eta_{\tau_d} \quad (31)$$

passes to a bijection from the set of equivalence classes of strong real forms of T^{Γ} of type J to set of equivalence classes of embeddings of T into strong real forms of G of type J . Here, the two embeddings $\eta_{\tau_1}, \eta_{\tau_2}$ are equivalent if there exists $g \in G$ such that $\text{Int}(g)$ is defined over \mathbb{R} and $z_{\tau_2}(\sigma) = g z_{\tau_1}(\sigma) \delta_q(g^{-1})$ (cf. [K1, p. 591]).

We conclude by making some cosmetic changes to the correspondences of (28). In our current context the correspondences may be written as

$$(\phi, \tau) \mapsto (\phi_d, \tau_d) \mapsto (\pi_d, \eta_{\tau_d}) \mapsto (\pi(\eta_{\tau_d}), \delta(\tau_d)). \quad (32)$$

In this notation the L-packet (30) becomes

$$\Pi_{\phi, J}^{\text{ABV}} = \left\{ (\pi(\eta_{\tau_d}), \delta(\tau_d)) : \tau \in (\vee G_{\phi}^J / (\vee G_{\phi}^J)^0)^{\wedge} \right\}. \quad (33)$$

Alternatively, we may highlight the torus ${}^d T$ and write

$$\Pi_{\phi, J}^{\text{ABV}} = \left\{ (\pi(\eta_{\tau_d}), \delta(\tau_d)) : \tau_d \in \left(({}^d T^{\vee\theta})^{\hat{J}} / ({}^d T^{\vee\theta})^{\hat{J}})^0 \langle \vee \alpha(-1) : \alpha \in R_{\mathbb{R}}({}^d L, {}^d T) \rangle \right)^{\wedge} \right\}. \quad (34)$$

Here, $({}^d T^{\vee\theta})^{\hat{J}}$ is the preimage of ${}^d T^{\vee\theta}$ under (29). The identification of the character groups follows from (24) and [ABV, Definition 10.10].

4 Tempered L-packets of Kaletha, Langlands and Shelstad

The scope of this section is narrower than the previous section. We begin with a *tempered* L-homomorphism $\phi : W_{\mathbb{R}} \rightarrow \vee G^{\Gamma}$ rather than an arbitrary

one. Moreover, we fix $J \subset Z(G)^{\text{tor}}$ to be a *finite* subgroup defined over \mathbb{R} . Otherwise, we continue with the same setup as Section 3. Our goal is to review the parameterization of the tempered L-packet $\Pi_{\phi, J}^{\text{KLS}}$ attached to ϕ and J as given in [K1, Section 5.6]. The presentation in [K1, Section 5.6] itself contains a review of the work of Shelstad in [S]. We shall only present the results of Shelstad and Kaletha necessary for the comparison with Section 3 later on.

We begin with an L-homomorphism as in (20) whose image is bounded. Let $y = \exp(\pi i \lambda) \phi(j)$ as before, and define ${}^\vee\theta = \text{Int}(y)$. Attached to ϕ is a root subsystem

$${}^\vee\Delta_\phi \subset R({}^\vee G, \mathcal{T}).$$

When ${}^\vee\Delta_\phi$ is non-empty, it is of Dynkin type $A_1 \times \cdots \times A_1$. The automorphism ${}^\vee\theta$ fixes every root in ${}^\vee\Delta_\phi$ and negates any root vector X_α in the root space of $\alpha \in {}^\vee\Delta_\phi$. Thinking of ${}^\vee\theta$ as a Cartan involution of ${}^\vee G$, we say that every root in ${}^\vee\Delta_\phi$ is imaginary and noncompact ([ABV, (12.5), (12.7)]). In addition, all roots $\alpha \in {}^\vee\Delta_\phi$ satisfy $\langle \lambda, {}^\vee\alpha \rangle = 0$, that is to say all roots are ϕ -singular ([ABV, Definition 12.4]).

Shelstad determines a new maximal torus of ${}^\vee G$, essentially by taking Cayley transforms of \mathcal{T} with respect to the roots of ${}^\vee\Delta_\phi$ ([K3, Section VI.7]). More precisely, for specific choices of root vectors $X_{\pm\alpha}$, Shelstad defines elements

$$c_\alpha = \exp\left(\frac{\pi i}{4}(X_\alpha + X_{-\alpha})\right), \quad \alpha \in {}^\vee\Delta_\phi. \quad (35)$$

For each $\alpha \in {}^\vee\Delta_\phi$ the element $w_\alpha = (c_\alpha)^2$ is a representative of the simple reflection of α in the Weyl group. By the properties of ${}^\vee\Delta_\phi$ listed above, these elements are seen to commute with one another, $\text{Ad}(c_\alpha)$ is seen to act trivially on λ , and ${}^\vee\theta(c_\alpha) = c_\alpha^{-1}$. Choose a positive system ${}^\vee\Delta_\phi^+$ and set

$$c = \prod_{\alpha \in {}^\vee\Delta_\phi^+} c_\alpha, \quad \mathcal{T}_1 = c\mathcal{T}c^{-1}. \quad (36)$$

We call the various actions induced by conjugation with c the *Cayley transform with respect to* ${}^\vee\Delta_\phi$. The roots in $c{}^\vee\Delta_\phi \subset R({}^\vee G, \mathcal{T}_1)$ are of the form $c\alpha = c_\alpha\alpha$ for $\alpha \in {}^\vee\Delta_\phi$, and we compute

$${}^\vee\theta(c\alpha) = c_\alpha^{-1}({}^\vee\theta\alpha) = c_\alpha(c_\alpha)^{-2}\alpha = -c\alpha.$$

In other words, the Cayley transform converts the imaginary roots of ${}^\vee\Delta_\phi$ into real roots $c{}^\vee\Delta_\phi$ in $R({}^\vee G, \mathcal{T}_1)$ ([ABV, (12.5)]).

The ϕ -singularity of the roots in ${}^\vee\Delta_\phi$ ensures that $c \cdot \lambda = \lambda$ and so $\phi(\mathbb{C}^\times) \subset \mathcal{T}_1$. In addition,

$$\phi(j) \cdot \mathcal{T}_1 = {}^\vee\theta(c \cdot \mathcal{T}) = c^{-1} \cdot {}^\vee\theta(\mathcal{T}) = c^{-1} \cdot \mathcal{T} = c \cdot c^{-2} \cdot \mathcal{T} = \mathcal{T}_1.$$

This proves that ϕ takes values in the group \mathcal{T}_1^Γ generated by \mathcal{T}_1 and $\phi(j)$ (or y). We define

$$\phi_1 : W_{\mathbb{R}} \rightarrow \mathcal{T}_1^\Gamma \quad (37)$$

to be the L-homomorphism given by changing the codomain of ϕ to \mathcal{T}_1^Γ . We could define ϕ_1 equally well by conjugating ϕ by c as in [K1, Section 5.6], and leaving \mathcal{T} unchanged. However, for the purposes of comparison later on, we prefer to record the changes effected by the Cayley transform in the maximal tori instead.

The representations in the the L-packet of G are obtained from representations in the L-packet of an intermediate Levi subgroup M_1 . The dual group ${}^\vee M_1$ of M_1 is defined by the root system

$$\begin{aligned} R({}^\vee M_1, \mathcal{T}_1) &= \{\alpha \in R({}^\vee G, \mathcal{T}_1) : {}^\vee \theta(\alpha) = -\alpha\} \\ &= \{c\alpha : {}^\vee \alpha \in R({}^\vee G, \mathcal{T}), \phi(j) \cdot \alpha = -c^2 \alpha\}. \end{aligned} \quad (38)$$

The group ${}^\vee M_1$ is a Levi subgroup of ${}^\vee G$ which corresponds to a Levi subgroup M_1 of G using the Borel pairs $T \subset B$ and $\mathcal{T}_1 \subset c\mathcal{B}c^{-1}$ ([B1, Section 3]). Furthermore, M_1 has a real structure defined by δ_q . Since M_1 is preserved by δ_q , we may define

$$M_1^\Gamma = M_1 \rtimes \langle \delta_q \rangle \subset G^\Gamma.$$

We continue in making our way to a description of the intermediate L-packet $\Pi_{\phi_1, J}^{M_1}$. As λ is not necessarily M_1 -regular, we make a detour by shifting to a regular infinitesimal character, describing the shifted L-packet, and then recovering $\Pi_{\phi_1, J}^{M_1}$ from the shifted L-packet. Let $\nu \in X_*(\mathcal{T}_1)$ be a strictly dominant element with respect to $c\mathcal{B}c^{-1} \cap {}^\vee M_1$ and let $\phi_\nu : W_\mathbb{R} \rightarrow \mathcal{T}_1^\Gamma$ be the L-homomorphism obtained from ϕ_1 by replacing λ with $\lambda + \nu$. (*cf.* (20)). Then ϕ_ν is a discrete L-parameter for M_1 . As in the previous section, [ABV, Proposition 13.10] pairs \mathcal{T}_1^Γ with a Cartan subgroup $T_1^\Gamma \subset M_1^\Gamma$. Again, as in the previous section, this inclusion into M_1^Γ yields an \mathbb{R} -embedding

$$\eta_{\mathcal{W}} : T_1 \rightarrow M_1 \quad (39)$$

where the \mathbb{R} -structure of T_1 and M_1 are given by δ_q . Moreover, this embedding together with the L-homomorphism ϕ_ν determines (the equivalence class of) a representation $\pi_\nu^{M_1}(\eta_{\mathcal{W}})$ which is generic with respect to the Whittaker datum \mathcal{W}_{M_1} inherited from \mathcal{W} ([ABV, Lemma 14.11]). The representation $\pi_\nu^{M_1}(\eta_{\mathcal{W}})$ is in the essential discrete series of $M_1(\mathbb{R}, \delta_q)$, and the pair $(\pi_\nu^{M_1}(\eta_{\mathcal{W}}), \delta_q)$ is a representation of a strong real form of type J (in fact it is pure).

Recall from (31) that there is a bijection between the set of equivalence classes of embeddings of T_1 into strong real forms of M_1 of type J and the set of equivalence classes of strong real forms of T_1^Γ of type J . Combining this bijection with bijection (17), we recover the L-packet

$$\Pi_{\phi_\nu, J}^{M_1} = \left\{ (\pi_\nu^{M_1}(\eta_{\tau_1}), \delta(\tau_1)) : \tau \in (({}^\vee M_1)_{\phi_\nu}^j / (({}^\vee M_1)_{\phi_\nu}^j)^0)^\wedge \right\}. \quad (40)$$

This is just the L-packet (33) for the group M_1 and the discrete L-parameter ϕ_ν . Nevertheless, some explanation of the terms is in order. The group $({}^\vee M_1)_{\phi_\nu}$ is the centralizer in ${}^\vee M_1$ of $\phi_\nu(W_\mathbb{R})$, and $({}^\vee M_1)_{\phi_\nu}^j$ is its preimage in the covering

group ${}^\vee M_1^{\hat{J}}$. (By Lemma 2.3, this covering is also the dual of $\overline{M}_1 = M_1/J$.) There is an isomorphism between $(\mathcal{T}_1^{\vee\theta})^{\hat{J}}/((\mathcal{T}_1^{\vee\theta})^{\hat{J}})^0$ and $({}^\vee M_1)_{\phi_\nu}^{\hat{J}}/(({}^\vee M_1)_{\phi_\nu}^{\hat{J}})^0$ (cf. (24)). The character τ_1 of the former group is identified with τ under this isomorphism.

The L-packet $\Pi_{\phi_1, J}^{M_1}$ is defined from $\Pi_{\phi_\nu, J}^{M_1}$ by translating the representations in $\Pi_{\phi_\nu, J}^{M_1}$ to the infinitesimal character λ using Jantzen-Zuckerman translation ([V1, Definition 4.5.7]). Each translate is either zero or an irreducible representation of a strong real form of type J . By [K1, p. 619], the characters of $({}^\vee M_1)_{\phi_\nu}^{\hat{J}}/(({}^\vee M_1)_{\phi_\nu}^{\hat{J}})^0$ which correspond to non-zero translates are precisely those characters which are trivial on the cosets of ${}^\vee\alpha(-1)$, where $\alpha \in c^\vee\Delta_\phi$ (cf. (23)). Consequently, the L-packet for the group M_1 and the merely tempered L-parameter ϕ_1 is

$$\Pi_{\phi_1, J}^{M_1} = \left\{ (\pi^{M_1}(\eta_{\tau_1}), \delta(\tau_1)) : \tau_1 \in \left((\mathcal{T}_1^{\vee\theta})^{\hat{J}}/((\mathcal{T}_1^{\vee\theta})^{\hat{J}})^0 \langle {}^\vee\alpha(-1) : \alpha \in c^\vee\Delta_\phi \rangle \right)^\wedge \right\}. \quad (41)$$

The representations in this set may either be thought of as the translates of the representations of (40), or equivalently as the standard tempered representations defined by ϕ_1 and the embeddings η_{τ_1} ([ABV, Proposition 11.18 (a)]).

The final step in the description of $\Pi_{\phi, J}^{\text{KLS}}$ is the parabolic induction

$$\pi(\eta_\tau) = \text{ind}_{M_1(\mathbb{R}, \delta(\tau_1))}^{G(\mathbb{R}, \delta(\tau_1))} \pi^{M_1}(\eta_{\tau_1}) \quad (42)$$

of each (equivalence class of a) representation occurring in (41). Shelstad proves that the resulting representations are irreducible and tempered. Observe that $\pi(\eta_{\tau_1})$ is a standard representation, so the notation conforms with that of the previous section. At last, we write the packet as

$$\Pi_{\phi, J}^{\text{KLS}} = \left\{ (\pi(\eta_{\tau_1}), \delta(\tau_1)) : \tau_1 \in \left((\mathcal{T}_1^{\vee\theta})^{\hat{J}}/((\mathcal{T}_1^{\vee\theta})^{\hat{J}})^0 \langle {}^\vee\alpha(-1) : \alpha \in c^\vee\Delta_\phi \rangle \right)^\wedge \right\}. \quad (43)$$

There is an obvious map from $(\mathcal{T}_1^{\vee\theta})^{\hat{J}}/((\mathcal{T}_1^{\vee\theta})^{\hat{J}})^0$ into ${}^\vee G_\phi^{\hat{J}}/({}^\vee G_\phi^{\hat{J}})^0$. By [K1, Proposition 5.9], this map is surjective with kernel generated by $\langle {}^\vee\alpha(-1) : \alpha \in c^\vee\Delta_\phi \rangle$. One may therefore also write

$$\Pi_{\phi, J}^{\text{KLS}} = \left\{ (\pi(\eta_{\tau_1}), \delta(\tau_1)) : \tau \in ({}^\vee G_\phi^{\hat{J}}/({}^\vee G_\phi^{\hat{J}})^0)^\wedge \right\}$$

where τ and τ_1 correspond under the isomorphism of character groups.

5 The comparison of tempered L-packets

In this section we are working under the assumptions of Section 4. In particular, the L-homomorphism ϕ as given in (20) is tempered, and the subgroup $J \subset Z(G)$ is finite and defined over \mathbb{R} .

5.1 A comparison of component groups

Consider the two groups

$$({}^d T^{\vee\theta})^{\hat{J}} / ({}^d T^{\vee\theta})^{\hat{J}0} \langle \vee \alpha(-1) : \alpha \in R_{\mathbb{R}}({}^d L, {}^d T) \rangle \quad (44)$$

and

$$(\mathcal{T}_1^{\vee\theta})^{\hat{J}} / ((\mathcal{T}_1^{\vee\theta})^{\hat{J}})^0 \langle \vee \alpha(-1) : \alpha \in c^{\vee} \Delta_{\phi} \rangle \quad (45)$$

appearing in (34) and (43) respectively. Each of these two groups is canonically isomorphic to ${}^{\vee} G_{\phi}^{\hat{J}} / ({}^{\vee} G_{\phi}^{\hat{J}})^0$ and therefore there is a canonical isomorphism between the two of them. The goal of this section is to describe the latter isomorphism without reference to ${}^{\vee} G_{\phi}^{\hat{J}} / ({}^{\vee} G_{\phi}^{\hat{J}})^0$.

Recall that ${}^d T$ was chosen to be a maximal torus of ${}^d L$ (21).

Lemma 5.1. *The maximal torus \mathcal{T}_1 is a subgroup of ${}^d L$.*

Proof. Recall that $\mathcal{T} \subset {}^d L$, $\mathcal{T}_1 = c\mathcal{T}c^{-1}$ (36), and $\langle \alpha, \lambda \rangle = 0$ for all $\alpha \in {}^{\vee} \Delta_{\phi}$. The final property and the definition of the Cayley transform imply that $\text{Int}(c) \circ \lambda = \lambda$. We must also prove that $\text{Int}(y)(c)$ also commutes with λ . This follows from $\text{Int}(y)(c) = {}^{\vee} \theta(c) = c^{-1}$. \square

We now see that both ${}^d T$ and \mathcal{T}_1 are maximal tori in ${}^d L$. The torus ${}^d T$ is defined by the property that ${}^d \mathfrak{t}^{-\vee\theta}$ is a maximal semisimple abelian subalgebra of ${}^d \mathfrak{t}^{-\vee\theta}$. If \mathcal{T}_1 satisfies the defining property of ${}^d T$ then we may take $\mathcal{T}_1 = {}^d T$ and the canonical isomorphism between (44) and (45) reduces to the identity map.

On the other hand, if \mathcal{T}_1 does not satisfy the defining property then $R({}^d L, \mathcal{T}_1)$ has an imaginary noncompact root β ([ABV, Lemma 12.10]), and the maximal torus $c_{\beta} \mathcal{T}_1 c_{\beta}^{-1} \subset {}^d L$ has a larger $(-\vee\theta)$ -fixed subalgebra. Proceeding recursively, we arrive to a product c' of Cayley transforms, and a maximal torus $c' \mathcal{T}_1 (c')^{-1} \subset {}^d L$ which satisfies the defining property of ${}^d T$. Without loss of generality,

$${}^d T = c' \mathcal{T}_1 (c')^{-1}. \quad (46)$$

The goal then is to describe the canonical isomorphism between (44) and (45) under this hypothesis.

Maintaining the dual notation only serves to clutter the exposition here. We shall therefore write the preliminary results in the context where G is a connected complex reductive algebraic group, θ is an involutive automorphism of G , and T is a θ -stable maximal subtorus of G .

Lemma 5.2. *Suppose $\beta \in R(G, T)$ is imaginary and noncompact with respect to θ . Let c_{β} be the Cayley transform with respect to β (35), and set $c_{\beta} \cdot T = c_{\beta} T c_{\beta}^{-1}$. Then*

- (a) $\ker(\beta) \subset T \cap (c_{\beta} \cdot T)$ and $\ker(\beta|_{T^{\theta}}) \subset (T \cap (c_{\beta} \cdot T))^{\theta}$.
- (b) $\ker(\beta|_{(T^{\theta})^0}) = ((c_{\beta} \cdot T)^{\theta})^0 \langle \vee (c_{\beta} \beta)(-1) \rangle$.

(c) The previous assertions induce a monomorphism

$$\ker(\beta|_{T^\theta}) / \ker(\beta|_{(T^\theta)^0}) \hookrightarrow (c_\beta \cdot T)^\theta / ((c_\beta \cdot T)^\theta)^0 \langle \vee(c_\beta\beta)(-1) \rangle.$$

(d) There is a natural isomorphism

$$T^\theta / (T^\theta)^0 \rightarrow \ker(\beta|_{T^\theta}) / \ker(\beta|_{(T^\theta)^0})$$

and so a monomorphism

$$T^\theta / (T^\theta)^0 \hookrightarrow (c_\beta \cdot T)^\theta / ((c_\beta \cdot T)^\theta)^0 \langle \vee(c_\beta\beta)(-1) \rangle.$$

Proof. Suppose $t \in T$ and $\beta(t) = 1$. Then $\text{Ad}(t)$ fixes both X_β and $X_{-\beta}$, and so $t = c_\beta t c_\beta^{-1} \in T \cap (c_\beta \cdot T)$. The first assertion follows. For (b), we note that $(T^\theta)^0 = (1 + \theta)T$ and $\theta(\beta) = \beta$. As a result

$$\begin{aligned} \ker(\beta|_{(T^\theta)^0}) &= \{t\theta(t) : t \in T, \beta(t\theta(t)) = 1\} \\ &= \{t\theta(t) : t \in T, \beta(t)^2 = 1\} \\ &= \{t\theta(t) : t \in T, \beta(t) = 1\} \cup \{t\theta(t) : t \in T, \beta(t) = -1\} \\ &= (1 + \theta)(\ker \beta) \cup \{t\theta(t) : t \in T, \beta(t) = -1\} \end{aligned} \quad (47)$$

We consider the group $(1 + \theta)(\ker \beta)$ first. Let us show that it is connected. Let Y be in the Lie algebra $\mathfrak{t} = (X_*(T) \otimes \mathbb{C}) / X_*(T)$ and denote the root of the Lie algebra corresponding to β by $d\beta$. Suppose $\exp(2\pi i Y) \in \ker \beta$. This is equivalent to $d\beta(Y) \in \mathbb{Z}$. We may decompose Y as

$$Y = \frac{d\beta(Y)}{2} \vee \beta + Y_\perp \in \frac{1}{2} \mathbb{Z} \vee \beta \oplus \ker d\beta.$$

Clearly, $\exp(\ker d\beta)$ is connected, so the component group of $\ker \beta$ is generated by elements of the form

$$\exp\left(2\pi i \frac{k}{2} \vee \beta\right) = \exp(\pi i k \vee \beta) = (\vee \beta(-1))^k, \quad k \in \mathbb{Z}.$$

The component group is thus generated by $\vee \beta(-1)$. We recall that β is imaginary so that $\theta \circ \vee \beta = \vee \beta$. This implies $(1 + \theta)(\vee \beta(-1)) = \vee \beta(-1)^2 = 1$, and we deduce that $(1 + \theta)(\ker \beta)$ is connected.

As $(1 + \theta)(\ker \beta)$ is connected, it is also a torus. The first assertion tells us that it is a subtorus of the torus $((c_\beta \cdot T)^\theta)^0$. The dimensions of the tori are equal as may be seen from

$$\begin{aligned} \dim(1 + \theta)(\ker \beta) &= \dim \ker(d\beta|_{\mathfrak{t}^\theta}) \\ &= (\dim \mathfrak{t}^\theta) - 1 \\ &= \dim(c_\beta \mathfrak{t})^\theta \\ &= \dim((c_\beta \cdot T)^\theta)^0 \end{aligned}$$

([K3, (6.65b), Proposition 6.69]). Consequently, the first set in (47) is $((c_\beta \cdot T)^\theta)^0$.

To compute the second set in (47), suppose $t \in T$ and $\beta(t) = -1$. Observe that ${}^\vee\beta(i) \in T$ and $\beta({}^\vee\beta(i)) = i^{\langle \beta, {}^\vee\beta \rangle} = -1$. Setting $t_1 = t {}^\vee\beta(i)^{-1}$, we see that $t = t_1 {}^\vee\beta(i)$, where $\beta(t_1) = 1$. Moreover,

$$t\theta(t) = (t_1\theta(t_1)) ({}^\vee\beta(i)\theta({}^\vee\beta(i))).$$

We have already proven that $(t_1\theta(t_1)) \in ((c_\beta \cdot T)^\theta)^0$. For the second term of the product, we compute $({}^\vee\beta(i)\theta({}^\vee\beta(i))) = {}^\vee\beta(-1)$, and since $\beta({}^\vee\beta(-1)) = (-1)^2 = 1$, we have

$${}^\vee\beta(-1) = c_\beta {}^\vee\beta(-1)c_\beta^{-1} = {}^\vee(c_\beta\beta)(-1).$$

This proves (b). Assertion (c) follows immediately.

For the isomorphism of (d), consider

$${}^\vee\beta(\beta(t)^{1/2}), \quad t \in T.$$

We compute

$$\beta\left({}^\vee\beta(\beta(t)^{1/2})\right) = \beta(t)^{\langle \beta, {}^\vee\beta \rangle/2} = \beta(t).$$

In addition, as β is imaginary, ${}^\vee\beta(\beta(t)^{1/2}) \in T^\theta$. In fact,

$${}^\vee\beta(\beta(t)^{1/2}) \in {}^\vee\beta(\mathbb{C}^\times) \subset (T^\theta)^0.$$

Suppose that $t \in T^\theta$ and set $t_1 = t {}^\vee\beta(\beta(t)^{-1/2})$. Then $t_1 \in T^\theta$ and $\beta(t_1) = 1$. It follows that $t(T^\theta)^0 = t_1(T^\theta)^0$ and in turn that

$$\begin{aligned} T^\theta / (T^\theta)^0 &\cong (\ker \beta|_{T^\theta})(T^\theta)^0 / (T^\theta)^0 \\ &\cong \ker \beta|_{T^\theta} / ((T^\theta)^0 \cap \ker \beta|_{T^\theta}) \\ &\cong \ker \beta|_{T^\theta} / \ker \beta|_{(T^\theta)^0}. \end{aligned}$$

The final monomorphism is now a consequence of this isomorphism and part (c). \square

We may apply Lemma 5.2 repeatedly. Suppose $\beta_1 \in R(G, T)$ and $\beta_2 \in R(G, c_{\beta_1} \cdot T)$ satisfy the hypotheses of Lemma 5.2 for the respective tori. Since $c_{\beta_1}\beta_1 \in R(G, c_{\beta_1} \cdot T)$ is real with respect to θ and β_2 is imaginary, the two roots are orthogonal. Consequently,

$$\beta_2 ({}^\vee(c_{\beta_1}\beta_1)(-1)) = (-1)^{\langle \beta_2, {}^\vee(c_{\beta_1}\beta_1) \rangle} = (-1)^0 = 1,$$

and

$${}^\vee(c_{\beta_1}\beta_1)(-1) = c_{\beta_2} {}^\vee(c_{\beta_1}\beta_1(-1)) c_{\beta_2}^{-1} \in c_{\beta_2}c_{\beta_1} \cdot T.$$

Similar arguments imply that $c_{\beta_2}c_{\beta_1}\beta_1 = c_{\beta_1}\beta_1$ is a real root in $R(G, c_{\beta_2}c_{\beta_1} \cdot T)$. Applying Lemma 5.2 twice, we obtain monomorphisms

$$\begin{aligned} T^\theta / (T^\theta)^0 &\hookrightarrow (c_{\beta_1} \cdot T)^\theta / ((c_{\beta_1} \cdot T)^\theta)^0 \langle {}^\vee(c_{\beta_1}\beta_1)(-1) \rangle \\ &\hookrightarrow (c_{\beta_2}c_{\beta_1} \cdot T)^\theta / ((c_{\beta_2}c_{\beta_1} \cdot T)^\theta)^0 \langle {}^\vee(c_{\beta_1}\beta_1)(-1), {}^\vee(c_{\beta_2}\beta_2)(-1) \rangle. \end{aligned}$$

Arguing inductively, we obtain the following corollary.

Corollary 5.3. *Suppose $c' \cdot T$ is the maximal torus obtained by iterated Cayley transforms $c' = c_{\beta_m} \cdots c_{\beta_1}$ from T by imaginary noncompact roots β_1, \dots, β_m with respect to θ . Then there is a natural monomorphism*

$$T^\theta / (T^\theta)^0 \hookrightarrow (c' \cdot T)^\theta / ((c' \cdot T)^\theta)^0 \langle {}^\vee(c_{\beta_j} \beta_j)(-1) : 1 \leq j \leq m \rangle.$$

Proposition 5.4. *Suppose c' in (46) is the iterated composition of Cayley transforms of imaginary noncompact roots β_1, \dots, β_m . Then the following inclusion diagram is commutative.*

$$\begin{array}{ccc}
\frac{\mathcal{T}_1^{\vee\theta}}{(\mathcal{T}_1^{\vee\theta})^0} & \hookrightarrow & \frac{{}^d\mathcal{T}^{\vee\theta}}{({}^d\mathcal{T}^{\vee\theta})^0 \langle {}^\vee(c_{\beta_j} \beta_j)(-1) : 1 \leq j \leq m \rangle} \\
\downarrow & & \downarrow \\
\frac{\mathcal{T}_1^{\vee\theta}}{(\mathcal{T}_1^{\vee\theta})^0 \langle {}^\vee\alpha(-1) : \alpha \in c^\vee \Delta_\phi \rangle} & \xrightarrow{\cong} & \frac{{}^d\mathcal{T}^{\vee\theta}}{({}^d\mathcal{T}^{\vee\theta})^0 \langle {}^\vee\alpha(-1) : \alpha \in R_{\mathbb{R}}({}^dL, {}^d\mathcal{T}) \rangle} \\
& \searrow \cong & \swarrow \cong \\
& & {}^\vee G_\phi / ({}^\vee G_\phi)^0
\end{array}$$

Proof. The top horizontal arrow is an application of Corollary 5.3 to the algebraic group dL containing the tori \mathcal{T}_1 and ${}^d\mathcal{T}$. The left vertical arrow is obvious. The right vertical arrow is clear once one recalls that the roots $c_{\beta_j} \beta_j$ are all real with respect to ${}^\vee\theta$. The bottom two arrows were addressed in Sections 3 and 4. The lower horizontal arrow is induced by the upper horizontal arrow since the roots of $c^\vee \Delta_\phi$ are real and are not altered by the imaginary Cayley transforms. In this way the diagram is commutative. The lower horizontal arrow is an isomorphism by virtue of the lower two isomorphisms. \square

We would like a version of Proposition 5.4 which encompasses the covering groups (29) for our finite central subgroup J . An application of Lemma 2.3 with T a maximal torus in G implies that ${}^\vee G^J = {}^\vee \overline{G}$, where $\overline{G} = G/J$. Let \mathcal{T}_1^J and ${}^d\mathcal{T}^J$ be the respective preimages of \mathcal{T}_1 and ${}^d\mathcal{T}$ under (29). These preimages are maximal tori in ${}^\vee \overline{G}$. There is an obvious bijection between $R({}^\vee G, \mathcal{T}_1)$ and $R({}^\vee \overline{G}, \mathcal{T}_1^J)$. These kinds of bijections allow one to lift the Cayley transforms on maximal tori in ${}^\vee G$ to Cayley transforms in ${}^\vee \overline{G}$. In particular, (46) lifts to an equation

$$\mathcal{T}_1^J = c' \cdot {}^d\mathcal{T}^J. \quad (48)$$

The following lemma is an extension of Lemma 5.2 to the setting of covers.

Lemma 5.5. *Suppose*

$$1 \rightarrow J \rightarrow G^J \rightarrow G \rightarrow 1$$

is a covering of G in which G^J is a connected reductive algebraic group with a finite central subgroup J . For any subgroup $H \subset G$ let H^J denote its preimage in G^J . Suppose $\beta \in R(G, T)$ is imaginary and noncompact with respect to θ and identify β with its corresponding root in $R(G^J, T^J)$. Let c_β be the Cayley transform with respect to β (35), and set $c_\beta \cdot T^J = c_\beta T^J c_\beta^{-1}$. Then

- (a) $(\ker(\beta|_{T^\theta}))^J \subset ((T \cap (c_\beta \cdot T))^\theta)^J$.
- (b) $(T^\theta)^J = (\ker(\beta|_{T^\theta}))^J \vee \beta(\mathbb{C}^\times)$
- (c) $(\ker(\beta|_{T^\theta}))^J \cap ((T^\theta)^J)^0 = (((c_\beta \cdot T)^\theta)^J)^0 \langle \vee (c_\beta \beta)(-1) \rangle$.
- (d) *The previous assertions induce a monomorphism*
- $$(\ker(\beta|_{T^\theta}))^J / (\ker(\beta|_{T^\theta}))^J \cap ((T^\theta)^J)^0 \hookrightarrow ((c_\beta \cdot T)^\theta)^J / (((c_\beta \cdot T)^\theta)^J)^0 \langle \vee (c_\beta \beta)(-1) \rangle.$$
- (e) *There is a natural isomorphism*

$$(T^\theta)^J / ((T^\theta)^J)^0 \rightarrow (\ker(\beta|_{T^\theta}))^J / (\ker(\beta|_{T^\theta}))^J \cap ((T^\theta)^J)^0$$

and so a monomorphism

$$(T^\theta)^J / ((T^\theta)^J)^0 \hookrightarrow ((c_\beta \cdot T)^\theta)^J / (((c_\beta \cdot T)^\theta)^J)^0 \langle \vee (c_\beta \beta)(-1) \rangle.$$

Proof. The first assertion follows from Lemma 5.2 (a). For (b) suppose $t \in (T^\theta)^J$. Let \bar{t} denote the image of t in T . By definition, \bar{t} is fixed by θ . Set $t_1 = t \vee \beta(\beta(t)^{-1/2})$. Then $\bar{t} = \bar{t}_1 \vee \beta(\beta(\bar{t})^{1/2})$, and following the arguments in the proof Lemma 5.2, we find $t_1 \in (\ker(\beta|_{T^\theta}))^J$. This implies $(T^\theta)^J \subset (\ker(\beta|_{T^\theta}))^J \vee \beta(\mathbb{C}^\times)$. The reverse inclusion follows from β being imaginary. By substituting (b) into the left-hand side of (c), we obtain

$$(\ker(\beta|_{T^\theta}))^J \cap ((T^\theta)^J)^0 = (\ker(\beta|_{T^\theta}))^J \cap ((\ker(\beta|_{T^\theta}))^J)^0 \vee \beta(\mathbb{C}^\times).$$

The intersection of $\vee \beta(\mathbb{C}^\times)$ with $(\ker(\beta|_{T^\theta}))^J$ is $\vee \beta(\pm 1)$. Furthermore, as seen in the proof of Lemma 5.2, $\vee \beta(\pm 1) = \vee (c_\beta \beta)(\pm 1)$. Thus,

$$(\ker(\beta|_{T^\theta}))^J \cap ((T^\theta)^J)^0 = ((\ker(\beta|_{T^\theta}))^J)^0 \langle \vee (c_\beta \beta)(-1) \rangle.$$

Clearly, $(\ker(\beta|_{T^\theta}))^J \supset (\ker(\beta|_{(T^\theta)_0}))^J$ and the two groups have the same dimension. This implies that $((\ker(\beta|_{T^\theta}))^J)^0 = ((\ker(\beta|_{(T^\theta)_0}))^J)^0$. Therefore Lemma 5.2 (b) implies

$$((\ker(\beta|_{T^\theta}))^J)^0 = \left((((c_\beta \cdot T)^\theta)^0 \langle \vee (c_\beta \beta)(-1) \rangle)^J \right)^0 = ((c_\beta \cdot T)^J)^0.$$

This proves (c).

Part (d) is immediate from parts (a) and (c). Part (e) follows the proof of Lemma 5.2 (d). \square

The following proposition ensues from Lemma 5.4 just as Proposition 5.3 ensues from Lemma 5.2.

Proposition 5.6. *Suppose c' in (48) is the iterated composition of imaginary noncompact roots β_1, \dots, β_m . Then the following inclusion diagram is commutative.*

$$\begin{array}{ccc}
\frac{(\mathcal{T}_1^{\vee\theta})^J}{((\mathcal{T}_1^{\vee\theta})^J)^0} \hookrightarrow & \xrightarrow{\quad} & \frac{({}^d\mathcal{T}^{\vee\theta})^J}{(({}^d\mathcal{T}^{\vee\theta})^J)^0 \langle \vee (c_{\beta_j} \beta_j) (-1) : 1 \leq j \leq m \rangle} \\
\downarrow & & \downarrow \\
\frac{(\mathcal{T}_1^{\vee\theta})^J}{((\mathcal{T}_1^{\vee\theta})^J)^0 \langle \vee \alpha(-1) : \alpha \in c \vee \Delta_\phi \rangle} & \xrightarrow{\cong} & \frac{({}^d\mathcal{T}^{\vee\theta})^J}{(({}^d\mathcal{T}^{\vee\theta})^J)^0 \langle \vee \alpha(-1) : \alpha \in R_{\mathbb{R}}({}^dL)^J, ({}^d\mathcal{T})^J \rangle} \\
& \searrow \cong & \swarrow \cong \\
& & (\vee G_\phi)^{\hat{J}} / ((\vee G_\phi)^{\hat{J}})^0
\end{array}$$

5.2 A comparison of strong real forms of type J

In Sections 3 and 4 the dual groups of (44) and (45) are placed in bijection with sets of strong real forms of type J . This requires a pairing between each of the two maximal tori ${}^d\mathcal{T}$ and \mathcal{T}_1 . In Section 3 we denoted the torus paired with ${}^d\mathcal{T}$ by $T \subset G$, and in Section 4 we denoted the torus paired with \mathcal{T}_1 by $T_1 \subset G$. Equation (46) relates ${}^d\mathcal{T}$ to \mathcal{T}_1 through Cayley transforms. We shall establish a parallel relationship between the paired tori T and T_1 . Once this is complete, the diagram dual to Proposition 5.6 together with (17) produce a map between strong real forms of T and T_1 .

We continue by reviewing the manner in which \mathcal{T}_1 (or any other maximal torus of $\vee G$) is paired with a maximal torus $T_1 \subset G$. This is presented in [ABV, Section 13-14]. We use the constructions of Section 4. Starting with the Cartan subgroup $\mathcal{T}_1^\Gamma \subset \vee G^\Gamma$ (cf. (37)), one chooses a positive system $R_{\mathbb{R}}^+(\vee G, \mathcal{T}_1)$, for the real roots with respect to $\vee\theta$, making $\lambda \in X_*(\mathcal{T})$ dominant. We arbitrarily fix a positive system $R_{i\mathbb{R}}^+(\vee G, \mathcal{T}_1)$ for the set of imaginary roots with respect to $\vee\theta$. The triple $(\mathcal{T}_1^\Gamma, R_{i\mathbb{R}}^+(\vee G, \mathcal{T}_1), R_{\mathbb{R}}^+(\vee G, \mathcal{T}_1))$ determines a unique *based Cartan subgroup* for the L-group $\vee G^\Gamma$ ([ABV, Definition 13.7, Proposition 13.8]). There is a pairing between this based Cartan subgroup and a *based Cartan subgroup* of (G^Γ, \mathcal{W}) ([ABV, Proposition 13.10]). The latter is a quadruple

$$(T_1^\Gamma, \mathcal{W}(T_1^\Gamma), R_{i\mathbb{R}}^+(G, T_1), R_{\mathbb{R}}^+(G, T_1)) \quad (49)$$

in which T_1^Γ is a Cartan subgroup of G^Γ ([ABV, Definition 12.1]), $\mathcal{W}(T_1^\Gamma)$ is a T_1 -conjugacy class of an element in $T_1^\Gamma - T_1$, and the last two terms are positive systems for the imaginary and real roots respectively, relative to any element in $\mathcal{W}(T_1^\Gamma)$. This quadruple satisfies additional properties ([ABV, Definition 13.5]), and is determined uniquely up to conjugacy by G . After possibly conjugating by an element of G , we may assume that $\mathcal{W}(T_1^\Gamma)$ is the T_1 -conjugacy class of the quasisplit strong real form δ_q so that $T_1^\Gamma = T_1 \rtimes \langle \delta_q \rangle$ (4).

The pairing between \mathcal{T}_1 and T_1 is an isomorphism

$$\zeta : \vee T_1 \rightarrow \mathcal{T}_1 \quad (50)$$

which transports the Galois action on T_1 compatibly to the Galois action of \mathcal{T}_1 , carries the roots of T_1 to the coroots of \mathcal{T}_1 , carries imaginary roots $R_{i\mathbb{R}}^+(G, T_1)$ to the positive real coroots of \mathcal{T}_1 , and carries the real roots $R_{\mathbb{R}}^+(G, T_1)$ to the positive imaginary coroots of \mathcal{T}_1 ([ABV, Definition 13.9]).

The pairing (50), together with the map (9), yields a bijection between the strong real forms of T_1 of type J and the group $\left((\mathcal{T}_1^{\vee\theta})^{\hat{J}} / ((\mathcal{T}_1^{\vee\theta})^{\hat{J}})^0 \right)^\wedge$ (cf. (17)).

By regarding the dual of (45) as subgroup of $\left((\mathcal{T}_1^{\vee\theta})^{\hat{J}} / ((\mathcal{T}_1^{\vee\theta})^{\hat{J}})^0 \right)^\wedge$, we obtain the strong real forms $\delta(\tau)$ of type J appearing in L-packet (43). We wish to compare these strong real forms $\delta(\tau)$ with those appearing in the L-packet (34).

To facilitate this comparison, we wish to express the maximal torus T , which is paired with ${}^d T$, as an iterated Cayley transform of T_1 . The first step in this direction is to construct a based Cartan subgroup from an iterated Cayley transform of T_1 parallel to (46). We shall achieve this one Cayley transform at a time.

Suppose $\beta \in R_{i\mathbb{R}}({}^d L, T_1)$ is an imaginary noncompact root. Then the pairing ζ sends β to a real coroot of T_1 . This coroot corresponds to a unique real root in $R_{\mathbb{R}}(G, T_1)$. To keep the notation from getting out of hand, we also denote this real root by β . The context should make it clear whether $\beta \in R_{i\mathbb{R}}({}^d L, T_1)$ or $\beta \in R_{\mathbb{R}}(G, T_1)$.

The root $\beta \in R_{\mathbb{R}}(G, T_1)$ is orthogonal to λ and satisfies a *parity condition* relative to the character of $T_1(\mathbb{R}, \delta_q)$ determined by (37) ([ABV, Proposition 13.12]). The parity condition is the one appearing in the Hecht-Schmid character identity theorem ([ABV, pp. 129-130]).

Since $\beta \in R_{\mathbb{R}}(G, T_1)$ is real, one may apply a Cayley transform $d_\beta \in G$ to obtain a maximal torus $d_\beta T_1 d_\beta^{-1} \subset G$ which is more compact than T_1 ([K3, (6.67)]). Computations similar to those in Section 4 reveal that $d_\beta \delta_q d_\beta^{-1} \delta_q^{-1} = d_\beta^2$ is a representative for the simple reflection w_β of β in the Weyl group, and therefore that δ_q normalizes $d_\beta T_1 d_\beta^{-1}$. Thus, we may define

$$(d_\beta T_1 d_\beta^{-1})^\Gamma = d_\beta T_1 d_\beta^{-1} \rtimes \langle \delta_q \rangle. \quad (51)$$

This is a Cartan subgroup of G^Γ ([ABV, Definition 12.1]). Let $R_{i\mathbb{R}}^+(G, d_\beta T_1 d_\beta^{-1})$ be any system of positive imaginary roots containing $R_{i\mathbb{R}}^+(G, T_1)$ and $d_\beta \beta$. Let $R_{\mathbb{R}}^+(G, d_\beta T_1 d_\beta^{-1})$ be any positive system of real roots. Changes in the choice of positive roots do not affect the constructions below, up to equivalence (see proof of [ABV, Proposition 13.13]).

The parity condition and the Hecht-Schmid character identity theorem are pertinent to the irreducible standard representations $\pi(\eta_{\tau_1})$ (42) appearing in the L-packet (43). The centralizer in $G(\mathbb{R}, \delta(\tau_1))$ of the split component of $(d_\beta T_1 d_\beta^{-1})(\mathbb{R})$ yields a real Levi subgroup $M_\beta \supsetneq M_1$ (38). The Hecht-Schmid character identity ([SV, pp. 264-265]) and the irreducibility of $\pi(\eta_{\tau_1})$ provide an identity of the form

$$\mathrm{ind}_{M_1(\mathbb{R}, \delta(\tau_1))}^{M_\beta(\mathbb{R}, \delta(\tau_1))} \pi^{M_1}(\eta_{\tau_1}) = \pi^{M_\beta}(\eta_{\tau_1}) \quad (52)$$

The term on the left is the parabolically induced character (equivalence class of a representation) of $\pi^{M_1}(\eta_{\tau_1})$ (cf. (41)). The term on the right is the character of an irreducible essential limit of discrete series representation on $M_\beta(\mathbb{R}, \delta(\tau_1))$. In addition, we have an equivalence

$$\pi(\eta_{\tau_1}) = \text{ind}_{M_\beta(\mathbb{R}, \delta(\tau_1))}^{G(\mathbb{R}, \delta(\tau_1))} \pi^{M_\beta}(\eta_{\tau_1})$$

of standard representations. The difference in perspective is not entirely apparent from our notation. The Hecht-Schmid identity converts the \mathbb{R} -embedding $\eta_{\tau_1} : T_1 \rightarrow G$ on the left into the identically denoted \mathbb{R} -embedding $\eta_{\tau_1} : d_\beta T_1 d_\beta^{-1} \rightarrow G$ on the right (see [K2, Theorem 14.71]).

Lemma 5.7. *The quadruple*

$$\left((d_\beta T_1 d_\beta^{-1})^\Gamma, \text{Int}(d_\beta T_1 d_\beta^{-1})(\delta_q), R_{\mathbb{R}}^+(G, d_\beta T_1 d_\beta^{-1}), R_{\mathbb{R}}^+(G, d_\beta T d_\beta^{-1}) \right)$$

is a based Cartan subgroup of (G^Γ, \mathcal{W}) .

Proof. We are required to verify conditions (a)-(e) of [ABV, Definition 13.5]. The only conditions which are not obviously satisfied are (c) and (e). Condition (c) states that every element in the $d_\beta T_1 d_\beta^{-1}$ -conjugacy class of δ_q (51) appears as the first entry in some triple in \mathcal{W} as given in Section 2. It suffices to show that δ_q appears as the first entry of a triple in \mathcal{W} , and this is true by (3).

Condition (e) states that every standard representation of $G(\mathbb{R}, \delta_q)$ induced from a character on $d_\beta T_1 d_\beta^{-1}(\mathbb{R})$, dominant with respect to $R_{\mathbb{R}}^+(G, d_\beta T_1 d_\beta^{-1})$, has a Whittaker model with respect to \mathcal{W} . The discussion on [ABV, pp. 161-162] reduces this condition to proving that the limit of discrete series representation $\pi^{M_\beta}(\eta_{\mathcal{W}})$ has a Whittaker model with respect to the Whittaker datum \mathcal{W}_{M_β} inherited from \mathcal{W} ((39), [ABV, Lemma 14.11]). Since (49) is a based Cartan subgroup for (G, \mathcal{W}) the same discussion entails that $\pi^{M_1}(\eta_{\mathcal{W}})$ has a Whittaker model with respect to \mathcal{W}_{M_1} . By (52) we have $\pi^{M_\beta}(\eta_{\mathcal{W}}) = \text{ind}_{M_1(\mathbb{R}, \delta(\tau))}^{M_\beta(\mathbb{R}, \delta(\tau))} \pi^{M_1}(\eta_{\mathcal{W}})$, and so ([ABV, Lemma 14.11]) implies that $\pi^{M_\beta}(\eta_{\mathcal{W}})$ has a Whittaker model with respect to \mathcal{W}_{M_β} . \square

We would now like to define a pairing between the based Cartan subgroup in Lemma 5.7 and a based Cartan subgroup containing $c_\beta \mathcal{T}_1 c_\beta^{-1}$. For this, we use the pairing ζ (50) between the based Cartan subgroups containing T_1 and \mathcal{T}_1 . The isomorphism ζ may be regarded as a pair of isomorphisms, $X^*(T_1) \cong X_*(\mathcal{T})$ and $X_*(T_1) \cong X^*(\mathcal{T}_1)$. Regarded in this manner we define

$$\zeta_\beta : d_\beta T d_\beta^{-1} \xrightarrow{\cong} c_\beta \mathcal{T}_1 c_\beta^{-1} \tag{53}$$

through the commutative diagrams

$$\begin{array}{ccc} X^*(T_1) & \xrightarrow{\zeta} & X_*(\mathcal{T}_1) \\ \text{Int}(d_\beta) \uparrow & & \downarrow \text{Int}(c_\beta) \\ X^*(d_\beta T_1 d_\beta^{-1}) & \xrightarrow{\zeta_\beta} & X_*(c_\beta \mathcal{T}_1 c_\beta^{-1}) \end{array} \quad \begin{array}{ccc} X_*(T_1) & \xrightarrow{\zeta} & X^*(\mathcal{T}_1) \\ \text{Int}(d_\beta) \uparrow & & \downarrow \text{Int}(c_\beta) \\ X_*(d_\beta T_1 d_\beta^{-1}) & \xrightarrow{\zeta_\beta} & X^*(c_\beta \mathcal{T}_1 c_\beta^{-1}) \end{array}$$

Recall that ζ carries the Galois action on T_1 compatibly to the Galois action on \mathcal{T}_1 . To be more precise, [ABV, Proposition 2.12] converts the antiholomorphic Galois action of conjugation by δ_q on T_1 into a holomorphic action $a_{T_1} \in \text{Aut}(T_1)$. In the present case,

$$a_{T_1}(\lambda_1) = \overline{\lambda_1 \circ \delta_q^{-1}}, \quad \lambda_1 \in X^*(T_1).$$

Conjugation by ${}^\vee\delta_q$ on \mathcal{T}_1 is already holomorphic and is denoted by $a_{\mathcal{T}_1} \in \text{Aut}(\mathcal{T}_1)$. The compatibility condition is

$$\zeta \circ a_{T_1} \circ \zeta^{-1} = a_{\mathcal{T}_1}.$$

An implicit consequence of this compatibility condition is that ζ carries real roots in $R(G, T_1)$ to imaginary coroots of $({}^\vee G, \mathcal{T}_1)$, and carries imaginary roots in $R(G, T_1)$ to real coroots of $({}^\vee G, \mathcal{T}_1)$.

Let $R_{\mathbb{R}}^+({}^\vee G, c_\beta \mathcal{T}_1 c_\beta^{-1})$ be the set of roots corresponding to the imaginary coroots $\zeta_\beta(R_{\mathbb{R}}^+(G, d_\beta T_1 d_\beta^{-1}))$. Let $R_{\mathbb{R}}^+({}^\vee G, c_\beta \mathcal{T}_1 c_\beta^{-1})$ be the set of roots corresponding to the real coroots $\zeta_\beta(R_{\mathbb{R}}^+(G, d_\beta T_1 d_\beta^{-1}))$.

Lemma 5.8. *The isomorphism ζ_β carries the Galois action of $(d_\beta T_1 d_\beta^{-1})^\Gamma$ compatibly to the Galois action on $(c_\beta \mathcal{T}_1 c_\beta^{-1})^\Gamma$. Moreover, this compatibility extends to a pairing ([ABV, Definition 13.9]) between the based Cartan subgroup of Lemma 5.7 and the based Cartan subgroup of ${}^\vee G$ determined by*

$$\left((c_\beta \mathcal{T}_1 c_\beta^{-1})^\Gamma, R_{\mathbb{R}}^+({}^\vee G, c_\beta \mathcal{T}_1 c_\beta^{-1}), R_{\mathbb{R}}^+({}^\vee G, c_\beta \mathcal{T}_1 c_\beta^{-1}) \right).$$

Proof. We compute for any $\lambda_1 \in X^*(T_1)$ that

$$\begin{aligned} (a_{d_\beta T_1 d_\beta^{-1}}(d_\beta \cdot \lambda_1)) \circ \text{Int}(d_\beta) &= \overline{\lambda_1 \circ \text{Int}(d_\beta^{-1}) \circ \delta_q^{-1} \circ \text{Int}(d_\beta)} \\ &= \overline{\lambda_1 \circ \text{Int}(w_\beta) \circ \delta_q^{-1}} \\ &= \overline{(w_\beta \cdot \lambda_1) \circ \delta_q^{-1}} \\ &= a_{T_1}(w_\beta \cdot \lambda_1). \end{aligned}$$

A similar, and slightly easier, computation shows that

$$\text{Int}(c_\beta^{-1}) \circ a_{c_\beta \mathcal{T}_1 c_\beta^{-1}} \circ \text{Int}(c_\beta) = a_{\mathcal{T}_1} \circ \text{Int}(w_\beta)$$

It follows from these two computations and the definition of ζ_β that

$$\begin{aligned} \zeta_\beta \circ a_{d_\beta T_1 d_\beta^{-1}} \circ \zeta_\beta^{-1} &= \text{Int}(c_\beta) \circ \zeta \circ a_{T_1} \circ \text{Int}(w_\beta) \circ \zeta^{-1} \circ \text{Int}(c_\beta^{-1}) \\ &= \text{Int}(c_\beta) \circ a_{\mathcal{T}_1} \circ \text{Int}(w_\beta) \circ \text{Int}(c_\beta^{-1}) \\ &= a_{c_\beta \mathcal{T}_1 c_\beta^{-1}}. \end{aligned}$$

This proves the compatibility of the Galois actions. The extension of the compatibility to a pairing follows from the choice of positive systems in the based Cartan subgroup containing $(c_\beta \mathcal{T}_1 c_\beta^{-1})^\Gamma$ (see the proof of [ABV, Proposition 13.10 (a)]). \square

Lemma 5.8 may be applied repeatedly in the context of (46). If c' in (46) is the iterated composition of imaginary noncompact roots β_1, \dots, β_m then the lemma tells us that there is a pairing between $d_{\beta_m} \cdots d_{\beta_1} T_1 (d_{\beta_m} \cdots d_{\beta_1})^{-1}$ and $c' \mathcal{T}_1 (c')^{-1} = {}^d T$. Let us denote $d_{\beta_m} \cdots d_{\beta_1} T_1 (d_{\beta_m} \cdots d_{\beta_1})^{-1}$ by $d' \cdot T_1$, so that $d' \cdot T_1$ is paired with ${}^d T$. This pairing and (17) allows us to identify the strong real forms of $d' \cdot T_1$ of type J with $\left(\frac{({}^d T^{\vee \theta})^{\hat{J}} / (({}^d T^{\vee \theta})^{\hat{J}})^0}{(({}^d T^{\vee \theta})^{\hat{J}})^0} \right)^\wedge$. Similarly the pairing between T_1 and \mathcal{T}_1 allows us to identify the strong real forms of T_1 of type J with $\left(\frac{(\mathcal{T}_1^{\vee \theta})^{\hat{J}} / ((\mathcal{T}_1^{\vee \theta})^{\hat{J}})^0}{((\mathcal{T}_1^{\vee \theta})^{\hat{J}})^0} \right)^\wedge$. We wish to make these identifications in conjunction with the diagram of Proposition 5.6.

Given a finite abelian group A , we denote $\text{Hom}(A, \mathbb{C}^\times)$ by A^\wedge . The next corollary is an application of $\text{Hom}(\cdot, \mathbb{C}^\times)$ to Proposition 5.6 and is a special instance of Pontryagin duality.

Corollary 5.9. *The following diagram, given by restriction from the diagram in Proposition 5.6, is commutative*

$$\begin{array}{ccc}
\left(\frac{(\mathcal{T}_1^{\vee \theta})^{\hat{J}}}{((\mathcal{T}_1^{\vee \theta})^{\hat{J}})^0} \right)^\wedge & \xleftarrow{\quad} & \left(\frac{({}^d T^{\vee \theta})^{\hat{J}}}{(({}^d T^{\vee \theta})^{\hat{J}})^0 \langle \vee (c_{\beta_j} \beta_j)(-1) : 1 \leq j \leq m \rangle} \right)^\wedge \\
\uparrow & & \uparrow \\
\left(\frac{(\mathcal{T}_1^{\vee \theta})^{\hat{J}}}{((\mathcal{T}_1^{\vee \theta})^{\hat{J}})^0 \langle \vee \alpha(-1) : \alpha \in c \vee \Delta_\phi \rangle} \right)^\wedge & \xleftarrow{\cong} & \left(\frac{({}^d T^{\vee \theta})^{\hat{J}}}{(({}^d T^{\vee \theta})^{\hat{J}})^0 \langle \vee \alpha(-1) : \alpha \in R_{\mathbb{R}}(({}^d L)^{\hat{J}}, ({}^d T)^{\hat{J}}) \rangle} \right)^\wedge \\
& \searrow \cong & \swarrow \cong \\
& \left(\frac{({}^\vee G_\phi)^{\hat{J}} / (({}^\vee G_\phi)^{\hat{J}})^0}{(({}^\vee G_\phi)^{\hat{J}})^0} \right)^\wedge &
\end{array}$$

Corollary 5.10. *Suppose τ_d is a character of $\frac{({}^d T^{\vee \theta})^{\hat{J}}}{(({}^d T^{\vee \theta})^{\hat{J}})^0 \langle \vee \alpha(-1) : \alpha \in R_{\mathbb{R}}(({}^d L)^{\hat{J}}, ({}^d T)^{\hat{J}}) \rangle}$ which corresponds to the unique character τ_1 of $\frac{(\mathcal{T}_1^{\vee \theta})^{\hat{J}}}{((\mathcal{T}_1^{\vee \theta})^{\hat{J}})^0 \langle \vee \alpha(-1) : \alpha \in c \vee \Delta_\phi \rangle}$ as in Corollary 5.9. Then the strong real form $\delta(\tau_d) \in G^\Gamma - G$ is equivalent to the strong real form $\delta(\tau_1) \in G^\Gamma - G$.*

Proof. First assume $J = \{1\}$ and $c' = c_\beta$ for a single imaginary noncompact root $\beta \in R({}^d L, {}^d T)$ (46). This places us in the setting of Lemma 5.8, that is $d' \cdot T_1$ is paired with ${}^d T = c_\beta \cdot \mathcal{T}_1$ through ζ_β (53), and $d' = d_\beta$ for real $\beta \in R(G, T_1)$. We regard τ_d as a character of $\frac{{}^d T^{\vee \theta}}{({}^d T^{\vee \theta})^0 \langle \vee (c_\beta \beta)(-1) \rangle}$ which is trivial on ${}^\vee \alpha(-1)$ for all $\alpha \in R_{\mathbb{R}}({}^d L, {}^d T)$. According to Lemma 5.2 and Proposition 5.4, the character τ_d is determined by its values on a set

$$\left\{ t_1 (\mathcal{T}_1^{\vee \theta})^0, \dots, t_\ell (\mathcal{T}_1^{\vee \theta})^0 \right\} \subset \mathcal{T}_1^{\vee \theta} / (\mathcal{T}_1^{\vee \theta})^0$$

in which $t_1, \dots, t_\ell \in \ker \beta$. In addition, the character τ_1 is determined by the equations

$$\tau_1(t_j (\mathcal{T}_1^{\vee \theta})^0) = \tau_d(t_j (\mathcal{T}_1^{\vee \theta})^0), \quad 1 \leq j \leq \ell. \quad (54)$$

The strong real form $\delta(\tau_d)$ is defined through the maps in (10) and the pairing ζ_β . To be more precise, the second map in (10) is defined by identifying τ_d with a character in $X^*({}^d T) = X^*(c_\beta \cdot \mathcal{T}_1)$ and sending this character to an element in $X_*(d_\beta \cdot T_1)$ under ζ_β (53). To be even more precise, the character τ_d is sent first to $\tau_d \circ \text{Int}(c_\beta) \in X^*(\mathcal{T}_1)$, then to $\lambda_1 \in X_*(T_1)$ under ζ , and finally to $\text{Int}(d_\beta) \circ \lambda_1 \in X_*(d_\beta \cdot T_1)$. The strong real form $\delta(\tau_d)$ is represented by the element

$$\exp(\pi i \text{Int}(d_\beta) \circ \lambda_1) \delta_q \in G^\Gamma. \quad (55)$$

Let us retrace some of these steps in view of the relationship (54) between τ_1 and τ_d . From $t_j \in \ker \beta$ for all $1 \leq j \leq \ell$, it follows in turn that $\text{Int}(c_\beta)t_j = t_j$, $\tau_d = \tau_d \circ \text{Int}(c_\beta)$, and $\tau_1 = \tau_d \circ \text{Int}(c_\beta) \in X^*(\mathcal{T}_1)$. The strong real form $\delta(\tau_1)$ is defined through (10) and the pairing ζ , so $\delta(\tau_1)$ is represented by the element

$$\exp(\pi i \lambda_1) \delta_q \in G^\Gamma, \quad (56)$$

for $\lambda_1 \in X_*(T_1)$ as above.

Hence, in comparing $\delta(\tau_1)$ with $\delta(\tau_d)$, we are reduced to comparing λ_1 with $\text{Int}(d_\beta) \circ \lambda_1$. For the latter comparison recall from (10) that $\sigma \circ \lambda_1 = -\lambda_1$. Since $\beta \in R(G, T_1)$ is real, we have

$$\langle \beta, \lambda_1 \rangle = \langle \beta \circ \sigma, \sigma \circ \lambda_1 \rangle = \langle \beta, -\lambda_1 \rangle$$

and $\langle \beta, \lambda_1 \rangle = 0$. This orthogonality implies $\text{Int}(d_\beta) \circ \lambda_1 = \lambda_1$ and so the respective representatives (55) and (56) of $\delta(\tau_d)$ and $\delta(\tau_1)$ are equal.

This proves the corollary when J is trivial and $c' = c_\beta$. The proof for non-trivial J follows the same argument except that one must replace Lemma 5.2 with Lemma 5.5, and replace the maps of (10) with those of (12). The proof for arbitrary c' is a proof by induction on the number of imaginary noncompact roots occurring in its definition. The details are left to the reader. \square

5.3 The proof of Theorem 1.1

We conclude by indicating how every representation in $\Pi_{\phi, J}^{\text{KLS}}$ (43) is equivalent to a unique representation in $\Pi_{\phi, J}^{\text{ABV}}$ (34). The main ideas have all been presented. All we have to do is recall them in the correct sequence.

We continue to work under the assumptions of Section 4 and fix a representation $(\pi(\eta_{\tau_1}), \delta(\tau_1)) \in \Pi_{\phi, J}^{\text{KLS}}$ of a strong real form given by

$$\tau_1 \in \left((\mathcal{T}_1^{\vee \theta})^{\hat{J}} / ((\mathcal{T}_1^{\vee \theta})^{\hat{J}})^0 \langle \vee \alpha(-1) : \alpha \in c^\vee \Delta_\phi \rangle \right)^\wedge.$$

We recall that $\pi(\eta_{\tau_1})$ is an irreducible standard representation of $G(\mathbb{R}, \delta(\tau))$. It is constructed from a character of $T_1(\mathbb{R})$ and the embedding $\eta_{\tau_1} : T_1(\mathbb{R}) \rightarrow G(\mathbb{R}, \delta(\tau_1))$ using cohomological and parabolic induction. Following the discussion after Lemma 5.8, we wish to transform these data to obtain an equivalent representation from an embedding $(d' \cdot T_1)(\mathbb{R}) \rightarrow G(\mathbb{R}, \delta(\tau))$. This is achieved

by a repeated application of Hecht-Schmid's character identity as follows. Recall $d' = d_{\beta_m} \cdots d_{\beta_1}$. The root $\beta_1 \in R(G, T_1)$ is real and satisfies the parity condition of the Hecht-Schmid character identity. The Hecht-Schmid character identity theorem then provides an embedding $(d_{\beta_1} T_1 d_{\beta_1}^{-1})(\mathbb{R}) \rightarrow G(\mathbb{R}, \delta(\tau_1))$ so that the resulting standard representation is equivalent to $\pi(\eta_{\tau_1})$. We repeat this process for the remaining real roots, which all satisfy requisite parity conditions. In the end, we obtain the irreducible standard representation obtained from an embedding $(d' \cdot T_1)(\mathbb{R}) \rightarrow G(\mathbb{R}, \delta(\tau_1))$. It is equivalent to the original representation $\pi(\eta_{\tau_1})$ so we keep this notation for it. From this perspective, the representation of the strong real form $(\pi(\eta_{\tau_1}), \delta(\tau_1))$ is determined by three data:

- the pairing between the maximal tori $d' \cdot T_1$ and $c' \cdot \mathcal{T}_1 = {}^d T$
- the embedding $\eta_{\tau_1} : d' \cdot T_1(\mathbb{R}) \rightarrow G(\mathbb{R}, \delta(\tau_1))$ and
- the L-homomorphism ϕ , whose image is contained in ${}^d T^\Gamma$.

These three data are precisely those which determine a representation in $\Pi_{\phi, J}^{\text{ABV}}$. Therefore $(\pi(\eta_{\tau_1}), \delta(\tau_1))$ is equivalent to the unique representation

$$(\pi(\eta_{\tau_d}), \delta(\tau_d)) \in \Pi_{\phi, J}^{\text{ABV}}$$

which indexed by

$$\tau_d \in \left(({}^d T^{\vee\theta})^{\hat{J}} / ({}^d T^{\vee\theta})^{\hat{J}}{}^0 \langle \vee \alpha(-1) : \alpha \in R_{\mathbb{R}}({}^d L, {}^d T) \rangle \right)^\wedge$$

such that $\delta(\tau_d) = \delta(\tau_1)$. By Corollary 5.10 the character τ_d is the unique character which corresponds to τ_1 in the diagram of Corollary 5.9. Moreover, the two characters, τ_1 and τ_d , map to the same character τ of $({}^\vee G_\phi)^{\hat{J}} / ({}^\vee G_\phi)^{\hat{J}}{}^0$. We may substitute τ for τ_1 or τ_d as we have in Sections 3-4. With this substitution, what we have proven is that every element $(\pi(\eta_{\tau_1}), \delta(\tau)) \in \Pi_{\phi, J}^{\text{KLS}}$ is equivalent to $(\pi(\eta_{\tau_d}), \delta(\tau)) \in \Pi_{\phi, J}^{\text{ABV}}$, and this is Theorem 1.1.

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