Projective relative unification through duality

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Abstract

Unification problems can be formulated and investigated in an algebraic setting, by identifying substitutions to modal algebra homomorphisms. This opens the door to applications of the notorious duality between Heyting or modal algebras and descriptive frames. Through substantial use of this correspondence, we give a necessary and sufficient condition for formulas to be projective. A close inspection of this characterization will motivate a generalization of standard unification, which we dub *relative unification*. Applying this result to a number of different logics, we then obtain new proofs of their projective – or non-projective – character. Aside from reproving known results, we show that the projective extensions of **K5** are exactly the extensions of **K45**. This resolves the open question of whether **K5** is projective.

1 Introduction

In a propositional language, substitutions can be defined as functions mapping variables to formulas. For reasons related to Unification Theory [4, Section 2], it is usually considered that such functions are almost everywhere equal to the identity function. As a result, one can see a substitution as a function $\sigma : \mathcal{L}_P \to \mathcal{L}_Q$ where $\mathcal{L}_P$ (resp. $\mathcal{L}_Q$) is the set of all formulas with variables in a finite set P (resp. Q), and satisfying ($\blacklozenge$) $\sigma(\circ(\varphi_1, \ldots, \varphi_n)) = \circ(\sigma(\varphi_1), \ldots, \sigma(\varphi_n))$ for all n-ary connectives $\circ$ of the language and all formulas $\varphi_1, \ldots, \varphi_n \in \mathcal{L}_P$. According to this point of view, which is the one usually considered within the context of modal and super-intuitionistic logics [5, 10, 15], two substitutions $\sigma : \mathcal{L}_P \to \mathcal{L}_Q$ and $\tau : \mathcal{L}_P \to \mathcal{L}_{Q'}$ are said to be equivalent with respect to a propositional logic $\mathbf{L}$ (in symbols $\sigma \simeq_{\mathbf{L}} \tau$) if for all $p \in P$, the formulas $\sigma(p)$ and $\tau(p)$ are $\mathbf{L}$-equivalent.

In the standard account of unification theory, a formula $\varphi \in \mathcal{L}_P$ is $\mathbf{L}$-unifiable if $\mathbf{L}$ contains instances of φ. In that case, any substitution $\sigma : \mathcal{L}_P \to \mathcal{L}_Q$ such that $\vdash_{\mathbf{L}} \sigma(\varphi)$ counts as a unifier of φ. This presentation, however, does not capture all the questions of interest in the area. In [3], Baader addresses unification in description logic $\mathcal{EL}$ with respect to a set $\mathcal{T}$ of so-called *general concept inclusions*. The point is that in some contexts, one is given a 'background theory' encoded by $\mathcal{T}$ and only looks to unify a formula φ *relatively* to $\mathcal{T}$, that is, to find an instance $\sigma(\varphi)$ of φ which is a local consequence of $\mathcal{T}$. In the case of a modal or super-intuitionistic logic $\mathbf{L}$,[1] the set $\mathcal{T}$ is replaced by a formula δ, and $\sigma(\varphi)$ is a local consequence of δ if it can be derived from δ using the axioms and rules of $\mathbf{L}$ (with the exception of uniform substitution). In this case we will call σ a unifier of φ *modulo* δ, which is in general a weaker condition than being a

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[1] Description and modal logics are closely related, see [2].

unifier of φ. However, if we take $\delta := \top$ then unifiers modulo δ are just unifiers in the standard sense, which means that relative unification is indeed a generalization of standard unification.

Furthermore, a $\mathbf{L}$-unifiable formula $\varphi \in \mathcal{L}$ is $\mathbf{L}$-projective if it possesses a projective unifier, that is to say a unifier σ such that $\varphi \vdash_{\mathbf{L}} \sigma(p) \leftrightarrow p$ holds for all $p \in P$. Such unifiers are interesting because they constitute by themselves minimal complete sets of unifiers [5, 10, 15]. For this reason, it is of the utmost importance to be able to determine if a given formula is $\mathbf{L}$-projective – and these observations remain valid in the relative unification setting.

Now, condition ($\blacklozenge$) may evoke homomorphism properties. Following this observation, Unification Theory was also formalized and studied in an algebraic setting [13, 29]. Indeed, let us consider the Lindenbaum-Tarski algebra[2] $\mathbf{A}_P$ obtained by taking the quotient of $\mathcal{L}_P$ modulo the relation $\equiv_{\mathbf{L}}$ of $\mathbf{L}$-equivalence. One can associate to a substitution $\sigma : \mathcal{L}_P \to \mathcal{L}_Q$ the map $\sigma^{\triangleright} : \mathbf{A}_P \to \mathbf{A}_Q$ by setting $\sigma^{\triangleright}([\varphi]_{\mathbf{L}}) := [\sigma(\varphi)]_{\mathbf{L}}$ for any formula $\varphi \in \mathcal{L}_P$, whose equivalence class modulo $\equiv_{\mathbf{L}}$ is denoted by $[\varphi]_{\mathbf{L}}$. In this perspective, condition ($\blacklozenge$) then truly expresses the homomorphic character of $\sigma^{\triangleright}$. Obviously, this association between substitutions and homomorphisms of Lindenbaum algebras is one-to-one modulo $\simeq_{\mathbf{L}}$: substitutions associated to the same homomorphism are equivalent modulo $\simeq_{\mathbf{L}}$. Then various properties of substitutions, such as being a unifier of a formula, admit an algebraic counterpart too.

In this paper, we combine this correspondence with a more traditional one, provided by *Duality Theory*. For any set P of variables, there is indeed a tight connection between the Lindenbaum algebra $\mathbf{A}_P$ and the canonical frame $\mathfrak{F}_P$ of $\mathbf{L}$ over P, determined by the set of all prime filters[3] on $\mathbf{A}_P$. Homomorphisms between Lindenbaum algebras are then in correspondence with bounded morphisms between canonical frames. See [7, Chapter 5], [8, Chapter 7 & 8] and [24, Chapter 4] for a general introduction to this subject. Given a finite set P of variables and a super-intuitionistic logic $\mathbf{L}$, we make essential use of this duality to construct a necessary and sufficient condition for $\varphi \in \mathcal{L}_P$ to be $\mathbf{L}$-projective modulo δ: the existence of a bounded morphism $f : \mathfrak{F}_P \to \mathfrak{F}_P$ such that the image of $\widehat{\delta}$ by f is contained in $\widehat{\varphi}$, and all elements of $\widehat{\varphi}$ are fixpoints of f, where $\widehat{\varphi}$ is the extension of φ in $\mathfrak{F}_P$. When $\mathbf{L}$ is a normal modal logic, we obtain the same result, but $\widehat{\delta}$ needs to be replaced by $\widehat{\delta}^{\infty}$, which denotes the set of all points lying in $\widehat{\Box^n \delta}$ for all $n \in \mathbb{N}$ – and likewise $\widehat{\varphi}$ needs to be replaced by $\widehat{\varphi}^{\infty}$. As we will see, this condition also sheds new light on relative unification, and provides an independent, unexpected motivation for this problem.

This paper is structured as follows. In Section 2, we introduce some basics of super-intuitionistic logics, modal logics, Unification Theory[4], and explains how to 'algebraize' unification problems. In Section 3 and Section 4, we develop some basics of Duality Theory in modal and super-intuitionistic logics, concentrating on the bijective correspondence between bounded morphisms of canonical frames and homomorphisms of Lindenbaum algebras. We then apply these tools to establish the above-mentioned necessary and sufficient condition for a formula to be projective. In Section 5, we use this characterization to investigate the projective character of the extensions of four selected logics: $\mathbf{LC}$, $\mathbf{K4}_n\mathbf{B}_k$, $\mathbf{K4D1}$, and $\mathbf{K5}$.

[2] Or Lindenbaum algebra for short.

[3] In the case of modal logics, the points of $\mathfrak{F}_P$ are usually defined as maximal $\mathbf{L}$-consistent sets of formulas instead of prime filters, but as explained in Section 5, this makes no difference.

[4] We usually distinguish between elementary unification and unification with parameters. In elementary unification, all variables are likely to be replaced by formulas when one applies a substitution [2]. In unification with parameters, some variables — called parameters — remain unchanged [27, Chapter 6]. In this paper, we only interest in elementary unification.

$p \to (q \to p)$
$(p \to (q \to r)) \to ((p \to q) \to (p \to r))$
$p \wedge q \to p$
$p \wedge q \to q$
$p \to (q \to p \wedge q)$
$p \to p \vee q$
$q \to p \vee q$
$(p \to r) \to ((q \to r) \to (p \vee q \to r))$
$\bot \to p$

From φ infer $\sigma(\varphi)$ (*uniform substitution*)
From φ and $\varphi \to \psi$ infer ψ (*modus ponens*)

Table 1: Axioms and rules of super-intuitionistic logics

2 Background

2.1 Some functional vocabulary

Let $f : X \to Y$ be a function. If $A \subseteq X$, we write $f[A] := \{f(x) \mid x \in A\}$. If $B \subseteq Y$, we write $f^{-1}[B] := \{x \in X \mid f(x) \in B\}$. We denote by $\mathrm{Im}\, f := f[X]$ the *image* of f. When $X = Y$, we denote by $\mathrm{fp}\, f := \{x \in X \mid f(x) = x\}$ the set of *fixpoints* of f. Given two functions $f : X \to Y$ and $g : Y \to Z$ we denote by $gf : X \to Z$ the *composition* of f and g, defined by $gf : x \mapsto g(f(x))$.

2.2 Intuitionistic logic

Let Prop be an infinite countable set of propositional variables. If $P \subseteq \mathrm{Prop}$ we define the intuitionistic language $\mathcal{L}^{\mathsf{int}}_P$ over P by the following grammar:

$$\varphi ::= p \mid \bot \mid (\varphi \wedge \varphi) \mid (\varphi \vee \varphi) \mid (\varphi \to \psi)$$

where $p \in P$. We write $\mathcal{L}^{\mathsf{int}} := \mathcal{L}^{\mathsf{int}}_{\mathrm{Prop}}$. We follow the standard rules for omission of parentheses. The abbreviations $\top$ and $\leq$ are defined as usual. Given $\varphi \in \mathcal{L}^{\mathsf{int}}$ we denote by $\mathrm{var}(\varphi)$ the set of variables occurring in φ. A *substitution* is a map $\sigma : \mathcal{L}^{\mathsf{int}}_P \to \mathcal{L}^{\mathsf{int}}_Q$ with $P, Q \subseteq \mathrm{Prop}$ finite and such that for all $\varphi, \psi \in \mathcal{L}^{\mathsf{int}}_P$ we have $\sigma(\bot) = \bot$, $\sigma(\varphi \wedge \psi) = \sigma(\varphi) \wedge \sigma(\psi)$, $\sigma(\varphi \vee \psi) = \sigma(\varphi) \vee \sigma(\psi)$, and $\sigma(\varphi \to \psi) = \sigma(\varphi) \to \sigma(\psi)$.

Definition 2.1 ([8, Sect. 2.6])**.** A *super-intuitionistic logic* is a set $\mathbf{L}$ of formulas containing the axioms and closed under the inferences rules described in Table 1. The smallest super-intuitionistic logic is denoted **IPC**.

Let $\mathbf{L}$ be a super-intuitionistic logic. Instead of $\varphi \in \mathbf{L}$ we may also write $\vdash_{\mathbf{L}} \varphi$. Given $\varphi, \psi \in \mathcal{L}^{\mathsf{int}}$, we write $\varphi \equiv_{\mathbf{L}} \psi$ in case $\vdash_{\mathbf{L}} \varphi \leftrightarrow \psi$. Then $\equiv_{\mathbf{L}}$ is an equivalence relation, and we denote by $[\varphi]_{\mathbf{L}}$ the equivalence class of φ modulo $\equiv_{\mathbf{L}}$. We call $\mathbf{L}$ *locally tabular* if for all finite sets $P \subseteq \mathrm{Prop}$, there are only finitely many equivalence classes over $\mathcal{L}^{\mathsf{int}}_P$ modulo $\equiv_{\mathbf{L}}$. Note that any extension of a locally tabular logic is also locally tabular. We call an *extension* of $\mathbf{L}$ any super-intuitionistic logic $\mathbf{L}'$ such that $\mathbf{L} \subseteq \mathbf{L}'$. If $\theta \in \mathcal{L}^{\mathsf{int}}$, we denote by $\mathbf{L} + \theta$ the smallest extension of $\mathbf{L}$ containing θ.

All propositional tautologies
$\Box(p \to q) \to (\Box p \to \Box q)$

From φ infer $\sigma(\varphi)$	(*uniform substitution*)
From φ and $\varphi \to \psi$ infer ψ	(*modus ponens*)
From φ infer $\Box\varphi$	(*generalization*)

Table 2: Axioms and rules of normal modal logics

Given $\varphi, \psi \in \mathcal{L}^{\mathsf{int}}$, we call a *derivation of ψ from φ in* $\mathbf{L}$ a sequence of formulas $\chi_0, \ldots, \chi_n \in \mathcal{L}^{\mathsf{int}}$ such that $\chi_n = \psi$ and for all $i \in \{0, \ldots, n\}$, at least one of the following conditions holds:

- $\chi_i \in \mathbf{L}$,
- $\chi_i = \varphi$,
- there exists $j, k \in \{0, \ldots, n\}$ such that $i > j, k$ and $\chi_k = \chi_j \to \chi_i$.

If there exists a derivation of ψ from φ in $\mathbf{L}$, we shall say that ψ is *deducible from φ in* $\mathbf{L}$, and write $\varphi \vdash_{\mathbf{L}} \psi$. A more concise characterization of derivable formulas is given by the following result, which is a weak form of the *Deduction Theorem* for intuitionistic logic. When $\mathbf{L} = \mathbf{IPC}$, see [8, Th. 2.42] for a proof of it, which can be easily adapted to the general case.

Proposition 2.2. *The following are equivalent:*

1. $\varphi \vdash_{\mathbf{L}} \psi$,
2. $\vdash_{\mathbf{L}} \varphi \to \psi$.

2.3 Modal logics

If $P \subseteq \mathrm{Prop}$ we define the modal language $\mathcal{L}^{\mathsf{mod}}_P$ over P by the following grammar:

$$\varphi ::= p \mid \bot \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \Box\varphi$$

where $p \in P$. We write $\mathcal{L}^{\mathsf{mod}} := \mathcal{L}^{\mathsf{mod}}_{\mathrm{Prop}}$. We follow the standard rules for omission of parentheses. The abbreviations $\top, \vee, \to, \leftrightarrow, \Diamond$ are defined as usual. If $n \in \mathbb{N}$ we define inductively $\Box^n\varphi$ and $\Box^{\leq n}\varphi$ by:

- $\Box^0\varphi := \varphi$ and $\Box^{\leq 0}\varphi := \varphi$,
- for all $n \in \mathbb{N}$, $\Box^{n+1}\varphi := \Box\Box^n\varphi$ and $\Box^{\leq n+1}\varphi := \Box\Box^n\varphi \wedge \Box^{\leq n}\varphi$.

We then define $\Diamond^n\varphi := \neg\Box^n\neg\varphi$ and $\Diamond^{\leq n}\varphi := \neg\Box^{\leq n}\neg\varphi$. Similarly to the intuitionistic case, a *substitution* is a map $\sigma : \mathcal{L}^{\mathsf{mod}}_P \to \mathcal{L}^{\mathsf{mod}}_Q$ with $P, Q \subseteq \mathrm{Prop}$ finite and such that for all $\varphi, \psi \in \mathcal{L}^{\mathsf{mod}}_P$ we have $\sigma(\bot) = \bot$, $\sigma(\neg\varphi) = \neg\sigma(\varphi)$, $\sigma(\varphi \wedge \psi) = \sigma(\varphi) \wedge \sigma(\psi)$, and $\sigma(\Box\varphi) = \Box\sigma(\varphi)$.

Definition 2.3. A *normal modal logic* is a set $\mathbf{L}$ of formulas containing the axioms and closed under the inferences rules described in Table 2. The smallest normal modal logic is denoted $\mathbf{K}$.

Let $\mathbf{L}$ be a normal modal logic. The notations $\vdash_{\mathbf{L}} \varphi$ and $\varphi \equiv_{\mathbf{L}} \psi$, as well as local tabularity, are defined as in the intuitionistic case[5]. A set Σ of formulas is $\mathbf{L}$-*consistent* if there are no formulas $\varphi_1, \ldots, \varphi_n \in \Sigma$ such that $\vdash_{\mathbf{L}} \neg(\varphi_1 \wedge \ldots \wedge \varphi_n)$. We call an *extension* of $\mathbf{L}$ any normal modal logic $\mathbf{L}'$ such that $\mathbf{L} \subseteq \mathbf{L}'$. If $\theta \in \mathcal{L}^{\mathsf{mod}}$, we denote by $\mathbf{L} + \theta$ the smallest extension of $\mathbf{L}$ containing θ.

Given $\varphi, \psi \in \mathcal{L}^{\mathsf{mod}}$, we call a *derivation of ψ from φ in* $\mathbf{L}$ a sequence of formulas $\chi_0, \ldots, \chi_n \in \mathcal{L}^{\mathsf{mod}}$ such that $\chi_n = \psi$ and for all $i \in \{0, \ldots, n\}$, at least one of the following conditions holds:

[5]Locally tabular modal logics possess interesting properties, in particular when it comes to decidability [25].

- $\chi_i \in \mathbf{L}$,
- $\chi_i = \varphi$,
- there exists $j, k \in \{0, \ldots, n\}$ such that $i > j, k$ and $\chi_k = \chi_j \to \chi_i$.
- there exists $j \in \{0, \ldots, n\}$ such that $i > j$ and $\chi_i = \Box\chi_j$.

If there exists a derivation of ψ from φ in $\mathbf{L}$, we shall say that ψ is *deducible from φ in* $\mathbf{L}$, and write $\varphi \vdash_{\mathbf{L}} \psi$. Modal logic also enjoys a Deduction Theorem, stated below (in its weak form again). When $\mathbf{L} = \mathbf{K}$, see [8, Th. 3.51] for a proof of it, which can be easily adapted to the general case. See [17] for an interesting discussion of this theorem.

Proposition 2.4. *The following are equivalent:*

1. $\varphi \vdash_{\mathbf{L}} \psi$,
2. *there exists* $k \in \mathbb{N}$ *such that* $\vdash_{\mathbf{L}} \Box^{\leq k}\varphi \to \psi$.

2.4 Relative unification

In this section, we let $\mathbf{L}$ be either a super-intuitionistic logic or a normal modal logic. Accordingly, the symbol $\mathcal{L}$ will denote the intuitionistic language or the modal language. Given $P \subseteq \mathrm{Prop}$ we denote by $\mathcal{S}_P$ the set of all substitutions from $\mathcal{L}_P$ to $\mathcal{L}_Q$, where Q ranges over the finite subsets of Prop. We then fix a formula $\delta \in \mathcal{L}$. The equivalence relation $\simeq^{\delta}_{\mathbf{L}}$ on $\mathcal{S}_P$ is defined by

$$\sigma \simeq^{\delta}_{\mathbf{L}} \tau \text{ if and only if } \delta \vdash_{\mathbf{L}} \sigma(p) \leftrightarrow \tau(p) \text{ for all } p \in P.$$

Then, we define the preorder $\preccurlyeq^{\delta}_{\mathbf{L}}$ on $\mathcal{S}_P$ by

$$\sigma \preccurlyeq^{\delta}_{\mathbf{L}} \tau \text{ iff there exists a substitution } \mu : \mathcal{L}_Q \to \mathcal{L}_R \text{ such that } \mu\sigma \simeq^{\delta}_{\mathbf{L}} \tau$$

where $\sigma : \mathcal{L}_P \to \mathcal{L}_Q$ and $\tau : \mathcal{L}_P \to \mathcal{L}_{Q'}$.

Given $\varphi \in \mathcal{L}_P$ we say that $\sigma : \mathcal{L}_P \to \mathcal{L}_Q$ is a *unifier of φ modulo δ* if we have $\delta \vdash_{\mathbf{L}} \sigma(\varphi)$. In addition, σ is called *concise* if $P = \mathrm{var}(\varphi)$. The formula φ is $\mathbf{L}$*-unifiable modulo δ* if there exists a unifier of φ modulo δ. A set $\mathcal{T}$ of concise unifiers of φ modulo δ is said to be *complete* if for all concise unifiers σ of φ modulo δ, there exists $\tau \in \mathcal{T}$ such that $\tau \preccurlyeq^{\delta}_{\mathbf{L}} \sigma$. In addition, we call $\mathcal{T}$ a *basis* if $\sigma \preccurlyeq^{\delta}_{\mathbf{L}} \tau$ implies $\sigma = \tau$ for all $\sigma, \tau \in \mathcal{T}$.

Proposition 2.5. *For all $\varphi \in \mathcal{L}$, if φ is* $\mathbf{L}$*-unifiable modulo δ then for all bases $\mathcal{T}$,$\mathcal{U}$ of concise unifiers of φ modulo δ, we have* $\mathtt{Card}(\mathcal{T}) = \mathtt{Card}(\mathcal{U})$ [13, Section 2].

In the sequel, when discussing standard unification, we will freely drop the part 'modulo δ' in case $\delta := \top$. For instance, 'φ is $\mathbf{L}$-unifiable' is a shorthand for 'φ is $\mathbf{L}$-unifiable modulo $\top$'. A central problem in unification theory is whether an $\mathbf{L}$-unifiable formula (modulo δ) possesses a basis of concise unifiers (modulo δ). When this is the case, Proposition 2.5 raises a more refined question, that is, how large is such a basis? This gives rise to a full classification of logics based on the cardinality of these bases.

Definition 2.6. Let φ be $\mathbf{L}$-unifiable modulo δ. We call φ $\mathbf{L}$*-nullary* modulo δ if φ has no basis of concise unifiers modulo δ. Otherwise, let $\mathcal{T}$ be a basis of concise unifiers of φ modulo δ. Then φ is said to be $\mathbf{L}$*-unitary*, $\mathbf{L}$*-finitary*, or $\mathbf{L}$*-infinitary* modulo δ, if $\mathcal{T}$ is a singleton, has finite cardinality ≥ 2, or is infinite, respectively.

Unification types are naturally sorted from 'best' to 'worst' in the following order: unitary, finitary, infinitary, nullary. The *unification type* of $\mathbf{L}$ is then defined as the worst type among the unification types of the $\mathbf{L}$-unifiable formulas; whereas the *relative unification type* of $\mathbf{L}$ is the worst type among the unification types modulo δ of the $\mathbf{L}$-unifiable formulas modulo δ, where δ ranges over $\mathcal{L}$.

Taking $\delta := \top$, it is apparent from this definition that the relative unification type of $\mathbf{L}$ is always worse than or equal to the unification type of $\mathbf{L}$. At the level of formulas, moving from standard to relative unification may dramatically impact the unification type, as illustrated by Example 2.9 below. A special case that deserves our attention is that of projective unifiers.

Definition 2.7. Let $\varphi \in \mathcal{L}_P$ with $P \subseteq$ Prop finite. We call a *φ-projective substitution* a substitution $\sigma : \mathcal{L}_P \to \mathcal{L}_P$ such that for all $p \in P$, we have $\varphi \vdash_{\mathbf{L}} \sigma(p) \leftrightarrow p$. We then say that φ is *$\mathbf{L}$-projective* modulo δ if there exists a φ-projective unifier of φ modulo δ. We shall say that $\mathbf{L}$ has *projective unification* (or that $\mathbf{L}$ is *projective*) if every $\mathbf{L}$-unifiable formula is $\mathbf{L}$-projective. Accordingly, we shall say that $\mathbf{L}$ has *relative projective unification* if for all $\delta \in \mathcal{L}$, every $\mathbf{L}$-unifiable formula modulo δ is $\mathbf{L}$-projective modulo δ.

Again, it is clear that if $\mathbf{L}$ has relative projective unification then $\mathbf{L}$ has projective unification. For an introduction to projective unification, see [5, 10, 15].

Example 2.8. We consider the logic $\mathbf{KD} := \mathbf{K} + \Diamond\top$. Let φ be a *positive* formula, that is, constructed by applying the connectives $\wedge$, $\vee$, $\Box$, $\Diamond$ only to propositional variables. Let p be an arbitrary variable, and $\sigma : \mathcal{L}_{\mathrm{var}(\varphi)} \to \mathcal{L}_{\{p\}}$ be the substitution defined by $\sigma(q) := p$ for all $q \in \mathrm{var}(\varphi)$. We can then prove by induction on φ that $p \vdash_{\mathbf{L}} \sigma(\varphi)$, that is, σ is a unifier of φ modulo p.

Example 2.9. We consider the logic $\mathbf{K}$. We know from Jeřábek [21] that the formula $\varphi := p \to \Box p$ is $\mathbf{K}$-unifiable, but has no basis of unifiers in $\mathbf{K}$ (and hence is $\mathbf{K}$-nullary). However, if we consider relative unification, there are many instances in which φ turn out to be $\mathbf{K}$-projective. A non-trivial example is given by $\delta := \Box\Box p$. Indeed, if σ is the substitution defined by $\sigma(p) := p \wedge \Box p$, it is easy to check that σ is a projective unifier of φ modulo δ.

Projective unifiers have been used by Ghilardi [14] to address unification in super-intuitionistic logics, and in particular to prove that **IPC** is finitary. In fact, Ghilardi has shown that if a formula φ is **IPC**-unifiable then it possesses a finite basis of unifiers, this basis being the set of projective unifiers of a finite set of projective formulas of modal degree at most equal to the modal degree of φ, having the same propositional variables as φ and implying φ in **IPC**. He also proved that the super-intuitionistic logics with unitary unification are exactly the extensions of De Morgan logic, namely **IPC** extended with the axiom of weak excluded middle $\neg p \vee \neg\neg p$. On a similar note, Wroński [31, 32] showed that the super-intuitionistic logics with projective unification are exactly the extensions of *Gödel-Dummett logic* $\mathbf{LC} := \mathbf{IPC} + (p \to q) \vee (q \to p)$. This result was partially recovered by Ghilardi [16, p. 10], through the use of duality theory (albeit via a different route than the one we present here).

On the side of modal logics, the logic $\mathbf{K}$ has been shown to be nullary by Jeřábek [21] (see Example 2.9 above). The logic **S5**, however, is known to be unitary since many years [2, 9]. The truth is that if $\sigma : \mathcal{L}_P \to \mathcal{L}_P$ is a **S5**-unifier of a formula φ then the substitution $\varepsilon : \mathcal{L}_P \to \mathcal{L}_P$ defined by $\varepsilon(p) := (\Box\varphi \wedge p) \vee (\Diamond\neg\varphi \wedge \sigma(p))$ is a projective unifier of φ in **S5**. Using the same techniques as in the intuitionistic case, Ghilardi [15] has proved that several transitive modal logics like **K4** or **S4** are finitary – see also [19] for a syntactic approach to unification in transitive modal logics. Recently, Dzik and Wojtylak [11] have proved that the projective extensions of **S4** are exactly the extensions of **S4.3**, a result improved by Kost [22] who has demonstrated that the projective extensions of **K4** are exactly the extensions of **K4D1**. Finally, Kostrzycka [23] has proved the projectivity of the modal logics of the form $\mathbf{K4}_n\mathbf{B}_k$ (see Section 5), which were first introduced by Jansana [20]. Whereas duality has been pervasive in some recent developments [6, 12], it has so far be present 'in disguise' only, and was to yet to be made explicit (as far as modal logic is concerned).

The proofs of propositions 2.10–2.13 below can be adapted *verbatim* from [15, Section 2] – in which they appear in the context of standard unification.

Proposition 2.10. *Let $\varphi \in \mathcal{L}_P$, and $\sigma : \mathcal{L}_P \to \mathcal{L}_P$ be a substitution. Then σ is φ-projective if and only if for all formulas $\psi \in \mathcal{L}_P$, we have $\varphi \vdash_{\mathbf{L}} \sigma(\psi) \leftrightarrow \psi$.*

Proposition 2.11. *Let $\delta \in \mathcal{L}$. For all formulas $\varphi \in \mathcal{L}_P$, for all φ-projective substitutions $\sigma : \mathcal{L}_P \to \mathcal{L}_P$ and for all unifiers $\tau : \mathcal{L}_P \to \mathcal{L}_Q$ of φ modulo δ, we have $\sigma \preccurlyeq^{\delta}_{\mathbf{L}} \tau$.*

Proposition 2.12. *Let $\delta \in \mathcal{L}$. If $\varphi \in \mathcal{L}$ is $\mathbf{L}$-projective modulo δ then φ is $\mathbf{L}$-unitary modulo δ.*

Proposition 2.13. *Let $\delta \in \mathcal{L}$. If $\mathbf{L}$ has projective unification modulo δ then $\mathbf{L}$ is unitary modulo δ.*

As a side note, the extension of a projective logic is, up to our knowledge, not necessarily projective.

Remark 2.14. If $\varphi \in \mathcal{L}$, there are obviously infinitely many subsets P of Prop such that $\varphi \in \mathcal{L}_P$. For this reason, many authors require a unifier of φ (modulo δ) to be of the form $\sigma : \mathcal{L}_{\mathrm{var}(\varphi)} \to \mathcal{L}_Q$ (or 'concise' in our terminology). In our setting, we also allow one to talk about the unifying or φ-projective character of any substitution of the form $\sigma : \mathcal{L}_P \to \mathcal{L}_Q$ with $\mathrm{var}(\varphi) \subseteq P$. This offers more flexibility, which will be helpful in Section 5.

As a result, our definition of $\mathbf{L}$-unifiable and $\mathbf{L}$-projective formulas are non-standard[6], but this is harmless. Indeed, if $\mathsf{var}(\varphi) \subseteq P \subseteq P'$ and $\sigma : \mathcal{L}_P \to \mathcal{L}_Q$ is a substitution, one can define the substitution $\sigma' : \mathcal{L}_{P'} \to \mathcal{L}_Q$ by setting $\sigma'(p) := \sigma(p)$ for all $p \in P$ and $\sigma'(p) := p$ for all $p \in P' \setminus P$, and it is clear that σ is a unifier of φ modulo δ (resp. is φ-projective) if and only if σ' is a unifier of φ modulo δ (resp. is φ-projective). Likewise, if $\mathsf{var}(\varphi) \subseteq P \subseteq P'$ and $\sigma' : \mathcal{L}_{P'} \to \mathcal{L}_Q$ is a substitution, let us denote by $\sigma := \sigma'_{|\mathcal{L}_P}$ the restriction of σ' to $\mathcal{L}_P$. Then σ' is a unifier of φ modulo δ (resp. is φ-projective) if and only if σ is a unifier of φ modulo δ (resp. is φ-projective).

2.5 Heyting algebras

The algebraic aspects of unification have already been investigated in e.g. [13, 29]. Here we give a lightweight, self-sufficient account of them. The goal of this section is essentially to state Proposition 2.18, a modest but inspiring starting point.

Definition 2.15. A *Heyting algebra* is a structure $\mathbf{A} = (A, 0, \wedge, \vee, \to)$ with $(A, 0, \wedge, \vee)$ a bounded distributive lattice and $\to : A^2 \to A$ an operator satisfying $a \wedge b \leq c \iff a \leq b \to c$ for all $a, b, c \in A$. For convenience we will identify $\mathbf{A}$ to its underlying set A. We denote by $1 := 0 \to 0$ the top element of $\mathbf{A}$.[7]

A *homomorphism* from a Heyting algebra $\mathbf{A}$ to a Heyting algebra $\mathbf{B}$ is a map $\alpha : \mathbf{A} \to \mathbf{B}$ such that for all $a, b \in \mathbf{A}$ we have $\alpha(0) = 0$, $\alpha(a \wedge b) = \alpha(a) \wedge \alpha(b)$, $\alpha(a \vee b) = \alpha(a) \vee \alpha(b)$, and $\alpha(a \to b) = \alpha(a) \to \alpha(b)$. We denote by $\mathrm{Ker}\, \alpha := \{(a, b) \in \mathbf{A}^2 \mid \alpha(a) = \alpha(b)\}$ the *kernel* of α.

Definition 2.16. Let $\mathbf{A}$ be a Heyting algebra. An equivalence relation $\sim$ on $\mathbf{A}$ is called a *congruence* on $\mathbf{A}$ if for all $a, a', b, b' \in \mathbf{A}$:

- $a \sim a'$ and $b \sim b'$ implies $(a \wedge b) \sim (a' \wedge b')$,
- $a \sim a'$ and $b \sim b'$ implies $(a \vee b) \sim (a' \vee b')$,
- $a \sim a'$ and $b \sim b'$ implies $(a \to b) \sim (a' \to b')$.

Then $\sim$ induces a *quotient algebra* $\mathbf{A}/\!\sim$ over the set of all equivalence classes of $\sim$. If for all $a \in \mathbf{A}$ we denote by $\pi(a)$ the equivalence class of a modulo $\sim$, we obtain a surjective homomorphism $\pi : \mathbf{A} \to \mathbf{A}/\!\sim$.

[6] Note that the unification type *does* remain standard, since it is defined with respect to concise unifiers only.

[7] In many places, we will deal with several Heyting algebras at the same time, and the symbol 1 will indifferently refer to the top element of all of them. Normally, this should not cause any ambiguity. The same convention will apply to modal algebras.

If $P \subseteq \mathrm{Prop}$, a particularly interesting Heyting algebra is the *Lindenbaum algebra* of $\mathbf{L}$ over P, defined as $\mathbf{A}_P := (\mathcal{L}_P^{\mathsf{int}}/\equiv_{\mathbf{L}}, 0, \wedge, \vee, \to)$ with:

- $0 := [\bot]_{\mathbf{L}}$,
- $[\varphi]_{\mathbf{L}} \wedge [\psi]_{\mathbf{L}} := [\varphi \wedge \psi]_{\mathbf{L}}$,
- $[\varphi]_{\mathbf{L}} \vee [\psi]_{\mathbf{L}} := [\varphi \vee \psi]_{\mathbf{L}}$,
- $[\varphi]_{\mathbf{L}} \to [\psi]_{\mathbf{L}} := [\varphi \to \psi]_{\mathbf{L}}$.

Notice that if $\mathbf{L}$ is locally tabular and P is finite, then $\mathbf{A}_P$ is finite. In all cases, every substitution $\sigma : \mathcal{L}_P^{\mathsf{int}} \to \mathcal{L}_Q^{\mathsf{int}}$ naturally induces a homomorphism $\sigma^\triangleright : \mathbf{A}_P \to \mathbf{A}_Q$ defined by $\sigma^\triangleright([\varphi]_{\mathbf{L}}) := [\sigma(\varphi)]_{\mathbf{L}}$. This object is well-defined since whenever $\varphi \equiv_{\mathbf{L}} \psi$, we have $\sigma(\varphi) \equiv_{\mathbf{L}} \sigma(\psi)$ as a consequence of the rule of uniform substitution. Conversely, one can recover σ from $\sigma^\triangleright$ (up to equivalence modulo $\simeq_{\mathbf{L}}$) since we have $\sigma(\varphi) \equiv_{\mathbf{L}} \psi$ for any $\psi \in \sigma^\triangleright([\varphi]_{\mathbf{L}})$. There is thus a one-to-one correspondence between homomorphisms and substitutions (up to equivalence modulo $\simeq_{\mathbf{L}}$). For convenience, we will then identify the two: the symbol σ will indifferently denote the substitution σ and the homomorphism $\sigma^\triangleright$.

Properties of substitutions can also be expressed algebraically. Given $\varphi \in \mathcal{L}_P^{\mathsf{int}}$, let $\equiv_\varphi^P$ be the least congruence on $\mathbf{A}_P$ such that $[\varphi]_{\mathbf{L}} \equiv_\varphi^P 1$. We then denote by $\pi_\varphi^P : \mathbf{A}_P \to \mathbf{A}_P/\equiv_\varphi^P$ the homomorphism associated to $\equiv_\varphi^P$ (as introduced in definition 2.16). Obviously the kernel of π_φ^P is $\equiv_\varphi^P$ itself. In the sequel, we will systematically omit the superscript ^{P} in the notations π_φ^P and $\equiv_\varphi^P$, as the type of these objects is generally clear from the context.

Proposition 2.17. *Given $\varphi, \psi, \theta \in \mathcal{L}_P^{\mathsf{int}}$, the following are equivalent:*

1. $\varphi \vdash_{\mathbf{L}} \psi \leftrightarrow \theta$,
2. $[\psi]_{\mathbf{L}} \equiv_\varphi [\theta]_{\mathbf{L}}$.

Proof. Suppose that $\varphi \vdash_{\mathbf{L}} \psi \leftrightarrow \theta$. Then $\vdash_{\mathbf{L}} \varphi \to (\psi \leftrightarrow \theta)$ by Proposition 2.2. Then $1 = [\varphi \to (\psi \leftrightarrow \theta)]_{\mathbf{L}}$, whence $[\varphi]_{\mathbf{L}} \leq [\psi \leftrightarrow \theta]_{\mathbf{L}}$. It follows that $\pi_\varphi([\varphi]_{\mathbf{L}}) \leq \pi_\varphi([\psi \leftrightarrow \theta]_{\mathbf{L}})$, and thus $1 = \pi_\varphi([\psi \leftrightarrow \theta]_{\mathbf{L}})$. Hence $\pi_\varphi([\psi]_{\mathbf{L}}) = \pi_\varphi([\theta]_{\mathbf{L}})$, or equivalently $[\psi]_{\mathbf{L}} \equiv_\varphi [\theta]_{\mathbf{L}}$.

Conversely, let us write $[\psi]_{\mathbf{L}} \sim [\theta]_{\mathbf{L}}$ whenever $\varphi \vdash_{\mathbf{L}} \psi \leftrightarrow \theta$. It is easily verified that $\sim$ is a congruence on $\mathbf{A}_P$, and that $[\varphi]_{\mathbf{L}} \sim 1$. By construction, $\equiv_\varphi$ is then included in $\sim$, and this proves the claim. □

We may now connect the projective or unifying character of a substitution to its algebraic properties.

Proposition 2.18. *Let $P, Q \subseteq \mathrm{Prop}$ finite, $\varphi \in \mathcal{L}_P^{\mathsf{int}}$ and $\delta \in \mathcal{L}_Q^{\mathsf{int}}$. Then:*

1. *A substitution $\sigma : \mathcal{L}_P^{\mathsf{int}} \to \mathcal{L}_Q^{\mathsf{int}}$ is a unifier of φ modulo δ iff* $\mathrm{Ker}\ \pi_\varphi \subseteq \mathrm{Ker}\ \pi_\delta\sigma$.
2. *A substitution $\sigma : \mathcal{L}_P^{\mathsf{int}} \to \mathcal{L}_P^{\mathsf{int}}$ is φ-projective iff $\pi_\varphi\sigma = \pi_\varphi$.*

Proof. 1. First observe that by Proposition 2.17, we have $\delta \vdash_{\mathbf{L}} \sigma(\varphi)$ iff $\sigma([\varphi]_{\mathbf{L}}) \equiv_\delta [\top]_{\mathbf{L}}$, iff $\pi_\delta\sigma([\varphi]_{\mathbf{L}}) = 1$. If $\pi_\delta\sigma([\varphi]_{\mathbf{L}}) = 1$, then $\mathrm{Ker}\ \pi_\delta\sigma$ is a congruence on $\mathbf{A}_P$ containing $([\varphi]_{\mathbf{L}}, 1)$, and so by construction it contains $\equiv_\varphi$. Conversely, if $\mathrm{Ker}\ \pi_\varphi \subseteq \mathrm{Ker}\ \pi_\delta\sigma$ then in particular $([\varphi]_{\mathbf{L}}, 1) \in \mathrm{Ker}\ \pi_\delta\sigma$ and therefore $\pi_\delta\sigma([\varphi]_{\mathbf{L}}) = 1$.

2. We have

$$\begin{array}{lll} & \sigma \text{ is } \varphi\text{-projective} & \\ \text{iff} & \forall \psi \in \mathcal{L}_P^{\mathsf{int}},\ \varphi \vdash_{\mathbf{L}} \sigma(\psi) \leftrightarrow \psi & \text{by Proposition 2.10} \\ \text{iff} & \forall \psi \in \mathcal{L}_P^{\mathsf{int}},\ [\sigma(\psi)]_{\mathbf{L}} \equiv_\varphi [\psi]_{\mathbf{L}} & \text{by Proposition 2.17} \\ \text{iff} & \forall \psi \in \mathcal{L}_P^{\mathsf{int}},\ \pi_\varphi\sigma([\psi]_{\mathbf{L}}) = \pi_\varphi([\psi]_{\mathbf{L}}) & \\ \text{iff} & \pi_\varphi\sigma = \pi_\varphi & \end{array}$$

□

2.6 Modal algebras

We now adapt the content of Section 2.5 to the case of modal algebras.

Definition 2.19. A *modal algebra* is a structure $\mathbf{A} = (A, 0, \neg, \wedge, \Box)$ with $(A, 0, \neg, \wedge)$ a Boolean algebra and $\Box : A \to A$ an operator satisfying $\Box 1 = 1$ (where $1 := \neg 0$ is the top element of $\mathbf{A}$) and $\Box(a \wedge b) = \Box a \wedge \Box b$ for all $a, b \in A$. For convenience we will identify $\mathbf{A}$ to its underlying set A.

A *homomorphism* from a modal algebra $\mathbf{A}$ to a modal algebra $\mathbf{B}$ is a map $\alpha : \mathbf{A} \to \mathbf{B}$ such that for all $a, b \in \mathbf{A}$ we have $\alpha(0) = 0$, $\alpha(\neg a) = \neg\alpha(a)$, $\alpha(a \wedge b) = \alpha(a) \wedge \alpha(b)$, and $\alpha(\Box a) = \Box\alpha(a)$. We denote by $\mathrm{Ker}\ \alpha := \{(a, b) \in \mathbf{A}^2 \mid \alpha(a) = \alpha(b)\}$ the *kernel* of α.

Definition 2.20. Let $\mathbf{A}$ be a modal algebra. An equivalence relation $\sim$ on $\mathbf{A}$ is called a *congruence* on $\mathbf{A}$ if for all $a, a', b, b' \in \mathbf{A}$:

- $a \sim a'$ implies $\neg a \sim \neg a'$,
- $a \sim a'$ and $b \sim b'$ implies $(a \wedge b) \sim (a' \wedge b')$,
- $a \sim a'$ implies $\Box a \sim \Box a'$.

The quotient algebra $\mathbf{A}/\!\sim$ and the surjection $\pi : \mathbf{A} \to \mathbf{A}/\!\sim$ associated to $\sim$ are then defined as in Definition 2.16.

We define the Lindenbaum algebra, the congruence $\equiv_\varphi$ and the homomorphism π_φ as in Section 2.5.

Proposition 2.21. *Given $\varphi, \psi, \theta \in \mathcal{L}_P^{\mathsf{mod}}$, the following are equivalent:*

1. $\varphi \vdash_{\mathbf{L}} \psi \leftrightarrow \theta$,
2. $[\psi]_{\mathbf{L}} \equiv_\varphi [\theta]_{\mathbf{L}}$.

Proof. Suppose that $\varphi \vdash_{\mathbf{L}} \psi \leftrightarrow \theta$. Then by Proposition 2.4 there exists $n \in \mathbb{N}$ such that $\vdash_{\mathbf{L}} \Box^{\leq n}\varphi \to (\psi \leftrightarrow \theta)$. Thus $\left[\Box^{\leq n}\varphi\right]_{\mathbf{L}} \leq [\psi \leftrightarrow \theta]_{\mathbf{L}}$, and it follows that $\pi_\varphi(\left[\Box^{\leq n}\varphi\right]_{\mathbf{L}}) \leq \pi_\varphi([\psi \leftrightarrow \theta]_{\mathbf{L}})$. It is easily proved by induction on n that $\pi_\varphi(\left[\Box^{\leq n}\varphi\right]_{\mathbf{L}}) = 1$, and therefore $\pi_\varphi([\psi \leftrightarrow \theta]_{\mathbf{L}}) = 1$ too. Hence $\pi_\varphi([\psi]_{\mathbf{L}}) = \pi_\varphi([\theta]_{\mathbf{L}})$, or equivalently $[\psi]_{\mathbf{L}} \equiv_\varphi [\theta]_{\mathbf{L}}$.

Conversely, let us write $[\psi]_{\mathbf{L}} \sim [\theta]_{\mathbf{L}}$ whenever $\varphi \vdash_{\mathbf{L}} \psi \leftrightarrow \theta$. It is easily verified that $\sim$ is a congruence on $\mathbf{A}_P$, and that $[\varphi]_{\mathbf{L}} \sim 1$. By construction, $\equiv_\varphi$ is then included in $\sim$, and this proves the claim. □

Proposition 2.22. *Let $P, Q \subseteq \mathrm{Prop}$ finite, $\varphi \in \mathcal{L}_P^{\mathsf{mod}}$ and $\delta \in \mathcal{L}_Q^{\mathsf{mod}}$. Then:*

1. *A substitution $\sigma : \mathcal{L}_P^{\mathsf{mod}} \to \mathcal{L}_Q^{\mathsf{mod}}$ is a unifier of φ modulo δ iff $\mathrm{Ker}\ \pi_\varphi \subseteq \mathrm{Ker}\ \pi_\delta\sigma$.*
2. *A substitution $\sigma : \mathcal{L}_P^{\mathsf{mod}} \to \mathcal{L}_P^{\mathsf{mod}}$ is φ-projective iff $\pi_\varphi\sigma = \pi_\varphi$.*

Proof. See the proof of Proposition 2.18. □

3 Duality for Heyting algebras

Now it is time to let duality play its role. In this section we introduce some rudiments of Duality Theory, and apply them to our setting. More precisely, we will use the duality between Heyting algebras and so-called general intuitionistic frames, which we will simply call intuitionistic frames. For more details we refer to [8, Chapter 8]. This investigation will ultimately lead to a characterization of projective formulas solely in terms of bounded morphisms.

Definition 3.1. An *intuitionistic frame* is a tuple $\mathfrak{F} = (X, \leq, \mathcal{B})$ with X a set of possible worlds, $\leq$ an order on X, and $\mathcal{B} \subseteq \mathcal{P}(X)$ such that:

- $\varnothing, X \in \mathcal{B}$,
- $A, B \in \mathcal{B}$ implies $A \cup B \in \mathcal{B}$,
- $A, B \in \mathcal{B}$ implies $A \cap B \in \mathcal{B}$,
- $A, B \in \mathcal{B}$ implies $A \Rightarrow B \in \mathcal{B}$, with

$$A \Rightarrow B := \{x \in X : \forall y \in X, \text{ if } x \leq y \text{ and } y \in A \text{ then } y \in B\}.$$

Further, we call $\mathfrak{F}$ *differentiated* if for all $x, y \in X$ such that $x \neq y$, there exists $A \in \mathcal{B}$ such that either $x \in A$ and $y \notin A$, or $x \notin A$ and $y \in A$. We call $\mathfrak{F}$ *compact* if for all $\mathcal{B}_1 \subseteq \mathcal{B}$ and $\mathcal{B}_0 \subseteq \{X \setminus A \mid A \in \mathcal{B}\}$, if $\bigcap(\mathcal{B}'_0 \cup \mathcal{B}'_1) \neq \varnothing$ whenever $\mathcal{B}'_0 \subseteq \mathcal{B}_0$ and $\mathcal{B}'_1 \subseteq \mathcal{B}_1$ are finite, then $\bigcap(\mathcal{B}_0 \cup \mathcal{B}_1) \neq \varnothing$. Finally, we say that $\mathfrak{F}$ is a *descriptive frame* if $\mathfrak{F}$ is compact and differentiated. If $\mathfrak{F} = (X, \leq, \mathcal{B})$ and $\mathfrak{F}' = (X', \leq', \mathcal{B}')$ are two intuitionistic frames, we call a *bounded morphism* from $\mathfrak{F}$ to $\mathfrak{F}'$ a map $f : X \to X'$ such that:

- if $x \leq y$ then $f(x) \leq' f(y)$,
- if $f(x) \leq' y'$ then there exists $y \in X$ such that $f(y) = y'$ and $x \leq y$,
- if $A' \in \mathcal{B}'$ then $f^{-1}[A'] \in \mathcal{B}$ (*continuity*).

Now, let $\mathbf{A}$ be a Heyting algebra. Given a set $F \subseteq \mathbf{A}$, we call F a *prime filter* on $\mathbf{A}$ if it satisfies the following conditions:

- $0 \notin F$,
- $a \in F$ and $a \leq b$ implies $b \in F$,
- $a, b \in F$ implies $a \wedge b \in F$,
- $a \vee b \in F$ implies $a \in F$ or $b \in F$.

Then the *dual* of $\mathbf{A}$ is the descriptive intuitionistic frame $\mathbf{A}^* := (X_{\mathbf{A}}, \leq_{\mathbf{A}}, \mathcal{B}_{\mathbf{A}})$ with:

- $X_{\mathbf{A}} := \{F \subseteq \mathbf{A} \mid F \text{ is a prime filter}\}$,
- $F \leq_{\mathbf{A}} F'$ whenever $F \subseteq F'$,
- $\mathcal{B}_{\mathbf{A}} := \{\{F \in X_{\mathbf{A}} \mid a \in F\} \mid a \in \mathbf{A}\}$.

Further, if $\alpha : \mathbf{A} \to \mathbf{B}$ is a homomorphism, we define a bounded morphism $\alpha^* : \mathbf{B}^* \to \mathbf{A}^*$ by

$$\alpha^*(F) := \alpha^{-1}[F].$$

If $\alpha : \mathbf{A} \to \mathbf{B}$ and $\beta : \mathbf{B} \to \mathbf{C}$ are two homomorphisms, the identity

$$(\beta\alpha)^* = \alpha^*\beta^*$$

is easily verified. One can prove that for all descriptive frames $\mathfrak{F}$ there exists a unique Heyting algebra $\mathbf{A}$ (up to isomorphism) such that $\mathfrak{F}$ and $\mathbf{A}^*$ are isomorphic. Likewise, if $f : \mathbf{B}^* \to \mathbf{A}^*$ is a bounded morphism, there exists a unique homomorphism $\alpha : \mathbf{A} \to \mathbf{B}$ such that $\alpha^* = f$[8]. In what follows we are going to make extensive use of this correspondence.

Naturally, if $\mathbf{L}$ is a super-intuitionistic logic, the dual of the Lindenbaum algebra of $\mathbf{L}$ is of central interest to us. So for all finite $P \subseteq \mathrm{Prop}$, we write $(X_{\mathbf{A}_P}, \leq_{\mathbf{A}_P}, \mathcal{B}_{\mathbf{A}_P}) = (X_P, \leq_P, \mathcal{B}_P)$ for simplicity, and denote by $\mathfrak{F}_P := (X_{\mathbf{A}_P}, \leq_{\mathbf{A}_P}, \mathcal{B}_{\mathbf{A}_P})$ the *canonical frame* of $\mathbf{L}$ over P. Given $\varphi \in \mathcal{L}^{\mathsf{int}}_P$, we also write $\widehat{\varphi} := \{F \in X_P \mid [\varphi]_{\mathbf{L}} \in F\}$, and we then see that $\mathcal{B}_P = \{\widehat{\varphi} \mid \varphi \in \mathcal{L}^{\mathsf{int}}_P\}$.

Note that the situation largely simplifies when we consider finite descriptive frames. In this case, the continuity condition of bounded morphisms holds for free. Indeed, consider two finite descriptive frames $\mathfrak{F} = (X, \leq, \mathcal{B})$ and $\mathfrak{F}' = (X', \leq', \mathcal{B}')$ and a map $f : X \to X'$. Then we can

[8]In categorical terms, we thus say that $(\cdot)^*$ is a *dual equivalence* between the category of Heyting algebras with homomorphisms and the category of descriptive intuitionistic frames with bounded morphisms.

see that $\mathcal{B} = \mathcal{P}(X)$, and so trivially we have $f^{-1}[A'] \in \mathcal{B}$ for all $A' \in \mathcal{B}'$. In particular, if $\mathbf{L}$ is locally tabular and P is finite, then X_P is finite. In the sequel, we will only address locally tabular logics, but the general account of duality remains of interest.

Now assume that P is finite and let $\varphi \in \mathcal{L}^{\mathsf{int}}_P$. In order to characterize the unifiable or projective character of φ, it is crucial to understand the behaviour of $\pi^*_\varphi : (\mathbf{A}_P/\equiv_\varphi)^* \to \mathbf{A}^*_P$. In the algebraic setting, the relevant information was contained in the kernel of π_φ[9]. In the dual setting, this information turns out to be carried by the image of π^*_φ, which we in fact prove to coincide with the set $\widehat{\varphi}$.

Lemma 3.2. *Let $F \in X_P$. Then the following are equivalent:*

1. *F is closed under $\equiv_\varphi$;*
2. *$F \in \mathrm{Im}\ \pi^*_\varphi$.*

Proof. From 1 to 2, suppose that F is closed under $\equiv_\varphi$. We introduce $G := \pi_\varphi[F]$ and prove that G is a prime filter on $\mathbf{A}_P/\equiv_\varphi$.

- Suppose that $0 \in G$. Then $0 = \pi_\varphi(a)$ for some $a \in F$. Hence $\pi_\varphi(0) = 0 = \pi_\varphi(a)$, which entails $0 \equiv_\varphi a$, and thus $0 \in F$ by assumption, contradicting the fact that F is a prime filter. Therefore $0 \notin G$.
- Suppose that $a' \in G$ and $a' \leq b'$. Then $a' = \pi_\varphi(a)$ for some $a \in F$. In addition, since π_φ is surjective, we have $b' = \pi_\varphi(b)$ for some $b \in \mathbf{A}_P$. From $a' \leq b'$ we obtain $a' = a' \wedge b'$, and thus $\pi_\varphi(a) = \pi_\varphi(a) \wedge \pi_\varphi(b) = \pi_\varphi(a \wedge b)$. Hence $a \equiv_\varphi a \wedge b$, and since $a \in F$ our assumption entails $a \wedge b \in F$. From $a \wedge b \leq b$ we obtain $b \in F$. Therefore $b' \in G$.
- Let $a', b' \in G$. We have $a' = \pi_\varphi(a)$ and $b' = \pi_\varphi(b)$ for some $a, b \in F$. Then $a' \wedge b' = \pi_\varphi(a \wedge b)$ with $a \wedge b \in F$. Therefore $a' \wedge b' \in G$.
- Let $a' \vee b' \in G$. Then $a' \vee b' = \pi_\varphi(c)$ for some $c \in F$. We also have $a' = \pi_\varphi(a)$ and $b' = \pi_\varphi(b)$ for some $a, b \in \mathbf{A}_P$. Thus $\pi_\varphi(c) = a' \vee b' = \pi_\varphi(a \vee b)$, that is, $c \equiv_\varphi a \vee b$. Since $c \in F$, we obtain that $a \vee b \in F$ by assumption. Since F is a prime filter we have either $a \in F$ or $b \in F$. Therefore, we have either $a' \in G$ or $b' \in G$.

To prove that $F \in \mathrm{Im}\ \pi^*_\varphi$ we then show that $\pi^*_\varphi(G) = F$, that is, $\pi^{-1}_\varphi[\pi_\varphi[F]] = F$. The inclusion from right to left is trivial. From left to right, suppose that $a \in \pi^{-1}_\varphi[\pi_\varphi[F]]$. Then $\pi_\varphi(a) \in \pi_\varphi[F]$, that is, $\pi_\varphi(a) = \pi_\varphi(b)$ for some $b \in F$. Since F is closed under $\equiv_\varphi$ we obtain $a \in F$ and we are done.
From 2 to 1, let $F \in \mathrm{Im}\ \pi^*_\varphi$. Then there exists a prime filter $G \in (\mathbf{A}_P/\equiv_\varphi)^*$ such that $F = \pi^*_\varphi(G) = \pi^{-1}_\varphi[G]$. If $a \in F$ and $a \equiv_\varphi b$ then $\pi_\varphi(b) = \pi_\varphi(a) \in G$, and therefore $b \in F$. This proves that F is closed under $\equiv_\varphi$. □

Proposition 3.3. *We have* $\mathrm{Im}\ \pi^*_\varphi = \widehat{\varphi}$.

Proof. Let $F \in \mathrm{Im}\ \pi^*_\varphi$. We have $[\varphi]_{\mathbf{L}} \equiv_\varphi 1$ with $1 \in F$, so by Lemma 3.2 we obtain $[\varphi]_{\mathbf{L}} \in F$. Conversely, let $F \in \widehat{\varphi}$. By Lemma 3.2, it suffices to prove that F is closed under $\equiv_\varphi$. So assume $[\psi]_{\mathbf{L}} \in F$ and $[\psi]_{\mathbf{L}} \equiv_\varphi [\theta]_{\mathbf{L}}$. By Proposition 2.17 we obtain $\varphi \vdash_{\mathbf{L}} \psi \leftrightarrow \theta$, and then by Proposition 2.2 it follows that $\vdash_{\mathbf{L}} \varphi \to (\psi \leftrightarrow \theta)$. Since $F \in \widehat{\varphi}$ we obtain $[\psi \leftrightarrow \theta]_{\mathbf{L}} \in F$, and since $[\psi]_{\mathbf{L}} \in F$ we conclude that $[\theta]_{\mathbf{L}} \in F$. □

With this result, we are then ready to transition from the algebraic setting to the dual setting.

Proposition 3.4. *Let $P, Q \subseteq \mathrm{Prop}$ finite, $\varphi \in \mathcal{L}^{\mathsf{int}}_P$ and $\delta \in \mathcal{L}^{\mathsf{int}}_Q$. Then:*

[9] In Proposition 2.18, the kernel of π_φ is not mentioned in item (2), but it still appears implicitly since $\pi_\varphi \sigma = \pi_\varphi$ can also be phrased as $\{(\sigma(a), a) \mid a \in \mathbf{A}_P\} \subseteq \mathrm{Ker}\ \pi_\varphi$.

1. *for any homomorphism* $\alpha : \mathbf{A}_P \to \mathbf{A}_Q$ *we have* $\mathrm{Ker}\ \pi_\varphi \subseteq \mathrm{Ker}\ \pi_\delta\alpha$ *iff* $\alpha^*[\widehat{\delta}] \subseteq \widehat{\varphi}$*;*
2. *for any homomorphism* $\alpha : \mathbf{A}_P \to \mathbf{A}_P$ *we have* $\pi_\varphi\alpha = \pi_\varphi$ *iff* $\widehat{\varphi} \subseteq \mathrm{fp}\ \alpha^*$*.*

Proof. Given $\alpha : \mathbf{A}_P \to \mathbf{A}_Q$, recall that we have $\alpha^* : X_Q \to X_P$.

1. Suppose that $\mathrm{Ker}\ \pi_\varphi \subseteq \mathrm{Ker}\ \pi_\delta\alpha$. Let $F \in \alpha^*[\widehat{\delta}]$. Then $F = \alpha^{-1}[G]$ for some $G \in \widehat{\delta}$. Further, we have $\pi_\varphi([\varphi]_{\mathbf{L}}) = 1 = \pi_\varphi(1)$, and thus $\pi_\delta\alpha([\varphi]_{\mathbf{L}}) = \pi_\delta\alpha(1)$ by assumption, that is, $\alpha([\varphi]_{\mathbf{L}}) \equiv_\delta 1$. From Proposition 3.3 and Lemma 3.2, we get that G is closed under $\equiv_\delta$, whence $\alpha([\varphi]_{\mathbf{L}}) \in G$. Therefore $[\varphi]_{\mathbf{L}} \in F$. This proves that $F \in \widehat{\varphi}$.

   Conversely, suppose that $\alpha^*[\widehat{\delta}] \subseteq \widehat{\varphi}$. First, we prove that $\pi_\delta\alpha([\varphi]_{\mathbf{L}}) = 1$. If not, then $\pi_\delta\alpha([\neg\varphi]_{\mathbf{L}}) \neq 0$, so by the Prime Filter Theorem [8, Th. 7.41] there exists a prime filter $H \in (\mathbf{A}_Q/\equiv_\delta)^*$ such that $\pi_\delta\alpha([\neg\varphi]_{\mathbf{L}}) \in H$. Let $G := \pi^*_\delta(H) \in X_Q$. We then have $\alpha([\neg\varphi]_{\mathbf{L}}) \in G$. Further, we have $\pi_\delta([\delta]_{\mathbf{L}}) = 1 \in H$, and thus $[\delta]_{\mathbf{L}} \in G$. Therefore $G \in \widehat{\delta}$, whence $\alpha^*(G) \in \widehat{\varphi}$ by assumption. But then $\alpha([\varphi]_{\mathbf{L}}) \in G$, a contradiction. Hence $\pi_\delta\alpha([\varphi]_{\mathbf{L}}) = 1 = \pi_\delta\alpha(1)$, which means that $\mathrm{Ker}\ \pi_\delta\alpha$ is a congruence containing $([\varphi]_{\mathbf{L}}, 1)$. By construction, we then obtain $\mathrm{Ker}\ \pi_\varphi \subseteq \mathrm{Ker}\ \pi_\delta\alpha$.
2. Suppose that $\pi_\varphi\alpha = \pi_\varphi$. Then $\alpha^*\pi^*_\varphi = \pi^*_\varphi$. Now let $F \in \widehat{\varphi}$. By Proposition 3.3 we have $F \in \mathrm{Im}\ \pi^*_\varphi$. Then there exists a prime filter $G \in (\mathbf{A}_P/\equiv_\varphi)^*$ such that $F = \pi^*_\varphi(G)$. Consequently, $\alpha^*(F) = \alpha^*\pi^*_\varphi(G) = \pi^*_\varphi(G) = F$, and this proves that $F \in \mathrm{fp}\ \alpha^*$.

   Conversely, suppose that $\widehat{\varphi} \subseteq \mathrm{fp}\ \alpha^*$. Given $G \in (\mathbf{A}_P/\equiv_\varphi)^*$ we have $\pi^*_\varphi(G) \in \widehat{\varphi}$ by Proposition 3.3 and thus $\pi^*_\varphi(G) \in \mathrm{fp}\ \alpha^*$. Hence $\alpha^*(\pi^*_\varphi(G)) = \pi^*_\varphi(G)$. This proves that $\alpha^*\pi^*_\varphi = \pi^*_\varphi$, and therefore $\pi_\varphi\alpha = \pi_\varphi$.

□

Finally, by combining these results with Proposition 2.18, we obtain the following characterization.

Theorem 3.5. *Let* $\varphi, \delta \in \mathcal{L}_P$ *with* $P \subseteq \mathrm{Prop}$ *finite. Then:*

1. φ *is* $\mathbf{L}$*-unifiable modulo* δ *if and only if there exists a bounded morphism* $f : \mathfrak{F}_Q \to \mathfrak{F}_P$ *such that* $P \subseteq Q$ *and* $f[\widehat{\delta}] \subseteq \widehat{\varphi}$*;*
2. φ *is* $\mathbf{L}$*-projective modulo* δ *if and only if there exists a bounded morphism* $f : \mathfrak{F}_P \to \mathfrak{F}_P$ *such that* $f[\widehat{\delta}] \subseteq \widehat{\varphi} \subseteq \mathrm{fp}\ f$*.*

Accordingly, we will call a *dual unifier* of φ modulo δ any bounded morphism $f : \mathfrak{F}_Q \to \mathfrak{F}_P$ such that $P \subseteq Q$ and $f[\widehat{\delta}] \subseteq \widehat{\varphi}$, and a *projective dual unifier* of φ modulo δ any bounded morphism $f : \mathfrak{F}_P \to \mathfrak{F}_P$ such that $f[\widehat{\delta}] \subseteq \widehat{\varphi} \subseteq \mathrm{fp}\ f$. Now, what would have happened if we had introduced dual unifiers in the context of standard unification only? Then, as $\widehat{\top} = X_P$, a dual unifier of φ would have been a bounded morphism $f : \mathfrak{F}_Q \to \mathfrak{F}_P$ such that $P \subseteq Q$ and $f[X_P] \subseteq \widehat{\varphi}$. The condition that $f[X_P] \subseteq \widehat{\varphi}$ means that f maps the *whole* frame $\mathfrak{F}_Q$ to *a part of* the frame $\mathfrak{F}_P$, and displays an obvious dissymmetry; whereas the condition that $f[\widehat{\delta}] \subseteq \widehat{\varphi}$ means that f maps *a part of* $\mathfrak{F}_Q$ to *a part of* $\mathfrak{F}_P$. Relative unification can thus be seen as a natural symmetrization of standard unification.

4 Duality for modal algebras

We now delve into the duality theory of modal algebras and general modal frames (again abbreviated as 'modal frames'). This section essentially follows the same lines as Section 3, with the necessary adjustments. For more details we refer to [7, Chapter 5], [8, Chapter 7] and [24, Chapter 4].

Definition 4.1. A *modal frame* is a tuple $\mathfrak{F} = (X, R, \mathcal{B})$ with X a set of possible worlds, R a binary relation on X, and $\mathcal{B} \subseteq \mathcal{P}(X)$ such that:

- $\varnothing \in \mathcal{B}$,
- $A \in \mathcal{B}$ implies $X \setminus A \in \mathcal{B}$,
- $A, B \in \mathcal{B}$ implies $A \cap B \in \mathcal{B}$,
- $A \in \mathcal{B}$ implies $\Box_R A \in \mathcal{B}$, with $\Box_R A := \{x \in X \mid \forall y \in X, xRy \Rightarrow y \in A\}$.

Further, we call $\mathfrak{F}$ *differentiated* if for all $x, y \in X$ such that $x \neq y$, there exists $A \in \mathcal{B}$ such that $x \in A$ and $y \notin A$. We call $\mathfrak{F}$ *tight* if for all $x, y \in X$ such that *not* xRy, there exists $A \in \mathcal{B}$ such that $x \in \Box_R A$ and $y \notin A$. We call $\mathfrak{F}$ *compact* if for all $\mathcal{C} \subseteq \mathcal{B}$, if $\bigcap \mathcal{C}' \neq \varnothing$ whenever $\mathcal{C}' \subseteq \mathcal{C}$ is finite, then $\bigcap \mathcal{C} \neq \varnothing$. Finally, we say that $\mathfrak{F}$ is a *descriptive frame* if $\mathfrak{F}$ is tight, compact and differentiated. If $\mathfrak{F} = (X, R, \mathcal{B})$ and $\mathfrak{F}' = (X', R', \mathcal{B}')$ are two modal frames, we call a *bounded morphism* from $\mathfrak{F}$ to $\mathfrak{F}'$ a map $f : X \to X'$ such that:

- if xRy then $f(x)Rf(y)$,
- if $f(x)R'y'$ then there exists $y \in X$ such that $f(y) = y'$ and xRy,
- if $A' \in \mathcal{B}'$ then $f^{-1}[A'] \in \mathcal{B}$.

Now let $\mathbf{A}$ be a modal algebra. The *dual* of $\mathbf{A}$ is the general frame $\mathbf{A}^* := (X_\mathbf{A}, R_\mathbf{A}, \mathcal{B}_\mathbf{A})$ with:

- $X_\mathbf{A} := \{F \subseteq \mathbf{A} \mid F \text{ is a prime filter}\}$,
- $F R_\mathbf{A} F'$ whenever $\forall a \in \mathbf{A}, \Box a \in F \Rightarrow a \in F'$,
- $\mathcal{B}_\mathbf{A} := \{\{F \in X \mid a \in F\} \mid a \in \mathbf{A}\}$.

Just as before, the map $\mathbf{A} \mapsto \mathbf{A}^*$ is a one-to-one correspondence from modal algebras to descriptive frames, up to isomorphism. Again, if $\alpha : \mathbf{A} \to \mathbf{B}$ is a homomorphism, we define a bounded morphism $\alpha^* : \mathbf{B}^* \to \mathbf{A}^*$ by $\alpha^*(F) := \alpha^{-1}[F]$, and this correspondence is also one-to-one and compatible with composition. Following earlier conventions, if $\mathbf{L}$ is a normal modal logic and $P \subseteq \mathrm{Prop}$, the dual of the Lindenbaum algebra of $\mathbf{L}$ over P will be denoted by $\mathfrak{F}_P := (X_P, R_P, \mathcal{B}_P)$ and called the *canonical frame* of $\mathbf{L}$ over P. Given $\varphi \in \mathcal{L}_P^{\mathsf{mod}}$, we also write $\widehat{\varphi} := \{F \in X_P \mid [\varphi]_\mathbf{L} \in F\}$.

Remark 4.2. The tight similarity between $\mathcal{L}_P^{\mathsf{mod}}$ and $\mathbf{A}_P$ materializes itself in a one-to-one correspondence between the *maximal* $\mathbf{L}$*-consistent subsets* of $\mathcal{L}_P^{\mathsf{mod}}$ [7, Section 4.2] and the prime filters of $\mathbf{A}_P$, realized by the mapping

$$\Gamma \mapsto \{[\varphi]_\mathbf{L} \mid \varphi \in \Gamma\}$$

where $\Gamma \subseteq \mathcal{L}_P^{\mathsf{mod}}$ is maximal $\mathbf{L}$-consistent. In fact, this correspondence induces an isomorphism between the frame (X_P, R_P) and the (traditional) canonical frame of $\mathbf{L}$ over P (as pointed out in [7, Section 5.3]). The justifies naming $\mathfrak{F}_P$ itself the canonical frame of $\mathbf{L}$ over P.

Now assume that P is finite and let $\varphi \in \mathcal{L}_P^{\mathsf{mod}}$. As before, we need to characterize the image of π_φ^*, though its expression is now a bit different. Indeed we show that it coincides with the set $\widehat{\varphi}^\infty := \bigcap_{n \in \mathbb{N}} \widehat{\Box^n \varphi}$.

Lemma 4.3. *Let $F \in X_P$. Then the following are equivalent:*

1. *F is closed under $\equiv_\varphi$;*
2. *$F \in \mathrm{Im}\, \pi_\varphi^*$.*

Proof. Same as Lemma 3.2. □

Proposition 4.4. *We have* $\mathrm{Im}\, \pi_\varphi^* = \widehat{\varphi}^\infty$.

Proof. Let $F \in \operatorname{Im} \pi_\varphi^*$. Given $n \in \mathbb{N}$, we have $[\Box^n\varphi]_{\mathbf{L}} \equiv_\varphi \Box^n 1 \equiv_\varphi 1$ with $1 \in F$, so by Lemma 4.3 we obtain $[\Box^n\varphi]_{\mathbf{L}} \in F$ and thus $F \in \widehat{\Box^n\varphi}$.

Conversely, let $F \in \widehat{\varphi}^\infty$. By Lemma 4.3, it suffices to prove that F is closed under $\equiv_\varphi$. So assume $[\psi]_{\mathbf{L}} \in F$ and $[\psi]_{\mathbf{L}} \equiv_\varphi [\theta]_{\mathbf{L}}$. By Proposition 2.21 we obtain $\varphi \vdash_{\mathbf{L}} \psi \leftrightarrow \theta$, and then by Proposition 2.4 there exists $n \in \mathbb{N}$ such that $\vdash_{\mathbf{L}} \Box^{\leq n}\varphi \to (\psi \leftrightarrow \theta)$. Since $F \in \widehat{\Box^{\leq n}\varphi}$ we obtain $[\psi \leftrightarrow \theta]_{\mathbf{L}} \in F$, and since $[\psi]_{\mathbf{L}} \in F$ we conclude that $[\theta]_{\mathbf{L}} \in F$. □

Proposition 4.5. *Let $P, Q \subseteq \operatorname{Prop}$ finite, $\varphi \in \mathcal{L}_P^{\mathsf{mod}}$ and $\delta \in \mathcal{L}_Q^{\mathsf{mod}}$. Then:*

1. *for any homomorphism $\alpha : \mathbf{A}_P \to \mathbf{A}_Q$ we have $\operatorname{Ker} \pi_\varphi \subseteq \operatorname{Ker} \pi_\delta\alpha$ iff $\alpha^*[\widehat{\delta}^\infty] \subseteq \widehat{\varphi}^\infty$;*
2. *for any homomorphism $\alpha : \mathbf{A}_P \to \mathbf{A}_P$ we have $\pi_\varphi\alpha = \pi_\varphi$ iff $\widehat{\varphi}^\infty \subseteq \operatorname{fp} \alpha^*$.*

Proof. Given $\alpha : \mathbf{A}_P \to \mathbf{A}_Q$, recall that we have $\alpha^* : X_Q \to X_P$.

1. Suppose that $\operatorname{Ker} \pi_\varphi \subseteq \operatorname{Ker} \pi_\delta\alpha$. Let $F \in \alpha^*[\widehat{\delta}^\infty]$. Then $F = \alpha^{-1}[G]$ for some $G \in \widehat{\delta}^\infty$. Let $n \in \mathbb{N}$. We have $\pi_\varphi([\Box^n\varphi]_{\mathbf{L}}) = 1 = \pi_\varphi(1)$, and thus $\pi_\delta\alpha([\Box^n\varphi]_{\mathbf{L}}) = \pi_\delta\alpha(1)$ by assumption, that is, $\alpha([\Box^n\varphi]_{\mathbf{L}}) \equiv_\delta 1$. From Proposition 4.4 and Lemma 4.3, we get that G is closed under $\equiv_\delta$ whence $\alpha([\Box^n\varphi]_{\mathbf{L}}) \in G$. Therefore $[\Box^n\varphi]_{\mathbf{L}} \in F$. This proves that $F \in \widehat{\varphi}^\infty$.

   Conversely, suppose that $\alpha^*[\widehat{\delta}^\infty] \subseteq \widehat{\varphi}^\infty$. First, we prove that $\pi_\delta\alpha([\varphi]_{\mathbf{L}}) = 1$. If not, then $\pi_\delta\alpha([\neg\varphi]_{\mathbf{L}}) \neq 0$, so by the Prime Filter Theorem there exists a prime filter $H \in (\mathbf{A}_Q/\equiv_\delta)^*$ such that $\pi_\delta\alpha([\neg\varphi]_{\mathbf{L}}) \in H$. Let $G := \pi_\delta^*(H) \in X_Q$. We then have $\alpha([\neg\varphi]_{\mathbf{L}}) \in G$. Further, for all $n \in \mathbb{N}$ we have $[\Box^n\delta]_{\mathbf{L}} \equiv_\delta 1$, so $\pi_\delta([\Box^n\delta]_{\mathbf{L}}) \in H$, and thus $[\Box^n\delta]_{\mathbf{L}} \in G$. Therefore $G \in \widehat{\delta}^\infty$, whence $\alpha^*(G) \in \widehat{\varphi}^\infty$ by assumption. But then $\alpha([\varphi]_{\mathbf{L}}) \in G$, a contradiction. Hence $\pi_\delta\alpha([\varphi]_{\mathbf{L}}) = 1 = \pi_\delta\alpha(1)$, which means that $\operatorname{Ker} \pi_\delta\alpha$ is a congruence containing $([\varphi]_{\mathbf{L}}, 1)$. By construction, we then obtain $\operatorname{Ker} \pi_\varphi \subseteq \operatorname{Ker} \pi_\delta\alpha$.
2. Suppose that $\pi_\varphi\alpha = \pi_\varphi$. Then $\alpha^*\pi_\varphi^* = \pi_\varphi^*$. Now let $F \in \widehat{\varphi}^\infty$. By Proposition 4.4 we have $F \in \operatorname{Im} \pi_\varphi^*$. Then there exists a prime filter $G \in (\mathbf{A}_P/\equiv_\varphi)^*$ such that $F = \pi_\varphi^*(G)$. Consequently, $\alpha^*(F) = \alpha^*\pi_\varphi^*(G) = \pi_\varphi^*(G) = F$, and this proves that $F \in \operatorname{fp} \alpha^*$.

   Conversely, suppose that $\widehat{\varphi}^\infty \subseteq \operatorname{fp} \alpha^*$. Given $G \in (\mathbf{A}_P/\equiv_\varphi)^*$ we have $\pi_\varphi^*(G) \in \widehat{\varphi}^\infty$ by Proposition 4.4 and thus $\pi_\varphi^*(G) \in \operatorname{fp} \alpha^*$. Hence $\alpha^*(\pi_\varphi^*(G)) = \pi_\varphi^*(G)$. This proves that $\alpha^*\pi_\varphi^* = \pi_\varphi^*$, and therefore $\pi_\varphi\alpha = \pi_\varphi$.

□

We finally arrive at a characterization similar to Theorem 3.5.

Theorem 4.6. *Let $\varphi, \delta \in \mathcal{L}_P^{\mathsf{mod}}$ with $P \subseteq \operatorname{Prop}$ finite. Then:*

1. *φ is $\mathbf{L}$-unifiable modulo δ if and only if there exists a bounded morphism $f : \mathfrak{F}_Q \to \mathfrak{F}_P$ such that $P \subseteq Q$ and $f[\widehat{\delta}^\infty] \subseteq \widehat{\varphi}^\infty$;*
2. *φ is $\mathbf{L}$-projective modulo δ if and only if there exists a bounded morphism $f : \mathfrak{F}_P \to \mathfrak{F}_P$ such that $f[\widehat{\delta}^\infty] \subseteq \widehat{\varphi}^\infty \subseteq \operatorname{fp} f$.*

As before, we will call a *dual unifier* of φ modulo δ any bounded morphism $f : \mathfrak{F}_Q \to \mathfrak{F}_P$ such that $P \subseteq Q$ and $f[\widehat{\delta}^\infty] \subseteq \widehat{\varphi}^\infty$, and a *projective dual unifier* of φ modulo δ any bounded morphism $f : \mathfrak{F}_P \to \mathfrak{F}_P$ such that $f[\widehat{\delta}^\infty] \subseteq \widehat{\varphi}^\infty \subseteq \operatorname{fp} f$.

5 Applications

In this section we delve into various applications of Theorem 3.5 and Theorem 4.6. In the intuitionistic case, we first reprove Wroński's result that the super-intuitionistic logics with relative projective unification are exactly the extensions of **LC**. Admittedly, the original proof

is already fairly concise, but our approach may reveal some new insights, and will also be a source of inspiration for Theorem 5.6. Recall that **LC** is defined as

$$\mathbf{LC} := \mathbf{IPC} + (p \to q) \vee (q \to p).$$

First we set some conventions. Let $\mathbf{L}$ be a super-intuitionistic logic and let $P \subseteq \mathrm{Prop}$. From now on we abstract away from the nature of the elements of X_P: we see them as points instead of prime filters, and denote them with the letters $x, y, z, \ldots$ Given $A \subseteq X_P$, we write $\downarrow A := \{x \in X_P \mid \exists y \in A, x \leq_P y\}$. For convenience, if $x \in X_P$ and $\varphi \in \mathcal{L}_P^{\mathsf{mod}}$ we let $x \Vdash \varphi$ stand for $x \in \widehat{\varphi}$.

Proposition 5.1 ([18])**.** *The logic* **LC** *is locally tabular.*

Proposition 5.2 ([30])**.** *Suppose that* $\mathbf{L}$ *is an extension of* **LC**. *Then for all* $P \subseteq \mathrm{Prop}$ *and* $x, y, z \in X_P$, *if* $x \leq_P y$ *and* $x \leq_P z$ *then either* $y \leq_P z$ *or* $z \leq_P y$.

Thus, the canonical frame (over finitely many variables) of **LC** and its extensions is a finite union of finite trees. This structure suggests a simple guideline to construct a projective dual unifier for a unifiable formula φ, namely: for every element x of one of these trees, map x to the first point above x that belongs to $\widehat{\varphi}$, when it exists; if no such point exists, map x to $f(x)$, where f is an arbitrary dual unifier of φ. This strategy allows us to establish Theorem 5.3 below.

Theorem 5.3. *Let* $\mathbf{L}$ *be a super-intuitionistic logic. The following are equivalent:*

1. $\mathbf{L}$ *is an extension of* **LC**,
2. $\mathbf{L}$ *has projective relative unification,*
3. $\mathbf{L}$ *has projective unification.*

Proof. From (1) to (2), assume that $\mathbf{LC} \subseteq \mathbf{L}$. Let $P \subseteq \mathrm{Prop}$ finite and let $\delta, \varphi \in \mathcal{L}_P^{\mathsf{int}}$ such that φ is $\mathbf{L}$-unifiable modulo δ. Then there exists a unifier $\sigma : \mathcal{L}_P^{\mathsf{int}} \to \mathcal{L}_Q^{\mathsf{int}}$ of φ modulo δ. Obviously we can assume $P \subseteq Q$. Then, as explained in Remark 2.14, we can construct a unifier $\sigma' : \mathcal{L}_Q^{\mathsf{int}} \to \mathcal{L}_Q^{\mathsf{int}}$ of φ. By Proposition 2.18 and Proposition 3.4 we then obtain a dual unifier $f := (\sigma')^* : \mathfrak{F}_Q \to \mathfrak{F}_Q$ of φ modulo δ (see Section 3). By construction of $\leq_Q$, it is immediate that $\widehat{\varphi}$ is *upward closed*, in the sense that $x \in \widehat{\varphi}$ and $x \leq_Q y$ implies $y \in \widehat{\varphi}$. To define a projective dual unifier $g : \mathfrak{F}_Q \to \mathfrak{F}_Q$ of φ modulo δ, we consider $x \in X_Q$ and proceed as follows:

(a) If $x \in \downarrow\widehat{\varphi}$ then we set $g(x)$ to be the smallest $y \in \widehat{\varphi}$ (with respect to $\leq_Q$) such that $x \leq_Q y$;

(b) otherwise, we set $g(x) := f(x)$.

This procedure described in case (a) is depicted in Figure 1 – but it requires some justification. By Proposition 5.2, the order $\leq_Q$ is total over $Y := \{y \in \widehat{\varphi} \mid x \leq_Q y\}$. In addition, Y is non-empty by assumption, and finite since $\mathbf{L}$ is locally tabular. Therefore Y admits a smallest element, which can safely be taken as $g(x)$. We now prove that g is a bounded morphism. First, suppose that $x \leq_Q y$. We consider two cases.

1. Suppose that $x \in \downarrow\widehat{\varphi}$, i.e., x falls in case (a). Then $x \leq_Q g(x)$ and $x \leq_Q y$, and by Proposition 5.2 we obtain either $g(x) \leq_Q y$ or $y \leq_Q g(x)$. Since $g(x) \in \widehat{\varphi}$, both cases lead to $y \in \downarrow\widehat{\varphi}$, whence $y \leq_Q g(y)$. Since $x \leq_Q y$ we then obtain $x \leq_Q g(y)$. Since $g(y) \in \widehat{\varphi}$, it follows by construction of $g(x)$ that $g(x) \leq_Q g(y)$.
2. Otherwise, x falls in case (b), and we have $g(x) = f(x)$. If $y \in \downarrow\widehat{\varphi}$, we have $x \in \downarrow\widehat{\varphi}$, a contradiction. Thus $y \notin \downarrow\widehat{\varphi}$ and $g(y) = f(y)$. Since $x \leq_Q y$ and f is a bounded morphism we obtain $f(x) \leq_Q f(y)$, and therefore $g(x) \leq_Q g(y)$.

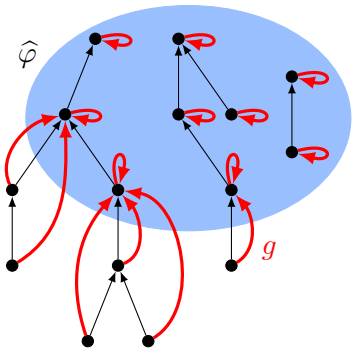

Figure 1: The construction of the dual projective unifier g (in bold arrows)

Now suppose that $g(x) \leq_Q y$. If x falls in case (a) then $x \leq_Q g(x)$, and by transitivity we obtain $x \leq_Q y$. In addition, $g(x) \in \widehat{\varphi}$ entails $y \in \widehat{\varphi}$. Thus $x \leq_Q y$ with $g(y) = y$, as desired. Otherwise, we have $g(x) = f(x)$. Then since f is a bounded morphism there exists $z \in X_Q$ such that $f(z) = y$ and $x \leq_Q z$. Since $x \notin {\downarrow}\widehat{\varphi}$ we obtain $z \notin {\downarrow}\widehat{\varphi}$ too, and therefore $g(z) = f(z) = y$. Further, since X_Q is finite, the continuity condition is immediately true for g, as explained in Section 3. It is also immediate that $g[\widehat{\delta}] \subseteq \widehat{\varphi}$. Finally, if $x \in \widehat{\varphi}$, then by construction we have $g(x) := x$, and this proves that $\widehat{\varphi} \subseteq \mathrm{fp}\ g$. We conclude that φ is $\mathbf{L}$-projective.

The implication from (2) to (3) is obvious. From (3) to (1), assume that $\mathbf{LC} \not\subseteq \mathbf{L}$. Let $P := \{p, q\}$. We prove that $\varphi := (p \to q) \vee (q \to p)$ is not projective. First, we have $\nvdash_{\mathbf{L}} \varphi$ by assumption. So by the Prime Filter Theorem [8, Th. 7.41], there exists $x \in X_P$ such that $x \nVdash (p \to q) \vee (q \to p)$. Then $x \nVdash p \to q$, whence $[q]_{\mathbf{L}} \notin y_0$ with $y_0 := \{a \in \mathbf{A}_P \mid \exists b \in x, b \wedge [p]_{\mathbf{L}} \leq a\}$, and so by the Prime Filter Theorem again there exists $y \in X_P$ such that $y_0 \subseteq y$ and $[q]_{\mathbf{L}} \notin y$. Therefore $x \leq_P y$, $y \Vdash p$ and $y \nVdash q$. Likewise, there exists $z \in X_P$ such that $x \leq_P z$, $z \Vdash q$ and $z \nVdash p$. Suppose toward a contradiction that there exists a projective dual unifier $f : \mathfrak{F}_P \to \mathfrak{F}_P$ of φ. Since $y \Vdash p$, we have $y \Vdash q \to p$, and therefore $y \in \widehat{\varphi}$ and $f(y) = y$. Since f is a bounded morphism and $x \leq_P y$, we obtain $f(x) \leq_P y$. Likewise, we can prove that $f(x) \leq_P z$. Then since $f(x) \in \widehat{\varphi}$ we have either $f(x) \Vdash p \to q$ or $f(x) \Vdash q \to p$. In the first case we obtain $y \Vdash q$, and in the second case we obtain $z \Vdash p$. Both outcomes are contradictions, and this concludes the proof. □

Moving to modal duality, we turn our attention to the following logics (with $n, k \geq 1$) :

$$\begin{array}{rcl}
\mathbf{K4} & := & \mathbf{K} + (\Diamond\Diamond p \to \Diamond p) \\
\mathbf{K5} & := & \mathbf{K} + (\Diamond p \to \Box\Diamond p) \\
\mathbf{K45} & := & \mathbf{K4} + (\Diamond p \to \Box\Diamond p) \\
\mathbf{K4D1} & := & \mathbf{K4} + \Box(\Box p \to q) \vee \Box(\Box q \to p) \\
\mathbf{K4}_n & := & \mathbf{K} + (\Diamond^{n+1} p \to \Diamond^{\leq n} p) \\
\mathbf{K4}_n\mathbf{D1}_n & := & \mathbf{K4}_n + \Box(\Box^{\leq n} p \to q) \vee \Box(\Box^{\leq n} q \to p) \\
\mathbf{K4}_n\mathbf{B}_k & := & \mathbf{K4}_n + (p \to \Box^{\leq k}\Diamond^{\leq k} p)
\end{array}$$

First, we recall some elementary facts and definitions. Let $\mathbf{L}$ be a normal modal logic and let $P \subseteq \mathrm{Prop}$. Given $X \subseteq X_P$, we write $R_P X := \{y \in X_P \mid \exists x \in X, x R_P y\}$ and $R_P^{-1} X := \{x \in X_P \mid \exists y \in X, x R_P y\}$. We then call X *upward closed* if $R_P X \subseteq X$, and *downward closed* if $R_P^{-1} X \subseteq X$. By recursion, we also define $R_P^0 := \{(x, x) \mid x \in X_P\}$ and $R_P^{n+1} := \{(x, z) \mid (x, y) \in R_P^n$ and $(y, z) \in R_P\}$ for all $n \in \mathbb{N}$. Then for all $n \in \mathbb{N}$ we write $R_P^{\leq n} := \bigcup_{k=0}^{n} R_P^k$.

Proposition 5.4. *Let* $\mathbf{L}$ *be a normal modal logic, and* $P \subseteq \mathrm{Prop}$ *be finite.*

1. *If* $\mathbf{K4} \subseteq \mathbf{L}$*, then* $\mathfrak{F}_P$ *is transitive.*
2. *If* $\mathbf{K5} \subseteq \mathbf{L}$*, then* $\mathfrak{F}_P$ *is* Euclidean*, that is, if* xR_Py *and* xR_Pz *then* yR_Pz*.*
3. *If* $\mathbf{K4D1} \subseteq \mathbf{L}$*, then* $\mathfrak{F}_P$ *is transitive and* strongly connected*, that is, if* xR_Py *and* xR_Pz*, then either* yR_Pz *or* zR_Py*.*
4. *If* $\mathbf{K4}_n \subseteq \mathbf{L}$*, then* $\mathfrak{F}_P$ *is* n-transitive*, that is,* $xR_P^{n+1}y$ *implies* $xR_P^{\leq n}y$*.*
5. *If* $\mathbf{K4}_n\mathbf{D1}_n \subseteq \mathbf{L}$*, then* $\mathfrak{F}_P$ *is* n*-transitive and* strongly n-connected*, that is, if* xR_Py *and* xR_Pz*, then either* $yR_P^{\leq n}z$ *or* $zR_P^{\leq n}y$*.*
6. *If* $\mathbf{K4}_n\mathbf{B}_k \subseteq \mathbf{L}$ *with* $n, k \geq 1$*, then* $\mathfrak{F}_P$ *is* n*-transitive and* k-symmetric*, that is,* $xR_P^{\leq k}y$ *implies* $yR_P^{\leq k}x$*.*

Proof.

1. This is a consequence of $\Diamond\Diamond p \to \Diamond p$ being a Sahlqvist formula [7, Sect. 3.6 & Th. 4.42].
2. Same as 1.
3. By 1, we already know that $\mathfrak{F}_P$ is transitive. Further, the formula $\Box(\Box p \to q) \vee \Box(\Box q \to p)$ is equivalent to the Sahlqvist formula $\Diamond(\Box p \wedge \neg q) \to \Box(\Box q \to p)$. By [7, Th. 4.42], it follows that $\Box(\Box p \to q) \vee \Box(\Box q \to p)$ is *valid* on $\mathfrak{F}_P$ (see [7, Def. 1.24]), which entails that $\mathfrak{F}_P$ is strongly connected [8, Prop 3.40].
4. Since $(\Diamond^{n+1}p \to \Diamond^{\leq n}p)$ is a Sahlqvist formula, it is valid on $\mathfrak{F}_P$, which entails that $\mathfrak{F}_P$ is n-transitive by [20, Prop. 3.5].
5. Easily adapted from 3.
6. By 4, we already know that $\mathfrak{F}_P$ is n-transitive. Further, $(p \to \Box^{\leq k}\Diamond^{\leq k}p)$ is a Sahlqvist formula, and thus valid on $\mathfrak{F}_P$. From [20, Prop. 4.3], it follows that $\mathfrak{F}_P$ is k-symmetric.

□

We first address the logic $\mathbf{K4}_n\mathbf{B}_k$ (and its extensions). Given a unifiable formula φ, the property of k-symmetry conveniently splits the canonical frame into two disconnected parts, namely $\widehat{\varphi}^\infty$ and its complement. Then, one can easily construct a projective dual unifier of φ as follows: map every point x to itself in case $x \in \widehat{\varphi}^\infty$, and to $f(x)$ otherwise, where f is an arbitrary dual unifier of φ. This allows us to recover Kostrzycka's result [23] in a very efficient manner.

Theorem 5.5. *Let* $n, k \geq 1$*. Every extension of* $\mathbf{K4}_n\mathbf{B}_k$ *has relative projective unification.*

Proof. Let $\mathbf{L}$ be an extension of $\mathbf{K4}_n\mathbf{B}_k$. Let $P \subseteq$ Prop finite, and let $\delta, \varphi \in \mathcal{L}_P^{\mathsf{mod}}$ such that φ is $\mathbf{L}$-unifiable modulo δ. Reasoning as above, we obtain the existence of a dual unifier $f : \mathfrak{F}_Q \to \mathfrak{F}_Q$ of φ modulo δ with $Q \subseteq$ Prop finite and $P \subseteq Q$.

We argue that $\widehat{\varphi}^\infty$ is both upward and downward closed for R_Q. For suppose xR_Qy. If $x \in \widehat{\varphi}^\infty$ then for all $i \in \mathbb{N}$ we have $x \in \widehat{\Box^{i+1}\varphi}$ and thus $y \in \widehat{\Box^i\varphi}$, and it follows that $y \in \widehat{\varphi}^\infty$. Conversely, suppose that $y \in \widehat{\varphi}^\infty$. Then since $\mathbf{K4}_n\mathbf{B}_k \subseteq \mathbf{L}$ and $xR_Q^{\leq k}y$ we have $yR_Q^{\leq k}x$. Since $\widehat{\varphi}^\infty$ is upward closed it is then clear that $x \in \widehat{\varphi}^\infty$. Now let us define $g : \mathfrak{F}_Q \to \mathfrak{F}_Q$ by

$$g(x) := \begin{cases} x & \text{if } x \in \widehat{\varphi}^\infty \\ f(x) & \text{otherwise} \end{cases}$$

for all $x \in X_Q$. We prove that g is a bounded morphism. First, assume that xR_Qy. We have seen that $x \in \widehat{\varphi}^\infty$ iff $y \in \widehat{\varphi}^\infty$, and we know that f is a bounded morphism, so $g(x)R_Qg(y)$ is immediate. Now suppose that $x \in X_Q$ and $g(x)R_Qy'$. If $x \in \widehat{\varphi}^\infty$ then $g(x) = x$ and xR_Qy', and thus $y' \in \widehat{\varphi}^\infty$ too, leading to $g(y') = y'$. Otherwise we have $x \notin \widehat{\varphi}^\infty$ and $g(x) = f(x)$

with $f(x)R_Qy'$. Since f is a bounded morphism we obtain the existence of $y \in X_Q$ such that $f(y) = y'$ and xR_Qy. Then $y \notin \widehat{\varphi}^\infty$, and thus $g(y) = y'$, as desired.

Now let $\widehat{\psi} \in \mathcal{B}_Q$. We have $g^{-1}[\widehat{\psi}] = (\widehat{\psi} \cap \widehat{\varphi}^\infty) \cup (f^{-1}[\widehat{\psi}] \setminus \widehat{\varphi}^\infty)$. Since $\vdash_{\mathbf{L}} \Box^{\leq n}p \to \Box^{n+1}p$ we have $\widehat{\varphi}^\infty = \widehat{\Box^{\leq n}\varphi} \in \mathcal{B}_Q$, and therefore $g^{-1}[\widehat{\psi}] \in \mathcal{B}_Q$ too. Finally it is immediate that $g[\widehat{\delta}^\infty] \subseteq \widehat{\varphi}^\infty \subseteq \text{fp } g$. We conclude that φ is **L**-projective. □

As mentioned in Section 2, Kost [22] showed that an extension of **K4** is projective if and only if it contains **K4D1**. In Theorem 5.6, we reprove a weaker version of the right-to-left implication, limited to locally tabular logics. In Theorem 5.7 we reprove the left-to-right implication. Our reasoning follows closely the lines of the proof of Theorem 5.3. This is no accident, as the canonical frame of **LC** and **K4D1** (and their extensions) share a very similar structure.

Theorem 5.6. *Every locally tabular extension of* **K4D1** *has projective relative unification.*

Proof. Let **L** be a locally tabular extension of **K4D1**. Let $P \subseteq \text{Prop}$ finite and let $\delta, \varphi \in \mathcal{L}_P^{\mathsf{mod}}$ such that φ is **L**-unifiable modulo δ. Reasoning as above, we obtain the existence of a dual unifier $f : \mathfrak{F}_Q \to \mathfrak{F}_Q$ of φ modulo δ with $Q \subseteq \text{Prop}$ finite and $P \subseteq Q$. To define $g : \mathfrak{F}_Q \to \mathfrak{F}_Q$, we consider $x \in X_Q$ and proceed as follows:

(a) if $x \in \widehat{\varphi}^\infty$ we set $g(x) := x$;

(b) otherwise, if $x \in R_Q^{-1}\widehat{\varphi}^\infty$ then we select $g(x)$ in the set $Y := \{y \in \widehat{\varphi}^\infty \mid xR_Qy\}$ so that $g(x)R_Qy$ for all $y \in Y$;

(c) otherwise, we set $g(x) := f(x)$.

Case (b) requires some justification. Since **K4D1** $\subseteq$ **L**, the frame $\mathfrak{F}_Q$ is strongly connected, so for all $y, z \in Y$ we have either yR_Qz or zR_Qy. In addition, Y is non-empty by assumption, and finite since **L** is locally tabular. This yields the existence of a 'smallest' element $g(x)$ with respect to R_Q – of course such an element is not necessarily unique. We now prove that g is a bounded morphism. First, suppose that xR_Qy. We examine each case for x.

1. If $x \in \widehat{\varphi}^\infty$, then $y \in \widehat{\varphi}^\infty$ too, and thus $g(x)R_Qg(y)$ is immediate.
2. Otherwise, suppose that $x \in R_Q^{-1}\widehat{\varphi}^\infty$. Then $xR_Qg(x)$ and xR_Qy, and since $\mathfrak{F}_Q$ is strongly connected we obtain either $g(x)R_Qy$ or $yR_Qg(x)$. Since $g(x) \in \widehat{\varphi}^\infty$ it follows respectively that $y \in \widehat{\varphi}^\infty$ or $y \in R_Q^{-1}\widehat{\varphi}^\infty$. In both cases we have $yR_Q^{\leq 1}g(y)$. Since xR_Qy and $\mathfrak{F}_Q$ is transitive we then obtain $xR_Qg(y)$. Since $g(y) \in \widehat{\varphi}^\infty$, it follows by construction of $g(x)$ that $g(x)R_Q^{\leq 1}g(y)$. If $g(x)R_Qg(y)$ we are done. Otherwise $g(x) = g(y)$. Since $xR_Qg(x)$ and $\mathfrak{F}_Q$ is strongly connected we also have $g(x)R_Qg(x)$. Therefore $g(x)R_Qg(y)$ holds as well.
3. Otherwise, we have $g(x) = f(x)$. If $y \in R_Q^{-1}\widehat{\varphi}^\infty$, we have $x \in R_Q^{-1}\widehat{\varphi}^\infty$ too by transitivity, a contradiction. Thus $y \notin R_Q^{-1}\widehat{\varphi}^\infty$ and $g(y) = f(y)$. Since xR_Qy and f is a bounded morphism we obtain $f(x)R_Qf(y)$, and therefore $g(x)R_Qg(y)$.

Now suppose that $g(x)R_Qy$. If x falls in case (a) or case (b) then $xR_Q^{\leq 1}g(x)$, and by transitivity we obtain xR_Qy. In addition, $g(x) \in \widehat{\varphi}^\infty$ entails $y \in \widehat{\varphi}^\infty$. Thus xR_Qy with $g(y) = y$, as desired. Otherwise, x falls in case (c), and we have $g(x) = f(x)$. Then since f is a bounded morphism there exists $z \in X_Q$ such that $f(z) = y$ and xR_Qz. Since $x \notin R_Q^{-1}\widehat{\varphi}^\infty$ we obtain $z \notin R_Q^{-1}\widehat{\varphi}^\infty$ too, and therefore $g(z) = f(z) = y$.

Again, the continuity of g follows from the fact that X_Q is finite. It is also immediate that $g[\widehat{\delta}^\infty] \subseteq \widehat{\varphi}^\infty \subseteq \text{fp } g$. We conclude that φ is **L**-projective. □

Theorem 5.7. *Every extension of* **K4** *with projective unification is also an extension of* **K4D1**.

Proof. Suppose that $\mathbf{K4} \subseteq \mathbf{L}$. By contraposition suppose that $\mathbf{K4D1} \not\subseteq \mathbf{L}$. We prove that $\varphi := \Box(\Box p \to q) \vee \Box(\Box q \to p)$ is not projective. Let $P := \{p, q\}$. First we have $\nvdash_{\mathbf{L}} \varphi$ by assumption. By the Prime Filter Theorem, there exists a prime filter $x \in X_P$ such that $x \Vdash \Diamond(\Box p \wedge \neg q) \wedge \Diamond(\Box q \wedge \neg p)$. Then, by the Existence Lemma [7, Lemma 4.20] (together with Remark 4.2), there exist $y, z \in X_P$ such that xR_Py, xR_Pz, $y \Vdash \Box p \wedge \neg q$ and $z \Vdash \Box q \wedge \neg p$. Suppose toward a contradiction that there exists a projective dual unifier $f : \mathfrak{F}_P \to \mathfrak{F}_P$ of φ.

Since $y \Vdash \Box p$ and $\mathbf{K4} \subseteq \mathbf{L}$, we have $y \Vdash \Box^{n+1} p$ and thus $y \Vdash \Box^n \Box(\Box q \to p)$ for all $n \in \mathbb{N}$. Therefore $y \in \widehat{\varphi}^\infty$ and $f(y) = y$. Since f is a bounded morphism and xR_Py, we obtain $f(x)R_Py$. Likewise, we can prove that $f(x)R_Pz$. Then since $f(x) \in \widehat{\varphi}^\infty$ we have in particular $f(x) \Vdash \varphi$, and thus either $f(x) \Vdash \Box(\Box p \to q)$ or $f(x) \Vdash \Box(\Box q \to p)$. In the first case we obtain $y \Vdash \Box p \to q$, and in the second case we obtain $z \Vdash \Box q \to p$. Both outcomes are contradictions, and this concludes the proof. □

Interestingly, the proof of Theorem 5.7 can easily be adapted to derive an analogous new result for extensions of $\mathbf{K4}_n$.

Theorem 5.8. *Every extension of* $\mathbf{K4}_n$ *with projective unification is also an extension of* $\mathbf{K4}_n\mathbf{D1}_n$.

Proof. Suppose that $\mathbf{K4}_n \subseteq \mathbf{L}$. By contraposition suppose that $\mathbf{K4}_n\mathbf{D1}_n \not\subseteq \mathbf{L}$. We prove that $\varphi := \Box(\Box^{\leq n} p \to q) \vee \Box(\Box^{\leq n} q \to p)$ is not projective. Let $P := \{p, q\}$. We have $\nvdash_{\mathbf{L}} \varphi$ by assumption, and arguing as above we obtain a prime filter $x \in X_P$ such that $x \Vdash \Diamond(\Box^{\leq n} p \wedge \neg q) \wedge \Diamond(\Box^{\leq n} q \wedge \neg p)$. Then there exist $y, z \in X_P$ such that xR_Py, xR_Pz, $y \Vdash \Box^{\leq n} p \wedge \neg q$ and $z \Vdash \Box^{\leq n} q \wedge \neg p$. Suppose that there exists a projective dual unifier $f : \mathfrak{F}_P \to \mathfrak{F}_P$ of φ.

Since $y \Vdash \Box^{\leq n} p$ and $\mathbf{K4}_n \subseteq \mathbf{L}$, we have $y \Vdash \Box^{k+1} p$ and thus $y \Vdash \Box^k \Box(\Box^{\leq n} q \to p)$ for all $k \in \mathbb{N}$. Therefore $y \in \widehat{\varphi}^\infty$ and $f(y) = y$. Since f is a bounded morphism and xR_Py, we obtain $f(x)R_Py$. Likewise, we can prove that $f(x)R_Pz$. Then since $f(x) \in \widehat{\varphi}^\infty$ we have either $f(x) \Vdash \Box(\Box^{\leq n} p \to q)$ or $f(x) \Vdash \Box(\Box^{\leq n} q \to p)$. In the first case we obtain $y \Vdash \Box^{\leq n} p \to q$, and in the second case we obtain $z \Vdash \Box^{\leq n} q \to p$. This yields a contradiction. □

Obviously Theorem 5.6 is weaker than Kost's result, but still covers a decent range of logics. In particular, it is enough to conclude that all extensions of **K45** are projective, since **K5** is locally tabular [26, Corollary 5]. We then refine this result by showing that the projective extensions of **K5** are, in fact, *exactly* the extensions of **K45**. We thus obtain a complete description of the landscape of projective logics above **K5**, which was only partially known prior to our work.

Theorem 5.9. *Let* $\mathbf{L}$ *be an extension of* **K5**. *The following are equivalent:*

1. $\mathbf{L}$ *is an extension of* **K45**,
2. $\mathbf{L}$ *has projective relative unification,*
3. $\mathbf{L}$ *has projective unification.*

Proof. We already know that (1) implies (2), and that (2) implies (3). Now, suppose that $\mathbf{K45} \not\subseteq \mathbf{L}$. We prove that $\varphi := \Diamond\Diamond p \to \Diamond p$ is not projective. Let $P := \{p\}$. First we have $\nvdash_{\mathbf{L}} \Diamond\Diamond p \to \Diamond p$ by assumption. Arguing as before, we obtain a prime filter $x \in X_P$ such that $x \Vdash \Diamond\Diamond p \wedge \Box\neg p$. Then there exists $y \in X_P$ such that xR_P^2y and $y \in \widehat{p}$. Suppose toward a contradiction that there exists a projective dual unifier $f : \mathfrak{F}_P \to \mathfrak{F}_P$ of φ. Then $f(x)R_P^2 f(y)$. Since y has a predecessor, it belongs to a final cluster (see [26] for a comprehensive description of Euclidean frames). Consequently $y \in \widehat{\varphi}^\infty$, and thus $f(y) = y$. Hence $f(x) \Vdash \Diamond\Diamond p$, and since $f(x) \in \widehat{\varphi}^\infty$ we obtain $f(x) \Vdash \Diamond p$. Therefore there exists $z \in X_P$ such that $f(x)R_Pz$ and $z \Vdash p$. Since f is a bounded morphism there exists $t \in X_P$ such that xR_Pt and $f(t) = z$. Again $t \in \widehat{\varphi}^\infty$ and therefore $f(t) = t$. Hence xR_Pz, contradicting $x \Vdash \Box\neg p$. This concludes the proof. □

6 Conclusion

In this paper, through substantial use of the duality between descriptive frames and modal algebras, we have given a necessary and sufficient condition for modal formulas to be projective. Applying it to the extensions of **IPC**, $\mathbf{K4}_n\mathbf{B}_k$ and **K4D1**, we have reproved known results obtained by Wroński [31, 32], Kostrzycka [23] and Kost [22]. Applying it to the extensions of **K5**, we have proved the new result that the projective extensions of **K5** are exactly the extensions of **K45**. In addition, we have generalized the problem of unification and proved all of these results in the more flexible framework of relative unification. While syntactic methods tend to involve technical twists, our proofs primarily exploit the relational properties of the canonical frame, and thus rely on visual intuitions. The trade-off is that duality requires more preliminary work and a heavier conceptual apparatus, but once this machinery is in place, it leads to fairly lightweight and concise proofs. At the end of the day, syntactic and duality-based techniques gets us to the same destination via radically different routes; though some of our results are not new, we hope that our approach will contribute to give a fresh perspective on the problem. Of course, this is only a first insight of what duality has to offer. Apart from the results about the unification types of modal logics mentioned in Section 5 and the new result about **K5** extensions, very little is known. For example, the unification types of $\mathbf{KD} := \mathbf{K} + \Diamond\top$ and $\mathbf{KT} := \mathbf{K} + \Box p \to p$ are just known to be non-unitary, seeing that the substitutions $\sigma_\top$ and $\sigma_\bot$ on the propositional variable p defined by $\sigma_\top(p) := \top$ and $\sigma_\bot(p) := \bot$ constitute both a basis of concise **KD**-unifiers and a basis of concise **KT**-unifiers of $\Box p \vee \Box\neg p$. This is an immediate consequence of the fact that **KD** and **KT** possess the modal disjunction property saying that for all formulas φ, ψ, if $\Box\varphi \vee \Box\psi$ is in **KD** (resp. **KT**) then either φ or ψ is in **KD** (resp. **KT**). To take another example, the unification types of $\mathbf{DAlt}_1 := \mathbf{KD} + \Diamond p \to \Box p$ and $\mathbf{KB} := \mathbf{K} + p \to \Box\Diamond p$ are not known either. Therefore, much remain to be done and further investigations are needed for obtaining, by means of our duality approach, the unification types of modal logics such as **KD**, **KT**, $\mathbf{DAlt}_1$ and **KB**. Finally, we have seen in Theorem 5.3 that all super-intuitionistic logics with projective unification also have relative projective unification, but it is unknown whether the same result holds for modal logics – a new open question that will hopefully be addressed in the future.

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