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► **To cite this version:**

Philippe Balbiani, Cigdem Gencer. Undecidability of admissibility in the product of two Alt logics. Logic Journal of the IGPL, 2023, 10.1093/jigpal/jzad021 . hal-04307720

HAL Id: hal-04307720

<https://hal.science/hal-04307720v1>

Submitted on 26 Nov 2023

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Undecidability of admissibility in the product of two **Alt** logics

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Abstract

The product of two **Alt** logics possesses the polynomial product finite model property and its membership problem is **coNP**-complete. Using a reduction from an undecidable domino-tiling problem, we prove that its admissibility problem is undecidable.

Keywords: Products of two modal logics. Admissibility of inference rules. Domino-tiling problems. Computability.

1 Introduction

An inference rule is admissible in a modal logic if the logic is closed with respect to applications of the rule. And the admissibility problem in a modal logic \mathbf{L} is to determine, given an inference rule $\frac{\Gamma}{\phi}$, whether for all substitutions σ , if $\sigma(\Gamma) \subseteq \mathbf{L}$ then $\sigma(\phi) \in \mathbf{L}$. In 1984, the admissibility problem has been proved by Rybakov [21] to be decidable in Intuitionistic Propositional Logic and in transitive modal logics such as **S4**. Algorithms deciding admissibility of inference rules in these modal logics have been proposed by Ghilardi [14] and Iemhoff [16]. The computational complexity of the admissibility problem in some of these modal logics has been shown by Jeřábek [17] to be **coNEXPTIME**-complete.

The ideas of Rybakov and the algorithms of Ghilardi and Iemhoff are applicable to a wide set of modal logics. Since the modal logics considered in [14, 16, 21] are all decidable, one may ask whether for all modal logics, the decidability of its membership problem ensures the decidability of its admissibility problem. In 1992, this question has been negatively answered by Chagrov [7] who has constructed a modal logic with

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a decidable membership problem and an undecidable admissibility problem. However, the modal logic constructed by Chagrov was rather *artificial*. Therefore, one may ask whether more *natural* modal logics exist with a decidable membership problem and an undecidable admissibility problem.

The effect of adding the universal modality to modal logics has been investigated by Hemaspaandra in 1996. In particular, for modal logics such as \mathbf{K} and $\mathbf{K4}$, it has been proved in [15] that enriching the modal language by the universal modality increases the complexity of the membership problem from \mathbf{PSPACE} -complete to $\mathbf{EXPTIME}$ -complete. Although the computability of the admissibility problem for \mathbf{K} is unknown, the admissibility problem for $\mathbf{K4}$ is known to be decidable since the pioneering work of Rybakov [22]. Hence, it came as a surprise when Wolter and Zakharyashev [24] proved that the admissibility problem for all modal logics between \mathbf{K} and $\mathbf{K4}$ is undecidable when the modal language is enriched by the universal modality.

Pushing further the envelope, one may ask whether modal logics exist with a membership problem in \mathbf{coNP} and an undecidable admissibility problem. The combination method on modal logics known as the product construction has been firstly investigated in [11]. For more on it, see [10, 19]. In this paper, we consider the product of two \mathbf{Alt} logics. This product has many interesting properties: it possesses the polynomial product finite model property; its membership problem is \mathbf{coNP} -complete. However, using a reduction from an undecidable domino-tiling problem, we prove that its admissibility problem is undecidable. We assume the reader is at home with the basic tools and techniques in modal logics. For more on them, see [6, 8, 18].

2 A domino-tiling problem

For all $I \in \mathbb{N}$, let $(I) = \{i: 1 \leq i \leq I\}$.

The following *domino-tiling problem* (Π) has been used for proving undecidability results about description logic \mathcal{ALCQIO} [20]. An *instance of* (Π) is a 7-tuple $(\Delta, V, H, \Delta_{up}, \Delta_{down}, \Delta_{right}, \Delta_{left})$ where

- Δ is a finite set of domino-types,
- V and H are binary relations on Δ ,
- $\Delta_{up}, \Delta_{down}, \Delta_{right}$ and Δ_{left} are subsets of Δ .

A *tiling of an instance* $(\Delta, V, H, \Delta_{up}, \Delta_{down}, \Delta_{right}, \Delta_{left})$ of (Π) is a triple (I, J, f) where

- $I, J \geq 1$,
- f is a function associating an element $f(i, j) \in \Delta$ to each $(i, j) \in (I) \times (J)$.

We shall say that a tiling (I, J, f) of an instance $(\Delta, V, H, \Delta_{up}, \Delta_{down}, \Delta_{right}, \Delta_{left})$ of (Π) is *correct* if the following conditions hold:

- for all $(i, j) \in (I - 1) \times (J)$, $(f(i, j), f(i + 1, j)) \in V$,
- for all $(i, j) \in (I) \times (J - 1)$, $(f(i, j), f(i, j + 1)) \in H$,
- for all $j \in (J)$, $f(I, j) \in \Delta_{up}$,
- for all $j \in (J)$, $f(1, j) \in \Delta_{down}$,
- for all $i \in (I)$, $f(i, J) \in \Delta_{right}$,
- for all $i \in (I)$, $f(i, 1) \in \Delta_{left}$.

Proposition 1 *It is undecidable to determine, given an instance $(\Delta, V, H, \Delta_{up}, \Delta_{down}, \Delta_{right}, \Delta_{left})$ of (Π) , whether there exists a correct tiling of $(\Delta, V, H, \Delta_{up}, \Delta_{down}, \Delta_{right}, \Delta_{left})$.*

Proof: See [20]. \dashv

3 Syntax and semantics

Let **VAR** be a countable set of *propositional variables* (x, y , etc). The *formulas* are inductively defined by the following rule:

- $\phi ::= x \mid \perp \mid \neg\phi \mid (\phi \vee \phi) \mid \Box_1\phi \mid \Box_2\phi$.

The other Boolean constructs are defined as usual. We adopt the standard rules for omission of the parentheses. The formulas $\Diamond_1\phi$ and $\Diamond_2\phi$ are the abbreviations defined as follows:

- $\Diamond_1\phi := \neg\Box_1\neg\phi$,
- $\Diamond_2\phi := \neg\Box_2\neg\phi$.

For all non-negative integers n , the formulas $\Box_1^n\phi$ and $\Box_2^n\phi$ are the abbreviations inductively defined as follows:

- $\Box_1^0\phi := \phi$,
- $\Box_2^0\phi := \phi$,
- $\Box_1^{n+1}\phi := \Box_1\Box_1^n\phi$,
- $\Box_2^{n+1}\phi := \Box_2\Box_2^n\phi$.

For all non-negative integers n , the formulas $\Diamond_1^n\phi$ and $\Diamond_2^n\phi$ are the abbreviations defined as follows:

- $\Diamond_1^n\phi := \neg\Box_1^n\neg\phi$,
- $\Diamond_2^n\phi := \neg\Box_2^n\neg\phi$.

A *substitution* is a function σ associating to each propositional variable x a formula $\sigma(x)$. For all formulas ϕ , let $\sigma(\phi)$ be the formula obtained from ϕ after having uniformly replaced each occurrence of x by $\sigma(x)$ for each propositional variable x . For all finite sets Γ of formulas, let $\sigma(\Gamma) = \{\sigma(\phi) : \phi \in \Gamma\}$. An *inference rule* is a pair $\frac{\Gamma}{\phi}$ consisting of a finite set Γ of formulas and a formula ϕ .

For all $I, J \in \mathbb{N}$, let $I \otimes J = \{(i, j) : 0 \leq i \leq I \text{ and } 0 \leq j \leq J\}$. For all $I, J \in \mathbb{N}$, let \ll be the well-founded partial order on $I \otimes J$ such that for all $(i, j), (k, l) \in I \otimes J$, $(i, j) \ll (k, l)$ if and only if either $i < k$ and $j \leq l$, or $i \leq k$ and $j < l$.

A *model* is a triple $\mathcal{M} = (I, J, v)$ where $I, J \in \mathbb{N}$ and v is a function associating a subset $v(x)$ of $I \otimes J$ to each $x \in VAR$. In this case, we shall say that \mathcal{M} is *based on I and J* . The *truth of a formula ϕ in a model $\mathcal{M} = (I, J, v)$ at $(i, j) \in I \otimes J$* (in symbols $(i, j) \models_{\mathcal{M}} \phi$) is inductively defined as follows:

- $(i, j) \models_{\mathcal{M}} x$ if and only if $(i, j) \in v(x)$,
- $(i, j) \not\models_{\mathcal{M}} \perp$,
- $(i, j) \models_{\mathcal{M}} \neg\phi$ if and only if $(i, j) \not\models_{\mathcal{M}} \phi$,
- $(i, j) \models_{\mathcal{M}} \phi \vee \psi$ if and only if $(i, j) \models_{\mathcal{M}} \phi$ or $(i, j) \models_{\mathcal{M}} \psi$,
- $(i, j) \models_{\mathcal{M}} \Box_1\phi$ if and only if if $i < I$ then $(i + 1, j) \models_{\mathcal{M}} \phi$,
- $(i, j) \models_{\mathcal{M}} \Box_2\phi$ if and only if if $j < J$ then $(i, j + 1) \models_{\mathcal{M}} \phi$.

A formula ϕ is said to be *true in a model $\mathcal{M} = (I, J, v)$* (in symbols $\models_{\mathcal{M}} \phi$) if for all $(i, j) \in I \otimes J$, $(i, j) \models_{\mathcal{M}} \phi$. We shall say that a finite set Γ of formulas is *true in a model $\mathcal{M} = (I, J, v)$* (in symbols $\models_{\mathcal{M}} \Gamma$) if for all $\phi \in \Gamma$, $\models_{\mathcal{M}} \phi$.

A formula ϕ is said to be *valid* (in symbols $\models \phi$) if for all models \mathcal{M} , $\models_{\mathcal{M}} \phi$. Obviously,

Proposition 2 *For all formulas ϕ , the following conditions are equivalent:*

- $\models \phi$,
- $\models \Box_1\phi$,
- $\models \Box_2\phi$.

From now on in this paper, we use Proposition 2 without explicitly mentioning it. We shall say that a finite set Γ of formulas is *valid* (in symbols $\models \Gamma$) if for all $\phi \in \Gamma$, $\models \phi$.

Proposition 3 *The set of all valid formulas is coNP-complete.*

Proof: See [10, Theorem 8.53]. \dashv

An inference rule $\frac{\Gamma}{\phi}$ is said to be *admissible* if for all substitutions σ , if $\models \sigma(\Gamma)$ then $\models \sigma(\phi)$.

Proposition 4 *The set of all admissible inference rules is undecidable.*

Proof: See Section 4. \dashv

4 Reduction of (Π)

Considering an instance $\mathcal{I} = (\Delta, V, H, \Delta_{up}, \Delta_{down}, \Delta_{right}, \Delta_{left})$ of (Π) , we will define an inference rule $R_{\mathcal{I}}$ such that $R_{\mathcal{I}}$ is not admissible if and only if there exists a correct tiling of \mathcal{I} . Let $\delta_1, \dots, \delta_a$ be a list of \mathcal{I} 's domino-types. We will use the propositional variables x_1, \dots, x_a in correspondence with its elements. We will also use the propositional variables y, z . Let us consider the following formulas:

- ϕ_1 : $\neg(x_b \wedge x_c)$ where $b, c \in (a)$ and $b \neq c$,
- ϕ_2 : $(x_b \rightarrow \Box_1 \bigvee \{x_c : c \in (a) \text{ and } (\delta_b, \delta_c) \in V\})$ where $b \in (a)$,
- ϕ_3 : $(x_b \rightarrow \Box_2 \bigvee \{x_c : c \in (a) \text{ and } (\delta_b, \delta_c) \in H\})$ where $b \in (a)$,
- ϕ_4 : $(y \wedge \Box_1 \perp \rightarrow \bigvee \{x_b : b \in (a) \text{ and } \delta_b \in \Delta_{up}\})$,
- ϕ_5 : $(y \wedge \neg z \rightarrow \Box_1(z \rightarrow \bigvee \{x_b : b \in (a) \text{ and } \delta_b \in \Delta_{down}\}))$,
- ϕ_6 : $(z \wedge \Box_2 \perp \rightarrow \bigvee \{x_b : b \in (a) \text{ and } \delta_b \in \Delta_{right}\})$,
- ϕ_7 : $(\neg y \wedge z \rightarrow \Box_2(y \rightarrow \bigvee \{x_b : b \in (a) \text{ and } \delta_b \in \Delta_{left}\}))$,
- ϕ_8 : $y \rightarrow \Box_1 y \wedge \Box_2 y$,
- ϕ_9 : $z \rightarrow \Box_1 z \wedge \Box_2 z$,
- ϕ_{10} : $\neg y \rightarrow \Box_1 \neg y$,
- ϕ_{11} : $\neg z \rightarrow \Box_2 \neg z$.

Let

- $\Gamma_{\mathcal{I}} = \{\phi_1, \phi_2, \phi_3, \phi_4, \phi_5, \phi_6, \phi_7, \phi_8, \phi_9, \phi_{10}, \phi_{11}\}$,
- $\psi_{\mathcal{I}} = \neg(\neg y \wedge \neg z \wedge \Diamond_2 y \wedge \Diamond_1 z)$.

Let $R_{\mathcal{I}} = \frac{\Gamma_{\mathcal{I}}}{\psi_{\mathcal{I}}}$.

Proposition 5 *If $R_{\mathcal{I}}$ is not admissible then there exists a correct tiling of \mathcal{I} .*

Proof: See Section 5. \dashv

Proposition 6 *If there exists a correct tiling of \mathcal{I} then $R_{\mathcal{I}}$ is not admissible.*

Proof: See Section 6. \dashv

Proposition 4 is an immediate consequence of Propositions 1, 5 and 6.

5 Proof of Proposition 5

Suppose $R_{\mathcal{I}}$ is not admissible. Hence, let σ be a substitution such that $\models \sigma(\Gamma_{\mathcal{I}})$ and $\not\models \sigma(\psi_{\mathcal{I}})$. Thus, let $\mathcal{M} = (I, J, v)$ be a model and $(i_0, j_0) \in I \otimes J$ be such that $(i_0, j_0) \not\models_{\mathcal{M}} \sigma(\psi_{\mathcal{I}})$. Without loss of generality, we can assume $(i_0, j_0) = (0, 0)$. Consequently, $(0, 0) \models_{\mathcal{M}} \neg\sigma(y)$, $(0, 0) \models_{\mathcal{M}} \neg\sigma(z)$, $(0, 0) \models_{\mathcal{M}} \diamond_2\sigma(y)$ and $(0, 0) \models_{\mathcal{M}} \diamond_1\sigma(z)$. Since $\models \sigma(\Gamma_{\mathcal{I}})$,

$$(1.1) \text{ for all } (i, j) \in I \otimes J, (i, j) \models_{\mathcal{M}} \sigma(\phi_1),$$

$$(1.2) \text{ for all } (i, j) \in I \otimes J, (i, j) \models_{\mathcal{M}} \sigma(\phi_2),$$

$$(1.3) \text{ for all } (i, j) \in I \otimes J, (i, j) \models_{\mathcal{M}} \sigma(\phi_3),$$

$$(1.4) \text{ for all } (i, j) \in I \otimes J, (i, j) \models_{\mathcal{M}} \sigma(\phi_4),$$

$$(1.5) \text{ for all } (i, j) \in I \otimes J, (i, j) \models_{\mathcal{M}} \sigma(\phi_5),$$

$$(1.6) \text{ for all } (i, j) \in I \otimes J, (i, j) \models_{\mathcal{M}} \sigma(\phi_6),$$

$$(1.7) \text{ for all } (i, j) \in I \otimes J, (i, j) \models_{\mathcal{M}} \sigma(\phi_7),$$

$$(1.8) \text{ for all } (i, j) \in I \otimes J, (i, j) \models_{\mathcal{M}} \sigma(\phi_8),$$

$$(1.9) \text{ for all } (i, j) \in I \otimes J, (i, j) \models_{\mathcal{M}} \sigma(\phi_9),$$

$$(1.10) \text{ for all } (i, j) \in I \otimes J, (i, j) \models_{\mathcal{M}} \sigma(\phi_{10}),$$

$$(1.11) \text{ for all } (i, j) \in I \otimes J, (i, j) \models_{\mathcal{M}} \sigma(\phi_{11}).$$

Since $(0, 0) \models_{\mathcal{M}} \diamond_2\sigma(y)$ and $(0, 0) \models_{\mathcal{M}} \diamond_1\sigma(z)$, $0 < I$ and $0 < J$. Moreover, $(0, 1) \models_{\mathcal{M}} \sigma(y)$ and $(1, 0) \models_{\mathcal{M}} \sigma(z)$. Since $(0, 0) \models_{\mathcal{M}} \neg\sigma(z)$, by (1.11), $(0, 0) \models_{\mathcal{M}} \Box_2\neg\sigma(z)$. Since $(0, 0) \models_{\mathcal{M}} \diamond_2\sigma(y)$, by (1.5), $(0, 0) \models_{\mathcal{M}} \diamond_2\Box_1(\sigma(z) \rightarrow \bigvee\{\sigma(x_b): b \in (a) \text{ and } \delta_b \in \Delta_{down}\})$. Since $(0, 0) \models_{\mathcal{M}} \diamond_1\sigma(z)$, by (1.9), $(0, 0) \models_{\mathcal{M}} \diamond_1\Box_2\sigma(z)$. Since $(0, 0) \models_{\mathcal{M}} \diamond_2\Box_1(\sigma(z) \rightarrow \bigvee\{\sigma(x_b): b \in (a) \text{ and } \delta_b \in \Delta_{down}\})$, $(0, 0) \models_{\mathcal{M}} \diamond_2\Box_1\bigvee\{\sigma(x_b): b \in (a)\}$. Hence, $(1, 1) \models_{\mathcal{M}} \bigvee\{\sigma(x_b): b \in (a)\}$.

Lemma 1 For all $i \in (I)$, $(i, 0) \models_{\mathcal{M}} \neg\sigma(y) \wedge \sigma(z)$.

Proof: By induction, using (1.9), (1.10), the fact that $(0, 0) \models_{\mathcal{M}} \neg\sigma(y)$ and the fact that $(1, 0) \models_{\mathcal{M}} \sigma(z)$. \dashv

Lemma 2 For all $j \in (J)$, $(0, j) \models_{\mathcal{M}} \sigma(y) \wedge \neg\sigma(z)$.

Proof: By induction, using (1.8), (1.11), the fact that $(0, 1) \models_{\mathcal{M}} \sigma(y)$ and the fact that $(0, 0) \models_{\mathcal{M}} \neg\sigma(z)$. \dashv

Lemma 3 For all $i \in (I)$ and for all $j \in (J)$, $(i, j) \models_{\mathcal{M}} \sigma(y) \wedge \sigma(z)$.

Proof: By induction, using (1.8), (1.9) and Lemmas 1 and 2. \dashv

Lemma 4 *Let $(i', j') \in (I) \times (J)$. There exists $b \in (a)$ such that $(i', j') \models_{\mathcal{M}} \sigma(x_b)$.*

Proof: By induction. Let $(i', j') \in (I) \times (J)$ be such that for all $(i'', j'') \in (I) \times (J)$, if $(i'', j'') \ll (i', j')$ then there exists $b \in (a)$ such that $(i'', j'') \models_{\mathcal{M}} \sigma(x_b)$.

Case $i' = 1$ and $j' = 1$: Since $(1, 1) \models_{\mathcal{M}} \bigvee \{\sigma(x_b) : b \in (a)\}$, $(1, 1) \models_{\mathcal{M}} \sigma(x_b)$ for some $b \in (a)$.

Case $i' > 1$: By induction hypothesis, let $b \in (a)$ be such that $(i' - 1, j') \models_{\mathcal{M}} \sigma(x_b)$. By (1.2), $(i' - 2, j' - 1) \models_{\mathcal{M}} \Box_1 \Box_2 \sigma(\phi_2)$. Hence, $(i' - 2, j' - 1) \models_{\mathcal{M}} \Box_1 \Box_2 (\sigma(x_b) \rightarrow \Box_1 \bigvee \{\sigma(x_c) : c \in (a) \text{ and } (\delta_b, \delta_c) \in V\})$. Thus, either $(i' - 1, j') \not\models_{\mathcal{M}} \sigma(x_b)$, or $(i', j') \models_{\mathcal{M}} \bigvee \{\sigma(x_c) : c \in (a) \text{ and } (\delta_b, \delta_c) \in V\}$. Since $(i' - 1, j') \models_{\mathcal{M}} \sigma(x_b)$, $(i', j') \models_{\mathcal{M}} \bigvee \{\sigma(x_c) : c \in (a) \text{ and } (\delta_b, \delta_c) \in V\}$. Consequently, $(i', j') \models_{\mathcal{M}} \sigma(x_c)$ for some $c \in (a)$.

Case $j' > 1$: By induction hypothesis, let $b \in (a)$ be such that $(i', j' - 1) \models_{\mathcal{M}} \sigma(x_b)$. By (1.3), $(i' - 1, j' - 2) \models_{\mathcal{M}} \Box_1 \Box_2 \sigma(\phi_3)$. Hence, $(i' - 1, j' - 2) \models_{\mathcal{M}} \Box_1 \Box_2 (\sigma(x_b) \rightarrow \Box_2 \bigvee \{\sigma(x_c) : c \in (a) \text{ and } (\delta_b, \delta_c) \in H\})$. Thus, either $(i', j' - 1) \not\models_{\mathcal{M}} \sigma(x_b)$, or $(i', j') \models_{\mathcal{M}} \bigvee \{\sigma(x_c) : c \in (a) \text{ and } (\delta_b, \delta_c) \in H\}$. Since $(i', j' - 1) \models_{\mathcal{M}} \sigma(x_b)$, $(i', j') \models_{\mathcal{M}} \bigvee \{\sigma(x_c) : c \in (a) \text{ and } (\delta_b, \delta_c) \in H\}$. Consequently, $(i', j') \models_{\mathcal{M}} \sigma(x_c)$ for some $c \in (a)$. \dashv

Lemma 5 *Let $(i', j') \in (I) \times (J)$. For all $b, c \in (a)$, if $(i', j') \models_{\mathcal{M}} \sigma(x_b)$ and $(i', j') \models_{\mathcal{M}} \sigma(x_c)$ then $b = c$.*

Proof: Let $b, c \in (a)$. Suppose $(i', j') \models_{\mathcal{M}} \sigma(x_b)$ and $(i', j') \models_{\mathcal{M}} \sigma(x_c)$. For the sake of the contradiction, suppose $b \neq c$. By (1.1), $(i' - 1, j' - 1) \models_{\mathcal{M}} \Box_1 \Box_2 \sigma(\phi_1)$. Since $b \neq c$, $(i' - 1, j' - 1) \models_{\mathcal{M}} \Box_1 \Box_2 \neg(\sigma(x_b) \wedge \sigma(x_c))$. Hence, either $(i', j') \not\models_{\mathcal{M}} \sigma(x_b)$, or $(i', j') \not\models_{\mathcal{M}} \sigma(x_c)$: a contradiction. \dashv

By Lemmas 4 and 5, for all $(i, j) \in (I) \times (J)$, let $b(i, j)$ be the unique $b \in (a)$ such that $(i, j) \models_{\mathcal{M}} \sigma(x_b)$. Let f be the function associating the element $\delta_{b(i, j)} \in \Delta$ to each $(i, j) \in (I) \times (J)$. Obviously, for all $(i, j) \in (I) \times (J)$, $(i, j) \models_{\mathcal{M}} \sigma(x_{b(i, j)})$.

Lemma 6 *(I, J, f) is a correct tiling of \mathcal{I} .*

Proof: We demonstrate for all $(i, j) \in (I - 1) \times (J)$, $(f(i, j), f(i + 1, j)) \in V$. Let $(i, j) \in (I - 1) \times (J)$. Hence, $1 \leq i \leq I - 1$ and $1 \leq j \leq J$. Thus, $(i - 1, j - 1) \in I \otimes J$. By (1.2), $(i - 1, j - 1) \models_{\mathcal{M}} \Box_1 \Box_2 \sigma(\phi_2)$. Consequently, $(i - 1, j - 1) \models_{\mathcal{M}} \Box_1 \Box_2 (\sigma(x_{b(i, j)}) \rightarrow \Box_1 \bigvee \{\sigma(x_c) : c \in (a) \text{ and } (\delta_{b(i, j)}, \delta_c) \in V\})$. Hence, either $(i, j) \not\models_{\mathcal{M}} \sigma(x_{b(i, j)})$, or $(i + 1, j) \models_{\mathcal{M}} \bigvee \{\sigma(x_c) : c \in (a) \text{ and } (\delta_{b(i, j)}, \delta_c) \in V\}$. Since $(i, j) \models_{\mathcal{M}} \sigma(x_{b(i, j)})$, $(i + 1, j) \models_{\mathcal{M}} \bigvee \{\sigma(x_c) : c \in (a) \text{ and } (\delta_{b(i, j)}, \delta_c) \in V\}$. Thus, $(i + 1, j) \models_{\mathcal{M}} \sigma(x_c)$ for some $c \in (a)$ such that $(\delta_{b(i, j)}, \delta_c) \in V$. Consequently, $b(i + 1, j) = c$. Hence, by definition of f , since

$$(\delta_{b(i,j)}, \delta_c) \in V, (f(i, j), f(i+1, j)) \in V.$$

We demonstrate for all $(i, j) \in (I) \times (J-1)$, $(f(i, j), f(i, j+1)) \in H$. Let $(i, j) \in (I) \times (J-1)$. Thus, $1 \leq i \leq I$ and $1 \leq j \leq J-1$. Consequently, $(i-1, j-1) \in I \otimes J$. By (1.3), $(i-1, j-1) \models_{\mathcal{M}} \Box_1 \Box_2 \sigma(\phi_3)$. Hence, $(i-1, j-1) \models_{\mathcal{M}} \Box_1 \Box_2 (\sigma(x_{b(i,j)}) \rightarrow \Box_2 \bigvee \{\sigma(x_c) : c \in (a) \text{ and } (\delta_{b(i,j)}, \delta_c) \in H\})$. Thus, either $(i, j) \not\models_{\mathcal{M}} \sigma(x_{b(i,j)})$, or $(i, j+1) \models_{\mathcal{M}} \bigvee \{\sigma(x_c) : c \in (a) \text{ and } (\delta_{b(i,j)}, \delta_c) \in H\}$. Since $(i, j) \models_{\mathcal{M}} \sigma(x_{b(i,j)})$, $(i, j+1) \models_{\mathcal{M}} \bigvee \{\sigma(x_c) : c \in (a) \text{ and } (\delta_{b(i,j)}, \delta_c) \in H\}$. Consequently, $(i, j+1) \models_{\mathcal{M}} \sigma(x_c)$ for some $c \in (a)$ such that $(\delta_{b(i,j)}, \delta_c) \in H$. Hence, $b(i, j+1) = c$. Thus, by definition of f , since $(\delta_{b(i,j)}, \delta_c) \in H$, $(f(i, j), f(i, j+1)) \in H$.

We demonstrate for all $j \in (J)$, $f(I, j) \in \Delta_{up}$. Let $j \in (J)$. Consequently, by Lemma 3, $(I, j) \models_{\mathcal{M}} \sigma(y)$. By (1.4), $(I-1, j-1) \models_{\mathcal{M}} \Box_1 \Box_2 \sigma(\phi_4)$. Hence, $(I, j) \models_{\mathcal{M}} \sigma(y) \wedge \Box_1 \perp \rightarrow \bigvee \{\sigma(x_b) : b \in (a) \text{ and } \delta_b \in \Delta_{up}\}$. Since $(I, j) \models_{\mathcal{M}} \sigma(y)$ and $(I, j) \models_{\mathcal{M}} \Box_1 \perp$, $(I, j) \models_{\mathcal{M}} \sigma(x_b)$ for some $b \in (a)$ such that $\delta_b \in \Delta_{up}$. Thus, by definition of f , $f(I, j) \in \Delta_{up}$.

We demonstrate for all $j \in (J)$, $f(1, j) \in \Delta_{down}$. Let $j \in (J)$. Consequently, by Lemmas 2 and 3, $(0, j) \models_{\mathcal{M}} \sigma(y) \wedge \neg \sigma(z)$ and $(1, j) \models_{\mathcal{M}} \sigma(z)$. By (1.5), $(0, j-1) \models_{\mathcal{M}} \Box_2 \sigma(\phi_5)$. Hence, $(0, j) \models_{\mathcal{M}} \sigma(y) \wedge \neg \sigma(z) \rightarrow \Box_1 (\sigma(z) \rightarrow \bigvee \{\sigma(x_b) : b \in (a) \text{ and } \delta_b \in \Delta_{down}\})$. Since $(0, j) \models_{\mathcal{M}} \sigma(y) \wedge \neg \sigma(z)$ and $(1, j) \models_{\mathcal{M}} \sigma(z)$, $(1, j) \models_{\mathcal{M}} \sigma(x_b)$ for some $b \in (a)$ such that $\delta_b \in \Delta_{down}$. Thus, by definition of f , $f(1, j) \in \Delta_{down}$.

We demonstrate for all $i \in (I)$, $f(i, J) \in \Delta_{right}$. Let $i \in (I)$. Consequently, by Lemma 3, $(i, J) \models_{\mathcal{M}} \sigma(z)$. By (1.6), $(i-1, J-1) \models_{\mathcal{M}} \Box_1 \Box_2 \sigma(\phi_6)$. Hence, $(i, J) \models_{\mathcal{M}} \sigma(z) \wedge \Box_2 \perp \rightarrow \bigvee \{\sigma(x_b) : b \in (a) \text{ and } \delta_b \in \Delta_{right}\}$. Since $(i, J) \models_{\mathcal{M}} \sigma(z)$ and $(i, J) \models_{\mathcal{M}} \Box_2 \perp$, $(i, J) \models_{\mathcal{M}} \sigma(x_b)$ for some $b \in (a)$ such that $\delta_b \in \Delta_{right}$. Thus, by definition of f , $f(i, J) \in \Delta_{right}$.

We demonstrate for all $i \in (I)$, $f(i, 1) \in \Delta_{left}$. Let $i \in (I)$. Consequently, by Lemmas 1 and 3, $(i, 0) \models_{\mathcal{M}} \neg \sigma(y) \wedge \sigma(z)$ and $(i, 1) \models_{\mathcal{M}} \sigma(y)$. By (1.7), $(i-1, 0) \models_{\mathcal{M}} \Box_1 \sigma(\phi_7)$. Hence, $(i, 0) \models_{\mathcal{M}} \neg \sigma(y) \wedge \sigma(z) \rightarrow \Box_2 (\sigma(y) \rightarrow \bigvee \{\sigma(x_b) : b \in (a) \text{ and } \delta_b \in \Delta_{left}\})$. Since $(i, 0) \models_{\mathcal{M}} \neg \sigma(y) \wedge \sigma(z)$ and $(i, 1) \models_{\mathcal{M}} \sigma(y)$, $(i, 1) \models_{\mathcal{M}} \sigma(x_b)$ for some $b \in (a)$ such that $\delta_b \in \Delta_{left}$. Thus, by definition of f , $f(i, 1) \in \Delta_{left}$. \dashv

Here finishes the proof of Proposition 5.

6 Proof of Proposition 6

Let (I, J, f) be a correct tiling of \mathcal{I} . Hence, the following conditions hold:

- for all $(i, j) \in (I-1) \times (J)$, $(f(i, j), f(i+1, j)) \in V$,
- for all $(i, j) \in (I) \times (J-1)$, $(f(i, j), f(i, j+1)) \in H$,

- for all $j \in (J)$, $f(I, j) \in \Delta_{up}$,
- for all $j \in (J)$, $f(1, j) \in \Delta_{down}$,
- for all $i \in (I)$, $f(i, J) \in \Delta_{right}$,
- for all $i \in (I)$, $f(i, 1) \in \Delta_{left}$.

Let σ be the substitution such that

- for all $b \in (a)$, $\sigma(x_b) = \bigvee \{ \diamond_1^{I-i} \square_1 \perp \wedge \diamond_2^{J-j} \square_2 \perp : (i, j) \in (I) \times (J) \text{ and } f(i, j) = \delta_b \}$,
- $\sigma(y) = \square_2^J \perp$,
- $\sigma(z) = \square_1^I \perp$.

Lemma 7 $\models \sigma(\Gamma_{\mathcal{I}})$.

Proof: For the sake of the contradiction, suppose $\not\models \sigma(\Gamma_{\mathcal{I}})$. Hence, let $\mathcal{M}' = (I', J', v')$ be a model and $(i', j') \in I' \otimes J'$ be such that at least one of the following conditions holds:

- (2.1) $(i', j') \not\models_{\mathcal{M}'} \square_1 \square_2 \sigma(\phi_1)$,
- (2.2) $(i', j') \not\models_{\mathcal{M}'} \square_1 \square_2 \sigma(\phi_2)$,
- (2.3) $(i', j') \not\models_{\mathcal{M}'} \square_1 \square_2 \sigma(\phi_3)$,
- (2.4) $(i', j') \not\models_{\mathcal{M}'} \square_1 \square_2 \sigma(\phi_4)$,
- (2.5) $(i', j') \not\models_{\mathcal{M}'} \square_2 \sigma(\phi_5)$,
- (2.6) $(i', j') \not\models_{\mathcal{M}'} \square_1 \square_2 \sigma(\phi_6)$,
- (2.7) $(i', j') \not\models_{\mathcal{M}'} \square_1 \sigma(\phi_7)$,
- (2.8) $(i', j') \not\models_{\mathcal{M}'} \sigma(\phi_8)$,
- (2.9) $(i', j') \not\models_{\mathcal{M}'} \sigma(\phi_9)$,
- (2.10) $(i', j') \not\models_{\mathcal{M}'} \sigma(\phi_{10})$,
- (2.11) $(i', j') \not\models_{\mathcal{M}'} \sigma(\phi_{11})$.

Case (2.1): Thus, $(i', j') \not\models_{\mathcal{M}'} \square_1 \square_2 \neg(\sigma(x_{b'}) \wedge \sigma(x_{c'}))$ for some $b', c' \in (a)$ such that $b' \neq c'$. Consequently, $i' < I'$, $j' < J'$, $(i' + 1, j' + 1) \models_{\mathcal{M}'} \sigma(x_{b'})$ and $(i' + 1, j' + 1) \models_{\mathcal{M}'} \sigma(x_{c'})$. Hence, $(i' + 1, j' + 1) \models_{\mathcal{M}'} \bigvee \{ \diamond_1^{I-i} \square_1 \perp \wedge \diamond_2^{J-j} \square_2 \perp : (i, j) \in (I) \times (J) \text{ and } f(i, j) = \delta_{b'} \}$ and $(i' + 1, j' + 1) \models_{\mathcal{M}'} \bigvee \{ \diamond_1^{I-i} \square_1 \perp \wedge \diamond_2^{J-j} \square_2 \perp : (i, j) \in (I) \times (J) \text{ and } f(i, j) = \delta_{c'} \}$. Let $(i_{b'}, j_{b'}) \in (I) \times (J)$ be such that $f(i_{b'}, j_{b'}) = \delta_{b'}$ and $(i' + 1, j' + 1) \models_{\mathcal{M}'} \diamond_1^{I-i_{b'}} \square_1 \perp \wedge \diamond_2^{J-j_{b'}} \square_2 \perp$ and $(i_{c'}, j_{c'}) \in (I) \times (J)$ be such that $f(i_{c'}, j_{c'}) = \delta_{c'}$ and $(i' + 1, j' + 1) \models_{\mathcal{M}'} \diamond_1^{I-i_{c'}} \square_1 \perp \wedge \diamond_2^{J-j_{c'}} \square_2 \perp$.

$\diamond_1^{I-i_{c'}} \square_1 \perp \wedge \diamond_2^{J-j_{c'}} \square_2 \perp$. Thus, $I - i_{b'} = I' - (i' + 1)$, $J - j_{b'} = J' - (j' + 1)$, $I - i_{c'} = I' - (i' + 1)$ and $J - j_{c'} = J' - (j' + 1)$. Consequently, $i_{b'} = i_{c'}$ and $j_{b'} = j_{c'}$. Since $f(i_{b'}, j_{b'}) = \delta_{b'}$ and $f(i_{c'}, j_{c'}) = \delta_{c'}$, $\delta_{b'} = \delta_{c'}$. Hence, $b' = c'$: a contradiction.

Case (2.2): Thus, $(i', j') \not\models_{\mathcal{M}'} \square_1 \square_2 (\sigma(x_{b'}) \rightarrow \square_1 \bigvee \{\sigma(x_c) : c \in (a) \text{ and } (\delta_{b'}, \delta_c) \in V\})$ for some $b' \in (a)$. Consequently, $i' < I'$, $j' < J'$, $(i' + 1, j' + 1) \models_{\mathcal{M}'} \sigma(x_{b'})$ and $(i' + 1, j' + 1) \not\models_{\mathcal{M}'} \square_1 \bigvee \{\sigma(x_c) : c \in (a) \text{ and } (\delta_{b'}, \delta_c) \in V\}$. Hence, $(i' + 1, j' + 1) \models_{\mathcal{M}'} \bigvee \{\diamond_1^{I-i} \square_1 \perp \wedge \diamond_2^{J-j} \square_2 \perp : (i, j) \in (I) \times (J) \text{ and } f(i, j) = \delta_{b'}\}$, $i' + 1 < I'$ and $(i' + 2, j' + 1) \not\models_{\mathcal{M}'} \bigvee \{\sigma(x_c) : c \in (a) \text{ and } (\delta_{b'}, \delta_c) \in V\}$. Let $(i_{b'}, j_{b'}) \in (I) \times (J)$ be such that $f(i_{b'}, j_{b'}) = \delta_{b'}$ and $(i' + 1, j' + 1) \models_{\mathcal{M}'} \diamond_1^{I-i_{b'}} \square_1 \perp \wedge \diamond_2^{J-j_{b'}} \square_2 \perp$. Thus, $I - i_{b'} = I' - (i' + 1)$ and $J - j_{b'} = J' - (j' + 1)$. Since $j' < J'$ and $i' + 1 < I'$, $i_{b'} < I$ and $j_{b'} \leq J$. Let $c' \in (a)$ be such that $f(i_{b'} + 1, j_{b'}) = \delta_{c'}$. Since $f(i_{b'}, j_{b'}) = \delta_{b'}$, $(\delta_{b'}, \delta_{c'}) \in V$. Since $(i' + 2, j' + 1) \not\models_{\mathcal{M}'} \bigvee \{\sigma(x_c) : c \in (a) \text{ and } (\delta_{b'}, \delta_c) \in V\}$, $(i' + 2, j' + 1) \not\models_{\mathcal{M}'} \sigma(x_{c'})$. Consequently, $(i' + 2, j' + 1) \not\models_{\mathcal{M}'} \bigvee \{\diamond_1^{I-i} \square_1 \perp \wedge \diamond_2^{J-j} \square_2 \perp : (i, j) \in (I) \times (J) \text{ and } f(i, j) = \delta_{c'}\}$. Since $f(i_{b'} + 1, j_{b'}) = \delta_{c'}$, either $(i' + 2, j' + 1) \not\models_{\mathcal{M}'} \diamond_1^{I-(i_{b'}+1)} \square_1 \perp$, or $(i' + 2, j' + 1) \not\models_{\mathcal{M}'} \diamond_2^{J-j_{b'}} \square_2 \perp$. Hence, either $I - (i_{b'} + 1) \neq I' - (i' + 2)$, or $J - j_{b'} \neq J' - (j' + 1)$. Thus, either $I - i_{b'} \neq I' - (i' + 1)$, or $J - j_{b'} \neq J' - (j' + 1)$: a contradiction.

Case (2.3): Consequently, $(i', j') \not\models_{\mathcal{M}'} \square_1 \square_2 (\sigma(x_{b'}) \rightarrow \square_2 \bigvee \{\sigma(x_c) : c \in (a) \text{ and } (\delta_{b'}, \delta_c) \in H\})$ for some $b' \in (a)$. Hence, $i' < I'$, $j' < J'$, $(i' + 1, j' + 1) \models_{\mathcal{M}'} \sigma(x_{b'})$ and $(i' + 1, j' + 1) \not\models_{\mathcal{M}'} \square_2 \bigvee \{\sigma(x_c) : c \in (a) \text{ and } (\delta_{b'}, \delta_c) \in H\}$. Thus, $(i' + 1, j' + 1) \models_{\mathcal{M}'} \bigvee \{\diamond_1^{I-i} \square_1 \perp \wedge \diamond_2^{J-j} \square_2 \perp : (i, j) \in (I) \times (J) \text{ and } f(i, j) = \delta_{b'}\}$, $j' + 1 < J'$ and $(i' + 1, j' + 2) \not\models_{\mathcal{M}'} \bigvee \{\sigma(x_c) : c \in (a) \text{ and } (\delta_{b'}, \delta_c) \in H\}$. Let $(i_{b'}, j_{b'}) \in (I) \times (J)$ be such that $f(i_{b'}, j_{b'}) = \delta_{b'}$ and $(i' + 1, j' + 1) \models_{\mathcal{M}'} \diamond_1^{I-i_{b'}} \square_1 \perp \wedge \diamond_2^{J-j_{b'}} \square_2 \perp$. Consequently, $I - i_{b'} = I' - (i' + 1)$ and $J - j_{b'} = J' - (j' + 1)$. Since $i' < I'$ and $j' + 1 < J'$, $i_{b'} \leq I$ and $j_{b'} < J$. Let $c' \in (a)$ be such that $f(i_{b'}, j_{b'} + 1) = \delta_{c'}$. Since $f(i_{b'}, j_{b'}) = \delta_{b'}$, $(\delta_{b'}, \delta_{c'}) \in H$. Since $(i' + 1, j' + 2) \not\models_{\mathcal{M}'} \bigvee \{\sigma(x_c) : c \in (a) \text{ and } (\delta_{b'}, \delta_c) \in H\}$, $(i' + 1, j' + 2) \not\models_{\mathcal{M}'} \sigma(x_{c'})$. Hence, $(i' + 1, j' + 2) \not\models_{\mathcal{M}'} \bigvee \{\diamond_1^{I-i} \square_1 \perp \wedge \diamond_2^{J-j} \square_2 \perp : (i, j) \in (I) \times (J) \text{ and } f(i, j) = \delta_{c'}\}$. Since $f(i_{b'}, j_{b'} + 1) = \delta_{c'}$, either $(i' + 1, j' + 2) \not\models_{\mathcal{M}'} \diamond_1^{I-i_{b'}} \square_1 \perp$, or $(i' + 1, j' + 2) \not\models_{\mathcal{M}'} \diamond_2^{J-(j_{b'}+1)} \square_2 \perp$. Thus, either $I - i_{b'} \neq I' - (i' + 1)$, or $J - (j_{b'} + 1) \neq J' - (j' + 2)$. Consequently, either $I - i_{b'} \neq I' - (i' + 1)$, or $J - j_{b'} \neq J' - (j' + 1)$: a contradiction.

Case (2.4): Hence, $(i', j') \not\models_{\mathcal{M}'} \square_1 \square_2 (\sigma(y) \wedge \square_1 \perp \rightarrow \bigvee \{\sigma(x_b) : b \in (a) \text{ and } \delta_b \in \Delta_{up}\})$. Thus, $i' < I'$, $j' < J'$, $(i' + 1, j' + 1) \models_{\mathcal{M}'} \sigma(y)$, $(i' + 1, j' + 1) \models_{\mathcal{M}'} \square_1 \perp$ and $(i' + 1, j' + 1) \not\models_{\mathcal{M}'} \bigvee \{\sigma(x_b) : b \in (a) \text{ and } \delta_b \in \Delta_{up}\}$. Consequently, $(i' + 1, j' + 1) \models_{\mathcal{M}'} \square_2 \perp$. Let $j \in (J)$ be such that $(i' + 1, j' + 1) \models_{\mathcal{M}'} \diamond_2^{J-j} \square_2 \perp$. Since $(i' + 1, j' + 1) \models_{\mathcal{M}'} \square_1 \perp$, $(i' + 1, j' + 1) \models_{\mathcal{M}'} \diamond_1^{I-I} \square_1 \perp \wedge \diamond_2^{J-j} \square_2 \perp$. Let $b' \in (a)$ be such that $f(I, j) = \delta_{b'}$. Hence, $\delta_{b'} \in \Delta_{up}$. Moreover, since $(i' + 1, j' + 1) \models_{\mathcal{M}'} \diamond_1^{I-I} \square_1 \perp \wedge \diamond_2^{J-j} \square_2 \perp$, $(i' + 1, j' + 1) \models_{\mathcal{M}'} \sigma(x_{b'})$. Thus, $(i' + 1, j' + 1) \models_{\mathcal{M}'} \bigvee \{\sigma(x_b) : b \in (a) \text{ and } \delta_b \in \Delta_{up}\}$: a contradiction.

Case (2.5): Consequently, $(i', j') \not\models_{\mathcal{M}'} \Box_2(\sigma(y) \wedge \neg\sigma(z) \rightarrow \Box_1(\sigma(z) \rightarrow \bigvee\{\sigma(x_b): b \in (a) \text{ and } \delta_b \in \Delta_{down}\}))$. Hence, $j' < J'$, $(i', j' + 1) \models_{\mathcal{M}'} \sigma(y)$, $(i', j' + 1) \models_{\mathcal{M}'} \neg\sigma(z)$ and $(i', j' + 1) \not\models_{\mathcal{M}'} \Box_1(\sigma(z) \rightarrow \bigvee\{\sigma(x_b): b \in (a) \text{ and } \delta_b \in \Delta_{down}\})$. Thus, $(i', j' + 1) \models_{\mathcal{M}'} \Box_2^\perp$. Let $j \in (J)$ be such that $(i', j' + 1) \models_{\mathcal{M}'} \diamond_2^{J-j} \Box_2^\perp$. Since $(i', j' + 1) \not\models_{\mathcal{M}'} \Box_1(\sigma(z) \rightarrow \bigvee\{\sigma(x_b): b \in (a) \text{ and } \delta_b \in \Delta_{down}\})$, $i' < I'$, $(i' + 1, j' + 1) \models_{\mathcal{M}'} \sigma(z)$ and $(i' + 1, j' + 1) \not\models_{\mathcal{M}'} \bigvee\{\sigma(x_b): b \in (a) \text{ and } \delta_b \in \Delta_{down}\}$. Since $(i', j' + 1) \models_{\mathcal{M}'} \diamond_2^{J-j} \Box_2^\perp$, $(i' + 1, j' + 1) \models_{\mathcal{M}'} \diamond_2^{J-j} \Box_2^\perp$. Since $(i', j' + 1) \models_{\mathcal{M}'} \neg\sigma(z)$, $(i', j' + 1) \models_{\mathcal{M}'} \neg\Box_1^\perp$. Consequently, $I \leq I' - i'$. Since $(i' + 1, j' + 1) \models_{\mathcal{M}'} \sigma(z)$, $(i' + 1, j' + 1) \models_{\mathcal{M}'} \Box_1^\perp$. Since $I \leq I' - i'$, $(i' + 1, j' + 1) \models_{\mathcal{M}'} \diamond_1^{I-1} \Box_1^\perp$. Since $(i' + 1, j' + 1) \models_{\mathcal{M}'} \diamond_2^{J-j} \Box_2^\perp$, $(i' + 1, j' + 1) \models_{\mathcal{M}'} \diamond_1^{I-1} \Box_1^\perp \wedge \diamond_2^{J-j} \Box_2^\perp$. Let $b' \in (a)$ be such that $f(1, j) = \delta_{b'}$. Hence, $\delta_{b'} \in \Delta_{down}$. Moreover, since $(i' + 1, j' + 1) \models_{\mathcal{M}'} \diamond_1^{I-1} \Box_1^\perp \wedge \diamond_2^{J-j} \Box_2^\perp$, $(i' + 1, j' + 1) \models_{\mathcal{M}'} \sigma(x_{b'})$. Thus, $(i' + 1, j' + 1) \models_{\mathcal{M}'} \bigvee\{\sigma(x_b): b \in (a) \text{ and } \delta_b \in \Delta_{down}\}$: a contradiction.

Case (2.6): Consequently, $(i', j') \not\models_{\mathcal{M}'} \Box_1 \Box_2(\sigma(z) \wedge \Box_2^\perp \rightarrow \bigvee\{\sigma(x_b): b \in (a) \text{ and } \delta_b \in \Delta_{right}\})$. Hence, $i' < I'$, $j' < J'$, $(i' + 1, j' + 1) \models_{\mathcal{M}'} \sigma(z)$, $(i' + 1, j' + 1) \models_{\mathcal{M}'} \Box_2^\perp$ and $(i' + 1, j' + 1) \not\models_{\mathcal{M}'} \bigvee\{\sigma(x_b): b \in (a) \text{ and } \delta_b \in \Delta_{right}\}$. Thus, $(i' + 1, j' + 1) \models_{\mathcal{M}'} \Box_1^\perp$. Let $i \in (I)$ be such that $(i' + 1, j' + 1) \models_{\mathcal{M}'} \diamond_1^{I-i} \Box_1^\perp$. Since $(i' + 1, j' + 1) \models_{\mathcal{M}'} \Box_2^\perp$, $(i' + 1, j' + 1) \models_{\mathcal{M}'} \diamond_1^{I-i} \Box_1^\perp \wedge \diamond_2^{J-j} \Box_2^\perp$. Let $b' \in (a)$ be such that $f(i, J) = \delta_{b'}$. Consequently, $\delta_{b'} \in \Delta_{right}$. Moreover, since $(i' + 1, j' + 1) \models_{\mathcal{M}'} \diamond_1^{I-i} \Box_1^\perp \wedge \diamond_2^{J-j} \Box_2^\perp$, $(i' + 1, j' + 1) \models_{\mathcal{M}'} \sigma(x_{b'})$. Hence, $(i' + 1, j' + 1) \models_{\mathcal{M}'} \bigvee\{\sigma(x_b): b \in (a) \text{ and } \delta_b \in \Delta_{right}\}$: a contradiction.

Case (2.7): Thus, $(i', j') \not\models_{\mathcal{M}'} \Box_1(\neg\sigma(y) \wedge \sigma(z) \rightarrow \Box_2(\sigma(y) \rightarrow \bigvee\{\sigma(x_b): b \in (a) \text{ and } \delta_b \in \Delta_{left}\}))$. Consequently, $i' < I'$, $(i' + 1, j') \models_{\mathcal{M}'} \neg\sigma(y)$, $(i' + 1, j') \models_{\mathcal{M}'} \sigma(z)$ and $(i' + 1, j') \not\models_{\mathcal{M}'} \Box_2(\sigma(y) \rightarrow \bigvee\{\sigma(x_b): b \in (a) \text{ and } \delta_b \in \Delta_{left}\})$. Hence, $(i' + 1, j') \models_{\mathcal{M}'} \Box_1^\perp$. Let $i \in (I)$ be such that $(i' + 1, j') \models_{\mathcal{M}'} \diamond_1^{I-i} \Box_1^\perp$. Since $(i' + 1, j') \not\models_{\mathcal{M}'} \Box_2(\sigma(y) \rightarrow \bigvee\{\sigma(x_b): b \in (a) \text{ and } \delta_b \in \Delta_{left}\})$, $j' < J'$, $(i' + 1, j' + 1) \models_{\mathcal{M}'} \sigma(y)$ and $(i' + 1, j' + 1) \not\models_{\mathcal{M}'} \bigvee\{\sigma(x_b): b \in (a) \text{ and } \delta_b \in \Delta_{left}\}$. Since $(i' + 1, j') \models_{\mathcal{M}'} \diamond_1^{I-i} \Box_1^\perp$, $(i' + 1, j' + 1) \models_{\mathcal{M}'} \diamond_1^{I-i} \Box_1^\perp$. Since $(i' + 1, j') \models_{\mathcal{M}'} \neg\sigma(y)$, $(i' + 1, j') \models_{\mathcal{M}'} \neg\Box_2^\perp$. Thus, $J \leq J' - j'$. Since $(i' + 1, j' + 1) \models_{\mathcal{M}'} \sigma(y)$, $(i' + 1, j' + 1) \models_{\mathcal{M}'} \Box_2^\perp$. Since $J \leq J' - j'$, $(i' + 1, j' + 1) \models_{\mathcal{M}'} \diamond_2^{J-1} \Box_2^\perp$. Since $(i' + 1, j' + 1) \models_{\mathcal{M}'} \diamond_1^{I-i} \Box_1^\perp$, $(i' + 1, j' + 1) \models_{\mathcal{M}'} \diamond_1^{I-i} \Box_1^\perp \wedge \diamond_2^{J-1} \Box_2^\perp$. Let $b' \in (a)$ be such that $f(i, 1) = \delta_{b'}$. Consequently, $\delta_{b'} \in \Delta_{left}$. Moreover, since $(i' + 1, j' + 1) \models_{\mathcal{M}'} \diamond_1^{I-i} \Box_1^\perp \wedge \diamond_2^{J-1} \Box_2^\perp$, $(i' + 1, j' + 1) \models_{\mathcal{M}'} \sigma(x_{b'})$. Hence, $(i' + 1, j' + 1) \models_{\mathcal{M}'} \bigvee\{\sigma(x_b): b \in (a) \text{ and } \delta_b \in \Delta_{left}\}$: a contradiction.

Case (2.8): Thus, $(i', j') \not\models_{\mathcal{M}'} \sigma(y) \rightarrow \Box_1 \sigma(y) \wedge \Box_2 \sigma(y)$. Consequently, $(i', j') \models_{\mathcal{M}'} \sigma(y)$ and either $(i', j') \not\models_{\mathcal{M}'} \Box_1 \sigma(y)$, or $(i', j') \not\models_{\mathcal{M}'} \Box_2 \sigma(y)$. Hence, either $(i', j') \models_{\mathcal{M}'} \Box_2^\perp$ and $(i', j') \not\models_{\mathcal{M}'} \Box_1 \Box_2^\perp$, or $(i', j') \models_{\mathcal{M}'} \Box_2^\perp$ and $(i', j') \not\models_{\mathcal{M}'} \Box_2 \Box_2^\perp$. In the former case, $i' < I'$ and $(i' + 1, j') \not\models_{\mathcal{M}'} \Box_2^\perp$. Since $(i', j') \models_{\mathcal{M}'} \Box_2^\perp$, $(i' + 1, j') \models_{\mathcal{M}'} \Box_2^\perp$: a contradiction. In the latter case, $j' < J'$ and $(i', j' + 1) \not\models_{\mathcal{M}'} \Box_2^\perp$. Since $(i', j') \models_{\mathcal{M}'} \Box_2^\perp$, $(i', j' + 1) \models_{\mathcal{M}'} \Box_2^\perp$: a contradiction.

Case (2.9): Thus, $(i', j') \not\models_{\mathcal{M}'} \sigma(z) \rightarrow \Box_1 \sigma(z) \wedge \Box_2 \sigma(z)$. Consequently, $(i', j') \models_{\mathcal{M}'} \sigma(z)$ and either $(i', j') \not\models_{\mathcal{M}'} \Box_1 \sigma(z)$, or $(i', j') \not\models_{\mathcal{M}'} \Box_2 \sigma(z)$. Hence, either $(i', j') \models_{\mathcal{M}'} \Box_1^I \perp$ and $(i', j') \not\models_{\mathcal{M}'} \Box_1 \Box_1^I \perp$, or $(i', j') \models_{\mathcal{M}'} \Box_1^I \perp$ and $(i', j') \not\models_{\mathcal{M}'} \Box_2 \Box_1^I \perp$. In the former case, $i' < I'$ and $(i' + 1, j') \not\models_{\mathcal{M}'} \Box_1^I \perp$. Since $(i', j') \models_{\mathcal{M}'} \Box_1^I \perp$, $(i' + 1, j') \models_{\mathcal{M}'} \Box_1^I \perp$: a contradiction. In the latter case, $j' < J'$ and $(i', j' + 1) \not\models_{\mathcal{M}'} \Box_1^I \perp$. Since $(i', j') \models_{\mathcal{M}'} \Box_1^I \perp$, $(i', j' + 1) \models_{\mathcal{M}'} \Box_1^I \perp$: a contradiction.

Case (2.10): Thus, $(i', j') \not\models_{\mathcal{M}'} \neg \sigma(y) \rightarrow \Box_1 \neg \sigma(y)$. Consequently, $(i', j') \models_{\mathcal{M}'} \neg \sigma(y)$ and $(i', j') \not\models_{\mathcal{M}'} \Box_1 \neg \sigma(y)$. Hence, $(i', j') \models_{\mathcal{M}'} \neg \Box_2^J \perp$ and $(i', j') \not\models_{\mathcal{M}'} \Box_1 \neg \Box_2^J \perp$. Thus, $i' < I'$ and $(i' + 1, j') \not\models_{\mathcal{M}'} \neg \Box_2^J \perp$. Consequently, $(i', j') \not\models_{\mathcal{M}'} \neg \Box_2^J \perp$: a contradiction.

Case (2.11): Hence, $(i', j') \not\models_{\mathcal{M}'} \neg \sigma(z) \rightarrow \Box_2 \neg \sigma(z)$. Thus, $(i', j') \models_{\mathcal{M}'} \neg \sigma(z)$ and $(i', j') \not\models_{\mathcal{M}'} \Box_2 \neg \sigma(z)$. Consequently, $(i', j') \models_{\mathcal{M}'} \neg \Box_1^I \perp$ and $(i', j') \not\models_{\mathcal{M}'} \Box_2 \neg \Box_1^I \perp$. Hence, $j' < J'$ and $(i', j' + 1) \not\models_{\mathcal{M}'} \neg \Box_1^I \perp$. Thus, $(i', j') \not\models_{\mathcal{M}'} \neg \Box_1^I \perp$: a contradiction. \dashv

Lemma 8 $\not\models \sigma(\psi_{\mathcal{I}})$.

Proof: Let $\mathcal{M} = (I, J, v)$ be a model based on I and J . Obviously, $(0, 0) \models_{\mathcal{M}} \neg \Box_2^J \perp$, $(0, 0) \models_{\mathcal{M}} \neg \Box_1^I \perp$, $(0, 0) \models_{\mathcal{M}} \Diamond_2 \Box_2^J \perp$ and $(0, 0) \models_{\mathcal{M}} \Diamond_1 \Box_1^I \perp$. Hence, $(0, 0) \models_{\mathcal{M}} \neg \sigma(y) \wedge \neg \sigma(z) \wedge \Diamond_2 \sigma(y) \wedge \Diamond_1 \sigma(z)$. Thus, $(0, 0) \not\models_{\mathcal{M}} \sigma(\psi_{\mathcal{I}})$. Consequently, $\not\models_{\mathcal{M}} \sigma(\psi_{\mathcal{I}})$. Hence, $\not\models \sigma(\psi_{\mathcal{I}})$. \dashv

Here finishes the proof of Proposition 6.

7 Conclusion

We have proved that the admissibility problem of the product of two **Alt** logics is undecidable.

In a modal logic **L**, the importance of the admissibility problem lies in its correspondence with the unifiability problem which is to determine, given a formula ϕ , whether there exists a substitution σ such that $\sigma(\phi) \in \mathbf{L}$ [1, 4, 9, 13]. In that case, ϕ is **L**-unifiable. Indeed, the correspondence between the admissibility problem in **L** and the unifiability problem in **L** can be expressed as follows: the formula ϕ is unifiable if and only if the inference rule $\frac{\{\phi\}}{\perp}$ is not admissible. Hence, in a modal logic **L**, an important question is the following: is the unifiability problem in **L** decidable?

In many modal logics, to solve the unifiability problem is easy. For instance, if $\mathbf{KD} \subseteq \mathbf{L}$ then the unifiability problem in **L** is in **NP**. Moreover, using a reduction to the problem of determining whether a given graph contains an Hamiltonian path, it has been proved that the **Alt**-unifiability problem is in **PSPACE** [5]. In other respect, by means of the so-called *universal model*, non-deterministic polynomial time algorithms for solving the unifiability problem in transitive modal logics such as **GL** have been

designed [12, 23].

However, in some other modal logics, to solve the unifiability problem is not a mere formality. For instance, the computability of the unifiability problem remains open in **K**. When parameters are allowed, the computability of the unifiability problem remains open as well in many modal logics containing **KD** [2, 3]. For instance, the computability of the unifiability problem is unknown in **DAIt**, a modal logic which has many interesting properties: it possesses the polynomial finite model property; its membership problem is **coNP**-complete.

Therefore, a challenging open question is whether the unifiability problem of the product of two **AIt** logics is undecidable. Other challenging open questions concern the admissibility problem and the unifiability problem in the products of two arbitrary modal logics, for instance the product of two **K** logics. Although this product does not possess the polynomial product finite model property, its membership problem is decidable. Has it an undecidable admissibility problem? Has it an undecidable unifiability problem?

Funding

The preparation of this paper has been supported by the Programme *Professeurs invités 2021* of *Université Toulouse III - Paul Sabatier* and the Project *RILA 48055QG* of *Bulgarian Minister of Education and Science, French Minister for Europe and Foreign Affairs* and *French Minister of Higher Education and Research*.

Acknowledgements

Special acknowledgement is heartily granted to Quentin Gougeon for many stimulating discussions about the subject of this paper. We also express our gratitude to Tinko Tinchev for his valuable remarks. Finally, we make a point of strongly thanking our referees for their feedback: their useful suggestions have been essential for improving the correctness and the readability of a preliminary version of this paper.

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