Well-posedness of the Prandtl equation without any structural assumption

Helge Dietert, David Gérard-Varet

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November 26, 2023

Abstract

We show the local in time well-posedness of the Prandtl equation for data with Gevrey 2 regularity in x and H^1 regularity in y. The main novelty of our result is that we do not make any assumption on the structure of the initial data: no monotonicity or hypothesis on the critical points. Moreover, our general result is optimal in terms of regularity, in view of the ill-posedness result of [9].

1 Introduction

We are interested in the 2D Prandtl equation

$$\partial_t U^P + U^P \partial_x U^P + V \partial_y U^P - \partial_y^2 U^P = \partial_t U^E + U^E \partial_x U^E, \quad \partial_x U^P + \partial_y V^P = 0, \tag{1}$$

set in the domain $\Omega = \mathbb{T} \times \mathbb{R}_+$, completed with boundary conditions

$$U^P|_{y=0} = V^P|_{y=0} = 0, \quad \lim_{y \to +\infty} U^P = U^E. \tag{2}$$

This equation is a degenerate Navier-Stokes model, introduced by Prandtl in 1904 to describe the boundary layer, which is the region of high velocity gradients that forms near solid boundaries in incompressible flows at high Reynolds number. It can be derived from the Navier-Stokes equation under the formal asymptotics

$$(u^\nu, v^\nu)(t,x,z) \approx (U^P(t,x,z/\sqrt{\nu}), \sqrt{\nu} V^P(t,x,z/\sqrt{\nu})), \quad (U^P, V^P) = (U^P, V^P)(t,x,y), \tag{3}$$

where ν is the inverse Reynolds number, and (u^ν, v^ν) is the Navier-Stokes solution. This asymptotics is supposed to apply to the flow in the boundary layer region: the typical scale $\sqrt{\nu}$ of the boundary layer in this model is inspired by the heat part of the Navier-Stokes equation. Away from the boundary, one rather expects an inviscid asymptotics of the type

$$(u^\nu, v^\nu)(t,x,z) = (u^E, v^E)(t,x,z),$$

where (u^E, v^E) is the solution of the Euler equation. In order to match the two asymptotic expansions, one must impose the condition

$$\lim_{y \to +\infty} U^P(t,x,y) = U^E(t,x) := u^E(t,x,0),$$

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which yields the boundary condition for $y \to \infty$ in (2). The other two boundary conditions at $y = 0$ express the usual no-slip condition at the boundary. We refer to [6] for a more detailed derivation. Let us stress that the pressure in the Prandtl model is independent of y: its value is given by the pressure in the Euler flow at $z = 0$. This explains the right-hand side of (1), which depends only on t, x, and is coherent with the third boundary condition in (2).

The Prandtl system (1)-(2) is very classical, as it appears in most textbooks on fluid dynamics. Still, it is well-known from physicists that its range of applications is narrow, due to underlying instabilities. Among those instabilities, one can mention the phenomenon of separation, which is related to the development of a reverse flow in the boundary layer [8, 4]. Another example is the so-called Tollmien-Schlichting wave, that is typical of viscous flows at high but finite Reynolds number [5, 15]. Of course, such instability mechanisms create difficulties at the PDE level, making the mathematical analysis of boundary layer theory an interesting topic. The two main problems that one needs to address are the well-posedness of the reduced model (1), and the validity of the asymptotics (3). We shall focus on the former in the present paper. About the validity of boundary layer expansions in the unsteady setting, there are many possible references, among which [31, 14, 32, 11, 26, 16]. About the steady setting, see the recent works [19, 10, 17].

To analyse the well-posedness of the Prandtl model is uneasy, even at the level of local in time smooth solutions. The key difference with Navier-Stokes is that there is no time evolution for the vertical velocity, which is recovered only through the divergence-free condition. Hence, the term $v\partial_y u$ can be seen as a first order nonlinear operator in x. Moreover, this operator is not skew-symmetric in H^s. As the diffusion in (1) is only transverse, this prevents the derivation of standard Sobolev estimates. The first rigorous study of the Prandtl equation goes back to Oleinik [29], who tackled the case of data $U|_{t=0}$ that are monotonic in y. She established local well-posedness of the system using the so-called Crocco transform, a tricky change of variables and unknowns. Let us stress that such monotonicity assumption excludes the phenomenon of reverse flow and therefore prevents boundary layer separation. More recently, the local well-posedness result of Oleinik was revisited using the standard Eulerian form of the equation, see [1, 27, 21] for the local theory in Sobolev spaces.

The analysis of non-monotonic data is much more recent, and has experienced some strong impetus over the last years. Surprisingly, it was shown in [9] that the Prandtl system is ill-posed in the Sobolev setting (*cf.* [13, 18, 24] for improvements). Specifically, paper [9] centers on the linearization of (1)-(2) around shear flows, given by $(U, V) = (U_s(y), 0)$. The linearized system reads

$$\begin{aligned} \partial_t u + U_s \partial_x u + U_s' v - \partial_y^2 u = 0, \\ \partial_x u + \partial_y v = 0, \\ u|_{y=0} = v|_{y=0} = 0, \quad \lim_{y \to +\infty} u = 0. \end{aligned} \tag{4}$$

In the case where U_s has one non-degenerate critical point, one can show that (4) has unstable solutions of the form $u(t, x, y) = \mathrm{e}^{\mathrm{i}kx} \mathrm{e}^{\sigma_k t} u_k(y)$ for k arbitrarily large and $\Re\sigma_k \sim \lambda\sqrt{k}$. Such high frequency instability forbids the construction of Sobolev solutions. To obtain positive results, one must start from initial data u_{in} that are strongly localized in Fourier, typically for which $|\hat{u}_0(k, y)| \lesssim \mathrm{e}^{-\delta|k|^\gamma}$ for some positive $\delta > 0$, $\gamma \leq 1$. Such localization condition corresponds to Gevrey regularity in x (Gevrey class $1/\gamma$). The first result in this direction is due to Sammartino and Caflisch [30], who established existence of local in time solutions in the analytic setting ($\gamma = 1$). See also the nice paper [22]. Note that the requirement for analyticity is natural in view of standard estimates. For instance, at the level of the linearized equation (4), one gets directly by testing against u that

$$\partial_t \|\hat{u}(t, k, \cdot)\|_{L^2_y} \leq C|k| \, \|\hat{u}(t, k, \cdot)\|_{L^2_y}$$

so that $\|\hat{u}(t,k,\cdot)\|_{L^2_y} \leq \mathrm{e}^{C|k|t}\|\hat{u}_0(k,\cdot)\|_{L^2_y}$. Hence, if $\|\hat{u}_0(k,\cdot)\|_{L^2_y} \lesssim \mathrm{e}^{-\delta|k|}$, a uniform control will be provided as long as $t \leq \delta/C$.

To relax the analyticity condition is much harder. In the special case where u_{in} *has for each value of* x *a single non-degenerate critical point in* y, the first author and N. Masmoudi proved the local well-posedness of system (1)-(2) for data that are in Gevrey class 7/4 with respect to x [12]. Well-posedness was extended to Gevrey class 2 in article [23], for data that are small perturbations of a shear flow with a single non-degenerate critical point. Note that this exponent (corresponding to $\gamma = 1/2$) is optimal in view of the instability mechanism of [9].

All the recent results mentioned above rely heavily on the structure of the initial data: monotonicity for the Sobolev setting, single non-degenerate critical points for the Gevrey setting. It is therefore natural to ask about the optimal regularity under which local well-posedness of the Prandtl equation holds, without additional structural assumption. This is the problem that we solve in the present paper: we establish the short-time well-posedness of the Prandtl equation for general data with Gevrey 2 regularity in x and H^1 regularity in y. We recall once more that such regularity framework is the best possible. Indeed, from the results of [9], high frequency modes k in x may experience exponential growth with rate $\sqrt{k}$. This means that to hope for short time stability, the amplitude of these modes should be $O(\mathrm{e}^{-C\sqrt{k}})$, which is the Fourier translation of a Gevrey 2 requirement.

2 Result

Let $\gamma \geq 1$, $\tau > 0$, $r \in \mathbb{R}$. For functions $f = f(x)$ of one variable, we define the Gevrey norm

$$|f|^2_{\gamma,\tau,r} = \sum_{j\in\mathbb{N}} \left(\frac{\tau^{j+1}(j+1)^r}{(j!)^\gamma}\right)^2 \|f^{(j)}\|^2_{L^2(\mathbb{T})} \tag{5}$$

and for functions $f = f(x,y)$ of two variables, the norm

$$\|f\|^2_{\gamma,\tau,r} = \sum_{j\in\mathbb{N}} \left(\frac{\tau^{j+1}(j+1)^r}{(j!)^\gamma}\right)^2 \|\partial_x^j f\|_j^2, \tag{6}$$

where $\|\cdot\|_j$, $j \geq 0$, denotes a family of weighted L^2 norms. Namely,

$$\|f\|_j^2 = \int_{\mathbb{T}\times\mathbb{R}^+} |f(x,y)|^2 \rho_j(y)\,\mathrm{d}x\,\mathrm{d}y, \tag{7}$$

where ρ_j, $j \geq 0$, is the family of weights given by

$$\rho_0(y) = (1+y)^{2m}, \quad \rho_j(y) = \frac{\rho_{j-1}(y)}{\left(1+\frac{y}{j^\alpha}\right)^2} = \rho_0(y)\prod_{k=1}^{j}\left(1+\frac{y}{k^\alpha}\right)^{-2}, \quad j \geq 1,$$

for fixed constants $\alpha \geq 0$ and $m \geq 0$ chosen later (m large enough and α matching the constraints found from the estimates). The need for this family of weights will be clarified later. Let us note that *locally in* y, this family of norms is comparable to more classical families such as

$$|||f|||^2_{\gamma,\tau,r} = \sum_{j\in\mathbb{N}} \left(\frac{\tau^{j+1}(j+1)^r}{(j!)^\gamma}\right)^2 \|\partial_x^j f\|^2_{L^2}. \tag{8}$$

For instance, for functions f which are zero for $|y| \geq M$, one has

$$\|f\|_{\gamma,\tau,r} \leq C_M|||f|||_{\gamma,\tau,r}, \quad |||f|||_{\gamma,\tau,r} \leq C_{M,\tau'}\|f\|_{\gamma,\tau',r} \text{ for any } \tau' > \tau.$$

The only difference is when y goes to infinity, where the family of weights ρ_j puts less constraints on the decay of the derivatives compared to a fixed weight ρ_0 for derivatives of any order.

With these spaces, we can now state our main result.

Theorem 1. *There exists m and α such that: for all $0 < \tau_1 < \tau_0$, $r \in \mathbb{R}$, for all $T_0 > 0$, for all U^E satisfying*

$$\sup_{[0,T_0]} |\partial_t U^E|_{2,\tau_0,r} + |U^E|_{2,\tau_0,r} < +\infty, \quad \sup_{[0,T_0]} \max_{l=0,\dots,3} \|\partial_t^l(\partial_t + U^E\partial_x)U^E\|_{H^{6-2l}(\mathbb{T})} < +\infty$$

for all $U^P_{\rm in}$ satisfying

$$\|U^P_{\rm in} - U^E|_{t=0}\|_{2,\tau_0,r} < +\infty, \quad \|(1+y)\partial_y U^P_{\rm in}\|_{2,\tau_0,r} < +\infty, \quad \|(1+y)^{m+6}\partial_y U^P_{\rm in}\|_{H^6(\mathbb{T}\times\mathbb{R}_+)} < +\infty$$

and under usual compatibility conditions (see the last remark below), there exists $0 < T \leq T_0$ and a unique solution U^P of (1)-(2) over $(0,T)$ with initial data $U^P_{\rm in}$ that satisfies

$$\sup_{t\in[0,T]} \|U^P(t) - U^E(t)\|^2_{2,\tau_1,r} + \sup_{t\in[0,T]} \|(1+y)\partial_y U^P(t)\|^2_{2,\tau_1,r} + \int_0^T \|(1+y)\partial_y^2 U^P(t)\|^2_{2,\tau_1,r}\,\mathrm{d}t < +\infty$$

Remarks.

- The main novelty of the theorem is that we reach the optimal Gevrey regularity although no structural assumption is made on the data: no monotonicity, or hypothesis on the number and order of the critical points is needed. Only Gevrey regularity of the data and natural compatiblity conditions are required.

- Our method of proof, explained below, is inspired by the hyperbolic part of the Prandtl equation. It is based on both a tricky change of unknown and appropriate choice of test function. This method would also allow to recover the Sobolev well-posedness of the hyperbolic version of the Prandtl system *by means of energy methods*. As far as we know, the well-posedness of this inviscid Prandtl equation had been only established in C^k spaces using the method of characteristics: see [20] for more. This part will be detailed elsewhere. In the case of the usual Prandtl equation studied here, our methodology has to be slightly modified to handle in an optimal way the diffusion term. Still, commutators are responsible for the loss of Sobolev regularity: only Gevrey 2 smoothness in x can be established.

- There is a loss on the Gevrey radius τ of the solutions through time, going from τ_0 to τ_1. This loss, which appears technical in the paper, is actually unavoidable. This is due to the instabilities described in [9]: exponential growth of perturbations at rate $\sqrt{k}$ causes a decay of the Gevrey radius linearly with time.

- Besides the regularity requirements mentioned in Theorem 1, the initial data must satisfy *compatibility conditions*. It is typical of parabolic problems in domains with boundaries, *cf.* [28, Chapter 3] for a general discussion. Here, the value of $U^P_{\rm in}$ and of some of its derivatives at $y = 0$ cannot be arbitrary: they must be related to U^E accordingly to the equation and to the amount of regularity asked for u (with respect to the y-variable). Let us note that locally near $y = 0$, most of our estimates only involve $U^P - U^E$ in $L^2_t H^2_y$ (not mentioning the Gevrey regularity in x). Such estimates could be carried with the single compatibility condition $U^P_{\rm in}|_{y=0} = 0$. Still, the low norm $\|(U^P, V^P)\|_{low}$ introduced in (19) involves more y-derivatives: its control through Lemma 15 implies therefore a few more compatibility conditions. For the sake of brevity, we do not provide their explicit expressions, and refer to [33, Proposition 2.3] for a detailed discussion on a variation of the Prandtl equation.

Outline of the strategy. As mentioned earlier, our analysis of the Prandtl equation relies on the identification of new controlled quantities because the usual unknown u and kinetic energy do not give enough information. To help to identify the relevant quantities, it is a good idea to start from the study of the linearized system (4). After Fourier transform in x and Laplace transform in time, we are left with the ODE

$$(\lambda + \mathrm{i}kU_s)\partial_y\Psi - \mathrm{i}kU_s'\Psi - \partial_y^3\Psi = u_{\mathrm{in}} \tag{9}$$

where Ψ corresponds to the Fourier-Laplace transform of the stream function. At high frequencies k, a natural idea (although not legitimate in the end) is to neglect the diffusion term. We are then left with the first order ODE

$$(\lambda + \mathrm{i}kU_s)\partial_y\Psi - \mathrm{i}kU_s'\Psi = u_{\mathrm{in}}. \tag{10}$$

We note that the standard estimate (based on taking $\partial_y\overline{\Psi}$ as a test function) yields a control of the type

$$\begin{aligned}\Re\lambda\,\|\partial_y\Psi\|_{L^2}^2 &\lesssim k\|\partial_y\Psi\|_{L^2}\|U_s'\Psi\|_{L^2} + \|u_{\mathrm{in}}\|_{L^2}\|\partial_y\Psi\|_{L^2}\\ &\lesssim k\|\partial_y\Psi\|_{L^2}^2 + \|u_{\mathrm{in}}\|_{L^2}\|\partial_y\Psi\|_{L^2}\end{aligned}$$

where the last line comes from the Hardy inequality (as soon as $|U_s'(y)| = \mathcal{O}(y^{-1})$ at infinity). Such bound ensures the solvability of the resolvent equation (10) only for $\lambda \sim k$. This in turn yields a semigroup bound of the type e^{Ckt}, only compatible with stability in the analytic setting.

To reach stability in lower regularity, an important point is to notice that the homogeneous equation has $\Psi_s = (\lambda + ikU_s)$ as a special solution. With the integrating factor method in mind, it is then natural to set $\Psi = (\lambda + ikU_s)\psi$. The first order equation (10) becomes

$$(\lambda + ikU_s)^2\partial_y\psi = u_{\mathrm{in}}$$

which is much better than the original formulation. Indeed, we can test the equation against $\phi = \frac{1}{\lambda+ikU_s}\partial_y\overline{\psi}$ to obtain a control of $\partial_y\psi$ in terms of u_{in}, and from there a control of Ψ for any $\lambda > 0$.

Back to the full resolvent equation (9) we find for the same unknown ψ

$$(\lambda + \mathrm{i}kU_s)^2\partial_y\psi - (\lambda + \mathrm{i}kU_s)\partial_y^3\psi = u_{\mathrm{in}} + [\lambda + \mathrm{i}kU_s, \partial_y^3]\psi.$$

Testing again against $\phi = \frac{1}{\lambda+ikU_s}\partial_y\overline{\psi}$, the LHS allows the control

$$\Re\lambda\,\|\partial_y\psi\|_{L^2}^2 + \|\partial_y^2\psi\|_{L^2}^2.$$

In the commutator at the RHS, the worst error term is $3\mathrm{i}k\partial_yU_s\partial_y^2\psi$, which is bounded as

$$C\frac{k}{|\Re\lambda|}\|\partial_y\psi\|_{L^2}\|\partial_y^2\psi\|_{L^2} \leq \frac{1}{2}\|\partial_y^2\psi\|_{L^2}^2 + \frac{C^2k^2}{|\Re\lambda|^2}\|\partial_y\psi\|_{L^2}^2.$$

We see that under the constraint $\Re\lambda \sim k^{2/3}$, the estimate can be closed, and this can be shown to imply short time stability for data with Gevrey regularity $3/2$. This estimate around a shear flow is detailed as Lemma 4.1 in [3].

In order to reach the optimal Gevrey exponent 2, we need to get rid of the commutator term containing $\partial_y^2\psi$, which comes with a worse control than $\partial_y\psi$. To do so, we change a bit our new unknown ψ: we now define ψ through the relation

$$\Psi = (\lambda + \mathrm{i}kU_s - \partial_y^2)\psi \tag{11}$$

including the diffusion term. Hence, (9) becomes

$$(\lambda + \mathrm{i}kU_s - \partial_y^2)^2\partial_y\psi + (\lambda + \mathrm{i}kU_s - \partial_y^2)(\mathrm{i}kU_s'\psi) - \mathrm{i}kU_s(\lambda + \mathrm{i}kU_s - \partial_y^2)\psi = u_{\mathrm{in}}.$$

Testing this time against the solution ϕ of $(\lambda + \mathrm{i}kU_s - \partial_y^2)\phi = \partial_y\overline{\psi}$ (again with the diffusion term), the LHS yields the same control, but the error term is now

$$\Re \int [\lambda + \mathrm{i}kU_s - \partial_y^2, \mathrm{i}kU_s']\psi\ \phi.$$

From the definition of ϕ it can be shown that $\|\phi\| \lesssim \lambda^{-1}\|\partial_y\psi\|$ so that the error can be bounded by

$$\frac{k}{|\Re\lambda|}\|\partial_y\psi\|_{L^2}^2.$$

The estimate can now be closed for $\Re\lambda \sim k^{1/2}$ yielding Gevrey regularity 2.

Obviously, such approach is no longer applicable as such to the nonlinear system (1)-(2): we not only lose the linearity of the equations, but the coefficients are no longer of shear flow type. They notably depend on t and x, which forbids an easy use of Fourier or Laplace transform. Rather than turning to the characterization of Gevrey spaces in the Fourier variable k, we consider norms based on the x-variable, see (5) and (6). Roughly, the idea is to work with time dependent norms, that is with the quantities

$$\|(U^P - U^E)(t)\|_{\gamma,\tau(t),r}, \quad \tau(t) = \tau_0\mathrm{e}^{-\beta t}.$$

By differentiating j-times the Prandtl equation, we can derive an equation on

$$u_j(t) := \frac{\tau(t)^{j+1}(j+1)^r}{(j!)^\gamma}\partial_x^j\left(U^P(t) - U^E(t)\right)$$

that can be written as

$$(\partial_t + \beta(j+1))u_j + U^P\partial_x u_j + V^P\partial_y u_j + v_j\partial_y U^P - \partial_y^2 u_j = F_j, \quad v_j = -\int_0^y \partial_x u_j. \tag{12}$$

Roughly, inspired by the shear flow case, the idea will be to introduce as a new unknown the solution $\psi_j = \int_0^y H_j$ of

$$(\partial_t + \beta(j+1) + U^P\partial_x - \partial_y^2)\psi_j = \int_0^y u_j\,\mathrm{d}z.$$

which is reminiscent of the Fourier relation (11). The test function ϕ_j should then solve the reverse equation

$$(-\partial_t + \beta(j+1) - U^P\partial_x - \partial_y^2)\phi_j = \partial_y\psi_j$$

and be solved backward in time. Performing the same estimate as in the shear flow case, we expect to find an inequality of the type

$$\beta(j+1)\|\partial_y\psi_j\|^2 + \|\partial_y^2\psi_j\|^2 \lesssim \frac{1}{\beta^3(j+1)^3}\|F_j\|^2 + \frac{1}{\beta^3(j+1)^3}\|\partial_x\partial_y\psi_j\|^2$$

By exploiting a relation of the form $\|\partial_x\partial_y\psi_j\| \sim j^\gamma\|\partial_y\psi_{j+1}\|$ (that needs to be shown!) and using that $\gamma \leq 2$, we will then be able to sum over j and establish for large enough β a control of $\sum_j \|\partial_y\psi_j\|^2$ in terms of $\sum_j \|F_j\|^2$.

In fact, in implementing this strategy, several refinements are necessary, and the relations satisfied by $\psi_j = \int_0^y H_j$ or ϕ_j need to be slightly modified. Particularly problematic is the term $V^P\partial_y$ because $V^P \sim -\partial_x U^E y$ increases linearly with y: this prevents from closing an energy estimate with a fixed weigth $\rho = \rho(y)$. This difficulty appears in various places in the literature on the Prandtl equation. This is for instance the reason why article [12] is limited to the special case $U^E = 0$ and decaying initial data. One can also mention [22], where this difficulty is overcome by a clever change of variables, which is reminiscent of the method of characteristics and allows to remove the bad part of the convection term from the equation. Energy estimates can then be established in these new coordinates x', y', and yield some local well-posedness result, with solutions that are analytic in x' and L^2 in y'. The disadvantage of this approach is that the regularity of the solution in the original variables x and y is no longer clear at positive times. Here, we stick to the eulerian variables, but overcome the difficulty by introducing the family of weights ρ_j, $j \geq 0$. These weights allow to trade a power of y against a derivative in x, which is appropriate to the commutator terms. Moreover, they put very little conditions on the derivatives of the solution, so that they provide a very general framework for well-posedness. Note that the specific expression of ρ_j is important: it could not be for instance replaced by the more natural guess $(1+y)^{2(m-j)}$, as commutators with the diffusion term would not be under control. Note also that the strategy used in [27], where Sobolev well-posedness is established under monotonicity assumptions by increasing the weight with the number of y-derivatives, does not extend to the Gevrey framework in variable x.

The plan of the paper is as follows. In the next section, we first collect several properties of the weight ρ_j. We then write the equations satisfied by the x-derivatives of the Prandtl solution in a form analogue to (12). This means that we put most of the nonlinear terms at the right-hand side, and consider those equations as linear. We finish the section by introducing the adapted quantities H_j and ϕ_j. The main section is Section 4: *a priori* Gevrey estimates for the linear equations are perfomed, that provide a control of the u_j's in terms of the nonlinear terms $F_j's$. Note that such estimates are obtained under a condition of the form $\beta > C(1+\|(U^P, V^P)\|_{low})^2$, where $\|(U^P, V^P)\|_{low}$ is a low regularity norm of the solution. The treatment of the nonlinearity F_j is then handled in Section 5. The last step in the derivation of *a priori* estimates is to recover the control of the low regularity norm $\|(U^P, V^P)\|_{low}$, see Section 6. Finally, issues regarding the construction and uniqueness of solutions are discussed in Section 7.

3 Preliminaries

The explicit form of the weights ρ_j is only needed in the Section 5. In the other parts, we just need a sufficient control of the logarithmic derivative (Lemma 2), a bound for antiderivatives (Lemma 3) and relate ρ_j to ρ_{j+1} (Lemma 4).

Lemma 2. *Let $m \geq 0$ and $\alpha \geq 0$. There exists a constant C_l such that for all $y \in \mathbb{R}^+$, $j \in \mathbb{N}$*

$$\left|\frac{\partial_y \rho_j(y)}{\rho_j(y)}\right| \leq \begin{cases} C_l(j+1)^{1-\alpha} & \text{if } \alpha < 1 \\ C_l \log(j+1) & \text{if } \alpha = 1 \\ C_l & \text{if } \alpha > 1 \end{cases}$$

and

$$(1+y)\left|\frac{\partial_y \rho_j(y)}{\rho_j(y)}\right| \leq C_l\,(j+1).$$

Proof. Given the explicit form of ρ_j, we can compute the logarithmic derivative of ρ directly as

$$\frac{\partial_y \rho_j}{\rho_j} = \partial_y \log \rho_j = \partial_y \log \rho_0 - 2\sum_{k=1}^{j} \partial_y \log\left(1 + \frac{y}{k^\alpha}\right) = \frac{2m}{1+y} - 2\sum_{k=1}^{j} \frac{1}{k^\alpha\left(1+\frac{y}{k^\alpha}\right)}.$$

From this expression the result follows directly. □

Lemma 3. *For $m > \frac{1}{2}$ introduce the constant*

$$C_m = \sqrt{\frac{1}{2m-1}}.$$

Then for all $\alpha \geq 0$, $j \in \mathbb{N}$ and all $f = f(y)$,

$$\sup_{y\geq 0}\left(\frac{\rho_j(y)}{\rho_0(y)}\right)^{1/2} \int_0^y |f(z)|\,\mathrm{d}z \leq C_m \|f\|_{L^2(\rho_j)}.$$

More generally, for $0 \leq n \leq j$ with $n < m - \frac{1}{2}$ one has

$$\sup_{y\geq 0}\left(\frac{\rho_j(y)}{\rho_n(y)}\right)^{1/2} \int_0^y |f(z)|\,\mathrm{d}z \leq C_{m-n} \|f\|_{L^2(\rho_j)}.$$

Eventually, for all $A = A(x,y)$ and $B = B(x,y)$, the following inequality holds:

$$\|A\int_0^y B(z)\,\mathrm{d}z\|_j \leq C_m \|A\|_{L^\infty_x L^2_y(\rho_0)} \|B\|_j.$$

Proof. Note that ρ_j/ρ_n for $j \geq n$ is non-increasing. Hence

$$\left(\frac{\rho_j(y)}{\rho_n(y)}\right)^{1/2} \int_0^y |f(z)|\,\mathrm{d}z \leq \int_0^y \left(\frac{\rho_j(z)}{\rho_n(z)}\right)^{1/2} |f(z)|\,\mathrm{d}z \leq \|f\|_{L^2(\rho_j)} \left(\int_0^y \frac{1}{\rho_n(z)}\,\mathrm{d}z\right)^{1/2},$$

where we used the Cauchy-Schwarz inequality in the second inequality.

As $\alpha \geq 0$ we find directly that

$$\frac{1}{\rho_n(y)} \leq (1+y)^{-2m} \prod_{k=1}^{j}\left(1+\frac{y}{k^\alpha}\right)^2 \leq \frac{1}{(1+y)^{2m-2n}}$$

whose integral over $y \in \mathbb{R}^+$ gives C^2_{m-n}. This proves the first and second bounds. The remaining estimate with A and B follows directly. □

The weights are decaying so that $\rho_j \leq \rho_k$ for $j \geq k$. As $\alpha \geq 0$, we have for $j \in \mathbb{N}$ that $(1+y)^2\rho_{j+1} \leq (j+1)^{2\alpha}\rho_j$ and $\rho_{j+1} \geq \frac{\rho_j}{(1+y)^2}$. This shows:

Lemma 4. *Let $\alpha \geq 0$. For $j \in \mathbb{N}$, $A = A(x,y)$ and $B = B(x,y)$ it holds that for*

$$\|A\|_{j+1} \leq \|A\|_j, \qquad \|(1+y)A\|_{j+1} \leq (j+1)^\alpha \|A\|_j$$

and

$$\left\|\frac{A}{(1+y)}\right\|_j \leq \|A\|_{j+1}, \qquad \|A\int_0^y B(z)\,\mathrm{d}z\|_j \leq C_m \|(1+y)A\|_{L^\infty_x L^2_y(\rho_0)} \|B\|_{j+1}.$$

Let us insist again that most parts of the proof would work with constant weight ρ instead of ρ_j. The dependency on j will be only needed to treat the commutator terms coming from $V^P\partial_y U^P$. The difficulty is that V^P grows like y as soon as U^E is non-constant. Here the crucial property that we will use is that we can control $\|(1+y)A\|_{j+1}$ by $(j+1)^\alpha\|A\|_j$.

The Prandtl equation is given for (U^P, V^P) with inhomogeneous boundary conditions at $y \to \infty$. In order to work with homogeneous boundary conditions at zero and infinity, we introduce

$$U^e(t,x,y) = (1-\mathrm{e}^{-y})U^E(t,x), \quad V^e(t,x,y) = -(y+\mathrm{e}^{-y}-1)\partial_x U^E(t,x)$$

and set $u = U^P - U^e$, $v = -\int_0^y \partial_x u = V^P - V^e$. Then,

$$\partial_t u + (u\partial_x + v\partial_y)u + (U^e\partial_x + V^e\partial_y)u + (u\partial_x + v\partial_y)U^e - \partial_y^2 u = f^e \tag{13}$$

where

$$f^e = \partial_t U^E + U^E\partial_x U^E - \partial_t U^e - U^e\partial_x U^e - V^e\partial_y U^e + \partial_y^2 U^e. \tag{14}$$

In the new variables (u,v) the boundary conditions are

$$u = v = 0 \quad \text{at } y = 0, \qquad \text{and } \lim_{y\to\infty} u = 0. \tag{15}$$

The condition at $y \to \infty$ will be encoded in the functional space of u.

To prove Theorem 1, the point is to obtain good estimates for Gevrey norms of u of type (6) for time-dependent radius $\tau = \tau(t)$. More precisely, we give ourselves parameters m, α, γ, r, to be fixed later, as well as the time-dependent radius $\tau(t) = \tau_0\mathrm{e}^{-\beta t}$, with $\beta > 0$ to be fixed later. Then, for any function $f = f(t,x)$ or $f = f(t,x,y)$ and $j \in \mathbb{N}$ we set

$$f_j(t,\cdot) := M_j\,\partial_x^j f(t,\cdot) \quad \text{with} \quad M_j := \frac{\tau(t)^{j+1}(j+1)^r}{(j!)^\gamma}.$$

Taking j derivatives in x of (13) and multiplying by M_j yields

$$\Big(\partial_t + \beta(j+1) + U^P\partial_x + (j+1)\partial_x U^P + V^P\partial_y - \partial_y^2\Big)u_j + \partial_y U^P v_j + j\partial_{xy}U^P\partial_x^{-1}v_j = F_j \tag{16}$$

where F_j collects all terms with less than j derivatives in x as well as the weighted derivative of the forcing f^e. It is given by

$$\begin{aligned}
F_j = f_j^e &+ M_j\left[u\partial_x, \partial_x^j\right]u + M_j\,\partial_x u\,\partial_x^j u \\
&+ M_j\left[\partial_y u, \partial_x^j\right]v + M_j j\,\partial_{xy}u\,\partial_x^{j-1}v + M_j v\,\partial_x^j\partial_y u \\
&+ M_j\left[U^e\partial_x, \partial_x^j\right]u + M_j j\,\partial_x U^e\,\partial_x^j u \\
&+ M_j\left[V^e\partial_y, \partial_x^j\right]u \\
&+ M_j\left[\partial_x U^e, \partial_x^j\right]u \\
&+ M_j\left[\partial_y U^e, \partial_x^j\right]v + M_j j\,\partial_{xy}U^e\,\partial_x^{j-1}v.
\end{aligned}$$

We now introduce our crucial auxiliary functions $H_j(t,x,y)$ defined by

$$\begin{aligned}
&\Big(\partial_t + \beta(j+1) + U^P\partial_x + (j+1)\partial_x U^p + V^P\partial_y - \partial_y^2\Big)\int_0^y H_j\,\mathrm{d}z = \int_0^y u_j\,\mathrm{d}z, \\
&H_j|_{t=0} = 0, \qquad \partial_y H_j|_{y=0} = 0, \qquad H_j|_{y\to\infty} = 0.
\end{aligned} \tag{17}$$

For the existence of H_j, one can consider (17) as a convection-diffusion equation for $A_j = \int_0^y H_j\,\mathrm{d}z$, with boundary conditions $A_j|_{y=0} = \partial_y A_j|_{y\to\infty} = 0$, which has a solution by the

classical theory of parabolic PDEs. The PDE (17) itself then implies that $\partial_y^2 A_j|_{y=0}$ so that taking $H_j = \partial_y A_j$ gives the required solution.

We further introduce the corresponding test functions ϕ_j by

$$\begin{aligned}&\left(-\partial_t + \beta(j{+}1) - U^P\partial_x + j\partial_x U^p - V^P\partial_y - \partial_y V^P - V^P\frac{\partial_y\rho_j}{\rho_j} - \left(\partial_y + \frac{\partial_y\rho_j}{\rho_j}\right)^2\right)\phi_j = H_j,\\ &\phi_j|_{t=T} = 0, \qquad \phi_j|_{y=0} = 0, \qquad \phi_j|_{y\to\infty} = 0.\end{aligned} \tag{18}$$

Note here that the operator acting on ϕ_j is the formal adjoint operator of the operator acting on $\int_0^y H_j\,\mathrm{d}z$ in (17), *with respect to the* $L^2(\rho_j)$ *scalar product, denoted* $\langle\,,\rangle_j$. This is a backward heat equation solved backward in time for $t \in [0,T]$.

Testing (18) against ϕ_j in $\|\cdot\|_j$ and integrating over $[t,T]$ yields

$$\begin{aligned}&\frac{1}{2}\|\phi_j(t)\|_j^2 + \beta(j{+}1)\int_t^T \|\phi_j(s)\|_j^2\,\mathrm{d}s + (j{+}\frac{1}{2})\int_t^T \langle\partial_x U^P\phi_j,\phi_j\rangle_j\,\mathrm{d}s\\ &-\frac{1}{2}\int_t^T \langle\left(\partial_y V^P + V^P\frac{\partial_y\rho_j}{\rho_j}\right)\phi_j,\phi_j\rangle_j\,\mathrm{d}s + \int_t^T\|\partial_y\phi_j(s)\|_j^2\,\mathrm{d}s + \int_t^T\langle\frac{\partial_y\rho_j}{\rho_j}\phi_j,\partial_y\phi_j\rangle_j\,\mathrm{d}s\\ &= \int_t^T\langle H_j,\phi_j\rangle_j\,\mathrm{d}s.\end{aligned}$$

Hence we find

$$\begin{aligned}&\frac{1}{2}\|\phi_j(t)\|_j^2 + \frac{3\beta(j{+}1)}{4}\int_t^T\|\phi_j(s)\|_j^2\,\mathrm{d}s + \frac{1}{2}\int_t^T\|\partial_y\phi_j(s)\|_j^2\,\mathrm{d}s\\ &\le \frac{1}{\beta(j{+}1)}\int_t^T\|H_j(s)\|_j^2\,\mathrm{d}s + \left((j{+}\frac{1}{2})\|\partial_x U^P\|_\infty + \frac{1}{2}\left\|\partial_y V^P + V^P\frac{\partial_y\rho_j}{\rho_j}\right\|_\infty + \frac{1}{2}\left\|\frac{\partial_y\rho_j}{\rho_j}\right\|_\infty^2\right)\int_t^T\|\phi_j(s)\|_j^2\,\mathrm{d}s.\end{aligned}$$

By Lemma 2, under the condition $\alpha \ge \frac{1}{2}$, we get the following control:

Lemma 5. *Fix* $m \ge 0$ *and* $\alpha \ge \frac{1}{2}$. *Then there exist a constant* $\mathcal{C} = \mathcal{C}(m,\alpha)$ *such that for all* $j \in \mathbb{N}$ *it holds that*

$$\|\phi_j(t)\|_j^2 + \beta(j{+}1)\int_t^T\|\phi_j(s)\|_j^2\,\mathrm{d}s + \int_t^T\|\partial_y\phi_j(s)\|_j^2\,\mathrm{d}s \le \frac{2}{\beta(j{+}1)}\int_t^T\|H_j(s)\|_j^2\,\mathrm{d}s$$

if

$$\beta \ge \mathcal{C}\left(1 + \|\partial_x U^P\|_\infty + \|\partial_y V^P\|_\infty + \left\|\frac{V^P}{1+y}\right\|_\infty\right).$$

Note that for $\alpha < \frac{1}{2}$, the term with $\|\frac{\partial_y\rho_j}{\rho_j}\|_\infty^2$ could not have been absorbed. This *a priori* estimate also ensure the existence of ϕ_j as solution of (18). A similar estimate holds for H_j which ensures the existence of H_j as solution of (17).

4 Linear estimates

In this section we analyse the linearised equation (16) and obtain an estimate for the solution in terms of the F_j containing the forcing and lower-order terms. For this, we shall first analyse (16) for a fixed j. We will obtain a control of H_j in terms of the forcing F_j and an error term $\partial_x H_j$, which will be shown to be approximately $(j{+}1)^\gamma H_{j+1}$. By summing over j, we will find the following control.

Lemma 6. *Fix $m > \frac{1}{2}, \frac{1}{2} \le \alpha \le \frac{1}{2} + \gamma, 1 \le \gamma \le 2, r \in \mathbb{R}$. Then there exists a constant $\mathcal{C} = \mathcal{C}(m, \alpha, \gamma, r)$ such that for all τ_1, β and T such that*

$$\beta \ge \mathcal{C}(1 + \|(U^P, V^P)\|_{low}) \, (1 + \frac{1}{\tau_1} + \|(U^P, V^P)\|_{low}) \text{ and } \tau(T) \ge \tau_1$$

the H_j's defined by (17) for solutions u_j's of (16) satisfy

$$\begin{aligned}
&\sum_{j=0}^{\infty} \beta^2 (j+1)^{2\gamma} \left[\int_0^T \|H_j(t)\|_j^2 \, \mathrm{d}t + \frac{1}{\beta(j+1)} \|H_j(T)\|_j^2 + \frac{1}{\beta(j+1)} \int_0^T \|\partial_y H_j\|_j^2 \, \mathrm{d}t \right] \\
&\le 16 \sum_{j=0}^{\infty} \left[\frac{(j+1)^{2\gamma-4}}{\beta^2} \int_0^T \|F_j(t)\|_j^2 \, \mathrm{d}t + \frac{(j+1)^{2\gamma-3}}{\beta} \|u_{\mathrm{in},j}\|_j^2 \right].
\end{aligned}$$

Here we use a low-order control of U^P and V^P in order to control the commutator error terms. From the required bounds, we define the low-order norm as

$$\begin{aligned}
\|(U^P, V^P)\|_{low} = \sup_{t \in [0,T]} \max \Big(& \max_{0 \le k \le 3} \|\partial_x^k U^P\|_\infty, \|\partial_x \partial_y^2 U^P\|_\infty, \|(1+y)\partial_y U^P\|_\infty, \|(1+y)\partial_y^2 U^P\|_\infty, \\
& \|(1+y)\partial_y U^P\|_{L_x^\infty L_y^2(\rho_0)}, \|\partial_{xy} U^P\|_{L_x^\infty L_y^2(\rho_0)}, \|\partial_{xxy} U^P\|_{L_x^\infty L_y^2(\rho_0)}, \\
& \|(1+y)^2 \partial_y^2 U^P\|_{L_x^\infty L_y^2(\rho_0)}, \|(1+y)\partial_x \partial_y^2 U^P\|_{L_x^\infty L_y^2(\rho_0)}, \quad \max_{0 \le k \le 2} \left\| \frac{\partial_x^k V^P}{1+y} \right\|_\infty \Big).
\end{aligned} \tag{19}$$

Although a main ingredient of our proof, the unknown H_j is less natural that the usual u_j, notably for the future treatment of the nonlinearity, which involves u_j and $\omega_j = \partial_y u_j$. This is why we shall we relate the control of H_j to u_j and show:

Proposition 7. *Fix $m > \frac{1}{2}, \frac{1}{2} \le \alpha \le \frac{1}{2} + \gamma, 1 \le \gamma \le 2, r \in \mathbb{R}$. Then there exist constants $C = C(m, \alpha, \gamma, r)$ and $\mathcal{C} = \mathcal{C}(m, \alpha, \gamma, r)$ such that for all τ_1, β and T such that*

$$\beta \ge \mathcal{C}(1 + \|(U^P, V^P)\|_{low}) \, (1 + \frac{1}{\tau_1} + \|(U^P, V^P)\|_{low}) \text{ and } \tau(T) \ge \tau_1$$

the solution u of (16) satisfies

$$\begin{aligned}
&\int_0^T \|u\|_{\gamma,\tau,r}^2 \, \mathrm{d}t + \sup_{t \in [0,T]} \frac{1}{\beta} \|u\|_{\gamma,\tau,r-\frac{\gamma}{2}}^2 + \int_0^T \frac{1}{\beta} \|(1+y)\omega\|_{\gamma,\tau,r+1-\gamma}^2 \, \mathrm{d}t \\
&\quad + \sup_{t \in [0,T]} \frac{1}{\beta^2} \|(1+y)\omega\|_{\gamma,\tau,r+\frac{1}{2}-\gamma}^2 + \frac{1}{\beta^2} \int_0^T \|(1+y)\partial_y \omega\|_{\gamma,\tau,r+\frac{1}{2}-\gamma}^2 \, \mathrm{d}t \\
&\le C \left[\frac{1}{\beta^2} \sum_{j=0}^{\infty} \int_0^T \frac{1}{(j+1)^{4-2\gamma}} \|F_j\|_j^2 \, \mathrm{d}t + \frac{1}{\beta^2} \sum_{j=0}^{\infty} \int_0^T \frac{1}{(j+1)^{2\gamma-1}} \|(1+y)F_j\|_j^2 \, \mathrm{d}t \right] \\
&\quad + \frac{C}{\beta^2} \sum_{j=0}^{\infty} \int_0^T \frac{1}{(j+1)^{2\gamma-1}} \|F_j|_{y=0}\|_{L_x^2}^2 \, \mathrm{d}t + C \left[\frac{1}{\beta} \|u_{\mathrm{in}}\|_{\gamma,\tau_0,r+\gamma-\frac{3}{2}}^2 + \frac{1}{\beta^2} \|(1+y)\omega_{\mathrm{in}}\|_{\gamma,\tau_0,r+\frac{1}{2}-\gamma}^2 \right].
\end{aligned}$$

For $\gamma \geq 5/4$ this is

$$\begin{aligned}
&\int_0^T \|u\|^2_{\gamma,\tau,r}\,\mathrm{d}t + \sup_{t\in[0,T]} \frac{1}{\beta}\|u\|^2_{\gamma,\tau,r-\frac{\gamma}{2}} + \int_0^T \frac{1}{\beta}\|(1+y)\omega\|^2_{\gamma,\tau,r+1-\gamma}\,\mathrm{d}t \\
&\quad + \sup_{t\in[0,T]} \frac{1}{\beta^2}\|(1+y)\omega\|^2_{\gamma,\tau,r+\frac{1}{2}-\gamma} + \frac{1}{\beta^2}\int_0^T \|(1+y)\partial_y\omega\|^2_{\gamma,\tau,r+\frac{1}{2}-\gamma}\,\mathrm{d}t \\
&\leq C\left[\frac{1}{\beta^2}\sum_{j=0}^{\infty}\int_0^T \frac{1}{(j+1)^{4-2\gamma}}\left\|\left(1+\frac{y}{(j+1)^{2\gamma-\frac{5}{2}}}\right)F_j\right\|_j^2\,\mathrm{d}t\right] \\
&\quad + \frac{C}{\beta^2}\sum_{j=0}^{\infty}\int_0^T \frac{1}{(j+1)^{2\gamma-1}}\|F_j|_{y=0}\|^2_{L^2_x}\,\mathrm{d}t + C\left[\frac{1}{\beta}\|u_{\mathrm{in}}\|^2_{\gamma,\tau_0,r+\gamma-\frac{3}{2}} + \frac{1}{\beta^2}\|(1+y)\omega_{\mathrm{in}}\|^2_{\gamma,\tau_0,r+\frac{1}{2}-\gamma}\right].
\end{aligned}$$

4.1 Estimate for H_j

We focus first on Lemma 6. The idea is to use the solution ϕ_j of (18) as a test function in (16). Taking the weighted scalar product and integrating over $[0,T]$, we find for the first term in (16):

$$\begin{aligned}
&\int_0^T \langle\Big(\partial_t + \beta(j+1) + U^P\partial_x + (j+1)\partial_x U^p + V^P\partial_y - \partial_y^2\Big)u_j, \phi_j\rangle_j\,\mathrm{d}t \\
&= -\langle u_{\mathrm{in},j}, \phi_j(0)\rangle_j \\
&+ \int_0^T \langle u_j, \left(-\partial_t + \beta(j+1) - U^P\partial_x + j\partial_x U^p - V^P\partial_y - \partial_y V^P - V^P\frac{\partial_y\rho_j}{\rho_j} - \left(\partial_y + \frac{\partial_y\rho_j}{\rho_j}\right)^2\right)\phi_j\rangle_j\,\mathrm{d}t \\
&= -\langle u_{\mathrm{in},j}, \phi_j(0)\rangle_j + \int_0^T \langle u_j, H_j\rangle_j\,\mathrm{d}t.
\end{aligned}$$

Note that there is no boundary term as u_j and ϕ_j vanish at the boundaries. Differentiating (17), we can replace u_j in the last integral and find

$$\begin{aligned}
&\int_0^T \langle u_j, H_j\rangle_j\,\mathrm{d}t \\
&= \int_0^T \langle\Big(\partial_t + \beta(j+1) + U^P\partial_x + (j+1)\partial_x U^p + V^P\partial_y - \partial_y^2\Big)H_j, H_j\rangle_j\,\mathrm{d}t \\
&\quad + \int_0^t \langle(\partial_y U^P\partial_x + j\partial_{xy}U^P)\int_0^y H_j\,\mathrm{d}z, H_j\rangle_j\,\mathrm{d}t \\
&\quad + \int_0^t \langle\partial_{xy}U^P\int_0^y H_j\,\mathrm{d}z, H_j\rangle_j\,\mathrm{d}t + \int_0^T \langle\partial_y V^P H_j, H_j\rangle_j\,\mathrm{d}t \\
&= \frac{1}{2}\|H_j(T)\|_j^2 + \beta(j+1)\int_0^T \|H_j(t)\|_j^2\,\mathrm{d}t + \int_0^T \|\partial_y H_j(t)\|_j^2\,\mathrm{d}t \\
&\quad + \int_0^t \langle(\partial_y U^P\partial_x + j\partial_{xy}U^P)\int_0^y H_j\,\mathrm{d}z, H_j\rangle_j\,\mathrm{d}t \\
&\quad + \int_0^t \langle\partial_{xy}U^P\int_0^y H_j\,\mathrm{d}z, H_j\rangle_j\,\mathrm{d}t + \int_0^T \langle\partial_y V^P H_j, H_j\rangle_j\,\mathrm{d}t \\
&\quad + (j+\frac{1}{2})\int_0^T \langle\partial_x U^P H_j, H_j\rangle_j\,\mathrm{d}t - \frac{1}{2}\int_0^T \langle\left(\partial_y V^P + V^P\frac{\partial_y\rho_j}{\rho_j}\right)H_j, H_j\rangle_j\,\mathrm{d}t + \int_0^T \langle\frac{\partial_y\rho_j}{\rho_j}H_j, H_j\rangle_j\,\mathrm{d}t.
\end{aligned}$$

By the boundary values of H_j there are again no boundary terms from partial integration in y. In the last expression, the first line contains the good controlled terms, the second line will

cancel the leading contribution from the bad terms $\partial_y U^P v_j + j\partial_{xy} U^P \partial_x^{-1} v_j$ (see below), while the last two lines collect the error terms.

Next, we compute the contribution from the terms with v_j using $v_j = -\partial_x \int_0^y u_j \,\mathrm{d}z$:

$$
\begin{aligned}
&\int_0^T \langle \partial_y U^P v_j, \phi_j \rangle_j \,\mathrm{d}t \\
&= -\int_0^T \langle \partial_y U^P \partial_x \left[\Big(\partial_t + \beta(j{+}1) + U^P \partial_x + (j{+}1)\partial_x U^p + V^P \partial_y - \partial_y^2 \Big) \int_0^y H_j \,\mathrm{d}z \right], \phi_j \rangle_j \,\mathrm{d}t \\
&= -\int_0^T \langle \partial_y U^P \Big(\partial_t + \beta(j{+}1) + U^P \partial_x + (j{+}1)\partial_x U^p + V^P \partial_y - \partial_y^2 \Big) \partial_x \int_0^y H_j \,\mathrm{d}z, \phi_j \rangle_j \,\mathrm{d}t \\
&\quad - \int_0^T \langle \partial_y U^P (\partial_x U^P \partial_x + (j{+}1)\partial_x^2 U^P + \partial_x V^P \partial_y) \int_0^y H_j \,\mathrm{d}z, \phi_j \rangle_j \,\mathrm{d}t \\
&= -\int_0^T \langle \Big(\partial_t + \beta(j{+}1) + U^P \partial_x + (j{+}1)\partial_x U^p + V^P \partial_y - \partial_y^2 \Big) \left[\partial_y U^P \partial_x \int_0^y H_j \,\mathrm{d}z \right], \phi_j \rangle_j \,\mathrm{d}t \\
&\quad + \int_0^T \langle \Big((\partial_t + U^P \partial_x + V^P \partial_y) \partial_y U^P - 2\partial_y^2 U^P \partial_y - \partial_y^3 U^P \Big) \partial_x \int_0^y H_j \,\mathrm{d}z, \phi_j \rangle_j \,\mathrm{d}t \\
&\quad - \int_0^T \langle \partial_y U^P (\partial_x U^P \partial_x + (j{+}1)\partial_x^2 U^P + \partial_x V^P \partial_y) \int_0^y H_j \,\mathrm{d}z, \phi_j \rangle_j \,\mathrm{d}t \\
&= -\int_0^T \langle \partial_y U^P \partial_x \int_0^y H_j \,\mathrm{d}z, H_j \rangle_j \,\mathrm{d}t \\
&\quad + \int_0^T \langle \Big((\partial_t + U^P \partial_x + V^P \partial_y) \partial_y U^P - 2\partial_y^2 U^P \partial_y - \partial_y^3 U^P \Big) \partial_x \int_0^y H_j \,\mathrm{d}z, \phi_j \rangle_j \,\mathrm{d}t \\
&\quad - \int_0^T \langle \partial_y U^P (\partial_x U^P \partial_x + (j{+}1)\partial_x^2 U^P + \partial_x V^P \partial_y) \int_0^y H_j \,\mathrm{d}z, \phi_j \rangle_j \,\mathrm{d}t
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_0^T \langle j\partial_{xy} U^P \partial_x^{-1} v_j, \phi_j \rangle_j \,\mathrm{d}t \\
&= -j \int_0^T \langle \partial_{xy} U^P \Big(\partial_t + \beta(j{+}1) + U^P \partial_x + (j{+}1)\partial_x U^p + V^P \partial_y - \partial_y^2 \Big) \int_0^y H_j \,\mathrm{d}z, \phi_j \rangle_j \,\mathrm{d}t \\
&= -j \int_0^T \langle \partial_{xy} U^P \int_0^y H_j \,\mathrm{d}z, H_j \rangle_j \,\mathrm{d}t \\
&\quad + j \int_0^T \langle \Big((\partial_t + U^P \partial_x + V^P \partial_y) \partial_{xy} U^P - 2\partial_x \partial_y^2 U^P \partial_y - \partial_x \partial_y^3 U^P \Big) \int_0^y H_j \,\mathrm{d}z, \phi_j \rangle_j \,\mathrm{d}t.
\end{aligned}
$$

In both cases the leading order term cancels. Hence collecting the terms we arrive at

$$
\begin{aligned}
&\frac{1}{2} \|H_j(T)\|_j^2 + \beta(j{+}1) \int_0^T \|H_j(t)\|_j^2 \,\mathrm{d}t + \int_0^T \|\partial_y H_j(t)\|_j^2 \,\mathrm{d}t \\
&\leq \int_0^T \langle F_j, \phi_j \rangle \,\mathrm{d}t + \langle u_{\mathrm{in},j}, \phi_j(0) \rangle_j + \int_0^T \sum_{i=1}^5 E_i \,\mathrm{d}t
\end{aligned}
$$

where $E_1, \dots, E_5$ collect the lower-order error terms as

$$
\begin{aligned}
E_1 &= -\langle \partial_{xy} U^P \int_0^y H_j \,\mathrm{d}z, H_j\rangle_j - \langle \partial_y V^P H_j, H_j\rangle_j,\\
E_2 &= -(j+\frac{1}{2})\langle \partial_x U^P H_j, H_j\rangle_j + \frac{1}{2}\langle \Big(\partial_y V^P + V^P \frac{\partial_y \rho_j}{\rho_j}\Big) H_j, H_j\rangle_j - \langle \frac{\partial_y \rho_j}{\rho_j} H_j, H_j\rangle_j,\\
E_3 &= -\langle \Big((\partial_t + U^P\partial_x + V^P\partial_y)\partial_y U^P - 2\partial_y^2 U^P \partial_y - \partial_y^3 U^P\Big)\partial_x \int_0^y H_j \,\mathrm{d}z, \phi_j\rangle_j,\\
E_4 &= \langle \partial_y U^P (\partial_x U^P \partial_x + (j+1)\partial_x^2 U^P + \partial_x V^P \partial_y) \int_0^y H_j \,\mathrm{d}z, \phi_j\rangle_j,\\
E_5 &= -j\langle \Big((\partial_t + U^P\partial_x + V^P\partial_y)\partial_{xy} U^P - 2\partial_x\partial_y^2 U^P \partial_y - \partial_x\partial_y^3 U^P\Big) \int_0^y H_j \,\mathrm{d}z, \phi_j\rangle_j.
\end{aligned}
$$

Here E_3 and E_4 contain the worst terms, as they involve x-derivatives of H_j. They are responsible for the Gevrey regularity requirement.

Assume $m \geq 0$, $\alpha \geq \frac{1}{2}$ and β large enough so that Lemma 5 applies. We can then estimate the forcing terms as

$$
\int_0^T \langle F_j, \phi_j\rangle_j \,\mathrm{d}t \leq \frac{2}{\beta^3 (j+1)^3} \int_0^T \|F_j(t)\|_j^2 \,\mathrm{d}t + \frac{\beta(j+1)}{4} \int_0^T \|H_j(t)\|_j^2 \,\mathrm{d}t
$$

and

$$
\langle u_{\mathrm{in},j}, \phi_j(0)\rangle_j \leq \frac{2}{\beta^2 (j+1)^2} \|u_{\mathrm{in},j}\|_j^2 + \frac{\beta(j+1)}{4} \int_0^T \|H_j(t)\|_j^2 \,\mathrm{d}t.
$$

Absorbing the terms with H_j we therefore find

$$
\begin{aligned}
&\|H_j(T)\|_j^2 + \beta(j+1) \int_0^T \|H_j(t)\|_j^2 \,\mathrm{d}t + 2\int_0^T \|\partial_y H_j(t)\|_j^2 \,\mathrm{d}t\\
&\leq \frac{4}{\beta^3 (j+1)^3} \int_0^T \|F_j(t)\|_j^2 \,\mathrm{d}t + \frac{4}{\beta^2 (j+1)^2} \|u_{\mathrm{in},j}\|_j^2 + 2\int_0^T \sum_{i=1}^5 E_i \,\mathrm{d}t.
\end{aligned}
$$

We now estimate the error terms, where we repeatedly use Lemma 3. For E_1 we find

$$
E_1 \leq \Big[C_m \|\partial_{xy} U^P\|_{L_x^\infty L_y^2(\rho_0)} + \|\partial_y V^P\|_\infty\Big] \|H_j\|_j^2.
$$

For E_2 we also use Lemma 2 and assume $\alpha \geq \frac{1}{2}$

$$
E_2 \leq \left[(j+\frac{1}{2})\|\partial_x U^P\|_\infty + \frac{1}{2}\|\partial_y V^P\|_\infty + C_l(j+1)\left(1 + \left\|\frac{V^P}{1+y}\right\|_\infty\right)\right] \|H_j\|_j^2.
$$

In the term E_3 we have terms with $\partial_x H_j$, which we want to estimate in $\|\cdot\|_{j+1}$ as they will be later controlled by H_{j+1}. Using Lemma 4 we find

$$
\begin{aligned}
E_3 &\leq C_m \|(1+y)(\partial_t + U^P\partial_x + V^P\partial_y - \partial_y^2)\partial_y U^P\|_{L_x^\infty L_y^2(\rho_0)} \|\partial_x H_j\|_{j+1} \|\phi_j\|_j\\
&\quad + 2\|(1+y)\partial_y^2 U^P\|_\infty \|\partial_x H_j\|_{j+1} \|\phi_j\|_j\\
&\leq 2\|(1+y)\partial_y^2 U^P\|_\infty \|\partial_x H_j\|_{j+1} \|\phi_j\|_j
\end{aligned}
$$

where we used the identity

$$
(\partial_t + U^P\partial_x + V^P\partial_y)\partial_y U^P - \partial_y^2 \partial_y U^P = 0. \tag{20}
$$

Similarly, we find for E_4 that

$$\begin{aligned}E_4 \le\ & C_m\|(1+y)\partial_y U^P \partial_x U^P\|_{L^\infty_x L^2_y(\rho_0)}\|\partial_x H_j\|_{j+1}\|\phi_j\|_j \\ & + (j+1)C_m\|\partial_y U^P \partial_x^2 U^P\|_{L^\infty_x L^2_y(\rho_0)}\|H_j\|_j\|\phi_j\|_j \\ & + \|\partial_y U^P \partial_x V^P\|_\infty \|H_j\|_j\|\phi_j\|\end{aligned}$$

And finally for E_5 we find

$$\begin{aligned}E_5 &\le \big(jC_m\|(\partial_t + U^P\partial_x + V^P\partial_y - \partial_y^2)\partial_{xy}U^P\|_{L^\infty_x L^2_y(\rho_0)} + 2j\|\partial_x\partial_y^2 U^P\|_\infty\big)\|H_j\|_j\|\phi_j\|_j \\ &\le \big(jC_m\|(\partial_x U^P\partial_x + \partial_x V^P\partial_y)\partial_y U^P\|_{L^\infty_x L^2_y(\rho_0)} + 2j\|\partial_x\partial_y^2 U^P\|_\infty\big)\|H_j\|_j\|\phi_j\|_j\end{aligned}$$

where we took again advantage of (20).

We collect the various factors in constants D_1, D_2, D_3 defined as folllows:

$$D_1 = 4\Big(\|(1+y)\partial_y^2 U^P\|_\infty + C_m\|(1+y)\partial_y U^P\partial_x U^P\|_{L^\infty_x L^2_y(\rho_0)}\Big)$$

and

$$\begin{aligned}D_2 = 2\Big((j+1)C_m\|\partial_y U^P\partial_x^2 U^P\|_{L^\infty_x L^2_y(\rho_0)} + \|\partial_y U^P\partial_x V^P\|_\infty \\ + jC_m\|(\partial_x U^P\partial_x + \partial_x V^P\partial_y)\partial_y U^P\|_{L^\infty_x L^2_y(\rho_0)} + 2j\|\partial_x\partial_y^2 U^p\|_\infty\Big)\end{aligned}$$

and

$$D_3 = 2\left(C_m\|\partial_{xy}U^P\|_{L^\infty_x L^2_y(\rho_0)} + \|\partial_y V^P\|_\infty + (j+\frac{1}{2})\|\partial_x U^P\|_\infty + \frac{1}{2}\|\partial_y V^P\|_\infty + C_l(j+1)\left(1 + \left\|\frac{V^P}{1+y}\right\|_\infty\right)\right).$$

Then

$$\begin{aligned}2\int_0^T \sum_{i=1}^5 E_i\,\mathrm{d}t &\le D_1\int_0^T \|\partial_x H_j\|_{j+1}\|\phi_j\|_j\,\mathrm{d}t + D_2\int_0^T \|H_j\|_j\|\phi_j\|_j\,\mathrm{d}t + D_3\int_0^T \|H_j\|_j^2\,\mathrm{d}t \\ &\le \frac{1}{4}\int_0^T \beta^3(j+1)^3\|\phi_j\|_j^2\,\mathrm{d}t + \frac{2D_1^2}{\beta^3(j+1)^3}\int_0^T \|\partial_x H_j\|_{j+1}^2\,\mathrm{d}t + \left(\frac{2D_2^2}{\beta^3(j+1)^3} + D_3\right)\int_0^T \|H_j\|_j^2\,\mathrm{d}t\end{aligned}$$

With Lemma 5 the ϕ integral can be estimated as

$$\frac{1}{4}\int_0^T \beta^3(j+1)^3\|\phi_j\|_j^2\,\mathrm{d}t \le \frac{1}{2}\beta(j+1)\int_0^T \|H_j(t)\|_j^2\,\mathrm{d}t$$

and thus can be absorbed in the LHS.

Here $\|(U^P, V^P)\|_{low}$ has been designed such that we can find numerical constants c_1, c_2, c_3 such that

$$\begin{aligned}D_1 &\le c_1(1 + \|(U^P, V^P)\|_{low})^2, \\ D_2 &\le c_2(j+1)\,(1 + \|(U^P, V^P)\|_{low})^2, \\ D_3 &\le c_3(j+1)\,(\|(U^P, V^P)\|_{low}).\end{aligned}$$

Combining all the estimates we arrive at the following lemma.

Lemma 8. *Assume $\alpha \ge \frac{1}{2}$ and $m > \frac{1}{2}$. Then there exist a constant $\mathcal{C} = \mathcal{C}(m, \alpha)$ such that for*

$$\beta \ge \mathcal{C}(1 + \|(U^P, V^P)\|_{U,low})$$

and $j \in \mathbb{N}$ the H_j defined by (17) for a solution u_j of (13) satisfy

$$\begin{aligned}
&2\|H_j(T)\|_j^2 + \beta(j+1)\int_0^T \|H_j(t)\|_j^2 \,\mathrm{d}t + 4\int_0^T \|\partial_y H_j(t)\|_j^2 \,\mathrm{d}t \\
&\leq \frac{8}{\beta^3(j+1)^3}\int_0^T \|F_j(t)\|_j^2 \,\mathrm{d}t + \frac{8}{\beta^2(j+1)^2}\|u_{\mathrm{in},j}\|_j^2 \\
&\quad + \frac{4c_1^2(1+\|(U^P,V^P)\|_{low})^4}{\beta^3(j+1)^3}\int_0^T \|\partial_x H_j\|_{j+1}^2 \,\mathrm{d}t.
\end{aligned}$$

Proof. Use the previous estimates. Note that the condition on β also implies that the hypothesis of Lemma 5 is satisfied by choosing $\mathcal{C}$ large enough. □

4.2 Relating $\partial_x H_j$ with H_{j+1}

To conclude the proof of Lemma 6, that will be achieved by summation of the previous estimate over j, we need first to control $\partial_x H_j$ by H_{j+1}.

Lemma 9. *Let $m > \frac{1}{2}$ and $\alpha \geq \frac{1}{2}$. Then there exist constants $\mathcal{C} = \mathcal{C}(m,\alpha)$ and $C = C(m,\alpha,r)$ such that for all τ_1, β and T with*

$$\beta \geq \mathcal{C}\left(1+\|(U^P,V^P)\|_{low}\right)^2, \quad \tau(T) \geq \tau_1,$$

it holds that

$$\begin{aligned}
&\int_0^T \|\partial_x H_j\|_{j+1}^2 \mathrm{d}t \\
&\leq C\frac{(j+1)^{2\gamma}}{\tau_1^2}\int_0^T \|H_{j+1}\|_{j+1}^2 \,\mathrm{d}t + C\frac{(j+1)^{2\alpha-2}}{\beta}\int_0^T \|\partial_y H_j\|_j^2 \,\mathrm{d}t + \frac{C}{\beta}\int_0^T \|H_j\|_j^2 \,\mathrm{d}t.
\end{aligned}$$

Proof. From the definition of u_j, it holds that $\partial_x u_j(t) = \left(\frac{j+2}{j+1}\right)^r \frac{(j+1)^\gamma}{\tau(t)} u_{j+1}(t)$. Hence we anticipate that

$$\partial_x H_j(t) \approx \left(\frac{j+2}{j+1}\right)^r \frac{(j+1)^\gamma}{\tau(t)} H_{j+1}(t).$$

Therefore we estimate the difference

$$\Delta_j := \partial_x H_j - \left(\frac{j+2}{j+1}\right)^r \frac{(j+1)^\gamma}{\tau(t)} H_{j+1}.$$

From equation (17) (used with indices j and $j+1$), we find that

$$\left(\partial_t + \beta(j+1) + U^P\partial_x + (j+2)\partial_x U^P + V^P\partial_y - \partial_y^2\right)\int_0^y \Delta_j \,\mathrm{d}z = -\left[(j+1)\partial_{xx}U^P + \partial_x V^P \partial_y\right]\int_0^y H_j \,\mathrm{d}z. \tag{21}$$

We stress that $\int_0^y \Delta_j \,\mathrm{d}z$ does not converge to zero at infinity, so that one can not perform L^2 estimates on this quantity. However, we can notice by Lemma 3 that

$$\begin{aligned}
\|\left(\tfrac{\rho_{j+1}}{\rho_0}\right)^{1/2}\int_0^y \Delta_j \,\mathrm{d}z\|_{L^2} &\leq \|\left(\tfrac{\rho_1}{\rho_0}\right)^{1/2}\|_{L_y^2}\,\|\left(\tfrac{\rho_{j+1}}{\rho_1}\right)^{1/2}\int_0^y \Delta_j \,\mathrm{d}z\|_{L_x^2 L_y^\infty} \\
&\leq C_{m-1}\|\Delta_j\|_{j+1} \\
&< +\infty.
\end{aligned} \tag{22}$$

The square integrable quantity $\delta_j = \big(\frac{\rho_{j+1}}{\rho_0}\big)^{1/2}\int_0^y \Delta_j\,\mathrm{d}z$ satisfies the equation

$$\begin{aligned}
&\Big(\partial_t + \beta(j+1) + U^P\partial_x + (j+2)\partial_x U^P + V^P\partial_y - \partial_y^2\Big)\delta_j\\
&= -(j+1)\partial_{xx}U^P\big(\frac{\rho_{j+1}}{\rho_0}\big)^{1/2}\int_0^y H_j\,\mathrm{d}z - \partial_x V^P\big(\frac{\rho_{j+1}}{\rho_0}\big)^{1/2}H_j\\
&\quad + V^P\partial_y\Big(\big(\frac{\rho_{j+1}}{\rho_0}\big)^{1/2}\Big)\big(\frac{\rho_{j+1}}{\rho_0}\big)^{-1/2}\delta_j - 2\partial_y\Big(\big(\frac{\rho_{j+1}}{\rho_0}\big)^{1/2}\Big)\Delta_j - \partial_y^2\Big(\big(\frac{\rho_{j+1}}{\rho_0}\big)^{1/2}\Big)\big(\frac{\rho_{j+1}}{\rho_0}\big)^{-1/2}\delta_j.
\end{aligned} \tag{23}$$

As in (22), we obtain

$$\|(j+1)\partial_{xx}U^P\big(\frac{\rho_{j+1}}{\rho_0}\big)^{1/2}\int_0^y H_j\,\mathrm{d}z\|_{L^2} \le C_{m-1}(j+1)\|\partial_{xx}U^P\|_\infty\|H_j\|_{j+1} \tag{24}$$

We also get

$$\|\partial_x V^P\big(\frac{\rho_{j+1}}{\rho_0}\big)^{1/2}H_j\|_{L^2} \le \|\frac{1}{1+y}\partial_x V^P\|_\infty\|H_j\|_{j+1}.$$

By Lemma 2, we find

$$\|V^P\partial_y\Big(\big(\frac{\rho_{j+1}}{\rho_0}\big)^{1/2}\Big)\big(\frac{\rho_{j+1}}{\rho_0}\big)^{-1/2}\delta_j\|_{L^2} \le \|\frac{1}{1+y}V^P\|_\infty C_l(j+1)\|\delta_j\|_{L^2}.$$

Using again Lemma 2 and the identity

$$\big(\frac{\rho_{j+1}}{\rho_0}\big)^{1/2}\Delta_j = \partial_y\delta_j - \partial_y\Big(\big(\frac{\rho_{j+1}}{\rho_0}\big)^{1/2}\Big)\big(\frac{\rho_{j+1}}{\rho_0}\big)^{-1/2}\delta_j$$

and defining

$$A_{j,\alpha} = \max((j+1)^{1-\alpha}, \log(j+1), 1) \tag{25}$$

we obtain

$$\|2\partial_y\Big(\big(\frac{\rho_{j+1}}{\rho_0}\big)^{1/2}\Big)\Delta_j\|_{L^2} \le 2C_l A_{j,\alpha}\|\partial_y\delta_j\|_{L^2} + 2C_l^2A_{j,\alpha}^2\|\delta_j\|_{L^2}.$$

Eventually,

$$\begin{aligned}
&\|\partial_y^2\Big(\big(\frac{\rho_{j+1}}{\rho_0}\big)^{1/2}\Big)\big(\frac{\rho_{j+1}}{\rho_0}\big)^{-1/2}\delta_j\|_{L^2}\\
&= \|\partial_y\Big(\sum_{k=1}^{j+1}\frac{1}{k^\alpha(1+\frac{y}{k^\alpha})}\big(\frac{\rho_{j+1}}{\rho_0}\big)^{1/2}\Big)\big(\frac{\rho_{j+1}}{\rho_0}\big)^{-1/2}\delta_j\|_{L^2} \le CA_{j,\alpha}^2\|\delta_j\|_{L^2}
\end{aligned}$$

for some constant $C = C(\alpha)$. The previous bounds combined with an energy estimate yield that for $\mathcal{C}$ large enough (we remind that $\alpha \ge \frac{1}{2}$):

$$\|\delta_j(T)\|_{L^2}^2 + \beta(j+1)\int_0^T\|\delta_j\|_{L^2}^2\,\mathrm{d}t + \int_0^T\|\partial_y\delta_j\|_{L^2}^2\,\mathrm{d}t \le (j+1)\int_0^T\|H_j\|_{j+1}^2\,\mathrm{d}t. \tag{26}$$

We can then take the x-derivative of equation (23) and proceed as above. For $\mathcal{C}$ large enough, we get

$$\begin{aligned}
&\|\partial_x\delta_j(T)\|_{L^2}^2 + \beta(j+1)\int_0^T\|\partial_x\delta_j\|_{L^2}^2\,\mathrm{d}t + \int_0^T\|\partial_x\partial_y\delta_j\|_{L^2}^2\,\mathrm{d}t\\
&\le (j+1)\int_0^T(\|\partial_x H_j\|_{j+1}^2 + \|H_j\|_{j+1}^2)\,\mathrm{d}t\\
&+ \int_0^T\Big(2\|\partial_x V^P\partial_y\Big(\big(\frac{\rho_{j+1}}{\rho_0}\big)^{1/2}\Big)\big(\frac{\rho_{j+1}}{\rho_0}\big)^{-1/2}\delta_j\|_{L^2} + 2(j+2)\|\partial_x^2U^P\delta_j\|_{L^2} + 2\|\partial_x V^P\partial_y\delta_j\|_{L^2}\Big)\|\partial_x\delta_j\|_{L^2}\,\mathrm{d}t
\end{aligned}$$

We then use that

$$\|\partial_x V^P \partial_y \Big((\frac{\rho_{j+1}}{\rho_0})^{1/2} \Big) (\frac{\rho_{j+1}}{\rho_0})^{-1/2} \delta_j \|_{L^2} \leq \Big\| \frac{\partial_x V^P}{1+y} \Big\|_\infty C_l(j+1) \|\delta_j\|_{L^2},$$

and

$$\|\partial_x V^P \partial_y \delta_j\|_{L^2} \leq \Big\| \frac{\partial_x V^P}{1+y} \Big\|_\infty C_l(j+1) \|\delta_j\|_{L^2} + \Big\| \frac{\partial_x V^P}{1+y} \Big\|_\infty \|\Delta_j\|_{j+1}$$

and the bound (26) to end up with

$$\begin{aligned} &\|\partial_x \delta_j(T)\|_{L^2}^2 + \frac{\beta(j+1)}{2} \int_0^T \|\partial_x \delta_j\|_{L^2}^2 \, \mathrm{d}t + \int_0^T \|\partial_x \partial_y \delta_j\|_{L^2}^2 \, \mathrm{d}t \\ &\leq 2(j+1) \int_0^T (\|\partial_x H_j\|_{j+1}^2 + \|H_j\|_{j+1}^2 + \|\Delta_j\|_{j+1}^2) \, \mathrm{d}t. \end{aligned} \tag{27}$$

To estimate directly Δ_j, we differentiate the equation (21) with respect to y, which gives

$$\begin{aligned} &\Big(\partial_t + \beta(j+1) + U^P \partial_x + (j+2) \partial_x U^p + V^P \partial_y + \partial_y V^P - \partial_y^2 \Big) \Delta_j \\ &= -(j+1) \partial_{xx} U^P H_j - (j+1) \partial_{xxy} U^P \int_0^y H_j - \partial_{xy} V^P H_j - \partial_x V^P \partial_y H_j \\ &\quad - \partial_y U^P \partial_x \int_0^y \Delta_j - (j+2) \partial_{xy} U^P \int_0^y \Delta_j. \end{aligned}$$

We take the $\langle \, , \rangle_{j+1}$ scalar product with Δ_j:

$$\begin{aligned} &\Big(\frac{1}{2} \partial_t + \beta(j+1) \Big) \|\Delta_j\|_{j+1}^2 - \Big[(j+2) \, \|\partial_x U^P\|_\infty + \frac{1}{2} \|\partial_y V^P\|_\infty + \frac{1}{2} \|V^P \frac{\partial_y \rho_{j+1}}{\rho_{j+1}}\|_\infty \Big] \|\Delta_j\|_{j+1}^2 \\ &\quad + \|\partial_y \Delta_j\|_{j+1}^2 - \langle \partial_y \Delta_j, \frac{\partial_y \rho_{j+1}}{\rho_{j+1}} \Delta_j \rangle_{j+1} \\ &\leq (j+1) \big(\|\partial_{xx} U^P\|_\infty + C_m \|\partial_{xxy} U^P\|_{L^\infty_x L^2_y(\rho_0)} + \|\partial_{xy} V^P\|_\infty \big) \|H_j\|_{j+1} \, \|\Delta_j\|_{j+1} \\ &\quad + \Big\| \frac{\partial_x V^P}{1+y} \Big\|_\infty \|(1+y) \partial_y H_j\|_{j+1} \, \|\Delta_j\|_{j+1} \\ &\quad + \big(\|\partial_y U^P \sqrt{\rho_0}\|_\infty \|\partial_x \delta_j\|_{L^2} + (j+2) \|\partial_{xy} U^P \sqrt{\rho_0}\|_\infty \|\delta_j\|_{L^2} \big) \, \|\Delta_j\|_{j+1}. \end{aligned}$$

By the 1d Sobolev imbedding theorem, we find that for a constant $C = C(m)$ it holds that

$$\|\partial_y U^P \sqrt{\rho_0}\|_\infty \leq C \|(U^P, V^P)\|_{low} \text{ and } \|\partial_{xy} U^P \sqrt{\rho_0}\|_\infty \leq C \|(U^P, V^P)\|_{low}.$$

Combining these last two inequalities with (26), (27) and the inequality

$$\|(1+y) \partial_y H_j\|_{j+1} \leq (j+1)^\alpha \|\partial_y H_j\|_j,$$

and taking $\mathcal{C}$ large enough, we obtain

$$\begin{aligned} &\|\Delta_j(T)\|_{j+1}^2 + \beta(j+1) \int_0^T \|\Delta_j\|_{j+1}^2 \, \mathrm{d}t + \int_0^T \|\partial_y \Delta_j\|_{j+1}^2 \, \mathrm{d}t \\ &\leq (j+1) \int_0^T \|H_j\|_{j+1}^2 \, \mathrm{d}t + (j+1)^{2\alpha - 1} \int_0^T \|\partial_y H_j\|_j^2 \, \mathrm{d}t + \frac{1}{\beta(j+1)} \int_0^T \|\partial_x H_j\|_{j+1}^2 \, \mathrm{d}t. \end{aligned} \tag{28}$$

Lemma 9 follows straightforwardly. □

Combining Lemmas 8 and 9, we will now prove Lemma 6.

Proof of Lemma 6. We choose $\mathcal{C}$ such that Lemmas 8 and 9 apply. We multiply the inequality in Lemma 8 by $\beta(j+1)^{2\gamma-1}$ and sum over j to get

$$\begin{aligned}
&\sum_{j=0}^{\infty} \beta^2(j+1)^{2\gamma}\left[\int_0^T \|H_j(t)\|_j^2\,\mathrm{d}t + \frac{1}{\beta(j+1)}\|H_j(T)\|_j^2 + \frac{1}{\beta(j+1)}\int_0^T \|\partial_y H_j(t)\|_j^2\,\mathrm{d}t\right]\\
&\le 8\sum_{j=0}^{\infty}\left[\frac{(j+1)^{2\gamma-4}}{\beta^2}\int_0^T \|F_j(t)\|_j^2\,\mathrm{d}t + \frac{(j+1)^{2\gamma-3}}{\beta}\|u_{\mathrm{in},j}\|_j^2\right]\\
&\quad + \sum_{j=0}^{\infty}\frac{4c_1^2(1+\|(U^P,V^P)\|_{low})^4}{\beta^2}(j+1)^{2\gamma-4}\int_0^T \|\partial_x H_j\|_{j+1}^2\,\mathrm{d}t.
\end{aligned}$$

Taking $\mathcal{C}$ large enough, we can then find by Lemma 9 a constant $C = C(m,\alpha,r)$ such that

$$\begin{aligned}
&\sum_{j=0}^{\infty}(j+1)^{2\gamma-4}\int_0^T \|\partial_x H_j\|_{j+1}^2\,\mathrm{d}t\\
&\le C\left(1+\frac{1}{\tau^2}\right)\sum_{j=0}^{\infty}(j+1)^{4\gamma-4}\int_0^T \|H_j\|_j^2\,\mathrm{d}t + \frac{C}{\beta}\sum_{j=0}^{\infty}(j+1)^{2(\gamma+\alpha)-6}\int_0^T \|\partial_y H_j\|_j^2\,\mathrm{d}t\\
&\le C\left(1+\frac{1}{\tau^2}\right)\sum_{j=0}^{\infty}(j+1)^{4\gamma-4}\left[\int_0^T \|H_j\|_j^2\,\mathrm{d}t + \frac{1}{\beta(j+1)}\int_0^T \|\partial_y H_j\|_j^2\,\mathrm{d}t\right].
\end{aligned}$$

We have used here that $\alpha \le \gamma + \frac{1}{2}$. Hence, the last term at the right-hand side can be absorbed if

$$\frac{1}{2}\beta^2(j+1)^{2\gamma} \ge \frac{4Cc_1^2(1+\|(U^P,V^P)\|_{low})^4}{\beta^2}\left(1+\frac{1}{\tau^2}\right)(j+1)^{4\gamma-4},$$

which can be ensured by a suitable large $\mathcal{C}$ if $\gamma \le 2$. □

4.3 Control of u_j and ω_j

We now relate the estimates on H_j to u_j and start with an estimate for the L^2 norm.

Lemma 10. *Let $m > \frac{1}{2}$ and $\alpha \ge \frac{1}{2}$. Then there exists a constant $\mathcal{C} = \mathcal{C}(m,\alpha)$ such that for*

$$\beta \ge \mathcal{C}\left(1+\|(U^P,V^P)\|_{low}\right)$$

and for any $\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4 > 0$ it holds that

$$\begin{aligned}
&\frac{1}{2}\int_0^T \|u_j\|_j^2\,\mathrm{d}t - \frac{\epsilon_1}{(j+1)^{2\gamma}}\int_0^T \|\partial_x u_j\|_{j+1}^2\,\mathrm{d}t - \frac{\epsilon_2}{\beta(j+1)^{\gamma}}\int_0^T \|\partial_y u_j\|_j^2\,\mathrm{d}t\\
&\quad - \frac{\epsilon_3}{4\beta(j+1)^{\gamma}}\|u_j(T)\|_j^2 - \frac{\epsilon_4}{\beta^2(j+1)^{2\gamma}}\int_0^T \|\partial_y^2 u_j(t)\|_j^2\,\mathrm{d}t\\
&\le \frac{\beta(j+1)^{\gamma}}{\epsilon_3}\|H_j(T)\|_j^2\\
&\quad + \left[16\beta^2(j+1)^2 + \frac{(j+1)^{2\gamma}}{\epsilon_1}C_m^2\|(1+y)\partial_y U^P\|_{L^\infty_{t,x}L^2_y(\rho_0)}^2 + \frac{\beta^2(j+1)^{2\gamma}}{4\epsilon_4}\right]\int_0^T \|H_j\|_j^2\,\mathrm{d}t\\
&\quad + \left[\frac{\beta(j+1)^{\gamma}}{4\epsilon_2} + 16C_l^2A_{j,\alpha}^2\right]\int_0^T \|\partial_y H_j\|_j^2\,\mathrm{d}t + \int_0^T \|H_j\|_j\|F_j\|_j\,\mathrm{d}t
\end{aligned}$$

where u_j is satisfying (16), $A_{j,\alpha}$ is defined in (25) and H_j is defined by (17).

Proof. Using the definition (17) of H_j we find

$$\int_0^T \|u_j(t)\|_j^2 \,\mathrm{d}t = \int_0^T \langle \Big(\partial_t + \beta(j{+}1) + U^P\partial_x + (j{+}1)\partial_x U^p + V^P\partial_y + \partial_y V^P - \partial_y^2\Big) H_j, u_j\rangle_j \,\mathrm{d}t + \int_0^T \langle (\partial_y U^P \partial_x + (j{+}1)\partial_{xy}U^P) \int_0^y H_j \,\mathrm{d}z, u_j\rangle_j \,\mathrm{d}t. \tag{29}$$

By the evolution equation (16) for u_j, the first term can be written (from the partial integration in y there is no boundary term as $u|_{y=0} = 0$)

$$\begin{aligned}
&\int_0^T \langle \Big(\partial_t + \beta(j{+}1) + U^P\partial_x + (j{+}1)\partial_x U^p + V^P\partial_y + \partial_y V^P - \partial_y^2\Big) H_j, u_j\rangle_j \,\mathrm{d}t \\
&= \langle H_j(T), u_j(T)\rangle_j + \int_0^T \langle H_j, (-\partial_t + \beta(j{+}1) - U^P\partial_x + j\partial_x U^P - V^P\partial_y - V^P\frac{\partial_y\rho_j}{\rho_j})u_j\rangle \\
&\quad + \int_0^T \langle \partial_y H_j, (\partial_y + \frac{\partial_y\rho_j}{\rho_j})u_j\rangle_j \\
&= \langle H_j(T), u_j(T)\rangle_j + \int_0^T \langle H_j, (2\beta(j{+}1) + (2j{+}1)\partial_x U^P - V^P\frac{\partial_y\rho_j}{\rho_j})u_j\rangle_j \,\mathrm{d}t \\
&\quad + \int_0^T \langle H_j, \partial_y U^P v_j + j\partial_{xy}U^P\partial_x^{-1}v_j\rangle_j \,\mathrm{d}t - \int_0^T \langle H_j, F_j\rangle_j \,\mathrm{d}t \\
&\quad + \int_0^T \langle \partial_y H_j, (\partial_y + \frac{\partial_y\rho_j}{\rho_j})u_j\rangle_j \,\mathrm{d}t - \int_0^T \langle H_j, \partial_y^2 u_j\rangle_j \,\mathrm{d}t.
\end{aligned}$$

The terms can now be bounded using Lemma 2:

$$\begin{aligned}
&\langle H_j, (2\beta(j{+}1) + (2j{+}1)\partial_x U^P - V^P\frac{\partial_y\rho_j}{\rho_j})u_j\rangle_j \\
&\le \left\| 2\beta(j{+}1) + (2j{+}1)\partial_x U^P - V^P\frac{\partial_y\rho_j}{\rho_j}\right\|_\infty \|H_j\|_j\|u_j\|_j \\
&\le (j{+}1)\left[2\beta + 2\|\partial_x U^P\|_\infty + C_l\left\|\frac{V^P}{1+y}\right\|_\infty\right]\|H_j\|_j\|u_j\|_j.
\end{aligned}$$

Recalling that $v_j = -\partial_x\int_0^y u_j\,\mathrm{d}z$ we find

$$\begin{aligned}
&\langle H_j, \partial_y U^P v_j + j\partial_{xy}U^P\partial_x^{-1}v_j\rangle_j \,\mathrm{d}t \\
&\le C_m\|(1{+}y)\partial_y U^P\|_{L^\infty_x L^2_y(\rho_0)}\|H_j\|_j\|\partial_x u_j\|_{j+1} + j\|\partial_{xy}U^P\|_{L^\infty_x L^2_y(\rho_0)}\|H_j\|_j\|u_j\|_j.
\end{aligned}$$

For the forcing terms we find

$$-\langle H_j, F_j\rangle_j \le \|H_j\|_j\|F_j\|_j.$$

The diffusion terms give

$$\begin{aligned}
&\langle \partial_y H_j, (\partial_y + \frac{\partial_y\rho_j}{\rho_j})u_j\rangle_j - \langle H_j, \partial_y^2 u_j\rangle_j \\
&\le \|\partial_y H_j\|_j\,\|\partial_y u_j\|_j + C_l A_{j,\alpha}\|\partial_y H_j\|_j\,\|u_j\|_j + \|H_j\|_j\|\partial_y^2 u_j\|_j.
\end{aligned}$$

The integrand in the second integral in (29) can be estimated as

$$\begin{aligned}
&\langle (\partial_y U^P\partial_x + (j{+}1)\partial_{xy}U^P)\int_0^y H_j\,\mathrm{d}z, u_j\rangle_j \\
&\le C_m\|(1{+}y)\partial_y U^P\|_{L^\infty_x L^2_y(\rho_0)}\|H_j\|_j\|\partial_x u_j\|_{j+1} + jC_m\|\partial_{xy}U^P\|_{L^\infty_x L^2_y(\rho_0)}\|H\|_j\|u\|_j.
\end{aligned}$$

Collecting the terms we find by choosing $\mathcal{C}$ large enough that

$$\begin{aligned}\int_0^T \|u_j\|^2 \,\mathrm{d}t \le \langle H_j(T), u_j(T)\rangle_j + 4\beta(j{+}1)\int_0^T \|H_j\|_j\|u\|_j \,\mathrm{d}t \\ + 2C_m\|(1{+}y)\partial_y U^P\|_{L^\infty_x L^2_y(\rho_0)}\int_0^T \|H_j\|_j\|\partial_x u_j\|_{j+1}\,\mathrm{d}t \\ + \int_0^T \Big(\|\partial_y H_j\|_j\,\|\partial_y u_j\|_j + C_l A_{j,\alpha}\|\partial_y H_j\|_j\,\|u_j\|_j + \|H_j\|_j\|\partial_y^2 u_j\|_j\Big)\,\mathrm{d}t \\ + \int_0^T \|H_j\|_j\|F_j\|_j\,\mathrm{d}t.\end{aligned}$$

Splitting the products gives the claimed estimate. □

The missing terms can be estimated by the evolution of u_j and $\omega_j = \partial_y u_j$. For u_j we find:

Lemma 11. *Let $m > \frac{1}{2}$ and $\alpha \ge \frac{1}{2}$. Then there exists a constant $\mathcal{C} = \mathcal{C}(m,\alpha)$ such that for*

$$\beta \ge \mathcal{C}\left(1 + \|(U^P, V^P)\|_{low}\right)$$

the solution u_j of (16) satisfies

$$\begin{aligned}&\frac{1}{2}\|u_j(T)\|_j^2 + \frac{1}{2}\int_0^T \|\partial_y u_j\|^2\,\mathrm{d}t \\ &\quad - 4\beta(j{+}1)^\gamma\int_0^T \|u_j\|_j^2\,\mathrm{d}t - \frac{C_m^2\|(1{+}y)\partial_y U^P\|^2_{L^\infty_x L^2_y(\rho_0)}}{\beta(j{+}1)^\gamma}\int_0^T \|\partial_x u_j\|_{j+1}^2\,\mathrm{d}t \\ &\le \frac{1}{2}\|u_{\mathrm{in},j}\|_j^2 + \frac{1}{\beta(j{+}1)^\gamma}\int_0^T \|F_j\|_j^2\,\mathrm{d}t.\end{aligned}$$

Proof. By (16) we find

$$\begin{aligned}\langle \partial_t u_j, u_j\rangle_j &= \langle\Big(-\beta(j{+}1) - U^P\partial_x - (j{+}1)\partial_x U^p - V^P\partial_y + \partial_y^2\Big)u_j, u_j\rangle_j \\ &\quad - \langle \partial_y U^P v_j + j\partial_{xy}U^P\partial_x^{-1}v_j, u_j\rangle_j + \langle F_j, u_j\rangle_j \\ &\le -\frac{1}{2}\|\partial_y u_j\|_j^2 + 4\beta(j{+}1)^\gamma\|u_j\|_j^2 + \frac{C_m^2\|(1{+}y)\partial_y U^P\|^2_{L^\infty_x L^2_y(\rho_0)}}{\beta(j{+}1)^\gamma}\|\partial_x u_{j+1}\|_{j+1}^2 \\ &\quad + \frac{1}{\beta(j{+}1)^\gamma}\|F_j\|_j^2,\end{aligned}$$

where there is no boundary term from the partial integration in y as u_j vanishes at the boundary and we used in the inequality that $\mathcal{C}$ can be chosen large enough. Integrating this over $[0,T]$ gives the claimed result. □

By differentiating (16) in y and find

$$\Big(\partial_t + \beta(j{+}1) + U^P\partial_x + (j{+}1)\partial_x U^p + V^P\partial_y + \partial_y V^P - \partial_y^2\Big)\omega_j + \partial_{yy}U^P v_j + j\partial_{xyy}U^P\partial_x^{-1}v_j + \partial_{xy}U^P u_j = \partial_y F_j. \tag{30}$$

This immediately yields the following control for ω_j.

Lemma 12. *Let $m > \frac{1}{2}$ and $\alpha \geq \frac{1}{2}$. Then there exists a constant $\mathcal{C} = \mathcal{C}(m, \alpha)$ such that for*

$$\beta \geq \mathcal{C}\left(1 + \|(U^P, V^P)\|_{low}\right)$$

the vorticity $\omega_j = \partial_y u_j$ satisfies

$$\begin{aligned}
&\|(1+y)\omega_j(T)\|_j^2 + \int_0^T \beta(j+1)\|(1+y)\omega_j\|_j^2 \,\mathrm{d}t + \int_0^T \|(1+y)\partial_y\omega_j\|_j^2 \,\mathrm{d}t \\
&\leq \frac{4C_m^2\|(1+y)^2\partial_{yy}U^P\|^2_{L^\infty_{t,x}L^2_y(\rho_0)}}{\beta(j+1)} \int_0^T \|\partial_x u_j\|_{j+1}^2 \,\mathrm{d}t \\
&+ \frac{4C_m^2(j+1)\|(1+y)\partial_{xyy}U^P\|^2_{L^\infty_{t,x}L^2_y(\rho_0)}}{\beta} \int_0^T \|u_j\|_j^2 \,\mathrm{d}t \\
&+ \|(1+y)\omega_{\mathrm{in},j}\|_j^2 + 4\int_0^T \|(1+y)F_j\|_j^2 \,\mathrm{d}t + 4\int_0^T \|F_j|_{y=0}\|^2_{L^2_x} \,\mathrm{d}t.
\end{aligned}$$

Proof. Integrate (30) against $(1+y)^2\omega_j$ in $\|\cdot\|_j$. This yields

$$\begin{aligned}
&\frac{1}{2}\partial_t\|(1+y)\omega_j\|_j^2 + \beta(j+1)\|(1+y)\omega_j\|_j^2 + \|(1+y)\partial_y\omega_j\|_j^2 \\
&\leq j\|\partial_x U^P\|_\infty\|(1+y)\omega_j\|_j^2 + \left\|V^P\left(\frac{\partial_y\rho_j}{\rho_j} + \frac{\partial_y(1+y)^2}{(1+y)^2}\right)\right\|_\infty \|(1+y)\omega_j\|_j^2 \\
&\quad + \|\omega_j|_{y=0}\|_{L^2_x}\|\partial_y\omega_j|_{y=0}\|_{L^2_x} \\
&\quad + \|(1+y)\partial_y\omega_j\|_j \left\|(1+y)\left(\frac{\partial_y\rho_j}{\rho_j} + \frac{\partial_y(1+y)^2}{(1+y)^2}\right)\omega_j\right\|_j \\
&\quad + C_m\|(1+y)^2\partial_{yy}U^P\|_{L^\infty_x L^2_y(\rho_0)}\|\partial_x u_j\|_{j+1}\|(1+y)\omega_j\|_j \\
&\quad + jC_m\|(1+y)\partial_{xyy}U^P\|_{L^\infty_x L^2_y(\rho_0)}\|u_j\|_j\|(1+y)\omega_j\|_j \\
&\quad + \|(1+y)F_j\|_j \left\|(1+y)\left(\partial_y + \frac{\partial_y\rho_j}{\rho_j} + \frac{\partial_y(1+y)^2}{(1+y)^2}\right)\omega_j\right\|_j,
\end{aligned}$$

where we find a boundary term from the diffusion and there is no boundary term from $V^P\partial_y$ because $V^P|_{y=0} = 0$.

From (16) we find $\partial_y\omega_j|_{y=0} = F_j|_{y=0}$. For $\omega_j|_{y=0}$ write

$$|\omega_j(y=0)| \leq \int_0^1 \left[\omega_j\sqrt{\rho_j} + \int_0^y |(\omega_j\sqrt{\rho_j})'| \,\mathrm{d}z\right] \mathrm{d}y$$

to get

$$\|\omega_j|_{y=0}\|^2_{L^2_x} \leq 2\left(1 + \left\|\frac{\partial_y\rho_j}{\rho_j}\right\|\right)^2 \|\omega_j\|_j^2 + 2\|\partial_y\omega_j\|_j^2.$$

By choosing $\mathcal{C}$ large enough and using that $\alpha \geq \frac{1}{2}$, the result follows after integration over time. ∎

Combining the results, we can conclude this section.

Proof of Proposition 7. Adding the control of Lemma 11 with a factor $\epsilon_3(j+1)^{-\gamma}/\beta$ and Lemma 12 with a factor $(j+1)^{1-2\gamma}/\beta^2$ to the inequality of Lemma 10 yields

$$
\begin{aligned}
&\left(\frac{1}{2} - 4\epsilon_3 - \frac{4C_m^2\|(1+y)\partial_{xyy}U^P\|^2_{L^\infty_{t,x}L^2_y(\rho_0)}}{(j+1)^{2\gamma-2}\beta^3}\right)\int_0^T \|u_j\|_j^2\,\mathrm{d}t \\
&\quad + \frac{\epsilon_3}{4\beta(j+1)^\gamma}\|u_j(T)\|_j^2 + \left(\frac{\epsilon_3}{2\beta(j+1)^\gamma} - \frac{\epsilon_2}{\beta(j+1)^\gamma}\right)\int_0^T \|\partial_y u_j\|_j^2\,\mathrm{d}t - \frac{\epsilon_4}{\beta^2(j+1)^{2\gamma}}\int_0^T \|\partial_y^2 u_j(t)\|_j^2\,\mathrm{d}t \\
&\quad - \left(\frac{\epsilon_1}{(j+1)^{2\gamma}} + \frac{\epsilon_3 C_m\|(1+y)\partial_y U^P\|^2_{L^\infty_{t,x}L^2_y(\rho_0)}}{\beta^2(j+1)^{2\gamma}} + \frac{4C_m^2\|(1+y)\partial_{xyy}U^P\|^2_{L^\infty_{t,x}L^2_y(\rho_0)}}{(j+1)^{2\gamma}\beta^3}\right)\int_0^T \|\partial_x u_j\|_{j+1}^2\,\mathrm{d}t \\
&\quad \frac{(j+1)^{1-2\gamma}}{\beta^2}\|(1+y)\omega_j(T)\|_j^2 + \int_0^T \frac{(j+1)^{2-2\gamma}}{\beta}\|(1+y)\omega_j\|_j^2\,\mathrm{d}t + \int_0^T \frac{(j+1)^{1-2\gamma}}{\beta^2}\|(1+y)\partial_y\omega_j\|_j^2\,\mathrm{d}t \\
&\le \frac{\beta(j+1)^\gamma}{\epsilon_3}\|H_j(T)\|_j^2 \\
&\quad + \left[16\beta^2(j+1)^2 + \frac{(j+1)^{2\gamma}}{\epsilon_1}C_m^2\|(1+y)\partial_y U^P\|^2_{L^\infty_t L^\infty_x L^2_y(\rho_0)} + \frac{\beta^2(j+1)^{2\gamma}}{4\epsilon_4}\right]\int_0^T \|H_j\|_j^2\,\mathrm{d}t \\
&\quad + \left[\frac{\beta(j+1)^\gamma}{4\epsilon_2} + 16C_l^2A_{j,\alpha}^2\right]\int_0^T \|\partial_y H_j\|_j^2\,\mathrm{d}t + \int_0^T \|H_j\|_j\|F_j\|_j\,\mathrm{d}t \\
&\quad + \frac{\epsilon_3}{2\beta(j+1)^\gamma}\|u_{\mathrm{in},j}\|_j^2 + \frac{\epsilon_3}{\beta^2(j+1)^{2\gamma}}\int_0^T \|F_j\|_j^2\,\mathrm{d}t \\
&\quad + \frac{(j+1)^{1-2\gamma}}{\beta^2}\|\omega_{\mathrm{in},j}\|_j^2 + \frac{4(j+1)^{1-2\gamma}}{\beta^2}\int_0^T \|(1+y)F_j\|_j^2\,\mathrm{d}t + \frac{4(j+1)^{1-2\gamma}}{\beta^2}\int_0^T \|F_j|_{y=0}\|_{L^2_x}^2\,\mathrm{d}t.
\end{aligned}
$$

Using that $\partial_x u_j(t) = \left(\frac{j+2}{j+1}\right)^r \frac{(j+1)^\gamma}{\tau(t)}u_{j+1}(t)$, we can sum over j and choose $\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4$ appropriately to arrive for $m > \frac{1}{2}, \alpha \ge \frac{1}{2}, \gamma \ge 1, \tau_1 > 0, r \in \mathbb{R}$ at the control

$$
\begin{aligned}
&\sum_{j=0}^\infty \left\{\int_0^T \|u_j\|_j^2\,\mathrm{d}t + \frac{1}{\beta(j+1)^\gamma}\|u_j(T)\|_j^2 + \frac{(j+1)^{2-2\gamma}}{\beta}\int_0^T \|(1+y)\omega_j\|_j^2\,\mathrm{d}t\right\} \\
&\quad + \sum_{j=0}^\infty \frac{(j+1)^{1-2\gamma}}{\beta^2}\left\{\|(1+y)\omega_j(T)\|_j^2 + \int_0^T \|(1+y)\partial_y\omega_j(t)\|_j^2\,\mathrm{d}t\right\} \\
&\le C\sum_{j=0}^\infty \left\{\beta(j+1)^\gamma\|H_j(T)\|_j^2 + \beta^2(j+1)^{2\gamma}\int_0^T \|H_j\|_j^2\,\mathrm{d}t + \beta(j+1)^\gamma\int_0^T \|\partial_y H_j\|_j^2\,\mathrm{d}t\right\} \\
&\quad + C\sum_{j=0}^\infty \left\{\frac{1}{\beta(j+1)^\gamma}\|u_{\mathrm{in},j}\|_j^2 + \frac{(j+1)^{1-2\gamma}}{\beta}\|(1+y)\omega_{\mathrm{in},j}\|_j^2\right\} \\
&\quad + C\sum_{j=0}^\infty \left\{\int_0^T \frac{1}{\beta^2(j+1)^{2\gamma}}\|F_j\|_j^2\,\mathrm{d}t + \frac{(j+1)^{1-2\gamma}}{\beta^2}\int_0^T \|(1+y)F_j\|_j^2\,\mathrm{d}t + \frac{(j+1)^{1-2\gamma}}{\beta^2}\int_0^T \|F_j|_{y=0}\|_{L^2_x}^2\,\mathrm{d}t.\right\}
\end{aligned}
\tag{31}
$$

if

$$
\beta \ge \mathcal{C}(1 + \|(U^P, V^P)\|_{low})\,(1 + \frac{1}{\tau_1} + \|(U^P, V^P)\|_{low}) \text{ and } \tau(T) \ge \tau_1
$$

where C and $\mathcal{C}$ are constant only depending on m, α, γ, r (and not τ_1).

Controlling H by Lemma 6 then yields the result for a fixed time T. Applying this estimate for all T in $[0, T^*]$ then shows the claimed estimate.

For $\gamma \geq 5/4$ we find that $(j+1)^{2\gamma-4} \geq (j+1)^{1-2\gamma}$ so that

$$(j+1)^{2\gamma-4}\|F_j\|_j^2 + (j+1)^{1-2\gamma}\|(1+y)F_j\|_j^2 \leq 2(j+1)^{2\gamma-4}\|(1+\frac{y}{(j+1)^{2\gamma-\frac{5}{2}}})F_j\|_j^2,$$

which proves the expression in this case. □

5 Nonlinear estimates

In order to close the estimate, we have to estimate F_j.

Proposition 13. *Fix the parameters m, α, γ, r and an additional parameter R such that*

$$\begin{aligned} &\gamma \in [\frac{3}{2}, 2], \quad \alpha \leq \gamma - 1, \quad m \geq \frac{2\gamma-1}{\alpha} + 1, \\ &r > 2\gamma, \quad R > 2\gamma + 1, \quad R \geq r + 3\gamma - 2. \end{aligned} \tag{32}$$

Then there exists a constant $C = C(m, \alpha, \gamma, r)$ such that for β, τ_1 and T with $\tau(T) \geq \tau_1$,

$$\begin{aligned} &\sum_{j=0}^{\infty} \frac{1}{(j+1)^{4-2\gamma}} \int_0^T \left\| \left(1 + \frac{y}{(j+1)^{2\gamma-\frac{5}{2}}}\right) F_j \right\|_j^2 \mathrm{d}t \\ &\leq 2\int_0^T \|(1+y)f_j^e\|^2_{\gamma,\tau,r-2+\gamma} \\ &+ \frac{C\beta}{\tau_1^4}\left[\sup_{[0,T]}\left(\|u\|^2_{\gamma,\tau,r-\frac{\gamma}{2}} + \frac{\|(1+y)\omega\|^2_{\gamma,\tau,r+\frac{1}{2}-\gamma}}{\beta} + |U^E|^2_{\gamma,\tau,R}\right)\right]\int_0^T \left[\|u\|^2_{\gamma,\tau,r} + \frac{\|(1+y)\omega\|^2_{\gamma,\tau,r+1-\gamma}}{\beta}\right]\mathrm{d}t. \end{aligned}$$

We restrict to the case of $\gamma \geq 3/2$ because we need $\alpha \geq \gamma - 1$ in order to control the terms $\partial_x^k u \partial_x^{l-k+1} u$ in F_j. Combined with the earlier requirement that $\alpha \geq 1/2$ this yields $\gamma \geq 3/2$.

Proof. Write $F_j = f_j^e + \sum_{i=1}^6 F_j^i$ with

$$\begin{aligned} F_j^1 &= M_j\left[u\partial_x, \partial_x^j\right]u + M_j(j+1)\,\partial_x u\,\partial_x^j u, \\ F_j^2 &= M_j\left[\partial_y u, \partial_x^j\right]v + M_j j\,\partial_{xy}u\,\partial_x^{j-1}v + M_j v\,\partial_x^j\partial_y u, \\ F_j^3 &= M_j\left[U^e\partial_x, \partial_x^j\right]u + M_j j\,\partial_x U^e\,\partial_x^j u, \\ F_j^4 &= M_j\left[V^e\partial_y, \partial_x^j\right]u, \\ F_j^5 &= M_j\left[\partial_x U^e, \partial_x^j\right]u, \\ F_j^6 &= M_j\left[\partial_y U^e, \partial_x^j\right]v + M_j j\,\partial_{xy}U^e\,\partial_x^{j-1}v. \end{aligned}$$

As $\gamma \geq 3/2$ and $\alpha \leq \gamma - 1$, we have $2\gamma - \frac{5}{2} \geq \alpha$, so that

$$\left\|\left(1 + \frac{y}{(j+1)^{2\gamma-\frac{5}{2}}}\right)F_j\right\|_j \leq \left\|\left(1 + \frac{y}{(j+1)^{\alpha}}\right)F_j\right\|_j$$

so that it suffices to bound the right-hand side.

Analysis of F_j^1. We write

$$F_j^1 = \sum_{l=2}^{\lfloor\frac{j+1}{2}\rfloor}\binom{j}{l}\frac{M_j}{M_l M_{j-l+1}}u_l u_{j-l+1} + \sum_{l=\lfloor\frac{j+1}{2}\rfloor+1}^{j-1}\binom{j}{l}\frac{M_j}{M_l M_{j-l+1}}u_l u_{j-l+1} =: F_{j,low}^1 + F_{j,high}^1.$$

For $F^1_{j,low}$, we notice that for $l \le \left\lfloor \frac{j+1}{2} \right\rfloor$ there exist a constant $C = C(r)$ with

$$\binom{j}{l} \frac{M_j}{M_l M_{j-l+1}} \le \frac{C}{\tau_1} \binom{j}{l}^{1-\gamma} \frac{(j+1)^\gamma}{(l+1)^r}.$$

This shows

$$\begin{aligned}
\frac{1}{(j+1)^{2-\gamma}} \left\| \left(1 + \frac{y}{(j+1)^\alpha}\right) F^1_{j,low} \right\|_j &\le \frac{C}{\tau_1} \sum_{l=2}^{\lfloor \frac{j+1}{2} \rfloor} \binom{j}{l}^{1-\gamma} \frac{(j+1)^{2\gamma-2}}{(l+1)^r} \|u_l u_{j-l+1}\|_{j-1} \\
&\le \frac{C}{\tau_1} \sum_{l=2}^{\lfloor \frac{j+1}{2} \rfloor} \binom{j}{l}^{1-\gamma} \frac{(j+1)^{2\gamma-2}}{(l+1)^r} \left\| \left(\frac{\rho_{j-1}}{\rho_{j-l+1}}\right)^{1/2} u_l \right\|_{L^\infty_{x,y}} \|u_{j-l+1}\|_{j-l+1}.
\end{aligned}$$

Note that for an absolute constant C_a,

$$\binom{j}{l}^{1-\gamma} (j+1)^{2\gamma-2} \le C_a \quad \text{for all } 2 \le l \le \left\lfloor \frac{j+1}{2} \right\rfloor. \tag{33}$$

From the 1d Sobolev embedding and Lemma 3, we find that for $n \le \min(m-1, l)$:

$$\begin{aligned}
\left\| \left(\frac{\rho_{j-1}}{\rho_{j-l+1}}\right)^{1/2} u_l \right\|_{L^\infty_{x,y}} &\le C_A \left\| \left(\frac{\rho_{j-1}}{\rho_{j-l+1}}\right)^{1/2} \partial_x u_l \right\|_{L^2_x L^\infty_y} \\
&\le C_A\, C_1 \sup_y \left(\frac{\rho_{j-1}\rho_n}{\rho_l \rho_{j-l+1}}\right)^{1/2} \|\partial_x \partial_y u_l\|_l \\
&\le \frac{C}{\tau_1} \sup_y \left(\frac{\rho_{j-1}\rho_n}{\rho_l \rho_{j-l+1}}\right)^{1/2} (l+1)^\gamma \, \|(1+y)\omega_{l+1}\|_{l+1}
\end{aligned}$$

where C_A is an absolute constant, C is a constant depending on m, r. Note that we used here Lemma 4 to bound $\|\omega_{l+1}\|_l$ by $\|(1+y)\omega_{l+1}\|_{l+1}$.The factor with the ρ is explicit:

$$\left(\frac{\rho_{j-1}\rho_n}{\rho_l \rho_{j-l+1}}\right)^{1/2} = \frac{\prod_{k=1}^{l}(1+\frac{y}{k^\alpha})}{\prod_{k=j-l+2}^{j-1}(1+\frac{y}{k^\alpha}) \prod_{k=1}^{n}(1+\frac{y}{k^\alpha})}.$$

For $l \le m-1$, we take $n = l$ and find that

$$\left(\frac{\rho_{j-1}\rho_n}{\rho_l \rho_{j-l+1}}\right)^{1/2} \le 1.$$

For $l > m-1$, we take $n = m-1$ and find that

$$\begin{aligned}
\left(\frac{\rho_{j-1}\rho_n}{\rho_l \rho_{j-l+1}}\right)^{1/2} &\le \frac{\prod_{k=1}^{l}(1+\frac{y}{k^\alpha})}{\prod_{k=j-l+2}^{j-m+2}(1+\frac{y}{k^\alpha}) \prod_{k=1}^{m-1}(1+\frac{y}{k^\alpha})} \\
&\le \left(\frac{(j-l+2)\cdots(j-m+2)}{m \cdots l}\right)^\alpha \\
&\le C \binom{j}{l}^\alpha (j+1)^{-\alpha(m-1)}
\end{aligned}$$

for a constant $C = C(m, \alpha)$ and using that $l \le \left\lfloor \frac{j+1}{2} \right\rfloor$.

Hence we find for a constant $C = C(m, \alpha, r)$ that

$$\frac{1}{(j+1)^{2-\gamma}} \left\| \left(1 + \frac{y}{(j+1)^\alpha}\right) F^1_{j,low} \right\|_j \le \frac{C}{\tau_1^2} \sum_{l=2}^{\lfloor \frac{j+1}{2} \rfloor} (l+1)^{\gamma-r} \|(1+y)\omega_{l+1}\|_{l+1} \|u_{j-l+1}\|_{j-l+1}$$

using that $1 - \gamma + \alpha \le 0$ and $2\gamma - 2 \le \alpha(m-1)$.

The discrete Young's convolution inequality implies for all $t \in [0, T]$ that

$$\begin{aligned}
&\sum_{j=0}^\infty \frac{1}{(j+1)^{4-2\gamma}} \left\| \left(1 + \frac{y}{(j+1)^\alpha}\right) F^1_{j,low} \right\|_j^2 \\
&\le \frac{\mathcal{C}}{\tau_1^4} \left(\sum_{l=0}^\infty (l+1)^{\gamma-r} \|(1+y)\omega_l\|_l \right)^2 \sum_{j=0}^\infty \|u_j\|_j^2 \\
&\le \frac{\mathcal{C}}{\tau_1^4} \left(\sum_{l=0}^\infty (l+1)^{4\gamma-1-2r} \right) \left(\sum_{l=0}^\infty (l+1)^{1-2\gamma} \|(1+y)\omega_l\|_l^2 \right) \left(\sum_{j=0}^\infty \|u_j\|_j^2 \right).
\end{aligned}$$

As $4\gamma - 1 - 2r < -1$, the first integral is finite. Hence we arrive at the required estimate

$$\sum_{j=0}^\infty \frac{1}{(j+1)^{4-2\gamma}} \int_0^T \|F^1_{j,low}\|_j^2 \,\mathrm{d}t \le \frac{C}{\tau_1^2} \sup_{t\in[0,T]} \|(1+y)\omega\|^2_{\gamma,\tau,r+1-\gamma} \int_0^T \|u\|^2_{\gamma,\tau,r} \,\mathrm{d}t$$

with a constant $C = C(m, \alpha, \gamma, r)$.

For the treatment of $F^1_{j,high}$ swap the roles of u_l and u_{j-l+1} so that

$$F^1_{j,high} = \sum_{l=2}^{j-\lfloor \frac{j+1}{2} \rfloor} \binom{j}{l-1} \frac{M_j}{M_l M_{j-l+1}} u_l u_{j-l+1}.$$

In the given range $l = 2, \dots, j - \left\lfloor \frac{j+1}{2} \right\rfloor$ we find

$$\binom{j}{l-1} \le \binom{j}{l}$$

so that it can be bounded as $F^1_{j,low}$.

Analysis of F^2_j. We write

$$\begin{aligned}
F^2_j &= -\sum_{l=2}^{\lfloor \frac{j+1}{2} \rfloor} \binom{j}{l} \frac{M_j}{M_l M_{j-l+1}} \partial_y u_l \, \partial_x^{-1} v_{j-l+1} - \sum_{l=\lfloor \frac{j+1}{2} \rfloor + 1}^{j-1} \binom{j}{l} \frac{M_j}{M_l M_{j-l+1}} \partial_y u_l \, \partial_x^{-1} v_{j-l+1} \\
&=: F^2_{j,low} + F^2_{j,high}
\end{aligned}$$

and note that it vanishes unless $j \ge 3$.

By $v_{j-l+1} = -\partial_x \int_0^y u_{j-l+1} \,\mathrm{d}z$ we find for $n \le \min(m-1, j-l+1)$ using the 1d Sobolev inequality and Lemma 3 that

$$\begin{aligned}
\left\| \left(1 + \frac{y}{(j+1)^\alpha}\right) \partial_y u_l \partial_x^{-1} v_{j-l+1} \right\|_j &\le \left\| \partial_y u_l \partial_x^{-1} v_{j-l+1} \right\|_{j-1} \\
&\le C_{m-n} \left\| \left(\frac{\rho_{j-1}\rho_n}{\rho_l \rho_{j-l+1}} \right)^{1/2} \partial_y u_l \right\|_{L^\infty_x L^2_y(\rho_l)} \|u_{j-l+1}\|_{j-l+1} \\
&\le \frac{\mathcal{C}}{\tau_1} \sup_y \left(\frac{\rho_{j-1}\rho_n}{\rho_l \rho_{j-l+1}} \right)^{1/2} (l+1)^\gamma \, \|(1+y)\omega_{l+1}\|_{l+1} \|u_{j-l+1}\|_{j-l+1}
\end{aligned}$$

for a constant $\mathcal{C} = \mathcal{C}(m, r)$.

In the range $l = 2, \dots, \left\lfloor \frac{j+1}{2} \right\rfloor$ for $F^2_{j,low}$ we find that $j - l + 1 \geq \frac{j+1}{2}$ and as we can assume that $j \geq 3$ we can always ensure that this is at least 2.

For $\frac{j+1}{2} \leq m - 1$, we can take $n = 2$ and find a constant $C = C(m, r)$ such that

$$\sup_y \left(\frac{\rho_{j-1}\rho_n}{\rho_l \rho_{j-l+1}} \right)^{1/2} \leq C$$

and otherwise we can taken $n = m - 1$ and find the same control as for $F^1_{j,low}$ as

$$\frac{1}{(j+1)^{2-\gamma}} \left\| \left(1 + \frac{y}{(j+1)^\alpha} \right) F^2_{j,low} \right\|_j \leq \frac{C}{\tau_1^2} \sum_{l=2}^{\lfloor \frac{j+1}{2} \rfloor} (l+1)^{\gamma - r} \|(1+y)\omega_{l+1}\|_{l+1} \|u_{j-l+1}\|_{j-l+1}$$

and we can conclude as for $F^1_{j,low}$.

For $F^2_{j,high}$ we find

$$F^2_{j,high} = - \sum_{l=2}^{j - \lfloor \frac{j+1}{2} \rfloor} \binom{j}{l-1} \frac{M_j}{M_l M_{j-l+1}} \partial_x^{-1} v_l \, \partial_y u_{j-l+1}.$$

For $n = \min(m - 1, l + 1)$ we find

$$\begin{aligned} \left\| \left(1 + \frac{y}{(j+1)^\alpha} \right) \partial_x^{-1} v_l \, \partial_y u_{j-l+1} \right\|_j &\leq \left\| \left(\frac{\rho_{j-1}}{\rho_{j-l}} \right)^{1/2} \partial_x^{-1} v_l \right\|_{L^\infty_{x,y}} \|(1+y)\omega_{j-l+1}\|_{j-l+1} \\ &\leq \frac{C}{\tau_1} \sup_y \left(\frac{\rho_{j-1}\rho_n}{\rho_{l+1}\rho_{j-l}} \right)^{1/2} (l+1)^\gamma \|u_{l+1}\|_{l+1} \|(1+y)\omega_{j-l+1}\|_{j-l+1}. \end{aligned}$$

For $l + 1 < m - 1$ we can find a constant $C = C(m)$ such that

$$\binom{j}{l-1} \leq \binom{j}{l} (j+1)^{-1}.$$

Using the stronger assumption $2\gamma - 1 \leq \alpha(m - 1)$, we can then conclude as in the treatment of $F^1_{j,low}$ that

$$\frac{1}{(j+1)^{2-\gamma}} \left\| \left(1 + \frac{y}{(j+1)^\alpha} \right) F^2_{j,high} \right\|_j \leq \frac{C}{\tau_1^2} \sum_{l=2}^{j - \lfloor \frac{j+1}{2} \rfloor} (l+1)^{\gamma - r} \|u_{l+1}\|_{l+1} \, (j+1)^{-1} \|(1+y)\omega_{j-l+1}\|_{j-l+1}.$$

Hence we find

$$\begin{aligned} &\sum_{j=0}^\infty \frac{1}{(j+1)^{4-2\gamma}} \left\| \left(1 + \frac{y}{(j+1)^\alpha} \right) F^2_{j,high} \right\|_j^2 \\ &\leq \frac{C}{\tau_1^4} \left(\sum_{l=0}^\infty (l+1)^{\gamma - r} \|u_l\|_l \right)^2 \sum_{j=0}^\infty (j+1)^{-2} \|(1+y)\omega_j\|_j^2 \\ &\leq \frac{C}{\tau_1^4} \left(\sum_{l=0}^\infty (l+1)^{3\gamma - 2r} \right) \left(\sum_{l=0}^\infty (l+1)^{-\gamma} \|u_l\|_l^2 \right) \left(\sum_{j=0}^\infty (j+1)^{-2} \|(1+y)\omega_j\|_j^2 \right). \end{aligned}$$

As $3\gamma - 2r < -1$, this gives the required estimate

$$\sum_{j=0}^{\infty} \frac{1}{(j+1)^{4-2\gamma}} \int_0^T \|F^2_{j,high}\|_j^2 \,\mathrm{d}t \le \frac{C}{\tau_1^4} \sup_{t\in[0,T]} \|u\|^2_{\gamma,\tau,r-\frac{\gamma}{2}} \int_0^T \|(1+y)\omega\|^2_{\gamma,\tau,r-1} \,\mathrm{d}t$$

with a constant $C = C(m,\alpha,\gamma,r)$. As $r - 1 \le r + 1 - \gamma$ this is the required control.
Analysis of F_j^3 and F_j^5. We write

$$\begin{aligned} F_j^3 + F_j^5 &= -\sum_{l=2}^{j} \binom{j}{l} \frac{M_j}{M_l M_{j-l+1}} U_l^e u_{j-l+1} - \sum_{l=1}^{j} \binom{j}{l} \frac{M_j}{M_{l+1} M_{j-l}} U_{l+1}^e u_{j-l} \\ &= -\sum_{l=2}^{\lfloor \frac{j+1}{2} \rfloor} \left[\binom{j}{l} + \binom{j}{l-1} \right] \frac{M_j}{M_l M_{j-l+1}} U_l^e u_{j-l+1} + \sum_{l=\lfloor \frac{j+1}{2} \rfloor + 1}^{j+1} \left[\binom{j}{l} + \binom{j}{l-1} \right] \frac{M_j}{M_l M_{j-l+1}} U_l^e u_{j-l+1} \\ &=: F^{3,5}_{j,low} + F^{3,5}_{j,high} \end{aligned}$$

with the convention that

$$\binom{j}{j+1} = 0.$$

Using the definition of U^e and the 1d Sobolev embedding theorem we find

$$\|U_l^e\|_{L^\infty_{x,y}} \le \|U_l^E\|_{L^\infty_x} \le \frac{C_s (l+1)^\gamma}{\tau_1} \|U_{l+1}^E\|.$$

As $l \ge 2$, this implies

$$\left\| \left(1 + \frac{y}{(j+1)^\alpha}\right) U_l^e u_{j-l+1} \right\|_j \le \frac{C_s (l+1)^\gamma}{\tau_1} \|U_{l+1}^E\| \, \|u_{j-l+1}\|_{j-l+1}.$$

For $l = 2, \dots, \left\lfloor \frac{j+1}{2} \right\rfloor$ we find for a constant $C = C(\gamma, r)$

$$\left[\binom{j}{l} + \binom{j}{l-1} \right] \frac{M_j}{M_l M_{j-l+1}} \le \frac{C}{\tau_1} \binom{j}{l}^{1-\gamma} \frac{(j+1)^\gamma}{(l+1)^r}$$

so that as $l \ge 2$

$$\frac{1}{(j+1)^{2-\gamma}} \left\| \left(1 + \frac{y}{(j+1)^\alpha}\right) F^{3,5}_{j,low} \right\|_j \le \frac{C}{\tau_1^2} \sum_{l=2}^{\lfloor \frac{j+1}{2} \rfloor} (l+1)^{\gamma - r} \|U_{l+1}^E\| \, \|u_{j-l+1}\|_{j-l+1}.$$

Hence we find

$$\begin{aligned} \sum_{j=0}^{\infty} \frac{1}{(j+1)^{4-2\gamma}} \left\| \left(1 + \frac{y}{(j+1)^\alpha}\right) F^{3,5}_{j,low} \right\|_j^2 &\le \frac{C}{\tau_1^4} \left(\sum_{l=0}^{\infty} (l+1)^{\gamma - r} \|U_l^E\| \right)^2 \sum_{j=0}^{\infty} \|u_j\|_j^2 \\ &\le \frac{C}{\tau_1^4} \left(\sum_{l=0}^{\infty} (l+1)^{2\gamma - 2R} \right) \left(\sum_{l=0}^{\infty} (l+1)^{2R-2r} \|U_l^E\|^2 \right) \left(\sum_{j=0}^{\infty} \|u_j\|_j^2 \right). \end{aligned}$$

As $2\gamma - R < -1$ this gives the bound

$$\int_0^T \sum_{j=0}^{\infty} \frac{1}{(j+1)^{4-2\gamma}} \left\| \left(1 + \frac{y}{(j+1)^\alpha}\right) F^{3,5}_{j,low} \right\|_j^2 \mathrm{d}t \le \frac{C}{\tau_1^4} \sup_{t\in[0,T]} |U^E|^2_{\gamma,\tau,R} \int_0^T \|u\|^2_{\gamma,\tau,r} \,\mathrm{d}t.$$

For $l = \left\lfloor \frac{j+1}{2} \right\rfloor + 1, \dots, j$ we find

$$\left[\binom{j}{l} + \binom{j}{l-1} \right] \frac{M_j}{M_l M_{j-l+1}} \leq \frac{C}{\tau_1} \binom{j}{l-1}^{1-\gamma} \frac{(l+1)^\gamma}{(j-l+1)^r}$$

so that

$$\frac{1}{(j+1)^{2-\gamma}} \left\| \left(1 + \frac{y}{(j+1)^\alpha}\right) F^{3,5}_{j,high} \right\|_j \leq \frac{C}{\tau_1^2} \sum_{l=\lfloor \frac{j+1}{2} \rfloor + 1}^{j} (l+1)^{3\gamma-2} \|U^E_{l+1}\| \, (j-l+1)^{-r} \|u_{j-l+1}\|_{j-l+1}.$$

Hence we find

$$\begin{aligned} \sum_{j=0}^{\infty} \frac{1}{(j+1)^{4-2\gamma}} \left\| \left(1 + \frac{y}{(j+1)^\alpha}\right) F^{3,5}_{j,high} \right\|_j^2 &\leq \frac{\mathcal{C}}{\tau_1^4} \left(\sum_{j=0}^{\infty} (j+1)^{-r} \|u_l\|_l \right)^2 \sum_{l=0}^{\infty} (l+1)^{6\gamma-4} \|U^E_{l+1}\| \\ &\leq \frac{\mathcal{C}}{\tau_1^4} \left(\sum_{j=0}^{\infty} (j+1)^{-2r} \right) \left(\sum_{j=0}^{\infty} \|u_l\|_l^2 \right) \left(\sum_{l=0}^{\infty} (l+1)^{6\gamma-4} \|U^E_{l+1}\|^2 \right). \end{aligned}$$

As $r > \frac{1}{2}$ this gives the bound

$$\int_0^T \sum_{j=0}^{\infty} \frac{1}{(j+1)^{4-2\gamma}} \left\| \left(1 + \frac{y}{(j+1)^\alpha}\right) F^{3,5}_{j,high} \right\|_j^2 \mathrm{d}t \leq \frac{\mathcal{C}}{\tau_1^4} \sup_{t \in [0,T]} |U^E|^2_{\gamma,\tau,r+3\gamma-2} \int_0^T \|u\|^2_{\gamma,\tau,r} \, \mathrm{d}t,$$

which is the required bound as $R \geq r + 3\gamma - 2$.

Analysis of F^4_j. This term is creating trouble with the integrability in y as $V^e \sim y$ and is the reason for most technical difficulties.

We write

$$\begin{aligned} F^4_j &= - \sum_{l=1}^{\lfloor \frac{j+1}{2} \rfloor} \binom{j}{l} \frac{M_j}{M_{l+1} M_{j-l}} \partial_x^{-1} V^e_{l+1} \partial_y u_{j-l} \; - \sum_{l=\lfloor \frac{j+1}{2} \rfloor + 1}^{j} \binom{j}{l} \frac{M_j}{M_{l+1} M_{j-l}} \partial_x^{-1} V^e_{l+1} \partial_y u_{j-l} \\ &=: F^4_{j,low} + F^4_{j,high}. \end{aligned}$$

As $l \geq 1$ we find

$$\begin{aligned} \left\| \left(1 + \frac{y}{(j+1)^\alpha}\right) \partial_x^{-1} V^e_{l+1} \partial_y u_{j-l} \right\|_j &\leq \left\| \frac{\partial_x^{-1} V^e_{l+1}}{1+y} \right\|_{L^\infty_{x,y}} \left\| \left(1 + \frac{y}{(j+1)^\alpha}\right) (1+y) \omega_{j-l} \right\|_j \\ &\leq \frac{C}{\tau} (l+1)^\gamma \|U^E_{l+2}\| \, \|(1+y)\omega_{j-l}\|_{j-l} \end{aligned}$$

where $C = C(r)$ is constant. In the last line we used the 1d Sobolev inequality and that

$$\sqrt{\frac{\rho_j}{\rho_{j-l}}} \left(1 + \frac{y}{(j+1)^\alpha}\right) \leq C.$$

For $l = 1, \dots, \left\lfloor \frac{j+1}{2} \right\rfloor$ we find

$$\binom{j}{l} \frac{M_j}{M_{l+1} M_{j-l}} \leq \frac{C}{\tau_1} \binom{j}{l}^{1-\gamma} (l+1)^{\gamma-r}$$

so that

$$\frac{1}{(j+1)^{2-\gamma}}\left\|\left(1+\frac{y}{(j+1)^{\alpha}}\right)F^4_{j,low}\right\|_j \leq \frac{C}{\tau_1^2}\sum_{l=1}^{\lfloor\frac{j+1}{2}\rfloor}\binom{j}{l}^{1-\gamma}(l+1)^{2\gamma-r}(j+1)^{\gamma-2}\|U^E_{l+2}\|\,\|(1+y)\omega_{j-l}\|_{j-l}$$
$$\leq \frac{C}{\tau_1^2}\sum_{l=1}^{\lfloor\frac{j+1}{2}\rfloor}(l+1)^{2\gamma-r}\|U^E_{l+2}\|\,(j+1)^{-1}\|(1+y)\omega_{j-l}\|_{j-l}.$$

Hence we find

$$\sum_{j=0}^{\infty}\frac{1}{(j+1)^{4-2\gamma}}\left\|\left(1+\frac{y}{(j+1)^{\alpha}}\right)F^4_{j,low}\right\|_j^2 \leq \frac{\mathcal{C}}{\tau_1^4}\left(\sum_{l=0}^{\infty}(l+1)^{2\gamma-r}\|U^E_{l+2}\|\right)^2\left(\sum_{j=0}^{\infty}(j+1)^{-2}\|(1+y)\omega_j\|_j^2\right)$$
$$\leq \frac{\mathcal{C}}{\tau_1^4}|U^E|^2_{\gamma,\tau,R}\left(\sum_{j=0}^{\infty}(j+1)^{-2}\|(1+y)\omega_j\|_j^2\right)$$

as $4\gamma-2R<-1$. This gives the bound

$$\int_0^T\sum_{j=0}^{\infty}\frac{1}{(j+1)^{4-2\gamma}}\left\|\left(1+\frac{y}{(j+1)^{\alpha}}\right)F^4_{j,low}\right\|_j^2\,\mathrm{d}t \leq \frac{\mathcal{C}}{\tau_1^4}\sup_{t\in[0,T]}|U^E|^2_{\gamma,\tau,R}\int_0^T\|(1+y)\omega\|^2_{\gamma,\tau,r-1}\,\mathrm{d}t,$$

which is the required bound as $-1\leq 1-\gamma$.

For $F^4_{j,high}$ we find

$$\frac{1}{(j+1)^{2-\gamma}}\left\|\left(1+\frac{y}{(j+1)^{\alpha}}\right)F^4_{j,high}\right\|_j \leq \frac{C}{\tau^2}\sum_{l=\lfloor\frac{j+1}{2}\rfloor+1}^{j}\frac{(l+1)^{3\gamma-2}}{(j-l+1)^r}\|U^E_{l+1}\|\,\|(1+y)\omega_{j-l}\|_{j-l}.$$

As $-1+\gamma-r<-\frac{1}{2}$ this gives the bound

$$\int_0^T\sum_{j=0}^{\infty}\frac{1}{(j+1)^{4-2\gamma}}\left\|\left(1+\frac{y}{(j+1)^{\alpha}}\right)F^4_{j,high}\right\|_j^2\,\mathrm{d}t \leq \frac{\mathcal{C}}{\tau_1^4}\sup_{t\in[0,T]}|U^E|^2_{\gamma,\tau,r+3\gamma-2}\int_0^T\|(1+y)\omega\|^2_{\gamma,\tau,r+1-\gamma}\,\mathrm{d}t.$$

As $R\geq r+3\gamma-2$ this is the required result.

Analysis of F^6_j. We write

$$F^6_j=-\sum_{l=2}^{\lfloor\frac{j+1}{2}\rfloor}\binom{j}{l}\frac{M_j}{M_lM_{j-l+1}}\partial_yU^e_l\partial_x^{-1}v_{j-l+1}-\sum_{l=\lfloor\frac{j+1}{2}\rfloor+1}^{j}\binom{j}{l}\frac{M_j}{M_lM_{j-l+1}}\partial_yU^e_l\partial_x^{-1}v_{j-l+1}=:F^6_{j,low}+F^6_{j,high}.$$

As ∂_yU^e is exponentially decaying, we find

$$\left\|\left(1+\frac{y}{(j+1)^{\alpha}}\right)\partial_yU^e_l\partial_x^{-1}v_{j-l+1}\right\|_j \leq C\|U^E_l\|_{L^\infty_x}\|u_{j-l+1}\|_{j-l+1}$$
$$\leq \frac{C(l+1)^{\gamma}}{\tau}\|U^E_{l+1}\|\,\|u_{j-l+1}\|_{j-l+1}.$$

For $F^6_{j,low}$ we find (using that $\binom{j}{l}\frac{M_j}{M_lM_{j-l+1}}\leq C(l+1)^{-r}$ for $l=2...\left\lfloor\frac{j+1}{2}\right\rfloor$):

$$\frac{1}{(j+1)^{2-\gamma}}\left\|\left(1+\frac{y}{(j+1)^{\alpha}}\right)F^6_{j,low}\right\|_j \leq \frac{C}{\tau_1^2}\sum_{l=2}^{\lfloor\frac{j+1}{2}\rfloor+1}(l+1)^{\gamma-r}\|U^E_{l+1}\|\,\|u_{j-l+1}\|_{j-l+1}.$$

As $\gamma - R < -\frac{1}{2}$ this gives the control

$$\int_0^T \sum_{j=0}^{\infty} \frac{1}{(j+1)^{4-2\gamma}} \left\| \left(1 + \frac{y}{(j+1)^{\alpha}}\right) F^6_{j,low} \right\|_j^2 \mathrm{d}t \leq \frac{\mathcal{C}}{\tau_1^4} \sup_{t\in[0,T]} |U^E|^2_{\gamma,\tau,R} \int_0^T \|u\|^2_{\gamma,\tau,r} \,\mathrm{d}t.$$

For $F^6_{j,high}$ we find

$$\frac{1}{(j+1)^{2-\gamma}} \left\| \left(1 + \frac{y}{(j+1)^{\alpha}}\right) F^6_{j,high} \right\|_j \leq \frac{C}{\tau_1^2} \sum_{l=\lfloor \frac{j+1}{2} \rfloor + 1}^{j} (l+1)^{2\gamma-2} \|U^E_{l+1}\| \, (j+1)^{\gamma-r} \|u_{j-l+1}\|_{j-l+1}.$$

As $\gamma - r < -\frac{1}{2}$ this gives the control

$$\int_0^T \sum_{j=0}^{\infty} \frac{1}{(j+1)^{4-2\gamma}} \left\| \left(1 + \frac{y}{(j+1)^{\alpha}}\right) F^6_{j,high} \right\|_j^2 \mathrm{d}t \leq \frac{\mathcal{C}}{\tau_1^4} \sup_{t\in[0,T]} |U^E|^2_{\gamma,\tau,r+1-\gamma} \int_0^T \|u\|^2_{\gamma,\tau,r} \,\mathrm{d}t,$$

which is the required control as $R \geq r + 1 - \gamma$. □

As a direct consequence of Proposition 7 and Proposition 13, we can state the following corollary, where we use that $F_j|_{y=0} = f^e_j|_{y=0}$ as u and v vanish at $y = 0$.

Corollary 14. *Fix the parameters m, α, γ, r, R as in (32) and $\alpha \geq 1/2$. There exists $\mathcal{C}$ and $\mathbf{C}$ such that for all β, τ_1, T with*

$$\beta \geq \mathcal{C}(1 + \|(U^P, V^P)\|_{low}) \, (1 + \frac{1}{\tau_1} + \|(U^P, V^P)\|_{low}), \qquad \text{and } \tau(T) \geq \tau_1$$

we have

$$\begin{aligned} |||u|||^2 \leq \mathbf{C} &\left[\frac{1}{\beta}\|u_{\mathrm{in}}\|^2_{\gamma,\tau_0,r+\gamma-\frac{3}{2}} + \frac{1}{\beta^2}\|(1+y)\omega_{\mathrm{in}}\|^2_{\gamma,\tau_0,r+\frac{1}{2}-\gamma}\right] \\ &+ \mathbf{C}\left[+\frac{1}{\beta^2}\int_0^T \|f^e_j|_{y=0}\|^2_{\gamma,\tau,r-2+\gamma} \,\mathrm{d}t + \frac{1}{\beta^2}\int_0^T \|(1+y)f^e_j\|^2_{\gamma,\tau,r-2+\gamma} \,\mathrm{d}t\right] \\ &+ \frac{\mathbf{C}}{\tau_1^4}\left(\frac{1}{\beta}|U^E|^2_{\gamma,\tau,R} + |||u|||^2\right)|||u|||^2 \end{aligned} \tag{34}$$

where

$$\begin{aligned} |||u|||^2 = \int_0^T \|u\|^2_{\gamma,\tau,r} \,\mathrm{d}t + \sup_{t\in[0,T]} \frac{1}{\beta}\|u\|^2_{\gamma,\tau,r-\frac{\gamma}{2}} + \frac{1}{\beta}\int_0^T \|(1+y)\omega\|^2_{\gamma,\tau,r+1-\gamma} \,\mathrm{d}t \\ + \sup_{t\in[0,T]} \frac{1}{\beta^2}\|(1+y)\omega\|^2_{\gamma,\tau,r+\frac{1}{2}-\gamma} + \frac{1}{\beta^2}\int_0^T \|(1+y)\partial_y\omega\|^2_{\gamma,\tau,r+\frac{1}{2}-\gamma} \,\mathrm{d}t \end{aligned} \tag{35}$$

6 Control of the low norm and final a priori estimate

Corollary 14, which shows an a priori bound on the Gevrey norm of u, was derived under a lower bound on β involving $\|(U^P, V^P)\|_{low}$. The last step is to see how this low norm relates to $|||u|||$. A convenient approach is to establish an additional estimate on a weighted Sobolev norm, namely

$$\|f\|^2_{\mathcal{H}^s} = \sum_{|\bar{\alpha}|\leq s} \int_{\mathbb{T}\times\mathbb{R}_+} |\partial^{\bar{\alpha}} f|^2 (1+y)^{2\bar{\alpha}_2} \rho_0(y) \,\mathrm{d}x \,\mathrm{d}y,$$

where the summation variable is the multiindex $\bar{\alpha} = (\bar{\alpha}_1, \bar{\alpha}_2)$. In this setting, we can state the following estimate.

Lemma 15. *Let $s \geq 3$ be an even integer, $m \geq s+2$, $\alpha \geq 0$, $r \in \mathbb{R}$, $\gamma \geq 1$, and define $|||u|||$ as in (35). Then, there exists C depending on s, m, α, γ, r such that*

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\omega^P\|_{\mathcal{H}^s}^2 + \|\partial_y\omega^P\|_{\mathcal{H}^s}^2 \leq C\|\omega^P\|_{\mathcal{H}^s}^s + C(1+\|U^E\|_{H^{s+1}(\mathbb{T})} + |||u|||)\|\omega^P\|_{\mathcal{H}^s}^2 + \sum_{l=0}^{\frac{s}{2}} \|\partial_t^l(\partial_t + U^E\partial_x U^E)\|_{H^{s-2l}}^2. \tag{36}$$

where $\omega^P = \partial_y U^P$.

Proof. A similar estimate was established in [27, Proposition 5.6], so that we will only explain the main steps. The starting point is the advection-diffusion equation on the vorticity

$$(\partial_t + U^P\partial_x + V^P\partial_y)\omega^P - \partial_y^2\omega^P = 0. \tag{37}$$

One applies ∂^α to the equation, test it against $(1+y)^{2\alpha_2}\rho_0\partial^\alpha\omega^P$, and sum over $|\alpha| \leq s$. Then,

$$\begin{aligned}\frac{1}{2}\partial_t\|\omega^P\|_{\mathcal{H}^s}^2 + \|\partial_y\omega^P\|_{\mathcal{H}^s}^2 \leq{}& \sum_{|\alpha|\leq s}\int V^P\partial_y((1+y)^{2\alpha_2}\rho_0)|\partial^\alpha\omega^P|^2 \\ &- \sum_{|\alpha|\leq s}\int [\partial^\alpha, (U^P\partial_x + V^P\partial_y)]\omega^P\,\partial^\alpha\omega^P(1+y)^{2\alpha_2}\rho_0 \\ &- \sum_{|\alpha|\leq s}\int_\Omega \partial_y((1+y)^{2\alpha_2}\rho_0)\partial_y\partial^\alpha\omega^P\,\partial^\alpha\omega^P - \sum_{|\alpha|\leq s}\int_{\{y=0\}}\partial_y\partial^\alpha\omega^P\,\partial^\alpha\omega^P.\end{aligned}$$

Using the equation on U^P, one can obtain recursively boundary conditions for the odd derivatives $\partial_y^{2k+1}\omega^P$, starting from the Neumann condition

$$\partial_y\omega^P|_{y=0} = -\partial_t U^E - U^E\partial_x U^E.$$

More precisely, the boundary data $\partial_y^{2k+1}\omega^P|_{y=0}$ can be expressed in terms of the data U^E and of products of mixed derivatives $\partial_x^{\rho_1}\partial_y^{\rho_2}\omega^P|_{y=0}$ with $\rho_2 \leq 2k-2$. We refer to [27, Lemma 5.9] for the expressions of these boundary conditions. This allows to establish the following bound, *cf* equations (5.20)-(5.22) in [27]:

$$-\sum_{|\alpha|\leq s}\int_{\{y=0\}}\partial_y\partial^\alpha\omega^P\,\partial^\alpha\omega^P \leq C_s\|\omega^P\|_{\mathcal{H}^s}^s + C_s\sum_{l=0}^{\frac{s}{2}}\|\partial_t^l(\partial_t + U^E\partial_x U^E)\|_{H^{s-2l}}^2 + \frac{1}{4}\|\partial_y\omega^P\|_{\mathcal{H}^s}^2.$$

The diffusion term does not raise any difficulty: we find

$$-\sum_{|\alpha|\leq s}\int_\Omega \partial_y((1+y)^{2\alpha_2}\rho_0)\partial_y\partial^\alpha\omega^P\,\partial^\alpha\omega^P \leq C\|\omega^P\|_{\mathcal{H}^s}^2 + \frac{1}{4}\|\partial_y\omega^P\|_{\mathcal{H}^s}^2.$$

where C depends on s and m. Also, through standard estimates, we find

$$\sum_{|\alpha|\leq s}\int V^P\partial_y((1+y)^{2\alpha_2}\rho_0)|\partial^\alpha\omega^P|^2 \leq C\|\omega^P\|_{\mathcal{H}^s}^3$$

and

$$\sum_{|\alpha|\leq s}\int [\partial^\alpha, U^P\partial_x]\omega^P\,\partial^\alpha\omega^P(1+y)^{2\alpha_2}\rho_0 \leq C\|\omega^P\|_{\mathcal{H}^s}^3.$$

The other part of the commutator is slightly more delicate. First, one can show that

$$\sum_{|\alpha|\leq s,\alpha_1\neq s}\int [\partial^\alpha, V^P\partial_y]\omega^P\,\partial^\alpha\omega^P(1+y)^{2\alpha_2}\rho_0 \leq C\|\omega^P\|^3_{\mathcal{H}^s}.$$

Note that the weight (that grows with the number of y-derivatives) allows to compensate for the linear growth in y of V^P. The success of this trick comes from the fact that we are interested here in Sobolev estimates (contrary to the former Gevrey estimates). When $\alpha_1 = s$, namely $\alpha = (s,0)$, one can show similarly that

$$\int [\partial^\alpha, V^P\partial_y]\omega^P\,\partial^\alpha\omega^P\rho_0 - \int \partial^\alpha V^P\partial_y\omega_P\,\partial^\alpha\omega^P\rho_0 \;\leq\; C\|\omega^P\|^3_{\mathcal{H}^s}.$$

However, the term where the s derivatives with respect to x apply to V^P can not be handled with usual manipulations. It is the well-known loss of x-derivative peculiar to the Prandtl equation: in particular, one cannot control $\|(1+y)^{-1}\partial_x^s V^P\|_{L^2(\rho_0)}$ by $\|\omega^P\|_{\mathcal{H}^s}$. This is where $|||u|||$ is involved. We find that

$$\begin{aligned}\int \partial^\alpha V^P\partial_y\omega_P\,\partial^\alpha\omega^P\rho_0 &\leq \|(1+y)^{-1}\partial_x^s V^P\|_\infty\,\|(1+y)\partial_y\omega^P\|_{L^2_xL^2(\rho_0)}\,\|\omega^P\|_{L^2_xL^2(\rho_0)}\\ &\leq C(\|U^E\|_{H^{s+1}} + |||u|||)\|\omega^P\|^2_{\mathcal{H}^s}\end{aligned}$$

using that $\|(1+y)^{-1}\partial_x^s V^P\|_\infty \leq C\|\partial_x^{s+1}u^P\|_\infty \leq C|||u|||$ as soon as $m \geq s+2$. Putting together the previous estimates yields the result. □

We conclude this section with

Proposition 16. *Let us fix $s = 6$, $m \geq s+2$, α, γ, r, R as in (32) and $\alpha \geq 1/2$. Further fix $\tau_1 > 0$. Let*

$$M_{\rm in} = 2\max(\mathbf{C},1)\left(\|u_{\rm in}\|^2_{\gamma,\tau_0,r+\gamma-\frac{3}{2}} + \|(1+y)\omega_{\rm in}\|^2_{\gamma,\tau_0,r+\frac{1}{2}-\gamma} + \|\omega^P|_{t=0}\|^2_{\mathcal{H}^s}\right)$$

where $\mathbf{C}$ is the constant appearing in Corollary 14. There exists β_* and T_* depending on τ_1, $M_{\rm in}$, on $\|\omega^P|_{t=0}\|_{\mathcal{H}^s}$, on $\sup_{[0,T_0]}|U^E|^2_{\gamma,\tau_0,R}$ and on various Sobolev norms of U^E, such that, for all $\beta > \beta_*$ and for all $T \leq T_*$ with $\tau(T) \geq \tau_1$: if $|||u|||^2 \leq \frac{2M_{\rm in}}{\beta}$, then $|||u|||^2 \leq \frac{3M_{\rm in}}{2\beta}$.

Proof. Let β, T such that $|||u|||^2 \leq \frac{2M_{\rm in}}{\beta} \leq 2M_{\rm in}$ (assuming $\beta \geq 1$). We first apply Lemma 15, which yields

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|\omega^P\|^2_{\mathcal{H}^s} + \|\partial_y\omega^P\|^2_{\mathcal{H}^s} &\leq C\|\omega^P\|^s_{\mathcal{H}^s} + C(1+\|U^E\|_{H^{s+1}(\mathbb{T})} + \sqrt{2M_{\rm in}})\|\omega^P\|^2_{\mathcal{H}^s}\\ &\quad + \sum_{l=0}^{\frac{s}{2}}\|\partial_t^l(\partial_t + U^E\partial_x U^E)\|^2_{H^{s-2l}}.\end{aligned} \tag{38}$$

Integrating this differential inequality shows

$$\sup_{t\in[0,T]}\|\omega^P(t)\|_{\mathcal{H}^s} \leq 2\|\omega^P|_{t=0}\|_{\mathcal{H}^s} \tag{39}$$

for $T \leq T_1$, where T_1 depends on $M_{\rm in}$, $\sup_{t\in[0,T_0]}\|U^e(t)\|_{H^{s+1}(\mathbb{T})}$, $\int_0^{T_0}\|\partial_t^l(\partial_t + U^E\partial_x U^E)\|^2_{H^{s-2l}}\,\mathrm{d}t$ and on $\|\omega^P|_{t=0}\|_{\mathcal{H}^s}$.

Standard Sobolev imbeddings imply that

$$\max_{0\le k\le 3}\|\partial_x^k U^P\|_\infty + \max_{0\le k\le 2}\left\|\frac{\partial_x^k V^P}{1+y}\right\|_\infty \le C\left(\max_{0\le k\le 3}\|\partial_x^k U^E\|_\infty + |||u|||\right).$$

As regards the other terms defining $\|(U^P,V^P)\|_{low}$, *cf* (19), they all involve ω^P and are controlled by $\|\omega^P\|_{\mathcal{H}^s}$ as soon as $s\ge 5$. Hence, it follows from (39) that

$$\|(U^P,V^P)\|_{low}\le K$$

for $T\le T_1$ and for some K depending on $M_{\rm in}$, $\|\omega^P|_{t=0}\|_{\mathcal{H}^s}$ and various norms of U^E. If we now choose

$$\beta_*\ge \mathcal{C}(1+K)\,(1+\frac{1}{\tau_1}+K),\qquad \text{and } \tau(T)\ge\tau_1$$

where $\mathcal{C}$ is the constant appearing in Corollary 14, we obtain for $\beta\ge\beta_*$:

$$|||u|||^2\le \frac{M_{\rm in}}{2\beta}+\frac{\mathbf{C}}{\beta^2}\int_0^T\|(1+y)f_j^e\|^2_{\gamma,\tau,r-2+\gamma}\,\mathrm{d}t+\frac{\mathbf{C}}{\beta\tau_1^4}\left(|U^E|^2_{\gamma,\tau,R}+2M_{\rm in}\right)|||u|||^2.$$

Taking β_* large enough so that

$$\frac{\mathbf{C}}{\beta_*\tau_1^4}\sup_{t\in[0,T_0]}\left(|U^E|^2_{\gamma,\tau_0,R}+2M_{\rm in}\right)\le\frac12$$

we get

$$|||u|||^2\le M_{\rm in}+\frac{2\mathbf{C}}{\beta^2}\int_0^T\|(1+y)f_j^e\|^2_{\gamma,\tau,r-2+\gamma}\,\mathrm{d}t+\frac{2\mathbf{C}}{\beta^2}\int_0^T\|f_j^e|_{y=0}\|^2_{\gamma,\tau,r-2+\gamma}\,\mathrm{d}t$$

If we take $T_*\le T_1$ such that $2\mathbf{C}\int_0^{T_*}\|(1+y)f_j^e\|^2_{\gamma,\tau,r-2+\gamma}\,\mathrm{d}t\le\frac12 M_{\rm in}$, the result follows. □

7 Existence and uniqueness

On the basis of the previous *a priori* estimates, we now complete the proof of Theorem 1: we construct a unique solution of (1)-(2) with data $U^P_{\rm in}$. This obviously amounts to constructing a unique solution of (13)-(15) with data $u_{\rm in}:=U^P_{\rm in}-U^e|_{t=0}$.

We fix $s=6$, $\gamma=2$. We take $m\ge s+2$ and $\alpha\ge\frac12$ that satisfy the inequalities in the first line of (32). Let $0<\tau_1<\tau_0$, $r\in\mathbb{R}$, $T_0>0$, and U^E, $U^P_{\rm in}=u_{\rm in}+U^e|_{t=0}$ satisfying the assumptions of the theorem. Let now (τ_0',τ_1') with $0<\tau_1<\tau_1'<\tau_0'<\tau_0$. Let r' and R' as in the second line of (32). As $\tau_0>\tau_0'$, we have

$$\|u_{\rm in}\|^2_{\gamma,\tau_0',r'+\gamma-\frac32}+\|(1+y)\omega_{\rm in}\|^2_{\gamma,\tau_0',r+\frac12-\gamma}\le C\left(\|u_{\rm in}\|^2_{\gamma,\tau_0,r}+\|(1+y)\omega_{\rm in}\|^2_{\gamma,\tau_0,r}\right)<+\infty$$

while

$$\|\omega^P|_{t=0}\|^2_{\mathcal{H}^s}\le C(\sup_{[0,T_0]}|U^E|_{2,\tau_0,r}+\|(1+y)^{m+6}\omega_{\rm in}\|_{H^6(\mathbb{T}\times\mathbb{R}_+)})<+\infty$$

and

$$\sup_{[0,T_0]}|U^E|_{2,\tau_0',R'}\le C\sup_{[0,T_0]}|U^E|_{2,\tau_0,r}<+\infty$$

for a constant C possibly depending on τ_0,τ_0',r,r',R'.

The idea is then to apply Proposition 16 to a solution of an approximate system, for which well-posedness is granted. Inspired by [27], we consider the regularized equation

$$\partial_t u + (u\partial_x + v\partial_y)u + (U^e_\epsilon \partial_x + V^e_\epsilon \partial_y)u + (u\partial_x + v\partial_y)U^e_\epsilon - \epsilon\partial_x^2 u - \partial_y^2 u = f^e_\epsilon, \tag{40}$$

adding a tangential diffusion $-\epsilon\partial_x^2 u$. The modified vector field $(U^e_\epsilon, V^e_\epsilon)$ takes the form

$$U^e_\epsilon = \partial_y(e^{-\epsilon y}(y + e^{-y} - 1))U^E_\epsilon, \quad V^e_\epsilon = -e^{-\epsilon y}(y + e^{-y} - 1)\partial_x U^E_\epsilon$$

where U^E_ϵ is an analytic approximation of U^E, converging to U^E in the norm $|\ |_{2,\tau_0,r}$ as $\epsilon \to 0$. Note that $(U^e_\epsilon, V^e_\epsilon)$ is still divergence-free, but has now fast decay in y, so that all difficulties generated by the linear growth of V^e vanish. Accordingly, the right-hand side f^e is modified into f^e_ϵ replacing U^E by U^E_ϵ, resp. (U^e, V^e) by $(U^e_\epsilon, V^e_\epsilon)$ in (14). Similarly, one regularizes the initial data to obtain some $u_{\mathrm{in},\epsilon}$ real analytic in x, y, with fast decay at infinity in y (and obeying suitable compatibility conditions).

One can show that system (40) is well-posed following classical methods for fully parabolic equations. For instance, for $T_{\epsilon,max}$ small enough, one can prove the existence of a Sobolev solution u_ϵ on $(0, T_{\epsilon,max})$ through a fixed point argument applied to

$$\mathcal{T}_\epsilon u(t) = e^{t(\epsilon\partial_x^2+\partial_y^2)}u_{\mathrm{in},\epsilon} + \int_0^t e^{(t-s)(\epsilon\partial_x^2+\partial_y^2)}F_\epsilon[u](s)ds$$

with $F_\epsilon[u] = f^e_\epsilon - (u\partial_x + v\partial_y)u - (U^e_\epsilon\partial_x + V^e_\epsilon\partial_y)u - (u\partial_x + v\partial_y)U^e_\epsilon$. Moreover, u^ϵ remains (real) analytic in (x, y) *as long as the Sobolev norm of* u_ϵ *does not blow up*, that is on $(0, T_{\epsilon,max})$. This property, related to the analytic regularization of the heat kernel is well-known, even in the more difficult context of the Navier-Stokes equation: see [7, 25, 2] and references therein.

We now claim that all *a priori estimates* obtained for a solution u of (13) can be established for u_ϵ solution of (40), uniformly in ϵ. For this, one just needs to adapt the definitions of the auxiliary quantities H_j and ϕ_j: we rather consider

$$\begin{aligned} &\left(\partial_t + \beta(j{+}1) + U^P\partial_x + (j{+}1)\partial_x U^p + V^P\partial_y - \epsilon\partial_x^2 - \partial_y^2\right)\int_0^y H_j\,\mathrm{d}z = \int_0^y u_j\,\mathrm{d}z, \\ &H_j|_{t=0} = 0, \qquad \partial_y H_j|_{y=0} = 0, \qquad H_j|_{y\to\infty} = 0. \end{aligned} \tag{41}$$

and

$$\begin{aligned} &\left(-\partial_t + \beta(j{+}1) - U^P\partial_x + j\partial_x U^p - V^P\partial_y - \partial_y V^P - V^P\frac{\partial_y\rho_j}{\rho_j} - \epsilon\partial_x^2 - \left(\partial_y + \frac{\partial_y\rho_j}{\rho_j}\right)^2\right)\phi_j = H_j, \\ &\phi_j|_{t=T} = 0, \qquad \phi_j|_{y=0} = 0, \qquad \phi_j|_{y\to\infty} = 0. \end{aligned} \tag{42}$$

The additional good terms coming from $-\epsilon\partial_x^2$ allow to control the extra commutator terms that it generates. Hence, we can apply Proposition 16 with τ_0', τ_1', r' and R' instead of τ_0, τ_1, r, and R. Let β_* and T_* given by the proposition (note that they are independent of ϵ). We then introduce

$$T_{\epsilon,*} = \sup\{T \le T_{\epsilon,max}, |||u|||^2 \le 2M_{\mathrm{in}}/\beta\}$$

where $\beta > \beta_*$ is fixed, and $|||u|||$ is defined in (35). Note that $|||u|||$ implicitly depends on T. By continuity in time of u^ϵ, one has $T_{\epsilon,*} > 0$. But from Proposition 16, one deduces easily that for any $T \le T_*$, $T_{\epsilon,max} \ge T_{\epsilon,*} \ge T$.

From there, by standard compactness arguments, one obtains a solution to the Prandtl system over $[0, T]$, with the regularity properties stated in the theorem. It remains to show

uniqueness. For this, we take two solutions u^1 and u^2 up to time T. The difference u^d then satisfies (from (13))

$$\partial_t u^d + u^1\partial_x u^d + u^d\partial_x u^2 + v^d\partial_y u^1 + v^2\partial_y u^d + (U^e\partial_x + V^e\partial_y)u^d + (u^d\partial_x + v^d\partial_y)U^e - \partial_y^2 u^d = f^{d,e}.$$

We then find for u_j^d that

$$\Big(\partial_t + \beta(j+1) + U^{1,P}\partial_x + (j+1)\partial_x U^{1,P} + V^{2,P}\partial_y - \partial_y^2\Big)u_j^d + \partial_y U^{1,P} v_j^d + j\partial_{xy}U^{1,P}\partial_x^{-1}v_j^d = F_j^d + \partial_x u^d u_j^d,$$

where again F_j^d consists of $f_j^{d,e}$ and mixed terms with less than j derivatives on u^1, u^2 or u^d. Comparing with (16), we see that the only difference is the replacement of (U^P, V^P) by $(U^{1,P}, V^{2,P})$. Let us stress that the latter field not being divergence-free is not an issue: none of the *a priori* estimates carried in Section 4 and Section 5 were using the fact that (U^P, V^P) was divergence-free. One can therefore obtain a similar Gevrey bound on u^d, under a lower bound on β (involving the low norms of $(U^{1,P}, V^{1,P})$ and $(U^{2,P}, V^{2,P})$). This provides a stability estimate which shows uniqueness. These considerations now finish the proof of our main result Theorem 1.

Acknowledgements

The authors thank Weiren Zhao for useful remarks on the first version of this manuscript. They acknowledge the support of the Université Sorbonne Paris Cité, through the funding "Investissements d'Avenir", convention ANR-11-IDEX-0005. H.D. is grateful to the People Programme (Marie Curie Actions) of the European Union's Seventh Framework Programme (FP7/2007-2013) under REA grant agreement n. PCOFUND-GA-2013-609102, through the PRESTIGE programme coordinated by Campus France. D.G.-V. acknowledges the support of the Institut Universitaire de France.

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