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On the convergence of bilateral ergodic averages

Christophe Cuny and Yves Derriennic
Université de Brest
Mathématiques (LMBA-UMR CNRS 6205)
6 av. Le Gorgeu
29238 Brest F.

Abstract

We study the almost sure convergence of bilateral ergodic averages for not necessarily integrable functions and relate it to the ones of the forward and backward averages, hence complementing results of Woś and the second named author. In the case of convergence, using results of Furstenberg on double recurrence, we prove oscillations of the bilateral ergodic averages around the limit.

1 Introduction

For a stationary random sequence, time running from $-\infty$ to $+\infty$, without any assumption of integrability, a question arises: what could be the link between convergence of symmetric bilateral averages, from time $-n$ to time $+n$, and convergence of unilateral averages, from time 0 to time n , usually considered in the ergodic theorem or the law of large numbers. The present paper is devoted to this question.

Birkhoff's ergodic theorem asserts that on a measure preserving dynamical system $(\mathbb{X}, \mathcal{X}, \mu, T)$ the averages $A_n^+ f = \frac{1}{n} \sum_{i=0}^{n-1} f \circ T^i$ of $f \in L^1$ converge μ a.e.. Still these averages converge too for many finite measurable functions which are not integrable. Their characterization is difficult since it cannot depend only on distributions. Assuming $\mu(\mathbb{X}) = 1$, and the transformation T invertible and ergodic as we shall always do in this paper,

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it was proved by Woś [W86] that backward averages $A_n^- f = \frac{1}{n} \sum_{i=0}^{n-1} f \circ T^{-i}$ of a measurable function, not necessarily integrable but finite, converge a.e. to a finite limit if and only if forward averages $A_n^+ f$ do and the limits are the same. This result was given a new approach in [D10] where furthermore it was shown that this equivalence fails if infinite limits are considered: $\lim A_n^+ f = +\infty$, $\limsup A_n^- f = +\infty$, $\liminf A_n^- f = -\infty$ a.e. may coexist.

Here, in the same context, bilateral averages $B_n f = \frac{1}{2n+1} \sum_{i=-n}^n f \circ T^i$ are considered for finite measurable functions. We prove that convergence a.e. of bilateral averages $B_n f$ is equivalent to convergence of forward and backward averages, $A_n^+ f$ and $A_n^- f$ together, the limit being the same, finite or not. This result holds even if we know the convergence of $B_n f$ *a priori* only on a subset of \mathbb{X} of positive measure. In all other cases $\limsup B_n f = +\infty$, $\liminf B_n f = -\infty$ a.e.. In [H10] another approach of the comparison of forward, backward and bilateral averages when they have finite limits on \mathbb{X} , is presented.

In the last part the phenomenon of infinite oscillations around the limit, well known for unilateral averages ([H76], see also [K85], [P83], [D10]) is established also for bilateral averages; this uses elements of Furstenberg's multirecurrence theory [F77].

Somehow these results show the absence of extra-effect of symmetry or compensation between future and past for a stationary sequence, even though the same statistical behavior occurs in both directions.

2 Preliminary approach

A first difficulty appearing in the study of the convergence of symmetric bilateral averages

$$B_n f = \frac{1}{2n+1} \sum_{i=-n}^n f \circ T^i = \frac{1}{2n+1} [f + \sum_{i=1}^n (f \circ T^{-i} + f \circ T^i)],$$

with f a finite measurable function, is the problem of the convergence of the residual terms $\frac{1}{n}(f \circ T^{-n} + f \circ T^n)$, $\frac{1}{n}(f \circ T^{-n} - f \circ T^{n+1})$. The convergence to 0 of the first one is a necessary condition for the convergence of the Cesàro averages defining $B_n f$. The convergence to 0 of the second one is a necessary condition for the invariance of the limit of $B_n f$, if it exists, since $B_n f - B_n f \circ T = \frac{1}{2n+1}(f \circ T^{-n} - f \circ T^{n+1})$. Thus the problem of the convergence of $\frac{1}{n} f \circ T^n$ is also present.

Because of the invariance of the measure the convergence in μ -measure of these residual terms to 0 is obvious hence for this type of convergence the invariance of the set of convergence and the invariance of the limit of averages are easy. The same is true for the limsup in μ -measure sense (that is always dominated by the limsup a.e.). But for a.e. convergence these questions are not obvious, in particular the invariance of the set of convergence of the sequence $B_n f$. As will be shown by Lemma 3.5 below, using Rokhlin towers it is not difficult to build, for any sequence $l(n)$ increasing to $+\infty$, a non negative measurable function f such that $\limsup \frac{1}{l(n)} f \circ T^n = +\infty$ a.e.; even with i.i.d. random variables, averages may converge in probability to 0 although the limsup and liminf are infinite.

If *a priori* the convergence a.e. of $B_n f$ is known on a non null subset $U \subset \mathbb{X}$, the convergence holds also in measure on U hence the sequence $B_n f$ must converge in measure to a constant on the whole space \mathbb{X} by ergodicity. Therefore the limit a.e. of $B_n f$ on U must be this same constant. Yet, at this point the divergence a.e. on U^c is not ruled out since the invariance of the set of convergence a.e. of the sequence $B_n f$ remains to be proved. If $\lim B_n f$ exists and is finite a.e. on U then $\lim \frac{1}{n} (f \circ T^{-n} + f \circ T^n) = 0$ a.e. on U ; if, at this point, we knew that this convergence implies $\lim \frac{1}{n} f \circ T^n = 0$ a.e., the invariance of $\lim \frac{1}{n} f \circ T^n$ being obvious, the invariance we wish for the set of convergence of $B_n f$ would follow. But this implication also requires a proof; it will be given in Part 4.

To overcome this difficulty in the next part, we shall use a direct close analysis of a typical trajectory of the dynamical system. Such a method was suggested by Benjamin Weiss to the second named author in a conversation some years ago; it is somehow similar to Woś's method to prove Theorem 2 in [W86].

Before going further we recall briefly here the results of [D10] that we shall need or extend.

We use the usual notation $f^+ = \max(f, 0)$ and $f^- = (-f)^+$.

Proposition 0. *Let f be a finite measurable real function defined on X . If $F = \sup_{n>0} \sum_{i=0}^{n-1} f \circ T^i < \infty$ a.e. the following equality, called "filling scheme equation" in [1], holds:*

$$f = -F^- + F^+ - F^+ \circ T \text{ a.e..}$$

From this equality the following Statements are deduced (we use the numbering of Part 4 in [D10]):

(b) if $\limsup |A_n^+ f| < \infty$ a.e. then $\lim A_n^+ f = \lim A_n^- f$ a.e.

(c) if $\limsup A_n^+ f = c \in \mathbb{R}$ and $\liminf A_n^+ f = -\infty$ a.e. then $\liminf A_n^- f = c$ and $\limsup A_n^- f = +\infty$ a.e.

(d) if $f \geq 0$, either $\lim \frac{1}{n} f \circ T^n = 0$ a.e. or $\liminf \frac{1}{n} f \circ T^n = 0$ a.e. and $\limsup \frac{1}{n} f \circ T^n = +\infty$ a.e.

(e) if $\limsup A_n^+ f = +\infty$, $\liminf A_n^+ f = -\infty$ a.e. it is possible that $\lim A_n^- f = +\infty$ a.e..

Remark. In [D10] Statement (e) was shown only for systems having a Bernoulli factor, using properties of sequences of i.i.d. random variables. Proposition 3.4, below, will show it for any ergodic dynamical system.

3 Main results

To begin with, here is an elementary lemma of additive combinatorics that we need in the sequel.

Lemma 3.1. *Let a be a positive integer and Δ a set of integers with $\Delta \subset [a, 5a)$. If $\text{card } \Delta/4a > 7/8$ then every integer $i \in [5a/2, 3a)$ can be written as $i = j - k$ with $j, k \in \Delta$ and $j - 2k \geq a$.*

Proof. Consider $i \in [5a/2, 3a)$ and the couples $(x, x + i)$ with $x \in [a, 5a)$ and $x + i - 2x = i - x \geq a$. These inequalities yield at once :

$$a \leq x \leq i - a < a + i \leq x + i \leq 2i - a < 5a.$$

Thus the two intervals $[a, i - a]$ and $[a + i, 2i - a]$ are disjoint, included in $[a, 5a)$, with the same length $i - 2a + 1$. If i could not be written as claimed by the lemma, Δ would contain at most one term of each couple $(x, x + i)$, hence $i - 2a + 1$ integers of the interval $[a, 5a)$ would be excluded from Δ ; we would get

$$\text{card } \Delta \leq 4a - (i - 2a + 1) < 7a/2, \text{ thus } \text{card } \Delta/4a < 7/8. \quad \square$$

Now a key idea of the paper appears in the next lemma.

Lemma 3.2. *Let f be a finite measurable function. If the set of points x such that $B_n f(x) \leq 0$ for n large enough has positive measure, i.e. if $\mu[\cup_{N>0} \cap_{n>N} \{B_n f \leq 0\}] > 0$ then $\sup_{n>0} \sum_{i=0}^{n-1} f \circ T^i < \infty$ a.e. on the whole space \mathbb{X} .*

Proof. Let us consider $V = \cap_{n \geq N} \{B_n f \leq 0\}$ with N large enough to get $\mu(V) > 0$. By ergodicity for an integer p large enough, $\mu(\cup_{p \geq i \geq 0} T^{-i} V) >$

7/8. Let us denote by $\mathcal{W}(x)$ the set of passage times in $(\cup_{p \geq i \geq 0} T^{-i}V)$ of the orbit of $x \in \mathbb{X}$.

By the ergodic theorem $\lim_n \frac{1}{n} \text{card}(\mathcal{W}(x) \cap [0, n]) = \mu(\cup_{p \geq i \geq 0} T^{-i}V)$ for a.e. $x \in \mathbb{X}$. For such an x , $\lim_a \frac{1}{4a} \text{card}(\mathcal{W}(x) \cap [a, 5a]) = \mu(\cup_{p \geq i \geq 0} T^{-i}V) > 7/8$.

Now assume that three integers a , k and j are such that $a > N + 4p$, k and $j \in \mathcal{W}(x)$ with $a \leq k < j < 5a$ and $j - 2k \geq a$. By definition of $\mathcal{W}(x)$ there exist two integers r and s between 0 and p , depending on k and j respectively, such that $T^{k+r}x \in V$ and $T^{j+s}x \in V$. By definition of V , since $k \geq a > N + 4p$, we have for any integer ζ such that $|\zeta| \leq 2p$:

$$(*) \quad \sum_{i=-k-r-\zeta}^{k+r+\zeta} f \circ T^i (T^{k+r}x) = \sum_{i=-\zeta}^{2k+2r+\zeta} f(T^i x) \leq 0.$$

Put $\xi = j - 2k + s - 2r - \zeta - 1$; since $j - 2k \geq a$, we get $\xi \geq N$ and:

$$(**) \quad \sum_{i=-\xi}^{\xi} f \circ T^i (T^{j+s}x) = \sum_{i=-\xi+j+s}^{\xi+j+s} f(T^i x) \leq 0.$$

Since $-\xi + j + s = 2k + 2r + \zeta + 1$, adding the two previous inequalities (*) and (**) we get $\sum_{i=-\zeta}^{\xi+j+s} f(T^i x) \leq 0$ where $\xi + j + s = 2(j - k) + 2(s - r) - \zeta - 1$. Now we can choose $\zeta = 2(s - r) \in [-2p, 2p]$ and we get

$$\sum_{i=-\zeta}^{2(j-k)-1} f(T^i x) \leq 0.$$

Then using Lemma 3.1 for a such that $\frac{1}{4a} \text{card}(\mathcal{W}(x) \cap [a, 5a]) > 7/8$ and $a > N + 4p$, we obtain for every $n \in [5a/2, 3a)$ two integers k and j with the properties required above and $n = j - k$, which leads to the inequalities

$$\sum_{i=0}^{2n-1} f(T^i x) \leq \sum_{i=-2p}^{2p} |f(T^i x)| \quad \text{and} \quad \sum_{i=0}^{2n} f(T^i x) \leq |f(x)| + \sum_{i=-2p}^{2p} |f(T^{i+1} x)|.$$

When a increases to ∞ the intervals $[5a/2, 3a)$ cover a half-line of integers and the desired result follows at once since p depends only on the set V .

Theorem 3.3. *Let f be a finite measurable function. If $\limsup B_n f < +\infty$ a.e. on a set of positive measure then $\lim A_n^+ f = \lim A_n^- f = \lim B_n f = c$ a.e. on the whole space \mathbb{X} i.e. these three sequences converge a.e. to the same constant c which may be $-\infty$.*

Proof. Consider a finite constant K such that $\mu \{ \limsup B_n f < K - \epsilon \} > 0$ with $\epsilon > 0$. Applying Lemma 3.2 to the function $f - K$ and the transformations T or T^{-1} we get $\limsup A_n^+ f \leq K$ and $\limsup A_n^- f \leq K$ a.e..

If $\limsup A_n^+ f = c > -\infty$ then either $\lim A_n^+ f = c$ or $\liminf A_n^+ f = -\infty$ a.e. by Statement (b). In the first case $\lim A_n^- f = c$ a.e. for the same reason, hence $\lim B_n f = c$ a.e.. The second case is impossible here since $\liminf A_n^+ f = -\infty$ would imply $\limsup A_n^- f = +\infty$ a.e. by Statement (c).

If $\lim A_n^+ f = -\infty$ a.e. two cases are again possible by Proposition 3.4 below (improvement of Statement (e)). Either $\lim A_n^- f = -\infty$ a.e. or $\limsup A_n^- f = +\infty$ and $\liminf A_n^- f = -\infty$ a.e.. In the first case obviously $\lim B_n f = -\infty$ a.e., and again the second case is here excluded, so the theorem is proved. \square

Before describing the possible behavior of the three sequences $A_n^+ f$, $A_n^- f$, $B_n f$ we shall give a complement to Statement (e) of [D10] recalled in Part 2, with a better argument valid for any non trivial ergodic dynamical system (avoiding the properties of stable probability laws used in [D10]).

Proposition 3.4. *On any non atomic ergodic system (\mathbb{X}, μ, T) there exist measurable functions f such that $\limsup A_n^+ f = +\infty$, $\liminf A_n^+ f = -\infty$ a.e. but $\lim A_n^- f = +\infty$ a.e..*

For the proof we need the following lemma.

Lemma 3.5. *On any non atomic ergodic system (\mathbb{X}, μ, T) and for every increasing sequence of integers $l(n)$ there exist measurable functions $v \geq 0$ such that $\limsup \frac{1}{l(n)} v \circ T^n = +\infty$ a.e..*

Proof. By Rokhlin's lemma (see [P83, p. 48]), for every integer n we can build a tower of height n^2 with basis U_n and complementary part V_n such that $\sum_n \mu(V_n) < \infty$: the sets $T^i U_n$ are pairwise disjoint for $0 \leq i \leq n^2$ and $V_n = (\cup_{0 \leq i \leq n^2} T^i U_n)^c$. Put $g_n = nl(n^2)$ on the roof of the n^{th} tower $T^{n^2} U_n$ and $g_n = 0$ elsewhere. Put $v = \sum_n g_n$; it is finite a.e. by Borel-Cantelli lemma since $\mu(U_n) \leq n^{-2}$. Again by Borel-Cantelli, a.e. $x \in \mathbb{X}$ belongs to all the towers except a finite number of them since $\sum_n \mu(V_n) < \infty$. If x belongs to the n^{th} tower there is an integer $i \leq n^2$ such that $T^i x \in T^{n^2} U_n$; thus $\max_{j \leq n^2} v(T^j x) \geq \max_{j \leq n^2} g_n(T^j x) \geq nl(n^2)$. Therefore $l(n^2)^{-1} \max_{j \leq n^2} v(T^j x) \geq n$ a.e. for n large enough, thus $\limsup \frac{1}{l(n)} \max_{j \leq n} v(T^j x) = +\infty$ a.e.. Since the sequence $l(n)$ is increasing, this implies at once the desired result.

Proof of Proposition 3.4. An example of a function f having the desired properties will be given by $f = u + v - v \circ T$ where u and v are non negative functions such that $\int u d\mu = +\infty$ and $\limsup \left(\frac{1}{n} v \circ T^n - A_n^+ u \right) = +\infty$ a.e.. Indeed, in this case we get $\lim A_n^- f = \lim (A_n^- u + \frac{1}{n} v \circ T^n) = +\infty$ a.e.. but $\limsup A_n^+ f = \limsup (A_n^+ u - \frac{1}{n} v \circ T^n) = +\infty$ a.e. (recall that since $v \geq 0$ is finite a.e. $\liminf \frac{1}{n} v \circ T^n = 0$ a.e.) and $\liminf A_n^+ f = -\limsup \left(\frac{1}{n} v \circ T^n - A_n^+ u \right) = -\infty$ a.e..

Let us build two functions u and v with these properties. By Lemma 3.5 there is a function $v \geq 0$ such that $\limsup n^{-2} (v \circ T^n) = +\infty$ a.e..

Let u be a non-negative function such that $u \notin L^1$ but $\sqrt{u} \in L^1$. By Birkhoff's ergodic theorem we know $\lim \frac{1}{n} \sum_0^{n-1} \sqrt{u} \circ T^i = l$ finite a.e.. Now for a non negative sequence x_i , if $\frac{1}{n} \sum_0^{n-1} x_i \rightarrow l$, then $\frac{x_n}{n} \rightarrow 0$ and

$$\frac{1}{n^2} \sum_0^{n-1} x_i^2 \leq \frac{\max_{i \leq n} x_i}{n} \left(\frac{1}{n} \sum_0^{n-1} x_i \right) \rightarrow 0.$$

Thus $\lim \frac{1}{n^2} \sum_0^{n-1} u \circ T^i = 0$ a.e..

For these two functions u and v , $\limsup \frac{1}{n} \left(\frac{1}{n} v \circ T^n - A_n^+ u \right) = +\infty$ a.e.

and *a fortiori* $\limsup \left(\frac{1}{n} v \circ T^n - A_n^+ u \right) = +\infty$ a.e.. So $f = u + v - v \circ T$ satisfies our claim. \square

Now we can describe synthetically the possible asymptotic behaviors of the sequences $A_n^+ f$, $A_n^- f$, $B_n f$, considered together.

Theorem 3.6. *On any ergodic dynamical system (\mathbb{X}, μ, T) , for a finite measurable function f there are only two typical situations:*

1) $\lim A_n^+ f = \lim A_n^- f = \lim B_n f = c$ a.e., i.e. the three sequences converge a.e. to the same constant which may be $\pm\infty$.

$$2) \begin{cases} \text{with } c \text{ any real value or } +\infty \\ \limsup A_n^+ f = c, \quad \liminf A_n^+ f = -\infty \\ \limsup A_n^- f = +\infty, \quad \liminf A_n^- f = c \\ \limsup B_n f = +\infty, \quad \liminf B_n f = -\infty \text{ a.e.} \end{cases}$$

(of course the situations derived from 2) by symmetry or sign changing can also occur).

In particular we see that the sequence of bilateral averages $B_n f$ converges a.e. to a finite or infinite limit if and only if the two sequences of unilateral

averages $A_n^+ f$ and $A_n^- f$ converge to the same limit. Moreover when the sequence $B_n f$ does not converge it cannot have a finite \limsup or \liminf , contrarily to $A_n^+ f$ or $A_n^- f$. If the system is non atomic taking two well chosen nonnegative functions u and v we have case 2) for $f = u - u \circ T + v \circ T - v$ with $c = +\infty$ and the convergence in measure to 0 of the three sequences.

The proof of Theorem 3.6 is an easy application of Theorem 4 together with results of [D10] recalled in Part 2.

4 Complementary results

It is natural to consider also asymmetric bilateral averages

$$B_n^q f = \frac{1}{2n+1+q} \sum_{i=-n}^{n+q} f \circ T^i \text{ with } q \text{ a fixed integer. Recall that } a \text{ priori}$$

the convergence of $\frac{1}{n} f \circ T^n$ is unknown. Lemma 3.2 can be reformulated for the asymmetric averages $B_n^q f$ with an entirely similar proof. Thus we get the following result parallel to Theorem 4:

Theorem 4.1. *Let f be a finite measurable function. If $\limsup B_n^q f < +\infty$ a.e. on a set of positive measure then*

$\lim A_n^+ f = \lim A_n^- f = \lim B_n f = \lim B_n^q f = c$ a.e. on the whole space \mathbb{X} with $c \in \mathbb{R}$ or $c = -\infty$. Convergences a.e. of $B_n f$ and $B_n^q f$ are equivalent.

Let us consider now the "residual" terms. *A priori* the invariance of the set of convergence a.e. of $B_n f$ was a problem since the convergence to 0 of $\frac{1}{n}(f \circ T^{-n} - f \circ T^{n+1})$ was unknown even on a part where the convergence of $B_n f$ could have been established. Now we can prove the following propositions.

Proposition 4.2. *For a finite measurable function f and a fixed integer q , if $\limsup \frac{1}{n} |f \circ T^{-n} - f \circ T^{n+q}| < +\infty$ a.e. on a set of positive measure then $\lim \frac{1}{n} f \circ T^n = 0$ a.e. on \mathbb{X} .*

Proof. Consider $B_n^{q-1}(f - f \circ T) = \frac{1}{2n+q}(f \circ T^{-n} - f \circ T^{n+q})$. Then the result follows from Theorem since $A_n^+(f - f \circ T) = \frac{1}{n}(f - f \circ T^n)$. \square

It appears less direct to get the same conclusion starting with $\frac{1}{n}(f \circ T^{-n} + f \circ T^n)$. We do not know how to deduce directly the next proposition from the results of Part 3. We shall give a proof similar to the one of Lemma 3.2.

Proposition 4.3. *Let f be a finite measurable function. If $\limsup \frac{1}{n} |f \circ T^{-n} + f \circ T^n| < +\infty$ a.e. on a set of positive measure then $\lim \frac{1}{n} f \circ T^n = 0$ a.e. on \mathbb{X} .*

First we need a lemma similar to Lemma 3.1 and even simpler, so we skip its proof.

Lemma 4.4. *Let a be a positive integer and Δ a set of integers with $\Delta \subset [2a, 3a)$. If $\text{card } \Delta/a > 3/4$ then every integer $i \in [a/4, a/2)$ can be written as $i = j - k$ with $j, k \in \Delta$.*

Proof of Proposition 4.3. With M and N large enough put $V = \bigcap_{n>N} \left\{ \frac{1}{n} |f \circ T^{-n} + f \circ T^n| < M \right\}$ and take p such that $\mu(\bigcup_{0 \leq i \leq p} T^{-i}V) > 3/4$. The set of passage times $\mathcal{W}(x)$ of the orbit of x in $(\bigcup_{0 \leq i \leq p} T^{-i}V)$ has an asymptotic density greater than $3/4$.

Assume that three integers a, k and j satisfy: $a > N + p$, k and $j \in \mathcal{W}(x)$ with $2a \leq k < j < 3a$. By definition of $\mathcal{W}(x)$ there exist non negative integers $r, s \leq p$ such that $T^{k+r}x \in V$ and $T^{j+s}x \in V$. For integers ζ and $\xi > N$ we have

$$|f(T^{k+r-\zeta}x) + f(T^{k+r+\zeta}x)| < \zeta M$$

and

$$|f(T^{j+s-\xi}x) + f(T^{j+s+\xi}x)| < \xi M.$$

The solutions ζ and ξ of $k + r + \zeta = j + s + \xi$ and $j + s - \xi = 2(j - k)$, are $\xi = 2k - j + s > a > N$ and $\zeta = \xi + j + s - k - r > a - r > N$. Hence the difference of the terms appearing in the two previous inequalities yields:

$$|f(T^{2(r-s)}x) - f(T^{2(j-k)}x)| < (\zeta + \xi)M < (6a + 3p)M$$

since $k + r - \zeta = 2(r - s)$ and $\zeta + \xi < 6a + 3p$.

By Lemma 4.4, since density of $\mathcal{W}(x) > 3/4$, for all a large enough every $n \in [a/4, a/2)$ can be written as $n = j - k$ with $k, j \in \mathcal{W}(x)$ and $2a < k < j < 3a$. Thus for such an n we get

$$\frac{1}{n} |f(T^{2n}x)| < \frac{4}{a} \left[(6a + 3p)M + \sum_{i=-2p}^{2p} |f(T^i x)| \right].$$

When a increases to ∞ the intervals $[a/4, a/2)$ cover a half-line of integers and we obtain

$$\limsup \frac{1}{n} |f \circ T^{2n}| < 24M < \infty \text{ a.e.}$$

since p depends only on the set V . By Statement (d), the desired result follows. \square

To conclude this part we extend to symmetric bilateral sums the (well-known) result that $\sup_{n \geq 0} |\sum_{i=0}^n f \circ T^i| < +\infty$ a.e. implies that f is a "bounded coboundary", that is $f = g - g \circ T$ with g a bounded measurable function. Since a precise reference seems difficult to give, we mention here that this result follows easily from Proposition 0: from $\sup_{n \geq 0} |\sum_{i=0}^n f \circ T^i| < +\infty$ a.e. we deduce $f = -F^- + F^+ - F^+ \circ T$ a.e.; then $\lim A_n^+ f = 0$ a.e. yields $F^- = 0$ and F^+ must be bounded since, otherwise, $\sup_{n \geq 0} |F^+ - F^+ \circ T^n| = +\infty$ a.e. by ergodicity.

Proposition 4.5. *Let f be a finite measurable function. If $\sup_{n \geq 0} |\sum_{i=-n}^n f \circ T^i| < +\infty$ a.e. on a set of positive measure then there exists a bounded measurable function g such that $f = g - g \circ T$.*

Proof. Let K be large enough for $\mu \left\{ \sup_{n \geq 0} |\sum_{i=-n}^n f \circ T^i| < K \right\} > 0$. Following the same argument as in Lemma 3.2 we get for a.e. $x \in \mathbb{X}$ and all n large enough $|\sum_{i=0}^{2n-1} f(T^i x)| < 2K + \sum_{i=-2p}^{2p} |f(T^i x)|$, where p is independent of x and n . Hence $\sup_{n \geq 0} |\sum_{i=0}^n f \circ T^i| < +\infty$ a.e. on \mathbb{X} . Then the conclusion follows from the result we just recalled above. \square

5 Infinite oscillations around the limit

For unilateral ergodic averages $A_n^+ f = \frac{1}{n} \sum_{i=0}^{n-1} f \circ T^i$ it is well known that infinite oscillations around the limit must occur; this was first established for L^1 -functions ([H76], see also [K85] §1.6.3 or [P83]). In [D10] it is proved for any measurable function for which the averages converge: the problem is first reduced to the case where f is a coboundary, that is of the form $f = g - g \circ T$ with g measurable, and then the result is an easy by-product of Poincaré's recurrence theorem. Here, for bilateral ergodic averages $B_n f$, we shall follow the same method but the conclusion will require some results of double recurrence due to Furstenberg [F77].

Theorem 5.1. *Let f be a finite measurable function. If $\lim B_n f = c$ a.e. with c finite, then the sequence $B_n f$ oscillates infinitely often around c a.e. (in the wide sense; that is the difference $B_n f - c$ cannot be ultimately strictly positive or ultimately strictly negative on a set of positive measure).*

Proof. Put $c = 0$. By Theorem 4, $\lim A_n^+ f = 0$ a.e..

Ad absurdum, suppose that for a.e. x in a set of positive measure, there exists N such that $B_n f(x) < 0$ for all $n > N$. Then by Lemma 3.2, we get $F = \sup_{n>0} \sum_{i=0}^{n-1} f \circ T^i < \infty$ a.e. on \mathbb{X} and by Proposition 0, $f = -F^- + F^+ - F^+ \circ T$ a.e.. Since $0 = \lim A_n^+ f = -\int F^- d\mu$ we obtain $F^- = 0$ a.e.. Therefore f must be a coboundary: $f = F^+ - F^+ \circ T$ and $B_n f = \frac{1}{2n+1} [F^+ \circ T^{-n} - F^+ \circ T^{n+1}]$.

But in this case it appears a contradiction with the following theorem, that we shall prove next, which completes the proof of Theorem 5.1. \square

This last theorem does not depend on the previous results, and might be of independent interest.

Theorem 5.2. *Let f be a measurable function on an ergodic invertible dynamical system (\mathbb{X}, μ, T) . The set of points x for which the strict inequality $f(T^{-n}x) < f(T^n x)$ holds for all n large enough, is negligible, that is*

$$\mu[\cup_{N>0} \cap_{n>N} \{f \circ T^{-n} < f \circ T^n\}] = 0.$$

The same statement holds for inequalities $f(T^{-n}x) < f(T^{n+q}x)$ with q fixed.

Before giving the proof we recall the elements of Furstenberg's multirecurrence theory for T and T^2 that we shall need ([F77], see the first three parts).

On an ergodic dynamical system (\mathbb{X}, μ, T) the averages $\frac{1}{n} \sum_{i=0}^{n-1} (u \circ T^i)(v \circ T^{2i})$ converge in $L^2(\mu)$ for every u and v bounded; to represent the limit the notion of maximal Kronecker factor is introduced. A factor of the ergodic dynamical system (\mathbb{X}, μ, T) is a Kronecker factor if it is a system (G, m, α) where G is a compact abelian "monothetic" group, m its Haar measure and $\alpha \in G$ acts on G by translation $\tau_\alpha : x \rightarrow x + \alpha$, the sequence $(n\alpha)_{n>0}$ being dense in G . For the maximal Kronecker factor there exists a positive linear operator π of $L^2(\mathbb{X}, \mu)$ onto $L^2(G, m)$ with $\pi(f \circ T) = \pi(f) \circ \tau_\alpha$, preserving the integral, such that for every bounded u, v and w the following identity, called Furstenberg's identity, holds:

$$\begin{aligned} & \lim_n \frac{1}{n} \sum_{i=0}^{n-1} \int_{\mathbb{X}} u(x)v(T^i x)w(T^{2i} x)d\mu(x) \\ &= \int \int_{G \times G} \pi u(z)\pi v(z + z')\pi w(z + 2z')dm(z)dm(z'). \end{aligned}$$

Here $2z = z + z$ defines a continuous endomorphism of the group G that we'll denote θ . The unique ergodicity of the Kronecker system is essential (see [F77] part 3, especially Lemma 3.4).

Proof of Theorem 5.2. If $\mu \{f = \text{ess sup } f\} > 0$ the result is easy. Indeed by ergodicity $T^{-n}x \in \{f = \text{ess sup } f\}$ for infinitely many n and a.e. x ; for them $f(T^{-n}x) \geq f(T^n x)$. If $\mu \{f = \text{ess inf } f\} > 0$ the dual argument applies.

Therefore we may assume $\mu \{f = \text{ess inf } f\} = 0$ or f unbounded below. In both cases there exist $a \in \mathbb{R}$ for which the sets $\{f \leq a\}$ have positive and arbitrarily small μ -measure.

Now let us take a fixed value $a \in \mathbb{R}$ with $0 < \mu \{f \leq a\} < 1$ (for the conclusion at the end of the proof we shall have to let $\mu \{f \leq a\} \rightarrow 0$).

Let E be the set considered in the statement :

$$E = \cup_{N>0} \cap_{n>N} \{f \circ T^{-n} < f \circ T^n\}.$$

Clearly if $x \in E$, for all n large enough $T^{-n}x \in \{f > a\}$ implies $T^n x \in \{f > a\}$. Thus $E \subset \cup_{N>0} E_N^a$ where

$$E_N^a = \cap_{n>N} (T^n \{f \leq a\} \cup T^{-n} \{f > a\}).$$

For every $n > N$, using the invariance of the measure we get :

$$\begin{aligned} & \mu [E_N^a \cap T^n \{f > a\} \cap T^{-n} \{f \leq a\}] \\ &= \mu [T^{-n} E_N^a \cap \{f > a\} \cap T^{-2n} \{f \leq a\}] = 0. \end{aligned}$$

Now we shall write Furstenberg's identity with $u = 1 - 1_{\{f \leq a\}}$, $v = 1_{E_N^a}$, $w = 1_{\{f \leq a\}}$. The two functions $\varphi_N^a = \pi v = \pi 1_{E_N^a}$ and $\psi^a = \pi w = \pi 1_{\{f \leq a\}}$ belong to $L^2(G, m)$ and satisfy $0 \leq \varphi_N^a \leq 1$ m -a.e., $\int_G \varphi_N^a dm = \mu(E_N^a)$, and $0 \leq \psi^a \leq 1$ m -a.e., $\int_G \psi^a dm = \mu \{f \leq a\}$. The identity together with the preceding equality yield:

$$\begin{aligned} 0 &= \lim_n \frac{1}{n} \sum_{i=0}^{n-1} \mu [T^{-i} E_N^a \cap \{f > a\} \cap T^{-2i} \{f \leq a\}] \\ &= \int \int_{G \times G} (1 - \psi^a(z)) \varphi_N^a(z + z') \psi^a(z + 2z') dm(z) dm(z') \end{aligned}$$

for every integer N .

Now put $\varphi = \pi 1_E$; it satisfies $\varphi \leq \pi 1_{E \setminus E_N^a} + \varphi_N^a$. Since $E \subset \cup_{N>0} E_N^a$, the sequence $(E_N^a)_N$ being increasing, we have

$\lim \int \pi 1_{E \setminus E_N^a} dm = \lim \mu (E \setminus E_N^a) = 0$, therefore we get

$$\int \int_{G \times G} (1 - \psi^a(z)) \varphi(z + z') \psi^a(z + 2z') dm(z) dm(z') = 0.$$

Then Haar measure properties yield :

$$\int_G \varphi(y) \int_G \psi^a(z)(1 - \psi^a(2y - z))dm(z)dm(y) = 0.$$

Now we shall argue by contradiction. Suppose $\mu(E) > 0$, which implies $m\{\varphi > 0\} > 0$. Since $0 \leq \varphi$ and $0 \leq \psi^a \leq 1$, for m -a.e. $y \in \{\varphi > 0\}$ we get $\psi^a(z)(1 - \psi^a(2y - z)) = 0$ for m -a.e. $z \in G$. Hence $\psi^a(z) > 0$ implies $\psi^a(2y - z) = 1$, that is $\{\psi^a > 0\} \subset (-A^a) + 2y$ where we denote $\{\psi^a = 1\} = A^a$. Thus $m(\{\psi^a > 0\}) = m(A^a)$ and $\psi^a = 1_{A^a}$ m -a.e.; moreover $A^a = (-A^a) + 2y$. In other words A^a is invariant by the map $z \rightarrow 2y - z$ for m -a.e. $y \in \{\varphi > 0\}$. Taking differences A^a is invariant by translations by elements of the set $(2\{\varphi > 0\}) - (2\{\varphi > 0\})$ and also of the generated subgroup. This subgroup is $2H = \theta H$ if we denote by H the subgroup generated by the set $\{\varphi > 0\} - \{\varphi > 0\}$. By our supposition $m\{\varphi > 0\} > 0$, the set $\{\varphi > 0\} - \{\varphi > 0\}$ is a neighborhood of 0 in G , so H is an open subgroup of G . At this point to deduce that the subgroup $2H$ is also open, a little digression is necessary.

The endomorphism $\theta : z \rightarrow 2z$ being continuous, its image $2G$ is a compact subgroup. Since G is monothetic the index of the subgroup $2G$ in G is 1 or 2 : $G = (2G) \cup (\alpha + 2G)$ the two classes being compact and equal or disjoint. Hence $2G$ is open. The subgroup H being open there is a smallest positive integer j such that $j\alpha \in H$ and the sequence $(nj\alpha)_n$ is dense in H , hence H is also monothetic. Since θ is also an endomorphism of H the subgroup $2H$ is open in H hence in G .

The index $[G : 2H]$ of the subgroup $2H$ in G is finite since $2H$ is open and G is compact, and $m(2H) = 1/[G : 2H]$. The set A^a is invariant by $2H$ hence it is union of classes of $2H$ and we obtain $\mu\{f \leq a\} = m(A^a) \geq 1/[G : 2H]$.

This is a contradiction since the subgroup $2H$ depends only on the set E and not on a , and, as was put at the beginning, $\mu\{f \leq a\}$ can be positive and arbitrarily small. Therefore $\mu(E) = 0$.

If instead of $f(T^{-n}x) < f(T^n x)$ we consider inequalities $f(T^{-n}x) < f(T^{n+q}x)$ with q fixed, the proof has to be slightly modified. The set E is replaced by

$$E_q = \cup_{N>0} \cap_{n>N} \{f \circ T^{-n} < f \circ T^{n+q}\}$$

and the sets E_N^a by

$$E_{N,q}^a = \cap_{n>N} (T^n \{f \leq a\} \cup T^{-n-q} \{f > a\}).$$

The key equality

$\int_G \varphi(y) \int_G \psi^a(z)(1 - \psi^a(2y - z))dm(z)dm(y) = 0$ becomes

$$\int_G \varphi_q(y) \int_G \psi^a(z)(1 - \psi^a(2y + q\alpha - z))dm(z)dm(y) = 0$$

with $\varphi_q = \pi 1_{E_q}$, since in Furstenberg's identity we have to replace $1_{\{f \leq a\}} = w$ by $1_{\{f \leq a\}} \circ T^q$ hence $\psi^a = \pi w$ by $\psi^a \circ \tau_{q\alpha}$. The analysis of this new equality remains almost the same: instead of $\{\psi^a > 0\} \subset (-A^a) + 2y$ we get $\{\psi^a > 0\} \subset (-A^a) + 2y + q\alpha$ and when we take differences of transformations $z \rightarrow 2y + q\alpha - z$, the term $q\alpha$ disappears. So Theorem 5.2 is completely proved. \square

Remarks. If the system is weakly mixing a shorter proof of Theorem 5.2 is possible.

In Theorems 5.1 and 5.2 the strict inequalities cannot be replaced by wide ones. As an example consider the group $G = \mathbb{Z}/3\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$ with $\alpha = (1, \epsilon)$ where ϵ is irrational; take $f(u, x) = 1$ if $u = 0$, 0 if $u = 1$, -1 if $u = 2$. Then it is clear that $B_n f = 0$ on $\{1\} \times \mathbb{R}/\mathbb{Z}$ for every $n \geq 0$.

In this work the assumption of ergodicity was only a simplification. For a non ergodic measure preserving dynamical system defined on a Lebesgue space, the results of this paper are easily reformulated using the ergodic decomposition of the invariant measure; in particular, without any change, Theorem 5.2 still holds.

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