



# Introduction to Higher Cubical Operads

Camell Kachour

## ► To cite this version:

| Camell Kachour. Introduction to Higher Cubical Operads. 2017. <hal-04306455>

**HAL Id: hal-04306455**

**<https://hal.science/hal-04306455v1>**

Preprint submitted on 25 Nov 2023

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



HAL Authorization

# Introduction to Higher Cubical Operads.

Camell Kachour

November 25, 2023

## Abstract

In this article, divided in two parts, we show how to build main aspects of the article [25] but with the cubical geometry. This first part is devoted to build the contractible  $\mathbb{S}$ -operad  $\mathbb{B}_C^0$  equipped with a cubical  $C^0$ -system, where  $\mathbb{S}$  is the monad of free strict cubical  $\infty$ -categories on cubical sets. Actions of this monad are on cubical sets with no notions of reflexivities (the classical and the connections) in order to be sure that it is cartesian (see [35]). In our approach, classical reflexivities plus connections, appear in the level of algebras. This operad is free on this  $C^0$ -system (which itself is a specific cubical pointed  $\mathbb{S}$ -collection). We exhibit some simple coherence cells of  $\mathbb{B}_C^0$  and show how they provide more richness compare to its globular analogue (see [25]).

In the second part of this article we use the cubical operad  $B_C^0$  of cubical weak  $\infty$ -categories (built in [34]) as a fundamental step to associate to any topological space  $X$  its fundamental cubical weak  $\infty$ -groupoids  $\Pi_\infty(X)$ , and this endows a functor  $\text{Top} \xrightarrow{\Pi_\infty(-)} \infty\text{-CGrp}$  which has a left adjoint functor  $CN_\infty$ . This pair of adjunction  $(CN_\infty, \Pi_\infty(-))$  should put an equivalence between the homotopy category of homotopy types and the homotopy category of  $\infty\text{-CGrp}$  of cubical weak  $\infty$ -groupoids with connections equipped with an adapted Quillen model structure. This was shown to be true but in the context of the Cisinski model structure on the category of cubical sets with connections (see [42]).

**Keywords.** cubical  $(\infty, n)$ -categories, weak cubical  $\infty$ -groupoids, homotopy types.

**Mathematics Subject Classification (2010).** 18B40, 18C15, 18C20, 18G55, 20L99, 55U35, 55P15.

## Contents

<b>1</b>	<b>Cubical sets</b>	<b>3</b>
1.1	The cubical category	3
1.2	Reflexive cubical sets	3
<b>2</b>	<b>The category of strict cubical <math>\infty</math>-categories</b>	<b>5</b>
2.1	Definition	7
2.2	The monad of cubical strict $\infty$ -categories	8
<b>3</b>	<b>The cubical higher operad of cubical weak <math>\infty</math>-categories</b>	<b>10</b>
3.1	The monoidal category of cubical pointed $\mathbb{S}$ -collections	10
3.2	Cubical contractions	11
3.3	The cubical operad of cubical weak $\infty$ -categories	13
<b>4</b>	<b>The Functor of Fundamental Cubical Weak <math>\infty</math>-Groupoids for Spaces</b>	<b>14</b>

<b>5</b>	<b>Cubical monoidal categories as Pseudo-algebras</b>	<b>16</b>
5.1	The cubical category . . . . .	16
<b>6</b>	<b>Cubical Higher Spans and Cubical Higher Cospans</b>	<b>19</b>
6.1	The pseudo-algebraic structure of $Span(C)$ . . . . .	19
6.2	$B_C^0$ -algebras and $B_C^0$ -coalgebras . . . . .	25
<b>7</b>	<b>Higher Cospans in <math>\mathbb{T}op</math></b>	<b>28</b>
7.1	The global object $box : I^\bullet$ . . . . .	28
7.2	$I^\bullet$ is a $B_C^0$ -coalgebra . . . . .	29
7.2.1	Composition systems on $\mathbb{COEND}(I^\bullet)$ . . . . .	29
7.2.2	Contractibility of $\mathbb{COEND}(I^\bullet)$ . . . . .	31
7.2.3	The fundamental morphism of operads . . . . .	33
7.2.4	Application for higher $K$ -theory . . . . .	34
<b>8</b>	<b>Importance of Coalgebraic structures for Globular and Cubical Higher Category Theory</b>	<b>35</b>
8.1	The Batanin's construction and the author's construction . . . . .	35
8.2	Steps toward the cubical weak $\infty$ -category of cubical weak $\infty$ -categories . . . . .	36

## Introduction

In this article, divided in two parts, we show how to build main aspects of the article [25] but with the cubical geometry. This first part is devoted to build the contractible  $\mathbb{S}$ -operad  $\mathbb{B}_C^0$  equipped with a cubical  $C^0$ -system, where  $\mathbb{S}$  is the monad of free strict cubical  $\infty$ -categories on cubical sets. Actions of this monad are on cubical sets with no notions of reflexivities (the classical and the connections) in order to be sure that it is cartesian (see [35]). In our approach, classical reflexivities plus connections, appear in the level of algebras. This operad is free on this  $C^0$ -system (which itself is a specific cubical pointed  $\mathbb{S}$ -collection). We exhibit some simple coherences cells of  $\mathbb{B}_C^0$  and show how they provide more richness compare to its globular analogue (see [25]). In the second part of this article (see [13]), we use it as a fundamental step to associate to any topological space  $X$  its fundamental cubical weak  $\infty$ -groupoids  $\Pi_\infty(X)$ , and this endows a functor  $Top \xrightarrow{\Pi_\infty(-)} \infty\text{-}\mathbb{C}Grp$  which has a left adjoint functor  $CN_\infty$ . This pair of adjunction  $(CN_\infty, \Pi_\infty(-))$  should put an equivalence between the homotopy category of homotopy types and the homotopy category of  $\infty\text{-}\mathbb{C}Grp$  of cubical weak  $\infty$ -groupoids equipped with an adapted Quillen model structure. This was shown to be true but in the context of the Cisinski model structure on the category of cubical sets with connections (see [42]).

**Acknowledgement.** This work was doing thanks to the beautiful ambiance I got during my two months visiting in the IHES. I especially want to mention Maxim Kontsevich whose questions on my work were deep and push me to clarify a lot of aspects of my work. I want also to mention Pierre Cartier, who patiently, has taken his time to understand my work and who shared with me many of his very deep point of views of Higher Category Theory. During my stay, Vasily Pestun has shared with me some deep questions which relate mathematics and theoretical physics. I was especially impressed by the huge imagination that a physicist can have, starting with simple questions in mathematics, but which answers is highly non-trivial and should open new mathematics, that mathematicians cannot see when "jailed" in mathematics. I cannot also forget my friendly discussions with Thibault Damour ! I want also to point out how Laurent Lafforgue, Fanny Kassel, Emmanuel Ullmo and Elisabeth Jasserand, where all very kind with me. Many other mathematicians or physicists open my eyes on other point of views of science. I want to mention Thiago Araujo (Strings theory), Yang Liu (Non-commutative geometry), and Benjamin Ward (Operads and homology). Last but not least, this work wouldn't exist without my natural interactions with Ross Street and Mark Weber.

# 1 Cubical sets

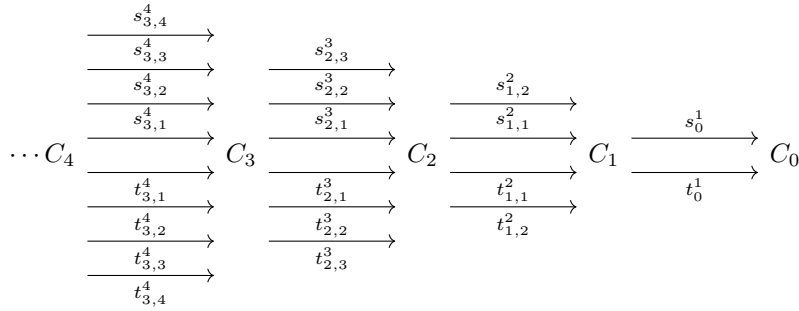
See also [11] for more references on cubical sets.

## 1.1 The cubical category

Consider the small category  $\mathbb{C}$  with integers  $\underline{n} \in \mathbb{N}$  as objects. Generators for  $\mathbb{C}$  are, for all  $\underline{n} \in \mathbb{N}$  given by *sources*  $\underline{n} \xrightarrow{s_{n-1,j}^{n-1}} \underline{n-1}$  for each  $j \in \{1, \dots, n\}$  and *targets*  $\underline{n} \xrightarrow{t_{n-1,j}^n} \underline{n-1}$  for each  $j \in \{1, \dots, n\}$  such that for  $1 \leq i < j \leq n$  we have the following cubical relations

- (i)  $s_{n-2,i}^{n-1} \circ s_{n-1,j}^n = s_{n-2,j-1}^{n-1} \circ s_{n-1,i}^n$ ,
- (ii)  $s_{n-2,i}^{n-1} \circ t_{n-1,j}^n = t_{n-2,j-1}^{n-1} \circ s_{n-1,i}^n$ ,
- (iii)  $t_{n-2,i}^{n-1} \circ s_{n-1,j}^n = s_{n-2,j-1}^{n-1} \circ t_{n-1,i}^n$ ,
- (iv)  $t_{n-2,i}^{n-1} \circ t_{n-1,j}^n = t_{n-2,j-1}^{n-1} \circ t_{n-1,i}^n$

These generators plus these relations give the small category  $\mathbb{C}$  called the *cubical category* that we may represent schematically with the low dimensional diagram :



and this category  $\mathbb{C}$  gives also the sketch  $\mathcal{E}_S$  of cubical sets used especially in 2.2, ?? and ?? to produce the monads  $\mathbb{S} = (S, \lambda, \mu)$ ,  $\mathbb{W} = (W, \eta, \nu)$  and  $\mathbb{W}^m = (W^m, \eta^m, \nu^m)$  on  $\mathbb{C}\text{Sets}$ , which algebras are respectively cubical strict  $\infty$ -categories, cubical weak  $\infty$ -categories and cubical weak  $(\infty, m)$ -categories.

**Definition 1** The category of cubical sets  $\mathbb{C}\text{Sets}$  is the category of presheaves  $[\mathbb{C}; \text{Sets}]$ . The terminal cubical set is denoted 1.  $\square$

Occasionally a cubical set shall be denoted with the notation

$$\mathcal{C} = (C_n, s_{n-1,j}^n, t_{n-1,j}^n)_{1 \leq j \leq n, n \in \mathbb{N}}$$

in case we want to point out its underlying structures.

## 1.2 Reflexive cubical sets

Reflexivity for cubical sets are of two sorts : one is "classical" in the sense that they are very similar to their globular analogue; thus we shall use the notation  $(1_{n+1,j}^n)_{n \in \mathbb{N}, j \in \{1, \dots, n\}}$  to denote these maps  $C(n) \xrightarrow{1_{n+1,j}^n} C(n+1)$  which formally behave like globular reflexivity ([15]); the others are called *connections* and are given by maps  $C(n) \xrightarrow{\Gamma} C(n+1)$  where the notation using the greek letter "*Gamma*" seems to be the usual notation.

However we do prefer to use instead the notation  $C(n) \xrightarrow{1_{n+1,j}^{n,\gamma}} C(n+1)$  ( $\gamma \in \{+, -\}$ ) in order to point out the reflexive nature of connections.

Consider the cubical category  $\mathbb{C}$ . For all  $n \in \mathbb{N}$  we add in it generators  $\underline{n-1} \xrightarrow{1_{n,j}^{n-1}} \underline{n}$  for each  $j \in \{1, \dots, n\}$  subject to the relations :

- (i)  $1_{n+1,i}^n \circ 1_{n,j}^{n-1} = 1_{n+1,j+1}^n \circ 1_{n,i}^{n-1}$  if  $1 \leq i \leq j \leq n$ ;
- (ii)  $s_{n-1,i}^n \circ 1_{n,j}^{n-1} = 1_{n-1,j-1}^{n-2} \circ s_{n-2,i}^{n-1}$  if  $1 \leq i < j \leq n$ ;
- (iii)  $s_{n-1,i}^n \circ 1_{n,j}^{n-1} = 1_{n-1,j}^{n-2} \circ s_{n-2,i-1}^{n-1}$  if  $1 \leq j < i \leq n$ ;
- (iv)  $s_{n-1,i}^n \circ 1_{n,j}^{n-1} = id(\underline{n-1})$  if  $i = j$ .

- (i)  $1_{n+1,i}^n \circ 1_{n,j}^{n-1} = 1_{n+1,j+1}^n \circ 1_{n,i}^{n-1}$  if  $1 \leq i \leq j \leq n$ ;
- (ii)  $t_{n-1,i}^n \circ 1_{n,j}^{n-1} = 1_{n-1,j-1}^{n-2} \circ t_{n-2,i}^{n-1}$  if  $1 \leq i < j \leq n$ ;
- (iii)  $t_{n-1,i}^n \circ 1_{n,j}^{n-1} = 1_{n-1,j}^{n-2} \circ t_{n-2,i-1}^{n-1}$  if  $1 \leq j < i \leq n$ ;
- (iv)  $t_{n-1,i}^n \circ 1_{n,j}^{n-1} = id(\underline{n-1})$  if  $i = j$ .

These generators and relations give the small category  $\mathbb{C}_{sr}$  called the *semireflexive cubical category* where a quick look at its underlying semireflexive structure is given by the following diagram :

$$\begin{array}{ccccccc}
 & & & & & \xrightarrow{1_{4,3}^3} & \\
 & & & & \xrightarrow{1_{3,2}^2} & & \\
 & & & \xrightarrow{1_{2,1}^1} & & \xrightarrow{1_{4,1}^3} & \\
 C_0 & \xrightarrow{1_1^0} & C_1 & \xrightarrow{1_{2,2}^1} & C_2 & \xrightarrow{1_{3,1}^2} & C_3 & \xrightarrow{1_{4,2}^3} & C_4 \cdots \\
 & & & \xrightarrow{1_{2,2}^1} & & \xrightarrow{1_{3,2}^2} & & \xrightarrow{1_{4,3}^3} & \\
 & & & & & \xrightarrow{1_{3,3}^2} & & \xrightarrow{1_{4,4}^3} & 
 \end{array}$$

**Definition 2** The category of semireflexive cubical sets  $\mathbb{C}_{sr}\text{Sets}$  is the category of presheaves  $[\mathbb{C}_{sr}; \text{Sets}]$ . The terminal semireflexive cubical set is denoted  $1_{sr}$  □

Consider the semireflexive cubical category  $\mathbb{C}_{sr}$ . For all integers  $n \geq 1$  we add in it generators  $\underline{n-1} \xrightarrow{1_{n,j}^{n-1,\gamma}} \underline{n}$  for each  $j \in \{1, \dots, n-1\}$  subject to the relations :

- (i) for  $1 \leq j < i \leq n$ ,  $1_{n+1,i}^{n,\gamma} \circ 1_{n,j}^{n-1,\gamma} = 1_{n+1,j+1}^{n,\gamma} \circ 1_{n,i}^{n-1,\gamma}$ ;
- (ii) for  $1 \leq i \leq n-1$ ,  $1_{n+1,i}^{n,\gamma} \circ 1_{n,i}^{n-1,\gamma} = 1_{n+1,i+1}^{n,\gamma} \circ 1_{n,i}^{n-1,\gamma}$ ;
- (iii) for  $1 \leq i, j \leq n$ ,  $\begin{cases} 1_{n+1,i}^{n,\gamma} \circ 1_{n,j}^{n-1} &= 1_{n+1,j+1}^{n,\gamma} \circ 1_{n,i}^{n-1,\gamma} & \text{if } 1 \leq i < j \leq n \\ &= 1_{n+1,j}^{n,\gamma} \circ 1_{n,i-1}^{n-1,\gamma} & \text{if } 1 \leq j < i \leq n \end{cases}$ ;
- (iv) for  $1 \leq j \leq n$ ,  $1_{n+1,j}^{n,\gamma} \circ 1_{n,j}^{n-1} = 1_{n+1,j}^{n,\gamma} \circ 1_{n,j}^{n-1}$ ;
- (v) for  $1 \leq i, j \leq n$ ,  
 $\begin{cases} s_{n-1,i}^{n-1,\gamma} \circ 1_{n,j}^{n-1,\gamma} &= 1_{n-1,j-1}^{n-2,\gamma} \circ s_{n-2,i}^{n-1} & \text{if } 1 \leq i < j \leq n-1 \\ &= 1_{n-1,j}^{n-2,\gamma} \circ s_{n-2,i-1}^{n-1} & \text{if } 2 \leq j+1 < i \leq n \end{cases}$ ;  
and  
 $\begin{cases} t_{n-1,i}^{n-1,\gamma} \circ 1_{n,j}^{n-1,\gamma} &= 1_{n-1,j-1}^{n-2,\gamma} \circ t_{n-2,i}^{n-1} & \text{if } 1 \leq i < j \leq n-1 \\ &= 1_{n-1,j}^{n-2,\gamma} \circ t_{n-2,i-1}^{n-1} & \text{if } 2 \leq j+1 < i \leq n \end{cases}$ ;
- (vi) for  $1 \leq j \leq n-1$ ,  $s_{n-1,j}^{n-1,-} \circ 1_{n,j}^{n-1,-} = s_{n-1,j+1}^{n-1,-} \circ 1_{n,j}^{n-1,-} = 1_{n-1}$  and  $t_{n-1,j}^{n-1,+} \circ 1_{n,j}^{n-1,+} = t_{n-1,j+1}^{n-1,+} \circ 1_{n,j}^{n-1,+} = 1_{n-1}$ ;
- (vii) for  $1 \leq j \leq n-1$ ,  $s_{n-1,j}^{n-1,+} \circ 1_{n,j}^{n-1,+} = s_{n-1,j+1}^{n-1,+} \circ 1_{n,j}^{n-1,+} = 1_{n-1,j}^{n-2} \circ s_{n-2,j}^{n-1}$ ;
- (viii) for  $1 \leq j \leq n-1$ ,  $t_{n-1,j}^{n-1,-} \circ 1_{n,j}^{n-1,-} = t_{n-1,j+1}^{n-1,-} \circ 1_{n,j}^{n-1,-} = 1_{n-1,j}^{n-2} \circ t_{n-2,j}^{n-1}$ .

These generators and relations give the small category  $\mathbb{C}_r$  called the *reflexive cubical category* and in it, connections have the following shape :

$$\begin{array}{ccccccc}
& & & & & & 1_{5,4}^{4,-} \\
& & & & & & \longrightarrow \\
& & & & & & 1_{5,3}^{4,-} \\
& & & & & & \longrightarrow \\
& & & & & & 1_{5,2}^{4,-} \\
& & & & & & \longrightarrow \\
& & & & & & 1_{5,1}^{4,-} \\
& & & & & & \longrightarrow \\
C_1 & \xrightarrow{1_{2,1}^{1,-}} & C_2 & \xrightarrow{1_{3,2}^{2,-}} & C_3 & \xrightarrow{1_{4,3}^{3,-}} & C_3 & \xrightarrow{1_{5,4}^{4,-}} & C_4 \cdots \\
& \xrightarrow{1_{2,1}^{1,+}} & & \xrightarrow{1_{3,1}^{2,-}} & & \xrightarrow{1_{4,2}^{3,-}} & & \xrightarrow{1_{5,3}^{4,-}} & \\
& & & \xrightarrow{1_{3,1}^{2,+}} & & \xrightarrow{1_{4,1}^{3,-}} & & \xrightarrow{1_{5,1}^{4,-}} & \\
& & & \xrightarrow{1_{3,2}^{2,+}} & & \xrightarrow{1_{4,1}^{3,+}} & & \xrightarrow{1_{5,1}^{4,+}} & \\
& & & & & \xrightarrow{1_{4,2}^{3,+}} & & \xrightarrow{1_{5,2}^{4,+}} & \\
& & & & & \xrightarrow{1_{4,3}^{3,+}} & & \xrightarrow{1_{5,3}^{4,+}} & \\
& & & & & & & \xrightarrow{1_{5,4}^{4,+}} & 
\end{array}$$

**Definition 3** The category of reflexive cubical sets  $\mathbb{C}_r\text{Sets}$  is the category of presheaves  $[\mathbb{C}_r; \text{Sets}]$ . The terminal reflexive cubical set is denoted  $1_r$  □

## 2 The category of strict cubical $\infty$ -categories

Cubical strict  $\infty$ -categories have been studied in [21, 27].

In [27] the authors proved that the category of cubical strict  $\infty$ -categories with cubical strict  $\infty$ -functors as morphisms is equivalent to the category of globular strict  $\infty$ -categories with globular strict  $\infty$ -functors as morphisms. Consider a cubical reflexive set

$$(C, (1_{n+1,j}^n)_{n \in \mathbb{N}, j \in \llbracket 1, n+1 \rrbracket}, (1_{n+1,j}^{n,\gamma})_{n \geq 1, j \in \llbracket 1, n \rrbracket})$$

equipped with partial operations  $(\circ_j^n)_{n \geq 1, j \in \llbracket 1, n \rrbracket}$  where if  $a, b \in C(n)$  then  $a \circ_j^n b$  is defined for  $j \in \{1, \dots, n\}$  if  $s_j^n(b) = t_j^n(a)$ . We also require these operations to follow the following axioms of positions :

(i) For  $1 \leq j \leq n$  we have :  $s_{n-1,j}^n(a \circ_j^n b) = s_{n-1,j}^n(a)$  and  $t_{n-1,j}^n(a \circ_j^n b) = t_{n-1,j}^n(a)$ ,

(ii)  $s_{n-1,i}^n(a \circ_j^n b) = \begin{cases} s_{n-1,i}^n(a) \circ_{j-1}^{n-1} s_{n-1,i}^n(b) & \text{if } 1 \leq i < j \leq n \\ s_{n-1,i}^n(a) \circ_j^{n-1} s_{n-1,i}^n(b) & \text{if } 1 \leq j < i \leq n \end{cases}$

(iii)  $t_{n-1,i}^n(a \circ_j^n b) = \begin{cases} t_{n-1,i}^n(a) \circ_{j-1}^{n-1} t_{n-1,i}^n(b) & \text{if } 1 \leq i < j \leq n \\ t_{n-1,i}^n(a) \circ_j^{n-1} t_{n-1,i}^n(b) & \text{if } 1 \leq j < i \leq n \end{cases}$

The following sketch  $\mathcal{E}_M$  of *axioms of positions* as above shall be used in 2.2 to justify the existence of the monad on  $\mathbb{C}\text{Sets}$  of cubical strict  $\infty$ -categories. It is important to notice that the sketch just below has only *one generation* which means that diagrams and cones involved in it are not build with previous data of other diagrams and cones.

- For  $1 \leq i < j \leq n$  we consider the following two cones :

$$\begin{array}{ccc} M_n \times_{M_{n-1,j}} M_n & \xrightarrow{\pi_{1,j}^n} & M_n \\ \pi_{0,j}^n \downarrow & & \downarrow s_{n-1,j}^n \\ M_n & \xrightarrow{t_{n-1,j}^n} & M_{n-1} \end{array} \quad \begin{array}{ccc} M_{n-1} \times_{M_{n-2,j-1}} M_{n-1} & \xrightarrow{\pi_{1,j-1}^{n-1}} & M_{n-1} \\ \pi_{0,j-1}^{n-1} \downarrow & & \downarrow s_{n-2,j-1}^{n-1} \\ M_{n-1} & \xrightarrow{t_{n-2,j-1}^{n-1}} & M_{n-2} \end{array}$$

and the following commutative diagram (definition of  $s_{n-1,i}^n \times_{j,j-1} s_{n-1,i}^n$ )

$$\begin{array}{ccccc} M_n \times_{M_{n-1,j}} M_n & \xrightarrow{\pi_{1,j}^n} & M_n & & \\ \pi_{0,j}^n \downarrow & \searrow s_{n-1,i}^n \times_{j,j-1} s_{n-1,i}^n & & \searrow s_{n-1,i}^n & \\ & M_{n-1} \times_{M_{n-2,j-1}} M_{n-1} & \xrightarrow{\pi_{1,j-1}^{n-1}} & M_{n-1} & \\ & \pi_{0,j-1}^{n-1} \downarrow & & \downarrow s_{n-2,j-1}^{n-1} & \\ M_n & \xrightarrow{s_{n-1,i}^n} & M_{n-1} & \xrightarrow{t_{n-2,j-1}^{n-1}} & M_{n-2} \end{array}$$

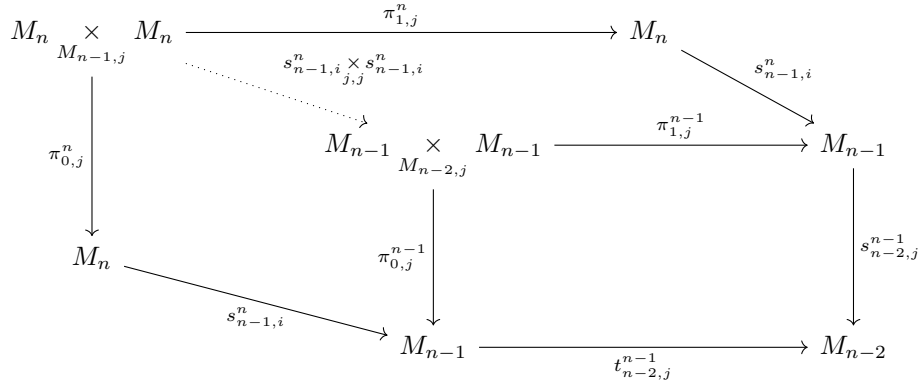
which gives the following commutative diagram

$$\begin{array}{ccc} M_n \times_{M_{n-1,j}} M_n & \xrightarrow{s_{n-1,i}^n \times_{j,j-1} s_{n-1,i}^n} & M_{n-1} \times_{M_{n-2,j-1}} M_{n-1} \\ \star_j^n \downarrow & & \downarrow \star_{j-1}^{n-1} \\ M_n & \xrightarrow{s_{n-1,i}^n} & M_{n-1} \end{array}$$

- For  $1 \leq j < i \leq n$  we consider the following two cones :

$$\begin{array}{ccc} M_n \times_{M_{n-1,j}} M_n & \xrightarrow{\pi_{1,j}^n} & M_n \\ \pi_{0,j}^n \downarrow & & \downarrow s_{n-1,j}^n \\ M_n & \xrightarrow{t_{n-1,j}^n} & M_{n-1} \end{array} \quad \begin{array}{ccc} M_{n-1} \times_{M_{n-2,j}} M_{n-1} & \xrightarrow{\pi_{1,j}^{n-1}} & M_{n-1} \\ \pi_{0,j}^{n-1} \downarrow & & \downarrow s_{n-2,j}^{n-1} \\ M_{n-1} & \xrightarrow{t_{n-2,j}^{n-1}} & M_{n-2} \end{array}$$

and the following commutative diagram (definition of  $s_{n-1,i}^n \times_{j,j} s_{n-1,i}^n$ )



The previous datas gives the following commutative diagram of axioms

$$\begin{array}{ccc}
M_n \times_{M_{n-1}} M_n & \xrightarrow{s_{n-1,i}^n \times_{j,j} s_{n-1,i}^n} & M_{n-1} \times_{M_{n-2}} M_{n-1} \\
\star_j^n \downarrow & & \downarrow \star_j^{n-1} \\
M_n & \xrightarrow{s_{n-1,i}^n} & M_{n-1}
\end{array}$$

and for  $1 \leq j \leq n$  we have the following commutative diagram of axioms

$$\begin{array}{ccc}
M_n \times_{M_{n-1}} M_n & \xrightarrow{\pi_1} & M_n \\
\star_j^n \downarrow & & \downarrow s_{n-1,j}^n \\
M_n & \xrightarrow{s_{n-1,j}^n} & M_{n-1}
\end{array}$$

which actually complete the description of  $\mathcal{E}_M$

**Definition 4** Cubical reflexive  $\infty$ -magmas are cubical reflexive set equipped with partial operations like just above which follow axioms of positions. A morphism between two cubical reflexive  $\infty$ -magmas is a morphism of their underlying cubical reflexive sets. The category of cubical reflexive  $\infty$ -magmas is noted  $\infty\text{-CMag}_r$ .

**Remark 1** Cubical  $\infty$ -magmas are poorer structure : they are cubical sets equipped with partial operations like above with these axioms of positions. A morphism between two cubical  $\infty$ -magmas is a morphism of their underlying cubical sets. The category of cubical  $\infty$ -magmas is noted  $\infty\text{-CMag}$   $\square$

## 2.1 Definition

Strict cubical  $\infty$ -categories are cubical reflexive  $\infty$ -magmas such that partials operations are associative and also we require the following axioms :



(i) The interchange laws :  $(a \circ_i^n b) \circ_j^n (c \circ_i^n d) = (a \circ_j^n c) \circ_i^n (b \circ_j^n d)$  whenever both sides are defined

(ii)  $1_{n+1,i}^n(a \circ_j^n b) = 1_{n+1,i}^n(a) \circ_{j+1}^{n+1} 1_{n+1,i}^n(b)$  if  $1 \leq i \leq j \leq n$   
 $1_{n+1,i}^n(a \circ_j^n b) = 1_{n+1,i}^n(a) \circ_j^{n+1} 1_{n+1,i}^n(b)$  if  $1 \leq j < i \leq n+1$

(iii)  $1_{n+1,i}^{n,\gamma}(a \circ_j^n b) = 1_{n+1,i}^{n,\gamma}(a) \circ_{j+1}^{n+1} 1_{n+1,i}^{n,\gamma}(b)$  if  $1 \leq i < j \leq n$   
 $1_{n+1,i}^{n,\gamma}(a \circ_j^n b) = 1_{n+1,i}^{n,\gamma}(a) \circ_j^{n+1} 1_{n+1,i}^{n,\gamma}(b)$  if  $1 \leq j < i \leq n$

(iv) First transport laws : for  $1 \leq j \leq n$

$$1_{n+1,j}^{n,+}(a \circ_j^n b) = \begin{bmatrix} 1_{n+1,j}^{n,+}(a) & 1_{n+1,j}^n(a) \\ 1_{n+1,j+1}^n(a) & 1_{n+1,j}^{n,+}(b) \end{bmatrix}$$

(v) Second transport laws : for  $1 \leq j \leq n$

$$1_{n+1,j}^{n,-}(a \circ_j^n b) = \begin{bmatrix} 1_{n+1,j}^{n,-}(a) & 1_{n+1,j+1}^n(b) \\ 1_{n+1,j}^n(b) & 1_{n+1,j}^{n,-}(b) \end{bmatrix}$$

(vi) for  $1 \leq j \leq n$ ,  $1_{n+1,i}^{n,+}(x) \circ_i^{n+1} 1_{n+1,i}^{n,-}(x) = 1_{n+1,i+1}^n(x)$  and  $1_{n+1,i}^{n,+}(x) \circ_{i+1}^{n+1} 1_{n+1,i}^{n,-}(x) = 1_{n+1,i}^n(x)$

The category  $\infty\text{-CCAT}$  of strict cubical  $\infty$ -categories is the full subcategory of  $\infty\text{-CMag}$ , spanned by strict cubical  $\infty$ -categories. A morphism in  $\infty\text{-CCAT}$  is called a *strict cubical  $\infty$ -functor*. We study it more specifically in ?? with the perspective to weakened it and to obtain cubical model of weak  $\infty$ -functors.

## 2.2 The monad of cubical strict $\infty$ -categories

In this section we describe cubical strict  $\infty$ -categories as algebras for a monad on  $\mathbb{C}\text{Sets}$ . We hope it to be a specific ingredient to compare globular strict  $\infty$ -categories with cubical strict  $\infty$ -categories

Consider the forgetful functor :  $\infty\text{-CCAT} \xrightarrow{U} \mathbb{C}\text{Sets}$  which associate to any strict cubical  $\infty$ -category its underlying cubical set and which associate to any strict cubical  $\infty$ -functor its underlying morphism of cubical sets.

**Proposition 1** *The functor  $U$  is right adjoint* □

Its left adjoint is denoted  $F$

PROOF We are going to use Sketch Theory as explain in [8] : Actually it is not difficult to see that the category  $\infty\text{-CCAT}$  and the category  $\mathbb{C}\text{Sets}$  are both projectively sketchable. Let us denote by  $\mathcal{E}_C$  the sketch of  $\infty\text{-CCAT}$  and  $\mathcal{E}_S$  the sketch of  $\mathbb{C}\text{Sets}$ . Main parts of  $\mathcal{E}_C$  are described just below and we see that  $\mathcal{E}_C$  contains  $\mathcal{E}_S$ , and that this inclusion induces a forgetful functor  $\infty\text{-CCAT} \xrightarrow{U} \mathbb{C}\text{Sets}$  which has a left adjunction thanks to the sheafification theorem of Foltz [10]. Now we have the commutative diagram

$$\begin{array}{ccc} \text{Mod}(\mathcal{E}_C) & \xrightarrow{\quad} & \text{Mod}(\mathcal{E}_S) \\ \text{iso} \downarrow & & \downarrow \text{iso} \\ \infty\text{-CCAT} & \xrightarrow{U} & \mathbb{C}\text{Sets} \end{array}$$

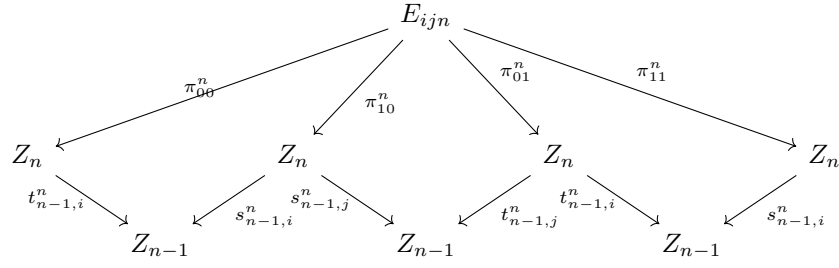
which shows that  $U$  is right adjoint.

The description of  $\mathcal{E}_C$  started with the description of  $\mathcal{E}_M$  in 2. We carry on to it in describing the sketch behind the interchange laws, which shall complete main parts of  $\mathcal{E}_C$  :

- In the first generation of  $\mathcal{E}_C$  we start with three cones :

$$\begin{array}{ccc}
Z_n \times_{Z_{n-1}, i} Z_n & \xrightarrow{\rho_{1,i}^n} & Z_n \\
\rho_{0,i}^n \downarrow & & \downarrow s_{n-1,i}^n \\
Z_n & \xrightarrow{t_{n-1,i}^n} & Z_{n-1}
\end{array}$$

$$\begin{array}{ccc}
Z_n \times_{Z_{n-1}, j} Z_n & \xrightarrow{\rho_{1,j}^n} & Z_n \\
\rho_{0,j}^n \downarrow & & \downarrow s_{n-1,j}^n \\
Z_n & \xrightarrow{t_{n-1,j}^n} & Z_{n-1}
\end{array}$$



- Then we consider the following commutative diagrams :

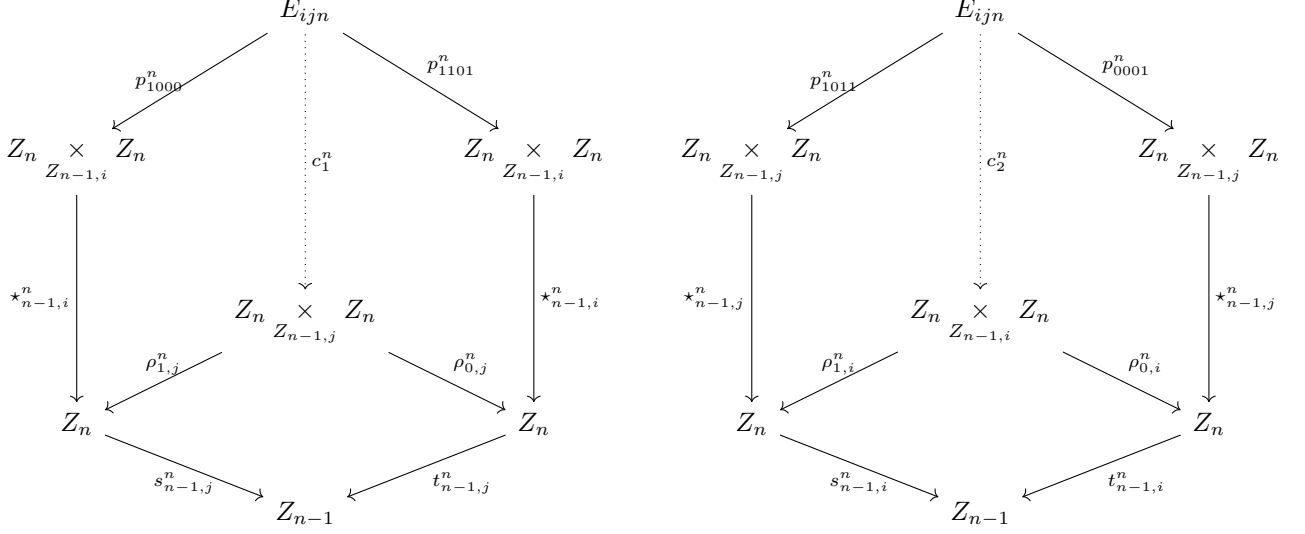
$$\begin{array}{ccccc}
E_{ijn} & & & & \\
& \searrow p_{1000}^n & \searrow \pi_{10}^n & & \\
& & Z_n \times_{Z_{n-1}, i} Z_n & \xrightarrow{\rho_{1,i}^n} & Z_n \\
& \searrow \pi_{00}^n & \downarrow \rho_{0,i}^n & & \downarrow s_{n-1,i}^n \\
& & Z_n & \xrightarrow{t_{n-1,i}^n} & Z_n
\end{array}$$

$$\begin{array}{ccccc}
E_{ijn} & & & & \\
& \searrow p_{1101}^n & \searrow \pi_{11}^n & & \\
& & Z_n \times_{Z_{n-1}, i} Z_n & \xrightarrow{\rho_{1,i}^n} & Z_n \\
& \searrow \pi_{01}^n & \downarrow \rho_{0,i}^n & & \downarrow s_{n-1,i}^n \\
& & Z_n & \xrightarrow{t_{n-1,i}^n} & Z_n
\end{array}$$

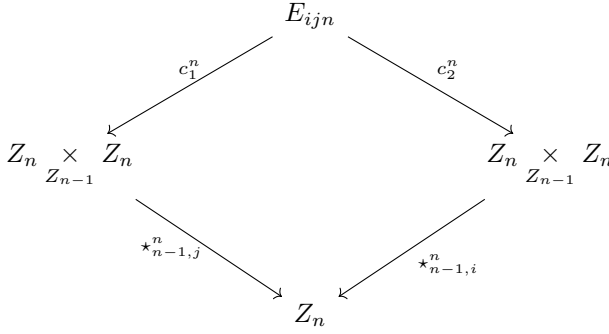
$$\begin{array}{ccccc}
E_{ijn} & & & & \\
& \searrow p_{1011}^n & \searrow \pi_{10}^n & & \\
& & Z_n \times_{Z_{n-1}, j} Z_n & \xrightarrow{\rho_{1,j}^n} & Z_n \\
& \searrow \pi_{11}^n & \downarrow \rho_{0,j}^n & & \downarrow s_{n-1,j}^n \\
& & Z_n & \xrightarrow{t_{n-1,j}^n} & Z_n
\end{array}$$

$$\begin{array}{ccccc}
E_{ijn} & & & & \\
& \searrow p_{0001}^n & \searrow \pi_{00}^n & & \\
& & Z_n \times_{Z_{n-1}, j} Z_n & \xrightarrow{\rho_{1,j}^n} & Z_n \\
& \searrow \pi_{01}^n & \downarrow \rho_{0,j}^n & & \downarrow s_{n-1,j}^n \\
& & Z_n & \xrightarrow{t_{n-1,j}^n} & Z_n
\end{array}$$

- We consider then (still in the first generation) the following two commutative diagrams :



- Finally we consider the following commutative diagram of interchange laws



The monad of strict cubical  $\infty$ -categories on cubical sets is denoted  $\mathbb{S} = (S, \lambda, \mu)$ . Here  $\lambda$  is the unit map of  $\mathbb{S} : 1_{\mathbb{C}\text{Sets}} \xrightarrow{\lambda} S$  and  $\mu$  is the multiplication of  $\mathbb{S} : S^2 \xrightarrow{\mu} S$

### 3 The cubical higher operad of cubical weak $\infty$ -categories

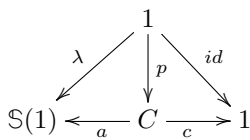
#### 3.1 The monoidal category of cubical pointed $\mathbb{S}$ -collections

In [35] we will prove that the free cubical strict  $\infty$ -categories monad  $\mathbb{S} = (S, \lambda, \mu)$  on  $\mathbb{C}\text{Sets}$  built in 2.2 is cartesian. Thanks to this cartesianess we can build the monoidal category  $\mathbb{S}\text{-Coll}_p$  of pointed  $\mathbb{S}$ -collections

If  $S$  is a cartesian monad on a category  $\mathcal{G}$  then  $S$ -collections are kind of  $S$ -graphs defined in [18], where their domains of arities is an object  $S(1)$  such that  $1$  is a terminal object of the category  $\mathcal{G}$ . The category of  $S$ -collections is denoted  $S\text{-Coll}$ . The category of pointed  $S$ -collections is also defined in [18] and is denoted  $S\text{-Coll}_p$ . In this section we accept the following result

**Conjecture** *The monad  $\mathbb{S} = (S, \lambda, \mu)$  (see 2.2) of strict cubical  $\infty$ -categories on cubical sets is cartesian*

Thus we can work with the locally finitely presentable category  $\mathbb{S}\text{-Coll}_p$  of pointed  $\mathbb{S}$ -collections ( $n \in \mathbb{N}$ ). An object of  $\mathbb{S}\text{-Coll}_p$  is denoted  $(C, a, c; p)$ , and described by a commutative diagram in  $\mathbb{C}\text{Sets}$



The category  $\mathbb{S}\text{-Coll}_p$  is monoidal and described in [18]. Monoids in it are  $\mathbb{S}$ -operads.

**Definition 5** The category of  $\mathbb{S}$ -operads is given by the category of monoids of the monoidal category  $\mathbb{S}\text{-Coll}_p$ . We denote it by  $\mathbb{C}Oper$ .  $\square$

Sometimes we shall call it *cubical operads* in order to make clear the geometry involved for this kind of higher operads.

### 3.2 Cubical contractions

Consider a pointed  $\mathbb{S}$ -collection  $(C, a, c; p)$ , and for each  $n \geq 1$  and for all integer  $k \geq 1$ , we define the following subsets of  $C_n \times C_n$

- $\underline{C}_n = \{(\alpha, \beta) \in C_n \times C_n : a_n(\alpha) = a_n(\beta)\}$
- $\underline{C}_{n,j}^s = \{(\alpha, \beta) \in C_n \times C_n : s_{n-1,j}^n(\alpha) = s_{n-1,j}^n(\beta) \text{ and } a_n(\alpha) = a_n(\beta)\}$
- $\underline{C}_{n,j}^t = \{(\alpha, \beta) \in C_n \times C_n : t_{n-1,j}^n(\alpha) = t_{n-1,j}^n(\beta) \text{ and } a_n(\alpha) = a_n(\beta)\}$

and also we consider  $\underline{C}_0 = \{(\alpha, \beta) \in C_0 \times C_0 : \alpha = \beta\}$

Then  $(C, a, c; p)$  is equipped with a *cubical contractibility structure* if they are extra structures given by maps :

$$([-; -]_{n+1,j}^n : \underline{C}_n \longrightarrow C_{n+1})_{n \in \mathbb{N}; j \in \{1, \dots, n+1\}}$$

$$([-; -]_{n+1,j}^{n,-} : \underline{C}_{n,j}^s \longrightarrow C_{n+1})_{n \geq 1; j \in \{1, \dots, n\}}, \quad ([-; -]_{n+1,j}^{n,+} : \underline{C}_{n,j}^t \longrightarrow C_{n+1})_{n \geq 1; j \in \{1, \dots, n\}}$$

such that

- If  $1 \leq i < j \leq n+1$ , then

$$s_{n,i}^{n+1}([\alpha, \beta]_{n+1,j}^n) = [s_{n-1,i}^n(\alpha), s_{n-1,i}^n(\beta)]_{n,j-1}^{n-1}, \text{ and } t_{n,i}^{n+1}([\alpha, \beta]_{n+1,j}^n) = [t_{n-1,i}^n(\alpha), t_{n-1,i}^n(\beta)]_{n,j-1}^{n-1}$$

- If  $1 \leq j < i \leq n+1$  then

$$s_{n,i}^{n+1}([\alpha, \beta]_{n+1,j}^n) = [s_{n-1,i-1}^n(\alpha), s_{n-1,i-1}^n(\beta)]_{n,j}^{n-1}, \text{ and } t_{n,i}^{n+1}([\alpha, \beta]_{n+1,j}^n) = [t_{n-1,i-1}^n(\alpha), t_{n-1,i-1}^n(\beta)]_{n,j}^{n-1}$$

- If  $i = j$  then

$$s_{n,i}^{n+1}([\alpha, \beta]_{n+1,j}^n) = \alpha \text{ and } t_{n,i}^{n+1}([\alpha, \beta]_{n+1,j}^n) = \beta$$

- $a_{n+1}([\alpha, \beta]_{n+1,j}^n) = 1_{n+1,j}^n(a_n(\alpha)) = 1_{n+1,j}^n(a_n(\beta)),$
- $\forall \alpha \in C_n, [\alpha, \alpha]_{n+1,j}^n = 1_{n+1,j}^n(\alpha).$

and such that

- for  $1 \leq j \leq n$  we have :
  - $s_{n,j}^{n+1}([\alpha; \beta]_{n+1,j}^{n,-}) = \alpha$  and  $s_{n,j+1}^{n+1}([\alpha; \beta]_{n+1,j}^{n,-}) = \beta$
  - $t_{n,j}^{n+1}([\alpha; \beta]_{n+1,j}^{n,+}) = \alpha$  and  $t_{n,j+1}^{n+1}([\alpha; \beta]_{n+1,j}^{n,+}) = \beta$
  - $s_{n,j}^{n+1}([\alpha; \beta]_{n+1,j}^{n,+}) = s_{n,j+1}^{n+1}([\alpha; \beta]_{n+1,j}^{n,+}) = [s_{n-1,j}^n(\alpha); s_{n-1,j}^n(\beta)]_{n,j}^{n-1}$
  - $t_{n,j}^{n+1}([\alpha; \beta]_{n+1,j}^{n,-}) = t_{n,j+1}^{n+1}([\alpha; \beta]_{n+1,j}^{n,-}) = [t_{n-1,j}^n(\alpha); t_{n-1,j}^n(\beta)]_{n,j}^{n-1}$
- for  $1 \leq i, j \leq n+1$ 
  - $s_{n,i}^{n+1}([\alpha; \beta]_{n+1,j}^{n,\gamma}) = \begin{cases} [s_{n-1,i}^n(\alpha); s_{n-1,i}^n(\beta)]_{n,j-1}^{n-1,\gamma} & \text{if } 1 \leq i < j \leq n \\ [s_{n-1,i-1}^n(\alpha); s_{n-1,i-1}^n(\beta)]_{n,j}^{n-1,\gamma} & \text{if } 2 \leq j+1 < i \leq n+1 \end{cases}$
  - $t_{n,i}^{n+1}([\alpha; \beta]_{n+1,j}^{n,\gamma}) = \begin{cases} [t_{n-1,i}^n(\alpha); t_{n-1,i}^n(\beta)]_{n,j-1}^{n-1,\gamma} & \text{if } 1 \leq i < j \leq n \\ [t_{n-1,i-1}^n(\alpha); t_{n-1,i-1}^n(\beta)]_{n,j}^{n-1,\gamma} & \text{if } 2 \leq j+1 < i \leq n+1 \end{cases}$
- $a_{n+1}([\alpha; \beta]_{n+1,j}^{n,\gamma}) = 1_{n+1,j}^{n,\gamma}(a_n(\alpha)) = 1_{n+1,j}^{n,\gamma}(a_n(\beta))$
- $\forall \alpha \in C_n, [\alpha, \alpha]_{n+1,j}^{n,\gamma} = 1_{n+1,j}^{n,\gamma}(\alpha)$ .

Such  $\mathbb{S}$ -collection  $(C, a, c; p)$  is called *contractible* where its contractibility structure is usually denoted by :

$$([-; -]_{n+1,j}^n)_{n \in \mathbb{N}; j \in \{1, \dots, n+1\}}, ([-; -]_{n+1,j}^{n,\gamma})_{n \geq 1; j \in \{1, \dots, n\}; \gamma \in \{-, +\}}$$

A morphism of pointed contractible  $\mathbb{S}$ -collections is given by a morphism of  $\mathbb{S}\text{-Coll}_p$  :

$$(C, a, c; p) \xrightarrow{f} (C', a', c'; p')$$

which preserves their structures of contractibility, i.e it is given by a map :

$$C \xrightarrow{f} C'$$

such that :

$$f([\alpha; \beta]_{n+1,j}^n) = [f(\alpha); f(\beta)]_{n+1,j}^n \text{ and } f([\alpha; \beta]_{n+1,j}^{n,\gamma}) = [f(\alpha); f(\beta)]_{n+1,j}^{n,\gamma}$$

The category of pointed contractible  $\mathbb{S}$ -collections is denoted  $\mathcal{CS}\text{-Coll}_p$ .

**Proposition 2 (G.M. Kelly)** *Let  $K$  be a locally finitely presentable category, and  $Mnd_f(K)$  the category of finitary monads on  $K$  and strict morphisms of monads. Then  $Mnd_f(K)$  is itself locally finitely presentable. If  $T$  and  $S$  are object of  $Mnd_f(K)$ , then the coproduct  $T \amalg S$  is algebraic, which means that  $K^T \times_K K^S$  is equal to  $K^T \amalg^S$  and the diagonal of the pullback square*

$$\begin{array}{ccc} K^T \times_K K^S & \xrightarrow{p_1} & K^S \\ p_2 \downarrow & & \downarrow U \\ K^T & \xrightarrow{V} & K \end{array}$$

is the forgetful functor  $K^T \amalg^S \rightarrow K$ . Furthermore the projections  $K^T \times_K K^S \rightarrow K^T$  and  $K^T \times_K K^S \rightarrow K^S$  are monadic.  $\square$

But the following forgetful functors are monadic :

$$\mathcal{COp} \xrightarrow{U} \mathbb{S}\text{-Coll}_p$$

$$\mathcal{CS}\text{-Coll}_p \xrightarrow{V} \mathbb{S}\text{-Coll}_p$$

Lets denote by  $\mathbb{B}$  the monad on  $\mathbb{S}\text{-Coll}_p$  which algebras are cubical higher operads, and denote by  $\mathbb{T}_V$  the monad on  $\mathbb{S}\text{-Coll}_p$  which algebras are pointed contractible  $\mathbb{S}$ -collections. We are in the situation of the above proposition, which shows that algebras of the sum  $\mathbb{B} \amalg \mathbb{T}_V$  is the following pullback in  $\mathbb{CAT}$  :

$$\begin{array}{ccc} \mathbb{C}\mathbb{O}\text{per} \times_{\mathbb{S}\text{-Coll}_p} \mathbb{C}\mathbb{S}\text{-Coll}_p & \xrightarrow{\quad} & \mathbb{C}\mathbb{S}\text{-Coll}_p \\ \downarrow & & \downarrow V \\ \mathbb{C}\mathbb{O}\text{per} & \xrightarrow{\quad U \quad} & \mathbb{S}\text{-Coll}_p \end{array}$$

This pullback is denoted  $\mathbb{C}\mathbb{C}\mathbb{O}\text{per}$  for short. It is the category of contractible cubical higher operads. The left adjoint functor  $F$  of the monadic forgetful functor :

$$\mathbb{C}\mathbb{C}\mathbb{O}\text{per} \xrightarrow{\quad W \quad} \mathbb{S}\text{-Coll}_p$$

gives free contractible cubical higher operads. In particular it gives the free contractible cubical higher operad  $B_C^0$  on the specific pointed  $\mathbb{S}$ -collection  $(C^0, a^0, c^0; p^0)$  that we shall describe in the next section. This operad  $B_C^0$  is the cubical analogue of the operad of Michael Batanin, the one which algebras are the globular weak  $\infty$ -categories. Similarly algebras for this cubical operad  $B_C^0$  are cubical weak  $\infty$ -categories.

### 3.3 The cubical operad of cubical weak $\infty$ -categories

*Cubical pasting diagrams* or *Cubical trees* are cells of the free cubical strict  $\infty$ -category  $\mathbb{S}(1)$  on the terminal object 1 of  $\mathbb{C}\mathbb{S}\text{ets}$ . For example  $1(n) \star_j^n 1(n)$  for  $j \in \{1, \dots, n\}$  are basic examples of cubical trees, and they are such that

$$s_{n-1,i}^n(1(n) \star_j^n 1(n)) = \begin{cases} 1(n-1) \star_{j-1}^{n-1} 1(n-1) & \text{if } 1 \leq i < j \leq n \\ 1(n-1) \star_j^{n-1} 1(n-1) & \text{if } 1 \leq j < i \leq n \end{cases}$$

Now we are going to build a specific pointed  $\mathbb{S}$ -collection  $(C^0, a^0, c^0; p^0)$  which is the underlying pointed  $\mathbb{S}$ -collection of the contractible cubical higher operad  $B_C^0$  which algebras are weak cubical  $\infty$ -categories. This collection is build as follow :

- $C^0(1)$  contains a cell  $u_1$  such that  $s_0^1(u_1) = t_0^1(u_1) = u_0$ , and for each integer  $n \geq 2$  we have an  $n$ -cell  $u_n \in C^0(n)$  which is such that :  $\forall j \in \{1, \dots, n\}$ ,  $s_{n-1,j}^n(u_n) = t_{n-1,j}^n(u_n) = u_{n-1}$ . Arities and coarities of such cells are easy :  $\forall n \in \mathbb{N}$ ,  $a_n^0(u_n) = c_n^0(u_n) = 1(n)$

- $C_n^0$  contains, for all  $n \geq 1$  and all  $j \in \{1, \dots, n\}$  an  $n$ -cell  $\mu_j^n$  which is such that :

$$\begin{aligned} & - \begin{cases} s_{n-1,j}^n(\mu_j^n) = u_{n-1}, t_{n-1,j}^n(\mu_j^n) = u_{n-1} \\ s_{n-1,i}^n(\mu_j^n) = t_{n-1,i}^n(\mu_j^n) = \mu_{j-1}^{n-1} & \text{if } 1 \leq i < j \leq n \\ s_{n-1,i}^n(\mu_j^n) = t_{n-1,i}^n(\mu_j^n) = \mu_j^{n-1} & \text{if } 1 \leq j < i \leq n \end{cases} \\ & - a_n(\mu_j^n) = 1(n) \star_j^n 1(n) \end{aligned}$$

- The pointing

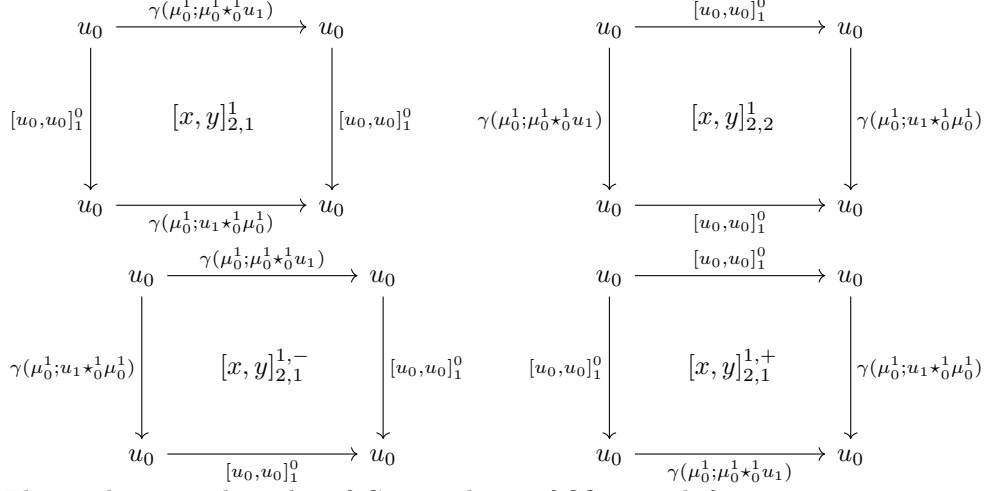
$$\begin{array}{ccccc} & & C^0 & & \\ & \swarrow a^0 & \uparrow p^0 & \searrow c^0 & \\ \mathbb{S}(1) & & 1 & & 1 \\ & \swarrow \lambda(1) & \uparrow id & \searrow & \end{array}$$

is given by  $p_n^0(1(n)) = u_n$ .

**Definition 6** The free contractible cubical higher operad  $B_C^0$  on the pointed  $\mathbb{S}$ -collection  $(C^0, a^0, c^0; p^0)$  described just above, is the operad for cubical weak  $\infty$ -categories. Its underlying monad is denoted  $(\mathbb{B}_C^0, \eta^0, \nu^0)$ <sup>1</sup>. The category of cubical weak  $\infty$ -categories is denoted  $B_C^0\text{-Alg}$   $\square$

<sup>1</sup>We use this short notation, but the reader has to have in mind that it means in particular that the underlying cubical set of the operad  $B_C^0$  is the value of this monad on the terminal object 1 of  $\mathbb{C}\mathbb{S}\text{ets}$

Let us give simple examples of cells in  $B_C^0$  : consider the 1-cells  $x = \gamma(\mu_0^1; \mu_0^1 \star_0^1 u_1)$  and  $y = \gamma(\mu_0^1; u_1 \star_0^1 \mu_0^1)$ . Because the couple  $(x, y)$  belongs to  $\underline{C}_{1,0}^- \cap \underline{C}_{1,0}^+$  we get the following 2-cells in  $B_C^0$  :



These coherences show that if  $G$  is an object of  $\mathbb{C}\text{Sets}$ , and if :

$$B_C^0(G) \xrightarrow{v} G$$

is a  $B_C^0$ -algebra, then for any string in  $G(1)$  :

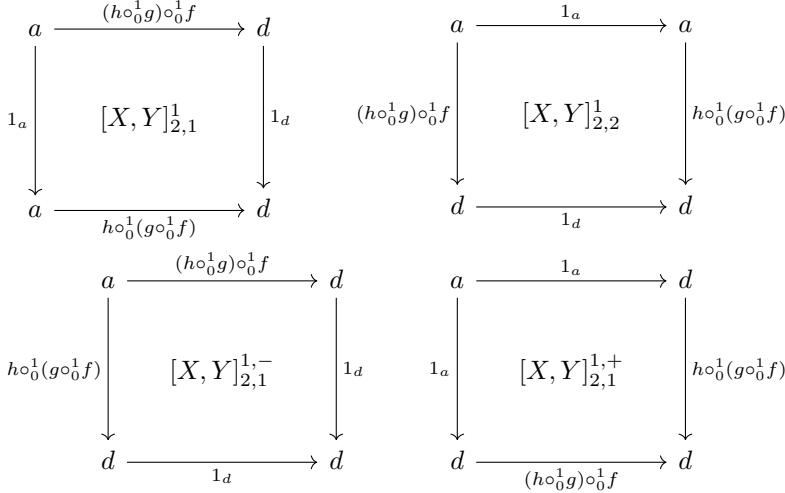
$$a \xrightarrow{f} b \xrightarrow{g} c \xrightarrow{h} d$$

where :

$$X = (h \circ_0^1 g) \circ_0^1 f := v(\gamma(\mu_0^1; \mu_0^1 \star_0^1 u_1); \eta^0(h) \star_0^1 \eta^0(g) \star_0^1 \eta^0(f))$$

$$Y = h \circ_0^1 (g \circ_0^1 f) := v(\gamma(\mu_0^1; u_1 \star_0^1 \mu_0^1); \eta^0(h) \star_0^1 \eta^0(g) \star_0^1 \eta^0(f))$$

the contractions of the operad  $B_C^0$  derive the following 2-cubes in  $G$  :



These simple examples show us how cubical weak  $\infty$ -categories provide more richness of coherences than the globular weak  $\infty$ -categories.

## 4 The Functor of Fundamental Cubical Weak $\infty$ -Groupoids for Spaces

20 years ago Michael Batanin in [25] had described the functor of fundamental globular weak  $\infty$ -groupoids for spaces in order to give a rigorous formulation of the Grothendieck conjecture on homotopy types [33] : in particular he built a functor from the category  $\mathbb{T}\text{op}$  of spaces to the category of globular weak  $\infty$ -groupoids. In order to do that he built an operadic approach of globular weak  $\infty$ -categories, that is his globular weak  $\infty$ -categories are algebra for a specific operad  $B_C^0$ . Two major steps for higher category theory were achieved in [25] :

- he builds a higher dimensional approach of non-symmetric operads  $\tilde{A}$  la Peter May;
- his definition of weak  $\infty$ -categories is more general than simplicial models of  $(\infty, 1)$ -categories. For example it is proved in [36] that some algebraic models of  $(\infty, 1)$ -categories are embedded in his weak  $\infty$ -categories.

In order to build the functor of fundamental globular weak  $\infty$ -groupoids for spaces he proved that the globular object  $D^\bullet$  in  $\mathbb{T}op$  consisting of topological disks :

$$D^0 \begin{array}{c} \xrightarrow{s_0^1} \\ \xrightarrow{t_0^1} \end{array} D^1 \begin{array}{c} \xrightarrow{s_1^2} \\ \xrightarrow{t_1^2} \end{array} D^2 \dots D^{n-1} \begin{array}{c} \xrightarrow{s_{n-1}^n} \\ \xrightarrow{t_{n-1}^n} \end{array} D^n \dots$$

is a  $B_C^0$ -coalgebra, which implication is the construction of the fundamental globular weak  $\infty$ -groupoid functor

$$\mathbb{T}op \xrightarrow{\Pi_\infty(-)} \infty\text{-Grp}$$

In [41] Tom Leinster gave a simplification of the original definition of higher operads by Michael Batanin. However the very important examples of (co)endomorphism globular operads are built very naturally with globular monoidal categories, and this is not clear for us that the  $\mathbb{T}$ -categorical framework of Leinster can capture such natural point of view of (co)endomorphism globular operads. It seems that in [41], he succeeded to define such (co)endomorphism globular operads through  $\mathbb{T}$ -categories, but only in the context of locally cartesian closed categories. For example if  $C$  is a category with pullbacks and if  $E$  is a global object in the monoidal globular category  $\text{Span}(C)$  consisting of globular higher spans in  $C$ , it is possible to define its globular endomorphism operad  $\text{END}(E)$  by using the theory of Batanin (see also [45]), but this is not clear for us how to get such operad  $\text{END}(E)$  with  $\mathbb{T}$ -categories. Thus in order to write the first part of the article [34] we used the Leinster approach to build the operad which algebras are cubical weak  $\infty$ -categories, but to define cubical higher operads of endomorphism we found that the cubical analogue of the globular monoidal categories was much more natural.

In this article, which is the second part of [34], we use the cubical operad  $B_C^0$  of cubical weak  $\infty$ -categories (built in [34]) as a fundamental step to associate to any topological space  $X$  its fundamental cubical weak  $\infty$ -groupoids  $\Pi_\infty(X)$ , and this endows a functor  $\mathbb{T}op \xrightarrow{\Pi_\infty(-)} \infty\text{-CGrp}$  which has a left adjoint functor  $CN_\infty$ . This pair of adjunction  $(CN_\infty, \Pi_\infty(-))$  should put an equivalence between the homotopy category of homotopy types and the homotopy category of  $\infty\text{-CGrp}$  of cubical weak  $\infty$ -groupoids with connections equipped with an adapted Quillen model structure. This was shown to be true but in the context of the Cisinski model structure on the category of cubical sets with connections (see [42]). It is also important to know that non-operadical approach have been considered in [28, 32] to define other higher groupoid constructions.

Important tools to build this functor  $\Pi_\infty(-)$  come from 2-category theory and especially thanks to the work of Mark Weber ([47, 48]) and Ross Street ([44]) : pseudo-algebras for 2-monads and a generalization of the  $\text{Span}$  construction have been successfully considered for this interaction between elementary 2-topos and cubical geometry. An important feature of this article is also to show how the 2-categorical tools developed in [44, 45, 47, 48] can lead to generalization of the original theory of Michael Batanin's higher operads. It is

Plan :

- In the first section we define *cubical monoidal categories* as pseudo  $\mathbb{S}$ -algebras, where  $\mathbb{S}$  is the 2-monad of free strict cubical monoidal categories on cubical categories.
- In the second section we state an important result of [35] which shows that for general situations the  $\text{Span}$ -construction leads to pseudo algebraic structure. Then we give a nice combinatorial description of the cubical (co)spans coming from Marco Grandis ([31]). Then we define (co)endomorphisms operads by using the 2-categorical point of view of Ross Street and Mark Weber in [44, 45, 47, 48]. Our 2-categorical point of view of (co)endomorphisms operads can be adapted in the general context of pseudo algebras, and this is very important for a 2-categorical generalisation of the theory of Batanin.
- In the third section we proved that the cocubical object "box" (as defined in [29]) in  $\mathbb{T}op$  :



$$\begin{array}{ccccccc}
& & & & & & \xrightarrow{s_{n-1,1}^n} \\
& & & & & & \cdots \\
& & & & & & \xrightarrow{s_{n-1,i}^n} \\
& & & & & & \cdots \\
& & & & & & \xrightarrow{s_{n-1,n-1}^n} \\
& & & & & & \cdots \\
I^0 & \xrightarrow{s_1^0} & I^1 & \xrightarrow{s_{1,2}^2} & I^2 & \xrightarrow{s_{2,3}^3} & I^3 & \xrightarrow{s_{3,4}^4} & I^4 \dots I^{n-1} & \xrightarrow{s_{n-1,n-1}^n} & I^n \dots \\
& \xrightarrow{t_1^0} & & \xrightarrow{s_{1,1}^2} & & \xrightarrow{s_{2,2}^3} & & \xrightarrow{s_{3,3}^4} & & \xrightarrow{s_{n-1,n-1}^n} & \\
& & & \xrightarrow{t_{1,1}^2} & & \xrightarrow{t_{2,1}^3} & & \xrightarrow{t_{3,4}^4} & & \xrightarrow{t_{n-1,n-1}^n} & \\
& & & \xrightarrow{t_{1,2}^2} & & \xrightarrow{t_{2,2}^3} & & \xrightarrow{t_{3,3}^4} & & \xrightarrow{t_{n-1,i}^n} & \\
& & & & & \xrightarrow{t_{2,3}^3} & & \xrightarrow{t_{3,2}^4} & & \xrightarrow{t_{n-1,1}^n} & \\
& & & & & & & \xrightarrow{t_{3,1}^4} & & & 
\end{array}$$

is a  $B_C^0$ -coalgebra, where  $B_C^0$  is the  $\mathbb{S}$ -operad which algebras are cubical weak  $\infty$ -categories. Then we show how to "glue" the  $K_i$ -functors of Quillen in order to obtain a functor :

$$\text{Rings} \xrightarrow{K_\infty} \infty\text{-}\mathbb{C}\text{Grp}$$

- The fourth and last section is a short "manifesto" for the following slogan : "*coalgebraic structures* govern different higher category theory". In particular we explain the main steps to get the cubical weak  $\infty$ -category of cubical weak  $\infty$ -categories.

## 5 Cubical monoidal categories as Pseudo-algebras

### 5.1 The cubical category

Consider the small category  $\mathbb{C}$  with integers  $\underline{n} \in \mathbb{N}$  as objects. Generators for  $\mathbb{C}$  are, for all  $\underline{n} \in \mathbb{N}$  given by *sources*  $\underline{n} \xrightarrow{s_{n-1,j}^n} \underline{n-1}$  for each  $j \in \{1, \dots, n\}$  and *targets*  $\underline{n} \xrightarrow{t_{n-1,j}^n} \underline{n-1}$  for each  $j \in \{1, \dots, n\}$  such that for  $1 \leq i < j \leq n$  we have the following cubical relations

- (i)  $s_{n-2,i}^{n-1} \circ s_{n-1,j}^n = s_{n-2,j-1}^{n-1} \circ s_{n-1,i}^n$ ,
- (ii)  $s_{n-2,i}^{n-1} \circ t_{n-1,j}^n = t_{n-2,j-1}^{n-1} \circ s_{n-1,i}^n$ ,
- (iii)  $t_{n-2,i}^{n-1} \circ s_{n-1,j}^n = s_{n-2,j-1}^{n-1} \circ t_{n-1,i}^n$ ,
- (iv)  $t_{n-2,i}^{n-1} \circ t_{n-1,j}^n = t_{n-2,j-1}^{n-1} \circ t_{n-1,i}^n$

These generators plus these relations give the small category  $\mathbb{C}$  called the *cubical category* that we may represent schematically with the low dimensional diagram :

$$\begin{array}{ccccccc}
& \xrightarrow{s_{3,4}^4} & & \xrightarrow{s_{2,3}^3} & & \xrightarrow{s_{1,2}^2} & & \xrightarrow{s_1^1} \\
& \xrightarrow{s_{3,3}^4} & & \xrightarrow{s_{2,2}^3} & & \xrightarrow{s_{1,1}^2} & & \\
& \xrightarrow{s_{3,2}^4} & & \xrightarrow{s_{2,1}^3} & & & & \\
& \xrightarrow{s_{3,1}^4} & & & & & & \\
\cdots C_4 & \xrightarrow{\quad} & C_3 & \xrightarrow{\quad} & C_2 & \xrightarrow{\quad} & C_1 & \xrightarrow{\quad} & C_0 \\
& \xrightarrow{t_{3,1}^4} & & \xrightarrow{t_{2,1}^3} & & \xrightarrow{t_{1,1}^2} & & \xrightarrow{t_0^1} \\
& \xrightarrow{t_{3,2}^4} & & \xrightarrow{t_{2,2}^3} & & \xrightarrow{t_{1,2}^2} & & \\
& \xrightarrow{t_{3,3}^4} & & \xrightarrow{t_{2,3}^3} & & & & \\
& \xrightarrow{t_{3,4}^4} & & & & & & 
\end{array}$$

and this category  $\mathbb{C}$  gives also the sketch  $\mathcal{E}_{\mathbb{S}}$  of cubical sets used especially in [38] to produce the monads  $\mathbb{S} = (S, \lambda, \mu)$ , which algebras are cubical strict  $\infty$ -categories.

**Definition 7** The category  $\mathbb{C}\text{Sets}$  of cubical sets is the category of presheaves  $[\mathbb{C}; \text{Sets}]$ . The terminal cubical set is denoted 1. □

**Definition 8** The 2-category  $\infty\text{-CCAT}$  of cubical categories is the 2-category of prestacks  $[\mathbb{C}; \text{CAT}]$ . The terminal cubical category is also denoted  $1$ .  $\square$

In particular it is shown in [38] that the category  $\infty\text{-CCat}$  of strict cubical  $\infty$ -categories is sketchable by a projective sketch. Thus we put the following definition of *cubical strict monoidal categories* :

**Definition 9** Strict cubical monoidal categories are internal cubical strict  $\infty$ -categories in  $\text{CAT}$ . They form a strict 2-category  $\text{CM}_s\mathbb{C}$  where :

- 0-cells are internal cubical strict  $\infty$ -categories in  $\text{CAT}$ ;
- 1-cells are internal cubical strict  $\infty$ -functors in  $\text{CAT}$ ;
- 2-cells are internal globular<sup>2</sup> strict  $\infty$ -natural transformations in  $\text{CAT}$ .  $\square$

In [38] we denoted by  $(S, \eta, \mu)$  the monad on  $\mathbb{C}\text{Sets}$  of cubical strict  $\infty$ -categories, and *cubical  $n$ -trees* are just  $n$ -cells of  $S(1)$ . We shall prove in [35] that this monad is cartesian, and we denote again by  $(S, \eta, \mu)$  its corresponding 2-monad on the 2-category  $\infty\text{-CCAT}$ . Also the following 2-forgetful functor is 2-monadic :

$\text{CM}_s\mathbb{C} \longrightarrow \infty\text{-CCAT}$ , because the forgetful functor  $\infty\text{-CCat} \longrightarrow \mathbb{C}\text{Sets}$  is monadic and the 2-functor  $\text{CAT}_{\text{pull}} \xrightarrow{\text{CAT}(-)} 2\text{-CAT}$ , which takes a category  $X$  with pullbacks to the 2-category  $\text{CAT}(X)$  of internal categories preserves (finite) limits, thus preserves adjunctions and Eilenberg-Moore constructions. Thus we prefer to denote  $S\text{-Alg}_s$  this 2-category  $\text{CM}_s\mathbb{C}$  of strict cubical monoidal categories. This 2-monad  $(S, \eta, \mu)$  gives weaker notions of algebras, and we recall it for any 2-monad  $(S, \eta, \mu)$  on a 2-category  $\mathcal{K}$  (see [26, 48]). In particular we shall need the notion of pseudo  $S$ -algebra in order to define *cubical monoidal categories* below.

**Definition 10** Let  $(S, \eta, \mu)$  be a 2-monad on a 2-category  $\mathcal{K}$ . A pseudo-algebra structure  $(a, \alpha_0, \alpha)$  on an object  $A \in \mathcal{K}$  is given by a 1-cell  $S(A) \xrightarrow{a} A$  and two invertible 2-cells in  $\mathcal{K}$  :

$$\begin{array}{ccc} S^2(A) & \xrightarrow{\mu(A)} & S(A) \\ S(a) \downarrow & \xRightarrow{\alpha} & \downarrow a \\ S(A) & \xrightarrow{a} & A \end{array} \quad \begin{array}{ccc} A & \xrightarrow{\eta_A} & S(A) \\ 1_A \swarrow & \xRightarrow{\alpha} & \searrow a \\ & A & \end{array}$$

such that the following equalities hold :

$$\begin{array}{c} \begin{array}{ccccc} & S^2(A) & \xrightarrow{\mu(A)} & S(A) & \\ \mu(S(A)) \nearrow & & & & \searrow a \\ S^3(A) & \xrightarrow{\quad} & S^2(A) & \xRightarrow{\alpha \uparrow} & A \\ S^2(a) \searrow & \xRightarrow{S(\alpha)} & \searrow S(a) & & \\ & S^2(A) & \xrightarrow{S(a)} & S(A) & \end{array} & = & \begin{array}{ccccc} & S^2(A) & \xrightarrow{\mu(A)} & S(A) & \\ \mu(S(A)) \nearrow & & & & \searrow a \\ S^3(A) & \xrightarrow{\quad} & S(A) & \xRightarrow{\alpha} & A \\ S^2(a) \searrow & \xRightarrow{S(a)} & S^2(A) & \xRightarrow{\alpha} & \\ & S^2(A) & \xrightarrow{S(a)} & S(A) & \end{array} \\ \\ \begin{array}{ccc} S(A) & \xrightarrow{1_{S(A)}} & S(A) \\ \downarrow 1_{S(A)} & \searrow & \nearrow \mu(A) \\ & S(\alpha_0) & S^2(A)_\alpha \\ \downarrow S(a) & \xRightarrow{\quad} & \downarrow a \\ S(A) & \xrightarrow{a} & A \end{array} = 1_a \end{array}$$

<sup>2</sup>that is they are 2-globes between two cubical strict  $\infty$ -functors, whereas cubical strict  $\infty$ -natural transformations are 2-cubes with faces, four cubical strict  $\infty$ -functors. See [38]

The triple  $(A, \alpha_0, \alpha)$  is called a pseudo  $\mathbb{S}$ -algebra. If  $\alpha_0$  is an identity the pseudo algebra is said to be *normal*. If  $\alpha_0$  and  $\alpha$  are identities then we recover the usual notion of  $\mathbb{S}$ -algebra, and in that case we say that  $A$  is equipped with a strict  $\mathbb{S}$ -algebra structure.

**Definition 11** Let  $(A, \alpha_0, \alpha)$  and  $(A', \alpha'_0, \alpha')$  two pseudo  $\mathbb{S}$ -algebras. A strong  $\mathbb{S}$ -morphism structure for a 1-cell

$$A \xrightarrow{f} A' \text{ is given by an invertible 2-cell : } \begin{array}{ccc} \mathbb{S}(A) & \xrightarrow{a} & A \\ \mathbb{S}(f) \downarrow & \xRightarrow{\bar{f}} & \downarrow f \\ \mathbb{S}(A') & \xrightarrow{a'} & A' \end{array}, \text{ such that we have the following}$$

equalities :

$$\begin{array}{c} \begin{array}{ccccc} & \mathbb{S}(A) & \xrightarrow{a} & A & \\ \mu(A) \nearrow & & & & \searrow a \\ \mathbb{S}^2(A) & \xrightarrow{\mathbb{S}(a)} & \mathbb{S}(A) & \xrightarrow{\bar{f}\uparrow} & A' \\ \searrow \mathbb{S}^2(f) & \xRightarrow{\mathbb{S}(\bar{f})} & \searrow \mathbb{S}(f) & & \nearrow a' \\ & \mathbb{S}(A') & \xrightarrow{\mathbb{S}(a')} & \mathbb{S}(A') & \end{array} & = & \begin{array}{ccccc} & \mathbb{S}(A) & \xrightarrow{a} & A & \\ \mu(A) \nearrow & & & & \searrow f \\ \mathbb{S}^2(A) & & & \mathbb{S}(A') & \xrightarrow{a'} A' \\ \searrow \mathbb{S}^2(f) & \xRightarrow{\mu(A')} & \searrow \mathbb{S}(A') & \xrightarrow{\alpha'} & \nearrow a' \\ & \mathbb{S}(A') & \xrightarrow{\mathbb{S}(a')} & \mathbb{S}(A') & \end{array} \end{array}$$

and

$$\begin{array}{ccc} \begin{array}{ccc} \mathbb{S}(A) & & \\ \eta(A) \nearrow & \searrow a & \\ A & & A \\ \downarrow f & & \downarrow f \\ A' & \xrightarrow{1_{A'}} & A' \end{array} & = & \begin{array}{ccc} \mathbb{S}(A) & & \\ \downarrow & \xRightarrow{\bar{f}} & \\ \mathbb{S}(A') & & \\ \alpha'_0 \uparrow \nearrow & & \searrow \\ A' & & A' \end{array} \end{array} = \begin{array}{ccc} A & \xrightarrow{\eta(A)} & \mathbb{S}(A) \\ 1_A \searrow & \xRightarrow{\alpha_0} & \swarrow a \\ & A & \\ & \downarrow f & \\ & A' & \end{array}$$

**Definition 12** Let  $f$  and  $f'$  be strong  $\mathbb{S}$ -morphisms :

$$(a, \alpha_0, \alpha) \xRightarrow[f']{f} (a', \alpha'_0, \alpha') .$$

A 2-cell  $f \xRightarrow{\psi} f'$  is an algebra 2-cell if the following equality holds :

$$\begin{array}{ccc} \begin{array}{ccc} \mathbb{S}(A) & \xrightarrow{a} & A \\ \mathbb{S}(f) \downarrow & \xRightarrow{\mathbb{S}(\psi)} & \downarrow f' \\ \mathbb{S}(A') & \xrightarrow{a'} & A' \end{array} & = & \begin{array}{ccc} \mathbb{S}(A) & \xrightarrow{a} & A \\ \mathbb{S}(f) \downarrow & \xRightarrow{\bar{f}} & \downarrow f' \\ \mathbb{S}(A') & \xrightarrow{a'} & A' \end{array} \end{array}$$

Let us denote by  $\text{Ps-}\mathbb{S}\text{-Alg}$  the 2-category which objects are pseudo  $\mathbb{S}$ -algebras, whose 1-cells are strong  $\mathbb{S}$ -morphisms and whose 2-cells are algebra 2-cells. The full sub-2-category of  $\text{Ps-}\mathbb{S}\text{-Alg}$  consisting of the normal pseudo-algebras is denoted  $\text{Ps}_0\text{-}\mathbb{S}\text{-Alg}$ , and the locally full sub-2-category of  $\text{Ps-}\mathbb{S}\text{-Alg}$  consisting of the strict algebras and strict morphisms is denoted  $\mathbb{S}\text{-Alg}_s$ .

**Remark 2**  $\text{Ps}_0\text{-}\mathbb{S}\text{-Alg}$  blabla bla : les  $\text{Psn}$  modelent les MG-cat, mais les  $\text{Psn}$  sont équivalentes au  $\text{Ps}$ . Donc on utilisera que les  $\text{Ps}$  pour modéliser les MC-cat.  $\square$

Now let us come back to the 2-monad  $\mathbb{S} = (S, \lambda, \mu)$  on the 2-category of cubical categories  $\infty\text{-CCAT}$  as described above, which strict 2-algebras are strict cubical monoidal categories.

**Definition 13** The 2-category of monoidal cubical categories consists of the 2-category  $\text{Ps-}\mathbb{S}\text{-Alg}$  of pseudo  $\mathbb{S}$ -algebras  $\square$

Also by using the theorem 5.1 and the theorem 5.12 of [26] we get the following biadjunction, similar to the one described in [45] :

**Corollary 1** The forgetful 2-functor  $U : \text{Ps-}\mathbb{S}\text{-Alg} \begin{array}{c} \xrightarrow{U} \\ \top \\ \xleftarrow{F} \end{array} \infty\text{-CCAT}$  such that :

- $\text{Ps-}\mathbb{S}\text{-Alg}$  is the 2-category of pseudo  $\mathbb{S}$ -algebras;
- $\infty\text{-CCAT}$  is the 2-category of cubical categories;
- $F$  builds the free strict monoidal cubical categories functor.

exhibits a biadjunction which restricts to a 2-adjunction on the strict monoidal cubical categories.  $\square$

Also we shall denote by  $\mathbb{S}\text{-Alg}_s \begin{array}{c} \xrightarrow{V} \\ \top \\ \xleftarrow{F} \end{array} \infty\text{-CCAT}$  the underlying strict 2-adjunction of this biadjunction.

## 6 Cubical Higher Spans and Cubical Higher Cospans

### 6.1 The pseudo-algebraic structure of $\text{Span}(C)$

Let us first recall the  $\text{Span}$  construction ([44, 48]) : for any small category  $C$  there is a 2-adjunction :

$$\mathbb{CAT} \begin{array}{c} \xleftarrow{\mathbb{EL}} \\ \perp \\ \xrightarrow{\text{Span}_C} \end{array} [C^{op}, \mathbb{CAT}]$$

where  $\text{Span}_C(\mathcal{E})(c) = [(C/c)^{op}, \mathcal{E}]$  and the category  $\mathbb{EL}(X)$  has the following definition :

- objects are pairs  $(c, x)$  where  $c \in C$  and  $x \in X(c)$ .
- morphisms :  $(c, x) \longrightarrow (d, y)$ , are pairs  $(f, \alpha)$  where  $d \xrightarrow{f} c$  is in  $C$  and  $X(f)(x) \xrightarrow{\alpha} y$  is in  $X(d)$ .
- compositions and identities come from  $C$  and the categories  $X(c)$ .

Suppose now that  $\mathbb{T} = (T, \eta, \mu)$  is a cartesian monad on  $[C^{op}, \text{Sets}]$ , and let us denote again by  $\mathbb{T} = (T, \eta, \mu)$  its extension to a 2-monad on  $[C^{op}, \mathbb{CAT}]$ . In fact, for any category  $\mathcal{E}$  with pullbacks it is proved in [35] that :

**Theorem 1 (Kachour, Weber)**  $\text{Span}_C(\mathcal{E})$  is a pseudo  $\mathbb{T}$ -algebra  $\square$

In fact we can dualize such construction and produce a similar result which says that  $\text{Cospan}_C(\mathcal{E})$  is a pseudo  $\mathbb{T}$ -algebra if  $\mathcal{E}$  is a category with pushouts, and these produce the following diagram of functors :

$$\begin{array}{ccccc} \mathbb{CAT}_{\text{push}} & \xrightarrow{j} & \mathbb{CAT} & & \\ \downarrow (-)^{op} & \searrow \text{Cospan}(-) & \downarrow (-)^{op} & \searrow \text{Cospan}(-) & \\ & \text{Ps-}\mathbb{S}\text{-Alg} & \xrightarrow{i} & \infty\text{-CCAT} & \\ & \nearrow \text{Span}(-) & \downarrow & \nearrow \text{Span}(-) & \\ \mathbb{CAT}_{\text{pull}} & \xrightarrow{k} & \mathbb{CAT} & & \end{array}$$

This result has two essential virtues : first it convince the reader that actually the structure behind the spans and the cospans construction are really of pseudo-algebraic nature; secondly it shows, and this is we believe the main fact, that probably not only globular and cubical higher category theory need such structures, but other useful higher category theory could need it.

However because of the "cubical scopes" of this article, we are going to describe cubical spans and cubical cospans in a more combinatorial way because this concrete point of view has the advantage to see it unpacked, and thus gives an accurate idea of what these cubical spans and cubical cospans looks like. This combinatorial description has been described first by Marco Grandis in [31], and it is instructive to compare it with the Batanin's combinatorial construction of globular spans and globular cospans [25]. The only new tools here are the connections on cubical (co)spans which are accurately describe.

In order to formalize cubical higher spans and cubical higher cospans we will use the *formal span category*  $V$  or the *formal cospan category*  $\Lambda$  used by Marco Grandis (see [31]). For simplicity we will explain only constructions for cubical higher spans, which use this small category  $V$  :

$$\begin{array}{ccc} & -1 & 1 \\ & \swarrow f & \searrow g \\ & 0 & \end{array}$$

because for cubical higher cospans, constructions are duals, and use the small category  $\Lambda$  :

$$\begin{array}{ccc} & 0 & \\ \swarrow f & & \nwarrow g \\ -1 & & 1 \end{array}$$

**Definition 14** Let  $C$  be a category. The category  $\text{Span}_n(C)$  of cubical  $n$ -spans in  $C$  is the category of functors  $[V^n; C]$  and natural transformations between them.  $\square$

The combinatoric description of the category  $V^n$  shall be useful : each objects of  $V^n$  are  $n$ -uplets  $(m_1, \dots, m_n) \in \{0, -1, 1\}^n$ . Also the category  $V^n$  underlies a  $n$ -cube structure, such that the object  $(0, \dots, 0)$  represents the  $n$ -face, and the  $n$ -uplets  $(m_1, \dots, m_n) \in \{0, -1, 1\}^n$  which countains exactly  $p$  integers  $m_j$  which are equal to zero, represent  $p$ -faces. Consider  $(m_1, \dots, m_n)$  a  $(p+1)$ -face and suppose  $m_{j_i} = 0$  for  $1 \leq i \leq p+1$ . Thus we get two morphisms in  $V^n$  :

$$(m_1, \dots, m_{j_i}, \dots, m_n) \xrightarrow[(m_1, \dots, m_{j_i-1}, g, m_{j_i+1}, \dots, m_n)]{(m_1, \dots, m_{j_i-1}, f, m_{j_i+1}, \dots, m_n)} (m_1, \dots, m_{j_i-1}, \hat{m}_{j_i}, m_{j_i+1}, \dots, m_n)$$

such that  $(m_1, \dots, m_{j_i-1}, f, m_{j_i+1}, \dots, m_n)$  switch the value  $m_{j_i}$  to the value  $\hat{m}_{j_i} = -1$  and  $(m_1, \dots, m_{j_i-1}, g, m_{j_i+1}, \dots, m_n)$  switch the value  $m_{j_i}$  to the value  $\hat{m}_{j_i} = 1$ .

**Remark 3** Intuitively such map  $(m_1, \dots, m_{j_i-1}, f, m_{j_i+1}, \dots, m_n)$  is a kind of  $s_{p,j_i}^{p+1}$  and the map  $(m_1, \dots, m_{j_i-1}, g, m_{j_i+1}, \dots, m_n)$  is a kind of  $t_{p,j_i}^{p+1}$ .  $\square$

In particular the following arrows in  $V^n$  :

$$(0, \dots, 0) \xrightarrow{(0, \dots, 0, f, 0, \dots, 0)} (0, \dots, 0, -1, 0, \dots, 0), \quad (0, \dots, 0) \xrightarrow{(0, \dots, 0, g, 0, \dots, 0)} (0, \dots, 0, 1, 0, \dots, 0)$$

shall be important for an accurate description of the projective cone below, when we will describe the pseudo-algebraic structure produced by cubical higher spans in a category with pullbacks.

Now we want to put a cubical category structure on cubical spans. For that we just recall the constructions of Marco Grandis (see [31]).

- The *formal source functor* is given by  $1 \xrightarrow{s} V$ , where  $1 = \{\star\}$  is the terminal category and  $s$  sends  $\star$  to  $-1$ . Similarly the *formal target functor* is given by  $1 \xrightarrow{t} V$  where  $t$  sends  $\star$  to  $1$ .

These give the *source functors*  $V^{n-1} \xrightarrow{s_{n-1,i}^n} V^n$ , given by  $s_{n-1,i}^n := V^{i-1} \times s \times V^{n-i}$  for  $1 \leq i \leq n$ ,

and the *target functors*  $V^{n-1} \xrightarrow{t_{n-1,i}^n} V^n$ , given by  $t_{n-1,i}^n := V^{i-1} \times t \times V^{n-i}$  for  $1 \leq i \leq n$ , and then we get the cubical category of spans in  $C$  :

$$\begin{array}{ccccccc}
 & \xrightarrow{s_{3,4}^4} & & \xrightarrow{s_{2,3}^3} & & \xrightarrow{s_{1,2}^2} & \xrightarrow{s_0^1} \\
 & \xrightarrow{s_{3,3}^4} & & \xrightarrow{s_{2,2}^3} & & \xrightarrow{s_{1,1}^2} & \\
 & \xrightarrow{s_{3,2}^4} & & \xrightarrow{s_{2,1}^3} & & & \\
 & \xrightarrow{s_{3,1}^4} & & & & & \\
 \cdots [V^4; C] & \xrightarrow{\quad} & [V^3; C] & \xrightarrow{\quad} & [V^2; C] & \xrightarrow{\quad} & [V; C] \xrightarrow{\quad} C \\
 & \xrightarrow{t_{3,1}^4} & & \xrightarrow{t_{2,1}^3} & & \xrightarrow{t_{1,1}^2} & \xrightarrow{t_0^1} \\
 & \xrightarrow{t_{3,2}^4} & & \xrightarrow{t_{2,2}^3} & & \xrightarrow{t_{1,2}^2} & \\
 & \xrightarrow{t_{3,3}^4} & & \xrightarrow{t_{2,3}^3} & & & \\
 & \xrightarrow{t_{3,4}^4} & & & & & 
 \end{array}$$

where for each  $1 \leq i \leq n$ ,  $s_{n-1,i}^n$  and  $t_{n-1,i}^n$  are functors :

$$[V^n; C] \xrightleftharpoons[t_{n-1,i}^n]{s_{n-1,i}^n} [V^{n-1}; C]$$

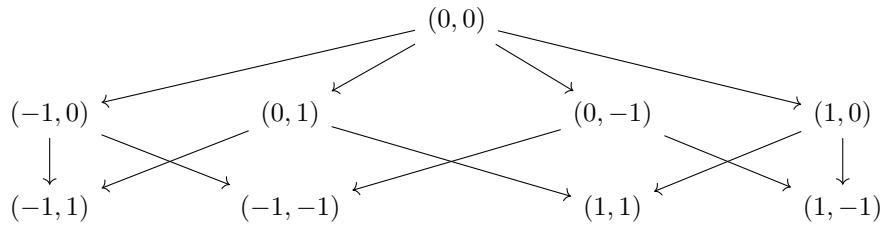
- The *formal reflexivity functor* is given by the unique functor  $V \xrightarrow{!} 1$ , and this gives for  $1 \leq i \leq n$  the *reflexivity functors*  $V^n \xrightarrow{1_{n,i}^{n-1}} V^{n-1}$ , given by  $1_{n,i}^{n-1} := V^{i-1} \times ! \times V^{n-i}$ , and then we get a reflexivity structure on the cubical category of spans in  $C$  :

$$\begin{array}{ccccccc}
 & & & & & \xrightarrow{1_{4,3}^3} & \\
 & & & & \xrightarrow{1_{3,2}^2} & \xrightarrow{1_{4,1}^3} & \\
 & & & \xrightarrow{1_{3,1}^2} & \xrightarrow{1_{3,3}^2} & \xrightarrow{1_{4,2}^3} & \\
 C \xrightarrow{1_0^0} [V^1; C] & \xrightarrow{1_{2,1}^1} & [V^2; C] & \xrightarrow{1_{2,2}^1} & [V^3; C] & \xrightarrow{1_{4,4}^3} & [V^4; C] \cdots
 \end{array}$$

where for each  $1 \leq i \leq n$ ,  $1_{n,i}^{n-1}$  is a functor :

$$[V^{n-1}; C] \xrightarrow{1_{n,i}^{n-1}} [V^n; C]$$

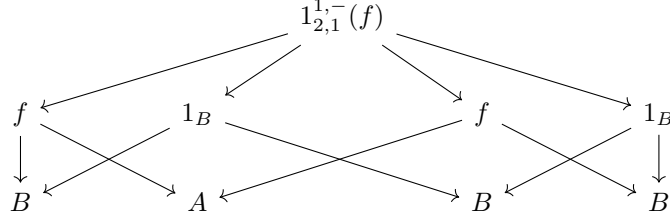
- Connections for cubical higher (co)spans are not defined in Grandis [31], thus we need to formalize it properly.  $V^2$  may be seen as the following cubical 2-span :



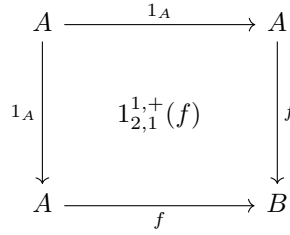
and if  $A \xrightarrow{f} B$  is an 1-cell, then the 2-cell :

$$\begin{array}{ccc}
 A & \xrightarrow{f} & B \\
 f \downarrow & & \downarrow 1_B \\
 B & \xrightarrow{1_B} & B
 \end{array}
 \quad 1_{2,1}^{1,-}(f)$$

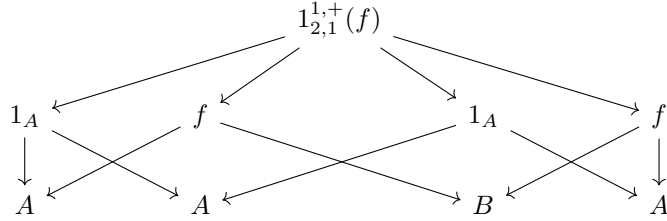
is represented by the following 2-span :



and the 2-cell



is represented by the following 2-span :



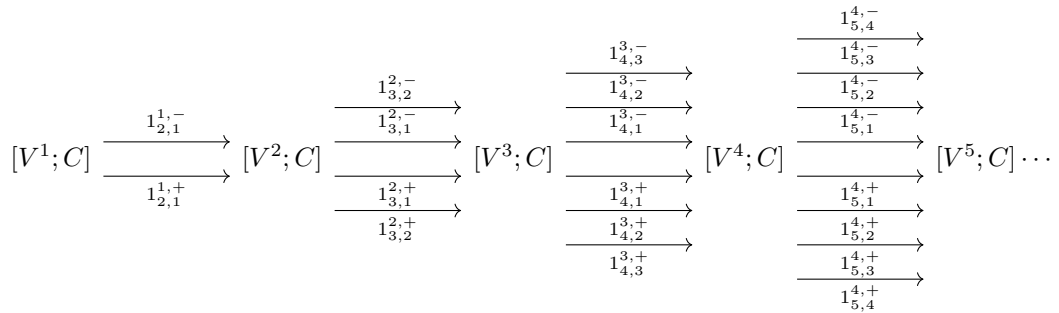
These show us how to formalise connections for cubical higher spans : the *formal connection functors* are thus given by :  $V^2 \xrightarrow{1^-} V$ ,  $V^2 \xrightarrow{1^+} V$  defined on objects<sup>3</sup> of  $V^2$  by

$$(0, 0), (-1, 0), (0, -1) \mapsto 0, \quad (0, 1), (1, 0), (-1, 1), (1, 1), (1, -1) \mapsto 1, \quad (-1, -1) \mapsto -1,$$

and

$$(0, 0), (0, 1), (1, 0) \mapsto 0, \quad (-1, 0), (0, -1), (-1, 1), (-1, -1), (1, -1) \mapsto -1, \quad (1, 1) \mapsto 1.$$

These give the *connection functors* :  $V^{n+1} \xrightarrow{1_{n+1,i}^{n,-}} V^n$ ,  $V^{n+1} \xrightarrow{1_{n+1,i}^{n,+}} V^n$ , given by  $1_{n+1,i}^{n,-} := V^{i-1} \times 1^- \times V^{n-i}$  and  $1_{n+1,i}^{n,+} := V^{i-1} \times 1^+ \times V^{n-i}$ , and then we get the structure of connections on the cubical category of spans in  $C$  :



<sup>3</sup>Of course, these definition on objects give the one on arrows of  $V^2$

where for each  $1 \leq i \leq n$ ,  $1_{n+1,i}^{n,-}$ ,  $1_{n+1,i}^{n,+}$  are functors :

$$[V^n; C] \begin{array}{c} \xrightarrow{1_{n+1,i}^{n,-}} \\ \xrightarrow{1_{n+1,i}^{n,+}} \end{array} [V^{n+1}; C]$$

Now suppose that  $C$  is a category equipped with pullbacks. This context allows to put a pseudo-algebra structure on cubical higher spans in  $C$ . In fact this cubical monoidal structure shall be given by these pullbacks. We will follow the definition of Grandis (see [31]) with a small variation on projective sketch. Our goal, for each  $n \geq 1$  and each  $1 \leq i \leq n$ , is to build functors :

$$[V^n; C] \times_{[V^{n-1}; C]} [V^n; C] \xrightarrow{\otimes_i^n} [V^n; C]$$

such that  $[V^n; C] \times_{[V^{n-1}; C]} [V^n; C]$  comes from the pullback :

$$\begin{array}{ccc} [V^n; C] \times_{[V^{n-1}; C]} [V^n; C] & \xrightarrow{\quad} & [V^n; C] \\ \downarrow & & \downarrow s_{n-1,i}^n \\ [V^n; C] & \xrightarrow{t_{n-1,i}^n} & [V^{n-1}; C] \end{array}$$

- First we consider the category  $V_2$  given by the following pushout :

$$\begin{array}{ccc} \{\star\} & \xrightarrow{s} & V \\ \downarrow t & & \downarrow k^+ \\ V & \xrightarrow{k^-} & V_2 \end{array}$$

Thus  $V_2$  is given by the category :

$$\begin{array}{ccccc} & a & & c & \\ & \swarrow & \searrow & \swarrow & \searrow \\ -1 & & b & & 1 \end{array}$$

that we extend to the category  $\hat{V}_2$  :

$$\begin{array}{ccccc} & & 0 & & \\ & \swarrow \cdots & & \cdots \searrow & \\ & a & & c & \\ & \swarrow & \searrow & \swarrow & \searrow \\ -1 & & b & & 1 \end{array}$$

Also the following subcategory  $W$  of  $V_2$  shall be considered :

$$\begin{array}{ccc} a & & c \\ & \searrow & \swarrow \\ & b & \end{array}$$

and the natural transformation  $\Delta(0)$  :

$$\begin{array}{ccc} & \Delta(0) & \\ & \curvearrowright & \\ W & \xrightarrow{F} & \hat{V}_2, \\ & \Downarrow \tau & \end{array}$$



where  $\Delta(0)$  is the constant functor with value 0. This allow to see the category  $\hat{V}_2$  as the category  $V_2$  equipped with a cone over  $W$ , that is  $\hat{V}_2$  is a projective sketch equipped with the cone  $\Delta(0) \xrightarrow{\tau} F$  ; also we have the concatenation functor :  $V \xrightarrow{k} \hat{V}_2$  which sends 0 to 0, and  $-1$  to  $-1$ , and finally 1 to 1, from the category  $V$  to the projective sketch  $\hat{V}_2$ . Now for each  $n \geq 1$  and each  $1 \leq i \leq n$ , consider the pushout diagram :

$$\begin{array}{ccc} V^{n-1} & \xrightarrow{s_{n-1,i}^n} & V^n \\ \downarrow t_{n-1,i}^n & & \downarrow k_i^+ \\ V^n & \xrightarrow{k_i^-} & V_i^n \end{array}$$

where  $k_i^- = V^{i-1} \times k^- \times V^{n-i}$ ,  $k_i^+ = V^{i-1} \times k^+ \times V^{n-i}$  and  $V_i^n = V^{i-1} \times V_2 \times V^{n-i}$ . The category  $V_i^n$  may be thought as the gluing of itself along the functors  $s_{n-1,i}^n$  and  $t_{n-1,i}^n$ , and also the category  $\hat{V}_i^n := V^{i-1} \times \hat{V}_2 \times V^{n-i}$  may be thought as the category  $V_i^n$  equipped with a cone over its following subdiagram :

$$\begin{array}{ccc} (0, \dots, 0) & & (0, \dots, 0) \\ \searrow (0, \dots, 0, f, 0, \dots, 0) & & \swarrow (0, \dots, 0, g, 0, \dots, 0) \\ (0, \dots, 0, 1, 0, \dots, 0) & \sim & (0, \dots, 0, -1, 0, \dots, 0) \end{array}$$

in this subdiagram the symbol  $\sim$  means the identification of  $(0, \dots, 0, 1, 0, \dots, 0)$  and  $(0, \dots, 0, -1, 0, \dots, 0)$  under the pushout, and the cone is formally described by the natural transformation :

$$\begin{array}{ccc} & V^{i-1} \times \Delta(0) \times V^{n-i} & \\ & \downarrow \tau = V^{i-1} \times \tau \times V^{n-i} & \\ V^{i-1} \times W \times V^{n-i} & \xrightarrow{V^{i-1} \times F \times V^{n-i}} & V^{i-1} \times \hat{V}_2 \times V^{n-i} \end{array}$$

- Now consider two cubical  $n$ -spans  $x$  and  $y$  such that  $s_{n-1,i}^n(x) = t_{n-1,i}^n(y)$  in category  $C$  equipped with pullbacks :

$$V^n \begin{array}{c} \xrightarrow{x} \\ \xrightarrow{y} \end{array} C.$$

We are in the following situation where we get the unique functor  $[x, y]_i$  :

$$\begin{array}{ccc} V^{n-1} & \xrightarrow{s_{n-1,i}^n} & V^n \\ \downarrow t_{n-1,i}^n & & \downarrow k_i^+ \\ V^n & \xrightarrow{k_i^-} & V_i^n \end{array} \quad \begin{array}{c} \xrightarrow{y} \\ \searrow [x, y]_i \\ \xrightarrow{x} \end{array} C$$

thus we get the functor  $[x, \hat{y}]_i : \hat{V}_i^n \xrightarrow{[x, \hat{y}]_i} C$  which is the extension of the functor  $[x, y]_i$  on the category  $\hat{V}_i^n$ , which sends the cone  $\tau_i = V^{i-1} \times \tau \times V^{n-i}$  to the following pullback in  $C$  :

$$\begin{array}{ccc}
 & \bullet & \\
 \swarrow & & \searrow \\
 x(0, \dots, 0) & & y(0, \dots, 0) \\
 \swarrow \scriptstyle x(0, \dots, 0, f, 0, \dots, 0) & & \searrow \scriptstyle y(0, \dots, 0, g, 0, \dots, 0) \\
 & x(0, \dots, 0, 1, 0, \dots, 0) = y(0, \dots, 0, -1, 0, \dots, 0) &
 \end{array}$$

- Thus we obtain the diagram :  $V^n \xrightarrow{k_i} \hat{V}_i^n \xrightarrow{[x, \hat{y}]_i} C$ , where  $k_i = V^{i-1} \times k \times V^{n-i}$  comes from the concatenation functor :  $V \xrightarrow{k} \hat{V}_2$ , and we put :  $y \otimes_i^n x = [x, \hat{y}]_i \circ k_i$ . As for globular higher spans, these tensor products on arrows comes from universality of these pullbacks. Thus for each  $n \geq 1$  and each  $1 \leq i \leq n$ , we built functors :

$$[V^n; C]_{[V^{n-1}; C]} \times_{[V^{n-1}; C]} [V^n; C] \xrightarrow{\otimes_i^n} [V^n; C]$$

which put on  $\text{Span}(C)$  a pseudo-algebra structure.

Of course the description of the pseudo-algebra  $\text{Cospan}(C)$ , where  $C$  is a category with pushouts, is obtained by dualizing these constructions.

## 6.2 $B_C^0$ -algebras and $B_C^0$ -coalgebras

**Definition 15** If  $\mathcal{C}$  is a cubical monoidal category then a *global object* of  $\mathcal{C}$  is given by a morphism :

$$1 \xrightarrow{E} \mathcal{C}$$

in the category  $\infty\text{-CCAT}$  of cubical categories. □

By the pseudo-universality of  $1 \xrightarrow{\eta(1)} \mathbb{S}(1)$  we get the following morphism  $[E]$  of cubical monoidal categories :

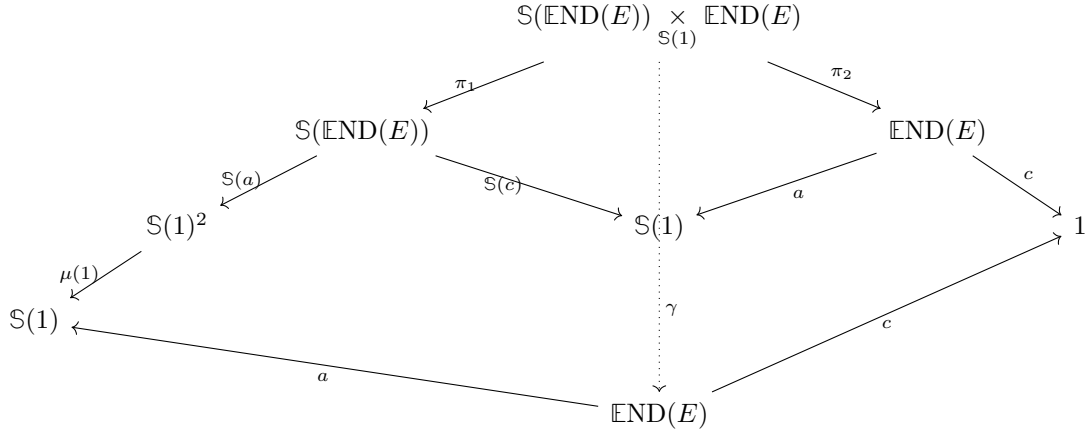
$$\begin{array}{ccc}
 \mathbb{S}(1) & \xrightarrow{[E]} & \mathcal{C} \\
 \eta(1) \uparrow & \nearrow E & \\
 1 & & 
 \end{array}$$

Now suppose :  $\mathbb{S}(\mathcal{C}) \xrightarrow{v} \mathcal{C}$ , is the structural map of the pseudo  $\mathbb{S}$ -algebra  $\mathcal{C}$ . It is important to notice that the freeness of  $\mathbb{S}(1)$  describes this extension  $[E]$  as the composition  $v \circ \mathbb{S}(E)$  :

$$\begin{array}{ccc}
 \mathbb{S}(1) & \xrightarrow{[E]} & \mathcal{C} \\
 \searrow \mathbb{S}(E) & & \nearrow v \\
 & \mathbb{S}(\mathcal{C}) & 
 \end{array}$$

This morphism  $[E]$  is denoted  $\mathbb{E}nd(E)$  for the case of the cubical monoidal category  $\mathcal{C} = \text{Span}(C)$  where  $C$  is a category with pullbacks; thus a global object in it :  $1 \xrightarrow{E} \text{Span}(C)$  produces such extension  $\mathbb{S}(1) \xrightarrow{\mathbb{E}nd(E)} \text{Span}(C)$ , and furthermore this morphism  $\mathbb{E}nd(E)$  contains all informations we need to define the  $\mathbb{S}$ -operad of endomorphism  $\mathbb{E}ND(E)$  associated to the global object  $E$  in  $\text{Span}(C)$  :

**Definition 16** For all  $n \in \mathbb{N}$ ,  $n$ -cells of  $\mathbb{E}ND(E)$  consist of elements of the set  $\text{hom}_{\text{Span}_n(C)}(\mathbb{E}nd(E)(t), E(n))$ , for each cubical  $n$ -tree  $t \in \mathbb{S}(1)$ . These  $n$ -cells form the set  $\mathbb{E}ND(E)(n)$ , and the corresponding cubical set  $\mathbb{E}ND(E)$  underlies an  $\mathbb{S}$ -operad where the multiplication of it is defined as follow : if  $(x, y)$  is an  $n$ -cell of  $\mathbb{S}(\mathbb{E}ND(E)) \times_{\mathbb{S}(1)} \mathbb{E}ND(E)$ , and is such that  $\mu(1)(\mathbb{S}(a)(x)) = t'$  and  $a(y) = t$  :



then  $\gamma(x; y)$  is given by the composition  $y \circ v(x)$  in  $\text{Span}_n(C)$  :

$$\mathbb{E}nd(E)(t') \xrightarrow{v(x)} \mathbb{E}nd(E)(t) \xrightarrow{y} E(n)$$

where

$$\mathbb{S}(\text{Span}(C)) \xrightarrow{v} \text{Span}(C)$$

is the structural map of the pseudo  $\mathbb{S}$ -algebra  $\text{Span}(C)$ ; the unity of it is given, for each  $n \in \mathbb{N}$ , by the singleton  $1_{E(n)} \in \text{hom}_{\text{Span}_n(C)}(E(n), E(n))$ . The axiom of associativity of the multiplication of  $\mathbb{E}ND(E)$  comes from the associativity of compositions of each categories  $\text{Span}_n(C)$  ( $n \in \mathbb{N}$ ), and we have the similar result for the axiom of unities.  $\square$

Also we have the following easy result :

**Corollary 2** A global object  $1 \xrightarrow{E} \text{Span}(C)$  is the same thing as to give a cubical object, still denoted by  $E$ , internal to the category  $C$  :  $\mathbb{C} \xrightarrow{E} C$   $\square$

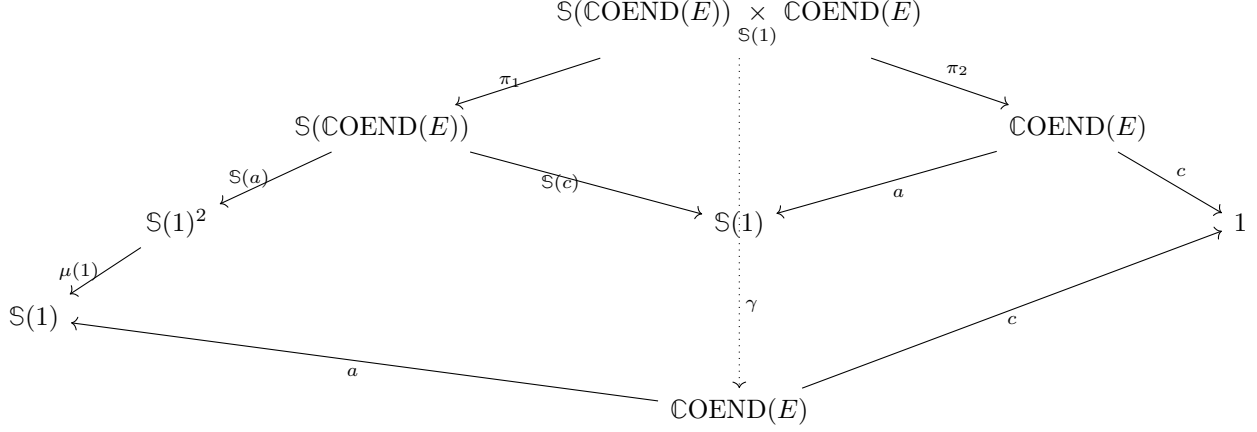
Now we are ready to define  $B_C^0$ -algebras :

**Definition 17** Consider a category  $C$  with pullbacks, plus a cubical object  $E$  in it :  $\mathbb{C} \xrightarrow{E} C$ .  $E$  is equipped with a  $B_C^0$ -algebra structure if there is a morphism of  $\mathbb{S}$ -operads :  $B_C^0 \xrightarrow{f} \mathbb{E}ND(E)$ .  $\square$

Operads of coendomorphisms and coalgebraic structures are defined similarly and dually, but because of their importance we prefer to give their precise dual definition : if  $C$  is a category with pushouts, thus  $\mathcal{C} = \text{Cospan}(C)$  is a cubical monoidal category, and if :  $1 \xrightarrow{E} \text{Cospan}(C)$  is a global object in it, then the corresponding extension  $\mathbb{C}oend(E)$  to  $\mathbb{S}(1)$  :  $\mathbb{S}(1) \xrightarrow{\mathbb{C}oend(E)} \text{Cospan}(C)$ , contains all informations we need to define the  $\mathbb{S}$ -operad of coendomorphism  $\mathbb{C}OEND(E)$  associated to the global object  $E$  in  $\text{Cospan}(C)$  :

<sup>4</sup>In this diagram  $\mathbb{S}$  is seen as a monad on the category of cubical sets. See [34] for the definition of  $\mathbb{S}$ -operads.

**Definition 18** For all  $n \in \mathbb{N}$ ,  $n$ -cells of  $\mathbb{C}\text{OEND}(E)$  consist of elements of the set  $\text{hom}_{\mathbb{C}\text{ospan}_n(C)}(E(n), \text{Coend}(E)(t))$ , for each cubical  $n$ -tree  $t \in \mathbb{S}(1)$ . These  $n$ -cells form the set  $\mathbb{C}\text{OEND}(E)(n)$ , and the corresponding cubical set  $\mathbb{C}\text{OEND}(E)$  underlies an  $\mathbb{S}$ -operad where the multiplication of it is defined as follow : if  $(x, y)$  is an  $n$ -cell of  $\mathbb{S}(\mathbb{C}\text{OEND}(E)) \times_{\mathbb{S}(1)} \mathbb{C}\text{OEND}(E)$ , and is such that  $\mu(1)(\mathbb{S}(a)(x)) = t'$  and  $a(y) = t$  :



then  $\gamma(x; y)$  is given by the composition  $y \circ v(x)$  in  $\text{Cospan}_n(C)$  :

$$\text{Coend}(E)(t') \xrightarrow{v(x)} \text{Coend}(E)(t) \xrightarrow{y} E(n)$$

where

$$\mathbb{S}(\text{Cospan}(C)) \xrightarrow{v} \text{Cospan}(C)$$

is the structural map of the pseudo  $\mathbb{S}$ -algebra  $\text{Cospan}(C)$ ; the unity of it is given, for each  $n \in \mathbb{N}$ , by the singleton  $1_{E(n)} \in \text{hom}_{\text{Cospan}_n(C)}(E(n), E(n))$ . The axiom of associativity of the multiplication of  $\mathbb{C}\text{OEND}(E)$  comes from the associativity of compositions of each categories  $\text{Cospan}_n(C)$  ( $n \in \mathbb{N}$ ), and we have the similar result for the axiom of unities.  $\square$

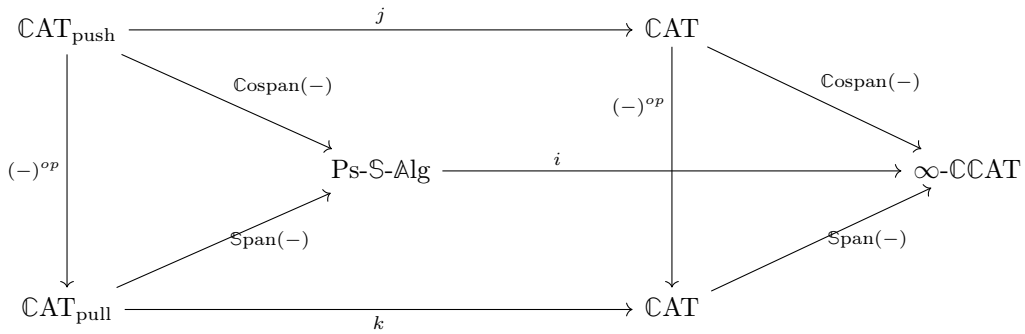
Also we have the following easy result :

**Corollary 3** A global object  $1 \xrightarrow{E} \text{Cospan}(C)$  is the same thing as to give a cocubical object, still denoted by  $E$ , internal to the category  $C$  :  $\mathbb{C}^{op} \xrightarrow{E} C$   $\square$

Now we are ready to define  $B_C^0$ -coalgebras :

**Definition 19** Consider a category  $C$  with pushouts, plus a cocubical object  $E$  in it :  $\mathbb{C}^{op} \xrightarrow{E} C$ .  $E$  is equipped with a  $B_C^0$ -coalgebra structure if there is a morphism of  $\mathbb{S}$ -operads :  $B_C^0 \xrightarrow{f} \mathbb{C}\text{OEND}(E)$ .  $\square$

Also it is easy to check that for each global object  $E$  of  $\text{Span}(C)$  where  $C$  has pullbacks, the construction of  $\mathbb{E}\text{ND}(E)$  endows a functor, and also for each global object  $E$  of  $\text{Cospan}(C)$  where  $C$  has pushouts, the construction of  $\mathbb{C}\text{OEND}(E)$  is also functorial. Recall from 6.1 that we got the following diagram of functors :



and we have the following result :

**Corollary 4** • If  $C \xrightarrow{F} D$  is a morphism of the category  $\mathbb{CAT}_{pull}$ , and if

$$\begin{array}{ccc} & 1 & \\ E \swarrow & & \searrow E' \\ \mathbb{S}pan(C) & \xrightarrow{\mathbb{S}pan(F)} & \mathbb{S}pan(D) \end{array}$$

is a morphism of the category  $(1 \downarrow i \circ \mathbb{S}pan(-))$ , then it produces the morphism of  $\mathbb{S}$ -operads :

$$\mathbb{E}ND(E) \xrightarrow{\mathbb{E}ND(\mathbb{S}pan(F))} \mathbb{E}ND(E')$$

Furthermore this construction is functorial and gives the functor

$$(1 \downarrow i \circ \mathbb{S}pan(-)) \xrightarrow{\mathbb{E}ND(-)} \mathbb{S}\text{-}\mathbb{O}per$$

• If  $C \xrightarrow{F} D$  is a morphism of the category  $\mathbb{CAT}_{push}$ , and if

$$\begin{array}{ccc} & 1 & \\ E \swarrow & & \searrow E' \\ \mathbb{C}ospan(C) & \xrightarrow{\mathbb{C}ospan(F)} & \mathbb{C}ospan(D) \end{array}$$

is a morphism of the category  $(1 \downarrow i \circ \mathbb{C}ospan(-))$ , then it produces the morphism of  $\mathbb{S}$ -operads :

$$\mathbb{C}OEND(E) \xrightarrow{\mathbb{C}OEND(\mathbb{C}ospan(F))} \mathbb{C}OEND(E')$$

Furthermore this construction is functorial and gives the functor

$$(1 \downarrow i \circ \mathbb{C}ospan(-)) \xrightarrow{\mathbb{C}OEND(-)} \mathbb{S}\text{-}\mathbb{O}per$$

## 7 Higher Cospans in $\mathbb{T}op$

### 7.1 The global object $box : I^\bullet$

Here  $I = [0, 1]$  is the usual interval of  $\mathbb{R}$ . Consider the following internal cocubical object in  $\mathbb{T}op$  :

$$\begin{array}{ccccccc} I^0 & \xrightarrow{s_1^0} & I^1 & \xrightarrow{s_{1,2}^2} & I^2 & \xrightarrow{s_{2,3}^3} & I^3 & \xrightarrow{s_{3,1}^4} & I^4 \dots I^{n-1} & \xrightarrow{s_{n-1,1}^n} & I^n \dots \\ & \xrightarrow{t_1^0} & & \xrightarrow{s_{1,1}^2} & & \xrightarrow{s_{2,2}^3} & & \xrightarrow{s_{3,2}^4} & & \xrightarrow{s_{n-1,i}^n} & \\ & & & \xrightarrow{t_{1,1}^2} & & \xrightarrow{s_{2,1}^3} & & \xrightarrow{s_{3,3}^4} & & \xrightarrow{s_{n-1,n-1}^n} & \\ & & & \xrightarrow{t_{1,2}^2} & & \xrightarrow{t_{2,1}^3} & & \xrightarrow{s_{3,4}^4} & & \xrightarrow{s_{n-1,n-1}^n} & \\ & & & & & \xrightarrow{t_{2,2}^3} & & \xrightarrow{t_{3,4}^4} & & \xrightarrow{t_{n-1,n-1}^n} & \\ & & & & & \xrightarrow{t_{2,3}^3} & & \xrightarrow{t_{3,3}^4} & & \xrightarrow{t_{n-1,i}^n} & \\ & & & & & & & \xrightarrow{t_{3,2}^4} & & \xrightarrow{t_{n-1,1}^n} & \\ & & & & & & & \xrightarrow{t_{3,1}^4} & & & \end{array}$$

defined by :

$$s_{n-1,i}^n(x_1, \dots, x_{i-1}, x_i, \dots, x_{n-1}) = (x_1, \dots, x_{i-1}, 0, x_i, \dots, x_{n-1})$$

$$t_{n-1,i}^n(x_1, \dots, x_{i-1}, x_i, \dots, x_{n-1}) = (x_1, \dots, x_{i-1}, 1, x_i, \dots, x_{n-1}).$$

This is a global object of the pseudo-algebra  $\mathbb{C}\text{ospan}(\mathbb{T}\text{op})$ . Following the notation in [29], this global object  $I^\bullet$  shall be called the *box object*. Thanks to the pseudo-universality of  $1 \xrightarrow{\eta(1)} \mathbb{S}(1)$  we get the following commutative (up to isomorphisms) diagram :

$$\begin{array}{ccc} \mathbb{S}(1) & \xrightarrow{\text{Coend}(I^\bullet)} & \mathbb{C}\text{ospan}(\mathbb{T}\text{op}) \\ \uparrow \eta(1) & \nearrow I^\bullet & \\ 1 & & \end{array}$$

and from the cubical monoidal functor  $\text{Coend}(I^\bullet)$  we get the  $\mathbb{S}$ -operad  $\mathbb{C}\text{OEND}(I^\bullet)$  (6.2). The next section is devoted to prove that  $I^\bullet$  is a  $B_C^0$ -coalgebra, i.e that the  $\mathbb{S}$ -operad  $\mathbb{C}\text{OEND}(I^\bullet)$  is contractible and is equipped with a composition system in the sense of cubical higher operads [34].

## 7.2 $I^\bullet$ is a $B_C^0$ -coalgebra

### 7.2.1 Composition systems on $\mathbb{C}\text{OEND}(I^\bullet)$

The cubical  $(n-1)$ -sphere  $\mathbb{S}_c^{n-1}$  is given by the sums :

$$\mathbb{S}_c^{n-1} := \coprod_{1 \leq i \leq n} (I^{i-1} \times \{0\} \times I^{n-i} \sqcup I^{i-1} \times \{1\} \times I^{n-i})$$

and we have the inclusion :  $\mathbb{S}_c^{n-1} \hookrightarrow I^n$

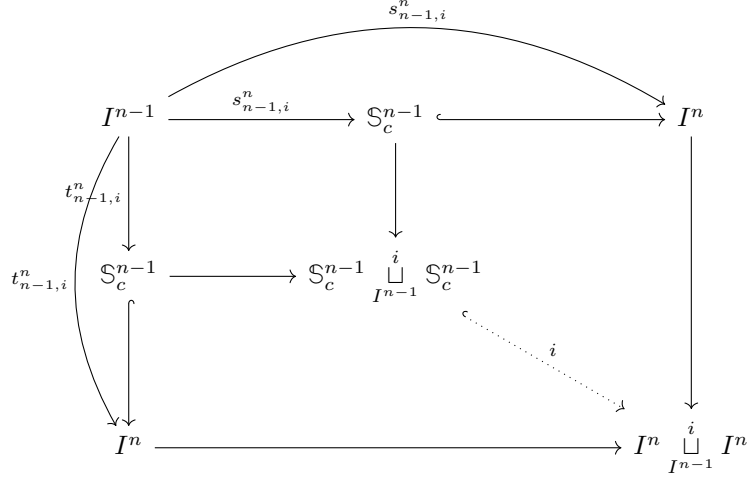
For all  $1 \leq i \leq n$  we are going to build by induction maps :

$$I^n \xrightarrow{\mu_i^n} I^n \sqcup_{I^{n-1}}^i I^n$$

such that  $I^n \sqcup_{I^{n-1}}^i I^n$  is the following pushout :

$$\begin{array}{ccc} I^{n-1} & \xrightarrow{s_{n-1,i}^n} & I^n \\ \downarrow t_{n-1,i}^n & & \downarrow \\ I^n & \xrightarrow{\quad} & I^n \sqcup_{I^{n-1}}^i I^n \end{array}$$

that is, we start with  $I^0 \xrightarrow{id} I^0$ , and we suppose that the maps  $I^{n-1} \xrightarrow{\mu_i^{n-1}} I^{n-1} \sqcup_{I^{n-2}}^i I^{n-1}$  are already defined for  $1 \leq i \leq n-1$ . We glue  $\mathbb{S}_c^{n-1}$  with itself along the same face and we obtain the inclusion  $i$  :



In order to build  $\mu_i^n$  we are going to build first its interior  $\mu_i^n : S_c^{n-1} \xrightarrow{\mu_i^n} S_c^{n-1} \sqcup_{I^{n-1}} S_c^{n-1}$ . It is defined by the following induction :

- If  $i = j$  then we put  $I^{i-1} \times \{0\} \times I^{n-i} \xrightarrow{id} I^{i-1} \times \{0\} \times I^{n-i}$ , where the identity map  $id$  sends the  $(n-1)$ -faces  $I^{i-1} \times \{0\} \times I^{n-i}$  of the first copy  $S_c^{n-1}$  in  $S_c^{n-1} \sqcup_{I^{n-1}} S_c^{n-1}$  to the  $(n-1)$ -faces  $I^{i-1} \times \{0\} \times I^{n-i}$  of the second copy  $S_c^{n-1}$ , and we put :  $I^{i-1} \times \{1\} \times I^{n-i} \xrightarrow{id} I^{i-1} \times \{1\} \times I^{n-i}$ , where the identity map  $id$  sends the  $(n-1)$ -faces  $I^{i-1} \times \{1\} \times I^{n-i}$  of the first copy  $S_c^{n-1}$  in  $S_c^{n-1} \sqcup_{I^{n-1}} S_c^{n-1}$  to the  $(n-1)$ -faces  $I^{i-1} \times \{1\} \times I^{n-i}$  of the second copy  $S_c^{n-1}$ .
- If  $1 \leq j < i \leq n$  then we put :  $I^{j-1} \times \{0\} \times I^{n-j} \xrightarrow{\mu_{i-1}^{n-1}} I^{j-1} \times \{0\} \times I^{n-j} \sqcup_{I^{n-2}}^{i-1} I^{j-1} \times \{0\} \times I^{n-j}$ ,  
and  $I^{j-1} \times \{1\} \times I^{n-j} \xrightarrow{\mu_{i-1}^{n-1}} I^{j-1} \times \{1\} \times I^{n-j} \sqcup_{I^{n-2}}^{i-1} I^{j-1} \times \{1\} \times I^{n-j}$ , where the codomains are given by the following pushout :

$$\begin{array}{ccc}
 I^{n-2} & \xrightarrow{s_{n-2,i-1}^{n-1}} & I^{n-1} \\
 \downarrow t_{n-2,i-1}^{n-1} & & \downarrow \\
 I^n & \longrightarrow & I^{n-1} \sqcup_{I^{n-2}}^{i-1} I^{n-1}
 \end{array}$$

- If  $1 \leq i < j \leq n$  then we put :  $I^{j-1} \times \{0\} \times I^{n-j} \xrightarrow{\mu_{j-1}^{n-1}} I^{j-1} \times \{0\} \times I^{n-j} \sqcup_{I^{n-2}}^{j-1} I^{j-1} \times \{0\} \times I^{n-j}$ ,  
and  $I^{j-1} \times \{1\} \times I^{n-j} \xrightarrow{\mu_{j-1}^{n-1}} I^{j-1} \times \{1\} \times I^{n-j} \sqcup_{I^{n-2}}^{j-1} I^{j-1} \times \{1\} \times I^{n-j}$ , where the codomains are given by the following pushout :

$$\begin{array}{ccc}
I^{n-2} & \xrightarrow{s_{n-2,j-1}^{n-1}} & I^{n-1} \\
\downarrow t_{n-2,j-1}^{n-1} & & \downarrow \\
I^n & \longrightarrow & I^{n-1} \sqcup_{I^{n-2}}^{j-1} I^{n-1}
\end{array}$$

Thus we obtain the desired extension  $\mu_i^n$  of  $\mu_i^{n-1}$  :

$$\begin{array}{ccccc}
I^n & & & & \\
\uparrow & \searrow \mu_i^n & & & \\
\mathbb{S}_c^{n-1} & \xrightarrow{\mu_i^{n-1}} & \mathbb{S}_c^{n-1} \sqcup_{I^{n-1}}^i \mathbb{S}_c^{n-1} & \xrightarrow{i} & I^n \sqcup_{I^{n-1}}^i I^n
\end{array}$$

### 7.2.2 Contractibility of $\text{COEND}(I^\bullet)$

Consider two maps in  $\mathbb{T}\text{op}$  :  $I^{n-1} \xrightarrow[f]{g} X$ , such that  $f$  and  $g$  are two  $(n-1)$ -cells of the operad  $\text{COEND}(I^\bullet)$ . Thus  $X$  is described as an iterated pushouts of the topological  $n$ -cubes  $I^n$  ( $n \in \mathbb{N}$ ) given by the global object  $I^\bullet$  in the pseudo-algebra  $\text{Cospan}(\mathbb{T}\text{op})$ ; and in particular  $X$  is contractible. We are going to build the contraction  $[f, g]_{n,j}^{n-1}$  by induction. Thus we suppose that for all  $1 \leq j \leq n-1$  the maps  $I^{n-1} \xrightarrow{[f, g]_{n-1,j}^{n-2}} X$  exist, and we start our induction with an easy choice of extension  $[f, g]_1^0$ , where  $f$  and  $g$  define here two points of  $X$  :  $I \xrightarrow{[f, g]_1^0} X$ . The contraction  $[f, g]_{n,j}^{n-1}$  is given by a continuous map  $I^n \xrightarrow{[f, g]_{n,j}^{n-1}} X$ . In order to do that, for all  $1 \leq j \leq n$ , we need first to define the map :  $\mathbb{S}_c^{n-1} \xrightarrow{\langle f, g \rangle_{n,j}^{n-1}} X$ . This map  $\langle f, g \rangle_{n,j}^{n-1}$  has the following definition :

- for  $i = j$  we put :  $I^{j-1} \times \{0\} \times I^{n-j} \xrightarrow{f} X$ , and  $I^{j-1} \times \{1\} \times I^{n-j} \xrightarrow{g} X$
- If  $1 \leq i < j \leq n$  then  $I^{i-1} \times \{0\} \times I^{n-i} \xrightarrow{[s_{n-2,i}^{n-1}(f), s_{n-2,i}^{n-1}(g)]_{n-1,j-1}^{n-2}} X$ , and  $I^{i-1} \times \{1\} \times I^{n-i} \xrightarrow{[t_{n-2,i}^{n-1}(f), t_{n-2,i}^{n-1}(g)]_{n-1,j-1}^{n-2}} X$
- If  $1 \leq j < i \leq n$  then  $I^{i-1} \times \{0\} \times I^{n-i} \xrightarrow{[s_{n-2,i-1}^{n-1}(f), s_{n-2,i-1}^{n-1}(g)]_{n-1,j}^{n-2}} X$ , and  $I^{i-1} \times \{1\} \times I^{n-i} \xrightarrow{[t_{n-2,i-1}^{n-1}(f), t_{n-2,i-1}^{n-1}(g)]_{n-1,j}^{n-2}} X$

then we obtain the desired extension :

$$\begin{array}{ccc}
I^n & & \\
\uparrow & \searrow [f, g]_{n,j}^{n-1} & \\
\mathbb{S}_c^{n-1} & \xrightarrow{\langle f, g \rangle_{n,j}^{n-1}} & X
\end{array}$$

Now consider two  $(n-1)$ -cells of  $\text{COEND}(I^\bullet)$  :  $I^{n-1} \xrightarrow[f]{g} X$ , such that for  $1 \leq j \leq n-1$  we have

$$f \circ s_{n-2,j}^{n-1} = g \circ s_{n-2,j}^{n-1} : I^{n-2} \xrightarrow{s_{n-2,j}^{n-1}} I^{n-1} \xrightarrow[f]{g} X,$$



that is  $s_{n-2,j}^{n-1}(f) = s_{n-2,j}^{n-1}(g)$ . We are going to build the contraction  $[f, g]_{n,j}^{n-1,-}$  by induction. Thus we suppose that for all  $1 \leq j \leq n-2$  the maps  $I^{n-1} \xrightarrow{[f,g]_{n-1,j}^{n-2,-}} X$  exist, and we start our induction with an easy choice of extension  $[f, g]_{2,1}^{1,-}$ , where  $f$  and  $g$  define here two paths in  $X$  :  $I^2 \xrightarrow{[f,g]_{2,1}^{1,-}} X$ . The map  $[f, g]_{n,j}^{n-1,-}$  is given by a continuous map  $I^n \xrightarrow{[f,g]_{n,j}^{n-1,-}} X$ . In order to do that, for all  $1 \leq j \leq n-1$ , we need first to define the map :  $\mathbb{S}_c^{n-1} \xrightarrow{\langle f, g \rangle_{n,j}^{n-1,-}} X$ . This map  $\langle f, g \rangle_{n,j}^{n-1,-}$  has the following definition :

- if  $i = j$  we put :  $I^{j-1} \times \{0\} \times I^{n-j} \xrightarrow{f} X$ , and  $I^j \times \{0\} \times I^{n-j-1} \xrightarrow{g} X$ ,  
and  $I^{j-1} \times \{1\} \times I^{n-j} \xrightarrow{[t_{n-2,j}^{n-1}(f), t_{n-2,j}^{n-1}(g)]_{n-1,j}^{n-2,-}} X$ , and  $I^j \times \{1\} \times I^{n-j-1} \xrightarrow{[t_{n-2,j}^{n-1}(f), t_{n-2,j}^{n-1}(g)]_{n-1,j}^{n-2,-}} X$ .
- If  $1 \leq i, j \leq n$  then we put :
  - if  $1 \leq i < j \leq n-1$  then  $I^{i-1} \times \{0\} \times I^{n-i} \xrightarrow{[s_{n-2,i}^{n-1}(f), s_{n-2,i}^{n-1}(g)]_{n-1,j-1}^{n-2,-}} X$ ,  
and  $I^{i-1} \times \{1\} \times I^{n-i} \xrightarrow{[t_{n-2,i}^{n-1}(f), t_{n-2,i}^{n-1}(g)]_{n-1,j-1}^{n-2,-}} X$ .
  - if  $2 \leq j+1 < i \leq n$  then  $I^{i-1} \times \{0\} \times I^{n-i} \xrightarrow{[s_{n-2,i-1}^{n-1}(f), s_{n-2,i-1}^{n-1}(g)]_{n-1,j}^{n-2,-}} X$ ,  
and  $I^{i-1} \times \{1\} \times I^{n-i} \xrightarrow{[t_{n-2,i-1}^{n-1}(f), t_{n-2,i-1}^{n-1}(g)]_{n-1,j}^{n-2,-}} X$

then we obtain the desired extension :

$$\begin{array}{ccc} & I^n & \\ \uparrow & \searrow [f, g]_{n,j}^{n-1,-} & \\ \mathbb{S}_c^{n-1} & \xrightarrow{\langle f, g \rangle_{n,j}^{n-1,-}} & X \end{array}$$

Now consider two  $(n-1)$ -cells of  $\mathbb{COEND}(I^\bullet)$  :  $I^{n-1} \xrightleftharpoons[g]{f} X$ , such that  $X$  is contractible, and such that for  $1 \leq j \leq n-1$  we have

$$f \circ t_{n-2,j}^{n-1} = g \circ t_{n-2,j}^{n-1} : I^{n-2} \xrightarrow{t_{n-2,j}^{n-1}} I^{n-1} \xrightleftharpoons[g]{f} X,$$

that is  $t_{n-2,j}^{n-1}(f) = t_{n-2,j}^{n-1}(g)$ . We are going to build the contraction  $[f, g]_{n,j}^{n-1,+}$  by induction. Thus we suppose that for all  $1 \leq j \leq n-2$  the maps  $I^{n-1} \xrightarrow{[f,g]_{n-1,j}^{n-2,+}} X$  exist, and we start our induction with an easy choice of extension  $[f, g]_{2,1}^{1,+}$ , where  $f$  and  $g$  define here two paths in  $X$  :  $I^2 \xrightarrow{[f,g]_{2,1}^{1,+}} X$ . The map  $[f, g]_{n,j}^{n-1,+}$  is given by a continuous map  $I^n \xrightarrow{[f,g]_{n,j}^{n-1,+}} X$ . In order to do that, for all  $1 \leq j \leq n-1$ , we need first to define the map :  $\mathbb{S}_c^{n-1} \xrightarrow{\langle f, g \rangle_{n,j}^{n-1,+}} X$ . This map  $\langle f, g \rangle_{n,j}^{n-1,+}$  has the following definition :

- if  $i = j$  we put :  $I^{j-1} \times \{1\} \times I^{n-j} \xrightarrow{f} X$ , and  $I^j \times \{1\} \times I^{n-j-1} \xrightarrow{g} X$ ,  
and  $I^{j-1} \times \{0\} \times I^{n-j} \xrightarrow{[s_{n-2,j}^{n-1}(f), s_{n-2,j}^{n-1}(g)]_{n-1,j}^{n-2,-}} X$ , and  $I^j \times \{0\} \times I^{n-j-1} \xrightarrow{[s_{n-2,j}^{n-1}(f), s_{n-2,j}^{n-1}(g)]_{n-1,j}^{n-2,-}} X$ ,
- If  $1 \leq i, j \leq n$  then we put :

- if  $1 \leq i < j \leq n-1$  then  $I^{i-1} \times \{0\} \times I^{n-i} \xrightarrow{[s_{n-2,i}^{n-1}(f), s_{n-2,i}^{n-1}(g)]_{n-1,j-1}^{n-2,+}} X$ ,  
and  $I^{i-1} \times \{1\} \times I^{n-i} \xrightarrow{[t_{n-2,i}^{n-1}(f), t_{n-2,i}^{n-1}(g)]_{n-1,j-1}^{n-2,+}} X$ .
- if  $2 \leq j+1 < i \leq n$  then  $I^{i-1} \times \{0\} \times I^{n-i} \xrightarrow{[s_{n-2,i-1}^{n-1}(f), s_{n-2,i-1}^{n-1}(g)]_{n-1,j}^{n-2,+}} X$ ,  
and  $I^{i-1} \times \{1\} \times I^{n-i} \xrightarrow{[t_{n-2,i-1}^{n-1}(f), t_{n-2,i-1}^{n-1}(g)]_{n-1,j}^{n-2,+}} X$ .

Then we obtain the desired extension :

$$\begin{array}{ccc}
 & I^n & \\
 \uparrow & \searrow [f,g]_{n,j}^{n-1,+} & \\
 \mathbb{S}_c^{n-1} & \xrightarrow{\langle f,g \rangle_{n,j}^{n-1,+}} & X
 \end{array}$$

and it is then straightforward to check the different axioms of contractions for such extensions  $[f,g]_{n,j}^{n-1,-}$ ,  $[f,g]_{n,j}^{n-1,+}$  and  $[f,g]_{n,j}^{n-1,+}$ . With 7.2.1 and 7.2.2 we thus have proved the :

**Theorem 2**  $I^\bullet$  is a  $B_C^0$ -coalgebra. □

### 7.2.3 The fundamental morphism of operads

Let us fix a topological space  $X \in \mathbb{Top}$ . From it we get a functor  $\mathbb{Top} \xrightarrow{\mathbb{Map}(-,X)} \mathbb{SET}^{op}$  in  $\mathbb{CAT}_{push}$ , thus from the functor :  $\mathbb{CAT}_{push} \xrightarrow{\mathbb{Cosp}(-)} \mathbb{Ps-S-Alg}$  we get the following morphism in  $\mathbb{Ps-S-Alg}$  :

$$\mathbb{Cosp}(\mathbb{Top}) \xrightarrow{\mathbb{Cosp}(\mathbb{Map}(-,X))} \mathbb{Cosp}(\mathbb{SET}^{op})$$

Also we have the functor :

$$(1 \downarrow i \circ \mathbb{Cosp}(-)) \xrightarrow{\mathbb{COEND}(-)} \mathbb{S-Oper}$$

which sends the following morphism  $\mathbb{Map}(-, X)$  of  $(1 \downarrow i \circ \mathbb{Cosp}(-))$  :

$$\begin{array}{ccc}
 & 1 & \\
 \swarrow I^\bullet & & \searrow \mathbb{Map}(I^\bullet, X)^{op} \\
 \mathbb{Cosp}(\mathbb{Top}) & \xrightarrow{\mathbb{Cosp}(\mathbb{Map}(-,X))} & \mathbb{Cosp}(\mathbb{SET}^{op})
 \end{array}$$

to the morphism of operads :

$$\mathbb{COEND}(I^\bullet) \xrightarrow{\mathbb{COEND}(\mathbb{Cosp}(\mathbb{Map}(-,X)))} \mathbb{COEND}(\mathbb{Map}(I^\bullet, X)^{op}) \simeq \mathbb{END}(\mathbb{Map}(I^\bullet, X))$$

this shows that  $\mathbb{Map}(I^\bullet, X)$  is an algebra for  $\mathbb{COEND}(I^\bullet)$ . But we proved in 7.2 that  $I^\bullet$  is also a  $B_C^0$ -coalgebra, which means that we have a morphism of operads :

$$B_C^0 \xrightarrow{!} \mathbb{COEND}(I^\bullet)$$

which shows that we have a morphism of operads :

$$B_C^0 \xrightarrow{\mathbb{COEND}(\mathbb{Cosp}(\mathbb{Map}(-,X))) \circ !} \mathbb{END}(\mathbb{Map}(I^\bullet, X))$$

that is the cubical set  $\mathbb{Map}(I^\bullet, X)$  :

$$\begin{array}{ccccccc}
\begin{array}{c} \xrightarrow{s_{n-1,n}^n} \\ \cdots \xrightarrow{s_{n-1,i}^n} \\ \cdots \xrightarrow{s_{n-1,1}^n} \\ \xrightarrow{t_{n-1,1}^n} \\ \cdots \xrightarrow{t_{n-1,i}^n} \\ \cdots \xrightarrow{t_{n-1,n}^n} \end{array} & \cdots \mathbb{M}\text{ap}(I^n, X) & \xrightarrow{\quad} & \mathbb{M}\text{ap}(I^{n-1}, X) \cdots \mathbb{M}\text{ap}(I^4, X) & \xrightarrow{\quad} & \mathbb{M}\text{ap}(I^3, X) & \xrightarrow{\quad} & \mathbb{M}\text{ap}(I^2, X) & \xrightarrow{\quad} & \mathbb{M}\text{ap}(I, X) & \xrightarrow{s_0^1} & \mathbb{M}\text{ap}(I^0, X) \\
& & & \begin{array}{c} \xrightarrow{s_{3,4}^4} \\ \xrightarrow{s_{3,3}^4} \\ \xrightarrow{s_{3,2}^4} \\ \xrightarrow{s_{3,1}^4} \\ \xrightarrow{t_{3,1}^4} \\ \xrightarrow{t_{3,2}^4} \\ \xrightarrow{t_{3,3}^4} \\ \xrightarrow{t_{3,4}^4} \end{array} & & \begin{array}{c} \xrightarrow{s_{2,3}^3} \\ \xrightarrow{s_{2,2}^3} \\ \xrightarrow{s_{2,1}^3} \\ \xrightarrow{t_{2,1}^3} \\ \xrightarrow{t_{2,2}^3} \\ \xrightarrow{t_{2,3}^3} \end{array} & & \begin{array}{c} \xrightarrow{s_{1,2}^2} \\ \xrightarrow{s_{1,1}^2} \\ \xrightarrow{t_{1,1}^2} \\ \xrightarrow{t_{1,2}^2} \end{array} & & \begin{array}{c} \xrightarrow{t_0^1} \end{array} & & 
\end{array}$$

is equipped with a structure of weak cubical  $\infty$ -category. This weak cubical  $\infty$ -category  $\Pi_\infty(X)$  is in fact a weak cubical  $\infty$ -groupoid (see [38] for the definition of cubical weak  $\infty$ -groupoids), called the *fundamental cubical weak  $\infty$ -groupoid* of  $X$ . Also if  $X \xrightarrow{f} Y$  is a continuous map between  $X$  and  $Y$ , then from our functorial constructions we get the following commutative diagram :

$$\begin{array}{ccc}
& & \mathbb{E}\text{ND}(\mathbb{M}\text{ap}(I^\bullet, X)) \\
& \nearrow \text{COEND}(\text{Cospan}(\mathbb{M}\text{ap}(-, X))) \circ ! & \\
B_C^0 & \xrightarrow{!} \text{COEND}(I^\bullet) & \\
& \searrow \text{COEND}(\text{Cospan}(\mathbb{M}\text{ap}(-, Y))) \circ ! & \\
& & \mathbb{E}\text{ND}(\mathbb{M}\text{ap}(I^\bullet, Y))
\end{array}
\quad \begin{array}{c} \downarrow \mathbb{E}\text{ND}(\mathbb{M}\text{ap}(I^\bullet, f)) \end{array}$$

which exhibits the *fundamental cubical weak  $\infty$ -groupoid functor* :

$$\mathbb{T}\text{op} \xrightarrow{\Pi_\infty(-)} \infty\text{-}\mathbb{C}\text{Grp}$$

which has a left adjoint functor  $CN_\infty$ . This pair of adjunction  $(CN_\infty, \Pi_\infty(-))$  should put an equivalence between the homotopy category of homotopy types and the homotopy category of  $\infty\text{-}\mathbb{C}\text{Grp}$  of cubical weak  $\infty$ -groupoids with connections equipped with an adapted Quillen model structure. This was shown to be true but in the context of the Cisinski model structure on the category of cubical sets with connections (see [42]).

Pursuing stacks : [33]

Comparaison globulaire-cubique : [27]

#### 7.2.4 Application for higher $K$ -theory

The functor  $\Pi_\infty(-)$  could be intuitively thought as the gluing of all the homotopy groups functor  $\pi_i$  together, and because the  $\pi_i$  are cohomologies,  $\Pi_\infty(-)$  could be thought as a *higher dimensional cohomology*, that is a functor between  $\infty$ -categories, or an  $\infty$ -functor which behave like cohomologies. It seems that such objects are of interest for the *Stolz-Teichner* program<sup>5</sup> [46] who try to investigate ideas from physic (TQFT=Topological Quantum Field Theory) through cohomologies, and also ETQFT (Extended TQFT) through higher dimensional cohomologies and vice-versa.

In this section we explain how to "glue" algebraic  $K_i$ -functors ( $i \in \mathbb{N}$ ) of Quillen :

$$\mathbb{R}\text{ings} \xrightarrow{K_i} \mathbb{T}\text{op}$$

into a single functor  $K_\infty$ , where here  $\mathbb{R}\text{ings}$  is the category of rings with unit.

<sup>5</sup>These ideas take their roots in the work of Graham Segal on Conformal Field Theory.

But first let us recall some basic facts which are defined more accurately in [43] : the functors  $K_i$  are defined by the composition :

$$\mathbb{Rings} \xrightarrow{\text{BGL}(-)^+ \times K_0(-)} \mathbb{Top} \xrightarrow{\pi_i(-)} \mathbb{Grp}$$

where for any rings  $R$  with unit :

- $\text{GL}(R) = \bigcup_{n=1}^{\infty} \text{GL}(n, R)$
- $\text{BGL}(R)$  is the classifying space of the group  $\text{GL}(R)$
- the  $+$ -construction on  $\text{BGL}(R)$  is taken relative to the perfect subgroup  $E(R)$  (elementary matrices) of  $\text{GL}(R)$
- $K_0(R)$  is given the discrete topology

Thus we get the functor  $K_{\infty}$  which is given by the composition :

$$\mathbb{Rings} \xrightarrow{\text{BGL}(-)^+ \times K_0(-)} \mathbb{Top} \xrightarrow{\Pi_{\infty}(-)} \infty\text{-}\mathbb{CGrp}$$

## 8 Importance of Coalgebraic structures for Globular and Cubical Higher Category Theory

### 8.1 The Batanin's construction and the author's construction

In the article [25], Michael Batanin has built the contractible operad  $B_C^0$  which algebras are globular weak  $\infty$ -categories. He also proved that the globular object  $D^{\bullet}$  in  $\mathbb{Top}$  consisting of topological disks :

$$D^0 \begin{array}{c} \xrightarrow{s_0^1} \\ \xrightarrow{t_0^1} \end{array} D^1 \begin{array}{c} \xrightarrow{s_1^2} \\ \xrightarrow{t_1^2} \end{array} D^2 \dots D^{n-1} \begin{array}{c} \xrightarrow{s_{n-1}^n} \\ \xrightarrow{t_{n-1}^n} \end{array} D^n \dots$$

is a  $B_C^0$ -coalgebra, which implication is the construction of the fundamental globular weak  $\infty$ -groupoid functor

$$\mathbb{Top} \xrightarrow{\Pi_{\infty}(-)} \infty\text{-}\mathbb{Grp}$$

In the other hand, in the article [37] the author built a coglobular object  $B_C^{\bullet}$

$$B_C^0 \begin{array}{c} \xrightarrow{s_0^1} \\ \xrightarrow{t_0^1} \end{array} B_C^1 \begin{array}{c} \xrightarrow{s_1^2} \\ \xrightarrow{t_1^2} \end{array} B_C^2 \dots B_C^{n-1} \begin{array}{c} \xrightarrow{s_{n-1}^n} \\ \xrightarrow{t_{n-1}^n} \end{array} B_C^n \dots$$

such that  $B_C^0$  is the contractible operad just above of Michael Batanin,  $B_C^1$  is the contractible operad which algebras are globular weak  $\infty$ -functors,  $B_C^2$  is the contractible operad which algebras are globular weak  $\infty$ -natural transformations, etc. Also we have the surprising fact : if  $B_C^{\bullet}$  is a  $B_C^0$ -coalgebra then it implies that the globular weak  $\infty$ -category of globular weak  $\infty$ -categories exists. We didn't prove yet this fact<sup>6</sup>, however this is an important improvement for globular higher category theory for two main reasons :

- in the beginning it was non-trivial to know why globular weak  $\infty$ -categories, globular weak  $\infty$ -functors, globular weak  $\infty$ -natural transformations, etc. organize in a globular weak  $\infty$ -category. Now we have replaced this very complex combinatorial question by a precise statement : the coendomorphism operad  $\text{COEND}(B_C^{\bullet})$  should be contractible<sup>7</sup>, like its topological little son<sup>8</sup>  $\text{COEND}(D^{\bullet})$ .

<sup>6</sup>Because of lack of financial support, because the author ignore how to fill applications, how to interact with persons in order to be enough persuasive, etc.

<sup>7</sup>which imply that it is equipped with a composition system. See [37]

<sup>8</sup>or little brother ...

- it brings a spectacular analogy between topological spaces and globular higher categories, which was hope by Grothendieck and Thomason. Let us gives a first smell of such analogy :

– Consider the following 1-cell in the operad  $\mathbb{COEND}(D^\bullet)$  of topological disks :

$$D^1 \xrightarrow{\mu_0^1} D^1 \sqcup_{D^0} D^1$$

and consider a topological space  $X$ . With these we get the following 1-cell of the fundamental weak  $\infty$ -groupoid  $\Pi_\infty(X)$  :

$$\mathbb{M}ap(D^1, X) \times_{\mathbb{M}ap(D^0, X)} \mathbb{M}ap(D^1, X) \xrightarrow{\circ_0^1} \mathbb{M}ap(D^1, X)$$

– Suppose that  $\mathbb{COEND}(B_C^\bullet)$  is contractible. It is then possible to consider the following 1-cell in the operad  $\mathbb{COEND}(B_C^\bullet)$  of operadical disks :

$$B_C^1 \xrightarrow{\mu_0^1} B_C^1 \sqcup_{B_C^0} B_C^1$$

and with this 1-cell of  $\mathbb{COEND}(B_C^\bullet)$ , we get the following 1-cell in the suspected globular weak  $\infty$ -category of globular weak  $\infty$ -categories :

$$\mathbb{A}lg(B_C^1)(0) \times_{\mathbb{A}lg(B_C^0)(0)} \mathbb{A}lg(B_C^1)(0) \xrightarrow{\circ_0^1} \mathbb{A}lg(B_C^1)(0)$$

which is the composition of globular weak  $\infty$ -functors !

## 8.2 Steps toward the cubical weak $\infty$ -category of cubical weak $\infty$ -categories

Consider the following internal cocubical object in a subcategory  $\mathcal{C}^9$  of the category  $\mathbb{M}nd$  of monads, such that  $\mathcal{C}$  has pushouts.

$$\begin{array}{ccccccc}
& & & & & & \xrightarrow{s_{n-1,1}^n} \\
& & & & & & \cdots \\
& & & & & & \xrightarrow{s_{n-1,i}^n} \\
& & & & & & \cdots \\
& & & & & & \xrightarrow{s_{n-1,n-1}^n} \\
B_C^0 & \xrightarrow{s_1^0} & B_C^1 & \xrightarrow{s_{1,2}^2} & B_C^2 & \xrightarrow{s_{2,3}^3} & B_C^3 & \xrightarrow{s_{3,1}^4} & B_C^4 & \cdots & B_C^{n-1} & \xrightarrow{s_{n-1,1}^n} & B_C^n \cdots \\
& \xrightarrow{t_1^0} & & \xrightarrow{s_{1,1}^2} & & \xrightarrow{s_{2,2}^3} & & \xrightarrow{s_{3,2}^4} & & & & \xrightarrow{s_{n-1,i}^n} & \\
& & & \xrightarrow{t_{1,1}^2} & & \xrightarrow{s_{2,1}^3} & & \xrightarrow{s_{3,3}^4} & & & & \xrightarrow{s_{n-1,n-1}^n} & \\
& & & \xrightarrow{t_{1,2}^2} & & \xrightarrow{s_{2,2}^3} & & \xrightarrow{s_{3,4}^4} & & & & \xrightarrow{t_{n-1,n-1}^n} & \\
& & & & & \xrightarrow{t_{2,1}^3} & & \xrightarrow{t_{3,4}^4} & & & & \cdots \\
& & & & & \xrightarrow{t_{2,2}^3} & & \xrightarrow{t_{3,3}^4} & & & & \xrightarrow{t_{n-1,i}^n} & \\
& & & & & \xrightarrow{t_{2,3}^3} & & \xrightarrow{t_{3,2}^4} & & & & \xrightarrow{t_{n-1,1}^n} & \\
& & & & & & & \xrightarrow{t_{3,1}^4} & & & & \xrightarrow{t_{n-1,1}^n} & \\
& & & & & & & & & & & \xrightarrow{t_{n-1,1}^n} & 
\end{array}$$

such that  $B_C^0$  is the  $\mathbb{S}^0$ -operad which algebras are cubical weak  $\infty$ -categories. Also the  $\mathbb{S}^1$ -operad  $B_C^1$  which algebras are cubical weak  $\infty$ -functors, the  $\mathbb{S}^2$ -operad  $B_C^2$  which algebras are cubical weak  $\infty$ -natural transformations, etc. where  $\mathbb{S}^1$  is the cartesian monad which algebras are cubical strict  $\infty$ -functors,  $\mathbb{S}^2$  is the cartesian monad which algebras are cubical strict  $\infty$ -natural transformations, etc. are not difficult to be built. For example the underlying combinatorics of the  $\mathbb{S}^1$ -collection of  $B_C^1$  comes easily from the monad of cubical weak  $\infty$ -functors as defined in [38] and the underlying combinatorics of the  $\mathbb{S}^2$ -collection of  $B_C^2$  comes easily from the monad of cubical weak  $\infty$ -natural transformations as defined in [38]. Also according to the cubical combinatorics it is straightforward to see that the cartesian monad  $\mathbb{S}^n$  of cubical strict  $n$ -transformations act on the category  $\mathbb{C}Sets^{2^n}$ , the cartesian product  $2^n$  times in  $\mathbb{CAT}$  of the category of cubical sets with itself. In order to build these contractible  $\mathbb{S}^n$ -operads  $B_C^n$  we have different technics to do it. We can use for example the

<sup>9</sup>Such subcategory exists according to a private communication with Ross Street and John Bourke who give me such accurate construction. We won't describe it here because of lack of time.

formalism of the  $\mathbb{T}$ -categorical stretchings as developed in [40], or we can use the theory of Garner [30] to build a fibrant replacement of the  $\mathbb{S}^n$ , or we can more classically just use the technology developed in [25, 34].

This internal cocubical object  $B^\bullet$  of  $\mathcal{C}$  is a global object of the pseudo-algebra  $\text{Cospan}(\mathcal{C})$ , where here we deal with cubical higher cospans. Thanks to the functor defined in 6.2

$$(1 \downarrow i \circ \text{Cospan}(-)) \xrightarrow{\text{COEND}(-)} \mathbb{S}\text{-Oper}$$

the following morphism  $\text{Alg}(-)$  of  $(1 \downarrow i \circ \text{Cospan}(-))$  :

$$\begin{array}{ccc} & 1 & \\ \swarrow I^\bullet & & \searrow \text{Alg}(B_C^\bullet)^{op} \\ \text{Cospan}(\mathcal{C}) & \xrightarrow{\text{Cospan}(\text{Alg}(-))} & \text{Cospan}(\text{SET}^{op}) \end{array}$$

is sent to the morphism of operads :

$$\text{COEND}(B_C^\bullet) \xrightarrow{\text{COEND}(\text{Cospan}(\text{Alg}(-)))} \text{COEND}(\text{Alg}(B_C^\bullet)^{op}) \simeq \mathbb{E}\text{ND}(\text{Alg}(B_C^\bullet)) .$$

This shows that  $\text{Alg}(B_C^\bullet)$  is an algebra for  $\text{COEND}(B_C^\bullet)$ . Now suppose  $B_C^\bullet$  is also a  $B_C^0$ -coalgebra. In fact we put the following conjecture :

**Conjecture** *The operad of coendomorphism  $\text{COEND}(B_C^\bullet)$  is contractible.*

Contractibility here means the cubical one, as developed in [34], where we consider contractions similar to their globular analogue, plus the "connections-contractions" which are for contractions what connections are for cubical  $\infty$ -categories.

If we accept this conjecture then it means that we have a morphism of operads :

$$B_C^0 \xrightarrow{!} \text{COEND}(B_C^\bullet)$$

which means that we have a morphism of operads :

$$B_C^0 \xrightarrow{\text{COEND}(\text{Cospan}(\text{Alg}(-))) \circ !} \mathbb{E}\text{ND}(\text{Alg}(B_C^\bullet))$$

which shows that the cubical set  $\text{Alg}(B_C^\bullet)(0)^{10}$  :

$$\begin{array}{ccccccc} \begin{array}{c} \xrightarrow{s_{n-1,n}^n} \\ \cdots \cdots \cdots \rightarrow \\ \xrightarrow{s_{n-1,i}^n} \\ \cdots \cdots \cdots \rightarrow \\ \xrightarrow{s_{n-1,1}^n} \\ \cdots \cdots \cdots \rightarrow \\ \xrightarrow{t_{n-1,1}^n} \\ \cdots \cdots \cdots \rightarrow \\ \xrightarrow{t_{n-1,i}^n} \\ \cdots \cdots \cdots \rightarrow \\ \xrightarrow{t_{n-1,n}^n} \end{array} & \xrightarrow{\quad} & \begin{array}{c} \xrightarrow{s_{3,4}^4} \\ \xrightarrow{s_{3,3}^4} \\ \xrightarrow{s_{3,2}^4} \\ \xrightarrow{s_{3,1}^4} \\ \xrightarrow{t_{3,1}^4} \\ \xrightarrow{t_{3,2}^4} \\ \xrightarrow{t_{3,3}^4} \\ \xrightarrow{t_{3,4}^4} \end{array} & \xrightarrow{\quad} & \begin{array}{c} \xrightarrow{s_{2,3}^3} \\ \xrightarrow{s_{2,2}^3} \\ \xrightarrow{s_{2,1}^3} \\ \xrightarrow{t_{2,1}^3} \\ \xrightarrow{t_{2,2}^3} \\ \xrightarrow{t_{2,3}^3} \end{array} & \xrightarrow{\quad} & \begin{array}{c} \xrightarrow{s_{1,2}^2} \\ \xrightarrow{s_{1,1}^2} \\ \xrightarrow{t_{1,1}^2} \\ \xrightarrow{t_{1,2}^2} \end{array} & \xrightarrow{\quad} & \begin{array}{c} \xrightarrow{s_0^1} \\ \xrightarrow{t_0^1} \end{array} & \xrightarrow{\quad} & \text{Alg}(B_C^0)(0) \end{array}$$

is equipped with a structure of weak cubical  $\infty$ -category. This is the cubical weak  $\infty$ -category of cubical weak  $\infty$ -categories. Like for globular higher category theory, we thus have an amazing analogy between topological spaces and cubical higher categories, up to these conjectures related to coalgebraicity. Thanks to it we can mimic the globular approach of weak Grothendieck  $\infty$ -topos as described in [39] to have a real smell of what is a cubical weak Grothendieck  $\infty$ -topos.

Cet article montre a quel point le materiel 2-categorique developper dans [44, 45, 47, 48] permet de generaliser la theorie de Batanin. And we can imagine other geometry for higher groupoids associated to topological spaces

<sup>10</sup>For each  $n \in \mathbb{N}$ ,  $\text{Alg}(B_C^n)(0)$  means the class of objects of the category  $\text{Alg}(B_C^n)$ .

## References

- [1] Michael Batanin, *Monoidal globular categories as a natural environment for the theory of weak- $n$ -categories*, Advances in Mathematics (1998), volume 136, pages 39–103 . [1](#), [2](#), [14](#), [20](#), [35](#), [37](#)
- [2] Fahd Ali Al-Agl and Ronald Brown and Richard Steiner, *Multiple Categories: The Equivalence of a Globular and a Cubical Approach*, Advances in Mathematics 170, 71–118 (2002) . [5](#), [34](#)
- [3] Brown, R., *A new higher homotopy groupoid: the fundamental globular  $\omega$ -groupoid of a filtered space*, Homology, Homotopy Appl. 10 (1) (2008) 327–343. 219 . [15](#)
- [4] Brown, R., *double modules?, double categories and groupoids, and a new homotopy double groupoid*, arXiv Math (0903.2627) (2009) 8 pp. .
- [5] Brown, R. and Higgins, P. J. *The equivalence of  $\infty$ -groupoids and crossed complexes*, Cahiers Topologie Géom. Différentielle 22 (4) (1981) 371–386. .
- [6] Brown, R. and Loday, J.-L. *Homotopical excision, and Hurewicz theorems for  $n$ -cubes of spaces*, Proc. London Math. Soc. (3) 54 (1) (1987) 176–192. .
- [7] Denis-Charles Cisinski, *Batanin higher groupoids and homotopy types*, Contemporary Mathematics (2007) volume 431, pages 171–186.
- [8] Laurent Coppey and Christian Lair, *Leçons de théorie des esquisses*, Université Paris VII, (1985). [8](#)
- [9] Eduardo Dubuc, *Adjoint triangles*, Lecture Notes in Mathematics (Springer-Verlag 1968) volume 61, pages 69–91.
- [10] F. Foltz, *Sur la catégorie des foncteurs dominés*, Cahiers de Topologie et de Géométrie Différentielle Catégorique (1969), volume 11(2), pages 101–130). [8](#)
- [11] Marco Grandis and Mauri, *Cubical sets and their site*, Theory Applic. Categories 11 (2003) 185–201. . [3](#)
- [12] Kamel Kachour, *Définition algébrique des cellules non-strictes*, Cahiers de Topologie et de Géométrie Différentielle Catégorique, volume 1 (2008), pages 1–68.
- [13] Camell Kachour, *Introduction to Higher Cubical Operads. Second Part : The Functor of Fundamental Cubical Weak  $\infty$ -Groupoids for Spaces* [2](#)
- [14] Camell Kachour and Mark Weber, *2-topos view for higher operads*, Work in progress [1](#), [2](#), [10](#), [15](#), [17](#), [19](#)
- [15] Camell Kachour, *Algebraic definition of weak  $(\infty, n)$ -categories*, Published in Theory and Applications of Categories (2015), Volume 30, No. 22, pages 775–807 [3](#)
- [16] Camell Kachour, *An algebraic approach to weak  $\omega$ -groupoids*, Australian Category Seminar, 14 September 2011. <http://web.science.mq.edu.au/groups/coact/seminar/cgi-bin/speaker-info.cgi?name=Camell+Kachour>
- [17] G.M.Kelly, *A unified treatment of transfinite constructions for free algebras, free monoids, colimits, associated sheaves, and so on*, Bulletin of the Australian Mathematical Society (1980), volume 22, pages 1–83.
- [18] Tom Leinster, *Higher Operads, Higher Categories*, London Mathematical Society Lecture Note Series, Cambridge University Press (2004), volume 298. [10](#)
- [19] Georges Maltsiniotis, *La catégorie cubique avec connections est une catégorie test stricte*, (preprint) (2009) 1–16 . [1](#), [2](#), [15](#), [34](#)
- [20] Timothy Porter,  *$n$ -types of simplicial groups and crossed  $n$ -cubes*, Topology 32 (1) (1993) 5–24.
- [21] Richard Steiner, *Thin fillers in the cubical nerves of omega-categories*, Theory Appl. Categ. 16 (2006) No. 8, 144–173 . [5](#)

- [22] Ross Street, *The petit topos of globular sets*, Journal of Pure and Applied Algebra. Volume 154, pages 299–315 (2000) . [15](#), [19](#), [37](#)
- [23] Mark Weber, *Yoneda structures from 2-toposes*, Applied Categorical Structures, Volume 15, pages 259–323 (2007). [15](#), [37](#)
- [24] Mark Weber, *Operads Within Monoidal Pseudo-algebras*, Applied Categorical Structures, Volume 13(5), page 389–420 (2005) . [15](#), [17](#), [19](#), [37](#)
- [25] Michael Batanin, *Monoidal globular categories as a natural environment for the theory of weak-n-categories*, Advances in Mathematics (1998), volume 136, pages 39–103 . [1](#), [2](#), [14](#), [20](#), [35](#), [37](#)
- [26] Robert Blackwell, Max Kelly and John Power, *Two-dimensional monad theory*, Journal of Pure and Applied Algebra (1989), Volume 59, pages 1–41.
- [27] Fahd Ali Al-Agl and Ronald Brown and Richard Steiner, *Multiple Categories: The Equivalence of a Globular and a Cubical Approach*, Advances in Mathematics 170, 71–118 (2002) . [5](#), [34](#)
- [28] Brown, R., *A new higher homotopy groupoid: the fundamental globular  $\omega$ -groupoid of a filtered space*, Homology, Homotopy Appl. 10 (1) (2008) 327–343. 219 . [15](#)
- [29] Ronald Brown and Philip Higgins and Rafael Sivera, *Non abelian Algebraic Topology*, Book, Tracts in Mathematics 15, European Mathematical Society .
- [30] Richard Garner, *Understanding the small object argument*. <https://arxiv.org/pdf/0712.0724.pdf>
- [31] Marco Grandis, *Higher cospans and weak cubical categories (Cospans in algebraic topology, I)*, Theory and Applications of Categories, Volume 18, Number 12, 2007, pp 321–347.
- [32] Marco Grandis, *Higher fundamental groupoids for spaces*, Topology and its Applications, Volume 129, 2003, pages 281–299.
- [33] Alexander Grothendieck, *Pursuing Stacks (A la poursuite des Champs)*, Manuscript, <https://thescrivener.github.io/PursuingStacks/>.
- [34] Camell Kachour, *Introduction to Higher Cubical Operads. First Part : The Cubical Operad of Cubical Weak  $\infty$ -Categories*. Prépublication de l’IHES, <http://preprints.ihes.fr/2017/M/M-17-19.pdf>
- [35] Camell Kachour and Mark Weber, *The structure of generalised higher spans*, Work in progress [1](#), [2](#), [10](#), [15](#), [17](#), [19](#)
- [36] Camell Kachour, *Algebraic definition of weak  $(\infty, n)$ -categories*, Published in Theory and Applications of Categories (2015), Volume 30, No. 22, pages 775–807
- [37] Camell Kachour, *Steps toward the weak higher category of weak higher categories in the globular setting*, Published in Categories and General Algebraic Structures with Applications, Volume 4, Number 1 (2015)
- [38] Camell Kachour, *Aspects of Cubical Higher Category Theory*, <https://arxiv.org/pdf/1702.00336.pdf>
- [39] Camell Kachour, *Globular perspective for Grothendieck  $\infty$ -topos and Grothendieck  $(\infty, n)$ -topos*, Topos À l’IHES, Conference, Short talk, 27 November 2015, <https://www.youtube.com/watch?v=topnX7dvENY>
- [40] Camell Kachour, *Aspects of Globular Higher Category Theory*, PhD thesis, Macquarie University, 2014
- [41] Tom Leinster, *Higher Operads, Higher Categories*, London Mathematical Society Lecture Note Series, Cambridge University Press (2004), volume 298.
- [42] Georges Maltsiniotis, *La catégorie cubique avec connections est une catégorie test stricte*, (preprint) (2009) 1–16 . [1](#), [2](#), [15](#), [34](#)
- [43] Jonathan Rosenberg, *Algebraic K-theory and Its Applications*, Book, Graduate Texts in Mathematics, Volume 147, Springer.



- [44] Ross Street, *The petit topos of globular sets*, Journal of Pure and Applied Algebra. Volume 154, pages 299–315 (2000) . [15](#), [19](#), [37](#)
- [45] Ross Street, *The role of Michael Batanin's monoidal globular categories*, Higher Category Theory', editors Ezra Getzler and Mikhail Kapranov, Contemporary Mathematics, Volume 230, A.M.S. 1998, pages 99–116.
- [46] Stephan Stolz and Peter Teichner, *Supersymmetric field theories and generalized cohomology*, Mathematical foundations of quantum field theory and perturbative string theory, Volume 83, Proceeding Symposium, Pure Mathematics, American Mathematical Society, 2011, pages 279–340.
- [47] Mark Weber, *Yoneda structures from 2-toposes*, Applied Categorical Structures, Volume 15, pages 259–323 (2007). [15](#), [37](#)
- [48] Mark Weber, *Operads Within Monoidal Pseudo-algebras*, Applied Categorical Structures, Volume 13(5), page 389–420 (2005) . [15](#), [17](#), [19](#), [37](#)

Camell KACHOUR  
 Department of Mathematics, Macquarie University  
 North Ryde, NSW 2109 Australia.  
 Phone: 00 612 9850 8942  
 Email: [camell.kachour@gmail.com](mailto:camell.kachour@gmail.com)