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Same decay rate of second order evolution equations with or without delay

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ABSTRACT

We consider abstract second order evolution equations with unbounded feedback with delay. If the delay term is small enough, we rigorously prove the fact that the system with delay has the same decay rate than the one without delay. Some old and new results easily follow.

1. Introduction

Time-delay appears in many applications (in biology, electrical engineering and mechanics, ...) [1–3], and in many cases, in particular for distributed parameter systems, even arbitrarily small delays in the feedback may destabilize the system, see for instance [4–12]. Therefore the stability issue of systems with delay is of theoretical and practical importance.

Different ad-hoc techniques have been recently developed to prove some exponential or polynomial decay of the energy of wave type equations with delay, we refer to [9,10,13–16] and the references citing them. In this paper, we rigorously prove the fact, accepted by many authors but not proved up to now, that if the delay term is small enough, then the system with delay has the same decay rate than the one without delay. The main (and simple) idea is to use a duality argument already used in [17, §2] in a fully different context. To our best knowledge, this idea was not used before in the context of problems with delay.

To be more precise, we first consider the abstract setting from [13] since it allows to consider a quite large class of wave type problems with time delay feedbacks. In this setting existence results were obtained in [13] under realistic assumptions that we keep. But stability results are proved in a fully different manner.

Before going on, let us present the abstract framework. Let H be a complex Hilbert space with norm and inner product denoted respectively by $\|.\|_H$ and $(.,.)_H$. Let $A : D(A) \to H$ be a self-adjoint positive operator with a compact inverse in H. Let $V := D(A^{\frac{1}{2}})$ be the domain of $A^{\frac{1}{2}}$. Denote by V' the dual space of V obtained by means of the inner product in H.

Further, for $i = 1, 2$, let U_i be a complex Hilbert space (which will be identified to its dual space) with norm and inner product denoted respectively by $\|.\|_{U_i}$ and $(.,.)_{U_i}$ and let $B_i \in \mathcal{L}(U_i, V')$.

In this paper we analyze stability properties of the closed loop system with a delay

$$\begin{cases} \ddot{u}(t) + Au(t) + B_1B_1^*\dot{u}(t) + B_2B_2^*\dot{u}(t-\tau) = 0,\ t > 0, \\ u(0) = u_0,\ \dot{u}(0) = u_1, \\ B_2^*\dot{u}(t-\tau) = f^0(t-\tau),\ 0 < t < \tau. \end{cases} \tag{1}$$

This system consists of adding an extra term $B_2B_2^*\dot{u}(t-\tau)$ to the stable system (without delay)

$$\begin{cases} \ddot{\omega}(t) + A\omega(t) + B_1B_1^*\dot{\omega}(t) = 0,\ t > 0, \\ \omega(0) = \omega_0,\ \dot{\omega}(0) = \omega_1. \end{cases} \tag{2}$$

Existence results for both systems are well-known, we refer to [16] for the details, see also Section 2 for some recalls. In [13], it was shown that the assumption

$$\exists\, \alpha \in (0,1),\ \forall u \in V,\ \left\|B_2^*u\right\|_{U_2}^2 \le \alpha \left\|B_1^*u\right\|_{U_1}^2 \tag{3}$$

guarantees the existence of a unique solution to (1) and decay of its energy. Recall that if (3) is not satisfied, dissipativeness may

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be lost and there exist cases where some unstable solutions to (1) may appear (see [9,10,12] for the wave equation). Hence this assumption seems realistic.

Our main result, proved in Section 3, concerns the relation between the decay rate of the energy of (2) and of (1). Under the additional assumption (3), in the case of exponential (resp. polynomial) decay of the energy of (2), we prove the same decay rate for the energy of (1). For less decay rate, we get almost the same result by using the frequency domain approach from [18]. This result allows to recover several recent results concerning exponential or polynomial decay of wave type equation with delayed feedback, but several new ones can be deduced since it mainly suffices to use the decay rate of the problem without delay. This fact is illustrated by many concrete examples in Section 4.

Let us finish this introduction with some notation used in the paper. Recall that the inner product (resp. norm) of H is denoted by $(\cdot,\cdot)_H$ (resp. $\|\cdot\|_H$). Similarly the inner product (resp. norm) of V will be denoted by $(\cdot,\cdot)_V$ (resp. $\|\cdot\|_V$). By $a \lesssim b$, we mean that there exists a constant $C > 0$ independent of a, b, such that $a \le Cb$. Finally $a \sim b$ means that both $a \lesssim b$ and $b \lesssim a$ hold.

2. Some recalls

Here we recall some results from [13] (see also [16]): By introducing the auxiliary variable $z(\rho,\, t) = B_2^*\dot{u}(t - \tau\rho)$ for $\rho \in (0,\, 1)$ and $t > 0$, problem (1) admits the following equivalent formulation

$$\begin{cases} \ddot{u}(t) + Au(t) + B_1B_1^*\dot{u}(t) + B_2z(1,\, t) = 0,\ t > 0 \\ \tau\frac{\partial z}{\partial t} + \frac{\partial z}{\partial \rho} = 0,\ t > 0,\ 0 < \rho < 1 \\ u(0) = u_0,\ \dot{u}(0) = u_1,\ z(\rho,\, 0) = f^0(-\tau\rho),\ 0 < \rho < 1 \\ z(0,\, t) = B_2^*\dot{u}(t),\ t > 0. \end{cases} \tag{4}$$

The Hilbert setting of this problem (or problem (1)) is the following one: First we introduce the Hilbert space

$$\mathcal{H} = V \times H \times L^2((0,\, 1),\, U_2),$$

equipped with the usual inner product

$$\left((u,\, v,\, z)^\top,\, (\tilde{u},\, \tilde{v},\, \tilde{z})^\top\right)_{\mathcal{H}} = (u,\, \tilde{u})_V + (v,\, \tilde{v})_H + \tau\eta\int_0^1 \left(z(\rho),\, \tilde{z}(\rho)\right)_{U_2} d\rho, \tag{5}$$

where η is a positive constant satisfying

$$1 < \eta < \frac{2}{\alpha} - 1, \tag{6}$$

that exists because $0 < \alpha < 1$

Then we define the operator $\mathcal{A}$ on $\mathcal{H}$ by

$$D(\mathcal{A}) := \{(u,\, v,\, z)^\top \in V \times V \times H^1((0,\, 1),\, U_2);\ z(0) = B_2^*v,\ Au + B_1B_1^*v + B_2z(1) \in H\},$$

and

$$\mathcal{A}\,(u,\, v,\, z)^\top = \left(v,\, -Au - B_1B_1^*v - B_2z(1),\, -\frac{1}{\tau}\frac{\partial z}{\partial \rho}\right)^\top,$$
$$\forall (u,\, v,\, z)^\top \in D(\mathcal{A}).$$

For further purposes, let us also recall the Hilbert setting of problem (2). Introduce the Hilbert space

$$\mathcal{H}_0 = V \times H,$$

equipped with the usual inner product

$$\left((u,\, v)^\top,\, (\tilde{u},\, \tilde{v})^\top\right)_{\mathcal{H}_0} = (u,\, \tilde{u})_V + (v,\, \tilde{v})_H\,. \tag{7}$$

Then we define the operator $\mathcal{A}_0$ on $\mathcal{H}_0$ by

$$D(\mathcal{A}_0) := \{(u,\, v)^\top \in V \times V;\ Au + B_1B_1^*v \in H\},$$

and

$$\mathcal{A}_0\,(u,\, v)^\top = \left(v,\, -Au - B_1B_1^*v\right)^\top,\ \forall (u,\, v)^\top \in D(\mathcal{A}_0).$$

It is shown in [13, Theorem 2.1] that system (1) is well-posed under the (weaker) assumption

$$\exists\, \alpha \in (0,\, 1],\ \forall u \in V,\ \left\|B_2^*u\right\|_{U_2}^2 \le \alpha\left\|B_1^*u\right\|_{U_1}^2. \tag{8}$$

Now we define the energy associated with system (1) (which corresponds to the inner product in $\mathcal{H}$) by

$$E(t) := \frac{1}{2}\left(\|u(t)\|_V^2 + \|\dot{u}(t)\|_H^2 + \tau\eta\int_0^1 \left\|B_2^*\dot{u}(t - \tau\rho)\right\|_{U_2}^2 d\rho\right).$$

It is shown in [13, Proposition 3.1] that under the assumption (3) this energy is non-increasing. This proposition actually equivalently proves that

$$\Re(\mathcal{A}U,\, U)_{\mathcal{H}} \lesssim -(\|B_1^*v\|_{U_1}^2 + \|z(1)\|_{U_2}^2),\ \forall U \in D(\mathcal{A}). \tag{9}$$

Note that a similar property holds for $\mathcal{A}_0$, namely, one has

$$\Re(\mathcal{A}_0(u,\, v)^\top,\, (u,\, v)^\top)_{\mathcal{H}_0} = -\|B_1^*v\|_{U_1}^2,\ \forall (u,\, v)^\top \in D(\mathcal{A}_0). \tag{10}$$

Under the assumption that V is compactly embedded into V', it was shown in [13, Proposition 3.4] that $i\mathbb{R} \cap \sigma(\mathcal{A}_0) = \emptyset$ if and only if $i\mathbb{R} \cap \sigma(\mathcal{A}) = \emptyset$.

3. Energy decay

In the whole section, we assume that condition (3) holds and that $i\mathbb{R} \cap \sigma(\mathcal{A}_0) = \emptyset$ (that guarantees that $i\mathbb{R} \cap \sigma(\mathcal{A}) = \emptyset$). Under these assumptions, our goal is to show that the decay of the energy of problem (1) is the same as the one of system (2). For that purpose, we use a duality argument used in [17, §2] in a fully different context. Let us first start with a useful identity.

Lemma 3.1. *For any $\xi \in \mathbb{R}$ and any $(u, v, z)^\top \in D(\mathcal{A})$, we consider the solution $(u^*, v^*)^\top \in D(\mathcal{A}_0)$ of*

$$(i\xi - \mathcal{A}_0)(u^*, v^*)^\top = (u, v)^\top. \tag{11}$$

Then we have

$$\|(u, v)^\top\|_{\mathcal{H}_0}^2 = -\left((i\xi - \mathcal{A})(u, v, z)^\top, (u^*, v^*, 0)^\top\right)_{\mathcal{H}} + 2(B_1^*v, B_1^*v^*)_{U_1} + (z(1), B_2^*v^*)_{U_2}. \tag{12}$$

Proof. By the definition of $\mathcal{A}$, we directly check that

$$\begin{aligned}&\left((i\xi - \mathcal{A})(u, v, z)^\top, (u^*, v^*, 0)^\top\right)_{\mathcal{H}} \\ &= i\xi(u, u^*)_V - (v, u^*)_V + i\xi(v, v^*)_H + (u, v^*)_V \\ &+ (B_1^*v, B_1^*v^*)_{U_1} + (z(1), B_2^*v^*)_{U_2}.\end{aligned}$$

Similarly, we have

$$\begin{aligned}&\left((u, v)^\top, (i\xi - \mathcal{A}_0)(u^*, v^*)^\top\right)_{\mathcal{H}_0} \\ &= -i\xi(u, u^*)_V - (u, v^*)_V - i\xi(v, v^*)_H + (v, u^*)_V \\ &+ (B_1^*v, B_1^*v^*)_{U_1}.\end{aligned}$$

Comparing these two identities, one directly obtains

$$\begin{aligned}&\left((u, v)^\top, (i\xi - \mathcal{A}_0)(u^*, v^*)^\top\right)_{\mathcal{H}_0} \\ &= -\left((i\xi - \mathcal{A})(u, v, z)^\top, (u^*, v^*, 0)^\top\right)_{\mathcal{H}} \\ &+ 2(B_1^*v, B_1^*v^*)_{U_1} + (z(1), B_2^*v^*)_{U_2}.\end{aligned}$$

The conclusion follows from (11). ∎

We go on with two properties of the resolvent of $\mathcal{A}$ on the imaginary axis.

Lemma 3.2. *For any $\xi \in \mathbb{R}$ and any $F = (f, g, h)^\top \in \mathcal{H}$, let $U = (u, v, z)^\top \in D(\mathcal{A})$ be the solution of*

$$(i\xi - \mathcal{A})U = F. \tag{13}$$

Then one has

$$\|B_1^* v\|_{U_1}^2 + \|z(1)\|_{U_2}^2 \lesssim \|F\|_{\mathcal{H}} \|U\|_{\mathcal{H}}, \tag{14}$$

as well as

$$\|U\|_{\mathcal{H}} \lesssim \|(u, v)^\top\|_{\mathcal{H}_0} + \|F\|_{\mathcal{H}}. \tag{15}$$

Proof. By (9), we have

$$\Re((i\xi - \mathcal{A})U, U)_{\mathcal{H}} \gtrsim \|B_1^* v\|_{U_1}^2 + \|z(1)\|_{U_2}^2.$$

We deduce (14) by Cauchy–Schwarz's inequality and (13).

For the second estimate we notice that (13) implies that z satisfies the differential equation

$$i\xi z + \frac{1}{\tau}\frac{\partial z}{\partial \rho} = h \text{ on } (0, 1),$$

hence z can be expressed as (compare with [13, page 424])

$$z(\rho) = e^{-i\xi\tau(\rho-1)} z(1) - \tau \int_\rho^1 e^{-i\xi\tau(\rho-\rho')} h(\rho')\, d\rho', \forall \rho \in [0, 1].$$

Consequently, using the triangular and Cauchy–Schwarz's inequalities, one has

$$\|z(\rho)\|_{U_2}^2 \le 2\|z(1)\|_{U_2}^2 + 2\tau \int_\rho^1 \|h(\rho')\|_{U_2}^2\, d\rho', \forall \rho \in [0, 1].$$

Integrating this estimate in $(0, 1)$, we deduce that

$$\|z\|_{L^2(0,1;U_2)}^2 \le 2\|z(1)\|_{U_2}^2 + 2\tau \|h\|_{L^2(0,1;U_2)}^2.$$

Using the estimate (14), we have found that

$$\|z\|_{L^2(0,1;U_2)}^2 \lesssim \|F\|_{\mathcal{H}} \|U\|_{\mathcal{H}} + \|h\|_{L^2(0,1;U_2)}^2.$$

This estimate and the definition of the norm of $\mathcal{H}$ directly imply that

$$\|U\|_{\mathcal{H}}^2 \lesssim \|(u, v)^\top\|_{\mathcal{H}_0}^2 + \|F\|_{\mathcal{H}} \|U\|_{\mathcal{H}} + \|h\|_{L^2(0,1;U_2)}^2.$$

By Young's inequality, we deduce that

$$\|U\|_{\mathcal{H}}^2 \lesssim \|(u, v)^\top\|_{\mathcal{H}_0}^2 + \|F\|_{\mathcal{H}}^2,$$

which immediately leads to (15). ■

We are ready to prove our main result.

Theorem 3.3. *Let the condition (3) be satisfied, that $i\mathbb{R} \cap \sigma(\mathcal{A}_0) = \emptyset$ and that the resolvent of $\mathcal{A}_0$ satisfies*

$$\|(i\xi - \mathcal{A}_0)^{-1}\| \lesssim M(|\xi|), \forall \xi \in \mathbb{R}, \tag{16}$$

where M is a continuous, positive, and non decreasing function from $[0, \infty)$ into itself. Then the resolvent of $\mathcal{A}$ satisfies

$$\|(i\xi - \mathcal{A})^{-1}\| \lesssim M(|\xi|), \forall \xi \in \mathbb{R}. \tag{17}$$

Proof. Fix $\xi \in \mathbb{R}$, $F = (f, g, h)^\top \in \mathcal{H}$, and let $U = (u, v, z)^\top \in D(\mathcal{A})$ be the solution of (13). Then we can consider $(u^*, v^*)^\top \in D(\mathcal{A}_0)$ solution of (11). We now notice that (10) implies

$$\begin{aligned}\|B_1^* v^*\|_{U_1}^2 &= \Re((i\xi - \mathcal{A}_0)(u^*, v^*)^\top, (u^*, v^*)^\top)_{\mathcal{H}_0} \\ &= \Re((u, v)^\top, (u^*, v^*)^\top)_{\mathcal{H}_0}.\end{aligned}$$

Cauchy–Schwarz's inequality then yields

$$\|B_1^* v^*\|_{U_1}^2 \le \|(u, v)^\top\|_{\mathcal{H}_0} \|(u^*, v^*)^\top\|_{\mathcal{H}_0}. \tag{18}$$

Recalling the identity (12), we may write

$$\begin{aligned}\|(u, v)^\top\|_{\mathcal{H}_0}^2 &= -\big(F, (u^*, v^*, 0)^\top\big)_{\mathcal{H}} \\ &\quad + 2(B_1^* v, B_1^* v^*)_{U_1} + (z(1), B_2^* v^*)_{U_2}.\end{aligned}$$

Now using again Cauchy–Schwarz's inequality and the assumption (3), we find

$$\begin{aligned}\|(u, v)^\top\|_{\mathcal{H}_0}^2 &\lesssim \|F\|_{\mathcal{H}} \|(u^*, v^*)^\top\|_{\mathcal{H}_0} \\ &\quad + (\|B_1^* v\|_{U_1} + \|z(1)\|_{U_2}) \|B_1^* v^*\|_{U_1}.\end{aligned}$$

Young's inequality and the estimates (14) and (18) lead to

$$\begin{aligned}\|(u, v)^\top\|_{\mathcal{H}_0}^2 &\lesssim \|F\|_{\mathcal{H}} \|(u^*, v^*)^\top\|_{\mathcal{H}} \\ &\quad + \frac{1}{\varepsilon}\|F\|_{\mathcal{H}} \|U\|_{\mathcal{H}} + \varepsilon \|(u, v)^\top\|_{\mathcal{H}_0} \|(u^*, v^*)^\top\|_{\mathcal{H}_0},\end{aligned}$$

for all $\varepsilon > 0$ (here and below, the involved constants are independent of ε). With the help of (15), we get

$$\begin{aligned}\|(u, v)^\top\|_{\mathcal{H}_0}^2 &\lesssim \frac{1}{\varepsilon}\|F\|_{\mathcal{H}}^2 + \|F\|_{\mathcal{H}} \|(u^*, v^*)^\top\|_{\mathcal{H}} \\ &\quad + \frac{1}{\varepsilon}\|F\|_{\mathcal{H}} \|(u, v)^\top\|_{\mathcal{H}_0} + \varepsilon \|(u, v)^\top\|_{\mathcal{H}_0} \|(u^*, v^*)^\top\|_{\mathcal{H}_0}.\end{aligned}$$

We finally use the assumption (16) that yields (recalling (11))

$$\|(u^*, v^*)^\top\|_{\mathcal{H}_0} \lesssim M(|\xi|) \|(u, v)^\top\|_{\mathcal{H}_0},$$

to conclude that

$$\begin{aligned}\|(u, v)^\top\|_{\mathcal{H}_0}^2 &\le \frac{C}{\varepsilon}\|F\|_{\mathcal{H}}^2 + C\|F\|_{\mathcal{H}} M(|\xi|) \|(u, v)^\top\|_{\mathcal{H}_0} \\ &\quad + \frac{C}{\varepsilon}\|F\|_{\mathcal{H}} \|(u, v)^\top\|_{\mathcal{H}_0} + C\varepsilon M(|\xi|) \|(u, v)^\top\|_{\mathcal{H}_0}^2,\end{aligned}$$

for some positive constant C independent of ε. Choosing $C\varepsilon M(|\xi|) = \frac{1}{2}$, we find

$$\|(u, v)^\top\|_{\mathcal{H}_0}^2 \lesssim M(|\xi|)\|F\|_{\mathcal{H}}^2 + \|F\|_{\mathcal{H}} M(|\xi|) \|(u, v)^\top\|_{\mathcal{H}_0}.$$

Again using Young's inequality, we arrive at

$$\|(u, v)^\top\|_{\mathcal{H}_0}^2 \lesssim M(|\xi|)\|F\|_{\mathcal{H}}^2 + M(|\xi|)^2 \|F\|_{\mathcal{H}}^2.$$

Since $M(\xi) \le (1 + M(|\xi|))^2$, we finally have found that

$$\|(u, v)^\top\|_{\mathcal{H}_0} \lesssim (1 + M(|\xi|)) \|F\|_{\mathcal{H}},$$

and we conclude by (15) that

$$\|U\|_{\mathcal{H}} \lesssim (1 + M(|\xi|)) \|F\|_{\mathcal{H}} \lesssim M(|\xi|) \|F\|_{\mathcal{H}}. \quad ■$$

This result combined with the well-known frequency domain approach yields the following decay rates of the semi-group generated by $\mathcal{A}$. We start with the exponential decay.

Corollary 3.4. *Let the condition (3) be satisfied, and assume that the semi-group $(e^{t\mathcal{A}_0})_{t\ge 0}$ is exponentially stable in $\mathcal{H}_0$. Then the semi-group $(e^{t\mathcal{A}})_{t\ge 0}$ is exponentially stable in $\mathcal{H}$.*

Proof. By a well-known result due to Huang and Prüss [19,20], $(e^{t\mathcal{A}_0})_{t\ge 0}$ is exponentially stable in $\mathcal{H}_0$ if and only if $i\mathbb{R} \cap \sigma(\mathcal{A}_0) = \emptyset$ and (16) with $M(x) = 1$ holds. Hence by Theorem 3.3, (17) holds with $M(x) = 1$, and again applying Huang/Prüss theorem to $\mathcal{A}$, we conclude. ■

For polynomial decays, we replace Huang/Prüss theorem by Borichev–Tomilov theorem [21, Theorem 2.4] to obtain the next result.

Corollary 3.5. *Let the condition (3) be satisfied, and assume that the semi-group $(e^{t\mathcal{A}_0})_{t\geq 0}$ is polynomially stable in $\mathcal{H}_0$, namely $i\mathbb{R}\cap\sigma(\mathcal{A}_0)=\emptyset$ and there exists a positive real number ℓ such that*

$$\|e^{t\mathcal{A}_0}U_0\| \lesssim t^{-\frac{1}{\ell}}\|U_0\|_{\mathcal{D}(\mathcal{A}_0)}, \quad \forall U_0\in\mathcal{D}(\mathcal{A}_0), \quad \forall t>1. \tag{19}$$

Then the semi-group $(e^{t\mathcal{A}})_{t\geq 0}$ is polynomially stable in $\mathcal{H}$, i.e.,

$$\|e^{t\mathcal{A}}U\| \lesssim t^{-\frac{1}{\ell}}\|U\|_{\mathcal{D}(\mathcal{A})}, \quad \forall U\in\mathcal{D}(\mathcal{A}), \quad \forall t>1. \tag{20}$$

Proof. Assuming that $i\mathbb{R}\cap\sigma(\mathcal{A}_0)=\emptyset$, by [21, Theorem 2.4], (19) holds if and only if (16) with $M(x)=x^{\ell}$ holds. As before the conclusion follows with the help of Theorem 3.3, and again applying [21, Theorem 2.4] to $\mathcal{A}$. ■

For lower decay, the equivalence between the semi-group decay rate and the asymptotic behavior of the resolvent on the imaginary axis is not guaranteed, but by taking advantage of a result due to Batty and Duyckaerts [18, Theorem 1.5] and our Theorem 3.3, we get as before the next corollary.

Corollary 3.6. *Let the condition (3) be satisfied, and assume that $i\mathbb{R}\cap\sigma(\mathcal{A}_0)=\emptyset$ and that (16) holds with a continuous, positive, and non decreasing function M from $[0,\infty)$ into itself. Then the semi-group $(e^{t\mathcal{A}})_{t\geq 0}$ has the following asymptotic decay in $\mathcal{H}$:*

$$\|e^{t\mathcal{A}}U\| \lesssim \frac{1}{M_{\log}^{-1}\left(\frac{t}{C}\right)}\|U\|_{\mathcal{D}(\mathcal{A})}, \quad \forall U\in\mathcal{D}(\mathcal{A}), \quad \forall t>1,$$

for some positive constant C, and $M_{\log}$ is defined by

$$M_{\log}(x)=M(x)\left(\log(1+M(x))+\log(1+x)\right), \forall x\geq 0.$$

4. Applications

In this section, we want to present some concrete examples of evolution equations for which we can apply the results of the previous section to obtain some decay rate of the associated energy. We mainly focused on weaker decay than the exponential one, because in most of the examples, if the system without delay has an exponentially decaying energy, then the corresponding system with a delay has also an exponentially decaying energy.

4.1. A $1-d$ wave equation with interior damping

We here consider a $1-d$ wave equation with a delay feedback

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2}(x,t)-\dfrac{\partial^2 u}{\partial x^2}(x,t)+\alpha_1\dfrac{\partial u}{\partial t}(\xi,t)\delta_\xi+\alpha_2\dfrac{\partial u}{\partial t}(\xi,t-\tau)\delta_\xi=0 \\ \quad 0<x<1, t>0, \\ u(0,t)=u(1,t)=0 \\ \quad t>0, \\ u(x,0)=u_0(x) \text{ and } u_t(x,0)=u_1(x) \\ \quad 0<x<1, \\ \dfrac{\partial u}{\partial t}(\xi,t-\tau)=f_0(t-\tau) \\ \quad 0<t<\tau, \end{cases} \tag{21}$$

where $\xi\in(0,1), \alpha_1>0, \alpha_2>0$, δ_ξ is the Dirac mass concentrated in the point $\xi\in(0,1)$ and $\tau>0$.

This system enters into our abstract framework, if we take $H=L^2(0,1)$, $V=H_0^1(0,1)$ and define the operator

$$A: D(A)\to H: \varphi\longmapsto -\frac{d^2\varphi}{dx^2},$$

where $D(A)=H^2(\Omega)\cap V$, which is self-adjoint and positive with a compact inverse in H. For $i=1$ or 2, we further define $U_i=\mathbb{R}$ and the operators B_i as

$$B_i:\mathbb{R}\to V'; \varphi\longmapsto \sqrt{\alpha_i}\varphi\delta_\xi, \tag{22}$$

where

$$\langle\delta_\xi, v\rangle_{V',V}=v(\xi), \forall v\in V. \tag{23}$$

The system can then be rewritten in the form (1) and trivially the relation (3) is equivalent to

$$\alpha_2<\alpha_1.$$

In [10, §7.1.2], it was shown that the system (21) is well posed when $\alpha_2\leq\alpha_1$, that the energy is non-increasing for $\alpha_2<\alpha_1$ and that an energy decay rate of t^{-2} holds when $\xi\in\mathcal{S}$, the set of all irrational numbers $\rho\in(0,1)$ such that the sequence $(a_n)_{n\geq 1}$ of its expansion $[0,a_1,\dots,a_n,\dots]$ as a continued fraction is bounded (roughly speaking this is the set all irrational numbers which are badly approximated by rational numbers, see [16, p. 29]). By Theorem 5.1 of [22], the decay energy rate of (21) when $\alpha_2=0$ (that is no delay occurs in the system) being in $\frac{1}{t^2}$, by Corollary 3.5 we recover the same decay rate as in [10, §7.1.2].

Let us finish this subsection by a result that to our best knowledge is new.

Corollary 4.1. *Under the assumptions of this subsection, if $\alpha_2<\alpha_1$, then for any $\varepsilon>0$, there exists a set $N_\varepsilon\subset(0,1)$ of measure 1, such that for all $\xi\in N_\varepsilon$ the energy of the system (21) decays with decay rate $t^{-\frac{1}{1+\varepsilon}}$, namely (20) holds with $\ell=2(1+\varepsilon)$.*

Proof. For any $\varepsilon>0$, [22, Theorem 5.1] yields a set $N_\varepsilon\subset(0,1)$ of measure 1, such that for all $\xi\in N_\varepsilon$, (19) holds with $\ell=2(1+\varepsilon)$ ($\mathcal{A}_0$ being the operator associated with (21) when $\alpha_2=0$). Applying Corollary 3.5, we get the same decay rate for system (21). ■

4.2. A $1-d$ Euler–Bernoulli beam with interior damping

We now consider the following $1-d$ Euler–Bernoulli beam equation with a delay term at $\xi\in(0,1)$:

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2}(x,t)+\dfrac{\partial^4 u}{\partial x^4}(x,t)+\alpha_1\dfrac{\partial u}{\partial t}(\xi,t)\delta_\xi+\alpha_2\dfrac{\partial u}{\partial t}(\xi,t-\tau)\delta_\xi=0 \\ \quad 0<x<1, t>0, \\ u(0,t)=u(1,t)=\dfrac{\partial^2 u}{\partial x^2}(0,t)=\dfrac{\partial^2 u}{\partial x^2}(1,t)=0 \\ \quad t>0, \\ u(x,0)=u_0(x) \text{ and } u_t(x,0)=u_1(x) \\ \quad 0<x<1, \\ \dfrac{\partial u}{\partial t}(\xi,t-\tau)=f_0(t-\tau) \\ \quad 0<t<\tau, \end{cases} \tag{24}$$

where u denotes the transverse displacement of the beam. As before $\alpha_1>0, \alpha_2>0$, δ_ξ is the Dirac mass concentrated at the point $\xi\in(0,1)$ and $\tau>0$. For this problem, we need to introduce $H=L^2(0,1)$, $V=H^2(0,1)\cap H_0^1(0,1)$ and the operator

$$A: D(A)\to H: \varphi\longmapsto \frac{d^4\varphi}{dx^4},$$

where $D(A) = \{\varphi \in H^4(0,1) \cap H_0^1(0;1), \frac{\partial^2 u}{\partial x^2}(0) = \frac{\partial^2 u}{\partial x^2}(1) = 0\}$. Again this operator A is self-adjoint and positive with a compact inverse in H. For $i = 1$ or 2, we still take $U_i = \mathbb{R}$ and define the operator B_i by (22).

With these notations, the system (24) can be rewritten in the form (1). Hence it is well posed when $\alpha_2 \le \alpha_1$ and the energy is non-increasing for $\alpha_2 < \alpha_1$.

Without delay (i.e., $\alpha_2 = 0$), the problem was analyzed in [22, Theorem 5.6], where it was shown that if $\xi \in \mathcal{S}$, then the energy of the system decays polynomially like t^{-2} and that for all $\varepsilon > 0$, there exists a set $M_\varepsilon \subset (0,1)$ of measure 1, such that for all $\xi \in M_\varepsilon$, the decay rate has the form $t^{-\frac{2}{1+\varepsilon}}$. If $\alpha_2 < \alpha_1$ and if $\xi \in \mathcal{S}$, by using the previous result and Corollary 3.5 we recover the same decay rate as in [13, Proposition 7.8]. For all $\varepsilon > 0$ and all $\xi \in M_\varepsilon$, application of Corollary 3.5 gives the decay rate in the form $t^{-\frac{2}{1+\varepsilon}}$, which seems to be a new result. More precisely, we can state the

Corollary 4.2. *Under the assumptions of this subsection, if $\alpha_2 < \alpha_1$, then for any $\varepsilon > 0$, there exists a set $M_\varepsilon \subset (0,1)$ of measure 1, such that for all $\xi \in M_\varepsilon$ the energy of the system (24) decays with decay rate $t^{-\frac{2}{1+\varepsilon}}$, namely*
(20) *holds with $\ell = 1 + \varepsilon$.*

4.3. The multi-dimensional wave equation with internal dampings

Let Ω be a bounded domain of $\mathbb{R}^n$, $n \ge 1$, with a sufficiently smooth boundary Γ. On this domain, we consider the wave equation with either two internal dampings or with boundary dampings: a standard one and a delayed one.

4.3.1. Internal dampings

Here we consider the following problem

$$\begin{cases} \frac{\partial^2 u}{\partial t^2}(x,t) - \Delta u(x,t) + \alpha_1(x)\frac{\partial u}{\partial t}(x,t) + \alpha_2(x)\frac{\partial u}{\partial t}(x,t-\tau) = 0 \\ \quad \text{in } \Omega \times (0,+\infty), \\ u = 0 \\ \quad \text{on } \Gamma \times (0,+\infty), \\ u(x,0) = u_0(x) \text{ and } \frac{\partial u}{\partial t}(x,0) = u_1(x) \\ \quad \text{in } \Omega, \\ \frac{\partial u}{\partial t}(x,t) = f(x,t) \\ \quad \text{in } \Omega \times (-\tau,0), \end{cases} \tag{25}$$

where $\alpha_1, \alpha_2 \in L^\infty(\Omega)$ are non negative functions.

To enter into our abstract framework, we set $H = L^2(\Omega)$, $V = H_0^1(\Omega)$, $D(A) = V \cap H^2(\Omega)$, and

$$A : D(A) \to H : \varphi \longmapsto -\Delta\varphi. \tag{26}$$

This operator A is self-adjoint and positive with a compact inverse in H. We further define $U_i = L^2(\Omega)$, $i = 1, 2$ and introduce the operators

$$B_i : L^2(\Omega) \to V : \varphi \longmapsto \sqrt{\alpha_i}\varphi.$$

In this setting, system (25) can be rewritten in the form (1). Furthermore a complete characterization of the condition (3) can be given, as shown in the next lemma.

Lemma 4.3. *In the setting of this subsection, (3) holds if and only if*

$$\exists\, \alpha \in (0,1),\ \alpha_2 \le \alpha\alpha_1 \quad \text{a. e. in } \Omega. \tag{27}$$

Proof. In our setting, as we have

$$\|B_i^* u\|_{U_i}^2 = \int_\Omega \alpha_i(x)|u(x)|^2\,dx,\ \forall u \in H_0^1(\Omega),$$

and since $H_0^1(\Omega)$ is dense in $L^2(\Omega)$, (3) holds if and only if

$$\exists\, \alpha \in (0,1),\ \int_\Omega (\alpha\alpha_1(x) - \alpha_2(x))|u(x)|^2\,dx \ge 0,\ \forall u \in L^2(\Omega). \tag{28}$$

Obviously (27) guarantees that (28) holds, so let us concentrate on the converse implication. First by setting

$$L_+^1(\Omega) := \{u \in L^1(\Omega) : u \ge 0 \text{ a.e. in } \Omega\},$$

we notice that

$$\{|u|^2 : u \in L^2(\Omega)\} = L_+^1(\Omega),$$

simply because $v \in L_+^1(\Omega)$ if and only if $u = \sqrt{v} \in L^2(\Omega)$.

Therefore, by setting $h = \alpha\alpha_1 - \alpha_2$, if the condition (28) holds, then

$$\int_\Omega h(x)v(x)\,dx \ge 0,\ \forall v \in L_+^1(\Omega).$$

By taking $v = h^-$ ($h = \alpha_1 - \alpha\alpha_2$ being in $L^\infty(\Omega)$, it is also in $L^1(\Omega)$) and using the splitting $h = h^+ - h^-$, we get

$$-\int_\Omega |h^-(x)|^2\,dx \ge 0,$$

which yields $h^- = 0$ and proves that $h = h^+$. ∎

Altogether under the additional assumption (27), problem (25) is well-posed and its energy is decaying.

According to Corollary 3.4, if (27) holds and if system (25) with $\alpha_2 = 0$ is exponentially stable, then system (24) is exponentially stable as well. In particular using the results from [23,24], we can state the following results that cover the results from [9,14,15].

Corollary 4.4. *In addition to the assumptions of this section, suppose that (27) holds and that*

$$\alpha_1 \ge a_0 > 0 \text{ a.e. in } \omega, \tag{29}$$

for some non empty subset ω of Ω. If one of the following assumptions holds
1. the boundary of Ω is of class C^∞ and ω satisfies the Geometric Control Condition (GCC). Recall [25] that the GCC can be formulated as follows: For a subset ω of Ω, we shall say that ω satisfies the Geometric Control Condition if there exists $T > 0$ such that every geodesic traveling at speed one issued from Ω at time $t = 0$ intersects ω before time T,
2. the boundary of Ω is of class C^2 and ω is a neighborhood of $\overline{\Gamma(x_0)}$ for some $x_0 \in \mathbb{R}^n$, where

$$\Gamma(x_0) = \{x \in \Gamma;\ m(x) \cdot \nu(x) > 0\}, \tag{30}$$

where $m(x) = (x - x_0)$, and $\nu(x)$ is the unit outward normal vector at $x \in \Gamma$,
3. Ω is convex or the boundary of Ω is of class $C^{1,1}$ and ω contains G, where G satisfies the condition (g, G) from [24, p. 1583].
Then system (24) is exponentially stable.

Similarly weaker decay rate for system (25) with $\alpha_2 = 0$ gives rise to the same decay rate for system (24) if (27) holds. Concerning polynomial decay rate, let us state the next results.

Corollary 4.5. *In addition to the assumptions of this subsection, suppose that (27) and (29) hold. Then the following hold*
1. If Ω is the unit square of $\mathbb{R}^2$ and

$$(a, b) \times (0, 1) \subset \omega,$$

for some $0 \leq a < b < 1$, then (20) holds with $\ell = 2$ if $a > 0$, while $\ell = 3/2$ if $a = 0$.
2. If Ω is a partially rectangular domain and ω contains the non-rectangular part of Ω, then (20) holds with $\ell = 2$.
3. If Ω and ω satisfy the assumptions of [26, §1.1], then there exists $\ell > 0$ such that (20) holds.

Proof. Both cases follow from Corollary 3.5 and the decay rate of system (25) with $\alpha_2 = 0$. For point 1, we use [27] for $a > 0$ and [28] for $a = 0$; for point 2, we take advantage of [29] (combined with [21, Theorem 2.4]), and finally for point 3, we use [26]. ■

Note that points 1 and 2 are new, while point 3 improves the polynomial decay rate in $t^{-\ell/2}$ obtained in [30] (where the first condition from (3.2) of [30] has to be replaced by (27)).

Logarithmic decay rate for system (25) can also be obtained.

Corollary 4.6. *In addition to the assumptions of this subsection, we suppose that (27) and (29) hold. If α_1 is smooth and not identically equal to zero and if the boundary of Ω is smooth, then*

$$\|e^{t\mathcal{A}}U\| \lesssim \frac{1}{\log t}\|U\|_{\mathcal{D}(\mathcal{A})}, \quad \forall U \in \mathcal{D}(\mathcal{A}), \quad \forall t > 1. \tag{31}$$

Proof. Under the above assumption on α_1 and on the boundary of Ω, it was shown in [31] that $i\mathbb{R} \cap \sigma(\mathcal{A}_0) = \emptyset$ and that (16) holds with $M(x) = e^{Cx}$, for some positive constant C. Consequently by Corollary 3.6, under the additional assumption (27), the semigroup generated by $\mathcal{A}$ satisfies (31) since $M_{\log}^{-1}(t) \sim \log t$, for t large (see [18, Example 1.6]). ■

Due to [32, Theorem 2.2], this last corollary remains valid if α_1 is only $L^\infty(\Omega)$, the boundary of Ω is only C^2 and the Dirichlet boundary conditions in (25) are replaced by the mixed boundary conditions:

$$u = 0 \text{ in } \Gamma_0 \times (0, +\infty),$$
$$\frac{\partial u}{\partial \nu} + pu = 0 \text{ in } \Gamma_1 \times (0, +\infty),$$

with $p \in C^1(\overline{\Gamma_1})$, $p \geq 0$, where Γ_0 and Γ_1 are two open sets of Γ such that $\Gamma = \overline{\Gamma_0} \cup \overline{\Gamma_1}$, $\Gamma_0 \cap \Gamma_1 = \emptyset$, and either Γ_0 is non empty or p is positive on a non empty set of Γ_1.

4.3.2. Boundary dampings

Here we consider the following wave equation with boundary dampings:

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2}(x,t) - \Delta u(x,t) = 0 \text{ in } \Omega \times (0, +\infty), \\ \dfrac{\partial u}{\partial \nu} + \alpha_1(x)\dfrac{\partial u}{\partial t}(x,t) + \alpha_2(x)\dfrac{\partial u}{\partial t}(x, t-\tau) = 0 \text{ in } \Gamma \times (0, +\infty), \\ u(x,0) = u_0(x) \text{ and } \dfrac{\partial u}{\partial t}(x,0) = u_1(x) \text{ in } \Omega, \\ \dfrac{\partial u}{\partial t}(x,t) = f(x,t) \text{ in } \Gamma \times (-\tau, 0), \end{cases} \tag{32}$$

where for $i = 1$ or 2, $\alpha_i \in L^\infty(\Gamma)$ and is non negative.

First, we rewrite this system in the form (1). For this purpose, we introduce $H = L^2(\Omega)$, $V = H^1(\Omega)$,

$$D(A) = \{u \in H^2(\Omega) : \frac{\partial u}{\partial \nu} = 0 \text{ on } \Gamma\},$$

and A defined by (26). As before the operator A is self-adjoint and positive with a compact inverse in H. Moreover we define $U_1 = U_2 = L^2(\Gamma)$, and for $i = 1$ or 2, and introduce the operator

$$B_i^* : V \to L^2(\Gamma) : \varphi \longmapsto \sqrt{\alpha_i}\gamma_0\varphi,$$

where γ_0 is the trace operator from $H^1(\Omega)$ to $L^2(\Gamma)$. Consequently B_i is given by

$$B_i : L^2(\Gamma) \to V' : y \longmapsto B_i y$$

with

$$\langle B_i y, \varphi\rangle_{V',V} = \int_\Gamma \gamma_0\varphi\sqrt{\alpha_i} y d\sigma, \forall \varphi \in V.$$

With these notations, system (32) can be rewritten in the form (1). As in the previous subsection (see Lemma 4.3), for our system the condition (3) holds if and only if

$$\exists \alpha \in (0,1),\ \alpha_2 \leq \alpha\alpha_1 \quad \text{a. e. on } \Gamma. \tag{33}$$

Therefore under this additional assumption, problem (32) is well-posed and its energy is decaying.

First many papers are devoted to the exponential decay of (32) with or without delay, using the results from [33–37] without delay and using Corollary 3.4, we recover for instance the results from [9,12,30]. On the other hand, to our best knowledge, there is no result about the polynomial or logarithmic decay of system (32). Let us then state some of them.

Corollary 4.7. *In addition to the assumptions of this subsection, suppose that (33) and that*

$$\alpha_1 \geq a_0 > 0 \text{ a.e. in } \theta,$$

for some non empty subset θ of Γ. Assume that Ω is a partially rectangular domain and θ contains $\partial W \cap \Gamma$, where W is the non-rectangular part of Ω, then (20) holds with $\ell = 4$ but in

$$\dot{\mathcal{H}} = \{(u, v, z) \in H^1(\Omega) \times L^2(\Omega) \times L^2((0,1), L^2(\Gamma));$$
$$\int_\Omega v\, dx + \int_\Gamma \alpha_1 u\, d\sigma(x) = 0\}.$$

Proof. This follows from Corollary 3.5 and the decay rate (19) with $\ell = 4$ of system (32) with $\alpha_2 = 0$ obtained in [38]. ■

Corollary 4.8. *In addition to the assumptions of this subsection, we suppose that Ω has a smooth boundary, α_1 is smooth and is not identically equal to zero, and that (33) holds. Then (31) holds.*

Proof. As Theorem 1 of [39] shows that (16) holds with $M(x) = e^{Cx}$, for some positive constant C, we conclude by Corollary 3.6. ■

Extension of this last result to second order operators with variable coefficients instead of the Laplace operator can be obtained with the help of the results from [32,40].

4.4. Weakly coupled and partially damped with boundary delay

Let $\Omega \subset \mathbb{R}^n$, $n \geq 1$ be an open bounded set with a boundary Γ of class C^2. We assume that Γ is divided into two open parts Γ_0 and Γ_1, i.e. $\Gamma = \overline{\Gamma_0} \cup \overline{\Gamma_1}$ such that $\Gamma_0 \cap \Gamma_1 = \emptyset$. In this domain, we consider the following weakly coupled and partially damped

with boundary feedbacks

$$\begin{cases} \frac{\partial^2 u}{\partial t^2}(x,t) - a\Delta u(x,t) + by(x,t) = 0 & \text{in } \Omega \times (0,+\infty), \\ \frac{\partial^2 y}{\partial t^2}(x,t) - \Delta y(x,t) + bu(x,t) = 0 & \text{in } \Omega \times (0,+\infty), \\ u = 0 & \text{on } \Gamma_0 \times (0,+\infty), \\ a\frac{\partial u}{\partial \nu} + \gamma u + \alpha_1 \frac{\partial u}{\partial t}(x,t) + \alpha_2 \frac{\partial u}{\partial t}(x,t-\tau) = 0 & \text{on } \Gamma_1 \times (0,+\infty), \\ y = 0 & \text{on } \Gamma \times (0,+\infty), \\ u(x,0) = u_0(x) \text{ and } \frac{\partial u}{\partial t}(x,0) = u_1(x) & \text{in } \Omega, \\ y(x,0) = y_0(x) \text{ and } \frac{\partial y}{\partial t}(x,0) = y_1(x) & \text{in } \Omega, \\ \frac{\partial u}{\partial t}(x,t) = f(x,t) & \text{in } \Gamma_1 \times (-\tau,0), \end{cases} \tag{34}$$

where a, γ are positive constants, b is a real number different from zero fixed small constant below and $\alpha_i \in L^\infty(\Gamma_1)$, for $i = 1$ and 2. Note that the dampings are only applied on the part Γ_1 of the boundary Γ in the first equation. The second equation is indirectly damped through the coupling between the two equations.

First, we rewrite this system in the form (1). For this purpose, we introduce $H = L^2(\Omega) \times L^2(\Omega)$,

$$V := H^1_{\Gamma_0}(\Omega) \times H^1_0(\Omega),$$

where

$$H^1_{\Gamma_0}(\Omega) = \{\varphi \in H^1(\Omega) : \varphi = 0 \text{ in } \Gamma_0\}.$$

We now introduce the following sesquilinear form a from $V \times V$ into $\mathbb{R}$ by

$$a((u,y),(v,z)) = \int_\Omega (a\nabla u \cdot \nabla \bar{v} + by\bar{v} + \nabla y \cdot \nabla \bar{z} + bu\bar{z})\, dx + \gamma \int_{\Gamma_1} u\bar{v}\, d\sigma(x), \forall (u,y),(v,z) \in V.$$

This sesquilinear form is actually an inner product in V if $|b| < \min\{\lambda_{\text{Dir}}, \lambda_{\text{mixed}}\}$, where λ_{Dir} and λ_{mixed} are respectively the first positive eigenvalues of the problems:

$$\begin{cases} -\Delta y = \lambda_{\text{Dir}} y \text{ in } \Omega, \\ y = 0 \text{ on } \Gamma, \end{cases}$$

and

$$\begin{cases} -a\Delta u = \lambda_{\text{mixed}} u \text{ in } \Omega, \\ u = 0 \text{ on } \Gamma_0, \\ a\frac{\partial u}{\partial \nu} + \gamma u = 0 \text{ on } \Gamma_1. \end{cases}$$

Indeed this is a consequence of the fact that they correspond to the minimum of the Rayleigh quotients:

$$\lambda_{\text{Dir}} = \min_{y \in H^1_0(\Omega), y \neq 0} \frac{\int_\Omega |\nabla u|^2\, dx}{\int_\Omega |u|^2\, dx},$$

$$\lambda_{\text{mixed}} = \min_{u \in H^1_{\Gamma_0}(\Omega), u \neq 0} \frac{\int_\Omega a|\nabla u|^2\, dx + \gamma \int_{\Gamma_1} |u|^2\, d\sigma(x)}{\int_\Omega |u|^2\, dx}.$$

Then one sets

$$D(A) = \{(u,y) \in H^2(\Omega) \cap H^1_{\Gamma_0}(\Omega) \times H^2(\Omega) \cap H^1_0(\Omega) \mid a\frac{\partial u}{\partial \nu} + \gamma u = 0 \text{ on } \Gamma_1\},$$

and

$$A : D(A) \to H : (u,y) \to (-a\Delta u + by, -\Delta y + bu),$$

and easily sees that for all $(u,y) \in D(A)$, one has

$$\int_\Omega A(u,y) \cdot (\bar{v}, \bar{z})\, dx = a((u,y),(v,z)), \forall (v,z) \in V.$$

This guarantees that the operator A is self-adjoint and positive with a compact inverse in H and that $D(A^{\frac{1}{2}}) = V$.

Moreover we define $U_1 = U_2 = L^2(\Gamma_1)$, and for $i = 1$ or 2, the operator

$$B_i^* : V \to L^2(\Gamma_1) : (u,y) \longmapsto \sqrt{\alpha_i}\gamma_{\Gamma_1} u,$$

where $\gamma_{\Gamma_1} u$ is the restriction to Γ_1 of the trace $\gamma_0 u$ of u.

In conclusion, system (34) can be rewritten in the form (1) and as before, it is well-posed and its energy is decaying, under the additional assumption

$$\exists \alpha \in (0,1),\ \alpha_2 \leq \alpha\alpha_1 \quad \text{a. e. on } \Gamma_1. \tag{35}$$

Since the system (34) with $\alpha_2 = 0$ is not exponentially stable (see [41, Theorem 2.2]), we cannot expect the exponential decay of (34). Let us now formulate some (new) results on polynomial decay rates.

Corollary 4.9. *In addition to the assumptions of this subsection, suppose that $\Gamma_1 = \Gamma(x_0)$, for some $x_0 \in \mathbb{R}^n$, and that (35) holds. Then the following hold*

1. If $a = 1$, then there exists $b_0 > 0$ small enough such that if b satisfies additionally $|b| < b_0$, then (20) holds with $\ell = 2$.

2. If Ω is a cube of $\mathbb{R}^n$, $n \leq 3$ and $a = \frac{1}{k^2}$ for some positive integer k, then there exists $b_0 > 0$ small enough such that if b satisfies additionally $|b| < b_0$, then (20) holds with $\ell = 2$.

3. If $\Omega = (0,d)$ is a non empty interval of $\mathbb{R}$, $x_0 = 0$, and $a \neq 1$, then there exists $b_0 > 0$ small enough such that if b satisfies additionally $|b| < b_0$, then (20) holds with ℓ given by

$$\ell = \begin{cases} 6 & \text{if } a \in \mathbb{Q}^+ \setminus \mathbb{Q}_s^+, \\ 10 & \text{if } a \in \mathbb{Q}_s^+, \\ 6+\varepsilon & \text{with } \varepsilon > 0, \text{ for a.e. positive } a \in \mathbb{R} \setminus \mathbb{Q}, \end{cases}$$

where $\mathbb{Q}^+ = \{q \in \mathbb{Q}; q > 0\}$ and $\mathbb{Q}_s^+ = \{q \in \mathbb{Q}^+; q = \frac{p^2}{q^2}$ for some $p, q \in \mathbb{N}\}$.

Proof. Both cases follow from Corollary 3.5 and the decay rate of (34) with $\alpha_2 = 0$. For point 1 (resp. 2 and 3), we use [42, Theorems 3.1] (resp. [41, Theorem 4.3] and [42, Theorems 4.2]). ■

Similar new results can be obtained when the boundary dampings are replaced by interior ones, using the results from [43–46] without delay.

CRediT authorship contribution statement

Gilbert Bayili: Conceptualization, Methodology. **Akram Ben Aissa:** Writing - original draft. **Serge Nicaise:** Writing - reviewing & editing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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