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Existence, Uniqueness and Stabilization of Solutions of a Generalized Telegraph Equation on Star Shaped Networks

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Abstract The existence, uniqueness, strong and exponential stability of a generalized telegraph equation set on one dimensional star shaped networks are established. It is assumed that a dissipative boundary condition is applied at all the external vertices and an improved Kirchhoff law at the common internal vertex is considered. First, using a general criteria of Arendt-Batty (see Arendt and Batty in *Trans. Am. Math. Soc.* 306(2):837–852, 1988), combined with a new uniqueness result, we prove that our system is strongly stable. Next, using a frequency domain approach, combined with a multiplier technique and the construction of a new multiplier satisfying some ordinary differential inequalities, we show that the energy of the system decays exponentially to zero.

Keywords Telegraph equation · Strong stability · Exponential stability · Frequency domain approach

1 Introduction

Over decades, telegraph equations have gained attention and interest among scientists due to their different applications in the transmission of electrical signals along transmission lines of all frequencies, in addition to many other physical, biological and engineering applications (see [4, 8, 17, 25, 31]). As a consequence, many mathematical models were set up, for

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instance in [15], a general and realistic situation was considered and a mathematical model of electromagnetic wave propagation in heterogeneous lossy coaxial cables was derived. Recently, referring to [29], S. Nicaise has considered the stabilization of the generalized telegraph equation set in a real interval (model on a cable from [15]):

$$\begin{cases} V_t + gV + aI_x + kW = 0, & \text{in } (0, L) \times (0, \infty), \\ I_t + rI + bV_x = 0, & \text{in } (0, L) \times (0, \infty), \\ W_t + cW = V, & \text{in } (0, L) \times (0, \infty), \end{cases} \quad (1.1)$$

with the following boundary conditions

$$V(0, t) = V(L, t) = 0, \quad t \in \mathbb{R}_+^*,$$

and the following initial conditions

$$V(x, 0) = V_0(x), \quad I(x, 0) = I_0(x), \quad W(x, 0) = W_0(x), \quad x \in (0, L),$$

where, a, b, c, r, k and g are all non-negative functions in $L^\infty(0, L)$ that verify some assumptions mentioned in [29], see (1.2) below for the exact conditions. The generalized telegraph equation is a coupling between the usual telegraph equation where the electric unknowns are V and I representing the electric potential and the electric current respectively with a first order differential equation of parabolic type involving an auxiliary variable W representing the non-local effects. In [29], the author was interested in studying the energy decay rate of system (1.1). In fact, under the additional condition that $r + g > 0$ in $(0, L)$, an exponential energy decay rate was established. Otherwise, he proved a polynomial energy decay rate of type t^{-1} for smooth initial data. Moreover, the obtained polynomial decay rate is optimal in the case $r = g = 0$. On the other hand, first order hyperbolic systems set on graphs, also called networks, appear in recent applications, like electrical circuits, arterial networks, networks of open channels, traffic flows on networks (see [2, 7, 10, 18–20, 22, 35, 40]); for heat, wave or beam equations on graphs, see [12, 24, 26–28, 36, 39] and the references there. In particular, referring to [30], the usual telegraph equation was considered on a network with a general class of transmission conditions, including reasonable ones (like the Kirchhoff law) so that with the dissipative boundary conditions at the exterior vertices, an exponential stability was established. Note that some exact controllability results or stability properties have been obtained in [6, 12, 21, 23, 33, 37, 39, 41] for instance. But to the best of our knowledge, the stability of the generalized telegraph equation on a network is an open problem. This motivates us to study this equation on star shaped graphs.

In the present work, we consider system (1.1) on a star shaped network \mathcal{S} made of $N + 1$ cables of equal length $L > 0$, with a positive integer N (see Fig. 1). For that purpose, we fix different real valued and non-negative functions $\mathbf{a} = (a_\ell)_{\ell=0}^N$, $\mathbf{b} = (b_\ell)_{\ell=0}^N$, $\mathbf{c} = (c_\ell)_{\ell=0}^N$, $\mathbf{k} = (k_\ell)_{\ell=0}^N$, $\mathbf{r} = (r_\ell)_{\ell=0}^N$, and $\mathbf{g} = (g_\ell)_{\ell=0}^N$ in $(L^\infty(0, L))^{N+1}$ satisfying the following assumption

$$a_\ell \geq \mu, \quad b_\ell \geq \mu, \quad c_\ell \geq \mu, \quad k_\ell + g_\ell \geq \mu \quad a.e \text{ in } (0, L), \quad \forall \ell = 0, \dots, N, \quad (1.2)$$

where $\mu > 0$ is a positive real number. These assumptions are in agreement with the physical setting from [14, 15].

On each edge e_ℓ with index $\ell \in \{0, \dots, N\}$, we consider the generalized telegraph equation

$$\begin{cases} V_{\ell,t} + g_\ell V_\ell + a_\ell I_{\ell,x} + k_\ell W_\ell = 0, & (x, t) \in (0, L) \times \mathbb{R}_+^*, \\ I_{\ell,t} + r_\ell I_\ell + b_\ell V_{\ell,x} = 0, & (x, t) \in (0, L) \times \mathbb{R}_+^*, \\ W_{\ell,t} + c_\ell W_\ell = V_\ell, & (x, t) \in (0, L) \times \mathbb{R}_+^*, \end{cases} \quad (1.3)$$

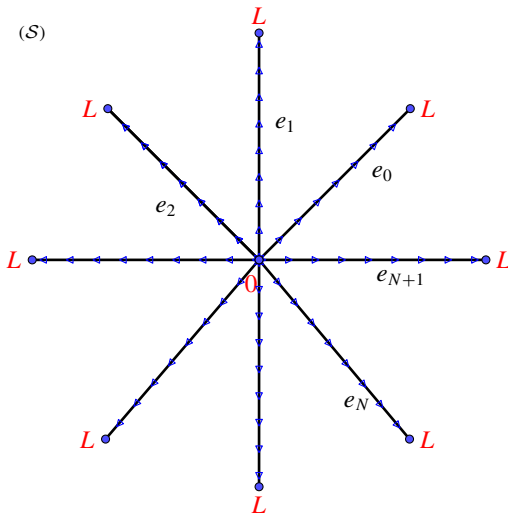


Fig. 1 A star-shaped network

where V_ℓ represents the electric potential, I_ℓ represents the electric current and W_ℓ represents the nonlocal effects. Moreover, we denote by $\mathbf{V} = (V_\ell)_{\ell=0}^N$, $\mathbf{I} = (I_\ell)_{\ell=0}^N$, and $\mathbf{W} = (W_\ell)_{\ell=0}^N$.

Let us mention that the case when the edges have different lengths L_ℓ , $\ell = 0, \dots, N$, namely system

$$\begin{cases} V_{\ell,t} + g_\ell V_\ell + a_\ell I_{\ell,x} + k_\ell W_\ell &= 0, & (x, t) \in (0, L_\ell) \times \mathbb{R}_+^*, \\ I_{\ell,t} + r_\ell I_\ell + b_\ell V_{\ell,x} &= 0, & (x, t) \in (0, L_\ell) \times \mathbb{R}_+^*, \\ W_{\ell,t} + c_\ell W_\ell &= V_\ell, & (x, t) \in (0, L_\ell) \times \mathbb{R}_+^*, \end{cases} \quad (1.4)$$

can be treated similarly since it can be reduced to (1.3) as follows. Indeed by performing on edge ℓ , the change of variable $x_\ell = \frac{L_\ell}{L}x$, with a fixed $L > 0$, and the change of unknowns

$$\hat{I}_\ell(x_\ell, t) = I\left(\frac{L_\ell}{L}x_\ell, t\right), \hat{V}_\ell(x_\ell, t) = V\left(\frac{L_\ell}{L}x_\ell, t\right), \hat{W}_\ell(x_\ell, t) = W\left(\frac{L_\ell}{L}x_\ell, t\right),$$

we transform system (1.4) into (1.3) where the coefficients a_ℓ and b_ℓ are changed into $\frac{L}{L_\ell}a_\ell$ and $\frac{L}{L_\ell}b_\ell$ respectively (and that continue to satisfy (1.2)).

System (1.3) is considered with the following dissipative boundary condition at the exterior vertices

$$V_\ell(L, t) - \alpha_\ell I_\ell(L, t) = 0, \quad \text{in } \mathbb{R}_+^*, \alpha_\ell > 0, \quad \forall \ell \in \{0, \dots, N\}, \quad (1.5)$$

and the transmission conditions from [3, §8.2] or [16] (called by these authors the improved Kirchhoff conditions) at the interior common vertex

$$\begin{cases} \sum_{k=1}^N Z_{\ell k} I_{k,t}(0, t) &= \frac{1}{\delta} (V_0(0, t) - V_\ell(0, t)), & \text{in } \mathbb{R}_+^*, \quad \forall \ell \in \{1, \dots, N\}, \\ V_{0,t}(0, t) &= \frac{-1}{\delta Y} \sum_{\ell=0}^N I_\ell(0, t) & \text{in } \mathbb{R}_+^*, \end{cases} \quad (1.6)$$

where, Y and δ are two positive constants and $Z = (Z_{\ell k})_{N \times N}$ is a symmetric, positive definite matrix. These boundary conditions are derived in [3, §8.2] or [16] by using the method of matched asymptotics starting from thin coaxial cables of thickness δ . The coefficient Y and the matrix Z are defined from the material properties of the medium and from 3D potentials defined in the reference domain.

Remark 1.1 If $r_\ell = 0$, then differentiating (1.5) with respect to t and using the second equation of (1.3), we exactly find the Neumann velocity feedback law

$$V_{\ell,t}(L, t) = -\alpha_\ell b_\ell V_{\ell,x}(L, t), \quad \text{in } \mathbb{R}_+^*, \forall \ell \in \{0, \dots, N\},$$

from [5, (4)] or [9, (4)].

By setting $v(t) = V_0(0, t)$ and $\eta(t) = (\eta_\ell(t))_{\ell=1}^N$ with $\eta_\ell(t) = \sum_{k=1}^N Z_{\ell k} I_k(0, t)$, System (1.3)-(1.6) is completed with the following initial conditions

$$V(x, 0) = V^0(x), \quad I(x, 0) = I^0(x), \quad W(x, 0) = W^0(x), \quad x \in (0, L)$$

and

$$v(0) = v^0, \quad \eta(0) = \eta^0.$$

As suggested before, our goal is then to find sufficient conditions on the functions $a_\ell, b_\ell, c_\ell, k_\ell, r_\ell$, and g_ℓ that guarantee that the above system is strongly stable and then exponentially stable.

The paper is organized as follows: First, in Sect. 2, we show that System (1.3)-(1.6) admits a unique solution in an appropriate Hilbert space using semi-group theory. Next, using a general criteria of Arendt-Batty (see [1]), we proved the strong stability of the system in the absence of the compactness of the resolvent. Finally, in Sect. 3, based on a frequency domain approach combined with an ad-hoc multiplier technique, we establish an exponential energy decay rate.

Let us finish this introduction with some notations used in the remainder of the paper: for all $p \in [1, \infty]$, $\mathbb{L}^p(\mathcal{S}) = L^p(0, L)^{N+1}$, $1 \leq p \leq \infty$ and $\mathbb{H}^1(\mathcal{S}) = H^1(0, L)^{N+1}$. The set $\mathcal{D}(0, L)$ is the set of smooth functions in $(0, L)$ with a compact support included into $(0, L)$ and let $\mathcal{D}'(0, L)$ be the associated space of distributions. Moreover, we denote by $(\cdot, \cdot)_{\mathbb{C}^N}$ the inner product in \mathbb{C}^N .

2 Well Posedness and Strong Stability

In this section, we will study the existence, uniqueness, regularity and asymptotic behavior of the solution of System (1.3)-(1.6).

2.1 Well Posedness of the Problem

First, we define the energy space

$$\mathcal{H} = (\mathbb{L}^2(\mathcal{S}))^3 \times \mathbb{C}^{N+1},$$

that is a Hilbert space equipped with the following inner product:

$$\begin{aligned} \langle (V, I, W, v, \eta)^\top, (V^*, I^*, W^*, v^*, \eta^*)^\top \rangle_{\mathcal{H}} &= \sum_{\ell=0}^N \int_0^L (\theta_\ell V_\ell \overline{V_\ell^*} + \beta_\ell I_\ell \overline{I_\ell^*} + \gamma_\ell W_\ell \overline{W_\ell^*}) dx \\ &\quad + \delta Y v \overline{v^*} + \delta (\eta, Z^{-1} \eta^*)_{\mathbb{C}^N} \end{aligned}$$

where $\theta = (\theta_\ell)_{\ell=0}^N, \beta = (\beta_\ell)_{\ell=0}^N, \gamma = (\gamma_\ell)_{\ell=0}^N \in \mathbb{L}^\infty(\mathcal{S})$ will be fixed later but are such that

$$\theta_\ell \geq \mu_0, \beta_\ell \geq \mu_0, \gamma_\ell \geq \mu_0 \text{ a.e in } \Omega, \quad \ell = 0, \dots, N, \tag{2.1}$$

for some $\mu_0 > 0$.

Next, we define the unbounded linear operator $\mathcal{A} : D(\mathcal{A}) \rightarrow \mathcal{H}$ by

$$\begin{aligned} D(\mathcal{A}) = \left\{ (V, I, W, v, \eta) \in \mathcal{H} : V \in \mathbb{H}^1(\mathcal{S}), I \in \mathbb{H}^1(\mathcal{S}), V_\ell(L) = \alpha_\ell I_\ell(L), \forall \ell \in \{0, \dots, N\}, \right. \\ \left. v = V_0(0), \text{ and } \eta = (\eta_\ell)_{\ell=1}^N \text{ with } \eta_\ell = \sum_{k=1}^N Z_{\ell k} I_k(0) \right\}, \end{aligned}$$

and

$$\begin{aligned} \mathcal{A} \begin{pmatrix} V \\ I \\ W \\ v \\ \eta \end{pmatrix} &= - \left(\mathbf{g} \otimes V + \mathbf{a} \otimes I_x + \mathbf{k} \otimes W, \mathbf{r} \otimes I + \mathbf{b} \otimes V_x, \mathbf{c} \otimes W - V, \frac{1}{\delta Y} \sum_{\ell=0}^N I_\ell(0), \right. \\ &\quad \left. \frac{1}{\delta} (V_\ell(0) - v)_{\ell=1}^N \right)^\top, \end{aligned}$$

where for two vector functions $\mathbf{P} = (p_\ell)_{\ell=0}^N \in \mathbb{L}^\infty(\mathcal{S})$, and $\mathbf{Q} = (q_\ell)_{\ell=0}^N \in \mathbb{L}^2(\mathcal{S})$, we set

$$\mathbf{P} \otimes \mathbf{Q} = (p_\ell q_\ell)_{\ell=0}^N,$$

while for $\mathbf{Q} = (q_\ell)_{\ell=0}^N \in \mathbb{H}^1(\mathcal{S})$,

$$\mathbf{Q}_x = (q_{\ell,x})_{\ell=0}^N.$$

Then, setting $U(t) = (V(\cdot, t), I(\cdot, t), W(\cdot, t), v(t), \eta(t))^\top$, System (1.3)-(1.6) can be formally written as a linear evolution equation in the space \mathcal{H}

$$\begin{cases} U_t = \mathcal{A}U, & t \in \mathbb{R}_+^*, \\ U(0) = U_0, \end{cases} \tag{2.2}$$

where $U_0 = (V_0, I_0, W_0, v_0, \eta_0)^\top$.

For this system, we have the following existence result.

Theorem 2.1 *Assume that condition (1.2) holds. Then, there exist $\theta, \beta, \gamma \in \mathbb{L}^\infty(\mathcal{S})$ satisfying (2.1) such that the operator \mathcal{A} generates a C_0 -semigroup of contractions $(e^{t\mathcal{A}})_{t \geq 0}$ on \mathcal{H} .*

Proof Using Lumer-Phillips Theorem (see [32, 38]), it is sufficient to prove that \mathcal{A} is a maximal dissipative operator so that \mathcal{A} generates a C_0 -semigroup of contractions on \mathcal{H} .

Step 1. Let us start by the dissipativity. Let $U = (V, I, W, v, \eta)^\top \in D(\mathcal{A})$ then, we have

$$\begin{aligned} \Re \langle \mathcal{A}U, U \rangle_{\mathcal{H}} &= \Re \left[- \sum_{\ell=0}^N \int_0^L \theta_\ell (g_\ell V_\ell + a_\ell I_{\ell,x} + k_\ell W_\ell) \overline{V}_\ell \, dx \right. \\ &\quad - \sum_{\ell=0}^N \int_0^L \beta_\ell (r_\ell I_\ell + b_\ell V_{\ell,x}) \overline{I}_\ell \, dx \\ &\quad \left. + \sum_{\ell=0}^N \int_0^L \gamma_\ell (V_\ell - c_\ell W_\ell) \overline{W}_\ell \, dx - V_0(0) \sum_{\ell=0}^N \overline{I}_\ell(0) - \sum_{\ell=1}^N (V_\ell(0) - V_0(0)) \overline{I}_\ell(0) \right] \\ &= - \sum_{\ell=0}^N \int_0^L (\theta_\ell g_\ell |V_\ell|^2 + \beta_\ell r_\ell |I_\ell|^2 + \gamma_\ell c_\ell |W_\ell|^2) \, dx - \Re \sum_{\ell=0}^N \int_0^L \theta_\ell a_\ell I_{\ell,x} \overline{V}_\ell \, dx \\ &\quad - \Re \sum_{\ell=0}^N \int_0^L \beta_\ell b_\ell V_{\ell,x} \overline{I}_\ell \, dx + \Re \sum_{\ell=0}^N \int_0^L (\gamma_\ell - \theta_\ell k_\ell) V_\ell \overline{W}_\ell \, dx - \Re V_0(0) \sum_{\ell=0}^N \overline{I}_\ell(0) \\ &\quad - \Re \sum_{\ell=1}^N (V_\ell(0) - V_0(0)) \overline{I}_\ell(0). \end{aligned}$$

Now, as in [29], we chose $\theta_\ell = a_\ell^{-1}$, $\beta_\ell = b_\ell^{-1}$ for all ℓ , and apply Green’s formula on the third term of the right-hand side, we obtain

$$\begin{aligned} \Re \langle \mathcal{A}U, U \rangle &= - \sum_{\ell=0}^N \int_0^L (a_\ell^{-1} g_\ell |V_\ell|^2 + b_\ell^{-1} r_\ell |I_\ell|^2 + \gamma_\ell c_\ell |W_\ell|^2) \, dx - \Re \sum_{\ell=0}^N \int_0^L I_{\ell,x} \overline{V}_\ell \, dx \\ &\quad + \Re \sum_{\ell=0}^N \int_0^L \overline{I}_{\ell,x} V_\ell \, dx - \Re \sum_{\ell=0}^N V_\ell(L) \overline{I}_\ell(L) + \Re \sum_{\ell=0}^N V_\ell(0) \overline{I}_\ell(0) \\ &\quad + \Re \sum_{\ell=0}^N \int_0^L (\gamma_\ell - \theta_\ell k_\ell) V_\ell \overline{W}_\ell \, dx - \Re V_0(0) \sum_{\ell=0}^N \overline{I}_\ell(0) \\ &\quad + \Re V_0(0) \sum_{\ell=1}^N \overline{I}_\ell(0) - \Re \sum_{\ell=1}^N V_\ell(0) \overline{I}_\ell(0) \\ &= - \sum_{\ell=0}^N \int_0^L (a_\ell^{-1} g_\ell |V_\ell|^2 + b_\ell^{-1} r_\ell |I_\ell|^2 + \gamma_\ell c_\ell |W_\ell|^2) \, dx \\ &\quad + \Re \sum_{\ell=0}^N \int_0^L (\gamma_\ell - \theta_\ell k_\ell) V_\ell \overline{W}_\ell \, dx - \sum_{\ell=0}^N \alpha_\ell |I_\ell(L)|^2. \end{aligned}$$

Finally, by estimating the second term of this right-hand side by Young’s inequality we get,

$$\Re \langle \mathcal{A}U, U \rangle \leq - \sum_{\ell=0}^N \int_0^L (a_\ell^{-1} g_\ell |V_\ell|^2 + b_\ell^{-1} r_\ell |I_\ell|^2 + \gamma_\ell c_\ell |W_\ell|^2) dx - \sum_{\ell=0}^N \alpha_\ell |I_\ell(L)|^2$$

$$+ \sum_{\ell=0}^N \int_0^L \frac{|\gamma_\ell - k_\ell \theta_\ell|}{2} (\epsilon_\ell |V_\ell|^2 + \frac{1}{\epsilon_\ell} |W_\ell|^2) dx,$$

for all $\epsilon(x) = (\epsilon_\ell(x))_{\ell=0}^N$ positive vector (i.e. $\epsilon_\ell(x) > 0, \forall \ell \in \{0, \dots, N\}, x \in (0, L)$).

Now, we need to find $\epsilon(x) = (\epsilon_\ell(x))_{\ell=0}^N$ and $\gamma = (\gamma_\ell)_{\ell=0}^N$ to satisfy for all $\ell \in \{0, \dots, N\}$:

$$\frac{|\gamma_\ell - k_\ell \theta_\ell| \epsilon_\ell}{2} - \theta_\ell g_\ell \leq 0$$

and

$$\frac{|\gamma_\ell - k_\ell \theta_\ell|}{2 \epsilon_\ell} - \gamma_\ell c_\ell \leq 0.$$

But according to the proof of Theorem 2.1 of [29] (see p. 3224-3225 in [29]) such a choice is always possible (we do not give the details and refer to that paper for the details). With such a choice, we find

$$\Re \langle \mathcal{A}U, U \rangle \leq - \frac{1}{2} \sum_{\ell=0}^N \int_0^L (\theta_\ell g_\ell |V_\ell|^2 + 2\beta_\ell r_\ell |I_\ell|^2 + \gamma_\ell c_\ell |W_\ell|^2) dx$$

$$- \sum_{\ell=0}^N \alpha_\ell |I_\ell(L)|^2 \leq 0, \tag{2.3}$$

and consequently, \mathcal{A} is dissipative.

Step 2. Now, let us go on with the maximality. Let $\lambda > 0$ be fixed. Given $F = (f_1, f_2, f_3, f_4, f_5)^\top \in \mathcal{H}$, we look for $U = (V, I, W, v, \eta)^\top \in D(\mathcal{A})$ solution of

$$(\lambda \mathcal{I} - \mathcal{A})U = F,$$

or equivalently,

$$\begin{cases} \lambda V_\ell + a_\ell I_{\ell,x} + g_\ell V_\ell + k_\ell W_\ell & = f_{\ell,1}, \quad \forall \ell \in \{0, \dots, N\}, \\ \lambda I_\ell + b_\ell V_{\ell,x} + r_\ell I_\ell & = f_{\ell,2}, \quad \forall \ell \in \{0, \dots, N\}, \\ \lambda W_\ell + c_\ell W_\ell - V_\ell & = f_{\ell,3}, \quad \forall \ell \in \{0, \dots, N\}, \\ \lambda v + \frac{1}{\delta Y} \sum_{\ell=0}^N I_\ell(0) & = f_4, \\ \lambda \eta_\ell + \frac{1}{\delta} (V_\ell(0) - v) & = f_{\ell,5}, \quad \forall \ell \in \{1, \dots, N\}. \end{cases} \tag{2.4}$$

Assume for the moment that a solution $U \in D(\mathcal{A})$ of (2.4) exists. Then, the third equation is equivalent to

$$W_\ell = \frac{V_\ell + f_{\ell,3}}{\lambda + c_\ell}, \text{ for all } \ell \in \{0, \dots, N\}. \tag{2.5}$$

Similarly, the last two equations of (2.4) allow to eliminate ν and η_ℓ , namely

$$f_4 - \frac{1}{\delta Y} \sum_{\ell=0}^N I_\ell(0) = \frac{\nu}{\lambda}, \tag{2.6}$$

and

$$\eta_\ell = \frac{f_{\ell,5} - \frac{1}{\delta}(V_\ell(0) - \nu)}{\lambda}, \text{ for all } \ell \in \{1, \dots, N\}. \tag{2.7}$$

Replacing (2.5) in the first equation of (2.4), we get

$$\begin{cases} V_\ell + a_{\ell,1}(\lambda)I_{\ell,x} = f_{\lambda,\ell}, \\ V_{\ell,x} + a_{\ell,2}(\lambda)I_\ell = g_{\lambda,\ell}, \end{cases} \tag{2.8}$$

where

$$\begin{cases} a_{\ell,1}(\lambda) = \frac{a_\ell}{\lambda + g_\ell + \frac{k_\ell}{\lambda + c_\ell}}, & a_{\ell,2}(\lambda) = \frac{\lambda + r_\ell}{b_\ell}, \\ f_{\lambda,\ell} = \frac{f_{\ell,1} - \frac{k_\ell f_{\ell,3}}{\lambda + c_\ell}}{\lambda + g_\ell + \frac{k_\ell}{\lambda + c_\ell}}, & g_{\lambda,\ell} = \frac{f_{\ell,2}}{b_\ell}, \end{cases} \tag{2.9}$$

for all $\ell \in \{0, \dots, N\}$.

Let $\psi = (\psi_\ell)_{\ell=0}^N \in \mathbb{H}^1(S)$. Multiplying the second equation of (2.8) by $\bar{\psi}_\ell$, integrating over $(0, L)$ and summing on $\ell \in \{0, \dots, N\}$, yields

$$\sum_{\ell=0}^N \int_0^L V_{\ell,x} \bar{\psi}_\ell dx + \sum_{\ell=0}^N \int_0^L a_{\ell,2}(\lambda) I_\ell \bar{\psi}_\ell dx = \sum_{\ell=0}^N \int_0^L g_{\lambda,\ell} \bar{\psi}_\ell dx. \tag{2.10}$$

Similarly, multiplying the first equation of (2.8) by $\bar{\psi}_{\ell,x}$, integrating over $(0, L)$ and summing on $\ell \in \{0, \dots, N\}$, gives

$$\sum_{\ell=0}^N \int_0^L V_\ell \bar{\psi}_{\ell,x} dx + \sum_{\ell=0}^N \int_0^L a_{\ell,1}(\lambda) I_{\ell,x} \bar{\psi}_{\ell,x} dx = \sum_{\ell=0}^N \int_0^L f_{\lambda,\ell} \bar{\psi}_{\ell,x} dx. \tag{2.11}$$

By integrating by parts the first term of the left hand side of (2.11), we obtain

$$\begin{aligned} & - \sum_{\ell=0}^N \int_0^L V_{\ell,x} \bar{\psi}_\ell dx + \sum_{\ell=0}^N V_\ell(L) \bar{\psi}_\ell(L) - \sum_{\ell=0}^N V_\ell(0) \bar{\psi}_\ell(0) \\ & + \sum_{\ell=0}^N \int_0^L a_{\ell,1}(\lambda) I_{\ell,x} \bar{\psi}_{\ell,x} dx = \sum_{\ell=0}^N \int_0^L f_{\lambda,\ell} \bar{\psi}_{\ell,x} dx. \end{aligned} \tag{2.12}$$

Then, adding (2.10) and (2.12) leads to

$$\begin{aligned} & \sum_{\ell=0}^N \int_0^L (a_{\ell,1}(\lambda) I_{\ell,x} \bar{\psi}_{\ell,x} + a_{\ell,2}(\lambda) I_\ell \bar{\psi}_\ell) dx + \sum_{\ell=0}^N V_\ell(L) \bar{\psi}_\ell(L) \\ & - \sum_{\ell=0}^N V_\ell(0) \bar{\psi}_\ell(0) = \sum_{\ell=0}^N \int_0^L (f_{\lambda,\ell} \bar{\psi}_{\ell,x} + g_{\lambda,\ell} \bar{\psi}_\ell) dx. \end{aligned} \tag{2.13}$$

401 By taking into account the boundary conditions in $D(\mathcal{A})$, in addition to (2.6) and (2.7), we
 402 find

$$\begin{aligned}
 403 \quad & \sum_{\ell=0}^N V_\ell(0) \bar{\psi}_\ell(0) = V_0(0) \bar{\psi}_0(0) + \sum_{\ell=1}^N V_\ell(0) \bar{\psi}_\ell(0) \\
 404 \quad & = V_0(0) \bar{\psi}_0(0) + \delta \sum_{\ell=1}^N f_{\ell,5} \bar{\psi}_\ell(0) \\
 405 \quad & \quad + V_0(0) \sum_{\ell=1}^N \bar{\psi}_\ell(0) - \delta \lambda \sum_{\ell=1}^N \left(\sum_{k=1}^N Z_{\ell k} I_k(0) \right) \bar{\psi}_\ell(0) \\
 406 \quad & = V_0(0) \sum_{\ell=0}^N \bar{\psi}_\ell(0) + \delta \sum_{\ell=1}^N f_{\ell,5} \bar{\psi}_\ell(0) \\
 407 \quad & \quad - \delta \lambda \sum_{\ell=1}^N \left(\sum_{k=1}^N Z_{\ell k} I_k(0) \right) \bar{\psi}_\ell(0) \\
 408 \quad & = \frac{f_4}{\lambda} \sum_{\ell=0}^N \bar{\psi}_\ell(0) - \frac{1}{\delta Y \lambda} \left(\sum_{\ell=0}^N I_\ell(0) \right) \left(\sum_{\ell=0}^N \bar{\psi}_\ell(0) \right) \\
 409 \quad & \quad + \delta \sum_{\ell=1}^N f_{\ell,5} \bar{\psi}_\ell(0) - \delta \lambda \sum_{\ell=1}^N \left(\sum_{k=1}^N Z_{\ell k} I_k(0) \right) \bar{\psi}_\ell(0)
 \end{aligned}$$

410 and

$$\sum_{\ell=0}^N V_\ell(L) \bar{\psi}_\ell(L) = \sum_{\ell=0}^N \alpha_\ell I_\ell(L) \bar{\psi}_\ell(L).$$

411 By replacing these two identities in (2.13), we arrive at

$$\begin{aligned}
 412 \quad & \sum_{\ell=0}^N \int_0^L (a_{\ell,1}(\lambda) I_{\ell,x} \bar{\psi}_{\ell,x} + a_{\ell,2}(\lambda) I_\ell \bar{\psi}_\ell) dx + \sum_{\ell=0}^N \alpha_\ell I_\ell(L) \bar{\psi}_\ell(L) \\
 413 \quad & + \frac{1}{\delta Y \lambda} \left(\sum_{\ell=0}^N I_\ell(0) \right) \left(\sum_{\ell=0}^N \bar{\psi}_\ell(0) \right) + \delta \lambda \sum_{\ell=1}^N \left(\sum_{k=1}^N Z_{\ell k} I_k(0) \right) \bar{\psi}_\ell(0) \quad (2.14) \\
 414 \quad & = \sum_{\ell=0}^N \int_0^L (f_{\lambda,\ell} \bar{\psi}_{\ell,x} + g_{\lambda,\ell} \bar{\psi}_\ell) dx + \frac{f_4}{\lambda} \sum_{\ell=0}^N \bar{\psi}_\ell(0) + \delta \sum_{\ell=1}^N f_{\ell,5} \bar{\psi}_\ell(0),
 \end{aligned}$$

415 for all $\psi \in \mathbb{H}^1(S)$. Now, we notice that this problem has a unique solution $I \in \mathbb{H}^1(S)$, by
 416 the Lax-Milgram Lemma. First, this left hand side is a continuous and coercive sesquilinear
 417 form on $\mathbb{H}^1(S)$, since by our assumptions (in particular on Z), there exists a positive
 418 constant $C(\lambda)$ (that depends on λ) such that

$$\sum_{\ell=0}^N \int_0^L (a_{\ell,1}(\lambda) |I_{\ell,x}|^2 + a_{\ell,2}(\lambda) |I_\ell|^2) dx + \sum_{\ell=0}^N \alpha_\ell |I_\ell(L)|^2 + \frac{1}{\delta Y \lambda} \left| \sum_{\ell=0}^N I_\ell(0) \right|^2$$

$$+ \delta\lambda(Z\mathbf{I}_*(0), \mathbf{I}_*(0))_{\mathbb{C}^N} \geq C(\lambda)\|I\|_{\mathbb{H}^1(\mathcal{S})}^2,$$

with $\mathbf{I}_*(0) = (I_\ell(0))_{\ell=1}^N$. Secondly, the right-hand side of (2.14) is a continuous form on $\mathbb{H}^1(\mathcal{S})$.

Now, according to the first equation of (2.8) we set

$$V_\ell = f_{\lambda,\ell} - a_{\ell,1}(\lambda)I_{\ell,x}, \tag{2.15}$$

that clearly belongs to $L^2(0, L)$, for all ℓ . Then, taking in (2.14) a test function ψ such that $\psi_\ell \in \mathcal{D}(0, L)$ for some $\ell = 0, \dots, N$ and $\psi_j = 0$, for all $j \neq \ell$, we find

$$\begin{aligned} & \int_0^L a_{\ell,1}(\lambda)I_{\ell,x}\bar{\psi}_{\ell,x} dx + \int_0^L a_{\ell,2}(\lambda)I_\ell\bar{\psi}_\ell dx \\ &= \int_0^L f_{\lambda,\ell}\bar{\psi}_{\ell,x} dx + \int_0^L g_{\lambda,\ell}\bar{\psi}_\ell dx. \end{aligned} \tag{2.16}$$

Eliminating $I_{\ell,x}$ from (2.15) and replacing it in (2.16), we get

$$-\int_0^L V_\ell\bar{\psi}_{\ell,x} dx + \int_0^L a_{\ell,2}(\lambda)I_\ell\bar{\psi}_\ell dx = \int_0^L g_{\lambda,\ell}\bar{\psi}_\ell dx. \tag{2.17}$$

Applying Green’s formula on the first integral of the left-hand side of (2.17), we obtain

$$V_{\ell,x} + a_{\ell,2}(\lambda)I_\ell = g_{\lambda,\ell}, \text{ in } \mathcal{D}'(0, L). \tag{2.18}$$

As $-a_{\ell,2}(\lambda)I_\ell + g_{\lambda,\ell} \in L^2(0, L)$, we deduce that $V_\ell \in H^1(0, L)$ and that (2.8) holds. Then, after defining \mathbf{W} by (2.5), ν by (2.6) and $\boldsymbol{\eta}$ by (2.7), it remains to check the boundary conditions appearing in $D(\mathcal{A})$. For this purpose, we introduce two Hilbert spaces

$$H_R^1 = \{u \in H^1(0, L); u(0) = 0\},$$

and

$$H_L^1 = \{u \in H^1(0, L); u(L) = 0\}.$$

Now, we first take different test functions ψ in (2.14). By Green’s formula and taking into account (2.8), we have

$$\begin{aligned} & \sum_{\ell=0}^N V_\ell(L)\bar{\psi}_\ell(L) - \sum_{\ell=0}^N V_\ell(0)\bar{\psi}_\ell(0) - \sum_{\ell=0}^N \alpha_\ell I_\ell(L)\bar{\psi}_\ell(L) \\ & - \frac{1}{\delta Y \lambda} \left(\sum_{\ell=0}^N I_\ell(0) \right) \left(\sum_{\ell=0}^N \bar{\psi}_\ell(0) \right) - \delta \lambda \sum_{\ell=1}^N \left(\sum_{k=1}^N Z_{\ell k} I_k(0) \right) \bar{\psi}_\ell(0) \\ & + \delta \sum_{\ell=1}^N f_{\ell,5} \bar{\psi}_\ell(0) + \frac{1}{\lambda} \sum_{\ell=0}^N f_4 \bar{\psi}_\ell(0) = 0, \quad \forall \psi \in \mathbb{H}^1(\mathcal{S}). \end{aligned} \tag{2.19}$$

501 First, we fix ℓ and take $\psi \in \prod_{\ell=0}^N H_R^1$ such that $\psi_\ell(L) = z$ arbitrary in \mathbb{C} and $\psi_j(L) = 0, \forall j \neq$
 502 ℓ , we get

$$(V_\ell(L) - \alpha_\ell I_\ell(L)) \bar{z} = 0, \forall z \in \mathbb{C}.$$

506 Consequently, $V_\ell(L) = \alpha_\ell I_\ell(L)$. Second, we fix ℓ and take $\psi \in \prod_{\ell=0}^N H_L^1$ such that $\psi_\ell(0) = z$
 507 arbitrary in \mathbb{C} and $\psi_j(0) = 0, \forall j \neq \ell$, we obtain

$$\left(-V_\ell(0) - \frac{1}{\delta Y \lambda} \left(\sum_{\ell=0}^N I_\ell(0) \right) - \delta \lambda \left(\sum_{k=1}^N Z_{\ell k} I_k(0) \right) + \delta f_{\ell,5} + \frac{1}{\lambda} f_4 \right) \bar{z} = 0, \forall z \in \mathbb{C}.$$

514 Using (2.6) and (2.7) we deduce that $\eta_\ell = \sum_{k=1}^N Z_{\ell k} I_k(0)$. Third, we fix ℓ and take $\psi \in \prod_{\ell=0}^N H_L^1$
 515 such that $\psi_\ell = 0, \forall \ell \neq 0$, and $\psi_0(0) = z$, arbitrary in \mathbb{C} we obtain

$$\frac{f_4}{\lambda} - \frac{1}{\delta Y \lambda} \sum_{\ell=0}^N I_\ell(0) = V_0(0).$$

522 Using (2.6), we deduce that $v = V_0(0)$. The proof of Theorem 2.1 is thus complete. □

526 As \mathcal{A} generates a C_0 semigroup on \mathcal{H} , problem (2.2) admits a unique solution given by

$$U(t) = e^{t\mathcal{A}} U_0, \quad t \geq 0.$$

530 Hence, the semi-group theory allows us to deduce the following existence and uniqueness
 531 results.

533 **Theorem 2.2** *Assume that condition (1.2) holds. Then, for any initial data $U_0 \in \mathcal{H}$, problem*
 534 *(2.2) admits a unique weak solution $U \in C^0([0, \infty); \mathcal{H})$. Moreover if $U_0 \in D(\mathcal{A})$, problem*
 535 *(2.2) admits a unique strong solution $U \in C^1([0, \infty); \mathcal{H}) \cap C^0([0, \infty); D(\mathcal{A}))$.*

537 As usual the energy associated with (2.2) is defined by

$$E(t) = \frac{1}{2} \| (V, I, W, v, \eta) \|_{\mathcal{H}}^2$$

542 and for $U = (V, I, W, v, \eta) \in C^1([0, \infty); \mathcal{H}) \cap C^0([0, \infty); D(\mathcal{A}))$, we have

$$E'(t) \leq -\frac{1}{2} \sum_{\ell=0}^N \int_0^L (\theta_\ell g_\ell |V_\ell|^2 + 2\beta_\ell r_\ell |I_\ell|^2 + \gamma_\ell c_\ell |W_\ell|^2) dx - \sum_{\ell=0}^N \alpha_\ell |I_\ell(L)|^2 \leq 0.$$

547 For such a solution, we then have

$$E(t_2) \leq E(t_1), \quad \forall 0 \leq t_1 \leq t_2,$$

in other words, our system is dissipative in the sense that its energy is non-increasing with respect to t . By the density of $D(\mathcal{A})$ into \mathcal{H} , this last property remains valid for weak solution $U \in C^0([0, \infty); \mathcal{H})$.

Now, we are able to study the strong stability of our system.

2.2 Strong Stability

To show the strong stability of the C_0 -semigroup of contractions $(e^{t\mathcal{A}})_{t \geq 0}$ we will rely on the following result due to Arendt-Batty [1].

Theorem 2.3 (Arendt-Batty [1]). *Let $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \rightarrow \mathcal{H}$ generates a C_0 -semigroup of contractions on the Hilbert space \mathcal{H} . If*

1) \mathcal{A} has no pure imaginary eigenvalues,

2) $\sigma(\mathcal{A}) \cap i\mathbb{R}$ is countable,

where $\sigma(\mathcal{A})$ denotes the spectrum of \mathcal{A} . Then, the C_0 -semigroup $(e^{t\mathcal{A}})_{t \geq 0}$ is strongly stable, namely

$$\lim_{t \rightarrow \infty} \|e^{t\mathcal{A}}U_0\|_{\mathcal{H}} = 0, \forall U_0 \in \mathcal{H}.$$

Let us now state the main result of this subsection.

Theorem 2.4 *Assume that assumption (1.2) holds. Then, the C_0 -semigroup $(e^{t\mathcal{A}})_{t \geq 0}$ is strongly stable on the energy space \mathcal{H} .*

Its proof is based on the use of Theorem 2.3. Since in our situation the resolvent of \mathcal{A} is not compact, we have to analyze its full spectrum on the imaginary axis. More precisely with the help of the following Lemmas, we will deduce that $\sigma(\mathcal{A}) \cap i\mathbb{R}$ is empty, hence the result.

Lemma 2.5 *Under the same condition of Theorem 2.4, we have $i\lambda - \mathcal{A}$ is injective for all $\lambda \in \mathbb{R}$, i.e.*

$$\ker(i\lambda\mathcal{I} - \mathcal{A}) = \{0\}, \quad \forall \lambda \in \mathbb{R}.$$

Proof Let $\lambda \in \mathbb{R}$ and $U = (V, I, W, v, \eta)^\top \in D(\mathcal{A})$ be such that

$$(i\lambda\mathcal{I} - \mathcal{A})U = 0,$$

equivalently,

$$\begin{cases} i\lambda V_\ell + a_\ell I_{\ell,x} + g_\ell V_\ell + k_\ell W_\ell & = 0, \quad \forall \ell \in \{0, \dots, N\}, \\ i\lambda I_\ell + b_\ell V_{\ell,x} + r_\ell I_\ell & = 0, \quad \forall \ell \in \{0, \dots, N\}, \\ i\lambda W_\ell + c_\ell W_\ell - V_\ell & = 0, \quad \forall \ell \in \{0, \dots, N\}, \\ i\lambda v + \frac{1}{\delta\gamma} \sum_{\ell=0}^N I_\ell(0) & = 0, \\ i\lambda \eta + \frac{1}{\delta} (V_\ell(0) - v) & = 0, \quad \forall \ell \in \{1, \dots, N\}. \end{cases} \tag{2.20}$$

Hence, as $\Re \langle \mathcal{A}U, U \rangle_{\mathcal{H}} = 0$, using inequality (2.3) and the fact that $c_\ell > 0$ and $\alpha_\ell > 0$, we get

$$g_\ell V_\ell = r_\ell I_\ell = W_\ell = I_\ell(L) = 0, \quad \forall \ell \in \{0, \dots, N\}. \tag{2.21}$$

601 Using the third identity of (2.20) we deduce that $V_\ell = 0$, for all $\ell \in \{0, \dots, N\}$. On the other
 602 hand, the first equation yields that $I_{\ell,x} = 0$, so I_ℓ is constant on $(0, L)$ for all $\ell \in \{0, \dots, N\}$. As
 603 $I_\ell(L) = 0$, we deduce that $I_\ell = 0$, for all $\ell \in \{0, \dots, N\}$. This implies that $v = V_0(0) = 0$ and
 604 $\eta_\ell = \sum_{k=1}^N Z_{\ell k} I_k(0) = 0$, for all $\ell \in \{1, \dots, N\}$. Hence $U = 0$ and the desired goal holds. \square
 605
 606

607 **Lemma 2.6** *Under the same condition of Theorem 2.4, we have $i\lambda - \mathcal{A}$ is surjective for all*
 608 $\lambda \in \mathbb{R}^*$, *i.e.*
 609

$$610 \mathcal{R}(i\lambda - \mathcal{A}) = \mathcal{H}, \quad \forall \lambda \in \mathbb{R}^*.$$

611
 612 *Proof* For any $\lambda \in \mathbb{R}^*$ and $\mathbf{F} = (f_1, f_2, f_3, f_4, f_5)^\top \in \mathcal{H}$, we look for a unique solution
 613 $\mathbf{U} = (\mathbf{V}, \mathbf{I}, \mathbf{W}, v, \boldsymbol{\eta})^\top \in D(\mathcal{A})$ of

$$614 (i\lambda \mathcal{I} - \mathcal{A})\mathbf{U} = \mathbf{F},$$

615
 616 or equivalently,
 617

$$618 \begin{cases} i\lambda V_\ell + a_\ell I_{\ell,x} + g_\ell V_\ell + k_\ell W_\ell & = f_{\ell,1}, \quad \forall \ell \in \{0, \dots, N\}, \\ i\lambda I_\ell + b_\ell V_{\ell,x} + r_\ell I_\ell & = f_{\ell,2}, \quad \forall \ell \in \{0, \dots, N\}, \\ i\lambda W_\ell + c_\ell W_\ell - V_\ell & = f_{\ell,3}, \quad \forall \ell \in \{0, \dots, N\}, \\ i\lambda v + \frac{1}{\delta Y} \sum_{\ell=0}^N I_\ell(0) & = f_4, \\ i\lambda \eta_\ell + \frac{1}{\delta} (V_\ell(0) - v) & = f_{\ell,5}, \quad \forall \ell \in \{1, \dots, N\}. \end{cases} \quad (2.22)$$

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 626 Suppose such a $\mathbf{U} = (\mathbf{V}, \mathbf{I}, \mathbf{W}, v, \boldsymbol{\eta})^\top \in D(\mathcal{A})$ exists, then
 627

$$628 W_\ell = \frac{V_\ell + f_{\ell,3}}{(i\lambda + c_\ell)}, \quad \forall \ell \in \{0, \dots, N\}, \quad (2.23)$$

$$629 v = \frac{f_4 - \frac{1}{\delta Y} \sum_{\ell=0}^N I_\ell(0)}{i\lambda}, \quad (2.24)$$

$$630 \eta_\ell = \frac{f_{\ell,5} - \frac{1}{\delta} (V_\ell(0) - v)}{i\lambda}, \quad \forall \ell \in \{1, \dots, N\}. \quad (2.25)$$

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 636
 637 Replace W_ℓ in the first and second identities of (2.22) to get
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$$639 \begin{cases} V_\ell + \frac{a_\ell}{(i\lambda + g_\ell + \frac{k_\ell}{i\lambda + c_\ell})} I_{\ell,x} & = \frac{(f_{\ell,1} - \frac{k_\ell f_{\ell,3}}{i\lambda + c_\ell})}{(i\lambda + g_\ell + \frac{k_\ell}{i\lambda + c_\ell})}, \\ \frac{i\lambda + r_\ell}{b_\ell} I_\ell + V_{\ell,x} & = \frac{f_{\ell,2}}{b_\ell}, \end{cases} \quad \ell \in \{0, \dots, N\}.$$

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 644 Then, to find a solution of (2.22) it is enough to find a solution of
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$$646 \begin{cases} V_\ell + a_{\ell,1}(i\lambda) I_{\ell,x} = f_{i\lambda,\ell}, \\ V_{\ell,x} + a_{\ell,2}(i\lambda) I_\ell = g_{i\lambda,\ell}, \end{cases} \quad (2.26)$$

647
 648
 649 where $a_{\ell,1}(\lambda)$, $a_{\ell,2}(\lambda)$, $f_{\lambda,\ell}$ and $g_{\lambda,\ell}$ are defined in (2.9).
 650

Now following the arguments of the proof of Theorem 2.1, we deduce that $\mathbf{I} \in \mathbb{H}^1(S)$ is a solution of

$$\begin{aligned} & \sum_{\ell=0}^N \int_0^L (a_{\ell,1}(i\lambda)I_{\ell,x}\bar{\psi}_{\ell,x} + a_{\ell,2}(i\lambda)I_{\ell}\bar{\psi}_{\ell}) dx + \sum_{\ell=0}^N \alpha_{\ell}I_{\ell}(L)\bar{\psi}_{\ell}(L) \\ & + \frac{1}{i\delta Y\lambda} \left(\sum_{\ell=0}^N I_{\ell}(0) \right) \left(\sum_{\ell=0}^N \bar{\psi}_{\ell}(0) \right) + i\delta\lambda \sum_{\ell=1}^N \left(\sum_{k=1}^N Z_{\ell k}I_k(0) \right) \bar{\psi}_{\ell}(0) \quad (2.27) \\ & = \sum_{\ell=0}^N \int_0^L (f_{i\lambda,\ell}\bar{\psi}_{\ell,x} + g_{i\lambda,\ell}\bar{\psi}_{\ell}) dx + \frac{f_4}{i\lambda} \sum_{\ell=0}^N \bar{\psi}_{\ell}(0) + \delta \sum_{\ell=1}^N f_{\ell,5}\bar{\psi}_{\ell}(0), \end{aligned}$$

for all $\psi \in \mathbb{H}^1(S)$. Here we note that Lax-Milgram Lemma cannot be applied because coercivity is not available. Therefore, we use a perturbation argument. For that purpose, let us introduce the sesquilinear form

$$\begin{aligned} b_{\lambda}(I, \psi) &= \sum_{\ell=0}^N \int_0^L a_{\ell,1}(i\lambda)I_{\ell,x}\bar{\psi}_{\ell,x} dx + \sum_{\ell=0}^N \int_0^L \frac{1}{b_{\ell}}I_{\ell}\bar{\psi}_{\ell} dx + \sum_{\ell=0}^N \alpha_{\ell}I_{\ell}(L)\bar{\psi}_{\ell}(L) \\ & + \frac{1}{i\delta Y\lambda} \left(\sum_{\ell=0}^N I_{\ell}(0) \right) \left(\sum_{\ell=0}^N \bar{\psi}_{\ell}(0) \right) + i\delta\lambda \sum_{\ell=1}^N \left(\sum_{k=1}^N Z_{\ell k}I_k(0) \right) \bar{\psi}_{\ell}(0), \forall I, \psi \in \mathbb{H}^1(S). \end{aligned}$$

This sesquilinear form b_{λ} is continuous and coercive on $\mathbb{H}^1(S)$, since for all ℓ

$$\Re(a_{\ell,1}(i\lambda)) = \frac{a_{\ell}(\lambda^2 g_{\ell} + c_{\ell}(c_{\ell}g_{\ell} + k_{\ell}))}{(c_{\ell}g_{\ell} + k_{\ell} - \lambda^2)^2 + \lambda^2(c_{\ell} + g_{\ell})^2} \geq C(\lambda),$$

where $C(\lambda)$ is a positive constant that depends only on λ . Then, by Lax-Milgram Lemma, the operator

$$B_{\lambda} : \mathbb{H}^1(S) \rightarrow (\mathbb{H}^1(S))^* : \mathbf{I} \rightarrow B_{\lambda}\mathbf{I},$$

with $B_{\lambda}\mathbf{I}(\psi) = b_{\lambda}(\mathbf{I}, \psi)$, is an isomorphism.

Now, let us set

$$R_{\lambda} : \mathbb{H}^1(S) \rightarrow (\mathbb{H}^1(S))^* : \mathbf{I} \rightarrow R_{\lambda}\mathbf{I},$$

with

$$R_{\lambda}\mathbf{I}(\psi) = \sum_{\ell=0}^N \int_0^L \frac{(i\lambda + r_{\ell} - 1)}{b_{\ell}}I_{\ell}\bar{\psi}_{\ell} dx.$$

As R_{λ} is a compact operator, we deduce that $B_{\lambda} + R_{\lambda}$ is a Fredholm operator of index zero from $\mathbb{H}^1(S)$ to $(\mathbb{H}^1(S))^*$.

Now by setting

$$L_{\lambda}(\psi) = \sum_{\ell=0}^N \int_0^L (g_{i\lambda,\ell}\bar{\psi}_{\ell} + f_{i\lambda,\ell}\bar{\psi}_{\ell,x}) dx + \delta \sum_{\ell=1}^N f_{\ell,5}\bar{\psi}_{\ell}(0) + \frac{1}{i\lambda} \sum_{\ell=0}^N f_4\bar{\psi}_{\ell}(0),$$

we notice that (2.27) is equivalent to

$$(B_{\lambda} + R_{\lambda})\mathbf{I} = L_{\lambda} \quad \text{in } (\mathbb{H}^1(S))^*. \quad (2.28)$$

701 Hence problem (2.27) admits a unique solution \mathbf{I} if and only if $B_\lambda + R_\lambda$ is invertible. $B_\lambda + R_\lambda$
 702 being a Fredholm operator it is enough to prove that $B_\lambda + R_\lambda$ is injective, i.e.,
 703

$$704 \quad \ker(B_\lambda + R_\lambda) = \{0\}.$$

705 Let us now fix $\mathbf{I} \in \ker(B_\lambda + R_\lambda)$, then it satisfies
 706

$$707 \quad \sum_{\ell=0}^N \int_0^L (a_{\ell,1}(i\lambda)I_{\ell,x}\bar{\psi}_{\ell,x} + a_{\ell,2}(i\lambda)I_\ell\bar{\psi}_\ell) dx + \sum_{\ell=0}^N \alpha_\ell I_\ell(L)\bar{\psi}_\ell(L)$$

$$708 \quad + \frac{1}{i\delta Y\lambda} \left(\sum_{\ell=0}^N I_\ell(0) \right) \left(\sum_{\ell=0}^N \bar{\psi}_\ell(0) \right)$$

$$709 \quad + i\delta\lambda \sum_{\ell=1}^N \left(\sum_{k=1}^N Z_{\ell k} I_k(0) \right) \bar{\psi}_\ell(0) = 0, \quad \forall \psi \in \mathbb{H}^1(\mathcal{S}).$$

710 Thus, if we set
 711

$$712 \quad V_\ell = -\frac{I_{\ell,x}}{a_{\ell,1}(i\lambda)}, \quad W_\ell = \frac{V_\ell}{i\lambda + c_\ell}$$

713 and
 714

$$715 \quad v = \frac{-\sum_{\ell=0}^N I_\ell(0)}{i\lambda\delta Y} \quad \text{and} \quad \eta_\ell = \frac{-V_\ell(0) + v}{i\lambda\delta},$$

716 we conclude that $(V, \mathbf{I}, W, v, \eta) \in D(\mathcal{A})$ is a solution of
 717

$$718 \quad (i\lambda - \mathcal{A})(V, \mathbf{I}, W, v, \eta)^\top = \mathbf{0}.$$

719 Using Lemma 2.5, we deduce that $V_\ell = I_\ell = W_\ell = v = \eta_\ell = 0$ for all $\ell \in \{0, \dots, N\}$.
 720

721 This shows that $B_\lambda + R_\lambda$ is invertible and therefore a unique solution $(I_\ell)_\ell \in \mathbb{H}^1(\mathcal{S})$ of
 722 (2.28) exists. At this stage, by setting $V_\ell = f_{i\lambda,\ell} - a_{\ell,1}(i\lambda)I_{\ell,x}$, we conclude as in the proof
 723 of Theorem 2.1 that $(V, \mathbf{I}, W, v, \eta) \in D(\mathcal{A})$ is a solution of (2.22) and the proof is thus
 724 complete. \square

725 It remains to show the surjectivity of \mathcal{A} , but first let us introduce the following Hilbert
 726 space
 727

$$728 \quad \mathcal{V}_* = \left\{ (V, \mathbf{I}) \in \mathbb{H}^1(\mathcal{S}) \times \mathbb{H}^1(\mathcal{S}) \text{ such that } V_\ell(L) = \alpha_\ell I_\ell(L), \forall \ell \in \{0, \dots, N\}, \right.$$

$$729 \quad \left. \sum_{\ell=0}^N I_\ell(0) = 0, \quad V_\ell(0) - V_0(0) = 0, \quad \forall \ell \in \{1, \dots, N\} \right\}.$$

730 **Lemma 2.7** Let $(\theta, \Lambda) \in \prod_{\ell=0}^N \mathcal{D}(0, L) \times \prod_{\ell=0}^N \mathcal{D}(0, L)$, then there exists $(\psi, \chi) \in \mathcal{V}_*$ such that
 731
 732

$$733 \quad \begin{cases} \psi_{\ell,x} = \theta_\ell, \\ \chi_{\ell,x} = \Lambda_\ell, \quad \forall \ell \in \{0, \dots, N\}. \end{cases} \quad (2.29)$$

Proof As (2.29) means that ψ_ℓ (resp. χ_ℓ) is a primitive of θ_ℓ (resp. Λ_ℓ), we have

$$\begin{aligned} \psi_\ell(x) &= \int_0^x \theta_\ell(y) dy + d_\ell, \\ \chi_\ell(x) &= \int_0^x \Lambda_\ell(y) dy + p_\ell, \end{aligned} \tag{2.30}$$

where d_ℓ, p_ℓ are two constants to be determined later on. Notice that $(\psi, \chi) \in \mathcal{V}_*$ if and only if

$$\begin{cases} \sum_{\ell=0}^N \chi_\ell(0) = 0, \\ \psi_\ell(0) = \psi_0(0), \quad \forall \ell \in \{1, \dots, N\}, \\ \psi_\ell(L) = \alpha_\ell \chi_\ell(L), \quad \forall \ell \in \{0, \dots, N\}. \end{cases} \tag{2.31}$$

Then, knowing that

$$\begin{aligned} \psi_\ell(0) &= d_\ell, \quad \psi_\ell(L) = \int_0^L \theta_\ell(y) dy + d_\ell, \\ \chi_\ell(0) &= p_\ell, \quad \chi_\ell(L) = \int_0^L \Lambda_\ell(y) dy + p_\ell, \end{aligned} \tag{2.32}$$

we see that (2.31) is equivalent to the following system with $2N + 2$ equations

$$\begin{cases} \sum_{\ell=0}^N p_\ell = 0, \\ d_\ell = d_0, \quad \forall \ell \in \{1, \dots, N\}, \\ \int_0^L \theta_\ell(y) dy + d_\ell = \alpha_\ell \left(\int_0^L \Lambda_\ell dy + p_\ell \right), \quad \forall \ell \in \{0, \dots, N\}. \end{cases} \tag{2.33}$$

By the second and third equations of (2.33) we find

$$p_\ell = \frac{1}{\alpha_\ell} \left(\int_0^L \theta_\ell(y) dy + d_0 \right) - \int_0^L \Lambda_\ell(y) dy, \quad \forall \ell \in \{0, \dots, N\}. \tag{2.34}$$

Then, replacing (2.34) in the first equation of (2.33) leads to

$$\sum_{\ell=0}^N \left(\frac{1}{\alpha_\ell} \int_0^L \theta_\ell(y) dy - \int_0^L \Lambda_\ell(y) dy \right) + \left(\sum_{\ell=0}^N \frac{1}{\alpha_\ell} \right) d_0 = 0. \tag{2.35}$$

As $\sum_{\ell=0}^N \frac{1}{\alpha_\ell}$ is positive, the existence of d_0 is guaranteed, which proves the Lemma. □

Lemma 2.8 *Under the same condition of Theorem 2.4, \mathcal{A} is surjective.*

Proof Let $\mathbf{F} = (f_1, f_2, f_3, f_4, f_5)^\top \in \mathcal{H}$, we look for a unique solution $\mathbf{U} = (\mathbf{V}, \mathbf{I}, \mathbf{W}, \nu, \boldsymbol{\eta})^\top \in D(\mathcal{A})$ of

$$-\mathcal{A}\mathbf{U} = \mathbf{F},$$

or equivalently,

$$\begin{cases} a_\ell I_{\ell,x} + g_\ell V_\ell + k_\ell W_\ell &= f_{\ell,1}, \\ b_\ell V_{\ell,x} + r_\ell I_\ell &= f_{\ell,2}, \\ c_\ell W_\ell - V_\ell &= f_{\ell,3}, \\ \frac{1}{\delta Y} \sum_{\ell=0}^N I_\ell(0) &= f_4, \\ \frac{1}{\delta} (V_\ell(0) - v) &= f_{\ell,5}. \end{cases} \tag{2.36}$$

Suppose such a $U = (V, I, W, v, \eta)^\top \in D(\mathcal{A})$ exists, then

$$W_\ell = \frac{V_\ell + f_{\ell,3}}{c_\ell}, \quad \forall \ell \in \{0, \dots, N\}.$$

Replace W_ℓ in the first and second identities of (2.36), we get

$$\begin{cases} V_\ell + a_{\ell,1}(0)I_{\ell,x} = f_{0,\ell}, \\ V_{\ell,x} + a_{\ell,2}(0)I_\ell = g_{0,\ell}, \end{cases} \tag{2.37}$$

where $a_{\ell,1}(0), a_{\ell,2}(0), f_{0,\ell}$ and $g_{0,\ell}$ are defined in (2.9).

We will proceed by dividing the proof into three steps.

Step 1. First, fix $(\phi, \varphi) \in \mathbb{H}^1(\mathcal{S}) \times \mathbb{H}^1(\mathcal{S})$, such that

$$\phi_\ell(0) = \delta f_{\ell,5} \quad \forall \ell \in \{1, \dots, N\}, \quad \phi_\ell = 0 \quad \text{on } [\epsilon_1, L] \text{ for some } \epsilon_1 \in (0, L), \quad \phi_0 = 0$$

and

$$\varphi_0(0) = \delta Y f_4, \quad \varphi_0 = 0 \text{ on } [\epsilon_2, L] \text{ for some } \epsilon_2 \in (0, L), \quad \varphi_\ell = 0 \quad \forall \ell \in \{1, \dots, N\}.$$

Then, setting $\widehat{V}_\ell = V_\ell - \phi_\ell$ and $\widehat{I}_\ell = I_\ell - \varphi_\ell$, by (2.37) we deduce that they satisfy

$$\begin{cases} \widehat{V}_\ell + a_{\ell,1}(0)\widehat{I}_{\ell,x} = \widehat{f}_{0,\ell}, \\ \widehat{V}_{\ell,x} + a_{\ell,2}(0)\widehat{I}_\ell = \widehat{g}_{0,\ell}, \end{cases} \tag{2.38}$$

where,

$$\widehat{f}_{0,\ell} = f_{0,\ell} - (\phi_\ell + a_{\ell,1}(0)\varphi_{\ell,x}) \quad \text{and} \quad \widehat{g}_{0,\ell} = g_{0,\ell} - (\phi_{\ell,x} + a_{\ell,2}(0)\varphi_\ell),$$

for all $\ell \in \{0, \dots, N\}$. We further notice that

$$\begin{cases} \sum_{\ell=0}^N \widehat{I}_\ell(0) &= 0, \\ \widehat{V}_\ell(0) - \widehat{V}_0(0) &= 0, \quad \forall \ell \in \{1, \dots, N\}, \\ \widehat{V}_\ell(L) &= \alpha_\ell \widehat{I}_\ell(L), \quad \forall \ell \in \{0, \dots, N\}. \end{cases} \tag{2.39}$$

851 **Step 2.** Let $(\psi, \chi) \in \mathcal{V}_*$. Multiplying the first and the second equation of (2.38) by $\bar{\chi}_{\ell,x}$ and
 852 $\bar{\psi}_{\ell,x}$ respectively, then integrating in $(0, L)$ gives

853
 854
$$\sum_{\ell=0}^N \int_0^L (\widehat{V}_\ell \bar{\chi}_{\ell,x} + a_{\ell,1}(0) \widehat{I}_{\ell,x} \bar{\chi}_{\ell,x}) dx + \sum_{\ell=0}^N \int_0^L (\widehat{V}_{\ell,x} \bar{\psi}_{\ell,x} + a_{\ell,2}(0) \widehat{I}_\ell \bar{\psi}_{\ell,x}) dx$$

 855
 856
$$= \sum_{\ell=0}^N \int_0^L (\widehat{f}_{0,\ell} \bar{\chi}_{\ell,x} + \widehat{g}_{0,\ell} \bar{\psi}_{\ell,x}) dx. \tag{2.40}$$

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861 Again the sesquilinear form from this left-hand side is not necessarily coercive, so again we
 862 introduce the sesquilinear form

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 864
$$q((\widehat{V}, \widehat{I}), (\psi, \chi)) = \sum_{\ell=0}^N \int_0^L a_{\ell,1}(0) \widehat{I}_{\ell,x} \bar{\chi}_{\ell,x} dx + \sum_{\ell=0}^N \int_0^L \widehat{V}_{\ell,x} \bar{\psi}_{\ell,x} dx$$

 865
 866
$$+ \sum_{\ell=0}^N \int_0^L (I_\ell \bar{\chi}_\ell + V_\ell \bar{\psi}_\ell) dx, \forall (\widehat{V}, \widehat{I}), (\psi, \chi) \in \mathcal{V}_*.$$

 867
 868
 869

870 This sesquilinear form q is continuous and coercive on \mathcal{V}_* since

871
 872
$$a_{\ell,1}(0) = \frac{a_\ell c_\ell}{(c_\ell g_\ell + k_\ell)} > 0, \forall \ell \in \{0, \dots, N\}.$$

 873
 874

875 Then, by Lax-Milgram Lemma, the operator

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 877
$$Q : \mathcal{V}_* \rightarrow (\mathcal{V}_*)^* : (\widehat{V}, \widehat{I}) \rightarrow Q(\widehat{V}, \widehat{I}),$$

 878

879 with

880
$$Q(\widehat{V}, \widehat{I})(\psi, \chi) = q((\widehat{V}, \widehat{I}), (\psi, \chi)),$$

881 is an isomorphism. Now, the remainder

882
 883
$$S : \mathcal{V}_* \rightarrow (\mathcal{V}_*)^* : (\widehat{V}, \widehat{I}) \rightarrow S(\widehat{V}, \widehat{I})(\psi, \chi),$$

 884

885 with

886
 887
$$S(\widehat{V}, \widehat{I})(\psi, \chi) = \sum_{\ell=0}^N \int_0^L (\widehat{V}_\ell \bar{\chi}_{\ell,x} + a_{\ell,2}(0) \widehat{I}_\ell \bar{\psi}_{\ell,x}) dx - \sum_{\ell=0}^N \int_0^L (I_\ell \bar{\chi}_\ell + V_\ell \bar{\psi}_\ell) dx,$$

 888
 889

890 is a compact operator. Hence, we deduce that $Q + S$ is a Fredholm operator of index zero
 891 from \mathcal{V}_* to $(\mathcal{V}_*)^*$.

892 By setting

893
 894
$$L(\psi, \chi) = \sum_{\ell=0}^N \int_0^L (\widehat{f}_{0,\ell} \bar{\chi}_{\ell,x} + \widehat{g}_{0,\ell} \bar{\psi}_{\ell,x}) dx,$$

 895
 896

897 we again see that (2.40) is equivalent to

898
 899
$$(Q + S)(\widehat{V}, \widehat{I}) = L \quad \text{in } (\mathcal{V}_*)^*. \tag{2.41}$$

 900

901 **Step 3.** Now, let us prove that the system (2.38)-(2.39) is equivalent to (2.41). First, let
 902 (θ, Λ) be an arbitrary test function, such that $\theta_{\ell'} = 0 \forall \ell' \in \{0, \dots, N\}$, and $\Lambda_{\ell} \in \mathcal{D}(0, L)$, for
 903 a fixed ℓ and $\Lambda_j = 0, \forall j \neq \ell$. Then, by Lemma 2.7, we deduce that there exists $(\psi, \chi) \in \mathcal{V}_*$
 904 such that

$$\begin{cases} \psi_{\ell,x} = 0, \\ \chi_{\ell,x} = \Lambda_{\ell}, \forall \ell \in \{0, \dots, N\}. \end{cases}$$

905 Replacing this (ψ, χ) in (2.41) we get,
 906

$$\int_0^L (\widehat{V}_{\ell} \Lambda_{\ell} + a_{\ell,1}(0) \widehat{I}_{\ell,x} \Lambda_{\ell}) dx = \int_0^L \widehat{f}_{0,\ell} \Lambda_{\ell} dx.$$

907 Then,
 908

$$\widehat{V}_{\ell} + a_{\ell,1}(0) \widehat{I}_{\ell,x} = \widehat{f}_{0,\ell} \text{ in } \mathcal{D}'(0, L),$$

909 which means that the first equation of (2.38) holds. Second, let (θ, Λ) be an arbitrary test
 910 function, such that $\Lambda_{\ell} = 0 \forall \ell \in \{0, \dots, N\}$, $\theta_{\ell} \in \mathcal{D}(0, L)$, for a fixed ℓ and $\theta_j = 0, \forall j \neq \ell$.
 911 Using Lemma 2.7 and following the same arguments as above we deduce that the second
 912 equation of (2.38) holds. In conclusion, the system (2.38)-(2.39) admits a unique solution
 913 $(\widehat{V}, \widehat{I})$ if and only if $Q + S$ is invertible. As $Q + S$ is a Fredholm operator, it is enough to
 914 prove that it is injective, i.e.,

$$\ker(Q + S) = \{0\}.$$

915 Let $(\widehat{V}, \widehat{I}) \in \ker(Q + S)$ then
 916

$$\begin{aligned} \sum_{\ell=0}^N \int_0^L (\widehat{V}_{\ell} \overline{\chi}_{\ell,x} + a_{\ell,1}(0) \widehat{I}_{\ell,x} \overline{\chi}_{\ell,x}) dx + \sum_{\ell=0}^N \int_0^L (\widehat{V}_{\ell,x} \overline{\psi}_{\ell,x} + a_{\ell,2}(0) \widehat{I}_{\ell} \overline{\psi}_{\ell,x}) dx \\ = 0, \forall (\psi, \chi) \in \mathcal{V}_*. \end{aligned} \tag{2.42}$$

917 Thus, if we set
 918

$$\widehat{W}_{\ell} = \frac{\widehat{V}_{\ell}}{c_{\ell}}, \widehat{v} = \widehat{V}_0(0) \text{ and } \widehat{\eta}_{\ell} = \sum_{k=0}^N Z_{\ell,k} \widehat{I}_k(0).$$

919 We conclude that $(\widehat{V}, \widehat{I}, \widehat{W}, \widehat{v}, \widehat{\eta}) \in D(\mathcal{A})$ is a solution of
 920

$$-\mathcal{A}(\widehat{V}, \widehat{I}, \widehat{W}, \widehat{v}, \widehat{\eta})^T = 0.$$

921 Using Lemma 2.5, we deduce that $\widehat{V}_{\ell} = \widehat{I}_{\ell} = \widehat{W}_{\ell} = \widehat{v} = \widehat{\eta}_{\ell} = 0$ for all $\ell \in \{0, \dots, N\}$.
 922

923 So, $Q + S$ is invertible and a unique solution $(\widehat{V}, \widehat{I}) \in \mathcal{V}_*$ of (2.41) exists. Then, by
 924 the previous arguments we deduce that (2.38) and (2.39) hold. As we set $\widehat{V}_{\ell} = V_{\ell} - \phi_{\ell}$ and
 925 $\widehat{I}_{\ell} = I_{\ell} - \varphi_{\ell}$, we deduce that a unique solution $(V, I) \in \mathbb{H}^1(S) \times \mathbb{H}^1(S)$ of (2.37) exists,
 926 and satisfies

$$\begin{cases} \sum_{\ell=0}^N I_{\ell}(0) & = \delta Y f_4, \\ V_{\ell}(0) - V_0(0) & = \delta f_{\ell,5}, \forall \ell \in \{1, \dots, N\}, \\ V_{\ell}(L) & = \alpha_{\ell} I_{\ell}(L), \forall \ell \in \{0, \dots, N\}. \end{cases}$$

Finally, by defining

$$W_\ell = \frac{V_\ell + f_{\ell,3}}{c_\ell} \quad \forall \ell \in \{0, \dots, N\}, \quad v = V_0(0), \quad \eta_\ell = \sum_{k=1}^N Z_{\ell k} I_k(0) \quad \forall \ell \in \{1, \dots, N\},$$

we deduce that a solution $(V, \mathbf{I}, \mathbf{W}, v, \boldsymbol{\eta}) \in D(\mathcal{A})$ of (2.36) exists. The proof is complete. \square

3 Exponential Energy Decay Rate

In this section, we will prove the exponential stability of System (1.3)-(1.6) based on a frequency domain approach. Here contrary to [29], we do not require that $r_\ell + g_\ell$ is uniformly bounded from below for each ℓ (see the assumption (25) in [29]) to get exponential decay of the energy but rather exploit the dissipative boundary conditions (1.5) at the exterior nodes. Hence our proof fully differs from the proof of Theorem 4.5.4. from [29]. We here need to combine the frequency domain approach with an ad-hoc multiplier method (see Lemma 3.5 above).

Our main result is the following one.

Theorem 3.1 *Assume that condition (1.2) holds. Also, assume that $a_\ell, b_\ell \in W^{1,\infty}(0, L)$, for all $\ell \in \{0, \dots, N\}$. Then, the semigroup $(e^{t\mathcal{A}})_{t \geq 0}$ is exponentially stable, i.e. there exist two positive constants M and $\epsilon > 0$ such that*

$$\|e^{t\mathcal{A}} \mathbf{U}_0\|_{\mathcal{H}} \leq M e^{-\epsilon t} \|\mathbf{U}_0\|_{\mathcal{H}}, \quad \forall t \geq 0, \quad \forall \mathbf{U}_0 \in \mathcal{H}.$$

Proof Following Huang [13] and Prüss [34], our C_0 -semigroup of contractions $(e^{t\mathcal{A}})_{t \geq 0}$ in \mathcal{H} is uniformly (exponentially) stable if and only if

$$i\mathbb{R} \subset \rho(\mathcal{A}), \tag{3.1}$$

and

$$\sup_{\lambda \in \mathbb{R}} \|(i\lambda - \mathcal{A})^{-1}\|_{L(\mathcal{H})} < \infty. \tag{3.2}$$

Since (3.1) was already proved, the proof of Theorem 3.1 is reduced to show that condition (3.2) holds. This is checked by using a contradiction argument. Indeed, suppose that (3.2) is false, then there exist a sequence of real number $\lambda_n \in \mathbb{R}$ and a sequence of vectors $\mathbf{U}_n = (V_n, \mathbf{I}_n, \mathbf{W}_n, v_n, \boldsymbol{\eta}_n)$ in $D(\mathcal{A})$, such that

$$|\lambda_n| \rightarrow \infty, \quad \|(V_n, \mathbf{I}_n, \mathbf{W}_n, v_n, \boldsymbol{\eta}_n)\|_{\mathcal{H}} = 1, \tag{3.3}$$

and

$$(i\lambda_n - \mathcal{A})\mathbf{U}_n \rightarrow 0 \text{ in } \mathcal{H} \quad \text{as } n \rightarrow \infty. \tag{3.4}$$

Our aim is to show that $\|(V_n, \mathbf{I}_n, \mathbf{W}_n, v_n, \boldsymbol{\eta}_n)\|_{\mathcal{H}} \rightarrow 0$. This condition permits to conclude a contradiction with (3.3). From now on, for simplicity we drop the index n .

Now, writing (3.4) in a detailed form we get

$$(i\lambda + g_\ell)V_\ell + a_\ell I_{\ell,x} + k_\ell W_\ell = f_{\ell,1} \rightarrow 0 \text{ in } L^2(0, L), \quad \forall \ell \in \{0, \dots, N\}, \tag{3.5}$$

$$(i\lambda + r_\ell)I_\ell + b_\ell V_{\ell,x} = f_{\ell,2} \rightarrow 0 \text{ in } L^2(0, L), \quad \forall \ell \in \{0, \dots, N\}, \tag{3.6}$$

$$(i\lambda + c_\ell)W_\ell - V_\ell = f_{\ell,3} \rightarrow 0 \text{ in } L^2(0, L), \quad \forall \ell \in \{0, \dots, N\}, \tag{3.7}$$

$$i\lambda V_0(0) + \frac{1}{\delta Y} \sum_{\ell=0}^N I_\ell(0) = f_4 \rightarrow 0 \text{ in } \mathbb{C}, \tag{3.8}$$

$$i\lambda \eta_\ell + \frac{1}{\delta} (V_\ell(0) - V_0(0)) = f_{\ell,5} \rightarrow 0 \text{ in } \mathbb{C}, \quad \forall \ell \in \{1, \dots, N\}. \tag{3.9}$$

Lemma 3.2 *Under the above assumptions, we have the following estimations for all $\ell \in \{0, \dots, N\}$:*

$$\sqrt{g_\ell} V_\ell = o(1) \text{ in } L^2(0, L), \tag{3.10}$$

$$I_\ell(L) = o(1) \text{ in } \mathbb{C}, \tag{3.11}$$

$$\sqrt{r_\ell} I_\ell = o(1) \text{ in } L^2(0, L), \tag{3.12}$$

$$W_\ell = o(1) \text{ in } L^2(0, L). \tag{3.13}$$

Proof Taking the inner product in \mathcal{H} of $(i\lambda - \mathcal{A})U$ with U and using (2.3) and Cauchy-Schwarz's inequality, we get

$$\begin{aligned} \frac{1}{2} \sum_{\ell=0}^N \int_0^L (\theta_\ell g_\ell |V_\ell|^2 + 2\beta_\ell r_\ell |I_\ell|^2 + \gamma_\ell c_\ell |W_\ell|^2) dx + \alpha_\ell |I_\ell(L)|^2 &\leq \Re \langle (i\lambda - \mathcal{A})U, U \rangle_{\mathcal{H}} \\ &\leq \| (i\lambda - \mathcal{A})U \|_{\mathcal{H}} \| U \|_{\mathcal{H}}. \end{aligned}$$

As $\| (i\lambda - \mathcal{A})U \|_{\mathcal{H}} \rightarrow 0$ and $\| U \|_{\mathcal{H}} = 1$, we obtain

$$\frac{1}{2} \int_0^L (\theta_\ell g_\ell |V_\ell|^2 + 2\beta_\ell r_\ell |I_\ell|^2 + \gamma_\ell c_\ell |W_\ell|^2) dx + \alpha_\ell |I_\ell(L)|^2 = o(1), \quad \forall \ell \in \{0, \dots, N\},$$

and the desired results hold. □

From (3.3), (3.6) and (3.7) we have

$$\frac{\|V_{\ell,x}\|_{L^2}}{\lambda} = O(1), \quad \frac{\|I_{\ell,x}\|_{L^2}}{\lambda} = O(1). \tag{3.14}$$

Lemma 3.3 *Under the above assumptions, for all $\ell \in \{0, \dots, N\}$, we have*

$$|V_\ell(0)| = O(1), \tag{3.15}$$

and

$$|I_\ell(0)| = O(1). \tag{3.16}$$

Proof As Z is an invertible matrix, $I_*(0) = Z^{-1}\eta$ and therefore, there exists $\zeta > 0$ such that

$$\|I_*(0)\|_{\mathbb{C}^N} \leq \zeta \|\eta\|_{\mathbb{C}^N}. \tag{3.17}$$

Then, using (3.3), we directly deduce that (3.16) holds for $\ell \in \{1, \dots, N\}$.

Now, multiply (3.6) by $2\bar{T}_\ell$, and then integrate over $(0, L)$, to obtain

$$\begin{aligned}
 2i \int_0^L \lambda V_\ell \bar{T}_\ell dx + 2 \int_0^L g_\ell V_\ell \bar{T}_\ell dx + 2 \int_0^L a_\ell I_{\ell,x} \bar{T}_\ell dx + 2 \int_0^L k_\ell W_\ell \bar{T}_\ell dx \\
 = 2 \int_0^L f_{\ell,1} \bar{T}_\ell dx.
 \end{aligned}
 \tag{3.18}$$

The second and fourth integral on the left-hand side of (3.18) converge to zero by (3.3), (3.10) and (3.13), similarly the right-hand side converge to zero, using (3.3) and the fact that $f_{\ell,1}$ converge to zero in $L^2(0, L)$. This yields

$$2i \int_0^L \lambda V_\ell \bar{T}_\ell dx + 2 \int_0^L a_\ell I_{\ell,x} \bar{T}_\ell dx = o(1).
 \tag{3.19}$$

Taking the real part of (3.19) and then applying Green’s formula on the second integral of the left side leads to,

$$-2 \Im \int_0^L \lambda V_\ell \bar{T}_\ell dx - \int_0^L a_{\ell,x} |I_\ell|^2 dx + a_\ell(L) |I_\ell(L)|^2 - a_\ell(0) |I_\ell(0)|^2 = o(1),
 \tag{3.20}$$

where, $\Im z$ denotes the imaginary part of a complex number z .

Similarly, multiply (3.7) by $2\bar{V}_\ell$, and then integrate over $(0, L)$, we get

$$2i \int_0^L \lambda I_\ell \bar{V}_\ell dx + 2 \int_0^L r_\ell I_\ell \bar{V}_\ell dx + 2 \int_0^L b_\ell V_{\ell,x} \bar{V}_\ell dx = 2 \int_0^L f_{\ell,2} \bar{V}_\ell dx.
 \tag{3.21}$$

The right hand-side of (3.21) converge to zero by (3.6) and (3.3), similarly the second integral of the left hand-side converge to zero by (3.3) and (3.12). This yields

$$2i \int_0^L \lambda I_\ell \bar{V}_\ell dx + 2 \int_0^L b_\ell V_{\ell,x} \bar{V}_\ell dx = o(1).
 \tag{3.22}$$

Again, taking the real part of (3.22) and then, applying Green’s formula in the second integral of the left hand side we get,

$$-2 \Im \int_0^L \lambda I_\ell \bar{V}_\ell dx - \int_0^L b_{\ell,x} |V_\ell|^2 dx + b_\ell(L) |V_\ell(L)|^2 - b_\ell(0) |V_\ell(0)|^2 = o(1).
 \tag{3.23}$$

Adding equations (3.20) and (3.23) leads to

$$\begin{aligned}
 - \int_0^L a_{\ell,x} |I_\ell|^2 dx - \int_0^L b_{\ell,x} |V_\ell|^2 dx + a_\ell(L) |I_\ell(L)|^2 - a_\ell(0) |I_\ell(0)|^2 \\
 + b_\ell(L) |V_\ell(L)|^2 - b_\ell(0) |V_\ell(0)|^2 = o(1), \quad \forall \ell \in \{0, \dots, N\}.
 \end{aligned}
 \tag{3.24}$$

Under the assumption that $a_\ell, b_\ell \in W^{1,\infty}(0, L)$, using (1.2), (3.3), (3.11), (3.17) and the fact that $V_\ell(L) = \alpha_\ell I_\ell(L)$, we deduce from (3.24) that

$$|V_\ell(0)| = O(1), \quad \forall \ell \in \{1, \dots, N\},
 \tag{3.25}$$

and as $|V_0(0)| = O(1)$ using (3.3), we conclude that (3.15) holds. On the other hand, for $\ell = 0$ and under the assumption that $a_0, b_0 \in W^{1,\infty}(0, L)$ and using (1.2), (3.3), (3.11) and

1101 as $V_0(L) = \alpha_0 I_0(L)$, we deduce from (3.24) that $|I_0(0)|^2 = O(1)$. Consequently, with (3.17),
 1102 we conclude that (3.16) holds. □

1103

1104 **Lemma 3.4** *Under the above assumptions, we have*

1105

$$1106 \quad v = V_0(0) = o(1), \tag{3.26}$$

1107

1108 *and*

1109

$$1110 \quad \eta_\ell = o(1) \quad \forall \ell \in \{1, \dots, N\}. \tag{3.27}$$

1111

1112 *Proof* Using (3.8), we have

1113

$$1114 \quad V_0(0) = -\frac{1}{i\lambda\delta Y} \sum_{\ell=0}^N I_\ell(0) + \frac{f_4}{i\lambda},$$

1115

1116 by (3.16) and the fact that $|f_4| = O(1)$, we deduce that $V_0(0) = o(1)$. Similarly, using (3.9),
 1117 we have

1118

$$1119 \quad \eta_\ell = -\frac{1}{i\lambda\delta} V_\ell(0) + \frac{1}{i\lambda\delta} V_0(0) + \frac{1}{i\lambda} f_{\ell,5},$$

1120

1121 by (3.15) and the fact that $|f_{\ell,5}| = O(1)$, we deduce that (3.27) holds. □

1122

1123 **Lemma 3.5** *Let $\ell \in \{0, \dots, N\}$ be fixed and assume that the functions $a_\ell, b_\ell \in W^{1,\infty}(0, L)$.
 1124 Then, there exists $h_\ell \in W^{1,\infty}(0, L)$ and a positive real number ρ_ℓ such that*

1125

$$1126 \quad h_\ell(0) = 0, \tag{3.28a}$$

1127

$$1128 \quad h_{\ell,x}(x) \geq \rho_\ell > 0, \text{ for a.a. } x \in (0, L), \tag{3.28b}$$

1129

$$1130 \quad h_{\ell,x}(x) \left(\frac{a_\ell(x)}{b_\ell(x)} \right) + h_\ell(x) \left(\frac{a_\ell(x)}{b_\ell(x)} \right)_x \geq 0, \text{ for a.a. } x \in (0, L). \tag{3.28c}$$

1131

1132 *Proof* We first notice that (3.28c) is equivalent to

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$$1134 \quad \frac{h_{\ell,x}(x)}{h_\ell(x)} + \frac{\left(\frac{a_\ell(x)}{b_\ell(x)} \right)_x}{\left(\frac{a_\ell(x)}{b_\ell(x)} \right)} \geq 0, \text{ for a.a. } x \in (0, L), \tag{3.29}$$

1135

1136 or equivalently

1137

$$1138 \quad \frac{h_{\ell,x}(x)}{h_\ell(x)} \geq \omega_\ell(x) := -\frac{\left(\frac{a_\ell(x)}{b_\ell(x)} \right)_x}{\left(\frac{a_\ell(x)}{b_\ell(x)} \right)}, \text{ for a.a. } x \in (0, L). \tag{3.30}$$

1139

1140 Since $\omega_\ell(x)$ is bounded on $(0, L)$, then if $h_\ell(x) = x$ near zero, there exists a positive real
 1141 number ϵ (sufficiently small) such that

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$$1143 \quad \frac{1}{x} \geq \omega_\ell(x), \quad \forall x \in (0, \epsilon]. \tag{3.31}$$

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1151 So, we define

$$1152 \quad M_\ell = \max_{x \in (0, L)} \omega_\ell(x),$$

1154 and we distinguish the following two cases:

1155 i) If $M_\ell \leq \frac{1}{L}$, then (3.31) holds with $\epsilon = L$, and we can choose $h_\ell(x) = x, \forall x \in (0, L)$. In
 1156 this case, (3.28a), (3.28b) and (3.28c) hold.

1157 ii) If $M_\ell > \frac{1}{L}$, then we choose $0 < \epsilon \leq \frac{1}{M_\ell}$, and define $h_\ell(x) = x, \forall x \leq \epsilon$, so that (3.31)
 1158 holds. Now, for $x \geq \epsilon$, we impose that

$$1159 \quad h_\ell(x) = \epsilon e^{\int_\epsilon^x m_\ell(y) dy}, \forall x \geq \epsilon, \tag{3.32}$$

1162 with

$$1164 \quad m_\ell(x) := \max\{\omega_\ell(x), \kappa\}, \tag{3.33}$$

1166 for some $\kappa > 0$. Notice that $\lim_{x \rightarrow \epsilon} h_\ell(x) = \epsilon$, i.e., h_ℓ is continuous on $(0, L)$. Now from the
 1167 expression (3.32), we have

$$1169 \quad h_{\ell,x}(x) = \epsilon m_\ell(x) e^{\int_\epsilon^x m_\ell(y) dy} = h_\ell(x) m_\ell(x), \forall x > \epsilon. \tag{3.34}$$

1171 As $m_\ell(x) \geq \omega_\ell(x)$, we deduce that (3.30) holds (i.e. (3.28c) holds). Finally, by (3.32), we
 1172 have that

$$1174 \quad h_\ell(x) \geq \epsilon, \forall x > \epsilon, \tag{3.35}$$

1175 which by (3.34) and (3.33) implies that

$$1177 \quad h_{\ell,x}(x) \geq \epsilon \kappa > 0, \forall x > \epsilon.$$

1179 In other words, (3.28b) holds for almost all $x \in (0, L)$. □

1181 **Lemma 3.6** *Under the above assumptions, we have for all $\ell \in \{0, \dots, N\}$,*

$$1183 \quad V_\ell = o(1) \text{ in } L^2(0, L). \tag{3.36}$$

1186 *Proof* Fix $\ell \in \{0, \dots, N\}$ and let $h_\ell \in W^{1,\infty}(0, L)$ be the function defined in Lemma 3.5.

1187 First, by multiplying (3.5) by $(2h_\ell \bar{V}_{\ell,x})$ and then integrating over $(0, L)$, we get

$$1189 \quad 2 \int_0^L h_\ell V_\ell \bar{V}_{\ell,x} dx - 2i \int_0^L h_\ell g_\ell V_\ell \frac{\bar{V}_{\ell,x}}{\lambda} dx - 2i \int_0^L a_\ell h_\ell \bar{V}_{\ell,x} \frac{I_{\ell,x}}{\lambda} dx$$

$$1190 \quad - 2i \int_0^L h_\ell k_\ell W_\ell \frac{\bar{V}_{\ell,x}}{\lambda} dx = -2i \int_0^L h_\ell f_{\ell,1} \frac{\bar{V}_{\ell,x}}{\lambda} dx. \tag{3.37}$$

1194 The second and fourth integral on the left hand-side of (3.37) converge to zero by (3.10),
 1195 (3.13) and (3.14). Similarly, the right hand-side converge to zero by (3.14) and the fact that
 1196 $f_{\ell,1}$ converge to zero in $L^2(0, L)$, for all ℓ . This yields

$$1198 \quad 2 \int_0^L h_\ell V_\ell \bar{V}_{\ell,x} dx - 2i \int_0^L a_\ell h_\ell \bar{V}_{\ell,x} \frac{I_{\ell,x}}{\lambda} dx = o(1). \tag{3.38}$$

1201 By eliminating $V_{\ell,x}$ from (3.6) in the second term of (3.38), we obtain

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$$2 \int_0^L h_\ell V_\ell \bar{V}_{\ell,x} dx - 2i \int_0^L \frac{a_\ell}{b_\ell} h_\ell \bar{f}_{\ell,2} \frac{I_{\ell,x}}{\lambda} dx + 2 \int_0^L \frac{a_\ell}{b_\ell} h_\ell \bar{I}_\ell I_{\ell,x} dx + 2i \int_0^L \frac{a_\ell}{b_\ell} h_\ell r_\ell \bar{I}_\ell \frac{I_{\ell,x}}{\lambda} dx = o(1). \tag{3.39}$$

1208 The second and fourth integral converge to zero by (3.14), (3.12) and the fact that $f_{\ell,2}$
1209 converge to zero in $L^2(0, L)$, $\forall \ell$. Now, by taking the real part of (3.39) and applying Green's
1210 formula on the remaining two integrals, we deduce

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$$- \int_0^L h_{\ell,x} |V_\ell|^2 dx + h_\ell(L) |V_\ell(L)|^2 - h_\ell(0) |V_\ell(0)|^2 - \int_0^L \left(\frac{a_\ell}{b_\ell} \right)_x h_\ell |I_\ell|^2 dx - \int_0^L \frac{a_\ell}{b_\ell} h_{\ell,x} |I_\ell|^2 dx + \frac{a_\ell(L)}{b_\ell(L)} h_\ell(L) |I_\ell(L)|^2 - \frac{a_\ell(0)}{b_\ell(0)} h_\ell(0) |I_\ell(0)|^2 = o(1). \tag{3.40}$$

1218 Then, due to the fact that $V_\ell(L) = \alpha_\ell I_\ell(L)$ and using (3.11) and (3.28a), we obtain

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$$\int_0^L h_{\ell,x} |V_\ell|^2 dx + \int_0^L \left(\left(\frac{a_\ell}{b_\ell} \right)_x h_\ell + \frac{a_\ell}{b_\ell} h_{\ell,x} \right) |I_\ell|^2 dx = o(1). \tag{3.41}$$

1222 By using (3.28b) and (3.28c), the result holds. □

1223 **Lemma 3.7** *Under the above assumptions, we have for all $\ell \in \{0, \dots, N\}$,*

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$$I_\ell = o(1) \quad \text{in } L^2(0, L). \tag{3.42}$$

1228 *Proof* First, multiplying (3.6) by $\frac{1}{b_\ell} \bar{I}_\ell$ and then integrating over $(0, L)$, we find

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$$\int_0^L \frac{i\lambda}{b_\ell} |I_\ell|^2 dx + \int_0^L \frac{r_\ell}{b_\ell} |I_\ell|^2 dx + \int_0^L V_{\ell,x} \bar{I}_\ell dx = \int_0^L \frac{f_{\ell,2}}{b_\ell} \bar{I}_\ell dx. \tag{3.43}$$

1233 The second integral of the left hand-side of (3.43) converge to zero by (3.3) and (3.13).
1234 Similarly, the integral of the right hand side converge to zero by (3.3) and the fact that $f_{\ell,2}$
1235 converge to zero in $L^2(0, L)$, for all ℓ . This yields

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$$\int_0^L \frac{i\lambda}{b_\ell} |I_\ell|^2 dx + \int_0^L V_{\ell,x} \bar{I}_\ell dx = o(1). \tag{3.44}$$

1240 Next, by applying Green's formula on the second integral of the left-hand side of (3.44),
1241 eliminating $I_{\ell,x}$ from (3.5) and then dividing by $i\lambda$, we obtain

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$$\int_0^L \frac{|I_\ell|^2}{b_\ell} dx - \int_0^L \frac{|V_\ell|^2}{a_\ell} dx + \int_0^L \frac{g_\ell}{i\lambda a_\ell} |V_\ell|^2 dx + \int_0^L \frac{k_\ell}{i\lambda a_\ell} V_\ell \bar{W}_\ell dx + \frac{V_\ell(L) \bar{I}_\ell(L)}{i\lambda} - \frac{V_\ell(0) \bar{I}_\ell(0)}{i\lambda} = \int_0^L V_\ell \frac{\bar{f}_{\ell,1}}{i\lambda a_\ell} dx. \tag{3.45}$$

1248 As $V_\ell(L) = \alpha_\ell I_\ell(L)$ and using (1.2), (3.3), (3.11), (3.13) and Lemmas 3.3 and 3.6, the result
1249 holds. □

Consequently, using (3.13), Lemma 3.4, Lemma 3.6 and Lemma 3.7, we deduce the desired contradiction. This ends the proof. \square

Remark 3.8 Contrary to [2, §5.6] (see also [11] for second order hyperbolic systems), our boundary conditions are chosen in such a way that there is no limit of stabilization, namely under our previous assumptions, exponential decay occurs for all lengths L but the decay rate ϵ in Theorem 3.1 may depend on L . If $r_\ell = g_\ell = 0$, we conjecture that this decay rate deteriorates as L goes to infinity.

4 Some Extensions and Open Problems

4.1 Other Dynamical Boundary Conditions

The standard Kirchhoff conditions at the interior common vertex are

$$\begin{cases} V_0(0, t) - V_\ell(0, t) = 0, & \text{in } \mathbb{R}_+^*, \forall \ell \in \{1, \dots, N\}, \\ \sum_{\ell=0}^N I_\ell(0, t) = 0 & \text{in } \mathbb{R}_+^*, \end{cases} \tag{4.1}$$

that corresponds to the limit case of (1.6) as δ goes to zero. These boundary conditions are stationary ones but can be also used instead of (1.6). More generally in the spirit of [18], we can mix up stationary and dynamical boundary conditions at the common vertex \mathbf{v} in the following way. First if $\mathbf{u}(\cdot, t) = ((\mathbf{V}_\ell(\cdot, t), \mathbf{I}_\ell(\cdot, t), \mathbf{W}_\ell(\cdot, t))_{\ell=0}^N)^\top$ represents the unknowns in (1.3), we denote by

$$\gamma_{\mathbf{v}}(\mathbf{u}(\cdot, t)) = ((\mathbf{V}_\ell(0, t))_{\ell=0}^N, (\mathbf{I}_\ell(0, t))_{\ell=0}^N)^\top$$

its trace at \mathbf{v} that is a vector in $\mathbb{C}^{2(N+1)}$. Obviously we only keep the trace of \mathbf{V}_ℓ and \mathbf{I}_ℓ since \mathbf{W}_ℓ is not regular enough in space (see the definition of $D(\mathcal{A})$). Then we fix a subspace $Y_{\mathbf{v}}$ of $\mathbb{C}^{2(N+1)}$, another subspace $Y_{\mathbf{v}}^{(d)}$ of $Y_{\mathbf{v}}$ and finally a linear operator $B_{\mathbf{v}}$ from $Y_{\mathbf{v}}$ to $Y_{\mathbf{v}}^{(d)}$ and set

$$x_{\mathbf{v}}(t) = P_{\mathbf{v}} \gamma_{\mathbf{v}}(\mathbf{u}(\cdot, t)),$$

where $P_{\mathbf{v}}$ is a projection from $\mathbb{C}^{2(N+1)}$ into $Y_{\mathbf{v}}^{(d)}$ (for an appropriate inner product of $\mathbb{C}^{2(N+1)}$).

Hence at \mathbf{v} , we can consider the boundary condition

$$\begin{cases} \gamma_{\mathbf{v}}(\mathbf{u}(\cdot, t)) \in Y_{\mathbf{v}}, & \text{in } \mathbb{R}_+^*, \\ x_{\mathbf{v},t}(t) = B_{\mathbf{v}} \gamma_{\mathbf{v}}(\mathbf{u}(\cdot, t)), & \text{in } \mathbb{R}_+^*. \end{cases} \tag{4.2}$$

The boundary conditions (1.6) and (4.1) enter in this setting. Namely in the first case, we need to take $Y_{\mathbf{v}} = \mathbb{C}^{2(N+1)}$ and

$$Y_{\mathbf{v}}^{(d)} = \{((v_\ell)_{\ell=0}^N, (i_\ell)_{\ell=0}^N)^\top : i_0 = v_\ell = 0, \forall \ell = 1, \dots, N\},$$

$P_{\mathbf{v}}$ is the orthogonal projection into $Y_{\mathbf{v}}^{(d)}$ with respect to the Euclidean inner product, and

$$B_{\mathbf{v}}((v_\ell)_{\ell=0}^N, (i_\ell)_{\ell=0}^N)^\top = \frac{1}{\delta} \left(\left(-\frac{1}{Y} \sum_{\ell=0}^N i_\ell, 0, \dots, 0 \right)^\top, \left(0, Z^{-1}(v_0 1_N - \mathbf{v}) \right)^\top \right)^\top,$$

where 1_N is the vector of \mathbb{C}^N with all entries equal to 1, while $\mathbf{v} = (v_\ell)_{\ell=1}^N$. On the contrary in the case (4.1), we simply need to take $Y_{\mathbf{v}}^{(d)} = \{0\}$ and

$$Y_{\mathbf{v}} = \{((v_\ell)_{\ell=0}^N, (i_\ell)_{\ell=0}^N)^\top : \sum_{\ell=0}^N i_\ell = 0 \text{ and } v_0 = v_\ell, \forall \ell = 1, \dots, N\}.$$

First, under some assumptions on $Y_{\mathbf{v}}$ and $B_{\mathbf{v}}$, one can show that (2.3) remains valid and therefore the associated operator \mathcal{A} is dissipative. For the maximality, in some particular situations, using the arguments of Sect. 2.1 (eliminating the variables W_ℓ and one of the other variables to obtain a variational formulation), we can prove that the associated operator is maximal dissipative under additional conditions on $Y_{\mathbf{v}}$ and $B_{\mathbf{v}}$ and therefore system (1.3) with the boundary conditions (1.5) and (4.2) is well-posed and is governed by a semigroup of contractions. In the general case, using a bounded perturbation argument as in [18, Theorem 3.3] we can prove that a bounded perturbation of \mathcal{A} is maximal dissipative under additional conditions on $Y_{\mathbf{v}}$ and $B_{\mathbf{v}}$ and therefore system (1.3) with the boundary conditions (1.5) and (4.2) is well-posed and is governed by a C_0 semigroup. But using the dissipativeness of \mathcal{A} , one can deduce that \mathcal{A} generates a semigroup of contractions.

Once the well-posedness of our system holds, one can analyze its stability. First, we notice that Lemma 2.5 remains valid because the proof presented before yields first $V_\ell = I_\ell = W_\ell = 0$, for all ℓ and hence $x_{\mathbf{v}} = P_{\mathbf{v}} \gamma_{\mathbf{v}}(\mathbf{u})$ will be zero as well. On the contrary, Lemma 2.6 is problematic because a compact perturbation argument cannot be invoked due to the third identity in (2.22). But if the surjectivity of $i\lambda - \mathcal{A}$ holds for all $\lambda \in \mathbb{R}$, then the system will be exponentially stable. Indeed, in this case the proof of Theorem 3.1 remains valid because the use of the multipliers h_ℓ from Lemma 3.5 allows to skip the use of the boundary condition (4.2).

In summary the exponential stability results for system (1.3) with the boundary conditions (1.5) and (4.2) are reduced to an existence result and the subjectivity of the resolvent on the imaginary axis.

4.2 General Networks

Obviously, we can consider system (1.3) on an arbitrary network \mathcal{G} with dissipative boundary conditions similar to (1.5) at exterior vertices (it is (1.5) if the edge having the exterior vertex \mathbf{v} as extremity is outgoing, otherwise it is $V_\ell(\mathbf{v}, t) + \alpha_\ell I_\ell(\mathbf{v}, t) = 0$) and boundary conditions like (4.2) at the interior vertices. As in the previous subsection, existence results can then be obtained under appropriate conditions on $Y_{\mathbf{v}}$ and $B_{\mathbf{v}}$. Obviously the surjectivity of $i\lambda - \mathcal{A}$ for all $\lambda \in \mathbb{R}$ will remain problematic, but here even its injectivity is not immediate.

The sole case for which injectivity of $i\lambda - \mathcal{A}$ holds for all $\lambda \in \mathbb{R}^*$ is the case of trees with the boundary conditions (1.6) or (4.1) at the interior vertices. Indeed, in this case the arguments of Lemma 2.5 allow to show that $V_\ell = I_\ell = W_\ell = 0$, for all edges ℓ of the last generation of the tree. But then using (1.6) or (4.1), we will get that $V_{\ell'}(\mathbf{v}') = I_{\ell'}(\mathbf{v}') = 0$ for all edges ℓ' of the penultimate generation and the vertex \mathbf{v}' in common with the last generation. This means that we can use again the arguments of Lemma 2.5 to these edges ℓ' and by iteration we will find the injectivity result. Finally, the exponential stability results for system (1.3) with the boundary conditions (1.5) at exterior vertices and boundary conditions (1.6) or (4.1) at interior vertices is not immediate even for trees.

Note that for general networks, as in [29], once $i\lambda - \mathcal{A}$ is bijective for all $\lambda \in \mathbb{R}$, exponential stability will be valid if we assume that $r_\ell + g_\ell$ is uniformly bounded from below for each ℓ (see the assumption (25) in [29]); without this condition, polynomial stability can

be achieved. Nevertheless the type of the decay rate of such systems on general networks remains largely an open problem. Does it remain exponential or polynomial? How many boundary conditions can we remove from the exterior nodes to preserve stability? What will happen if we add some damping terms localized at the interior nodes?

Our proof of Theorem 3.1 is non constructive (since it uses a contradiction argument), hence it does not give the dependency of the decay rate with respect to the involved parameters, hence its degeneracy as one parameter tends to zero or to infinity (see Remark 3.8) cannot be detected. Therefore, it would be interesting to characterize such dependencies by performing some spectral analysis for instance. In power grids, failure of some parts can occur from time to time, this corresponds to switch off one of the boundary controllers ($\alpha_\ell = 0$) from time to time. As in [12] for the wave equation, is it possible to find sufficient conditions on the controllers to achieve exponential decay? All of these problems are challenging and merit to be investigated.

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