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EXPONENTIAL AND POLYNOMIAL STABILITY RESULTS FOR NETWORKS OF ELASTIC AND THERMO-ELASTIC RODS

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Abstract. In this paper, we investigate a network of elastic and thermo-elastic materials. On each thermo-elastic edge, we consider two coupled wave equations such that one of them is damped via a coupling with a heat equation. On each elastic edge (undamped), we consider two coupled conservative wave equations. Under some conditions, we prove that the thermal damping is enough to stabilize the whole system. If the two waves propagate with the same speed on each thermo-elastic edge, we show that the energy of the system decays exponentially. Otherwise, a polynomial energy decay is attained. Finally, we present some other boundary conditions and show that under sufficient conditions on the lengths of some elastic edges, the energy of the system decays exponentially on some particular networks similar to the ones considered in [18].

1. **Introduction.** Thermoelasticity is a principle concerned with predicting the thermo-mechanical behaviour of elastic solids. Understanding such a principle is needed by many engineers to design different materials. Thus, several scientists were motivated to study the thermoelastic system described by the coupling between the mechanical vibration and the heat (thermal) effect of materials. Mathematically, a linear one-dimensional thermo-elastic system satisfied by a thermoelastic bar $(0, L)$ is represented by the following two equations:

$$\begin{cases} u_{tt} - u_{xx} + \alpha\theta_x &= 0, & \text{in } (0,L)\times(0,\infty), \\ \theta_t - \theta_{xx} + \alpha u_{tx} &= 0, & \text{in } (0,L)\times(0,\infty), \end{cases} \tag{1.1}$$

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with the initial conditions

$$u(x,0)=u_0, u_t(x,0)=u_1, \theta(x,0)=\theta_0, \qquad x\in(0,L), \tag{1.2}$$

where, u is the displacement, θ is the temperature deviation from the reference temperature and the mechanical-thermal coupling α is a positive constant. The existence and asymptotic behavior of the solution of the linear thermo-elastic system was firstly studied in [5] but, no decay rate was given. In the one dimensional case, the stabilization of the linear thermo-elastic system satisfied by thermo-elastic materials (damped by thermal effect) with various boundary conditions was investigated by several authors. We will recall some of these results. In [8], the author considered the stabilization of system (1.1)-(1.2) on a thermo-elastic rod (see Figure 1) with u and θ satisfying the Dirichlet and Neumann condition respectively (or vice versa). He succeeded in proving the exponential stability of the system. More precisely, the author established the following energy estimate: There exist two positive constants M and ϵ such that

$$E(t)\leq Me^{-\epsilon t}E(0), \quad \forall t>0. \tag{1.3}$$

Similarly, when u and θ satisfy both the Dirichlet condition, it was shown that the estimate (1.3) still holds in [13]. Then, the method of [13] was extended in [4] to prove (1.3) when $u_x-\alpha\theta$ satisfies Dirichlet condition on both ends and $\theta_x(0)=\theta_x(L)=0$ or $\theta_x(0)=0, \theta(L)=0$.

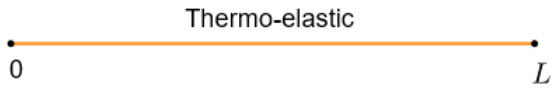

FIGURE 1. A thermoelastic rod

Later on, the importance of damping and controlling the vibrations of materials composed of both elastic (undamped) and thermo-elastic (damped by thermal effect) parts appears in several physical applications and consequently in several mathematical papers. The main questions that received the interest of the researchers is the kind of stability of the thermo-elastic system on such composite materials and how should the thermo-elastic damping be localized to get the best decay rate or what is the energy decay rate in different localizations of the thermal damping? Such questions were answered in several ways. For example, in [14], it was considered a one dimensional body which is configurated in $[0,L_3]\subset\mathbb{R}$ and for a given $L_1<L_2$ in $]0,L_3[$, they assumed that the material is thermo-elastic over $]0,L_1[\cup]L_2,L_3[$ and elastic over $]L_1,L_2[$ (see Figure 2). The authors proved that the whole system is exponentially stable, i.e, (1.3) holds.

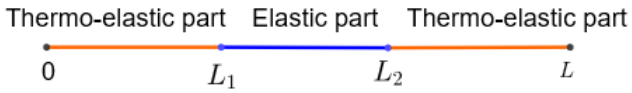

FIGURE 2. An elastic/thermo-elastic transmission problem

Then, in [6], the authors considered the stabilization of a transmission problem for the thermo-elastic system with local thermal effect which is effective only over the

interval $[0, L_0], L_0 \in [0, L]$, see Figure 3. This corresponds to the following system:

$$\begin{cases} u_{tt} - u_{xx} + \alpha\theta_x &= 0, & \text{in } (0, L_0) \times (0, \infty), \\ \theta_t - \theta_{xx} + \alpha u_{tx} &= 0, & \text{in } (0, L_0) \times (0, \infty), \\ \mathsf{v}_{tt} - \mathsf{v}_{xx} &= 0, & \text{in } (L_0, L) \times (0, \infty), \end{cases} \tag{1.4}$$

with the initial conditions

$$u(x, 0) = u_0, u_t(x, 0) = u_1, \theta(x, 0) = \theta_0, \mathsf{v}(x, 0) = \mathsf{v}_0, \mathsf{v}_t(x, 0) = \mathsf{v}_1, \quad x \in (0, L),$$

where u is the displacement in the thermo-elastic part, v is the displacement in the elastic part and θ is the temperature difference from a reference value. The system is completed with the following boundary conditions

$$u(0, t) = \mathsf{v}(L, t) = \theta(0, t) = 0, \quad t \in (0, \infty),$$

and the following transmission conditions

$$u_x(L_0, t) - \alpha\,\theta(L_0, t) = \mathsf{v}_x(L_0, t) \text{ and } \theta_x(L_0, t) = 0.$$

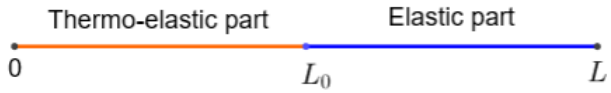

FIGURE 3. An elastic/thermo-elastic transmission problem

The authors proved that the localized dissipation due to the thermal effect is strong enough to prove the exponential decay to zero of the energy. We also refer to [11] and [15] and the references therein for the study of the stabilization of multi-dimensional linear thermo-elastic systems.

On the other hand, there are only few publications on the stabilization of networks of thermo-elastic materials. Let us recall some of these results. In [1], an exponential stability was proved on a network of thermo-elastic materials under both Fourier's law and Cattaneo's law. In [18], the author studied the stability problem of a thermo-elastic system on particular cases of networks of elastic and thermo-elastic materials (see Figure 4).

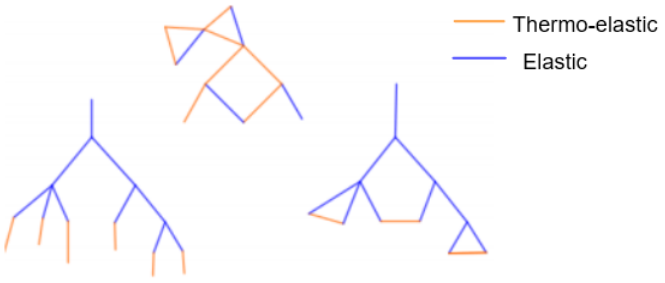

FIGURE 4. Elastic/therm-elastic networks

Under the continuity condition of the displacement, the Neumann condition for

the temperature at the internal nodes, and the balance condition, an exponential stability was proved (see also, [19] for the network of elastic and thermo-elastic beams). Later on the authors in [7] discussed the asymptotic behaviour of a transmission problem of the thermo-elastic system on star shaped networks of elastic and thermo-elastic rods (see Figure 5).

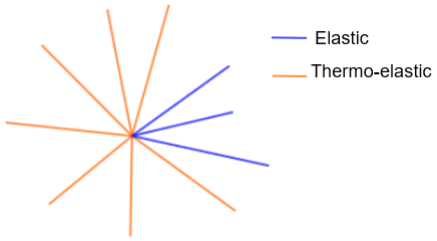

FIGURE 5. Elastic/thermo-elastic star shaped network

The uniform exponential decay rate was proved by a frequency domain analysis when only one purely elastic edge was present. Otherwise, a polynomial decay rate was deduced under a suitable irrationality condition on the lengths of the rods when more than one purely elastic edge is involved. After the review of these results that investigated the stabilization of a thermo-elastic system composed of the coupling between one wave equation and a heat equation, a remarkable question can be asked. What happens if we consider a network of elastic and thermo-elastic materials such that:

• On the thermo-elastic edges, we have a system of two wave equations coupled by velocity, such that one wave equation is coupled to a heat equation with a thermal effect.
• On the purely undamped elastic edges, we have only a system of two conservative wave equations coupled by velocity.

Hence our main question is the following one: Will the dissipation due to the thermal effect be also strong enough to prove the exponential stability of the energy of the whole system? To the best of our knowledge, the answer to this question remains an open problem. Therefore, our aim is to solve this open question.

In this work, we investigate the stabilization of the above described transmission problem on networks of elastic and thermo-elastic materials. We prove the exponential stability of the whole system under the condition that the two waves propagate with the same speed on all the thermo-elastic edges of the network. On the other hand, if there exists an exterior thermo-elastic edge such that the two waves propagate with different speed on this edge, we show the polynomial stability of the whole system. Our main tool is a frequency domain approach, namely to prove the exponential stability we use a result due to Huang [10] and Prüss [17] and to show the polynomial stability we use a result due to Borichev and Tomilov [3].

Now, let us introduce some notations needed to formulate the problem under consideration, refer to [20] and [1] for more details. Let $\mathcal{N}$ be a network embedded in the Euclidean space $\mathbb{R}^m, m \in \mathbb{N}^*$, with n vertices $\mathcal{V} = \{a_0, a_2, .., a_{n-1}\}$ and N edges $E = \{e_1, .., e_N\}$, with $\mathcal{I}(\mathcal{N}) = \{1, .., N\}$, the set of indices of edges. Each

edge e_j is a curve, parametrized by

$$\pi_j : [0, \ell_j] \to e_j : x_j \to \pi_j(x_j). \tag{1.5}$$

The degree of a vertex is the number of incident edges at the vertex. A vertex with degree 1 is called an exterior vertex. On the other hand, a vertex with degree greater than 1 is called an interior vertex.

We assume that the network is made of thermo-elastic edges and elastic ones, this means that $\mathcal{I}(\mathcal{N})$ is split up into $\mathcal{I}(\mathcal{N}) = \mathcal{I}_{\rm te} \cup \mathcal{I}_{\rm e}$, with $\mathcal{I}_{\rm e} \cap \mathcal{I}_{\rm te} = \emptyset$, in other words, $\mathcal{I}_{\rm te}$ (resp. $\mathcal{I}_{\rm e}$) is the set of thermo-elastic (resp. elastic) edges.
We further denote by:

$V_{\rm ext}$:= set of exterior vertices of $\mathcal{N}$.

$V_{\rm int}$:= set of interior vertices of $\mathcal{N}$.

$\mathcal{I}(a_k)$:= set of indices of edges incident to a_k.

$\mathcal{I}_{\rm te}(a_k)$:= set of indices of thermo-elastic edges adjacent to a_k.

$\mathcal{I}_{\rm e}(a_k)$:= set of indices of elastic edges incident to a_k.

$\mathcal{I}_{\rm ext}$:= set of indices of edges adjacent to an exterior vertex of $\mathcal{N}$.

The incidence matrix $D = (d_{kj})_{n\times N}$ of $\mathcal{N}$ is defined by

$$d_{kj} = \begin{cases} 1 & \text{if } \pi_j(\ell_j) = a_k, \\ -1 & \text{if } \pi_j(0) = a_k, \\ 0 & \text{otherwise,} \end{cases} \tag{1.6}$$

and for a function $f : \mathcal{N} \to \mathbb{C}$, we set $f^j = f \circ \pi_j$ its restriction to the edge e_j. For simplicity, we will write $f = (f^1, .., f^N)$ and we will denote $f^j(x) = f^j(\pi_j(x))$ for any x in $(0, \ell_j)$. We consider a network of elastic and thermo-elastic materials that coincides with the graph $\mathcal{N}$. We assume that $\mathcal{N}$ contains at least one thermoelastic edge, that $V_{\rm ext} \neq \emptyset$, that every maximal subgraph of elastic edges is a tree whose all of its exterior vertices except one are attached to thermo-elastic edges and that every maximal subgraph of thermo-elastic edges is not a circuit.

Let $u^j = u^j(x,t)$ and $y^j = y^j(x,t)$ be the functions describing the displacement at time t of the edge e_j, $j \in \mathcal{I}(\mathcal{N})$ and $\theta^j = \theta^j(x,t)$ be the temperature difference to a fixed reference temperature of $e_j, j \in \mathcal{I}_{\rm te}$ at time t.

Our system is described as follows:
• On every thermo-elastic edge ($j \in \mathcal{I}_{\rm te}$) the following equations hold:

$$\begin{cases} u^j_{tt} - u^j_{xx} + \alpha_j \theta^j_x - \beta_j y^j_t &= 0 \quad \text{in} \quad (0,\ell_j)\times(0,\infty), \\ y^j_{tt} - \rho_j y^j_{xx} + \beta_j u^j_t &= 0 \quad \text{in} \quad (0,\ell_j)\times(0,\infty), \\ \theta^j_t - \kappa_j \theta^j_{xx} + \alpha_j u^j_{tx} &= 0 \quad \text{in} \quad (0,\ell_j)\times(0,\infty), \end{cases} \tag{1.7}$$

where α_j, ρ_j, κ_j and β_j are positive constants.
• On every elastic edge ($j \in \mathcal{I}_{\rm e}$) one has:

$$\begin{cases} u_{tt}^j - u_{xx}^j - \beta_j y_t^j & = 0 \quad \text{in} \quad (0,\ell_j)\times(0,\infty), \\ y_{tt}^j - \rho_j y_{xx}^j + \beta_j u_t^j & = 0 \quad \text{in} \quad (0,\ell_j)\times(0,\infty), \end{cases} \tag{1.8}$$

where β_j and ρ_j are positive constants.

We assume that the initial data on the network $\mathcal{N}$ are

$$\begin{cases} u^j(x,0) = u_0^j(x),\ u_t^j(x,0) = u_1^j(x), & \forall j \in \mathcal{I}(\mathcal{N}), \\ y^j(x,0) = y_0^j(x),\ y_t^j(x,0) = y_1^j(x), & \forall j \in \mathcal{I}(\mathcal{N}), \\ \theta^j(x,0) = \theta_0^j(x), & \forall j \in \mathcal{I}_{\text{te}}(\mathcal{N}). \end{cases} \tag{1.9}$$

We denote by V'_{ext}(resp. V'_{int}) the set of exterior (resp. interior) nodes of maximal subgraphs of thermo-elastic edges. Then, the boundary condition on $\mathcal{N}$ are described as follows:
The displacement and temperature satisfies the Dirichlet boundary condition,

$$\begin{cases} u^j(a_k,t) = 0, & j \in \mathcal{I}(a_k),\ \ a_k \in V_{\text{ext}}, \\ y^j(a_k,t) = 0, & j \in \mathcal{I}(a_k),\ \ a_k \in V_{\text{ext}}, \\ \theta^j(a_k,t) = 0, & j \in \mathcal{I}_{\text{te}}(a_k),\ a_k \in V'_{\text{ext}}. \end{cases} \tag{1.10}$$

The displacement and temperature are continuous,

$$\begin{cases} u^j(a_k,t) = u^\ell(a_k,t), & j,\ell \in \mathcal{I}(a_k),\ \ a_k \in V_{\text{int}}, \\ y^j(a_k,t) = y^\ell(a_k,t), & j,\ell \in \mathcal{I}(a_k),\ \ a_k \in V_{\text{int}}, \\ \theta^j(a_k,t) = \theta^\ell(a_k,t), & j,\ell \in \mathcal{I}_{\text{te}}(a_k),\ a_k \in V'_{\text{int}}. \end{cases} \tag{1.11}$$

The system satisfies the balance condition on y at every interior node,

$$\sum_{j\in\mathcal{I}(a_k)} d_{kj}\rho_j y_x^j(a_k,t) = 0, \quad a_k \in V_{\text{int}}. \tag{1.12}$$

The system satisfies the following balance conditions on u and θ,

$$\begin{cases} \displaystyle\sum_{j\in\mathcal{I}_{\text{te}}(a_k)} d_{kj}\kappa_j\theta_x^j(a_k,t) = 0, \quad a_k \in V'_{\text{int}}, \\ \displaystyle\sum_{j\in\mathcal{I}_{\text{te}}(a_k)} d_{kj}(u_x^j(a_k,t) - \alpha_j\theta^j(a_k,t)) + \sum_{j\in\mathcal{I}_{\text{e}}(a_k)} d_{kj}u_x^j(a_k,t) = 0, \quad a_k \in V_{\text{int}}. \end{cases} \tag{1.13}$$

Remark that $\alpha_j > 0$ and $\kappa_j > 0$, for all $j \in \mathcal{I}_{\text{te}}$ while, on each elastic edge only two conservative wave equations hold, i.e, the two wave equations on each elastic edge are neither coupled to a heat equation nor affected by a thermal damping. Hence for $j \in \mathcal{I}_{\text{e}}$, we may set $\alpha_j = \kappa_j = 0$. From time to time, this will allow us to unify some arguments by not distinguishing between elastic and thermoelastic edges.

The paper is organized as follows. In Section 2, we prove that system (1.7)-(1.13) admits a unique solution in an appropriate Hilbert space using semi-group theory. Next, in Section 3, using a general criteria of Arendt-Batty [2], we discuss the strong stability of the system. In Section 4, under the condition that the two waves propagate with the same speed on each thermo-elastic edge of the network, we prove the exponential stability of the system using a frequency domain approach combined with a multiplier technique. Otherwise, we establish a polynomial decay. Finally, in Section 5, we present the Neumann boundary condition at the interior nodes of some particular networks, some of which being considered in [18]. We show

that under some sufficient conditions, the same results as the ones from Section 4 hold.

2. **Well-posedness.** In this section, we will study the existence, uniqueness and regularity of the solution of system (1.7)-(1.13), using a semigroup approach. First, denote by

$$\mathbb{L}^2 = \prod_{j=1}^{N} L^2(0,\ell_j), \quad \mathbb{H}^m = \prod_{j=1}^{N} H^m(0,\ell_j),\ m=1,2,$$

and

$$\mathbb{V} = \prod_{j\in\mathcal{I}_{\rm te}} L^2(0,\ell_j), \quad \mathbb{V}^m = \prod_{j\in\mathcal{I}_{\rm te}} H^m(0,\ell_j),\ m=1,2.$$

Set

$$\begin{aligned}\mathbb{H}_0^1 = \big\{u=(u^j)_{j\in\mathcal{I}(\mathcal{N})}\in\mathbb{H}^1 /\ u^j(a_k)=0, \forall j\in\mathcal{I}(a_k), a_k\in V_{\rm ext}\\ \text{and}\ \ u^j(a_k)=u^\ell(a_k),\ \forall j,\ell\in\mathcal{I}(a_k), a_k\in V_{\rm int}\big\}.\end{aligned} \tag{2.1}$$

We define the energy space $\mathcal{H}$ associated with system (1.7)-(1.13), by

$$\mathcal{H} = \mathbb{H}_0^1\times\mathbb{L}^2\times\mathbb{H}_0^1\times\mathbb{L}^2\times\mathbb{V} \tag{2.2}$$

equipped with the following inner product:

$$\begin{aligned}(U,\tilde{U})_{\mathcal{H}} = \sum_{j=1}^{N}\int_0^{\ell_j}(u_x^j\overline{\tilde{u}_x^j} + v^j\overline{\tilde{v}^j} + \rho_j y_x^j\overline{\tilde{y}_x^j} + z^j\overline{\tilde{z}^j})\,dx\\ + \sum_{j\in\mathcal{I}_{\rm te}}\int_0^{\ell_j}\theta^j\overline{\tilde{\theta}^j}\,dx,\end{aligned} \tag{2.3}$$

for all $U=(u,v,y,z,\theta)$, $\tilde{U}=(\tilde{u},\tilde{v},\tilde{y},\tilde{z},\tilde{\theta})\in\mathcal{H}$. Next, we define the unbounded linear operator $\mathcal{A}$ associated to system (1.7)-(1.13) by

$$\mathcal{A}\begin{pmatrix}u\\v\\y\\z\\\theta\end{pmatrix} = \left(\begin{pmatrix}v^j\\ u_{xx}^j-\alpha_j\theta_x^j+\beta_j z^j\\ z^j\\ \rho_j y_{xx}^j-\beta_j v^j\\ \kappa_j\theta_{xx}^j-\alpha_j v_x^j\end{pmatrix}\right)_{j\in\mathcal{I}(\mathcal{N})} \tag{2.4}$$

whose domain $D(\mathcal{A})$ is given by

$$D(\mathcal{A}) = \big\{(u,v,y,z,\theta)\in\mathcal{H}\cap[\mathbb{H}^2\times\mathbb{H}_0^1\times\mathbb{H}^2\times\mathbb{H}_0^1\times\mathbb{V}^2] \text{ satisfying (2.5) below}\big\},$$

$$\begin{cases}\theta^j(a_k)=0,\ \ j\in\mathcal{I}_{\rm te}(a_k),\quad a_k\in V'_{\rm ext},\\ \theta^j(a_k)=\theta^\ell(a_k),\quad j,\ell\in\mathcal{I}_{\rm te}(a_k),\ \ a_k\in V'_{\rm int},\\ \displaystyle\sum_{j\in\mathcal{I}_{\rm te}(a_k)} d_{kj}\kappa_j\theta_x^j(a_k)=0,\ \ a_k\in V'_{\rm int},\\ \displaystyle\sum_{j\in\mathcal{I}(a_k)} d_{kj}(u_x^j(a_k)-\alpha_j\theta^j(a_k))=0,\quad a_k\in V_{\rm int},\\ \displaystyle\sum_{j\in\mathcal{I}(a_k)} d_{kj}\rho_j y_x^j(a_k)=0,\ \ a_k\in V_{\rm int}.\end{cases} \tag{2.5}$$

If (u, y, θ) is a regular solution of (1.7)-(1.9), then by setting $U = (u, u_t, y, y_t, \theta)$, we can rewrite this system as the following evolution equation:

$$U_t = \mathcal{A}U, \quad U(0) = U_0, \tag{2.6}$$

where $U_0 = (u_0, u_1, y_0, y_1, \theta_0)$.

The energy associated with system (1.7)-(1.13) is given by

$$E(t) = \frac{1}{2}\sum_{j=1}^{N}\int_0^{\ell_j}(|u_x^j|^2 + |u_t^j|^2 + \rho_j|y_x^j|^2 + |y_t^j|^2)\,dx + \frac{1}{2}\sum_{j\in\mathcal{I}_{\text{te}}}\int_0^{\ell_j}|\theta_j|^2\,dx \tag{2.7}$$

and we will see that

$$E'(t) = -\sum_{j\in\mathcal{I}_{\text{te}}}\int_0^{\ell_j}\kappa_j|\theta_x^j|^2\,dx, \tag{2.8}$$

for regular solutions. Hence, the system is dissipative in the sense that its energy is non-increasing.

Theorem 2.1. *The unbounded linear operator $\mathcal{A}$ associated with system (1.7)-(1.13) generates a C_0-semigroup of contractions on $\mathcal{H}$.*

Proof. Using Lumer-Phillips Theorem (see [16]), it is sufficient to prove that $\mathcal{A}$ is a maximal dissipative operator so that $\mathcal{A}$ generates a C_0-semigroup of contractions on $\mathcal{H}$. First, let $U = (u, v, y, z, \theta) \in D(\mathcal{A})$. We have,

$$\begin{aligned}\text{Re}(\mathcal{A}U, U)_{\mathcal{H}} = \text{Re}\Big[&\sum_{j=1}^{N}\int_0^{\ell_j} v_x^j\overline{u_x^j}\,dx + \sum_{j=1}^{N}\int_0^{\ell_j}(u_{xx}^j - \alpha_j\theta_x^j + \beta_j z^j)\overline{v^j}\,dx\\ &+\sum_{j=1}^{N}\int_0^{\ell_j}\rho_j z_x^j\overline{y_x^j}\,dx + \sum_{j=1}^{N}\int_0^{\ell_j}(\rho_j y_{xx}^j - \beta_j v^j)\overline{z^j}\,dx + \sum_{j\in\mathcal{I}_{\text{te}}}\int_0^{\ell_j}(\kappa_j\theta_{xx}^j - \alpha_j v_x^j)\overline{\theta^j}\,dx\Big].\end{aligned} \tag{2.9}$$

Using Green's formula, boundary and transmission conditions (1.10)-(1.13), we get

$$\text{Re}(\mathcal{A}U, U)_{\mathcal{H}} = -\sum_{j\in\mathcal{I}_{\text{te}}}\int_0^{\ell_j}\kappa_j|\theta_x^j|^2\,dx \le 0. \tag{2.10}$$

Thus, the operator $\mathcal{A}$ is dissipative. Now, in order to prove that $\mathcal{A}$ is maximal it is sufficient to show that $R(I - \mathcal{A}) = \mathcal{H}$. So, for $F = (f, \tilde{f}, g, \tilde{g}, h) \in \mathcal{H}$, we look for $U \in D(\mathcal{A})$ such that

$$(I - \mathcal{A})U = F. \tag{2.11}$$

Equivalently, for all $j \in \mathcal{I}(\mathcal{N})$,

$$u^j - v^j = f^j, \tag{2.12}$$

$$v^j - u_{xx}^j + \alpha_j\theta_x^j - \beta_j z^j = \tilde{f}^j, \tag{2.13}$$

$$y^j - z^j = g^j, \tag{2.14}$$

$$z^j - \rho_j y_{xx}^j + \beta_j v^j = \tilde{g}^j, \tag{2.15}$$

$$\theta^j - \kappa_j\theta_{xx}^j + \alpha_j v_x^j = h^j. \tag{2.16}$$

Assume that $U \in D(\mathcal{A})$ exists, then by using equation (2.12) and (2.14) we obtain for all $j \in \mathcal{I}(\mathcal{N})$,

$$v^j = u^j - f^j, \tag{2.17}$$

$$z^j = y^j - g^j. \tag{2.18}$$

Inserting (2.17)-(2.18) in equations (2.13), (2.15) and (2.16), we get the following system for all $j \in \mathcal{I}(\mathcal{N})$,

$$u^j - u^j_{xx} + \alpha_j \theta^j_x - \beta_j y^j = F^j_1, \tag{2.19}$$

$$y^j - \rho_j y^j_{xx} + \beta_j u^j = F^j_2, \tag{2.20}$$

$$\theta^j - \kappa_j \theta^j_{xx} + \alpha_j u^j_x = F^j_3 \tag{2.21}$$

where, $F^j_1 = \tilde{f}^j + f^j - \beta_j g^j$, $\quad F^j_2 = \tilde{g}^j + g^j + \beta_j f^j$, $\quad F^j_3 = h^j + \alpha_j f^j_x$.

Set

$$\begin{aligned}\mathcal{X} = \big\{(\varphi_1, \varphi_2, \varphi_3) \in \mathbb{H}^1_0 \times \mathbb{H}^1_0 \times \mathbb{V}^1 / \ \varphi^j_3(a_k) = 0,\ j \in \mathcal{I}_{\text{te}}(a_k),\ a_k \in V'_{\text{ext}} \\ \text{and } \varphi^j_3(a_k, t) = \varphi^\ell_3(a_k, t), \quad j, \ell \in \mathcal{I}_{\text{te}}(a_k),\ \ a_k \in V'_{\text{int}}\big\}.\end{aligned} \tag{2.22}$$

Let $(\varphi_1, \varphi_2, \varphi_3) \in \mathcal{X}$, multiply (2.19) by $\overline{\varphi^j_1}$, (2.20) by $\overline{\varphi^j_2}$ and (2.21) by $\overline{\varphi^j_3}$, then integrate over $(0, \ell_j)$ we get,

$$\begin{aligned}\int_0^{\ell_j} u^j \overline{\varphi^j_1}\, dx - \int_0^{\ell_j} u^j_{xx} \overline{\varphi^j_1}\, dx + \int_0^{\ell_j} \alpha_j \theta^j_x \overline{\varphi^j_1}\, dx - \int_0^{\ell_j} \beta_j y^j \overline{\varphi^j_1}\, dx \\ = \int_0^{\ell_j} F^j_1 \overline{\varphi^j_1}\, dx,\end{aligned} \tag{2.23}$$

$$\int_0^{\ell_j} y^j \overline{\varphi^j_2}\, dx - \int_0^{\ell_j} \rho_j y^j_{xx} \overline{\varphi^j_2}\, dx + \int_0^{\ell_j} \beta_j u^j \overline{\varphi^j_2}\, dx = \int_0^{\ell_j} F^j_2 \overline{\varphi^j_2}\, dx, \tag{2.24}$$

and

$$\int_0^{\ell_j} \theta^j \overline{\varphi^j_3}\, dx - \int_0^{\ell_j} \kappa_j \theta^j_{xx} \overline{\varphi^j_3}\, dx + \int_0^{\ell_j} \alpha_j u^j_x \overline{\varphi^j_3}\, dx = \int_0^{\ell_j} F^j_3 \overline{\varphi^j_3}\, dx. \tag{2.25}$$

Applying Green's formula on the second and third term of (2.23) and taking the sum over $\mathcal{I}(\mathcal{N})$, we obtain using (1.13),

$$\begin{aligned}\sum_{j=1}^N \int_0^{\ell_j} u^j \overline{\varphi^j_1}\, dx + \sum_{j=1}^N \int_0^{\ell_j} u^j_x \overline{\varphi^j_{1,x}}\, dx - \sum_{j=1}^N \int_0^{\ell_j} \alpha_j \theta^j \overline{\varphi^j_{1,x}}\, dx \\ - \sum_{j=1}^N \int_0^{\ell_j} \beta_j y^j \overline{\varphi^j_1}\, dx = \sum_{j=1}^N \int_0^{\ell_j} F^j_1 \overline{\varphi^j_1}\, dx.\end{aligned} \tag{2.26}$$

Again applying Green's formula on the second term of (2.24) and taking the sum over $\mathcal{I}(\mathcal{N})$, using (1.12), we get

$$\begin{aligned}\sum_{j=1}^N \int_0^{\ell_j} y^j \overline{\varphi^j_2}\, dx + \sum_{j=1}^N \int_0^{\ell_j} \rho_j y^j_x \overline{\varphi^j_{2,x}}\, dx + \sum_{j=1}^N \int_0^{\ell_j} \beta_j u^j \overline{\varphi^j_2}\, dx \\ = \sum_{j=1}^N \int_0^{\ell_j} F^j_2 \overline{\varphi^j_2}\, dx.\end{aligned} \tag{2.27}$$

Similarly, applying Green's formula on the second term of (2.25) and taking the sum over $\mathcal{I}(\mathcal{N})$, condition (1.13) yields that,

$$\sum_{j=1}^{N}\int_0^{\ell_j}\theta^j\overline{\varphi_3^j}\,dx+\sum_{j=1}^{N}\int_0^{\ell_j}\kappa_j\theta_x^j\overline{\varphi_{3,x}^j}\,dx+\sum_{j=1}^{N}\int_0^{\ell_j}\alpha_j u_x^j\overline{\varphi_3^j}\,dx = \sum_{j=1}^{N}\int_0^{\ell_j}F_3^j\overline{\varphi_3^j}\,dx. \tag{2.28}$$

Adding equations (2.26), (2.27) and (2.28) we obtain

$$a((u,y,\theta),(\varphi_1,\varphi_2,\varphi_3))=L(\varphi_1,\varphi_2,\varphi_3),\quad \forall(\varphi_1,\varphi_2,\varphi_3)\in\mathcal{X}, \tag{2.29}$$

where,

$$\begin{aligned}a((u,y,\theta),(\varphi_1,\varphi_2,\varphi_3)) &= \sum_{j=1}^{N}\int_0^{\ell_j}u^j\overline{\varphi_1^j}\,dx+\sum_{j=1}^{N}\int_0^{\ell_j}u_x^j\overline{\varphi_{1,x}^j}\,dx-\sum_{j=1}^{N}\int_0^{\ell_j}\alpha_j\theta^j\overline{\varphi_{1,x}^j}\,dx\\ &-\sum_{j=1}^{N}\int_0^{\ell_j}\beta_j y^j\overline{\varphi_1^j}\,dx+\sum_{j=1}^{N}\int_0^{\ell_j}y^j\overline{\varphi_2^j}\,dx+\sum_{j=1}^{N}\int_0^{\ell_j}\rho_j y_x^j\overline{\varphi_{2,x}^j}\,dx\\ &+\sum_{j=1}^{N}\int_0^{\ell_j}\beta_j u^j\overline{\varphi_2^j}\,dx+\sum_{j=1}^{N}\int_0^{\ell_j}\theta^j\overline{\varphi_3^j}\,dx+\sum_{j=1}^{N}\int_0^{\ell_j}\kappa_j\theta_x^j\overline{\varphi_{3,x}^j}\,dx\\ &+\sum_{j=1}^{N}\int_0^{\ell_j}\alpha_j u_x^j\overline{\varphi_3^j}\,dx\end{aligned}$$

and

$$L(\varphi_1,\varphi_2,\varphi_3)=\sum_{j=1}^{N}\int_0^{\ell_j}F_1^j\overline{\varphi_1^j}\,dx+\sum_{j=1}^{N}\int_0^{\ell_j}F_2^j\overline{\varphi_2^j}\,dx+\sum_{j=1}^{N}\int_0^{\ell_j}F_3^j\overline{\varphi_3^j}\,dx.$$

As a is a continuous, coercive form on $\mathcal{X}\times\mathcal{X}$ and L is a continuous form on $\mathcal{X}$, then using Lax-Milgram Theorem there exists a unique solution $(u,y,\theta)\in\mathcal{X}$ of (2.29). Now, take in (2.29) the test function $(\varphi_1,0,0)$ such that $\varphi_1^j\in C_c^\infty(0,\ell_j)$, for some fixed $j\in\mathcal{I}(\mathcal{N})$ and $\varphi_1^k=0$ for all $k\neq j$, we obtain

$$\begin{aligned}&\int_0^{\ell_j}u^j\,\overline{\varphi_1^j}\,dx+\int_0^{\ell_j}u_x^j\overline{\varphi_{1,x}^j}\,dx-\int_0^{\ell_j}\alpha_j\theta^j\overline{\varphi_{1,x}^j}\,dx\\ &-\int_0^{\ell_j}\beta_j y^j\overline{\varphi_1^j}\,dx=\int_0^{\ell_j}F_1^j\overline{\varphi_1^j}\,dx,\quad\forall\varphi_1^j\in C_c^\infty(0,\ell_j)\text{ for a fixed }j.\end{aligned} \tag{2.30}$$

Applying Green's formula on the second and third term of (2.30) we get

$$\begin{aligned}&\int_0^{\ell_j}u^j\,\overline{\varphi_1^j}\,dx-\int_0^{\ell_j}u_{xx}^j\overline{\varphi_1^j}\,dx+\int_0^{\ell_j}\alpha_j\theta_x^j\overline{\varphi_1^j}\,dx-\int_0^{\ell_j}\beta_j y^j\overline{\varphi_1^j}\,dx\\ &=\int_0^{\ell_j}F_1^j\overline{\varphi_1^j}\,dx,\quad\forall\varphi_1^j\in C_c^\infty(0,\ell_j)\text{ for a fixed }j.\end{aligned} \tag{2.31}$$

This implies that

$$u^j-u_{xx}^j+\alpha_j\theta_x^j-\beta_j y^j=F_1^j,\ \text{ in }\mathcal{D}'(0,\ell_j)$$

where, $\mathcal{D}'(0,\ell_j)$ is the associated space of distributions.
As $u^j+\alpha_j\theta_x^j-\beta_j y^j-F_1^j\in L^2(0,\ell_j)$, we deduce that $u^j\in H^2(0,\ell_j)$. Similarly, we

can prove that

$$\begin{aligned} y^j - \rho_j y^j_{xx} + \beta_j u^j &= F^j_2, \\ \theta^j - \kappa_j \theta^j_{xx} + \alpha_j u^j_x &= F^j_3 \end{aligned} \tag{2.32}$$

and $y^j, \theta^j \in H^2(0, \ell_j)$. Now, it remains to prove the transmission conditions in (1.12)-(1.13). For that aim, fix $a_k \in V_{\text{int}}$. Let,

$$\varphi^j_1 = \begin{cases} \frac{x}{\ell_j}, & \text{if } j \in \mathcal{I}(a_k) \text{ and } \pi_j(\ell_j) = a_k, \\ \frac{\ell_j - x}{\ell_j}, & \text{if } j \in \mathcal{I}(a_k) \text{ and } \pi_j(0) = a_k, \\ 0, & \text{if } j \notin \mathcal{I}(a_k). \end{cases} \tag{2.33}$$

Then, take in (2.29), a test function $(\varphi_1, 0, 0) \in \mathcal{X}$, apply Green's formula and take into account (2.19)-(2.21), to get

$$\sum_{j \in \mathcal{I}(a_k)} d_{kj} u^j_x(a_k) - \sum_{j \in \mathcal{I}(a_k)} d_{kj} \alpha_j \theta^j(a_k) = 0. \tag{2.34}$$

Similarly, by taking in (2.29) the test function $(0, \varphi_1, 0) \in \mathcal{X}$ then, using Green's formula and taking into account (2.19)-(2.21) we obtain (1.12). Finally, we fix $a_k \in V'_{\text{int}}$, take $(0, 0, \varphi_1)$ in (2.29), apply Green's formula and take into account (2.19)-(2.21), we get (1.13). By defining v^j by (2.17) and z^j by (2.18), for all $j \in \mathcal{I}(\mathcal{N})$, we deduce that $(u, v, y, z, \theta) \in D(\mathcal{A})$ a solution of (2.12)-(2.16) exists and the desired goal is attained. □

As $\mathcal{A}$ generates a C_0-semigroup of contractions $(e^{t\mathcal{A}})_{t \geq 0}$ (see [16]), we have the following result:

Theorem 2.2. *(Existence and uniqueness of the solution)*
(1) If $U_0 = (u_0, u_1, y_0, y_1, \theta_0) \in D(\mathcal{A})$, then problem (2.6) admits a strong unique solution $U = (u, v, y, z, \theta)$ satisfying

$$U \in C^1(\mathbb{R}_+, \mathcal{H}) \cap C^0(\mathbb{R}_+, D(\mathcal{A})).$$

(2) If $U_0 = (u_0, u_1, y_0, y_1, \theta_0) \in \mathcal{H}$, then problem (2.6) admits a unique weak solution $U = (u, v, y, z, \theta)$ satisfying

$$U \in C^0(\mathbb{R}_+, \mathcal{H}).$$

3. **Strong stability.** In this section, we will give sufficient conditions that guarantee the strong stability of the system (1.7)-(1.13) in the sense that the energy $E(t)$, of the associated system decreases to zero as t tends to infinity. To show the strong stability of the C_0-semigroup of contractions $(e^{t\mathcal{A}})_{t \geq 0}$ we will rely on the following result obtained by Arendt-Batty [2].

Theorem 3.1. *(Arendt-Batty [2]). Let $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$ generates a C_0-semigroup of contractions on the Hilbert space $\mathcal{H}$. If*
1) $\mathcal{A}$ has no pure imaginary eigenvalues,
2) $\sigma(\mathcal{A}) \cap i\mathbb{R}$ is countable where, $\sigma(\mathcal{A})$ is the spectrum of $\mathcal{A}$.

Then, the C_0-semigroup $(e^{t\mathcal{A}})_{t \geq 0}$ is strongly stable.

Now, we are in position to state the main result of this subsection.

Theorem 3.2. *Consider the system (1.7)-(1.13) on $\mathcal{N}$. Assume additionally that one of the following conditions holds,*

1) Each maximal subgraph of thermo-elastic edges has an exterior vertex that belongs to V_{ext} .

2) There exists a maximal subgraph of thermo-elastic edges with no exterior vertices that belong to V_{ext} and $\beta_j = \beta$, for all $j \in \mathcal{I}(\mathcal{N})$.

Then

$$i\mathbb{R} \subset \rho(\mathcal{A}), \tag{S1}$$

and therefore $\lim_{t\to\infty} E(t) \to 0$.

Proof. Using Sobelev embedding Theorem, we deduce that $(I-\mathcal{A})^{-1}$ is a compact operator. Then, the spectrum $\sigma(\mathcal{A})$ of $\mathcal{A}$ is reduced to its discrete spectrum $\sigma_p(\mathcal{A})$. Hence, using Arendt-Batty Theorem [2], it is sufficient to prove that $\sigma_p(\mathcal{A})\cap i\mathbb{R} = \emptyset$, since it implies that (S1) holds. Let $\lambda \in \mathbb{R}$ and $U = (u, v, y, z, \theta) \in D(\mathcal{A})$ be such that

$$\mathcal{A}U = i\lambda U,$$

equivalently, for all $j \in \mathcal{I}(\mathcal{N})$ we have,

$$v^j = i\lambda u^j, \tag{3.1}$$
$$u^j_{xx} - \alpha_j\theta^j_x + \beta_j z^j = i\lambda v^j, \tag{3.2}$$
$$z^j = i\lambda y^j, \tag{3.3}$$
$$\rho_j y^j_{xx} - \beta_j v^j = i\lambda z^j, \tag{3.4}$$
$$\kappa_j\theta^j_{xx} - \alpha_j v^j_x = i\lambda\theta^j. \tag{3.5}$$

Eliminating v^j (resp. z^j) using (3.1) (resp. (3.3)) and inserting them in (3.2), (3.4) and (3.5) we get the following system for all $j \in \mathcal{I}(\mathcal{N})$,

$$\lambda^2 u^j + u^j_{xx} - \alpha_j\theta^j_x + i\lambda\beta_j y^j = 0, \tag{3.6}$$
$$\lambda^2 y^j + \rho_j y^j_{xx} - i\lambda\beta_j u^j = 0, \tag{3.7}$$
$$\kappa_j\theta^j_{xx} - i\lambda\alpha_j u^j_x - i\lambda\theta^j = 0. \tag{3.8}$$

Since we have

$$\sum_{j\in\mathcal{I}_{\rm te}} \kappa_j \int_0^{\ell_j} |\theta^j_x|^2\, dx = \mathrm{Re}(\mathcal{A}U, U)_{\mathcal{H}} = \mathrm{Re}(i\lambda U, U)_{\mathcal{H}} = 0,$$

we deduce that

$$\theta^j_x = 0,\ \forall j \in \mathcal{I}_{\rm te}. \tag{3.9}$$

Thus, θ^j is constant for all $j \in \mathcal{I}_{\rm te}$. But, using the fact that every maximal subgraph of thermo-elastic edges is not a circuit and using (1.10) and (1.11), we deduce that

$$\theta^j = 0,\ \forall j \in \mathcal{I}_{\rm te}. \tag{3.10}$$

Suppose that $\lambda = 0$. Then, (3.6), (3.7) and (3.9) implies that

$$\begin{cases} u^j_{xx} = 0, & \forall j \in \mathcal{I}(\mathcal{N}), \\ \rho_j y^j_{xx} = 0, & \forall j \in \mathcal{I}(\mathcal{N}). \end{cases} \tag{3.11}$$

Multiplying the first equation and second equation of (3.11) by $\overline{u^j}$ and $\overline{y^j}$, respectively. Then, integrating over $(0,\ell_j)$, summing over $j\in\mathcal{I}(\mathcal{N})$ and applying Green's formula, we get

$$\begin{cases}\displaystyle\sum_{j=1}^{N}\int_0^{\ell_j}|u_x^j|^2\,dx-\sum_{j=1}^{N}u_x^j\overline{u^j}\Big|_0^{\ell_j}=0,\\[2ex]\displaystyle\sum_{j=1}^{N}\int_0^{\ell_j}\rho_j|y_x^j|^2\,dx-\sum_{j=1}^{N}\rho_j y_x^j\overline{y^j}\Big|_0^{\ell_j}=0.\end{cases}\tag{3.12}$$

But using (3.10) and the boundary conditions (1.10)-(1.13), the boundary terms are zero, hence (3.12) becomes

$$\begin{cases}\displaystyle\sum_{j=1}^{N}\int_0^{\ell_j}|u_x^j|^2\,dx=0,\\[2ex]\displaystyle\sum_{j=1}^{N}\int_0^{\ell_j}\rho_j|y_x^j|^2\,dx=0.\end{cases}\tag{3.13}$$

By the fact that $\rho_j>0$, for all $j\in\mathcal{I}(\mathcal{N})$, we obtain that $u_x^j=y_x^j=0$, for all $j\in\mathcal{I}(\mathcal{N})$. Again, by (1.10), (1.11) and using the fact that $V_{\text{ext}}\neq\emptyset$, we deduce that $u^j=y^j=0$, for all $j\in\mathcal{I}(\mathcal{N})$. Consequently, using (3.1) and (3.3), we conclude that $v^j=z^j=0$, for all $j\in\mathcal{I}(\mathcal{N})$ and therefore, $U=0$.

Now, suppose that $\lambda\neq 0$. We will distinguish between two cases.

Case 1. Assume that each maximal subgraph of thermo-elastic edges has an exterior vertex that belongs to V_{ext}. Using (3.10) and (3.8), we have

$$u_x^j=0,\ \forall j\in\mathcal{I}_{\text{te}}.\tag{3.14}$$

This means that u^j is constant for all $j\in\mathcal{I}_{\text{te}}$. But, using (1.10), (1.11) and the fact that every maximal subgraph of thermo-elastic edges has an exterior vertex that belongs to V_{ext}, we deduce that $u^j=0$, for all $j\in\mathcal{I}_{\text{te}}$. Thus, by (3.1), we have $v^j=0$ and by (3.6), (3.14) and (3.9), we obtain that $y^j=0$, for all $j\in\mathcal{I}_{\text{te}}$. Consequently, by (3.3), we get $z^j=0$, for all $j\in\mathcal{I}_{\text{te}}$. Hence, $u^j=v^j=y^j=z^j=\theta^j=0$ on both ends of e_j, for all $j\in\mathcal{I}_{\text{te}}$.
Now, let e_j be an elastic edge attached only to thermo-elastic edge. As e_j is identified by $[0,\ell_j]$, assume that ℓ_j is the extremity in common with the thermo-elastic edge. Then, using (1.11), (3.10), (1.12) and (1.13), we have the following system

$$\begin{cases}\lambda^2u^j+u_{xx}^j+i\lambda\beta_j y^j &=0,\\ \lambda^2y^j+\rho_j y_{xx}^j-i\lambda\beta_j u^j &=0,\\ u^j(\ell_j)=y^j(\ell_j)=0,\\ u_x^j(\ell_j)=y_x^j(\ell_j)=0.\end{cases}\tag{3.15}$$

Let

$$\tilde{u}^j=\begin{cases}u^j, & \text{on } (0,\ell_j),\\ 0, & \text{on } (\ell_j,\ell_j+1),\end{cases}$$

and

$$\tilde{y}^j = \begin{cases} y^j, & \text{on } (0,\ell_j), \\ 0, & \text{on } (\ell_j,\ell_j+1). \end{cases}$$

Then, using the boundary conditions of (3.15), we deduce that $(\tilde{u}^j,\tilde{y}^j)$ belongs to $H^2(0,\ell_j+1)\times H^2(0,\ell_j+1)$ and satisfies the first two equations of (3.15). Consequently, using Theorem 2.5 of [9], we deduce that $\tilde{u}^j = \tilde{y}^j = 0$ on $(0,\ell_j+1)$ and hence, $u^j = y^j = 0$ on $(0,\ell_j)$. Then, $v^j = z^j = 0$ by equation (3.1) and (3.3) respectively. We repeat this technique to every elastic edge connected only to thermo-elastic edges and we proceed by iteration the same method on each maximal subgraph of purely elastic edges (from the leaves to the root), so that $u^j = v^j = y^j = z^j = 0$, for all $j \in \mathcal{I}(\mathcal{N})$.

Case 2. Assume that there exists a maximal subgraph of thermo-elastic edges with no exterior vertices that belong to V_{ext} and $\beta_j = \beta$, for all $j \in \mathcal{I}(\mathcal{N})$. First, notice that (3.14) holds and thus,

$$u^j_{xx} = 0,\ \forall j \in \mathcal{I}_{\text{te}}. \tag{3.16}$$

Then, using (3.16), (3.9) and the fact that $\lambda \neq 0, \beta_j = \beta$, equation (3.6) becomes

$$\lambda u^j + i\beta y^j = 0,\ \forall j \in \mathcal{I}_{\text{te}}. \tag{3.17}$$

Differentiating (3.17) twice with respect to x and using (3.16), we deduce that

$$y^j_{xx} = 0,\ \forall j \in \mathcal{I}_{\text{te}}. \tag{3.18}$$

Then, using (3.18) and as $\lambda \neq 0, \beta_j = \beta$, (3.7) becomes

$$\lambda y^j - i\beta u^j = 0,\ \forall j \in \mathcal{I}_{\text{te}}. \tag{3.19}$$

Eliminating u^j from (3.17) and replacing it in (3.19) we obtain

$$(\lambda^2 - \beta^2) y^j = 0,\ \forall j \in \mathcal{I}_{\text{te}}. \tag{3.20}$$

Then, for $\lambda \neq \pm\beta$ we deduce that $y^j = 0$, for all $j \in \mathcal{I}_{\text{te}}$ and thus by equation (3.19) we get $u^j = 0$, for all $j \in \mathcal{I}_{\text{te}}$. Again, we proceed using unique continuation Theorem from [9] and iteration technique used in **Case 1** to conclude that $u^j = v^j = y^j = z^j = 0$, for all $j \in \mathcal{I}(\mathcal{N})$.

On the other hand, if $\lambda = \pm\beta$. Without loss of generality, assume that $\lambda = \beta$. First, using (3.19), we have

$$y^\ell = iu^\ell, \forall \ell \in \mathcal{I}_{\text{te}} \tag{3.21}$$

and thus using (3.14), (3.21) implies that

$$y^\ell_x = iu^\ell_x = 0, \forall \ell \in \mathcal{I}_{\text{te}}. \tag{3.22}$$

Our aim is to prove that

$$\begin{cases} u^j_{xx} = 0,\ \forall j \in \mathcal{I}(\mathcal{N}), \\ y^j_{xx} = 0,\ \forall j \in \mathcal{I}(\mathcal{N}). \end{cases} \tag{3.23}$$

This would end the proof as in the case $\lambda = 0$. As (3.16) and (3.18) hold, it is enough to prove that (3.23) holds for each elastic edge. Let e_j be an elastic edge attached to a thermo-elastic edge at the vertex a_k, where a_k is a leaf of a maximal subgraph of elastic edges. As $\lambda = \beta$, then (3.6) and (3.7) lead to

$$\beta^2 u^j + u^j_{xx} + i\beta^2 y^j = 0, \tag{3.24}$$

$$\beta^2 y^j + \rho_j y^j_{xx} - i\beta^2 u^j = 0. \tag{3.25}$$

By eliminating u^j from (3.24) and inserting it in (3.25), we obtain the following equation

$$y^j_{xxxx} + \beta^2 \frac{(\rho_j + 1)}{\rho_j} y^j_{xx} = 0. \tag{3.26}$$

Moreover, using (3.25), (1.11) and (3.21), we have

$$y^j_{xx}(a_k) = 0 \tag{3.27}$$

and using (3.25), (1.12), (1.13), (3.10) and (3.22), we get

$$y^j_{xxx}(a_k) = 0. \tag{3.28}$$

Consequently, by setting $Z^j = y^j_{xx}$ and using (3.26)-(3.28), we have the following system

$$\begin{cases} Z^j_{xx} + \beta^2 \dfrac{(\rho_j + 1)}{\rho_j} Z^j = 0, \\ Z^j(a_k) = 0, \\ Z^j_x(a_k) = 0. \end{cases} \tag{3.29}$$

Therefore, $Z^j = 0$ and then $y^j_{xx} = 0$. This means that y^j_x is constant. But using (1.12) and (3.22), we deduce that $y^j_x(a_k) = 0$. Hence, $y^j_x = 0$. Therefore, using (3.25), we obtain that $y^j = iu^j$ and then $y^j_x = iu^j_x = 0$. Again, by iteration on each maximal subgraph of purely elastic edges (from the leaves to the root), we repeat the same procedure to prove that (3.23) holds. Whenever (3.23) is attained, we can proceed as the case $\lambda = 0$ which finishes the proof. The same procedure can be used in the case $\lambda = -\beta$. □

Let us finish this section by introducing some notations that will be used in the next section.

Let $\mathcal{I}'_{\text{ext}}$ denotes the set of indices of edges adjacent to a vertex in V'_{ext} and $\mathcal{G}'_{\text{int}}$ denotes the set of indices of edges adjacent to two vertices in V'_{int}.

4. Energy decay rates.

Take an arbitrary network $\mathcal{N}$ for which the System (1.7)-(1.13) is stable. In this section, we will prove that under the condition that the two coupled wave equations propagate with the same speed on each thermo-elastic edge, i.e., $\rho_j = 1$, for all $j \in \mathcal{I}_{\text{te}}$, and using a frequency domain approach combined with a multiplier method, the energy of the system decays exponentially to zero. Otherwise, if there exist $j \in \mathcal{I}_{\text{te}} \cap \mathcal{I}'_{\text{ext}}$ such that $\rho_j \neq 1$, we prove a polynomial decay rate of type $t^{-1/3}$, see ([3, 17]). The main results are presented in Theorem 4.1 and Theorem 4.11.

4.1. Exponential stability.

Theorem 4.1. *Let $\mathcal{N}$ be an arbitrary network for which the operator $\mathcal{A}$ associated with System (1.7)-(1.13) satisfies (S1). If $\rho_j = 1$, for all $j \in \mathcal{I}_{te}$, then the energy of the system decays exponentially in $\mathcal{H}$. In other words, there exist two positive constants M and ϵ such that*

$$\|e^{t\mathcal{A}} x_0\|_{\mathcal{H}} \leq M e^{-\epsilon t} \|x_0\|_{\mathcal{H}}, \quad \forall\ t > 0,\ \forall\ x_0 \in \mathcal{H}.$$

Proof. Following Huang [10] and Prüss [17], the C_0-semigroup of contractions $(e^{t\mathcal{A}})_{t\geq 0}$ on $\mathcal{H}$ is exponentially stable if and only if (S1) and

$$\limsup_{|\lambda|\to\infty} \|(i\lambda - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} < \infty \qquad (S2)$$

hold. As we have assumed that (S1) is satisfied, it remains to prove that condition (S2) is satisfied. To this aim, we proceed by a contradiction argument. Suppose that (S2) does not hold, then there exist a sequence of real numbers $\lambda_n \in \mathbb{R}$ and a sequence of vectors $U_n = (u_n, v_n, y_n, z_n, \theta_n) \in D(\mathcal{A})$ such that

$$|\lambda_n| \longrightarrow +\infty, \qquad \|U_n\|_{\mathcal{H}} = \|(u_n, v_n, y_n, z_n, \theta_n)\|_{\mathcal{H}} = 1, \tag{4.1}$$

and

$$(i\lambda_n - \mathcal{A})U_n = (f_n, \tilde{f}_n, g_n, \tilde{g}_n, h_n) \longrightarrow 0 \quad \text{in } \mathcal{H}, \tag{4.2}$$

are satisfied.
In what follows, we drop the index n for simplicity.

Now by detailing (4.2), we get for all $j \in \mathcal{I}(\mathcal{N})$

$$i\lambda u^j - v^j = f^j \to 0 \text{ in } H^1(0, \ell_j), \tag{4.3}$$
$$i\lambda v^j - u^j_{xx} + \alpha_j \theta^j_x - \beta_j z^j = \tilde{f}^j \to 0 \text{ in } L^2(0, \ell_j), \tag{4.4}$$
$$i\lambda y^j - z^j = g^j \to 0 \text{ in } H^1(0, \ell_j), \tag{4.5}$$
$$i\lambda z^j - \rho_j y^j_{xx} + \beta_j v^j = \tilde{g}^j \to 0 \text{ in } L^2(0, \ell_j), \tag{4.6}$$
$$i\lambda \theta^j - \kappa_j \theta^j_{xx} + \alpha_j v^j_x = h^j \to 0 \text{ in } L^2(0, \ell_j). \tag{4.7}$$

Then, by eliminating v^j and z^j from equations (4.3) and (4.5) respectively, (4.3)-(4.7) imply

$$\lambda^2 u^j + u^j_{xx} - \alpha_j \theta^j_x + i\lambda\beta_j y^j = \beta_j g^j - \tilde{f}^j - i\lambda f^j, \tag{4.8}$$
$$\lambda^2 y^j + \rho_j y^j_{xx} - i\lambda\beta_j u^j = -\beta_j f^j - \tilde{g}^j - i\lambda g^j, \tag{4.9}$$
$$i\lambda\theta^j - \kappa_j \theta^j_{xx} + i\lambda\alpha_j u^j_x = h^j + \alpha_j f^j_x, \tag{4.10}$$

where, $\rho_j = 1$, for all $j \in \mathcal{I}_{\text{te}}$. Now, we will proceed by dividing the proof into different Lemmas.

Lemma 4.2. *Under all above assumptions, we have*

$$\int_0^{\ell_j} |\theta^j_x|^2 \, dx = o(1), \ \forall j \in \mathcal{I}_{te}. \tag{4.11}$$

Proof. Taking the inner product in $\mathcal{H}$ of equation (4.2) with the uniformly bounded sequence $U = (u, v, y, z, \theta)$, we get

$$\sum_{j \in \mathcal{I}_{\text{te}}} \kappa_j \int_0^{\ell_j} |\theta^j_x|^2 \, dx = -\text{Re}((i\lambda I - \mathcal{A})U, U)_{\mathcal{H}} = o(1).$$

As $\kappa_j > 0, \forall j \in \mathcal{I}_{\text{te}}$, it follows that

$$||\theta^j_x||^2_{L^2(0,\ell_j)} = o(1), \ \forall j \in \mathcal{I}_{\text{te}}.$$

□

Using (4.3), (4.5) and (4.1), we have for all $j \in \mathcal{I}(\mathcal{N})$

$$||\lambda u^j||_{L^2(0,\ell_j)} = O(1), \ ||u^j_x||_{L^2(0,\ell_j)} = O(1), \tag{4.12}$$
$$||\lambda y^j||_{L^2(0,\ell_j)} = O(1), \ ||y^j_x||_{L^2(0,\ell_j)} = O(1). \tag{4.13}$$

Also, using (4.8), (4.9) and (4.10), we have for all $j \in \mathcal{I}(\mathcal{N})$

$$\left|\left|\frac{u^j_{xx}}{\lambda}\right|\right|_{L^2(0,\ell_j)} = O(1), \ \left|\left|\frac{y^j_{xx}}{\lambda}\right|\right|_{L^2(0,\ell_j)} = O(1), \ \left|\left|\frac{\theta^j_{xx}}{\lambda}\right|\right|_{L^2(0,\ell_j)} = O(1). \tag{4.14}$$

Lemma 4.3. *Under all above assumptions, we have for all $j \in \mathcal{I}_{te}$*

$$\lambda u^j(\ell_j) = O(1), \ u^j_x(\ell_j) = O(1), \tag{4.15}$$
$$\lambda u^j(0) = O(1), \ u^j_x(0) = O(1), \tag{4.16}$$
$$\lambda y^j(\ell_j) = O(1), \ y^j_x(\ell_j) = O(1), \tag{4.17}$$
$$\lambda y^j(0) = O(1), \ y^j_x(0) = O(1). \tag{4.18}$$

Proof. For all $j \in \mathcal{I}_{\text{te}}$, let Φ^j be a function in $W^{1,\infty}(0,\ell_j)$, then multiply (4.8) by $2\Phi^j\overline{u^j_x}$, integrate over $(0,\ell_j)$, take the real part and apply Green's formula, we get

$$\begin{aligned} &-\int_0^{\ell_j} \Phi^j_x|\lambda u^j|^2\,dx + \Phi^j|\lambda u^j|^2\Big|_{x=0}^{x=\ell_j} - \int_0^{\ell_j} \Phi^j_x|u^j_x|^2\,dx + \Phi^j|u^j_x|^2\Big|_{x=0}^{x=\ell_j} \\ &\qquad -2\text{Re}\,\alpha_j\int_0^{\ell_j}\theta^j_x\Phi^j\overline{u^j_x}\,dx + \text{Re}\,2i\int_0^{\ell_j}\beta_j\Phi^j\lambda y^j\overline{u^j_x}\,dx \\ &= 2\text{Re}\int_0^{\ell_j}\beta_j\Phi^j g^j\overline{u^j_x}\,dx - 2\text{Re}\int_0^{\ell_j}\tilde{f}^j\Phi^j\overline{u^j_x}\,dx - 2\text{Re}\int_0^{\ell_j} i\lambda f^j\Phi^j\overline{u^j_x}\,dx. \end{aligned} \tag{4.19}$$

Using (4.12), (4.14) and (4.11) we obtain,

$$\Phi^j|\lambda u^j|^2\Big|_{x=0}^{x=\ell_j} + \Phi^j|u^j_x|^2\Big|_{x=0}^{x=\ell_j} = O(1) - 2\text{Re}\int_0^{\ell_j} i\lambda f^j\Phi^j\overline{u^j_x}\,dx. \tag{4.20}$$

But,

$$\begin{aligned} -2\text{Re}\int_0^{\ell_j} i\lambda f^j\Phi^j\overline{u^j_x}\,dx = 2\text{Re}\int_0^{\ell_j} i\lambda f^j\Phi^j_x\overline{u^j}\,dx + 2\text{Re}\int_0^{\ell_j} i\lambda\Phi^j\overline{u^j}f^j_x\,dx \\ -2\text{Re}(i\lambda f^j\Phi^j\overline{u^j})\Big|_{x=0}^{x=\ell_j}. \end{aligned} \tag{4.21}$$

Using (4.12) and the fact that f^j converges to zero in $H^1(0,\ell_j)$, (4.21) becomes

$$-2\text{Re}\int_0^{\ell_j} i\lambda f^j\Phi^j\overline{u^j_x}\,dx = -2\text{Re}(i\lambda f^j\Phi^j\overline{u^j})\Big|_{x=0}^{x=\ell_j} + o(1). \tag{4.22}$$

Let $\Phi^j = x$, for all $j \in \mathcal{I}_{\text{te}}$. Using Young's inequality, we get

$$\left|2\text{Re}\int_0^{\ell_j} i\lambda f^j\Phi^j\overline{u^j_x}\,dx\right| \le \ell_j^2\epsilon_j|\lambda u^j(\ell_j)|^2 + \frac{|f^j(\ell_j)|^2}{\epsilon_j} + o(1). \tag{4.23}$$

Recalling the Gagliardo-Nirenberg inequality [12]: For all $\ell > 0$, there are two positive constants C_1 and C_2 depending on ℓ such that for any Ψ in $H^1(0,\ell) \subset C([0,\ell])$,

$$||\Psi||_{L^\infty(0,\ell)} \le C_1||\Psi_x||^{1/2}_{L^2(0,\ell)}||\Psi||^{1/2}_{L^2(0,\ell)} + C_2||\Psi||_{L^2(0,\ell)}. \tag{4.24}$$

Applying (4.24) to $\Psi = f^j$ and using the fact that f^j converge to zero in $H^1(0,\ell_j)$, we deduce that $f^j(\ell_j) = o(1)$. Thus, (4.23) yields that

$$\left|2\mathrm{Re}\int_0^{\ell_j} i\lambda f^j \Phi^j \overline{u_x^j}\, dx\right| \leq \ell_j^2\epsilon_j|\lambda u^j(\ell_j)|^2 + o(1). \tag{4.25}$$

By inserting (4.25) in (4.20) and as $\Phi^j = x$, we obtain

$$(\ell_j - \ell_j^2\epsilon_j)|\lambda u^j(\ell_j)|^2 + \ell_j|u_x^j(\ell_j)|^2 = O(1).$$

By taking $\epsilon_j = \frac{1}{2\ell_j}$, we deduce that (4.15) holds. Similarly, by taking $\Phi^j = (x-\ell_j)$, for all $j \in \mathcal{I}_{\mathrm{te}}$, we conclude that (4.16) holds. Also, multiplying (4.9) by $2x\overline{y_x^j}$ and $2(x-\ell_j)\overline{y_x^j}$ respectively, we deduce that (4.17) and (4.18) hold. □

Lemma 4.4. *Under all above assumptions, we have for all $j \in \mathcal{I}_{te}$,*

$$\int_0^{\ell_j} |u_x^j|^2\, dx = o(1), \quad \int_0^{\ell_j} |\theta^j|^2\, dx = o(1). \tag{4.26}$$

Proof. Let $j \in \mathcal{I}_{\mathrm{te}}$. Multiply (4.10) by $\frac{\overline{u_x^j}}{\lambda}$ and integrate over $(0,\ell_j)$ we get

$$\begin{aligned} i\int_0^{\ell_j} \theta^j\overline{u_x^j}\, dx - \int_0^{\ell_j} \kappa_j\theta_{xx}^j\frac{\overline{u_x^j}}{\lambda}\, dx + i\alpha_j\int_0^{\ell_j}|u_x^j|^2\, dx = \int_0^{\ell_j} h^j\frac{\overline{u_x^j}}{\lambda}\, dx \\ +\alpha_j\int_0^{\ell_j} f_x^j\frac{\overline{u_x^j}}{\lambda}\, dx. \end{aligned} \tag{4.27}$$

Applying Green's formula on the first and second term of (4.27) and using (4.12), we obtain

$$\begin{aligned} -i\int_0^{\ell_j}\theta_x^j\overline{u^j}\, dx + i\theta^j\overline{u^j}\Big|_{x=0}^{x=\ell_j} + \int_0^{\ell_j}\kappa_j\theta_x^j\frac{\overline{u_{xx}^j}}{\lambda}\, dx - \kappa_j\theta_x^j\frac{\overline{u_x^j}}{\lambda}\Big|_{x=0}^{x=\ell_j} \\ +i\,\alpha_j\int_0^{\ell_j}|u_x^j|^2\, dx = o(1). \end{aligned}$$

Using Cauchy-Schwarz inequality, (4.1), (4.11) and (4.14), we have

$$i\,\alpha_j\int_0^{\ell_j}|u_x^j|^2\, dx + i\theta^j\overline{u^j}\Big|_{x=0}^{x=\ell_j} - \kappa_j\theta_x^j\frac{\overline{u_x^j}}{\lambda}\Big|_{x=0}^{x=\ell_j} = o(1). \tag{4.28}$$

Then, by applying Gagliardo-Nirenberg inequality for $\Psi = \frac{\theta^j}{\sqrt{\lambda}}$, $\Psi = \frac{\theta_x^j}{\sqrt{\lambda}}$ and again using (4.11), (4.14), (4.15), (4.16), we deduce that (4.28) yields

$$i\,\alpha_j\int_0^{\ell_j}|u_x^j|^2\, dx = o(1).$$

Taking the imaginary part and using the fact that α_j is a positive constant for all $j \in \mathcal{I}_{\mathrm{te}}$, we deduce that

$$\int_0^{\ell_j}|u_x^j|^2\, dx = o(1). \tag{4.29}$$

Now, multiply (4.10) by $\frac{\overline{\theta^j+\alpha_j u_x^j}}{\lambda}$ and integrate over $(0,\ell_j)$, we get

$$\begin{aligned} i\int_0^{\ell_j}|\theta^j+\alpha_j u_x^j|^2\,dx-\int_0^{\ell_j}\frac{\kappa_j\theta_{xx}^j}{\lambda}(\overline{\theta^j+\alpha_j u_x^j})\,dx=\int_0^{\ell_j}\frac{h^j}{\lambda}(\overline{\theta^j+\alpha_j u_x^j})\,dx\\ +\int_0^{\ell_j}\frac{\alpha_j f_x^j}{\lambda}(\overline{\theta^j+\alpha_j u_x^j})\,dx. \end{aligned} \tag{4.30}$$

By applying Green's formula on the second term of (4.30), using Cauchy- Schwarz inequality on the integrals of the right hand side, (4.1) and the fact that h^j and f_x^j converge to zero in $L^2(0,\ell_j)$, we obtain

$$\begin{aligned} i\int_0^{\ell_j}|\theta^j+\alpha_j u_x^j|^2\,dx+\int_0^{\ell_j}\kappa_j\frac{|\theta_x^j|^2}{\lambda}\,dx+\int_0^{\ell_j}\alpha_j\kappa_j\theta_x^j\frac{\overline{u_{xx}^j}}{\lambda}\,dx\\ -\frac{\kappa_j\theta_x^j}{\sqrt{\lambda}}\left(\overline{\frac{\theta^j}{\sqrt{\lambda}}+\frac{\alpha_j u_x^j}{\sqrt{\lambda}}}\right)\Bigg|_{x=0}^{x=\ell_j}=o(1). \end{aligned} \tag{4.31}$$

Again, by using Gagliardo-Nirenberg inequality for $\Psi=\frac{\theta^j}{\sqrt{\lambda}}$, $\Psi=\frac{\theta_x^j}{\sqrt{\lambda}}$ and $\Psi=\frac{u_x^j}{\sqrt{\lambda}}$, we deduce that the boundary term in (4.31) converges to zero. Moreover, using (4.11) and (4.14), the second and third terms of (4.31) converge to zero. Consequently, using (4.29), we conclude (4.26). □

Lemma 4.5. *Under all above assumptions, we have for all $j\in\mathcal{I}_{te}$*

$$\int_0^{\ell_j}|\lambda u^j|^2\,dx=o(1)\ \text{and}\ \int_0^{\ell_j}|v^j|^2\,dx=o(1). \tag{4.32}$$

Proof. Let $j\in\mathcal{I}_{\text{te}}$. Multiply (4.8) by $\overline{u^j}$, integrate over $(0,\ell_j)$ and apply Green's formula, we get

$$\begin{aligned} \int_0^{\ell_j}|\lambda u^j|^2\,dx-\int_0^{\ell_j}|u_x^j|^2\,dx+u_x^j\overline{u^j}\Big|_{x=0}^{x=\ell_j}-\int_0^{\ell_j}\alpha_j\theta_x^j\overline{u^j}\,dx\\ +\int_0^{\ell_j}i\beta_j\lambda y^j\overline{u^j}\,dx=\int_0^{\ell_j}\beta_j g^j\overline{u^j}\,dx-\int_0^{\ell_j}\tilde{f}^j\overline{u^j}\,dx-i\lambda\int_0^{\ell_j}f^j\overline{u^j}\,dx. \end{aligned}$$

Using Cauchy-Schwarz inequality, (4.26), (4.15), (4.16), (4.11), (4.12) and (4.13), we deduce that

$$\int_0^{\ell_j}|\lambda u^j|^2\,dx=o(1).$$

Using (4.3) we conclude that (4.32) holds. □

As a conclusion, we have for every $j\in\mathcal{I}_{te}$

$$\begin{aligned} v^j&\to 0,\ \text{ in }\ L^2(0,\ell_j),\\ u^j&\to 0,\ \text{ in }\ H^1(0,\ell_j),\\ \theta^j&\to 0,\ \text{ in }\ H^1(0,\ell_j). \end{aligned}$$

Lemma 4.6. *Assume that all above assumptions hold. Then, for every thermo-elastic edge, we have*

$$\lambda u^j(\ell_j) = o(1),\ u_x^j(\ell_j) = o(1),\ Re\big(i\lambda f^j(\ell_j)u^j(\ell_j)\big) = o(1), \tag{4.33}$$

$$\lambda u^j(0) = o(1),\ u_x^j(0) = o(1),\ Re\big(i\lambda f^j(0)u^j(0)\big) = o(1), \tag{4.34}$$

$$\theta^j(0) = o(1), \theta^j(\ell_j) = o(1). \tag{4.35}$$

Proof. By the proof of Lemma 4.3, for all $j \in \mathcal{I}_{\text{te}}$, and any Φ^j in $W^{1,\infty}(0,\ell_j)$, (4.19) holds. Then, using (4.26), (4.32), (4.11) and (4.13) we obtain

$$\Phi^j|\lambda u^j|^2\Big|_{x=0}^{x=\ell_j} + \Phi^j|u_x^j|^2\Big|_{x=0}^{x=\ell_j} = o(1) - 2\text{Re}\int_0^{\ell_j} i\lambda f^j\Phi^j\overline{u_x^j}\,dx. \tag{4.36}$$

Then, by taking $\Phi^j = x$, for all $j \in \mathcal{I}_{\text{te}}$, and using (4.25), we deduce that (4.36) becomes,

$$(\ell_j - \ell_j^2\epsilon_j)|\lambda u^j(\ell_j)|^2 + \ell_j|u_x^j(\ell_j)|^2 = o(1).$$

Taking $\epsilon_j = \frac{1}{2\ell_j}$, we deduce that

$$\lambda u^j(\ell_j) = o(1) \text{ and } u_x^j(\ell_j) = o(1).$$

Consequently, by (4.36) and (4.22), we conclude that (4.33) holds. Similarly, by taking $\Phi^j = (x - \ell_j)$, for all $j \in \mathcal{I}_{\text{te}}$, we conclude that (4.34) holds. On the other hand, applying Gagliardo-Nirenberg inequality for $\Psi = \theta^j$, using (4.26) and (4.11) we deduce that (4.35) holds. □

Lemma 4.7. *Under all above assumptions, we have for all $j \in \mathcal{I}_{te}$,*

$$\int_0^{\ell_j} |y_x^j|^2\,dx = o(1). \tag{4.37}$$

Proof. Multiply (4.8) by $\frac{\overline{y_{xx}^j}}{\lambda}$, then integrate over $(0,\ell_j)$ we get

$$\begin{aligned}\int_0^{\ell_j} \lambda u^j\overline{y_{xx}^j}\,dx + \int_0^{\ell_j} u_{xx}^j\frac{\overline{y_{xx}^j}}{\lambda}\,dx - \int_0^{\ell_j}\alpha_j\theta_x^j\frac{\overline{y_{xx}^j}}{\lambda}\,dx + i\int_0^{\ell_j}\beta_j y^j\overline{y_{xx}^j}\,dx \\ = \int_0^{\ell_j}\beta_j g^j\frac{\overline{y_{xx}^j}}{\lambda}\,dx - \int_0^{\ell_j}\tilde{f}^j\frac{\overline{y_{xx}^j}}{\lambda}\,dx - i\int_0^{\ell_j} f^j\overline{y_{xx}^j}\,dx.\end{aligned}$$

Using Cauchy-Schwarz inequality, (4.11), (4.14) and the fact that g^j and $\tilde{f}^j$ converge to zero in $L^2(0,\ell_j)$, we obtain

$$\int_0^{\ell_j} \lambda u^j\overline{y_{xx}^j}\,dx + \int_0^{\ell_j} u_{xx}^j\frac{\overline{y_{xx}^j}}{\lambda}\,dx + i\int_0^{\ell_j}\beta_j y^j\overline{y_{xx}^j}\,dx = o(1) - i\int_0^{\ell_j} f^j\overline{y_{xx}^j}\,dx. \tag{4.38}$$

Applying Green's formula on the first and third term of the left hand side and on the integral of the right hand side, we get

$$\begin{aligned}-\int_0^{\ell_j}\lambda u_x^j\overline{y_x^j}\,dx + \lambda u^j\overline{y_x^j}\Big|_{x=0}^{x=\ell_j} + \int_0^{\ell_j} u_{xx}^j\frac{\overline{y_{xx}^j}}{\lambda}\,dx - i\int_0^{\ell_j}\beta_j|y_x^j|^2\,dx \\ + i\beta_j y^j\overline{y_x^j}\Big|_{x=0}^{x=\ell_j} = i\int_0^{\ell_j} f_x^j\overline{y_x^j}\,dx - if^j\overline{y_x^j}\Big|_{x=0}^{x=\ell_j} + o(1).\end{aligned}$$

But, using (4.33), (4.34), (4.17), (4.18), (4.13) and the fact that f^j converge to zero in $H^1(0,\ell_j)$, we deduce that

$$-\int_0^{\ell_j}\lambda u_x^j\overline{y_x^j}\,dx+\int_0^{\ell_j}u_{xx}^j\overline{\frac{y_{xx}^j}{\lambda}}\,dx-i\int_0^{\ell_j}\beta_j|y_x^j|^2\,dx=o(1). \tag{4.39}$$

Similarly, multiplying (4.9) by $\overline{\frac{u_{xx}^j}{\lambda}}$, integrating over $(0,\ell_j)$ and using the fact that $\rho_j=1$ for all $j\in\mathcal{I}_{\text{te}}$, we get

$$\begin{aligned}&\int_0^{\ell_j}\lambda y^j\overline{u_{xx}^j}\,dx+\int_0^{\ell_j}y_{xx}^j\overline{\frac{u_{xx}^j}{\lambda}}\,dx-i\int_0^{\ell_j}\beta_j u^j\overline{u_{xx}^j}\,dx\\&=-\int_0^{\ell_j}\beta_j f^j\overline{\frac{u_{xx}^j}{\lambda}}\,dx-\int_0^{\ell_j}\tilde{g}^j\overline{\frac{u_{xx}^j}{\lambda}}\,dx-i\int_0^{\ell_j}g^j\overline{u_{xx}^j}\,dx.\end{aligned}$$

Using Cauchy-Schwarz inequality, (4.14) and the fact that f^j and $\tilde{g}^j$ converge to zero in $L^2(0,\ell_j)$, we obtain

$$\int_0^{\ell_j}\lambda y^j\overline{u_{xx}^j}\,dx+\int_0^{\ell_j}y_{xx}^j\overline{\frac{u_{xx}^j}{\lambda}}\,dx-i\int_0^{\ell_j}\beta_j u^j\overline{u_{xx}^j}\,dx=o(1)-i\int_0^{\ell_j}g^j\overline{u_{xx}^j}\,dx. \tag{4.40}$$

Applying Green's formula on the first and third term of the left hand side and on the integral of the right hand side, we obtain

$$\begin{aligned}&-\int_0^{\ell_j}\lambda y_x^j\overline{u_x^j}\,dx+\lambda y^j\overline{u_x^j}\Big|_{x=0}^{x=\ell_j}+\int_0^{\ell_j}y_{xx}^j\overline{\frac{u_{xx}^j}{\lambda}}\,dx+i\int_0^{\ell_j}\beta_j|u_x^j|^2\,dx\\&\qquad-i\beta_j u^j\overline{u_x^j}\Big|_{x=0}^{x=\ell_j}=i\int_0^{\ell_j}g_x^j\overline{u_x^j}\,dx-ig^j\overline{u_x^j}\Big|_{x=0}^{x=\ell_j}+o(1).\end{aligned}$$

But, using (4.17), (4.18), (4.26), (4.33), (4.34), and the fact that g^j coverges to zero in $H^1(0,\ell_j)$, we deduce that

$$-\int_0^{\ell_j}\lambda y_x^j\overline{u_x^j}\,dx+\int_0^{\ell_j}y_{xx}^j\overline{\frac{u_{xx}^j}{\lambda}}\,dx=o(1). \tag{4.41}$$

Taking the imaginary part of equations (4.39) and (4.41) then, adding the two resulting equations, we conclude that

$$\int_0^{\ell_j}|y_x^j|^2\,dx=o(1).$$

and the result holds. □

Lemma 4.8. *Under all above assumptions, we have*

$$\int_0^{\ell_j}|\lambda y^j|^2\,dx=o(1),\quad\forall j\in\mathcal{I}_{te}. \tag{4.42}$$

Proof. Multiply (4.9) by $\overline{y^j}$ then, integrate over $(0,\ell_j)$ and apply Green's formula, we get

$$\begin{aligned}&\int_0^{\ell_j}|\lambda y^j|^2\,dx-\int_0^{\ell_j}|y_x^j|^2\,dx+y^j\overline{y_x^j}\Big|_{x=0}^{x=\ell_j}-i\int_0^{\ell_j}\beta_j\lambda u^j\overline{y^j}\,dx\\&=-\int_0^{\ell_j}\beta_j f^j\overline{y^j}\,dx-\int_0^{\ell_j}\tilde{g}^j\overline{y^j}\,dx-i\int_0^{\ell_j}g^j\lambda\overline{y^j}\,dx.\end{aligned}$$

But, using Cauchy-Schwarz inequality, (4.12), (4.13), (4.17), (4.18), (4.37) and the fact that f^j, g^j converge to zero in $H^1(0,\ell_j)$ and $\tilde{g}^j$ converges to zero in $L^2(0,\ell_j)$, we deduce that (4.42) holds. □

Lemma 4.9. *Assume that all above assumptions hold. Then, for every thermo-elastic edge, we have*

$$\lambda y^j(\ell_j) = o(1),\ y_x^j(\ell_j) = o(1),\ Re\big(i\lambda g^j(\ell_j) y^j(\ell_j)\big) = o(1), \tag{4.43}$$

$$\lambda y^j(0) = o(1),\ y_x^j(0) = o(1),\ Re\big(i\lambda g^j(0) y^j(0)\big) = o(1). \tag{4.44}$$

Proof. The proof is the same as the one of Lemma 4.3 or Lemma 4.6, using (4.37) and (4.42), the result holds. □

Lemma 4.10. *Under all above assumptions, for each elastic edge we have*

$$\int_0^{\ell_j} |\lambda u^j|^2\, dx = o(1),\ \int_0^{\ell_j} |u_x^j|^2\, dx = o(1), \tag{4.45}$$

$$\int_0^{\ell_j} |\lambda y^j|^2\, dx = o(1),\ \int_0^{\ell_j} |y_x^j|^2\, dx = o(1). \tag{4.46}$$

Proof. Let e_j be an elastic edge attached to a thermoelastic one at an interior vertex a_k, where a_k is a leaf of a maximal subgraph of elastic edges. Recall that $\alpha_j = 0$, and let $\Phi^j \in W^{1,\infty}(0,\ell_j)$. Multiply (4.8) by $2\Phi^j \overline{u_x^j}$ then integrate over $(0,\ell_j)$, take the real part and apply Green's formula, we obtain

$$\begin{aligned} -\int_0^{\ell_j} \Phi_x^j |\lambda u^j|^2\, dx + \Phi^j |\lambda u^j|^2 \Big|_{x=0}^{x=\ell_j} - \int_0^{\ell_j} \Phi_x^j |u_x^j|^2\, dx + \Phi^j |u_x^j|^2 \Big|_{x=0}^{x=\ell_j} \\ +\mathrm{Re}\ 2i \int_0^{\ell_j} \lambda \beta_j \Phi^j y^j \overline{u_x^j}\, dx = 2\mathrm{Re} \int_0^{\ell_j} \beta_j \Phi^j g^j \overline{u_x^j}\, dx \\ -2\mathrm{Re} \int_0^{\ell_j} \tilde{f}^j \Phi^j \overline{u_x^j}\, dx - 2\mathrm{Re} \int_0^{\ell_j} i\lambda f^j \Phi^j \overline{u_x^j}\, dx. \end{aligned}$$

Again applying Green's formula on the fifth term of the left hand side and on the third term of the right hand side, we get

$$\begin{aligned} -\int_0^{\ell_j} \Phi_x^j |\lambda u^j|^2\, dx + \Phi^j |\lambda u^j|^2 \Big|_{x=0}^{x=\ell_j} - \int_0^{\ell_j} \Phi_x^j |u_x^j|^2\, dx + \Phi^j |u_x^j|^2 \Big|_{x=0}^{x=\ell_j} \\ -\mathrm{Re}\ 2i \int_0^{\ell_j} \lambda \overline{u^j} \beta_j \Phi^j y_x^j\, dx - \mathrm{Re}\ 2i \int_0^{\ell_j} \lambda \overline{u^j} \beta_j \Phi_x^j y^j\, dx + \mathrm{Re}\ (2i\beta_j \Phi^j \lambda y^j \overline{u^j}) \Big|_{x=0}^{x=\ell_j} \\ = 2\mathrm{Re} \int_0^{\ell_j} \beta_j \Phi^j g^j \overline{u_x^j}\, dx - 2\mathrm{Re} \int_0^{\ell_j} \tilde{f}^j \Phi^j \overline{u_x^j}\, dx + \mathrm{Re}\ 2i \int_0^{\ell_j} \lambda \overline{u^j} f_x^j \Phi^j\, dx \\ +\mathrm{Re}\ 2i \int_0^{\ell_j} f^j \Phi_x^j \lambda \overline{u^j}\, dx - 2\mathrm{Re}(i\lambda f^j \Phi^j \overline{u^j}) \Big|_{x=0}^{x=\ell_j}. \end{aligned} \tag{4.47}$$

But using Cauchy-Schwarz inequality, (4.12) and (4.13), we deduce that

$$\mathrm{Re}\ 2i \int_0^{\ell_j} \lambda \overline{u^j} \beta_j \Phi_x^j y^j\, dx = o(1). \tag{4.48}$$

Also, using Cauchy-Schwarz inequality, (4.12), (4.13) and the fact that f^j converge to zero in $H^1(0,\ell_j)$ and $\tilde{f}^j, g^j$ converge to zero in $L^2(0,\ell_j)$, we have

$$\begin{aligned} 2\mathrm{Re}\int_0^{\ell_j}\beta_j\Phi^j g^j\overline{u_x^j}\,dx - 2\mathrm{Re}\int_0^{\ell_j}\tilde{f}^j\Phi^j\overline{u_x^j}\,dx + \mathrm{Re}\;2i\int_0^{\ell_j}\lambda\overline{u^j}f_x^j\Phi^j\,dx \\ +\mathrm{Re}\;2i\int_0^{\ell_j}f^j\Phi_x^j\lambda\overline{u^j}\,dx = o(1). \end{aligned} \tag{4.49}$$

Inserting (4.48) and (4.49) in the identity (4.47), we get

$$\begin{aligned} -\int_0^{\ell_j}\Phi_x^j|\lambda u^j|^2\,dx + \Phi^j|\lambda u^j|^2\Big|_{x=0}^{x=\ell_j} - \int_0^{\ell_j}\Phi_x^j|u_x^j|^2\,dx + \Phi^j|u_x^j|^2\Big|_{x=0}^{x=\ell_j} \\ +\mathrm{Re}\,(2i\beta_j\Phi^j\lambda y^j\overline{u^j})\Big|_{x=0}^{x=\ell_j} + 2\,\mathrm{Im}\int_0^{\ell_j}\lambda\overline{u^j}\beta_j\Phi^j y_x^j\,dx = -2\mathrm{Re}(i\lambda f^j\Phi^j\overline{u^j})\Big|_{x=0}^{x=\ell_j}. \end{aligned} \tag{4.50}$$

Similarly, multiply (4.9) by $2\Phi^j\overline{y_x^j}$, integrate over $(0,\ell_j)$, take the real part and apply Green's formula, we obtain

$$\begin{aligned} -\int_0^{\ell_j}\Phi_x^j|\lambda y^j|^2\,dx + \Phi^j|\lambda y^j|^2\Big|_{x=0}^{x=\ell_j} - \int_0^{\ell_j}\rho_j\Phi_x^j|y_x^j|^2\,dx + \rho_j\Phi^j|y_x^j|^2\Big|_{x=0}^{x=\ell_j} \\ -\mathrm{Re}\;2i\int_0^{\ell_j}\lambda\beta_j\Phi^j u^j\overline{y_x^j}\,dx = -\mathrm{Re}\;2\int_0^{\ell_j}\beta_j f^j\Phi^j\overline{y_x^j}\,dx \\ -\mathrm{Re}\;2\int_0^{\ell_j}\tilde{g}^j\Phi^j\overline{y_x^j}\,dx - \mathrm{Re}\;2i\int_0^{\ell_j}\lambda g^j\Phi^j\overline{y_x^j}\,dx. \end{aligned}$$

Applying Green's formula on the last integral of the right hand side, we get

$$\begin{aligned} -\int_0^{\ell_j}\Phi_x^j|\lambda y^j|^2\,dx + \Phi^j|\lambda y^j|^2\Big|_{x=0}^{x=\ell_j} - \int_0^{\ell_j}\rho_j\Phi_x^j|y_x^j|^2\,dx + \rho_j\Phi^j|y_x^j|^2\Big|_{x=0}^{x=\ell_j} \\ -\mathrm{Re}\;2i\int_0^{\ell_j}\lambda\beta_j\Phi^j u^j\overline{y_x^j}\,dx = -\mathrm{Re}\;2\int_0^{\ell_j}\beta_j f^j\Phi^j\overline{y_x^j}\,dx \\ -\mathrm{Re}\;2\int_0^{\ell_j}\tilde{g}^j\Phi^j\overline{y_x^j}\,dx + \mathrm{Re}\;2i\int_0^{\ell_j}\lambda g_x^j\Phi^j\overline{y^j}\,dx + \mathrm{Re}\;2i\int_0^{\ell_j}\lambda g^j\Phi_x^j\overline{y^j}\,dx \\ -2\;\mathrm{Re}(i\lambda g^j\Phi^j\overline{y^j})\Big|_{x=0}^{x=\ell_j}. \end{aligned} \tag{4.51}$$

But, using (4.13) and the fact that g^j converge to zero in $H^1(0,\ell_j)$ and $f^j, \tilde{g}^j$ converge to zero in $L^2(0,\ell_j)$, we conclude that

$$\begin{aligned} -\mathrm{Re}\;2\int_0^{\ell_j}\beta_j f^j\Phi^j\overline{y_x^j}\,dx - \mathrm{Re}\;2\int_0^{\ell_j}\tilde{g}^j\Phi^j\overline{y_x^j}\,dx + \mathrm{Re}\;2i\int_0^{\ell_j}\lambda g_x^j\Phi^j\overline{y^j}\,dx \\ +\mathrm{Re}\;2i\int_0^{\ell_j}\lambda g^j\Phi_x^j\overline{y^j}\,dx = o(1). \end{aligned} \tag{4.52}$$

Then, inserting (4.52) in equation (4.51), we get

$$-\int_0^{\ell_j}\Phi_x^j|\lambda y^j|^2\,dx+\Phi^j|\lambda y^j|^2\Big|_{x=0}^{x=\ell_j}-\int_0^{\ell_j}\rho_j\Phi_x^j|y_x^j|^2\,dx+\rho_j\Phi^j|y_x^j|^2\Big|_{x=0}^{x=\ell_j}$$
$$+2\ \mathrm{Im}\int_0^{\ell_j}\lambda\beta_j\Phi^j u^j\overline{y_x^j}\,dx=o(1)-2\ \mathrm{Re}(i\lambda g^j\Phi^j\overline{y^j})\Big|_{x=0}^{x=\ell_j}. \tag{4.53}$$

Without loss of generality, assume that $\pi_j(a_k)=0$ and let $\Phi^j=x-\ell_j$ (otherwise, let $\Phi^j=x$). Then, adding the two equations (4.50) and (4.53), using Lemma 4.6, Lemma 4.9, the fact that u,y,f and g satisfy the continuity conditions in (1.11) and u,θ and y satisfy the balance conditions (1.12)-(1.13), we deduce that

$$\int_0^{\ell_j}|\lambda u^j|^2\,dx+\int_0^{\ell_j}|u_x^j|^2\,dx$$
$$+\int_0^{\ell_j}|\lambda y^j|^2\,dx+\int_0^{\ell_j}\rho_j|y_x^j|^2\,dx=o(1).$$

Consequently, as $\rho_j>0$ for all $j\in\mathcal{I}_{\mathrm{e}}$, (4.45) and (4.46) hold. Repeating the same technique of Lemma 4.6 and Lemma 4.9, we conclude that

$$\lambda u^j(0)=o(1),\ u_x^j(0)=o(1),\ \mathrm{Re}\big(i\lambda f^j(0)u^j(0)\big)=o(1),$$
$$\lambda y^j(0)=o(1),\ y_x^j(0)=o(1),\ \mathrm{Re}\big(i\lambda g^j(0)y^j(0)\big)=o(1),$$
$$\lambda u^j(\ell_j)=o(1),\ u_x^j(\ell_j)=o(1),\ \mathrm{Re}\big(i\lambda f^j(\ell_j)u^j(\ell_j)\big)=o(1),$$
$$\lambda y^j(\ell_j)=o(1),\ y_x^j(\ell_j)=o(1),\ \mathrm{Re}\big(i\lambda g^j(\ell_j)y^j(\ell_j)\big)=o(1).$$

Then, by iteration on each maximal subgraph of purely elastic edges (from leaves to the root), we prove that

$$\int_0^{\ell_j}|\lambda u^j|^2\,dx+\int_0^{\ell_j}|u_x^j|^2\,dx+\int_0^{\ell_j}|\lambda y^j|^2\,dx+\int_0^{\ell_j}|y_x^j|^2\,dx=o(1), \tag{4.54}$$

for all $j\in\mathcal{I}_{\mathrm{e}}$. □

In conclusion, using Lemmas 4.4, 4.5, 4.7, 4.8 and 4.10, we conclude that $||U||_{\mathcal{H}}=o(1)$, which contradicts (4.1). □

Remark 1. Examples of networks for which (S1) holds are given by Theorem 3.2.

Remark 2. If there exists an elastic edge ($j\in\mathcal{I}_{\mathbf{e}}$) such that $\rho_j\neq 1$ then, using Lemma 4.10, we show that Theorem 4.1 holds (i.e., the energy of the system decays exponentially to zero). But, if there exists a thermo-elastic edge ($j\in\mathcal{I}_{\mathbf{te}}$) such that $\rho_j\neq 1$ then, it seems that the energy of the system does not decay exponentially, but polynomially (see Theorem 4.11 below).

4.2. **Polynomial stability.**

Theorem 4.11. *Let $\mathcal{N}$ be an arbitrary network for which the operator $\mathcal{A}$ associated with System (1.7)-(1.13) satisfies (S1). Assume that there exists $j\in\mathcal{I}_{te}\cap\mathcal{I}'_{ext}$ such that $\rho_j\neq 1$. Then, the energy of the system satisfies*

$$E(t)\leq\frac{C}{t^{1/3}}||U_0||^2_{D(\mathcal{A})},\quad\forall U_0\in D(\mathcal{A}),\quad t>0, \tag{4.55}$$

for some positive constant $C > 0$.

Proof. Following Borichev-Tomilov [3], the C_0-semigroup of contractions $(e^{t\mathcal{A}})_{t\geq 0}$ on $\mathcal{H}$ is polynomially stable if and only if (S1) and

$$\limsup_{|\lambda|\to\infty} \frac{1}{\lambda^6}\|(i\lambda - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} < \infty \qquad (S2)$$

hold. As we have assumed that (S1) is satisfied, it remains to prove that condition (S2) is satisfied. To this aim, we proceed by a contradiction argument. Suppose that (S2) does not hold, then there exist a sequence of real numbers $\lambda_n \in \mathbb{R}$ and a sequence of vectors $U_n = (u_n, v_n, y_n, z_n, \theta_n) \in D(\mathcal{A})$ such that (4.1) and

$$\lambda^6(i\lambda_n - \mathcal{A})U_n = (f_n, \tilde{f}_n, g_n, \tilde{g_n}, h_n) \longrightarrow 0 \quad \text{in } \mathcal{H} \tag{4.56}$$

are satisfied.

In what follows, we drop the index n for simplicity.

Now by detailing (4.56), we get for all $j \in \mathcal{I}(\mathcal{N})$

$$\lambda^6(i\lambda u^j - v^j) = f^j \to 0 \text{ in } H^1(0,\ell_j), \tag{4.57}$$
$$\lambda^6(i\lambda v^j - u^j_{xx} + \alpha_j\theta^j_x - \beta_j z^j) = \tilde{f}^j \to 0 \text{ in } L^2(0,\ell_j), \tag{4.58}$$
$$\lambda^6(i\lambda y^j - z^j) = g^j \to 0 \text{ in } H^1(0,\ell_j), \tag{4.59}$$
$$\lambda^6(i\lambda z^j - \rho_j y^j_{xx} + \beta_j v^j) = \tilde{g}^j \to 0 \text{ in } L^2(0,\ell_j), \tag{4.60}$$
$$\lambda^6(i\lambda\theta^j - \kappa_j\theta^j_{xx} + \alpha_j v^j_x) = h^j \to 0 \text{ in } L^2(0,\ell_j). \tag{4.61}$$

Then, by eliminating v^j and z^j using (4.57) and (4.59) respectively, (4.57)-(4.61) becomes

$$\lambda^2 u^j + u^j_{xx} - \alpha_j\theta^j_x + i\lambda\beta_j y^j = \frac{\beta_j g^j}{\lambda^6} - \frac{\tilde{f}^j}{\lambda^6} - i\frac{f^j}{\lambda^5}, \tag{4.62}$$
$$\lambda^2 y^j + \rho_j y^j_{xx} - i\lambda\beta_j u^j = -\frac{\beta_j f^j}{\lambda^6} - \frac{\tilde{g}^j}{\lambda^6} - i\frac{g^j}{\lambda^5}, \tag{4.63}$$
$$i\lambda\theta^j - \kappa_j\theta^j_{xx} + i\lambda\alpha_j u^j_x = \frac{h^j}{\lambda^6} + \frac{\alpha_j f^j_x}{\lambda^6}. \tag{4.64}$$

First, our aim is to prove that for each thermo-elastic edge, we have

$$\begin{cases} v^j \to 0, & \text{in } L^2(0,\ell_j), \\ u^j \to 0, & \text{in } H^1(0,\ell_j), \\ \theta^j \to 0, & \text{in } H^1(0,\ell_j), \\ z^j \to 0, & \text{in } L^2(0,\ell_j), \\ y^j \to 0, & \text{in } H^1(0,\ell_j). \end{cases} \tag{4.65}$$

Following the same proof of Lemmas 4.4, 4.5, 4.7, 4.8, we can prove that (4.65) holds for all $j \in \mathcal{I}_{\text{te}}$ with $\rho_j = 1$. Hence, we only need to prove that (4.65) holds for all $j \in \mathcal{I}_{\text{te}} \cap \mathcal{I}'_{\text{ext}}$ with $\rho_j \neq 1$. We will proceed by dividing the proof into different Lemmas.

Lemma 4.12. *Under all above assumptions, we have*

$$\int_0^{\ell_j} |\theta^j_x|^2\, dx = \frac{o(1)}{\lambda^6},\ \forall j \in \mathcal{I}_{te}. \tag{4.66}$$

Proof. Same proof as the one of Lemma 4.2. □

Using (4.1), (4.57) and (4.59), we can easily deduce that (4.12)-(4.13) holds, for all $j \in \mathcal{I}(\mathcal{N})$. Also, using (4.62), (4.63) and (4.64) we conclude (4.14) for all $j \in \mathcal{I}(\mathcal{N})$.

Lemma 4.13. *Under all above assumptions,* (4.15)-(4.18) *holds.*

Proof. Let Φ^j be a function in $W^{1,\infty}(0,\ell_j)$, for all $j \in \mathcal{I}_{\text{te}}$. Multiplying (4.62) by $2\Phi^j\overline{u_x^j}$, integrating over $(0,\ell_j)$ then taking the real part and applying Green's Formula, we obtain

$$\begin{aligned} -\int_0^{\ell_j} \Phi_x^j|\lambda u^j|^2\,dx + \Phi^j|\lambda u^j|^2\Big|_{x=0}^{x=\ell_j} - \int_0^{\ell_j} \Phi_x^j|u_x^j|^2\,dx + \Phi^j|u_x^j|^2\Big|_{x=0}^{x=\ell_j} \\ -2\text{Re }\alpha_j\int_0^{\ell_j}\theta_x^j\Phi^j\overline{u_x^j}\,dx + \text{Re }2i\int_0^{\ell_j}\beta_j\lambda y^j\Phi^j\overline{u_x^j}\,dx \qquad (4.67) \\ = 2\text{Re}\int_0^{\ell_j}\frac{\beta_j\Phi^j g^j\overline{u_x^j}}{\lambda^6}\,dx - 2\text{Re}\int_0^{\ell_j}\frac{\tilde{f}^j\Phi^j\overline{u_x^j}}{\lambda^6}\,dx - 2\text{Re}\int_0^{\ell_j} i\frac{f^j\Phi^j\overline{u_x^j}}{\lambda^5}\,dx. \end{aligned}$$

Using (4.12), (4.13) and (4.66) we get,

$$\Phi^j|\lambda u^j|^2\Big|_{x=0}^{x=\ell_j} + \Phi^j|u_x^j|^2\Big|_{x=0}^{x=\ell_j} = O(1).$$

Let $\Phi^j = x$, for all $j \in \mathcal{I}_{\text{te}}$. We deduce that (4.15) holds. Similarly, by taking $\Phi^j = (x-\ell_j)$, we conclude (4.16). Also, multiplying (4.9) by $2x\overline{y_x^j}$ and $2(x-\ell_j)\overline{y_x^j}$ respectively, we deduce that (4.17) and (4.18) hold. □

Lemma 4.14. *Under all above assumptions, we have*

$$\int_0^{\ell_j}|u_x^j|^2\,dx = \frac{o(1)}{\lambda^2},\quad \forall\, j \in \mathcal{I}_{te}\cap\mathcal{I}'_{ext}. \qquad (4.68)$$

Proof. Let $j \in \mathcal{I}_{\text{te}}\cap\mathcal{I}'_{\text{ext}}$. Multiply (4.64) by $\frac{\overline{u_x^j}}{\lambda}$, and integrate over $(0,\ell_j)$, we get

$$\begin{aligned} i\int_0^{\ell_j}\theta^j\overline{u_x^j}\,dx - \int_0^{\ell_j}\kappa_j\theta_{xx}^j\frac{\overline{u_x^j}}{\lambda}\,dx + i\int_0^{\ell_j}\alpha_j|u_x^j|^2\,dx \\ = \int_0^{\ell_j}\frac{h^j\overline{u_x^j}}{\lambda^7}\,dx + \int_0^{\ell_j}\frac{\alpha_j f_x^j\overline{u_x^j}}{\lambda^7}\,dx. \end{aligned} \qquad (4.69)$$

Applying Green's formula on the first and second term of (4.69), using Cauchy-Schwarz inequality, (4.12) and the fact that f^j converge to zero in $H^1(0,\ell_j)$ and h^j converge to zero in $L^2(0,\ell_j)$, we obtain

$$\begin{aligned} -i\int_0^{\ell_j}\theta_x^j\overline{u^j}\,dx + i\theta^j\overline{u^j}\Big|_{x=0}^{x=\ell_j} + \int_0^{\ell_j}\kappa_j\theta_x^j\frac{\overline{u_{xx}^j}}{\lambda}\,dx - \kappa_j\frac{\theta_x^j}{\lambda}\overline{u_x^j}\Big|_{x=0}^{x=\ell_j} \\ +i\int_0^{\ell_j}\alpha_j|u_x^j|^2\,dx = \frac{o(1)}{\lambda^7}. \end{aligned} \qquad (4.70)$$

But, using Cauchy-Schwarz inequality, (4.66), (4.12) and (4.14), we deduce that

$$\int_0^{\ell_j}\theta_x^j\overline{u^j}\,dx = \frac{o(1)}{\lambda^4}, \qquad (4.71)$$

and

$$\int_0^{\ell_j} \theta_x^j \overline{\frac{u_{xx}^j}{\lambda}}\, dx = \frac{o(1)}{\lambda^3}. \tag{4.72}$$

On the other hand, by applying Gagliardo-Nirenberg inequality to $\Psi = \theta^j$ and $\Psi = \frac{\theta_x^j}{\lambda}$, using Poincaré inequality, (4.66) and (4.14) we deduce that

$$\theta^j(0) = \frac{o(1)}{\lambda^3}, \;\; \theta^j(\ell_j) = \frac{o(1)}{\lambda^3}, \;\; j \in \mathcal{I}_{\rm te} \cap \mathcal{I}'_{\rm ext}, \tag{4.73}$$

and

$$\frac{\theta_x^j(0)}{\lambda} = \frac{o(1)}{\lambda^2}, \;\; \frac{\theta_x^j(\ell_j)}{\lambda} = \frac{o(1)}{\lambda^2}, \;\; j \in \mathcal{I}_{\rm te} \cap \mathcal{I}'_{\rm ext}. \tag{4.74}$$

Using (4.71), (4.72), (4.73), (4.74), (4.15) and (4.16) we conclude that (4.68) holds. □

Lemma 4.15. *Under all above assumptions, we have for all $j \in \mathcal{I}_{te} \cap \mathcal{I}'_{ext}$*

$$\int_0^{\ell_j} |\lambda u^j|^2\, dx = o(1) \text{ and } \int_0^{\ell_j} |v^j|^2\, dx = o(1). \tag{4.75}$$

Proof. Let $j \in \mathcal{I}_{\rm te} \cap \mathcal{I}'_{\rm ext}$. Multiply (4.62) by $\overline{u^j}$, integrate over $(0, \ell_j)$ and apply Green's formula, we get

$$\int_0^{\ell_j} |\lambda u^j|^2\, dx - \int_0^{\ell_j} |u_x^j|^2\, dx + u_x^j \overline{u^j}\Big|_{x=0}^{x=\ell_j} - \int_0^{\ell_j} \alpha_j \theta_x^j \overline{u^j}\, dx$$
$$+ \int_0^{\ell_j} i\beta_j \lambda y^j \overline{u^j}\, dx = \int_0^{\ell_j} \frac{\beta_j g^j \overline{u^j}}{\lambda^6}\, dx - \int_0^{\ell_j} \frac{\tilde{f}^j \overline{u^j}}{\lambda^6}\, dx - i\int_0^{\ell_j} \frac{f^j \overline{u^j}}{\lambda^5}\, dx.$$

Using Cauchy-Schwarz inequality, (4.12), (4.13), (4.66), (4.68), (4.15) and (4.16) we deduce that

$$\int_0^{\ell_j} |\lambda u^j|^2\, dx = o(1).$$

Using (4.57), we conclude that (4.75) holds. □

Lemma 4.16. *Assume that all above assumptions hold. Then, for $j \in \mathcal{I}_{te} \cap \mathcal{I}'_{ext}$, we have*

$$\lambda u^j(\ell_j) = o(1), \;\; u_x^j(\ell_j) = o(1), \tag{4.76}$$
$$\lambda u^j(0) = o(1), \;\; u_x^j(0) = o(1), \tag{4.77}$$
$$\theta^j(0) = o(1), \theta^j(\ell_j) = o(1). \tag{4.78}$$

Proof. By the proof of Lemma 4.13, for all $j \in \mathcal{I}_{\rm te} \cap \mathcal{I}'_{\rm ext}$, and any Φ^j in $W^{1,\infty}(0, \ell_j)$, (4.67) holds. Then, using (4.68), (4.75), (4.66) and (4.13) we obtain

$$\Phi^j |\lambda u^j|^2\Big|_{x=0}^{x=\ell_j} + \Phi^j |u_x^j|^2\Big|_{x=0}^{x=\ell_j} = o(1).$$

Then, by taking $\Phi^j = x$, for all $j \in \mathcal{I}_{\rm te} \cap \mathcal{I}'_{\rm ext}$, we deduce that (4.76) holds. Similarly, by taking $\Phi^j = (x - \ell_j)$, for all $j \in \mathcal{I}_{\rm te} \cap \mathcal{I}'_{\rm ext}$, we conclude that (4.77) holds. On the other hand, by (4.73), we directly have (4.78). □

Lemma 4.17. *Under all above assumptions, we have for all $j \in \mathcal{I}_{te} \cap \mathcal{I}'_{ext}$*

$$\int_0^{\ell_j} |\lambda y^j|^2 \, dx = o(1) \text{ and } \int_0^{\ell_j} |z^j|^2 \, dx = o(1). \tag{4.79}$$

Proof. Let $j \in \mathcal{I}_{\text{te}} \cap \mathcal{I}'_{\text{ext}}$. Multiply equation (4.63) by $\lambda\overline{u^j}$, then integrate over $(0, \ell_j)$, we get

$$\begin{aligned} &\int_0^{\ell_j} \lambda^3 y^j \overline{u^j} \, dx + \rho_j \int_0^{\ell_j} y^j_{xx} \lambda \overline{u^j} \, dx - i \int_0^{\ell_j} \beta_j |\lambda u^j|^2 \, dx \\ &= -\int_0^{\ell_j} \frac{\beta_j f^j \overline{u^j}}{\lambda^5} \, dx - \int_0^{\ell_j} \frac{\tilde{g}^j \overline{u^j}}{\lambda^5} \, dx - i \int_0^{\ell_j} \frac{g^j \overline{u^j}}{\lambda^4} \, dx. \end{aligned} \tag{4.80}$$

Applying Green's formula on the second term of (4.80), using Cauchy-Schwarz inequality and (4.12), we obtain

$$\int_0^{\ell_j} \lambda^3 y^j \overline{u^j} \, dx - \rho_j \int_0^{\ell_j} y^j_x \lambda \overline{u^j_x} \, dx + \rho_j y^j_x \lambda \overline{u^j} \Big|_{x=0}^{x=\ell_j} - i \int_0^{\ell_j} \beta_j |\lambda u^j|^2 \, dx = o(1).$$

But, using (4.76), (4.77), (4.68) (4.75), (4.17), (4.18) and (4.13), we deduce that

$$\int_0^{\ell_j} \lambda^3 y^j \overline{u^j} \, dx = o(1). \tag{4.81}$$

Similarly, multiply (4.62) by $\lambda\overline{y^j}$ then integrate over $(0, \ell_j)$ and apply Green's formula, we obtain

$$\begin{aligned} &\int_0^{\ell_j} \lambda^3 u^j \overline{y^j} \, dx - \int_0^{\ell_j} u^j_x \lambda \overline{y^j_x} \, dx + u^j_x \lambda \overline{y^j} \Big|_{x=0}^{x=\ell_j} - \int_0^{\ell_j} \alpha_j \theta^j_x \lambda \overline{y^j} \, dx \\ &+ i\beta_j \int_0^{\ell_j} |\lambda y^j|^2 \, dx = \int_0^{\ell_j} \frac{\beta_j g^j \overline{y^j}}{\lambda^5} \, dx - \int_0^{\ell_j} \frac{\tilde{f}^j \overline{y^j}}{\lambda^5} \, dx - i \int_0^{\ell_j} \frac{f^j \overline{y^j}}{\lambda^4} \, dx. \end{aligned}$$

Consequently, using (4.68), (4.76), (4.77), (4.17), (4.18) and (4.66) we get

$$\int_0^{\ell_j} \lambda^3 u^j \overline{y^j} \, dx + i\beta_j \int_0^{\ell_j} |\lambda y^j|^2 \, dx = o(1).$$

Then, taking the imaginary part of the above equality, using (4.81) and the fact that β_j is a positive constant for all $j \in \mathcal{I}_{\text{te}} \cap \mathcal{I}'_{\text{ext}}$, we deduce that

$$\int_0^{\ell_j} |\lambda y^j|^2 \, dx = o(1).$$

Finally, by (4.60), we conclude that(4.79) holds. □

Lemma 4.18. *Under all above assumptions, we have for all $j \in \mathcal{I}_{te} \cap \mathcal{I}'_{ext}$*

$$\int_0^{\ell_j} |y^j_x|^2 \, dx = o(1). \tag{4.82}$$

Proof. Multiply (4.63) by $\overline{y^j}$, integrate over $(0,\ell_j)$ and apply Green's formula, we get

$$\int_0^{\ell_j} |\lambda y^j|^2\,dx - \rho_j\int_0^{\ell_j}|y_x^j|^2\,dx + \rho_j y_x^j\overline{y^j}\Big|_{x=0}^{x=\ell_j} - i\int_0^{\ell_j}\beta_j u^j\overline{y^j}\,dx$$
$$= -\int_0^{\ell_j}\frac{\beta_j f^j\overline{y^j}}{\lambda^6}\,dx - \int_0^{\ell_j}\frac{\tilde{g}^j\overline{y^j}}{\lambda^6}\,dx - i\int_0^{\ell_j}\frac{g^j\overline{y^j}}{\lambda^5}\,dx.$$

Then, using (4.79), (4.17), (4.18) and (4.13), we deduce that (4.82) holds. □

Now, following the same proof of Lemma 4.13 and Lemma 4.16, we deduce that for all $j\in\mathcal{I}_{\rm te}\cap\mathcal{I}'_{\rm ext}$, we have

$$\begin{cases}\lambda y^j(\ell_j)=o(1),\ y_x^j(\ell_j)=o(1),\\ \lambda y^j(0)=o(1),\ y_x^j(0)=o(1).\end{cases}\tag{4.83}$$

Hence, using (4.83), Lemma 4.16 and by iteration on each maximal subgraph of purely elastic edges (from leaves to root), the results of Lemma 4.10 hold for every elastic edge. Finally, using Lemmas 4.14, 4.15, 4.17, 4.18 and 4.10, we conclude that $||U||_{\mathcal{H}}=o(1)$, which contradicts (4.1). □

Remark 3. 1) If every maximal subgraph of thermo-elastic edges is composed of maximum three thermo-elastic edges and there exist $j\in\mathcal{I}_{\rm te}\cap\mathcal{G}'_{\rm int}$ such that $\rho_j\neq 1$ then, the same result of Theorem 4.11 holds. The key step to prove that result is to show that (4.73) still holds. Due to (1.11) and as (4.73) holds for all $j\in\mathcal{I}_{\rm te}\cap\mathcal{I}'_{\rm ext}$, we deduce that (4.73) is achieved for all $j\in\mathcal{I}_{\rm te}$ (in particular for $j\in\mathcal{I}_{\rm te}\cap\mathcal{G}'_{\rm int}$). Hence, the desired result is attained.

2) If we replace condition (1.13) by the following condition

$$\begin{cases}\displaystyle\sum_{j\in\mathcal{I}(a_k)} d_{kj}u_x^j(a_k,t)=0,\ a_k\in V_{\rm int},\\ \displaystyle\sum_{j\in\mathcal{I}_{\rm te}(a_k)} d_{kj}(\alpha_j u_t^j(a_k,t)-\kappa_j\theta_x^j(a_k,t))=0,\ a_k\in V'_{\rm int}.\end{cases}\tag{4.84}$$

Then, we can also prove that the same results of Theorem 4.1 and Theorem 4.11 hold for system {(1.7)-(1.12),(4.84)}. Notice that due to condition (4.84), a slight gain appears when proving the system {(1.7)-(1.12),(4.84)} is strongly stable. In other words, the energy associated with system {(1.7)-(1.12),(4.84)} converges to zero if one of the following conditions holds,

i) Each maximal subgraph of thermo-elastic edges contains at least one interior vertex or contains an exterior vertex that belongs to $V_{\rm ext}$.

ii) There exists a maximal subgraph of thermo-elastic edges that contains no interior vertices and contains no exterior vertex that belongs to $V_{\rm ext}$ and $\beta_j=\beta$, for all $j\in\mathcal{I}(\mathcal{N})$.

Idea of the proof. In comparison with Theorem 3.2, it is enough to prove that if each maximal subgraph of thermo-elastic edges contains at least one interior vertex, then the energy associated with system {(1.7)-(1.12),(4.84)} converges to zero. Due to the fact that $(I-\mathcal{A}_*)^{-1}$ is compact where, $\mathcal{A}_*=\mathcal{A}_{|_{D(\mathcal{A}_*)}}$ is the operator

associated with system {(1.7)-(1.12),(4.84)}, we have $\sigma(\mathcal{A}_*) = \sigma_p(\mathcal{A}_*)$. Following the same proof as the one of Theorem 3.2, let $\lambda \in \mathbb{R}$ and $U = (u, v, y, z, \theta) \in D(\mathcal{A}_*)$ be such that

$$\mathcal{A}_* U = i\lambda U.$$

As $\mathcal{A}_* = \mathcal{A}_{|_{D(\mathcal{A}_*)}}$, we obtain, (3.1)-(3.10). If $\lambda = 0$, we proceed exactly as in the proof of Theorem 3.2. Otherwise, if $\lambda \neq 0$ then, using (3.10) and (3.8) we have (3.14), which means that u^j is constant for every $j \in \mathcal{I}_{\text{te}}$ but, due to the fact that each maximal subgraph of thermo-elastic edges contains at least one interior vertex and using (3.9), (3.1) and (1.11), the balance condition (4.84) asserts that $u^j(a_k) = 0$ for some $j \in \mathcal{I}_{\text{te}}(a_k), a_k \in V'_{\text{int}}$. Again, using (1.11) we deduce that $u^j = 0$, for all $j \in \mathcal{I}_{\text{te}}$. Finally, we proceed exactly as **Case 1** of the proof of Theorem 3.2 to reach the desired aim.

5. **Stabilization of thermo-elastic system with Neumann boundary condition at the interior nodes of some particular networks.** In this section, we investigate the stabilization of a thermo-elastic system with Neumann boundary condition at the interior nodes of some particular networks (composed of elastic and thermoelastic materials) similar to the particular networks considered in [18]. In the first case, we consider trees (G_1) for which all exterior edges (except one) are thermo-elastic. In the second case, we consider the path $(\mathcal{P})$ composed of two exterior elastic edges and an interior thermoelastic edge. In the third case, we consider (G_2), trees of elastic materials, whose leaves (exterior nodes of the last generation) are connected to thermoelastic materials as follows: the thermoelastic body connects two leaves issued from the same vertex, with the condition that each leaf is connected to only one thermoelastic body.

In fact, the considered networks (G_1), $(\mathcal{P})$ and (G_2) are particular graphs of the general networks covered in Section 1-Section 4. Notice that, if we apply the boundary conditions of Section 1 on these particular networks, we can deduce that the stabilization of the thermo-elastic system on (G_1), $(\mathcal{P})$ and (G_2) is achieved when θ satisfies Dirichlet condition on each end of every thermo-elastic edge (see Section 1-Section 4). In this section, we discuss the stabilization of the thermo-elastic system on these particular networks such that θ satisfies Neumann boundary condition at each interior node connected to a thermo-elastic edge (see Figure 6).

The system is described as follows:

• On every thermo-elastic edge ($j \in \mathcal{I}_{\text{te}}$) the following equations hold:

$$\begin{cases} u^j_{tt} - u^j_{xx} + \alpha_j \theta^j_x - \beta_j y^j_t &= 0 \quad \text{in} \quad (0, \ell_j) \times (0, \infty), \\ y^j_{tt} - \rho_j y^j_{xx} + \beta_j u^j_t &= 0 \quad \text{in} \quad (0, \ell_j) \times (0, \infty), \\ \theta^j_t - \kappa_j \theta^j_{xx} + \alpha_j u^j_{tx} &= 0 \quad \text{in} \quad (0, \ell_j) \times (0, \infty), \end{cases} \tag{5.1}$$

where α_j, ρ_j, κ_j and β_j are positive constants.

• On every elastic edge ($j \in \mathcal{I}_{\text{e}}$) one has:

$$\begin{cases} u^j_{tt} - u^j_{xx} - \beta_j y^j_t &= 0 \quad \text{in} \quad (0, \ell_j) \times (0, \infty), \\ y^j_{tt} - \rho_j y^j_{xx} + \beta_j u^j_t &= 0 \quad \text{in} \quad (0, \ell_j) \times (0, \infty), \end{cases} \tag{5.2}$$

where β_j and ρ_j are positive constants.

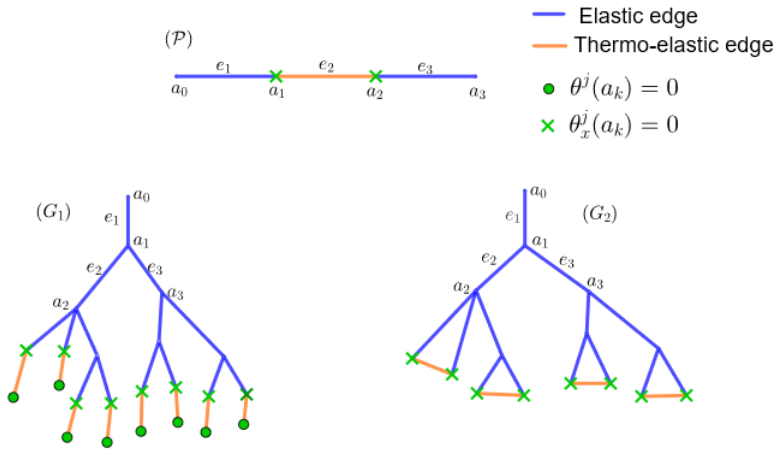

FIGURE 6. Elastic/thermo-elastic networks

We assume that the initial data on the network $\mathcal{N}$ are

$$\begin{cases} u^j(x,0) = u_0^j(x),\ u_t^j(x,0) = u_1^j(x), & \forall j \in \mathcal{I}(\mathcal{N}), \\ y^j(x,0) = y_0^j(x),\ y_t^j(x,0) = y_1^j(x), & \forall j \in \mathcal{I}(\mathcal{N}), \\ \theta^j(x,0) = \theta_0^j(x), & \forall j \in \mathcal{I}_{\rm te}(\mathcal{N}). \end{cases} \tag{5.3}$$

The boundary conditions of system (5.1)-(5.3) on the considered networks will be as follows.
The system satisfies the Dirichlet boundary condition for the displacement and temperature at the exterior nodes,

$$\begin{cases} u^j(a_k,t) = 0, & j \in \mathcal{I}(a_k),\ \ a_k \in V_{\rm ext}, \\ y^j(a_k,t) = 0, & j \in \mathcal{I}(a_k),\ \ a_k \in V_{\rm ext}, \\ \theta^j(a_k,t) = 0, & j \in \mathcal{I}_{\rm te}(a_k),\ \ a_k \in V_{\rm ext}. \end{cases} \tag{5.4}$$

The displacement is continuous at every interior node,

$$\begin{cases} u^j(a_k,t) = u^\ell(a_k,t), & j,\ell \in \mathcal{I}(a_k),\ \ a_k \in V_{\rm int}, \\ y^j(a_k,t) = y^\ell(a_k,t), & j,\ell \in \mathcal{I}(a_k),\ \ a_k \in V_{\rm int}. \end{cases} \tag{5.5}$$

The temperature satisfies the Neumann condition at the interior nodes,

$$\theta_x^j(a_k,t) = 0, \quad j \in \mathcal{I}_{\rm te}(a_k),\ \ a_k \in V_{\rm int}. \tag{5.6}$$

The system satisfies the balance condition at every interior node,

$$\begin{cases} \displaystyle\sum_{j\in\mathcal{I}_{\rm te}(a_k)} d_{kj}(u_x^j(a_k,t) - \alpha_j\theta^j(a_k,t)) + \sum_{j\in\mathcal{I}_{\rm e}(a_k)} d_{kj}u_x^j(a_k,t) = 0,\ \ a_k \in V_{\rm int}, \\ \displaystyle\sum_{j\in\mathcal{I}(a_k)} d_{kj}\rho_j y_x^j(a_k,t) = 0,\ \ a_k \in V_{\rm int}. \end{cases} \tag{5.7}$$

Mainly, we find sufficient conditions on the lengths of the purely elastic edges attached to the thermo-elastic ones so that the system is strongly stable and then exponentially stable on the above described networks.

Here the energy space $\mathcal{H}_1$ is given by

$$\mathcal{H}_1 = \big\{(u, u_t, y, y_t, \theta) \in \mathbb{H}_0^1 \times \mathbb{L}^2 \times \mathbb{H}_0^1 \times \mathbb{L}^2 \times \mathbb{V} \text{ satisfying } (5.9)\big\}, \tag{5.8}$$

$$\int_0^{\ell_j} (\alpha_j u_x^j + \theta^j)\, dx = 0, \ \forall\, j \in \mathcal{G}_{\text{int}} \cap \mathcal{I}_{\text{te}} \tag{5.9}$$

where, $\mathcal{G}_{\text{int}}$ is the set of indices of edges adjacent to two interior vertices.

Remark that it follows from the third equation of (5.1) and from the Neumann condition in (5.6) that for all $j \in \mathcal{G}_{\text{int}} \cap \mathcal{I}_{\text{te}}$, we have $\int_0^{\ell_j} (\alpha_j u_{tx}^j + \theta_t^j)\, dx = 0$, which implies that $\int_0^{\ell_j} (\alpha_j u_x^j + \theta^j)\, dx$ is conservative in time. Hence, if this quantity is zero at time $t = 0$, it will remain zero at all time $t > 0$, hence without loss of generality, we assume that (5.9) holds.

The Hilbert space $\mathcal{H}_1$ is equipped with the inner product given in (2.3).
Next, we define the unbounded linear operator $\mathcal{A}_1$ by:

$$\mathcal{A}_1 \begin{pmatrix} u \\ v \\ y \\ z \\ \theta \end{pmatrix} = \left(\begin{pmatrix} v^j \\ u_{xx}^j - \alpha_j \theta_x^j + \beta_j z^j \\ z^j \\ \rho_j y_{xx}^j - \beta_j v^j \\ \kappa_j \theta_{xx}^j - \alpha_j v_x^j \end{pmatrix} \right)_{j \in \mathcal{I}(\mathcal{N})} \tag{5.10}$$

with domain

$$D(\mathcal{A}_1) = \big\{(u, v, y, z, \theta) \in \mathcal{H}_1 \cap [\mathbb{H}^2 \times \mathbb{H}_0^1 \times \mathbb{H}^2 \times \mathbb{H}_0^1 \times \mathbb{V}^2], \text{ satisfying } (5.11)\big\}$$

$$\begin{cases} \theta^j(a_k) = 0, \quad j \in \mathcal{I}_{\text{te}}(a_k), \quad a_k \in V_{\text{ext}}, \\ \theta_x^j(a_k) = 0, j \in \mathcal{I}_{\text{te}}(a_k), \quad a_k \in V_{\text{int}}, \\ \displaystyle\sum_{j \in \mathcal{I}(a_k)} d_{kj}(u_x^j(a_k) - \alpha_j \theta^j(a_k)) = 0, \quad a_k \in V_{\text{int}}, \\ \displaystyle\sum_{j \in \mathcal{I}(a_k)} d_{kj} \rho_j y_x^j(a_k) = 0, \quad a_k \in V_{\text{int}}. \end{cases} \tag{5.11}$$

Let Ω denotes the set of indices of purely elastic edges attached to thermo-elastic edges in the network $\mathcal{N}$. The main results of this section are stated in the following Theorems:

Theorem 5.1. *The unbounded linear operator $\mathcal{A}_1$ generates a C_0-semigroup of contractions on $\mathcal{H}_1$.*

Proof. The same proof as the one of Theorem 2.1 implies that $\mathcal{A}_1$ is a maximal dissipative operator. Then, using Lumer-Phillips Theorem (see [16]), $\mathcal{A}_1$ generates a C_0-semigroup of contractions $(e^{t\mathcal{A}_1})_{t \geq 0}$ on $\mathcal{H}_1$. □

Theorem 5.2. *Consider the system (5.1)-(5.7) on $\mathcal{N}$. Assume that one of the following conditions holds:*

1) $\mathcal{N}$ is the graph G_1,

2) $\mathcal{N}$ is the graph $\mathcal{P}$, $\beta_j = \beta, \rho_j = 1, \forall j \in \mathcal{I}(\mathcal{P})$, and there exists $j \in \{1,3\}$ such that

$$\ell_j \neq \frac{m\pi}{\sqrt{2}\beta_j}, \quad \forall m \in \mathbb{N}^*, \tag{5.12}$$

3) $\mathcal{N}$ is the graph G_2, $\beta_j = \beta, \rho_j = 1, \forall j \in \mathcal{I}(G_2)$, and in every circuit C, for the unique $j,k \in \Omega$ such that e_j and e_k are edges of C, we have

$$\sin(\sqrt{2}\beta_j\ell_j) + \sin(\sqrt{2}\beta_k\ell_k) \neq 0. \tag{5.13}$$

Then, $i\mathbb{R} \subset \rho(\mathcal{A}_1)$ and therefore the C_0-semigroup of contractions $(e^{t\mathcal{A}_1})_{t\geq 0}$ is strongly stable.

Proof. As before $(I - \mathcal{A}_1)^{-1}$ being compact, then $\sigma(\mathcal{A}_1) = \sigma_p(\mathcal{A}_1)$. Thus, it is sufficient to prove that $\sigma_p(\mathcal{A}_1) \cap i\mathbb{R} = \emptyset$. Let $\lambda \in \mathbb{R}$ and $U = (u, v, y, z, \theta) \in D(\mathcal{A}_1)$ be such that

$$\mathcal{A}_1 U = i\lambda U,$$

equivalently, for all $j \in \mathcal{I}(\mathcal{N})$ we have (3.1)-(3.5). Using (3.1), (3.3) to eliminate v^j and z^j, we obtain (3.6)-(3.8). Further, we easily check that

$$\sum_{j\in\mathcal{I}_{\text{te}}} \kappa_j \int_0^{\ell_j} |\theta_x^j|^2 \, dx = \text{Re}(\mathcal{A}_1 U, U)_{\mathcal{H}_1} = \text{Re}(i\lambda U, U)_{\mathcal{H}_1} = 0.$$

This implies that (3.9) holds. Thus, θ^j is constant for all $j \in \mathcal{I}_{\text{te}}$.
If $\lambda = 0$ then, using (3.6), (3.8) and (3.9), we have

$$\begin{cases} u_{xx}^j = 0, & \forall j \in \mathcal{I}(\mathcal{N}), \\ u_{xx}^j - \alpha_j \theta_x^j = 0, & \forall j \in \mathcal{I}(\mathcal{N}), \\ \rho_j y_{xx}^j = 0, & \forall j \in \mathcal{I}(\mathcal{N}), \end{cases} \tag{5.14}$$

where $\alpha_j = 0, \forall j \in \mathcal{I}_e$.

Multiplying the second equation and third equation of (5.14) by $\overline{u^j}$ and $\overline{y^j}$, respectively. Then, integrating over $(0, \ell_j)$, summing over $j \in \mathcal{I}(\mathcal{N})$ and applying Green's formula, we get

$$\begin{cases} -\displaystyle\sum_{j=1}^N \int_0^{\ell_j} |u_x^j|^2 \, dx + \sum_{j=1}^N u_x^j \overline{u^j}\Big|_0^{\ell_j} + \sum_{j=1}^N \int_0^{\ell_j} \alpha_j \theta^j \overline{u_x^j} \, dx - \sum_{j=1}^N \alpha_j \theta^j \overline{u^j}\Big|_0^{\ell_j} = 0, \\ \displaystyle\sum_{j=1}^N \int_0^{\ell_j} \rho_j |y_x^j|^2 \, dx - \sum_{j=1}^N \rho_j y_x^j \overline{y^j}\Big|_0^{\ell_j} = 0. \end{cases}$$

But, using (5.4), (5.5), (5.7) and (5.9), we obtain for all $j \in \mathcal{I}(\mathcal{N})$,

$$\begin{cases} -\displaystyle\sum_{j=1}^N \int_0^{\ell_j} |u_x^j|^2 \, dx - \sum_{j\in\mathcal{I}_{\text{te}}} \int_0^{\ell_j} |\theta^j|^2 \, dx = 0, \\ \displaystyle\sum_{j=1}^N \int_0^{\ell_j} \rho_j |y_x^j|^2 \, dx = 0. \end{cases}$$

This implies that $u_x^j = y_x^j = 0$, for all $j \in \mathcal{I}(\mathcal{N})$ and $\theta^j = 0$, for all $j \in \mathcal{I}_{\text{te}}$. Again, by (5.4)-(5.7), we deduce that $u^j = y^j = 0$, for all $j \in \mathcal{I}(\mathcal{N})$. Consequently, using (3.1) and (3.3), we conclude that $U = 0$.

Now, assume that $\lambda \neq 0$. We will proceed by distinguishing different cases:

Case i. Assume that $\mathcal{N}$ is the graph G_1. Then, the proof in this case is exactly the same as the proof of Case 1 in Theorem 3.2. In fact, under the boundary conditions of Section 5, on each thermo-elastic edge e_j of (G_1), θ^j satisfies the Dirichlet boundary condition on one end and the Neumann boundary condition on the other end. While, under the boundary conditions of Section 1, θ^j satisfies the Dirichlet boundary condition on the both ends of each thermo-elastic edge e_j of (G_1). This shows that on networks like (G_1), if θ^j satisfies Dirichlet boundary condition on only one end of each thermo-elastic edge e_j, then it is enough to prove that the system is strongly stable.

Now, before proceeding the other cases (**Case ii** and **Case iii** below), remark that for an arbitrary network $\mathcal{N}$ with $\beta_j = \beta, \rho_j = 1$, for all $j \in \mathcal{I}(\mathcal{N})$, and using (3.8) and (3.9), (3.16)-(3.20) hold for all $j \in \mathcal{I}_{\text{te}}$. Then, for $\lambda \neq \pm\beta$ we have $y^j = 0$, for all $j \in \mathcal{I}_{\text{te}}$ and thus by equation (3.19) we get $u^j = 0$, for all $j \in \mathcal{I}_{\text{te}}$. Then, by (3.8), we obtain that $\theta^j = 0$, for all $j \in \mathcal{I}_{\text{te}}$. Again, by proceeding using unique continuation Theorem in [9] and iteration technique used in Case 1 of the proof of Theorem 3.2, we conclude that $u^j = v^j = y^j = z^j = 0$, for all $j \in \mathcal{I}(\mathcal{N})$. So, it is enough to treat the remaining cases (**Case ii** and **Case iii** below) with $\lambda = \pm\beta$. Without loss of generality, assume that $\lambda = \beta$.

Case ii. Assume that $\mathcal{N}$ is the path $\mathcal{P}$, composed of two exterior elastic edges and an interior thermo-elastic edge, $\beta_j = \beta, \rho_j = 1$, for all $j \in \mathcal{I}(\mathcal{P})$ and there exists $j \in \{1, 3\}$ such that (5.12) holds. Without loss of generality, assume that $\pi_1(0) = a_0, \pi_2(0) = a_1$ and $\pi_3(0) = a_2$. By differentiating (3.8) with respect to x and by using (3.9) we deduce that, for the thermoelastic edge e_2, we have

$$u_{xx}^2 = 0. \tag{5.15}$$

But, using (3.17) and dividing by $\lambda = \beta$, we also have $y^2 = iu^2$. Thus, u^2 and y^2 can be written in the following form

$$y^2 = ax + b, \quad u^2 = -iax - ib, \quad \text{for some } a, b \in \mathbb{C}. \tag{5.16}$$

Moreover, using (3.9), (5.16) and (5.9), we can write

$$\theta^2 = i\alpha_2 a. \tag{5.17}$$

For the elastic edges $\{e_1, e_3\}$, (3.6)-(3.8) becomes,

$$\begin{cases} \beta^2 u^j + u_{xx}^j + i\beta^2 y^j = 0, & \text{on } (0, \ell_j),\ j = 1, 3, \\ \beta^2 y^j + y_{xx}^j - i\beta^2 u^j = 0, & \text{on } (0, \ell_j),\ j = 1, 3. \end{cases} \tag{5.18}$$

Using (5.4)-(5.7), (5.16) and (5.17), we have the following boundary conditions:

$$\begin{cases} u^1(0) = y^1(0) = 0, \\ u^1(\ell_1) = u^2(0) = -ib,\ y^1(\ell_1) = y^2(0) = b, \\ u_x^1(\ell_1) = u_x^2(0) - \alpha_2\theta^2(0) = -i(1 + \alpha_2^2)a,\ y_x^1(\ell_1) = y_x^2(0) = a \end{cases} \tag{5.19}$$

and

$$\begin{cases} u^3(\ell_3) = y^3(\ell_3) = 0, \\ u^3(0) = u^2(\ell_2) = -ia\ell_2 - ib,\ y^3(0) = y^2(\ell_2) = a\ell_2 + b, \\ u_x^3(0) = u_x^2(\ell_2) - \alpha_2\theta^2(\ell_2) = -i(1+\alpha_2^2)a,\ y_x^3(0) = y_x^2(\ell_2) = a. \end{cases} \tag{5.20}$$

Consequently, (5.18), (5.19) and (5.20) leads to the following system

$$\Phi_x^j = A\Phi^j,\ j = 1, 3 \tag{5.21}$$

where,

$$\Phi^j = \begin{pmatrix} u^j \\ u_x^j \\ y^j \\ y_x^j \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -\beta^2 & 0 & -i\beta^2 & 0 \\ 0 & 0 & 0 & 1 \\ i\beta^2 & 0 & -\beta^2 & 0 \end{pmatrix}.$$

The solution of (5.21) is given by

$$\Phi^j = e^{Ax}\ \Phi^j(0),\ j = 1, 3. \tag{5.22}$$

But using (5.19) and (5.20), we have

$$\Phi^1(0) = \begin{pmatrix} 0 \\ u_x^1(0) \\ 0 \\ y_x^1(0) \end{pmatrix};\ \ \Phi^1(\ell_1) = \begin{pmatrix} -ib \\ -i(1+\alpha_2^2)a \\ b \\ a \end{pmatrix}, \tag{5.23}$$

and

$$\Phi^3(\ell_3) = \begin{pmatrix} 0 \\ u_x^3(\ell_3) \\ 0 \\ y_x^3(\ell_3) \end{pmatrix};\ \ \Phi^3(0) = \begin{pmatrix} -ia\ell_2 - ib \\ -i(1+\alpha_2^2)a \\ a\ell_2 + b \\ a \end{pmatrix}. \tag{5.24}$$

Since we have

$$\Phi^1(0) = e^{-A\ell_1}\Phi^1(\ell_1) \text{ and } \Phi^3(\ell_3) = e^{A\ell_3}\Phi^3(0), \tag{5.25}$$

using (5.23), (5.24) and technical computations, we obtain

$$\begin{cases} \dfrac{i\sqrt{2}\alpha_2^2 a}{4\beta}\sin(\sqrt{2}\beta\ell_1) + ia\ell_1 - ib + \dfrac{i\alpha_2^2\ell_1 a}{2} = 0, \\ \dfrac{\sqrt{2}\alpha_2^2 a}{4\beta}\sin(\sqrt{2}\beta\ell_1) - a\ell_1 + b - \dfrac{\alpha_2^2\ell_1 a}{2} = 0, \end{cases} \tag{5.26}$$

and

$$\begin{cases} \dfrac{-i\sqrt{2}\alpha_2^2 a}{4\beta}\sin(\sqrt{2}\beta\ell_3) - ia(\ell_2+\ell_3) - ib - \dfrac{i\alpha_2^2\ell_3 a}{2} = 0, \\ \dfrac{-\sqrt{2}\alpha_2^2 a}{4\beta}\sin(\sqrt{2}\beta\ell_3) + a(\ell_2+\ell_3) + b + \dfrac{\alpha_2^2\ell_3 a}{2} = 0. \end{cases} \tag{5.27}$$

Multiplying the second equation of (5.26) (and (5.27) repectively) by i, then adding the resulting equation to the first equation of (5.26) (and (5.27) repectively) we obtain,

$$\frac{-i\alpha_2^2 a}{\beta\sqrt{2}}\sin(\sqrt{2}\beta\ell_j)=0,\ \forall j\in\{1,3\}. \tag{5.28}$$

So, if there exists $j\in\{1,3\}$ such that $\ell_j\neq\dfrac{m\pi}{\sqrt{2}\beta}$, $\forall m\in\mathbb{N}^*$, we deduce that $a=0$. Consequently, by (5.26) or (5.27) we deduce that $b=0$ and hence $u^2=y^2=0$ and $\theta^2=0$. By proceeding using unique continuation Theorem in [9] and iteration technique used in Case 1 of the proof of Theorem 3.2, we deduce that $u^j=y^j=v^j=z^j=0,\ j=1,2,3$. The same procedure can be used in the case $\lambda=-\beta$ so that the desired goal holds.

Case iii. Assume that $\mathcal{N}$ is the graph G_2, $\beta_j=\beta,\rho_j=1$, for all $j\in\mathcal{I}(G_2)$ and that in every circuit C, for the unique $j,k\in\Omega$ such that e_j and e_k are edges of C, (5.13) holds. Notice that for all $j\in\mathcal{I}_{\rm te}$, we have (3.21) holds. As in the proof of **Case 2** of Theorem 3.2, our aim is to prove (5.14). This would end the proof as in the case $\lambda=0$. But since, (3.9), (3.16) and (3.18) hold, then it is enough to prove that (5.14) holds for every elastic edge. First, for a fixed circuit C of G_2 without loss of generality, we may use the parametrizations from Figure 7.

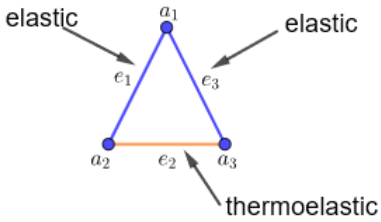

FIGURE 7. A circuit and its parametrizations: $\pi_1(0)=a_1,\ \pi_2(0)=a_2,$ and $\pi_3(0)=a_3$.

First, notice that for the thermoelastic edge e_2, (5.16) and (5.17) hold. Also, for the elastic edges $\{e_1,e_3\}$, (5.21) holds with

$$\Phi^1(\ell_1)=\begin{pmatrix}-ib\\ -i(1+\alpha_2^2)a\\ b\\ a\end{pmatrix}\ \text{and}\ \Phi^3(0)=\begin{pmatrix}-ia\ell_2-ib\\ -i(1+\alpha_2^2)a\\ a\ell_2+b\\ a\end{pmatrix}. \tag{5.29}$$

Our aim is to find $\{u^1(0),y^1(0),u^3(\ell_3),y^3(\ell_3)\}$. For that purpose, we use (5.25) to find $\Phi^1(0)$ and $\Phi^3(\ell_3)$.

Then, using (5.29) and technical computations, we have

$$\begin{cases}u^1(0)=\frac{i\sqrt{2}\alpha_2^2 a}{4\beta}\sin(\sqrt{2}\beta\ell_1)+ia\ell_1-ib+\frac{i\alpha_2^2\ell_1 a}{2},\\ \\ y^1(0)=\frac{\sqrt{2}\alpha_2^2 a}{4\beta}\sin(\sqrt{2}\beta\ell_1)-a\ell_1+b-\frac{\alpha_2^2\ell_1 a}{2}.\end{cases}$$

This implies that

$$y^1_{xx}(0)=\beta^2(y^1(0)-iu^1(0))=\frac{\alpha_2^2 a\beta}{\sqrt{2}}\sin(\sqrt{2}\beta\ell_1). \tag{5.30}$$

Similarly, using (5.29) and technical computations, we have

$$\begin{cases} u^3(\ell_3) = \frac{-i\sqrt{2}\alpha_2^2 a}{4\beta}\sin(\sqrt{2}\beta\ell_3) - ia(\ell_2+\ell_3) - ib - \frac{i\alpha_2^2\ell_3 a}{2}, \\ \\ y^3(\ell_3) = \frac{-\sqrt{2}\alpha_2^2 a}{4\beta}\sin(\sqrt{2}\beta\ell_3) + a(\ell_2+\ell_3) + b + \frac{\alpha_2^2\ell_3 a}{2}. \end{cases}$$

This implies that

$$y^3_{xx}(0) = \beta^2(y^3(0) - iu^3(0)) = \frac{-\alpha_2^2 a\beta}{\sqrt{2}}\sin(\sqrt{2}\beta\ell_3). \tag{5.31}$$

On the other hand, using (5.4)-(5.7), we have $u^1(0) = u^3(\ell_3)$ and $y^1(0) = y^1(\ell_3)$. Then,

$$y^1_{xx}(0) = y^3_{xx}(\ell_3). \tag{5.32}$$

This means that

$$\frac{\alpha_2^2 a\beta}{\sqrt{2}}\Big(\sin(\sqrt{2}\beta\ell_1) + \sin(\sqrt{2}\beta\ell_3)\Big) = 0.$$

Notice that if $\sin(\sqrt{2}\beta\ell_1) + \sin(\sqrt{2}\beta\ell_3) \neq 0$, then, $a = 0$, i.e., $\theta^2 = 0$. Using (3.8), we obtain that $u^2_x = 0$. Again, repeating the same strategy in every circuit of $\mathcal{N}$ and using the fact that (5.13) holds, we deduce that

$$\theta^j = 0, \ \forall j \in \mathcal{I}_{\text{te}}. \tag{5.33}$$

Hence, $u^j_x = 0$, for all $j \in \mathcal{I}_{\text{te}}$ and (3.22) holds. Consequently, using (5.4)-(5.7), (5.33), (3.21)-(3.22) and using iteration method from the leaves to the root, we prove that every elastic edge satisfies (3.29). This implies that $y^j_{xx} = 0$, for all $j \in \mathcal{I}_{\text{e}}$. Finally, we can proceed as the case $\lambda = 0$. This finishes the proof. The same procedure is used in the case $\lambda = -\beta$ so that the desired goal holds. □

Theorem 5.3. *Let $\mathcal{N}$ be an arbitrary network for which the operator $\mathcal{A}_1$ associated with System (5.1)-(5.7) satisfies $i\mathbb{R} \subset \rho(\mathcal{A}_1)$. If $\rho_j = 1$, for all $j \in \mathcal{I}(\mathcal{N})$, then the energy of the system decays exponentially in $\mathcal{H}_1$. In other words, there exist two positive constants M and ϵ such that*

$$\|e^{t\mathcal{A}_1}x_0\|_{\mathcal{H}_1} \leq Me^{-\epsilon t}\|x_0\|_{\mathcal{H}_1}, \quad \forall\ t > 0,\ \forall\ x_0 \in \mathcal{H}_1.$$

Proof. Same proof as the one of Theorem 4.1 holds. □

Remark 4. Examples of networks for which $i\mathbb{R} \subset \rho(\mathcal{A}_1)$ are given by Theorem 5.2.

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