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# Event-triggered boundary control of an unstable reaction diffusion PDE with input delay

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## Abstract

In chemical, biological, or population (epidemiological) processes the feedback action may be considerably delayed by time-consuming chemical measurements or biological tests. With such large delays on the control action in mind, and motivated by the fact that in some of these systems only piecewise-constant inputs can be applied between time instants at which measurements trigger changes in control, we consider the problem of event-triggered stabilization of 1-D reaction-diffusion PDE systems with input delay. The approach relies on reformulating the delay problem as an actuated transport PDE which cascades into the reaction-diffusion PDE, and on the emulation of backstepping control. The paper proposes a static (state-dependent) triggering condition which establishes the time instants at which the control value needs to be updated. It is shown that under the proposed event-triggered boundary control, there exists a minimal dwell-time (independent of the initial conditions) between two triggering times which allows to guarantee the well-posedness of the closed-loop system, and the exponential stability. The stability analysis is based on Input-to-State stability theory for PDEs and small-gain arguments. A simulation example is presented to validate the theoretical results.

*Key words:* reaction-diffusion systems, first order hyperbolic equation, backstepping control design, event-triggered control, small-gain analysis.

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## 1 Introduction

### 1.1 Literature on stabilization of reaction-diffusion PDEs with delay

Several physical phenomena arising in biology [31], chemistry [13], spatial ecology [1], etc, are described by reaction-diffusion partial differential equations (PDEs). In these applications, the feedback action may be considerably delayed by e.g., time-consuming chemical measurements or biological tests. The presence of delay in the inputs may imply situations where there is a substantial time lag between an event occurring and its impact being felt in the system. This phenomenon is often observed in processes involving propagation, transport, communication delays, distribution of chemicals in biological tissues as highlighted in [30,37,11]. In real-world scenarios, particularly during a pandemic prior to the availability of commercial rapid tests, both individuals and public health authorities would typically wait a couple of days to receive test results. This waiting period can be categorized as a common sensor delay. Subsequently, it would

take even more time for authorities to reach a consensus on a public health response. Similarly, certain medications or vaccine (control actions) require days to manifest their effects. All of these scenarios exemplify input delays. Henceforth, the delays of the inputs have to be taken into account in the control design, as they may induce instabilities in the closed-loop system [37].

The stabilization of reaction-diffusion PDEs with an arbitrary level of instability and under arbitrarily long input delay is a challenging problem, first formulated and solved in [27] using the *backstepping method for PDEs* as the stabilization of a hyperbolic (transport) PDE which cascades into the reaction-diffusion PDE. Since then, control design for delay compensation (including known or unknown constant/time-varying delays) has evolved considerably and several results have been proposed for reaction-diffusion PDEs, see, e.g., [12,14,32,23,35,4,44] and the references therein.

PDE backstepping [28] makes use of a Volterra transformation to map the PDE system into a suitable target PDE system on which one can perform Lyapunov stability analysis. An alternative method for stabilization of a parabolic PDE with input delay is *modal decomposition* [32,23,4] which relies on separating a finite-dimensional unstable part from a

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stable infinite-dimensional part of the PDE. Then, one applies the classical predictor-based techniques to the finite-dimensional system and uses for example, spectral analysis, the pole-shifting theorem, and Lyapunov-based techniques. The application of a backstepping approach in boundary control design offers several key advantages. It enables the expression of designs applicable to an entire class of systems, regardless of the specific plant's unstable eigenvalues. Moreover, it allows for extension to parameter-adaptive use, with the real-time parameter estimation, where one does not know the number of unstable eigenvalues in the unknown plant. Additionally, this approach demonstrates the potential to achieve enhanced type of convergence, including the desirable outcomes of finite or prescribed-time convergence [2,10]. Both of the aforementioned methods have been the object of further advances, which include, on the one hand, the Fredholm backstepping control for coupled parabolic PDEs with input/output delays [3], and on the other hand, finite-dimensional observer-based control design for parabolic PDEs with delays and sampled-data (using spectral reduction and LMIs-based stability conditions) [22], among others. More specifically, in [22], the authors design a finite-dimensional observer-based control, for the reaction–diffusion equation under fast-varying input (known or unknown) and output delayed measurements, thanks to the modal decomposition approach. Furthermore, this technique incorporates an LMI-based time-regularized dynamic event-triggering mechanism, designed to reduce the workload of the network, addressing an open problem that persists with the new approach developed in this paper.

### 1.2 Event-triggered control of PDEs

In some systems modeled by reaction-diffusion PDEs, only piecewise-constant inputs can be applied between time instants, yet when the arising controller is continuous (e.g., boundary controllers designed by the backstepping method). Therefore, the issue of implementing the controller in a sampled-and-hold fashion has to be carefully studied. To that end, sampled-data and *event-triggered control (ETC)* can offer suitable approaches to be adopted towards digital realizations and realistic ways for the actuation.

Event-triggered control is a computer control strategy that only updates the control value when the system needs attention while preserving stability and performance. It has been particularly relevant in controlled networked systems because of the efficient usage of computational and communication resources such as power usage, bandwidth, etc (see, e.g., [38,15,29] and the references therein in the framework of finite-dimensional systems).

ETC for PDEs has gained a lot of interest during the last few years, and several contributions have since then been proposed for wide classes of PDEs. For hyperbolic PDEs, [7] proposes an output feedback event-triggered boundary controller for 1-D linear hyperbolic systems of conservation laws through Lyapunov techniques. Using the backstepping approach, [8] and [5] introduce dynamic triggering conditions to the event-triggered boundary controllers for the stabilization of coupled  $2 \times 2$  linear hyperbolic systems by full-state feedback and output feedback, respectively. The methodology is further employed and advanced in [41] and [43], the

latter proposing an event-triggered adaptive control for coupled hyperbolic PDEs. The results on event-triggered control using backstepping-based methods have been applied to load-moving cable systems [40] and traffic flow control on connected roads [6].

For parabolic PDEs, [36] proposes a decentralized event-triggered control to reduce the number of transmitted measurements, while [24] builds on modal decomposition and comes up with sampled-data and observer-based event triggered boundary control for 1-D reaction-diffusion systems in the presence of time-varying input delays. Such a contribution includes a novel switching-based dynamic triggering condition depending on the finite modes of the estimated state and a suitable time regularization, allowing the avoidance of the Zeno phenomenon. On the other hand, using Input-to-State stability (ISS) properties for PDEs and small gain arguments, [9] proposes a backstepping-based full-state feedback ETC strategy for a 1-D reaction-diffusion system with constant parameters and Dirichlet boundary actuation. In addition, [34] proposes an observer-based event-triggered backstepping boundary control in the case of Robin boundary actuation. The ETC strategy includes a dynamic triggering condition under which it is possible to obtain a dwell-time, thus avoiding the Zeno phenomenon. Moreover, [42] extends the results of [21] and [34] and proposes a novel adaptive event-triggered boundary control for a parabolic PDE-ODE system with uncertain parameters, whereas [33] goes further with applications to the Stefan problem. Event-triggered control strategies for other classes of PDEs (including abstract infinite-dimensional systems [39]) are reported in [16] for nonlinear Korteweg–de Vries (KdV) under averaged measurements, [25] for the damped linear wave equation, and [26] for the damped linear Schrödinger equation, to mention a few.

### 1.3 Contributions

In this paper, we propose an event-triggered control scheme of the backstepping-based controller originally introduced in [27]. Our contribution extends the results of [9] to the case of delayed input, reformulates the problem as a parabolic-transport hyperbolic PDE-PDE, and provides a suitable state-dependent event-triggering condition. The choice of the space norms ( $L^2$ -norm for the reaction-diffusion PDE and supremum-norm for the hyperbolic PDE) is crucial in the design of the triggering policy and for the stability analysis, which is based on Input-to-State stability and small-gain arguments. We prove the existence of a uniform minimal dwell time (independent of the initial condition) between two consecutive triggering time instants, thus avoiding the Zeno Phenomenon. Consequently, we guarantee the existence and uniqueness of solutions to the closed-loop system and the global exponential stability of the closed-loop system.

### 1.4 Organization

The structure of the present work is as follows: Section 2 is devoted to the presentation of the class of reaction-diffusion parabolic systems, some preliminaries on stability and backstepping boundary control and the notion of existence and uniqueness of solutions. Section 3 provides the

event-triggered boundary control and the main results. Section 4 provides a numerical example to illustrate the main results. Finally, conclusions and perspectives are given in Section 5.

**Notation**  $\mathbb{R}_+$  will denote the set of nonnegative real numbers. Let  $S \subseteq \mathbb{R}^n$  be an open set and let  $A \subseteq \mathbb{R}^n$  be an open set that satisfies  $S \subseteq A \subseteq \bar{S}$ . By  $C^0(A; \Omega)$ , we denote the class of continuous functions on  $A$ , which take values in  $\Omega \subseteq \mathbb{R}$ . By  $C^k(A; \Omega)$ , where  $k \geq 1$  is an integer, we denote the class of functions on  $A$ , which takes values in  $\Omega$  and has continuous derivatives of order  $k$ .  $L^2(0, 1)$  denotes the equivalence class of Lebesgue measurable functions  $f : [0, 1] \rightarrow \mathbb{R}$  such that  $\|f\| = \left(\int_0^1 |f(x)|^2 dx\right)^{1/2} < \infty$ . Let  $u : \mathbb{R}_+ \times [0, 1] \rightarrow \mathbb{R}$  be given.  $u(t, \cdot)$  denotes the profile of  $u$  at certain  $t \geq 0$ , for all  $x \in [0, 1]$ . For an interval  $J \subseteq \mathbb{R}_+$ , the space  $C^0(J; L^2(0, 1))$  is the space of continuous mappings  $J \ni t \mapsto u(t, \cdot) \in L^2(0, 1)$ .  $H^2(0, 1)$  denotes the Sobolev space of functions  $f \in L^2(0, 1)$  with square integrable (weak) first and second-order derivatives  $f'(\cdot), f''(\cdot) \in L^2(0, 1)$ .

A function  $f : J \rightarrow \mathbb{R}$  is called right continuous on  $J$  if for every  $s \in J$  and  $\varepsilon > 0$  there exists  $\delta(\varepsilon, s) > 0$  such that for all  $\tau \in J$  with  $s \leq \tau < s + \delta(\varepsilon, s)$  it holds that  $|f(\tau) - f(s)| < \varepsilon$ . A right continuous function  $f : J \rightarrow \mathbb{R}$  is piecewise  $C^1$  on  $J$  (and we denote it as  $\mathcal{C}_{rpw}^1(J, \mathbb{R})$ ) if there exists a finite set  $B \subset J$  such that  $f$  is  $C^1$  on  $J \setminus B$  and all meaningful limits  $\lim_{h \rightarrow 0^+} (\dot{f}(s+h))$ ,  $\lim_{h \rightarrow 0^+} (\dot{f}(s-h))$ ,  $\lim_{h \rightarrow 0^+} (f(s+h))$ ,  $\lim_{h \rightarrow 0^+} (f(s-h))$  exist for all  $s \in J \setminus B$  and are finite. The sup-norm is defined by  $\|f\|_\infty = \max_{x \in J} (|f(x)|)$ .

$I_m(\cdot)$ ,  $J_m(\cdot)$  with  $m \in \mathbb{Z}$ , denote the modified Bessel and (nonmodified) Bessel functions of the first kind.

## 2 Preliminaries and problem description

Let us consider the following scalar reaction-diffusion PDE with known constant input delay  $D > 0$ :

$$u_t(t, x) = u_{xx}(t, x) + \lambda u(t, x), \quad (1)$$

$$u(t, 0) = 0, \quad (2)$$

$$u(t, 1) = U(t - D), \quad (3)$$

where  $\lambda \in \mathbb{R}$ . As in [27], we pose this delay problem as an actuated transport PDE (modeling the delay phenomenon) which cascades into the boundary of the reaction-diffusion PDE,

$$u_t(t, x) = u_{xx}(t, x) + \lambda u(t, x), \quad (4)$$

$$u(t, 0) = 0, \quad (5)$$

$$u(t, 1) = v(t, 0), \quad (6)$$

$$v_t(t, x) = \frac{1}{D} v_x(t, x), \quad (7)$$

$$v(t, 1) = U(t). \quad (8)$$

$(t, x) \in \mathbb{R}_+ \times [0, 1]$ , where  $u(t, \cdot)$  and  $v(t, \cdot)$  are respectively, the reaction-diffusion PDE and the transport PDE states at time  $t$ , with initial conditions  $u(0, x) = u_0(x)$ , and  $v(0, x) =$

$v_0(x)$ , for all  $x \in [0, 1]$  (where  $u_0, v_0$  are given functions, belonging to appropriate spaces to be specified later from Subsection 2.2).  $U(t) \in \mathbb{R}$  is the control input.

The solution of the input delay dynamics is given as  $v(t, x) = v_0(\frac{1}{D}t + x)$  for  $t \leq D(1 - x)$  and  $v(t, x) = U(t + D(x - 1))$  for  $t \geq D(1 - x)$ .

When  $U(t) \equiv 0$  and the reaction parameter  $\lambda > \pi^2$ , the system (4)–(8) is unstable; whereas for  $\lambda \leq \pi^2$ , (4)–(8) is asymptotically stable. In [27] the following continuous-time controller (nominal boundary feedback as we will call it in the sequel) was obtained by the Backstepping approach:

$$U(t) = \int_0^1 \gamma(1, y) u(t, y) dy + D \int_0^1 q(1, y) v(t, y) dy, \quad (9)$$

where

$$\gamma(x, y) = 2 \sum_{n=1}^{\infty} e^{D(\lambda - n^2 \pi^2)x} \sin(n\pi y) \int_0^1 \sin(n\pi \zeta) k(1, \zeta) d\zeta, \quad (10)$$

with

$$k(x, y) = -\lambda y \frac{I_1\left(\sqrt{\lambda(x^2 - y^2)}\right)}{\sqrt{\lambda(x^2 - y^2)}}, \quad (11)$$

on  $\mathcal{T} := \{(x, y) : 0 \leq y \leq x \leq 1\}$ , where  $I_n(\cdot)$  denotes the modified Bessel function of first kind. In addition

$$q(x, y) = -\gamma_y(x - y, 1). \quad (12)$$

**Remark 1.** In [27], the continuous-time controller (9) obtained by the Backstepping approach is used to guarantee the global exponential stability of the closed-loop system (4) in  $H^1$ -norm. More precisely, we recall that if the initial conditions are such that  $(u_0, v_0) \in L^2(0, 1) \times H^1(1, 1 + D)$ , then the system has a unique solution

$$(u(t, \cdot), v(t, \cdot)) \in C((0, \infty), L^2(0, 1) \times H^1(1, 1 + D))$$

and there exists a positive continuous function  $M : \mathbb{R}^2 \rightarrow \mathbb{R}_+$  such that

$$V(t) \leq M(\lambda, D) e^{cD} V(0) e^{-\min(2, c)t}, \quad \forall t \geq 0$$

for any  $c > 0$ , where

$$V(t) = \int_0^1 u^2(t, x) dx + \int_1^{1+D} (v^2(t, x) + v_x^2(t, x)) dx.$$

### 2.1 Emulation of the boundary controller

Under the emulation approach<sup>1</sup>, the boundary controller is perfectly known (i.e., the nominal control given in (9)). We

<sup>1</sup> We recall that in the context of event-triggered control frameworks, there are typically two approaches: emulation and co-design. The emulation approach involves having a predefined controller and then designing the event-triggering mechanism to work with it. On the other hand, the co-design approach entails the simultaneous design of both the controller and the event-triggering mechanism.

aim at stabilizing closed-loop system (4)–(8) on events while updating the controller (9) at certain time  $\{t_j\}_j$  defined by an event-triggered mechanism. To that end, we consider the following event-triggered boundary control:

$$U_d(t) = \int_0^1 \gamma(1,y)u(t_j,y)dy + D \int_0^1 q(1,y)v(t_j,y)dy, \quad (13)$$

for all  $t \in [t_j, t_{j+1})$ . The updates times  $\{t_j\}_j$  form an increasing sequence and are such that the value of the control is held constant between two successive events and is updated when some triggering condition (state-dependent triggering condition) is verified. Thus, the boundary value of the state is modified as  $v(t, 1) = U_d(t)$ , for all  $t \in [t_j, t_{j+1})$ ,  $j \geq 0$ . Note that  $U_d(t) = U(t) + d(t)$  with  $U(t)$  given by (9) and  $d$  given by:

$$\begin{aligned} d(t) &= \int_0^1 \gamma(1,y)(u(t_j,y) - u(t,y))dy \\ &+ D \int_0^1 q(1,y)(v(t_j,y) - v(t,y))dy. \end{aligned} \quad (14)$$

where  $d$  can be viewed as an actuation deviation (or input holding error).

Therefore, the control problem we aim at handling can be reformulated as follows:

$$u_t(t,x) = u_{xx}(t,x) + \lambda u(t,x), \quad (15)$$

$$u(t,0) = 0, \quad (16)$$

$$u(t,1) = v(t,0), \quad (17)$$

$$v_t(t,x) = \frac{1}{D}v_x(t,x), \quad (18)$$

$$v(t,1) = U_d(t), \quad (19)$$

with  $U_d(t)$  being defined in (13) for all  $t \in [t_j, t_{j+1})$ ,  $j \geq 0$ .

## 2.2 Well-posedness issues

Similar to [7] and [19], in this paper we deal with a linear hyperbolic equation subject to a discontinuous boundary input (see (19)). The discontinuous signal gets into the reaction-diffusion PDE through the boundary, i.e., (6). Consequently, the well-posedness study requires to extend the case in [9] along with [7,19] in order to be able to construct the solution for the closed-loop PDE-PDE system. This is done by means of the following proposition:

**Proposition 1.** *For every initial data  $v_0 \in \mathcal{C}_{rpw}^1([0,1], \mathbb{R})$  and  $u_0 \in L^2(0,1)$ , there exist unique solutions  $v, u$  to (15)-(19) with the following properties:*

- $v$  is the unique solution to (18)-(19) in the sense of characteristics on  $[0, \lim_{j \rightarrow \infty}(t_j)) \times [0, 1]$ . Moreover, for all  $t \in [0, \lim_{j \rightarrow \infty}(t_j))$ ,  $v(t, \cdot) \in \mathcal{C}_{rpw}^1([0, 1], \mathbb{R})$  and for all  $x \in [0, 1]$ ,  $v(\cdot, x) \in \mathcal{C}_{rpw}^1([0, \lim_{j \rightarrow \infty}(t_j)), \mathbb{R})$ .
- $u \in C^0([0, \lim_{j \rightarrow \infty}(t_j)); L^2(0, 1))$  with  $u(t, \cdot) \in C^2([0, 1])$  for  $t \in (0, \lim_{j \rightarrow \infty}(t_j))$  and  $u \in C^1(\tilde{I} \times [0, 1])$  where  $\tilde{I} = [0, \lim_{j \rightarrow \infty}(t_j)) \setminus \{t_j : j = 0, 1, 2, \dots\}$  which also satisfies (15)-(17) for  $t \in \tilde{I}$ .

*Proof.* Let us focus first on the  $v$ -system of (18)-(19). Following similar arguments as in [7], let us define for  $k \in \mathbb{N}$ , the interval  $\Delta_k := [kD, (k+1)D) \subset [0, \lim_{j \rightarrow \infty}(t_j))$ , where  $D$  is the time for the transport equation with velocity  $1/D$  to cross the spatial domain  $[0, 1]$ . By the method of characteristics (see e.g., [18]), the explicit solution of (18)-(19), for a given initial data  $v(kD, \cdot) \in \mathcal{C}_{rpw}^1([0, 1], \mathbb{R})$  is as follows:

$$v(t,x) = \begin{cases} v\left(kD, \frac{1}{D}(t - kD) + x\right), & kD \leq t < kD + D(1-x) \\ U_d(t + D(x-1)), & kD + D(1-x) \leq t \leq (k+1)D. \end{cases} \quad (20)$$

for all  $t \in \Delta_k$ ,  $k \in \mathbb{N}$ . It follows then, from (20), that  $v$  is well-defined on  $\Delta_k \times [0, 1]$ . Moreover, by definition of  $U_d$  (being a piecewise constant function and assuming  $U_d(t) \in \mathcal{C}_{rpw}^1(\Delta_k, \mathbb{R})$ ) we have that  $U_d(t + D(x-1))$  belongs to  $\mathcal{C}_{rpw}^1$  with respect to  $t$  and belongs to  $\mathcal{C}_{rpw}^1$  with respect to  $x$ . In addition,  $v\left(kD, \frac{1}{D}(t - kD) + x\right)$  belongs to  $\mathcal{C}_{rpw}^1$  with respect to  $t$  and belongs to  $\mathcal{C}_{rpw}^1$  with respect to  $x$ . Therefore, from (20) it holds that  $v(t, \cdot) \in \mathcal{C}_{rpw}^1([0, 1], \mathbb{R})$  for all  $t \in \Delta_k$  and  $v(\cdot, x) \in \mathcal{C}_{rpw}^1(\Delta_k, \mathbb{R})$  for all  $x \in [0, 1]$ . This yields  $v(t, 0) \in \mathcal{C}_{rpw}^1(\Delta_k, \mathbb{R})$  which constitutes an allowable boundary input for the  $u$ -system. Indeed, since it is piecewise continuous with the required regularity properties, we can apply [20, Theorem 4.10] for system (15)-(17) on the interval  $\Delta_k$ . We obtain then, that for any initial data  $u(kD, \cdot) \in L^2(0, 1)$ , there exists a unique function  $u \in C^0(\Delta_k; L^2(0, 1))$  with  $u(t, \cdot) \in C^2([0, 1])$  for  $t \in \Delta_k \setminus \{kD, (k+1)D\}$  and  $u \in C^1(\tilde{I}_k \times [0, 1])$  where  $\tilde{I}_k = \Delta_k \setminus \{t_j : j = 0, 1, 2, \dots\}$ ,  $k \in \mathbb{N}$  which also satisfies (15)-(17) for  $t \in \tilde{I}_k$ .

Therefore, by the step-by-step method, we can construct the solution for all  $[0, \lim_{j \rightarrow \infty}(t_j))$ , i.e., i) solutions  $v$  on  $[0, \lim_{j \rightarrow \infty}(t_j)) \times [0, 1]$  such that for all  $t \in [0, \lim_{j \rightarrow \infty}(t_j))$ ,  $v(t, \cdot) \in \mathcal{C}_{rpw}^1([0, 1], \mathbb{R})$  and for all  $x \in [0, 1]$ ,  $v(\cdot, x) \in \mathcal{C}_{rpw}^1([0, \lim_{j \rightarrow \infty}(t_j)), \mathbb{R})$ ; ii)  $u \in C^0([0, \lim_{j \rightarrow \infty}(t_j)); L^2(0, 1))$  with  $u(t, \cdot) \in C^2([0, 1])$  for  $t \in (0, \lim_{j \rightarrow \infty}(t_j))$  and  $u \in C^1(\tilde{I} \times [0, 1])$  where  $\tilde{I} = [0, \lim_{j \rightarrow \infty}(t_j)) \setminus \{t_j : j = 0, 1, 2, \dots\}$  which also satisfies (15)-(17) for  $t \in \tilde{I}$ . This concludes the proof. ■

## 2.3 Backstepping transformation and target system

Since we need to assess the impact of the deviation  $d(t)$  to the closed-loop system under the event-triggered implementation, we use a backstepping transformations so that we can work on a target system with desired stability properties and that exhibits the deviation  $d(t)$  at the boundary. The backstepping transformation is defined as follows:

$$w(t,x) = u(t,x) - \int_0^x k(x,y)u(t,y)dy, \quad (21)$$

$$z(t,x) = v(t,x) - D \int_0^x q(x,y)v(t,y)dy - \int_0^1 \gamma(x,y)u(t,y)dy, \quad (22)$$

$x \in [0, 1]$ , with  $\gamma(x, y)$ ,  $k(x, y)$  and  $q(x, y)$  given, respectively, by (10), (11) and (12). Hence, the system (15)-(19) is transformed into the following target system:

$$w_t(t, x) = w_{xx}(t, x), \quad (23)$$

$$w(t, 0) = 0, \quad (24)$$

$$w(t, 1) = z(t, 0), \quad (25)$$

$$z_t(t, x) = \frac{1}{D} z_x(t, x), \quad (26)$$

$$z(t, 1) = d(t), \quad (27)$$

with initial conditions

$$w_0(x) = u_0(x) - \int_0^x k(x, y) u_0(y) dy, \quad (28)$$

$$z_0(x) = v_0(x) - \int_0^1 \gamma(x, y) u_0(y) dy - D \int_0^x q(x, y) v_0(y) dy. \quad (29)$$

Notice that when  $d(t) = 0$ , one has that the target system is globally exponential stable.

It is worth recalling that the backstepping transformation (21)-(22) is invertible. The inverse transformation is given by

$$u(t, x) = w(t, x) + \int_0^x l(x, y) w(t, y) dy, \quad (30)$$

$$v(t, x) = z(t, x) + \int_0^1 \delta(x, y) w(t, y) dy + D \int_0^x p(x, y) z(t, y) dy, \quad (31)$$

where

$$l(x, y) = -\lambda y \frac{J_1\left(\sqrt{\lambda(x^2 - y^2)}\right)}{\sqrt{\lambda(x^2 - y^2)}}, \quad (32)$$

on  $\mathcal{T} := \{(x, y) : 0 \leq y \leq x \leq 1\}$  where  $J_n(\cdot)$  denotes the Bessel function of first kind,

$$\delta(x, y) = 2 \sum_{n=1}^{\infty} e^{-Dn^2 \pi^2 x} \sin(n\pi y) \int_0^1 \sin(n\pi \zeta) l(1, \zeta) d\zeta, \quad (33)$$

and

$$p(x, y) = -\delta_y(x - y, 1). \quad (34)$$

Using the inverse transformation, we can rewrite (13) and (14) as a function of the states  $w$  and  $z$ , i.e.,

$$U_d(t) = \int_0^1 \delta(1, y) w(t_j, y) dy + D \int_0^1 p(1, y) z(t_j, y) dy, \quad (35)$$

and

$$d(t) = \int_0^1 \delta(1, y) (w(t_j, y) - w(t, y)) dy + D \int_0^1 p(1, y) (z(t_j, y) - z(t, y)) dy \quad (36)$$

for all  $t \in [t_j, t_{j+1})$ ,  $j \geq 0$ .

### 3 Event-triggered boundary control and main results

In this section we introduce the event-triggered boundary control and the main results: avoidance of the Zeno phenomenon and the exponential stability of the event-triggered controlled system. The event-triggered boundary control considered in this paper involves a triggering condition (which determines the time instant at which the controller needs to be updated) and the backstepping boundary feedback which is applied as Zero-Order Hold. The proposed event-triggering condition is based on the evolution of the magnitude of the actuation deviation and the evolution of the norms (on suitable functional spaces) of both the reaction-diffusion and transport PDE states.

#### 3.1 Definition of the event-triggered boundary control

Let  $\beta > 0$  be a design parameter and define the following set:

$$E(t_j) := \{t > t_j : |d(t)| > \beta \max_{t_j \leq s \leq t} (\|w(s, \cdot)\|_{L^2}) + \beta \max_{t_j \leq s \leq t} (\|z(s, \cdot)\|_{\infty})\} \quad (37)$$

where  $w(t, \cdot)$  and  $z(t, \cdot)$  are the solution of (23)-(27) for all  $t \geq t_j$  and  $d(t)$  is defined by (36).

The event-triggered boundary control is defined by considering the following components:

I) (The event-triggering condition) The times of the events  $t_j \geq 0$  with  $t_0 = 0$  form a finite or countable set of times which is determined by the following rules for some  $j \geq 0$ :

- a) if  $E(t_j) = \emptyset$  then the set of the times of the events is  $\{t_0, \dots, t_j\}$ .
- b) if  $E(t_j) \neq \emptyset$ , then the next event time is given by:

$$t_{j+1} := \inf E(t_j). \quad (38)$$

II) (the control action) The boundary feedback law,

$$U_d(t) = \int_0^1 \delta(1, y) w(t_j, y) dy + D \int_0^1 p(1, y) z(t_j, y) dy, \quad (39)$$

for all  $t \in [t_j, t_{j+1})$ .

**Remark 2.** Notice that in definition of the event-triggering condition in (37)-(38), we use the  $L_{\infty}$ - norm for the transport PDE subsystem (26)-(27). It is worth recalling that in [27], a Lyapunov-based stability analysis is performed using the  $H^1$ -norm for the transport PDE subsystem (instead of the  $L^2$ -norm) mainly due to the unboundedness of the operator in the interconnection parabolic-transport PDEs. One may indeed have trace terms that cannot be estimated using the  $L^2$ -norm. Therefore,  $H^1$ -norm turned out to be suitable in that work (see Remark 1). Nevertheless, in this event-triggered framework, working with Lyapunov-based techniques using the  $H^1$ -norm may not be appropriate as this needs to have regularity on the initial data and solutions to be at least absolutely continuous. This may not be possible in the present setting as we deal with piecewise-constant input signals in

the transport PDE and discontinuities propagating through the spatial domain. Therefore, a  $L_\infty$ - norm is suitable for the transport PDE subsystem, thanks to which it is possible: i) to rely on the required regularity for the analysis for the well-posedness of the overall closed-loop system (as established in Subsection 2.2), ii) to obtain a suitable ISS estimate allowing a subsequent small-gain-based stability analysis (see Section 3.3).

### 3.2 Avoidance of the Zeno phenomenon and well-posedness

**Theorem 1.** *Consider the closed-loop system (15)-(19) with the event-triggered boundary control (37)-(39) with  $\beta > 0$  be given. Then, there exists a minimal dwell-time between two triggering times, i.e. there exists a constant  $\tau > 0$  (independent of the initial conditions  $u_0, v_0$ ) such that  $t_{j+1} - t_j \geq \tau$ , for all  $j \geq 0$ .*

*Proof.* Let us focus on the deviation of actuation given in (36), expressed in terms of the dynamics of the target system (23)-(27) and the kernels of the inverse transformation. Proposition 1 in conjunction with the backstepping transformations (21)-(22) allow to assert that target system is well-posed as  $d(t)$  can be proved to belong to  $\mathcal{C}_{r,pw}^1([0, \lim_{j \rightarrow \infty}(t_j)], \mathbb{R})$ . Following similar arguments as in [19, Section 3], it can be further proved that the following differential equation holds, for  $t \in (t_j, t_{j+1})$ ,  $j \geq 0$ :

$$\begin{aligned} \dot{d}(t) &= - \int_0^1 \delta(1,y) w_t(t,y) dy - D \int_0^1 p(1,y) z_t(t,y) dy \\ &= - \int_0^1 \delta(1,y) w_{yy}(t,y) dy - \int_0^1 p(1,y) z_y(t,y) dy \\ &= - \delta(1,1) w_y(t,1) + \delta(1,0) w_y(t,0) + \delta_y(1,1) w(t,1) \\ &\quad - \delta_y(1,0) w(t,0) - p(1,1) z(t,1) + p(1,0) z(t,0) \\ &\quad - \int_0^1 \delta_{yy}(1,y) w(t,y) dy + \int_0^1 p_y(1,y) z(t,y) dy. \end{aligned} \quad (40)$$

Knowing that  $w(t,0) = 0$ , and from (33)-(34), one has  $\delta(1,1) = \delta(1,0) = 0$  and  $-\delta_y(1,1) = p(1,0)$ . Hence, from (40), we get

$$\begin{aligned} \dot{d}(t) &= -p(1,1)d(t) - \int_0^1 \delta_{yy}(1,y) w(t,y) dy \\ &\quad + \int_0^1 p_y(1,y) z(t,y) dy. \end{aligned} \quad (41)$$

Moreover, the following inequality holds for  $t \in (t_j, t_{j+1})$ :

$$\begin{aligned} |\dot{d}(t)| &\leq a_0 |d(t)| + \int_0^1 |\delta_{yy}(1,y) w(t,y)| dy \\ &\quad + \int_0^1 |p_y(1,y) z(t,y)| dy. \end{aligned} \quad (42)$$

with

$$a_0 := p(1,1) = \frac{\lambda}{2} + \frac{\lambda^2}{8}, \quad (43)$$

which can be computed by using (32) and (34) together with the fact that  $\frac{d}{d\rho}(\rho^{-1}J_1(\rho)) = -\rho^{-1}J_2(\rho)$  and  $\lim_{\rho \rightarrow 0} \frac{J_n(\rho)}{\rho^n} = \frac{1}{2^n n!}$ .

Using the absolute continuity of  $d(t)$  on  $(t_j, t_{j+1})$ , we get, for all  $t \in [t_j, t_{j+1})$

$$\begin{aligned} |d(t)| &\leq \exp(a_0(t-t_j)) |d(t_j)| + \int_{t_j}^t \exp(a_0(t-s)) \\ &\quad \times \left( \int_0^1 |\delta_{yy}(1,y) w(s,y)| dy + \int_0^1 |p_y(1,y) z(s,y)| dy \right) ds. \end{aligned} \quad (44)$$

Using the Cauchy-Schwarz inequality and the fact  $d(t_j) = 0$ , we have from (44) the following estimate:

$$\begin{aligned} |d(t)| &\leq \left( \int_0^1 |p_y(t,y)| dy \right) \int_{t_j}^t \exp(a_0(t-s)) \|z(s,\cdot)\|_\infty ds \\ &\quad + \|\delta_{yy}(1,\cdot)\|_{L^2} \int_{t_j}^t \exp(a_0(t-s)) \|w(s,\cdot)\|_{L^2} ds. \end{aligned} \quad (45)$$

Moreover, it holds for all  $t \in [t_j, t_{j+1})$

$$\begin{aligned} |d(t)| &\leq \Gamma_{a_0}(t-t_j) \left( a_1 \max_{t_j \leq s \leq t} (\|z(s,\cdot)\|_\infty) \right. \\ &\quad \left. + a_2 \max_{t_j \leq s \leq t} (\|w(s,\cdot)\|_{L^2}) \right), \end{aligned} \quad (46)$$

where

$$\Gamma_{a_0}(s) := \frac{1}{a_0} (\exp(a_0(s)) - 1) > 0, \quad (47)$$

$$a_1 := \int_0^1 |p_y(1,y)| dy, \quad (48)$$

$$a_2 := \|\delta_{yy}(1,\cdot)\|_{L^2}. \quad (49)$$

Using (46) and assuming that an event is triggered at  $t = t_{j+1}$ , we have

$$\begin{aligned} |d(t_{j+1})| &\leq \Gamma_{a_0}(t_{j+1}-t_j) \left( a_1 \max_{t_j \leq s \leq t_{j+1}} (\|z(s,\cdot)\|_\infty) \right. \\ &\quad \left. + a_2 \max_{t_j \leq s \leq t_{j+1}} (\|w(s,\cdot)\|_{L^2}) \right), \end{aligned} \quad (50)$$

which, together with the definition of the event-triggering condition (37)-(38), yield the following inequality:

$$\begin{aligned} &\beta \left( \max_{t_j \leq s \leq t_{j+1}} (\|z(s,\cdot)\|_\infty) + \max_{t_j \leq s \leq t_{j+1}} (\|w(s,\cdot)\|_{L^2}) \right) \\ &\leq \Gamma_{a_0}(t_{j+1}-t_j) \left( a_1 \max_{t_j \leq s \leq t_{j+1}} (\|z(s,\cdot)\|_\infty) \right. \\ &\quad \left. + a_2 \max_{t_j \leq s \leq t_{j+1}} (\|w(s,\cdot)\|_{L^2}) \right), \end{aligned} \quad (51)$$

Therefore, from (51), we obtain

$$0 < \frac{\beta}{\max\{a_1, a_2\}} \leq \Gamma_{a_0}(t_{j+1}-t_j). \quad (52)$$

Using the definition (47) and from (52), we can conclude, for all  $j \geq 0$

$$t_{j+1} - t_j \geq \frac{1}{a_0} \ln \left( 1 + \frac{a_0 \beta}{\max\{a_1, a_2\}} \right) =: \tau > 0, \quad (53)$$

which is a minimal dwell-time (independent on the initial conditions). This concludes the proof. ■

Theorem 1 allows to conclude that  $\lim_{j \rightarrow +\infty} (t_j) = +\infty$  and therefore we can apply Proposition 1 to finally get the following well-posedness result of the closed-loop system (15)-(19).

**Corollary 1.** *For every initial data  $v_0 \in \mathcal{C}_{rpw}^1([0, 1], \mathbb{R})$  and  $u_0 \in L^2(0, 1)$ , there exist unique solutions  $v, u$  to (15)-(19) with the following properties:*

- $v$  is the unique solution to (18)-(19) on  $\mathbb{R}_+ \times [0, 1]$ . Moreover, for all  $t \in \mathbb{R}_+$ ,  $v(t, \cdot) \in \mathcal{C}_{rpw}^1([0, 1], \mathbb{R})$  and for all  $x \in [0, 1]$ ,  $v(\cdot, x) \in \mathcal{C}_{rpw}^1(\mathbb{R}_+, \mathbb{R})$ .
- $u \in C^0(\mathbb{R}_+; L^2(0, 1))$  with  $u(t, \cdot) \in C^2([0, 1])$  for  $t \in \mathbb{R}_+$  and  $u \in C^1(I \times [0, 1])$  where  $I = \mathbb{R}_+ \setminus \{t_j : j = 0, 1, 2, \dots\}$  which also satisfies (15)-(17) for  $t \in I$ .

*Proof.* It is an immediate consequence of Proposition 1 and Theorem 1 (which guarantees that no Zeno solution can appear). ■

### 3.3 Stability result

In this section, we derive the exponential stability result for the closed-loop system (15)-(19). To that end, we seek an Input-to-State stability property of the target system (23)-(27) with respect to the deviation  $d(t)$ , and we follow small-gain arguments.

**Theorem 2.** *Let  $\beta > 0$  be a design parameter (involved in the triggering condition (38)) that is selected in such a way that the following condition is fulfilled:*

$$\beta < \frac{\exp(-D)}{\left(1 + \frac{\sqrt{3}}{3}\right)}. \quad (54)$$

*Then, the closed-loop system (15)-(19) with event-triggered boundary control (38)-(39) is globally exponentially stable. More specifically, there exist constants  $M, \xi > 0$  such that:*

$$\|u(t, \cdot)\|_{L^2} + \|v(t, \cdot)\|_{\infty} \leq M \exp(-\xi t) (\|u_0\|_{L^2} + \|v_0\|_{\infty}), \quad (55)$$

for all  $t \geq 0$ .

*Proof.* By virtue of condition (54), there exist constants  $\varepsilon, \mu > 0$ , such that

$$\beta \left( \frac{1}{\sqrt{3}} (1 + \varepsilon)^3 \exp(\mu D) + (1 + \varepsilon)^2 \exp(\mu D) \right) \exp(D) < 1. \quad (56)$$

The existence of  $\varepsilon$ , and  $\mu > 0$  is guaranteed since the function  $h_1(\varepsilon, \mu) := \beta \left( \frac{1}{\sqrt{3}} (1 + \varepsilon)^3 \exp(\mu D) + (1 + \varepsilon)^2 \exp(\mu D) \right) \exp(D)$  is continuous at  $(0, 0)$  and satisfies  $h_1(0, 0) < 1$ . Condition (56), in turn, implies the following condition:

$$\beta \left( (1 + \varepsilon)^2 \exp(\mu D) \right) \exp(D) < 1. \quad (57)$$

Applying the results in [20, Chapters 3.2 and 6.4], the following ISS estimates for the solutions of the target system (23)-(27) can be obtained:

$$\|w(t, \cdot)\|_{L^2} \leq \exp(-\pi^2 t) \|w_0\|_{L^2} + \frac{1}{\sqrt{3}} \max_{0 \leq s \leq t} (|z(s, 0)|), \quad (58)$$

and

$$\|z(t, \cdot)\|_{\infty} \leq \exp(-\mu(t - D)) \exp(D) \|z_0\|_{\infty} + \exp(\mu D) \exp(D) \max_{0 \leq s \leq t} (|d(s)|), \quad (59)$$

for all  $t \geq 0$  and  $\mu > 0$ . Moreover, using [20, Lemma 7.1], we guarantee that there exists  $\xi > 0$  such that the following fading memory estimates hold for all  $t \geq 0$ :

$$\|w(t, \cdot)\|_{L^2} \exp(\xi t) \leq \|w_0\|_{L^2} + \frac{1}{\sqrt{3}} (1 + \varepsilon) \max_{0 \leq s \leq t} (\|z(s, \cdot)\|_{\infty} \exp(\xi s)), \quad (60)$$

$$\|z(t, \cdot)\|_{\infty} \exp(\xi t) \leq \exp(D) \|z_0\|_{\infty} + \exp(\mu D) \exp(D) (1 + \varepsilon) \max_{0 \leq s \leq t} (|d(s)| \exp(\xi s)). \quad (61)$$

We define the following quantities for all  $t \geq 0$ :

$$\|w\|_{[0, t]} := \max_{0 \leq s \leq t} (\|w(s, \cdot)\|_{L^2} \exp(\xi s)), \quad (62)$$

$$\|z\|_{[0, t]} := \max_{0 \leq s \leq t} (\|z(s, \cdot)\|_{\infty} \exp(\xi s)). \quad (63)$$

Using (60)-(61) and the definitions (62)-(63), we get for all  $t \geq 0$ ,

$$\|w\|_{[0, t]} \leq \|w_0\|_{L^2} + \frac{1}{\sqrt{3}} (1 + \varepsilon) \|z\|_{[0, t]}, \quad (64)$$

$$\|z\|_{[0, t]} \leq \exp(D) \|z_0\|_{\infty} + \exp(\mu D) \exp(D) (1 + \varepsilon) \max_{0 \leq s \leq t} (|d(s)| \exp(\xi s)). \quad (65)$$

From the definition of the event-triggering condition (37)-(38), events are triggered to guarantee, for all  $t_j \geq 0$  and  $t \geq t_j$ ,

$$|d(t)| \leq \beta \max_{t_j \leq s \leq t} (\|w(s, \cdot)\|_{L^2}) + \beta \max_{t_j \leq s \leq t} (\|z(s, \cdot)\|_{\infty}). \quad (66)$$

Notice that (66) can be read as e.g.,  $|d(t)| \leq \exp(-\pi(t - t_j)) |d(t_j)| + \beta \max_{t_j \leq s \leq t} (\|w(s, \cdot)\|_{L^2}) + \beta \max_{t_j \leq s \leq t} (\|z(s, \cdot)\|_{\infty})$ , knowing that that  $|d(t_j)| = 0$ . Using again the fading memory estimate [20, Lemma 7.1] (provided  $|d(t)|$  being locally bounded which is indeed guaranteed by the triggering law), then the following inequality holds for all  $t \geq 0$ :

$$|d(t)| \leq \exp(-\xi t) |d(0)| + \beta (1 + \varepsilon) \max_{0 \leq s \leq t} (\|w(s, \cdot)\|_{L^2} \exp(-\xi(t - s))) + \beta (1 + \varepsilon) \max_{0 \leq s \leq t} (\|z(s, \cdot)\|_{\infty} \exp(-\xi(t - s))). \quad (67)$$



with  $\xi$ ,  $\varepsilon$  as in (60)-(61). Hence, we obtain the following estimate (since  $|d(0)| = 0$ ):

$$\begin{aligned} |d(t)| \exp(\xi t) &\leq \beta(1+\varepsilon) \max_{0 \leq s \leq t} (\|w(s, \cdot)\|_{L^2} \exp(\xi s)) \\ &\quad + \beta(1+\varepsilon) \max_{0 \leq s \leq t} (\|z(s, \cdot)\|_{\infty} \exp(\xi s)). \end{aligned} \quad (68)$$

Using definitions (62)-(63), we get

$$\begin{aligned} \max_{0 \leq s \leq t} (|d(s)| \exp(\xi s)) &\leq \beta(1+\varepsilon) \|w\|_{[0,t]} \\ &\quad + \beta(1+\varepsilon) \|z\|_{[0,t]}. \end{aligned} \quad (69)$$

Therefore, using (69) along with (64)-(65), we get

$$\|w\|_{[0,t]} \leq \|w_0\|_{L^2} + \frac{1}{\sqrt{3}}(1+\varepsilon) \|z\|_{[0,t]}, \quad (70)$$

and

$$\begin{aligned} \|z\|_{[0,t]} &\leq \exp(D) \|z_0\|_{\infty} \\ &\quad + \exp(\mu D) \exp(D) (1+\varepsilon)^2 \beta \|w\|_{[0,t]} \\ &\quad + \exp(\mu D) \exp(D) (1+\varepsilon)^2 \beta \|z\|_{[0,t]}. \end{aligned} \quad (71)$$

From (71) and since (57) holds, we have

$$\begin{aligned} \|z\|_{[0,t]} &\leq (1 - \beta(1+\varepsilon)^2 \phi)^{-1} \exp(D) \|z_0\|_{\infty} \\ &\quad + \beta(1+\varepsilon)^2 \phi (1 - \beta(1+\varepsilon)^2 \phi)^{-1} \|w\|_{[0,t]}, \end{aligned} \quad (72)$$

where

$$\phi := \exp((1+\mu)D). \quad (73)$$

Then,

$$\begin{aligned} \|w\|_{[0,t]} &\leq \|w_0\|_{L^2} + \frac{1}{\sqrt{3}}(1+\varepsilon) (1 - \beta(1+\varepsilon)^2 \phi)^{-1} \exp(D) \|z_0\|_{\infty} \\ &\quad + \frac{1}{\sqrt{3}} \beta(1+\varepsilon)^3 \phi (1 - \beta(1+\varepsilon)^2 \phi)^{-1} \|w\|_{[0,t]}. \end{aligned} \quad (74)$$

From (56), one has

$$\beta \frac{1}{\sqrt{3}} (1+\varepsilon)^3 \exp(\mu D) \exp(D) < 1 - \beta(1+\varepsilon)^2 \exp(\mu D) \exp(D)$$

which can be rewritten, using (73), as follows:

$$\beta \frac{1}{\sqrt{3}} (1+\varepsilon)^3 \phi < 1 - \beta(1+\varepsilon)^2 \phi.$$

Moreover, from (57) it holds that  $(1 - \beta(1+\varepsilon)^2 \phi)^{-1} > 0$ , thus one obtains

$$\begin{aligned} \beta \frac{1}{\sqrt{3}} (1+\varepsilon)^3 \phi (1 - \beta(1+\varepsilon)^2 \phi)^{-1} &< 1, \\ 1 - \beta \frac{1}{\sqrt{3}} (1+\varepsilon)^3 \phi (1 - \beta(1+\varepsilon)^2 \phi)^{-1} &> 0, \end{aligned}$$

that is,  $1 - \beta(1+\varepsilon)^3 \psi > 0$  where

$$\psi := \frac{1}{\sqrt{3}} \phi (1 - \beta(1+\varepsilon)^2 \phi)^{-1}. \quad (75)$$

Therefore, from (74), one has

$$\begin{aligned} \|w\|_{[0,t]} &\leq (1 - \beta(1+\varepsilon)^3 \psi)^{-1} \|w_0\|_{L^2} \\ &\quad + \frac{1}{\sqrt{3}} (1+\varepsilon) (1 - \beta(1+\varepsilon)^3 \psi)^{-1} \\ &\quad \times (1 - \beta(1+\varepsilon)^2 \phi)^{-1} \exp(D) \|z_0\|_{\infty}, \end{aligned} \quad (76)$$

On the other hand, from (70) and (72), we have

$$\begin{aligned} \|z\|_{[0,t]} &\leq (1 - \beta(1+\varepsilon)^2 \phi)^{-1} \exp(D) \|z_0\|_{\infty} \\ &\quad + \beta(1+\varepsilon)^2 \phi (1 - \beta(1+\varepsilon)^2 \phi)^{-1} \|w_0\|_{L^2} \\ &\quad + \frac{1}{\sqrt{3}} \beta(1+\varepsilon)^3 \phi (1 - \beta(1+\varepsilon)^2 \phi)^{-1} \|z\|_{[0,t]}, \end{aligned} \quad (77)$$

and since (56) holds, then

$$\begin{aligned} \|z\|_{[0,t]} &\leq (1 - \beta(1+\varepsilon)^3 \psi)^{-1} (1 - \beta(1+\varepsilon)^2 \phi)^{-1} \\ &\quad \times \exp(D) \|z_0\|_{\infty} + \beta(1+\varepsilon)^2 (1 - \beta(1+\varepsilon)^3 \psi)^{-1} \\ &\quad \times \phi (1 - \beta(1+\varepsilon)^2 \phi)^{-1} \|w_0\|_{L^2}. \end{aligned} \quad (78)$$

Combining (76) and (78), we get

$$\begin{aligned} \|w\|_{[0,t]} + \|z\|_{[0,t]} &\leq (1 - \beta(1+\varepsilon)^3 \psi)^{-1} \\ &\quad \times \left( 1 + \beta(1+\varepsilon)^2 \phi (1 - \beta(1+\varepsilon)^2 \phi)^{-1} \right) \|w_0\|_{L^2} \\ &\quad + (1 - \beta(1+\varepsilon)^3 \psi)^{-1} (1 - \beta(1+\varepsilon)^2 \phi)^{-1} \\ &\quad \times (1 + \frac{1}{\sqrt{3}}(1+\varepsilon)) \exp(D) \|z_0\|_{\infty}. \end{aligned} \quad (79)$$

Hence,

$$\begin{aligned} \|w(t, \cdot)\|_{L^2} + \|z(t, \cdot)\|_{\infty} &\leq M_0 \exp(-\xi t) \|w_0\|_{L^2} \\ &\quad + M_0 (1 + \frac{1}{\sqrt{3}}(1+\varepsilon)) \exp(D) \exp(-\xi t) \|z_0\|_{\infty}, \end{aligned} \quad (80)$$

with  $M_0 := (1 - \beta(1+\varepsilon)^3 \psi)^{-1} (1 - \beta(1+\varepsilon)^2 \phi)^{-1}$ ; furthermore

$$\|w(t, \cdot)\|_{L^2} + \|z(t, \cdot)\|_{\infty} \leq M_1 \exp(-\xi t) (\|w_0\|_{L^2} + \|z_0\|_{\infty}), \quad (81)$$

with  $M_1 := M_0 (1 + \frac{1}{\sqrt{3}}(1+\varepsilon)) \exp(D)$ .

Next, we use the estimates of the backstepping transformations, i.e.,

$$\|w(t, \cdot)\|_{L^2} \leq \tilde{k} \|u(t, \cdot)\|_{L^2}, \quad (82)$$

$$\|u(t, \cdot)\|_{L^2} \leq \tilde{l} \|w(t, \cdot)\|_{L^2}, \quad (83)$$

$$\|z(t, \cdot)\|_\infty \leq \tilde{\gamma}\|u(t, \cdot)\|_{L^2} + \tilde{q}\|v(t, \cdot)\|_\infty, \quad (84)$$

$$\|v(t, \cdot)\|_\infty \leq \tilde{\delta}\|w(t, \cdot)\|_{L^2} + \tilde{p}\|z(t, \cdot)\|_\infty, \quad (85)$$

with  $\tilde{k} := 1 + \left(\int_0^1 \left(\int_0^x |k(x, y)|^2 dy\right) dx\right)^{1/2}$ ,  $\tilde{l} := 1 + \left(\int_0^1 \left(\int_0^x |l(x, y)|^2 dy\right) dx\right)^{1/2}$  and  $\tilde{\gamma} := \|\gamma(x, \cdot)\|_{L^2}$ ,  $\tilde{q} := 1 + D \max_{0 \leq x \leq 1} \int_0^x |q(x, y)| dy$ ,  $\tilde{\delta} := \|\delta(x, \cdot)\|_{L^2}$  and  $\tilde{p} := 1 + D \max_{0 \leq x \leq 1} \int_0^x |p(x, y)| dy$ . Hence, from (81), along with (82)-(83) and (84)-(85), we finally obtain, for all  $t \geq 0$

$$\|u(t, \cdot)\|_{L^2} + \|v(t, \cdot)\|_\infty \leq M \exp(-\xi t) (\|u_0\|_{L^2} + \|v_0\|_\infty), \quad (86)$$

with  $M := M_1 M_2 M_3$ , where  $M_1$  is as in (81),  $M_2 := \max\{\tilde{l} + \tilde{\delta}\}$ ,  $\tilde{p}\}$  and  $M_3 := \max\{\tilde{k} + \tilde{\gamma}\}$ ,  $\tilde{q}\}$ . This concludes the proof. ■

**Remark 3.** The small-gain condition  $\beta \left(\frac{1+\sqrt{3}}{\sqrt{3}}\right) \exp(D) < 1$  from (54) is a delay-dependent condition which involves also the parameter  $\beta$  of the triggering condition (38). Notice that the larger  $D$ , the smaller  $\beta$  should be chosen to preserve the theoretical guarantees. This implies sampling faster, thus the boundary control input is updated more often. It is worth mentioning, however, that larger values of  $\beta$  can be taken (eventually violating (54)) and may be used in practice since the obtained estimates are conservative.

#### 4 Numerical simulations

We illustrate the results by considering the reaction-diffusion PDE (15)-(19) with  $\lambda = 12$ , input delay  $D = 0.5$ , and initial condition  $u_0(x) = \sum_{n=1}^3 \frac{\sqrt{2}}{n} \sin(n\pi x) + 3(x^2 - x^3)$ ,  $v_0(x) = 0$ ,  $x \in [0, 1]$ . We run simulations on a time horizon  $T = 1$ . For the numerical simulations: i) we implement an implicit Euler scheme for the parabolic subsystems (15)-(17) and (23)-(25). The discretization with respect to space and time is done with steps  $\Delta_x = 1 \times 10^{-3}$  and  $\Delta_t = 1 \times 10^{-3}$ , respectively. ii) We compute the numerical solution of the hyperbolic subsystems (18)-(19) and (26)-(27) by means of their explicit solutions, i.e.,

$$v(t, x) = \begin{cases} v\left(0, \frac{1}{D}t + x\right), & 0 \leq t < D(1-x) \\ U_d(t + D(x-1)), & t \geq D(1-x), \end{cases} \quad (87)$$

$$z(t, x) = \begin{cases} z\left(0, \frac{1}{D}t + x\right), & 0 \leq t < D(1-x) \\ d(t + D(x-1)), & t \geq D(1-x). \end{cases} \quad (88)$$

We stabilize the system on events under the event-triggered control (38)-(39) where the parameter  $\beta = 0.05$  is selected according to (54) in Theorem 2. Conditions (56)-(57) (used just in the stability analysis) are also verified with e.g.,  $\varepsilon = 0.1$  and  $\mu = 0.1$ . Since the event-triggering condition is monitored in terms of the states of the target system (23)-(27) and the kernel of the inverse transformation, their numerical solutions can also be found according to (21)-(22), along with the explicit expressions (32)-(34). In addition, using (53), we compute the minimal dwell-time  $\tau = 8.7 \times 10^{-3}$ .

Figures 1 and 2 show the numerical solution of the closed-loop system (15)-(19) with continuous-time boundary control (9) and with event-triggered control (38)-(39), respectively. Figure 3 shows the numerical solutions of the target system (23)-(27). We can observe on the right that the piecewise-continuous signal  $d(t)$  appears at the boundary and the discontinuities propagate along the spatial domain of the transport PDE. The time-evolution of control functions under the continuous and event-triggered case is shown in Figure 4. The control value is kept constant between event times and updated according to the triggering law. We obtained in total 27 updates within the considered time horizon.

#### 5 Conclusion

In this paper, an event-triggered boundary control is proposed for the stabilization of a 1-D reaction diffusion equation with input delay. The delay is treated as a transport PDE, thus the problem is reformulated as a cascade PDE-PDE controlled system. We performed emulation on the backstepping control and proposed a state dependent event-triggering mechanism. The existence of a minimal dwell-time between two triggering times is proved in order to avoid the Zeno behavior. Henceforth, we ensured the well-posedness of the closed-loop system and the global exponential stability. In future work, we may consider an observer-based event-based control for more complex coupled reaction-diffusion systems with varying coefficients and subject to input/output delays inspired by the results of [3]. Besides, we expect to extend the results of [17] to the delay compensated event-triggered gain scheduling for the reaction-diffusion system with time- and space varying reaction coefficients.

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#### References

- [1] R.S. Cantrell and C. Cosner. *Spatial ecology via reaction-diffusion equations*. John Wiley & Sons, 2004.
- [2] J.-M. Coron and H.-M. Nguyen. Null controllability and finite time stabilization for the heat equations with variable coefficients in space in one dimension via backstepping approach. *Archive for Rational Mechanics and Analysis*, 225(3):993–1023, 2017.
- [3] J. Deutscher and J. Gabriel. Fredholm backstepping control of coupled linear parabolic pdes with input and output delays. *IEEE Transactions on Automatic Control*, 65(7), 2020.
- [4] I.-A. Djebour, T. Takahashi, and J. Valein. Feedback stabilization of parabolic systems with input delay. *Mathematical Control and Related Fields*, 12(2):405–420, 2022.
- [5] N. Espitia. Observer-based event-triggered boundary control of a linear  $2 \times 2$  hyperbolic systems. *Systems & Control Letters*, 138:104668, 2020.
- [6] N. Espitia, J. Auriol, H. Yu, and M. Krstic. Traffic flow control on cascaded roads by event-triggered output feedback. *International Journal of Robust and Nonlinear Control*, 2022.

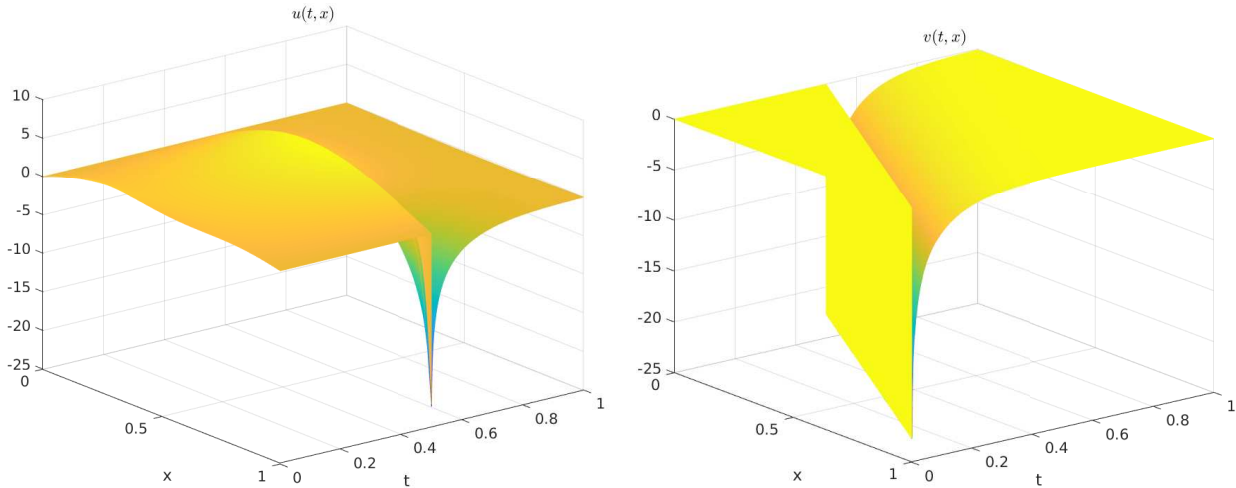


Figure 1. Numerical solutions of the closed-loop system with  $\lambda = 12$ , delay  $D = 0.5$ , initial condition  $u_0(x) = \sum_{n=1}^3 \frac{\sqrt{2}}{n} \sin(n\pi x) + 3(x^2 - x^3)$ ,  $v_0(x) = 0$ ,  $x \in [0, 1]$  and under the continuous-time boundary control (9). The parabolic PDE subsystem is depicted on the left and the transport PDE subsystem is depicted on the right.

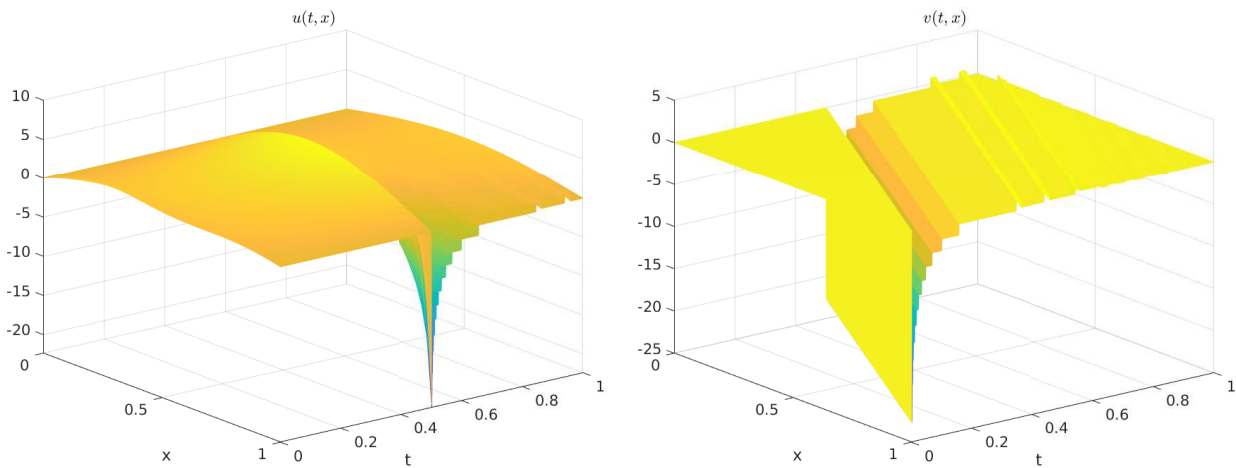


Figure 2. Numerical solutions of the closed-loop system with  $\lambda = 12$ , delay  $D = 0.5$ , initial condition  $u_0(x) = \sum_{n=1}^3 \frac{\sqrt{2}}{n} \sin(n\pi x) + 3(x^2 - x^3)$ ,  $v_0(x) = 0$ ,  $x \in [0, 1]$  and under the event-triggered control (38)-(39) with  $\beta = 0.05$ . The parabolic PDE subsystem is depicted on the left and the transport PDE subsystem is depicted on the right. The piecewise-constant signal appears at the boundary and the discontinuities propagate along the spatial domain.

[7] N. Espitia, A. Girard, N. Marchand, and C. Prieur. Event-based control of linear hyperbolic systems of conservation laws. *Automatica*, 70:275 – 287, 2016.

[8] N. Espitia, A. Girard, N. Marchand, and C. Prieur. Event-based boundary control of a linear  $2 \times 2$  hyperbolic system via backstepping approach. *IEEE Transactions on Automatic Control*, 63(8):2686–2693, 2017.

[9] N. Espitia., I. Karafyllis., and M. Krstic. Event-triggered boundary control of constant-parameter reaction–diffusion pdes: A small-gain approach. *Automatica*, 128:109562, 2021.

[10] N. Espitia, A. Polyakov, D. Efimov, and W. Perruquetti. Boundary time-varying feedbacks for fixed-time stabilization of constant-parameter reaction-diffusion systems. *Automatica*, 103, 2019.

[11] E. Fridman. *Introduction to time-delay systems: Analysis and control*. Springer, 2014.

[12] E. Fridman and Y. Orlov. Exponential stability of linear distributed parameter systems with time-varying delays. *Automatica*, 45(1):194–201, 2009.

[13] B.A. Grzybowski. *Chemistry in motion: reaction-diffusion systems for micro-and nanotechnology*. John Wiley & Sons, 2009.

[14] T. Hashimoto and M. Krstic. Stabilization of reaction diffusion equations with state delay using boundary control input. *IEEE Transactions on Automatic Control*, 61(12):4041–4047, 2016.

[15] W.P.M.H Heemels, K.H. Johansson, and P. Tabuada. An introduction to event-triggered and self-triggered control. In *2012 IEEE 51st IEEE Conference on Decision and Control (CDC)*, pages 3270–3285. IEEE, 2012.

[16] W. Kang, L. Baudouin, and E. Fridman. Event-triggered control of Korteweg–de–Vries equation under averaged measurements. *Automatica*, 123:109315, 2021.

[17] I. Karafyllis, N. Espitia, and M. Krstic. Event-triggered gain scheduling of reaction-diffusion pdes. *SIAM Journal on Control and Optimization*, 59(3):2047–2067, 2021.

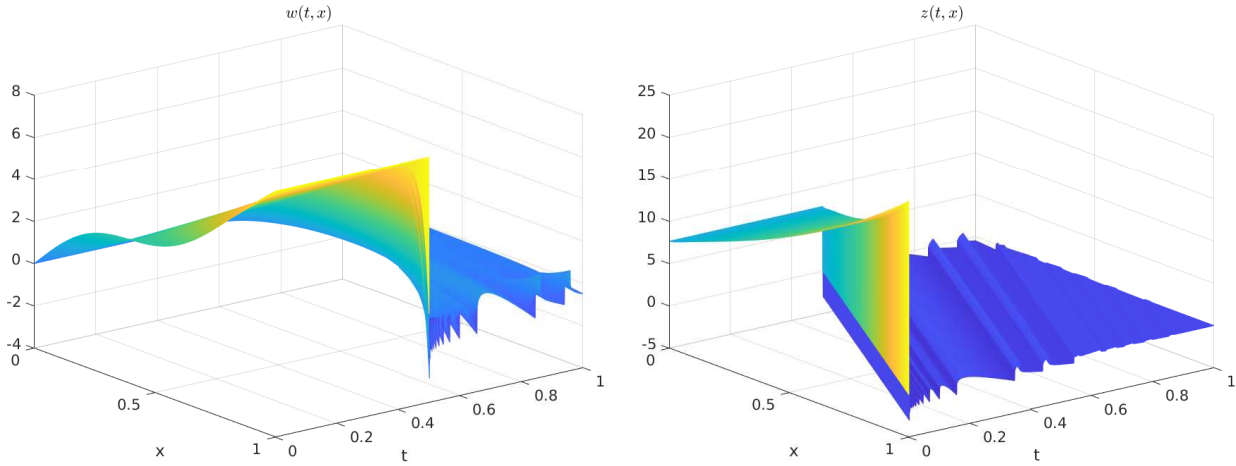


Figure 3. Numerical solutions of the target system (23)-(27). The parabolic PDE subsystem is depicted on the left and the transport PDE subsystem is depicted on the right. The piecewise-continuous signal  $d(t)$  appears at the boundary and the discontinuities propagate along the spatial domain.

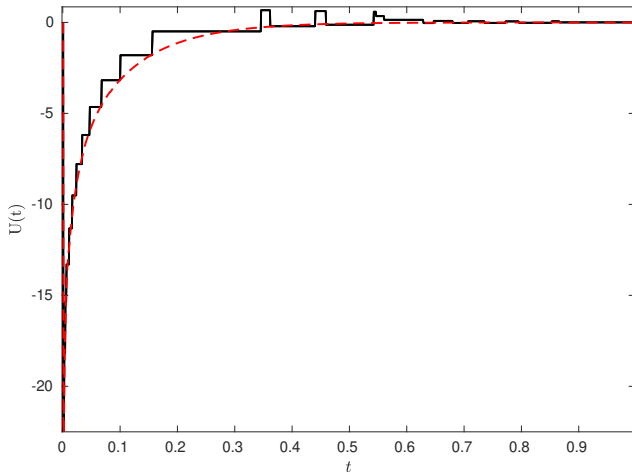


Figure 4. Time-evolution of the continuous-time boundary control (9) (red dashed line) and the event-triggered boundary control (38)-(39) (black line).

[18] I. Karafyllis and M. Krstic. On the relation of delay equations to first-order hyperbolic partial differential equations. *ESAIM: Control, Optimisation and Calculus of Variations*, 20:894–923, 2014.

[19] I. Karafyllis and M. Krstic. Sampled-data boundary feedback control of 1-D Hyperbolic PDEs with non-local terms. *Systems & Control Letters*, 17:68–75, 2017.

[20] I. Karafyllis and M. Krstic. *Input-to-State Stability for PDEs*. Springer-Verlag, London (Series: Communications and Control Engineering), 2019.

[21] I. Karafyllis, M. Krstic, and K. Chrysafi. Adaptive boundary control of constant-parameter reaction–diffusion PDEs using regulation-triggered finite-time identification. *Automatica*, 103:166–179, 2019.

[22] R. Katz and E. Fridman. Delayed finite-dimensional observer-based control of 1d parabolic PDEs via reduced-order LMIs. *Automatica*, 142(110341), 2022.

[23] R. Katz, E. Fridman, and A. Selivanov. Network-based boundary observer-controller design for 1D heat equation. In *2019 IEEE 58th Conference on Decision and Control (CDC)*, pages 2151–2156. IEEE, 2019.

[24] R. Katz, E. Fridman, and A. Selivanov. Boundary delayed observer-controller design for reaction–diffusion systems. *IEEE Transaction on Automatic Control*, 66:275 – 282, 2021.

[25] F. Koudohode, L. Baudouin, and S. Tarbouriech. Event-based control for the damped linear wave equation. *Automatica*, 146, 2022.

[26] F. Koudohode, L. Baudouin, and S. Tarbouriech. Event-based control of a damped linear Schrödinger equation. In *2022 European Control Conference (ECC)*, pages 2099–2104, 2022.

[27] M. Krstic. Control of an unstable reaction–diffusion PDE with long input delay. *Systems & Control Letters*, 58(10-11):773–782, 2009.

[28] M. Krstic. *Delay Compensation for Nonlinear, Adaptive, and PDE Systems*. Birkhäuser, 2009.

[29] T. Liu, P. Zhang, and Z.-P. Jiang. *Robust event-triggered control of nonlinear systems*. Springer, 2020.

[30] M. Malek-Zavarei and M. Jamshidi. *Time-delay systems: analysis, optimization and applications*. Elsevier Science Inc., 1987.

[31] B. Perthame. Parabolic equations in biology. In *Parabolic Equations in Biology*, pages 1–21. Springer, 2015.

[32] C. Prieur and E. Trélat. Feedback stabilization of a 1-d linear reaction–diffusion equation with delay boundary control. *IEEE Transactions on Automatic Control*, 64(4):1415–1425, 2019.

[33] B. Rathnayake and M. Diagne. Event-based boundary control of one-phase Stefan Problem: A static triggering approach. In *Proc of 2022 American Control Conference (ACC)*, 2022.

[34] B. Rathnayake, M. Diagne, N. Espitia, and I. Karafyllis. Observer-based event-triggered boundary control of a class of reaction-diffusion pdes. *IEEE Transaction on Automatic Control*, 67(6):2905–2917, 2021.

[35] H. Sano. Neumann boundary stabilization of one-dimensional linear parabolic systems with input delay. *IEEE Transactions on Automatic Control*, 63(9), 2018.

[36] A. Selivanov and E. Fridman. Distributed event-triggered control of diffusion semilinear PDEs. *Automatica*, 68:344–351, 2016.

[37] S. Sen, P. Ghosh, S.S Riaz, and D.S. Ray. Time-delay-induced instabilities in reaction-diffusion systems. *Physical Review E*, 80:046212, Oct 2009.

[38] P. Tabuada. Event-triggered real-time scheduling of stabilizing control tasks. *IEEE Transactions on Automatic Control*, 52(9):1680–1685, 2007.

- [39] M. Wakaiki and H. Sano. Event-triggered control of infinite-dimensional systems. *SIAM Journal on Control and Optimization*, 58(2):605–635, 2020.
- [40] J. Wang and M. Krstic. Adaptive event-triggered PDE control for load- moving cable systems. *Automatica*, 129, 2021.
- [41] J. Wang and M. Krstic. Event-triggered output-feedback backstepping control of sandwiched hyperbolic PDE systems. *IEEE Transactions on Automatic Control*, pages 1–1, 2021.
- [42] J. Wang and M. Krstic. Event-triggered adaptive control of a parabolic PDE-ODE cascade with piecewise-constant inputs and identification. *IEEE Transactions on Automatic Control*, 2022.
- [43] J. Wang and M. Krstic. Event-triggered adaptive control of coupled hyperbolic PDEs with piecewise-constant inputs and identification. *IEEE Transactions on Automatic Control*, pages 1–1, 2022.
- [44] S. Wang, J. Qi, and M. Diagne. Adaptive boundary control of reaction–diffusion pdes with unknown input delay. *Automatica*, 134, 2021.