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Claude Boutin, Antoine Rallu, Stephane Hans. Large scale modulation of high frequency acoustic waves in periodic porous media. *Journal of the Acoustical Society of America*, 2012, 132 (6), pp.3622-3636. 10.1121/1.4763553 . hal-04302190

HAL Id: hal-04302190

<https://hal.science/hal-04302190>

Submitted on 23 Nov 2023

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Large scale modulation of high frequency acoustic waves in periodic porous media

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(Received 9 May 2012; revised 19 September 2012; accepted 1 October 2012)

This paper deals with the description of the modulation at large scale of high frequency acoustic waves in gas saturated periodic porous media. High frequencies mean local dynamics at the pore scale and therefore absence of scale separation in the usual sense of homogenization. However, although the pressure is spatially varying in the pores (according to periodic eigenmodes), the mode amplitude can present a large scale modulation, thereby introducing another type of scale separation to which the asymptotic multi-scale procedure applies. The approach is first presented on a periodic network of inter-connected Helmholtz resonators. The equations governing the modulations carried by periodic eigenmodes, at frequencies close to their eigenfrequency, are derived. The number of cells on which the carrying periodic mode is defined is therefore a parameter of the modeling. In a second part, the asymptotic approach is developed for periodic porous media saturated by a perfect gas. Using the “multicells” periodic condition, one obtains the family of equations governing the amplitude modulation at large scale of high frequency waves. The significant difference between modulations of simple and multiple mode are evidenced and discussed. The features of the modulation (anisotropy, width of frequency band) are also analyzed.

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PACS number(s): 43.20.Jr, 43.35.Cg, 43.20.Bi, 43.20.Fn [KVVH]

Pages: 3622–3636

I. INTRODUCTION

This paper investigates the occurrence and modeling of modulation at large scale of high frequency acoustic waves in gas saturated *periodic* porous media. The interest of understanding this phenomenon lies in the fact that despite the short acoustic wavelengths, the acoustic field presents large correlation lengths.

High frequencies mean local dynamics at the pores scale and therefore absence of scale separation in the usual sense of homogenization^{1–3} because the pressure is varying spatially in the periodic microstructure. However, focusing on dynamics associated with eigenmodes defined on periodic domains (that will be specified later on), the mode amplitude can vary at large scale while the scale separation disappears for the pressure. This situation introduces another type of scale separation on which the multi-scale asymptotic method can be performed. It is remarkable that this mathematical method based on double scale, asymptotic expansions, periodicity, variational formulation, and Fredholm alternative also applies here despite the differences in the physical meaning. This idea is in the same spirit as the study proposed by Craster *et al.*⁴ in the context of composite elastic media with some difference in the theoretical implementation.

Clearly, acoustics in porous media may be seen as a peculiar case of waves in heterogeneous media. It presents nevertheless specificities that are generally not met in other media:

- (1) The solid phase is motionless because of its large stiffness and density (compared to the gas).
- (2) The gas phase is homogeneous and described by the scalar field of pressure.

This situation reduces the “technical” complexity involved in the general study of waves in periodic media (inhomogeneous phases in motion described by a vector field of displacement) without missing the physical insight. Simultaneously, it enables investigation of, in a simple (but non-trivial) manner, the main three-dimensional (3D) phenomena occurring in such periodic media. This permits pointing out the similarities and dissimilarities with the reference case of the 1D mass-spring chain.

It is also worth mentioning that a porous media consisting of a periodic network of interconnected Helmholtz resonators is a simple physical model (may be the simplest) that actually matches in practice the theoretical assumptions of mass-spring models with short range interactions. Consequently, those media may be indeed excellent candidates to perform experiments either in view of validating theory or for engineering applications of the theoretical results.

In the first part, the physical principle of this approach is introduced on a 1D network of interconnected Helmholtz resonators. Equations governing modulations carried by a given eigenmode, at frequencies close to its eigenfrequency, are derived. As the local dynamic state is involved through the periodic eigenmode, the number of cells constituting the domain of definition of the periodic mode becomes a parameter of the modeling.

In the second part, the same question is addressed regarding periodic porous media saturated by a perfect gas.

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The asymptotic approach enables us to derive the governing equation of modulation, valid in a narrow frequency band centered around each eigenfrequency. It is established that the nature of the modulation differs significantly whether simple or multiple modes. Further, the whole periodic configurations of cells (number and distribution) have to be considered. The use of the periodicity condition on “multi”-cells provides the whole family of large modulation phenomena.

II. PERIODIC NETWORK OF HELMHOLTZ RESONATORS

As an introduction to the topic, consider an idealized periodic porous medium the channels of which being constituted by interconnected Helmholtz resonators. Each resonator is made of a “chamber” connected to a much smaller duct. For the sake of simplicity, all the resonators are assumed to be identical, and their connections are actually in a single direction (see Fig. 1). Hence the medium consists of unidirectional identical and parallel channels of periodic shape, and each channel appears as a periodic chain of large pores (chamber of length l_0 and volume V) linked by small constricted ducts (of section s , volume v , and negligible length compared to l_0). Thus it is sufficient to focus on a single channel to derive the description of the propagation of acoustic wave through the whole medium.

Acoustics of this medium can be described by the usual equivalent fluid (or dynamic permeability) approach of porous media, provided that the scale separation assumption is satisfied.^{3,5,6} This latter applies for large wavelengths, corresponding to frequencies significantly lower than the eigenfrequency of the resonator f_0 . In the sequel, we focus on the frequency range $O(f_0)$, where the usual approach becomes irrelevant.

According to the classic simplified analysis of Helmholtz resonators, (i) the thermal and viscous dissipation

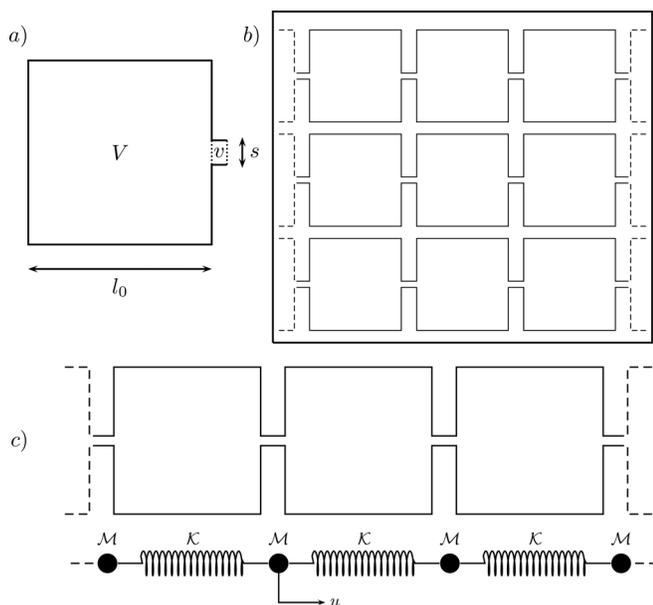


FIG. 1. (a) Unit Helmholtz resonator; (b) porous media made of 1-D interconnected resonators; (c) equivalent 1-D spring-mass chain.

effects are neglected, (ii) the gas in the chamber is compressed adiabatically and quasistatically (thus the investigated frequency range is restricted by an upper limit), and (iii) the gas in the duct suffer negligible compression. Conveniently, to describe perturbations from the equilibrium state (P^e and ρ^e are the ambient pressure and gas density at equilibrium), the motion u of the gas at the aperture of the chamber (and in the duct) is taken as acoustic variable. With these assumptions, a channel may be sketched by a line of mass-less spring \mathcal{K} of length l_0 alternating with rigid point mass \mathcal{M} (see Fig. 1). The actuator mass of the resonator is the mass of the gas in the duct, $\mathcal{M} = \rho^e v$. The resonator spring \mathcal{K} is defined from the adiabatic compressibility of the gas box: When the mass \mathcal{M} moves of u , a volume us is injected in the chamber and the perturbation of pressure is $P = \gamma P^e us/V$. Hence the mass undergoes a force Ps , from which we deduce the equivalent spring of the gas chamber

$$\mathcal{K} = \frac{Ps}{u} = \frac{\gamma P^e s^2}{V}.$$

The resonance frequency of a single resonator results from the interaction between the chamber compressibility and the duct inertia. Denoting by $\mathcal{C} = \sqrt{\gamma P^e / \rho^e}$ the sound velocity, f_0 is given by

$$\omega_0 = 2\pi f_0 = \sqrt{\frac{\mathcal{K}}{\mathcal{M}}} = \sqrt{\frac{\gamma P^e}{\rho^e}} \sqrt{\frac{s^2}{Vv}} = \mathcal{C} \sqrt{\frac{s^2}{Vv}}.$$

For the purpose of this section, we focus on 1D “Helmholtz chains.” However, using the same principles, 2D or 3D “Helmholtz periodic networks” made of identical or different resonators (i.e., chambers and/or ducts) can be described by spring-mass lattices. The dynamics of these systems involves richest phenomena, and one may refer to Craster *et al.*⁷ in whose study the description of such lattices is addressed.

A. Harmonic dynamics of a chain of resonators

Wave propagation in 1D spring-mass system has been widely studied in the literature since Brillouin and Parodi,⁸ see a recent review.⁹ Owing to the simple 1D discrete and periodic geometry, an exact dispersion equation (relating wave number and frequency) is available whatever the frequency range. However, the dispersion can be established in different manners, either considering the irreducible period Ω_0 (of size l_0 and constituted by one spring one mass), or the double cells period $\Omega_2 = \Omega_0 \cup \Omega_0$ or the multiple (e.g., p) cells period $\Omega_p = \cup_p \Omega_0$. Those different but equivalent formulations make use of an analogue of the Born–von Karman (BvK) periodic boundary conditions. Such approaches are well-known in mathematics¹⁰ and physics.¹¹

1. Dispersion relation deduced from the irreducible period Ω_0

In the purpose of studying harmonic wave propagation at the frequency ω , the motion u_n of n th mass, i.e., the mass of the n th cell Ω_0 , is written in the following form:

$$u_n = U_0 \exp(i(\omega t - \kappa_1 n l_0)), \quad (1)$$

where $\kappa_1(\omega)$ is the wave number. Note that any value $\kappa_1 l_0 + 2q\pi$, q integer, would lead to identical motions u_n . Hence, $\kappa_1 l_0$ is defined modulo- 2π , and it is sufficient to focus on the first Brillouin zone defined by $\text{Re}(\kappa_1 l_0) \in [-\pi, \pi]$ [moreover, the discrete nature of the system with massless springs avoids waves of wavelength shorter than $l_0/2$ (cf. Shannon's theorem)]. The balance of forces on the n th mass, reads

$$\mathcal{K}(u_{n+1} - u_n) - \mathcal{K}(u_n - u_{n-1}) = -\mathcal{M}\omega^2 u_n. \quad (2)$$

Then replacing u_n by its expression (1) and dividing by $\mathcal{K}u_n$ yields

$$e^{-i\kappa_1 l_0} - 2 + e^{i\kappa_1 l_0} = -\left(\frac{\omega}{\omega_0}\right)^2.$$

This relation defines the exact dispersion relation, valid for any frequency:

$$4\sin^2\left(\frac{\kappa_1(\omega)l_0}{2}\right) = \left(\frac{\omega}{\omega_0}\right)^2; \quad \text{Re}(\kappa_1 l_0) \in [-\pi, \pi]. \quad (3)$$

From Eq. (1), large wavelengths correspond to $\kappa_1 l_0 \simeq 0$, and according to Eq. (3), this situation is reached at low frequency, i.e., $\omega \ll \omega_0$. Expanding Eq. (3) near zero provides

$$|\kappa_1|l_0 = \frac{\omega}{\omega_0} \left(1 + \frac{1}{6} \left(\frac{\omega}{2\omega_0}\right)^2 + \dots\right) \quad \text{when} \quad \frac{\omega}{\omega_0} \rightarrow 0. \quad (4)$$

Considering the first term only, $|\kappa_1| \simeq \omega/\sqrt{\mathcal{K}l_0^2/\mathcal{M}}$ when $\omega \ll \omega_0$. Thus the dispersion relation matches the usual description of long wave propagation in a 1D medium of lineic elastic modulus $\mathcal{K}l_0$ and lineic mass \mathcal{M}/l_0 . In particular Eq. (1) indicates that $u_{n+1} = u_n e^{-i\kappa_1 l_0} = u_n(1 - i\kappa_1 l_0 + \dots)$, so successive masses follow almost the same motion as expected from the usual scale separation assumption. The ω^2 -corrector in Eq. (4) traduces the weak dynamic effect at cell scale and is consistent with the usual description of Rayleigh scattering.

Dispersion equation (3) also implies that the wave number κ_1 increases as ω increases, until $\omega = 2\omega_0$, for which $\kappa_1(2\omega_0) = \pi/l_0$. In that case, Eq. (1) indicates that $u_{n+1} = u_n e^{-i\kappa_1 l_0} = -u_n$, thus successive masses experience opposite motions. Obviously, this does not fit in the framework of usual scale separation assumption.

For $\omega > 2\omega_0$, $\kappa_1(\omega)$ becomes imaginary, then the dynamic phenomena are attenuated and confined in a length of the order or smaller (as ω increases) than l_0 . In the sequel of this section, only frequencies $\omega \leq 2\omega_0$ will be considered.

2. Double cells period Ω_2

Reconsider now the wave propagation problem set on the double cells period Ω_2 constituted of two masses, two springs, thus of length $l = 2l_0$. Denote by u_n and v_n , the motions of the two successive masses of the n th period Ω_2 (see Fig. 2). By convention, the mass at the left extremity on

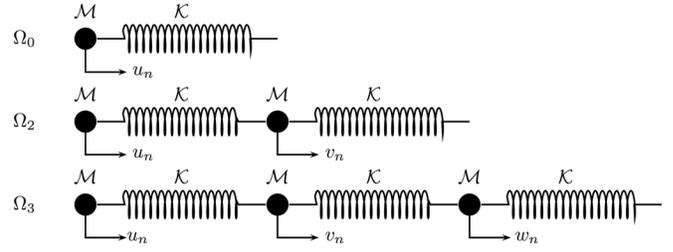


FIG. 2. Definition of variables for simple and multiple cells period. (a) Irreducible period Ω_0 , (b) double cells period Ω_2 , (c) triple cells period Ω_3 .

the n th period is chosen as *leading node*, and its motion u_n is the *leading variable*, while the motion v_n of the “inner” mass (inner node of Ω_2) involves the internal dynamic in the period. Using the double cells as reference period leads us to consider u_n under the form (as in the preceding text, the first Brillouin zone is sufficient for a comprehensive analysis)

$$u_n = U_0 \exp(i(\omega t - \kappa_2 n(2l_0))); \quad \text{Re}(2\kappa_2 l_0) \in [-\pi, \pi]. \quad (5)$$

The dynamic equilibrium of the inner mass experiencing motion v_n reads

$$\mathcal{K}(u_{n+1} - v_n) - \mathcal{K}(v_n - u_n) = -\mathcal{M}\omega^2 v_n,$$

therefore,

$$v_n = \frac{u_{n+1} + u_n}{2 - (\omega/\omega_0)^2}. \quad (6)$$

Similarly, the dynamic equilibrium of the mass at the n th leading node experiencing the motion u_n gives the following relation:

$$(v_n - 2u_n + v_{n-1}) = -(\omega/\omega_0)^2 u_n, \quad (7)$$

and substituting v_n by its expression in function of u_n and u_{n+1} , we get

$$(u_{n+1} - 2u_n + u_{n-1}) = \left(\frac{\omega}{\omega_0}\right)^2 \left(\left(\frac{\omega}{\omega_0}\right)^2 - 4\right) u_n. \quad (8)$$

Then replacing u_n by its expression (5) and dividing by u_n provides the dispersion relation for $\kappa_2(\omega)$:

$$4\sin^2(\kappa_2(\omega)l_0) = \mathcal{X}(4 - \mathcal{X}), \quad \mathcal{X} = (\omega/\omega_0)^2; \quad \text{Re}(2\kappa_2 l_0) \in [-\pi, \pi]. \quad (9)$$

Because the same mechanical systems are described by Ω_0 and Ω_2 , both descriptions must be identical. Therefore as the n th leading node of the Ω_2 -chain coincides with the $2n$ th node of the Ω_0 -chain, the identification of motions u_n (Ω_2 period) and u_{2n} (Ω_0 period) gives

$$(\kappa_2 - \kappa_1)(2l_0) = 2q\pi; \quad q \text{ integer.}$$

Now, considering the restricted intervals of variation of κ_1 and κ_2 defined by the first Brillouin zones, we obtain

$$\begin{aligned} \kappa_2 &= \kappa_1 \quad \text{when } |\kappa_1|l_0 \leq \pi/2; \\ |\kappa_2| &= |\kappa_1| - \pi/l_0 \quad \text{when } \pi/2 \leq |\kappa_1|l_0 \leq \pi. \end{aligned}$$

This result is consistent with the dispersion equations (3) and (9) governing κ_1 and κ_2 , respectively. Indeed it is easy to check that if $4 \sin^2(\theta) = a$ then $4 \sin^2(2\theta) = a(a-4)$. Large scale evolution of the motions of the *leading* nodes correspond to $\kappa_2 l_0 \simeq 0$, see Eq. (5). Dispersion equation (9) shows that such situations arise in two frequency ranges:

- (1) $\omega \ll \omega_0$, hence $\kappa_2 = \kappa_1$. This is the low frequency case already encountered in the description based on Ω_0 .
- (2) $\omega \simeq 2\omega_0$, hence $|\kappa_1| \simeq \pi/l_0$ thus $|\kappa_2| = |\kappa_1| - \pi/l_0$. Expanding eq. (9) around $2\omega_0$ gives

$$(\kappa_2 l_0)^2 \simeq 4 - \left(\frac{\omega}{\omega_0}\right)^2 \quad \text{when } \omega \rightarrow 2\omega_0. \quad (10)$$

From Eq. (5), $u_{n+1} = u_n(1 - i\kappa_2(2l_0) + \dots)$, so that successive *leading* masses follow almost the same motion, while Eq. (7) indicates $v_n \simeq -(u_n + u_{n+1})/2$, meaning that the motions of the inner and the leading masses are almost opposite.

As expected, we find again the same description using Ω_0 or Ω_2 . However, the double cells approach enables us to identify a large scale phenomenon at high frequency, *included but masked* in the simple cell approach. The interpretation lies in the fact that $2\omega_0$ is the eigenfrequency of the double cells period Ω_2 with periodic boundary conditions [this becomes obvious by setting $u_n = u_{n+1}$ in Eq. (6) and $v_n = v_{n+1}$ in Eq. (7), the eigenmode shape being defined by $v_n = -u_n$]. Therefore the large scale phenomenon driven by $\kappa_2(\omega \simeq 2\omega_0)$ corresponds to large modulations carried by the periodic eigenmode. This means that the scale separation concerns the *amplitude* of the mode instead of the amplitude of the variable itself, as thought usually. For this reason, the situation of large modulation of high frequency waves departs from the usual sense given to the scale separation assumption.

3. Multicells period Ω_p

The same analysis is carried on with multiple cells period Ω_p made of p masses and p springs of length $l = pl_0$. The n th period contains the leading node (of motion u_n), and $p-1$ inner nodes (see Fig. 2 with $p=3$). Following the same method, we write

$$u_n = U_0 \exp(i(\omega t - \kappa_p n(pl_0))); \quad \text{Re}(p\kappa_p l_0) \in [-\pi, \pi].$$

From the balance equations of the p nodes, we deduce the following dispersion relation:

$$\begin{aligned} 4\sin^2\left(\frac{\kappa_p(\omega) pl_0}{2}\right) &= -(\beta^{-p/2} - \beta^{p/2})^2; \\ \beta &= -\frac{2 - \left(\frac{\omega}{\omega_0}\right)^2 + i\frac{\omega}{\omega_0} \sqrt{4 - \left(\frac{\omega}{\omega_0}\right)^2}}{2}; \quad |\beta| = 1. \end{aligned} \quad (11)$$

The usual long wave propagation is obtained for $\kappa_p pl_0 \simeq 0$ and $\omega \ll \omega_0$.

The modulation situations correspond to $\kappa_p \simeq 0$ for higher frequencies bands. To identify those bands, note from Eq. (11) that κ_p vanishes for $\beta^p = 1$, i.e., for $\beta_J = e^{\pm i2\pi J/p}$, $0 \leq J \leq p/2$. From the expression of β , we deduce that $\kappa_p(\omega_J) = 0$ for the series of frequency ω_J defined by:

$$\begin{aligned} \text{Re}(\beta_J) &= \cos(2\pi J/p) = 1 - \frac{1}{2} \left(\frac{\omega_J}{\omega_0}\right)^2, \\ \text{i.e., } 4\sin^2(\pi J/p) &= \left(\frac{\omega_J}{\omega_0}\right)^2. \end{aligned} \quad (12)$$

In other words, coming back to the Ω_0 -based dispersion equation (3):

$$\begin{aligned} \kappa_p(\omega_J) &= 0 \quad \text{for } \left(\frac{\omega_J}{\omega_0}\right)^2 = 4\sin^2(\pi J/p) \quad \text{means} \\ \kappa_1(\omega_J) &= \pm \frac{\pi 2J}{l_0 p}; \quad 0 \leq J \leq p/2. \end{aligned}$$

For these specific values of $\kappa_1(\omega_J)$, according to Eq. (1), the motions in the chain are Ω_p periodic. Therefore frequencies ω_J are the eigenfrequency of the period Ω_p (with periodic conditions) and the motions of the inner nodes define the eigenmodes. Note also that $\omega_J = \omega_{-J}$, and therefore all the modes are double except when p is even, the simple mode at $\omega_{p/2} = 2\omega_0$.

Remark: In this specific 1D case, the periodic boundary conditions applied to Ω_p for the modes J result in a “ $e^{\pm i2\pi J/(pl_0)}$ -phase shifted” boundary condition between two successive nodes of Ω_0 . For instance, the periodic eigenmode of Ω_2 corresponds to “antiperiodic” conditions in Ω_0 .

As an example of application, consider the triple cell period Ω_3 ($p=3$, $l=3l_0$). It exists only a double mode ($J=1$) at the frequency: $\omega_1/\omega_0 = 2\sin^2(\pi/3) = \sqrt{3}$ (the normalized mode shapes are, with evident notations for the three mass period, $(2/\sqrt{3}, -1/\sqrt{6}, -1/\sqrt{6})$ and $(0, 1/\sqrt{2}, -1/\sqrt{2})$). Therefore when $\omega \rightarrow \omega_1$, $\kappa_3(\omega) \rightarrow 0$, $\beta(\omega) \rightarrow \beta_1 = e^{i2\pi/3}$. Then expanding Eq. (11) around ω_1 gives

$$\begin{aligned} (\kappa_3(\omega) 3l_0)^2 &= -(\beta^{-3}(\omega) - 2 + \beta^3(\omega)) \\ &\simeq -9\left(\frac{\beta(\omega)}{\beta_1} - 1\right)^2. \end{aligned}$$

Furthermore, from the expression of β :

$$\frac{\beta(\omega)}{\beta_1} - 1 \simeq 2i \frac{\omega - \omega_1}{\omega_0},$$

and finally the modulation length of this mode reads

$$(\kappa_3 l_0)^2 \simeq 4 \left(\frac{\omega}{\omega_0} - \sqrt{3}\right)^2 \quad \text{when } \omega \rightarrow \omega_1 = \sqrt{3}\omega_0. \quad (13)$$

Figure 3 displays the fundamental dispersion curve established for single cell Ω_0 i.e., $\kappa_1(\omega)$, and the dispersion

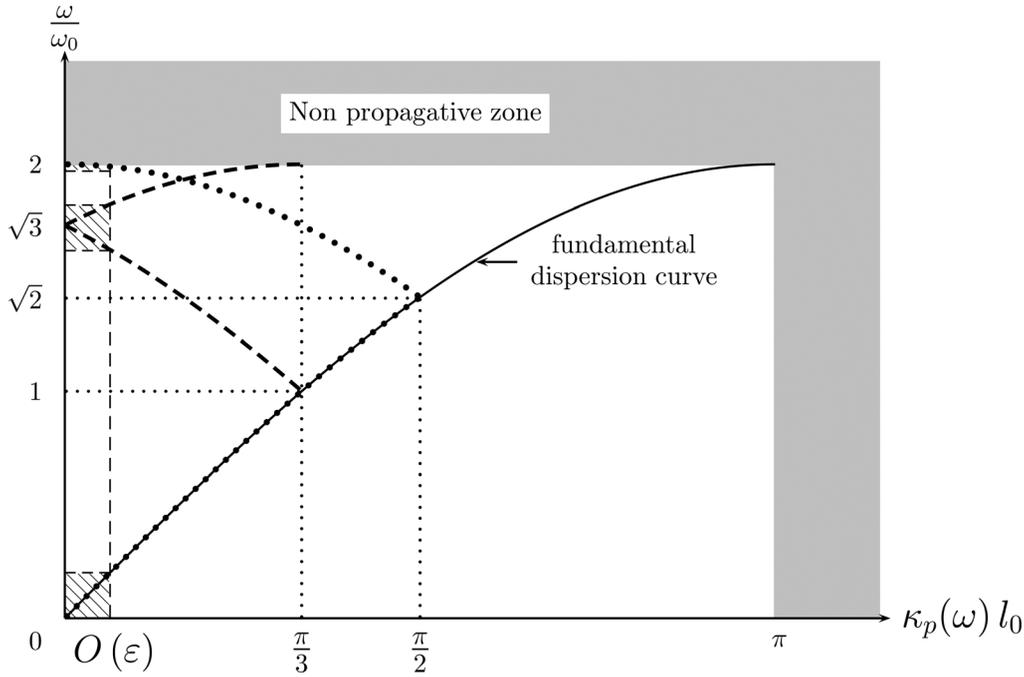


FIG. 3. Dispersion relations determined by considering Ω_0 (in plain), Ω_2 (in dotted), and Ω_3 (in dashed) in the first fundamental Brillouin zone. For each considered period, the large scale modulation phenomena occurs in hatches frequency bands corresponding to vanishing values κ_p , $p \neq 1$.

curve related to double and triple cells, Ω_2 and Ω_3 , i.e., $\kappa_2(\omega)$, $\kappa_3(\omega)$. According to the folding rule,^{10,11} the dispersion $\kappa_2(\omega)$ matches the fundamental dispersion curve for $\kappa_2 l_0 / \pi \leq 1/2$ and its specular reflection with respect to the axis $\kappa_1 l_0 / \pi = 1/2$. The dispersion $\kappa_3(\omega)$ matches (i) the fundamental dispersion curve, (ii) its specular reflection with respect to the axis $\kappa_1 l_0 / \pi = 1/3$, and (iii) a direct translation of the fundamental dispersion curve. By construction, the folding of the fundamental dispersion curve results in zero valued modulation numbers κ_p , $p \neq 1$ at the eigen frequencies of Ω_p . Consequently, in the vicinity of these frequencies, large-scale modulations occur.

Those results evidence the possibility of large modulation of high frequency waves in frequency bands centered around the periodic eigenmodes frequencies arising in the multicell period. The existence of two different scales, related to the modes and modulation lengths, suggests the use of upscaling method. This aspect is addressed in the next section.

B. Large modulation of high frequency waves in a periodic network of resonators

The scale separation between the multicells period Ω_p of length $l = p l_0$, and the modulation length L naturally introduces the scale ratio $\varepsilon = l/L = p l_0 / L = p \varepsilon_0$. Hereafter, a macroscopic description is derived using the asymptotic approach of discrete periodic media^{12,13} adapted to multicells period and high frequencies. The method is briefly summarized in the following paragraph.

1. Discrete homogenization method

Following the previous analysis, among the p masses of n th period Ω_p , the motion u_n of the left extremity is arbitrarily

chosen as *leading* variable located at the leading node (the motions of the $p - 1$ other masses are related to the leading variables through the local balances in Ω_p) (cf. Fig. 2). The modulations of the mode amplitude (denoted by A) at large scale are evidenced on the leading variables. These latter variables match the values taken at the leading nodes by a continuous function of the macroscopic variable x . Assuming the convergence when ε tends to zero, the leading motions “carrying” the modulations are expanded in powers of ε ,

$$u_n = u(x = nl) = A^{(0)}(nl) + \varepsilon A^{(1)}(nl) + \varepsilon^2 A^{(2)}(nl) + \dots \quad (14)$$

The frequency is expanded *around the eigenfrequency* of one among the periodic eigenmodes of Ω_p (say K th mode),

$$\omega = \omega_K + \varepsilon \omega^{(1)} + \varepsilon^2 \omega^{(2)} + \dots$$

As the size of the period is small compared with the modulation length, the difference of the motions between neighboring leading nodes is expressed using Taylor’s series, which introduces the macroscopic derivatives. For instance ($''$ stands for double spatial derivative),

$$\frac{u_{n+1} - 2u_n + u_{n-1}}{l^2} = u''(x = nl) + O(l^2 u''''(x = nl)).$$

As the motions of the inner masses are related to those of the leading masses, it only remains to express the balance equations at the leading nodes. Then introducing in these latter, the expansions and the Taylor series of variables u_n , leads to a series of equations at different powers of ε that are resolved until the macroscopic description is obtained (the present study focuses on the leading order only). This procedure is

presented hereafter for periods constituted by one, two, and three cells.

2. Macroscopic description based on irreducible period Ω_0

For irreducible period Ω_0 ($p = 1$), begin with the balance equation (2):

$$\mathcal{K}(u_{n+1} - 2u_n + u_{n-1}) = -\mathcal{M}\omega^2 u_n.$$

Substituting the discrete values u_n by the continuous function $u(x)$, gives on each node $x = nl$:

$$(u''(x) + O(l^2 u''''(x))) l^2 = -\left(\frac{\omega}{\omega_0}\right)^2 u(x). \quad (15)$$

Now because of the scale separation, u slowly varies according to the macroscopic scale, and therefore $O(u'') = O(u/L^2)$, $O(u''') = O(u/L^4)$. Then the balance of both terms of Eq. (15) implies that

$$O\left(\frac{u}{L^2} l^2\right) = O\left(\left(\frac{\omega}{\omega_0}\right)^2 u\right) \quad \text{hence} \quad \omega = O(\varepsilon\omega_0).$$

Consequently, the expansion of the frequency takes the form $\omega = \varepsilon\omega^{(1)} + \varepsilon^2\omega^{(2)} + \dots$. Then introducing the expansion (14) in (15) and separating ε -power, we obtain at the leading ε^2 order:

$$(A^{(0)}(x))'' L^2 = -\left(\frac{\omega^{(1)}}{\omega_0}\right)^2 A^{(0)}(x).$$

This leading order result matches the classic wave propagation equation (A stands for $A^{(0)}$, and coincides with u in this case):

$$\mathcal{K} l_0^2 A''(x) + \mathcal{M}\omega^2 A(x) = 0. \quad (16)$$

Note that the scale separation only applies at low frequency because Ω_0 does not present eigenmode except the “degenerated” mode displaying a uniform motion at zero frequency.

3. Macroscopic description for double-cell period Ω_2

Double cells period Ω_2 ($p = 2$, $l = 2l_0$) presents a unique and *simple* periodic mode at the eigenfrequency $2\omega_0$. In the previous section, the study of the balance of inner and leading nodes provided Eqs. (6) and (8), which can be rewritten with the continuous variables in the form:

$$v(x) = \frac{2u(x) + O(lu'(x))}{2 - \left(\frac{\omega}{\omega_0}\right)^2}, \quad (17)$$

$$(u''(x) + O(l^2 u''''(x))) l^2 = \left(\frac{\omega}{\omega_0}\right)^2 \left(\left(\frac{\omega}{\omega_0}\right)^2 - 4\right) u(x). \quad (18)$$

The assumed scale separation imposes once again that $O(u'' l^2) = O(\varepsilon^2 u)$, $O(u'''' l^4) = O(\varepsilon^4 u)$, so that the balance of elastic and inertial terms implies

$$\left(\frac{\omega}{\omega_0}\right)^2 \left(\left(\frac{\omega}{\omega_0}\right)^2 - 4\right) = O(\varepsilon^2).$$

Focusing on high frequency, the asymptotic expansion procedure requires that $\omega = 2\omega_0 + \varepsilon^2\omega^{(2)}$, which specifies a narrow frequency band $O(\varepsilon^2\omega_0)$ centered on the eigenfrequency of Ω_2 . Then introducing the expansions (14) in Eqs. (17) and (18) and separating ε -powers, we obtain at the leading orders:

$$\begin{aligned} v^{(0)}(x) &= -A^{(0)}(x); \\ (A^{(0)}(x))'' L^2 &= 16 \left(\frac{\omega^{(2)}\omega_0}{\omega_0^2}\right) A^{(0)}(x). \end{aligned}$$

Therefore the motions of the inner node and the leading node are opposite, and the modulation A (that stands here for $A^{(0)}$) of the Ω_2 -eigenmode, at the leading order, is governed by

$$\mathcal{K} l_0^2 A'' + \mathcal{M}(-\omega^2 + (2\omega_0)^2) A = 0. \quad (19)$$

4. Macroscopic description for triple-cell period Ω_3

Triple cells period $p = 3$ Ω_2 ($p = 3$, $l = 3l_0$) presents a *double* periodic mode at the eigenfrequency $\omega = \sqrt{3}\omega_0$ [cf. Eq. (12) with $J = 1$, $p = 3$] (this latter mode, derived with periodic conditions on Ω_3 could not be derived with antiperiodic conditions on Ω_0). The balance equations of both inner nodes, the motions of which are v_n and w_n (see Fig. 2), reads

$$\begin{cases} -\mathcal{M}\omega^2 v_n = \mathcal{K}(u_n - 2v_n + w_n) \\ -\mathcal{M}\omega^2 w_n = \mathcal{K}(v_n - 2w_n + u_{n+1}). \end{cases} \quad (20)$$

Solving the system (20) yields the motions v_n and w_n in function of the leading nodes u_n and u_{n+1} (note that the eigenfrequency of the double periodic mode is a root of the denominator)

$$\begin{cases} v_n = \frac{u_n}{1 - (\omega/\omega_0)^2} + \frac{u_{n+1} - u_n}{(1 - (\omega/\omega_0)^2)(3 - (\omega/\omega_0)^2)} \\ w_n = \frac{u_{n+1}}{1 - (\omega/\omega_0)^2} - \frac{u_{n+1} - u_n}{(1 - (\omega/\omega_0)^2)(3 - (\omega/\omega_0)^2)}. \end{cases} \quad (21)$$

Moreover, the balance equation of the leading node reads

$$-\mathcal{M}\omega^2 u_n = \mathcal{K}(w_{n-1} - 2u_n + v_n). \quad (22)$$

Substituting the expressions of inner nodes motions (21) and introducing the continuous variables $u(x)$ gives

$$(u''(x) + O(l^2 u''''(x))) l^2 = -\left(\frac{\omega}{\omega_0}\right)^2 \left(\left(\frac{\omega}{\omega_0}\right)^2 - 3\right)^2 u(x).$$

As in the preceding cases, $O(u''^2) = O(\varepsilon^2 u)$, and the asymptotic expansion procedure at high frequency requires that $\omega = \sqrt{3} \omega_0 + \varepsilon \omega^{(1)}$, which corresponds to a frequency band $O(\varepsilon \omega_0)$ centered on the eigenfrequency of the double periodic eigenmode of Ω_3 . Then introducing the expansions (14) in Eqs. (21) and (22) and separating ε -powers, we obtain (i) the expression of the inner motions at the leading order (contrary to the simple periodic mode in Ω_2 , the frequency still appears in the inner nodes motions at the leading order):

$$\begin{cases} v^{(0)}(x) = -\frac{1}{2} \left(A^{(0)}(x) - \frac{\sqrt{3}}{2} l_0 \frac{\omega_0}{(\omega - \sqrt{3} \omega_0)} \left(A^{(0)}(x) \right)' \right) \\ w^{(0)}(x) = -\frac{1}{2} \left(A^{(0)}(x) + \frac{\sqrt{3}}{2} l_0 \frac{\omega_0}{(\omega - \sqrt{3} \omega_0)} \left(A^{(0)}(x) \right)' \right), \end{cases}$$

and (ii) the equation of modulation (below, A stands for $A^{(0)}$) of this *double* periodic mode that occurs at frequencies lower than the edge of the Brillouin zone (i.e., $2\omega_0$):

$$\mathcal{K} l_0^2 A'' + 4\mathcal{M}(\omega - \sqrt{3} \omega_0)^2 A = 0. \quad (23)$$

Finally, looking for solution of Eqs. (16), (19), and (23) in the form

$$A(x) = U_0 \exp(i(\omega t - \kappa_p(\omega)x))$$

gives, respectively, at the leading order

$$\begin{aligned} (\kappa_1(\omega) l_0)^2 &= \left(\frac{\omega}{\omega_0} \right)^2 \quad \text{for } \omega \ll \omega_0, \\ (\kappa_2(\omega) l_0)^2 &= 4 - \left(\frac{\omega}{\omega_0} \right)^2 \quad \text{for } \omega \simeq 2\omega_0, \\ (\kappa_3(\omega) l_0)^2 &= 4 \left(\frac{\omega}{\omega_0} - \sqrt{3} \right)^2 \quad \text{for } \omega \simeq \sqrt{3} \omega_0. \end{aligned}$$

These leading order expressions coincide with Eqs. (4), (10), and (13), derived from the general dispersion equation (11) when $\kappa_p \simeq 0$. Note the essential differences in the frequency terms in the three cases.

Low frequency description matches the usual wave phenomena. The high frequency descriptions give the equations of modulation phenomena, that differ in nature if the mode is simple or double (these feature are discussed in details in Secs. III C and III D).

Hence the exact description of modulation phenomena can be approached independently via an asymptotic method. An illustration is given on Fig. 4, where the spatial modulation at the frequency $\omega = (\sqrt{3} - 0.01)\omega_0$ is determined on a system of 160 resonators. In the next section, the general principles of this approach developed for 1D discrete systems (with a single degree of freedom in the irreducible cell), are transposed to 3D continuous porous media (where the irreducible cell presents an infinity of degree of freedom).

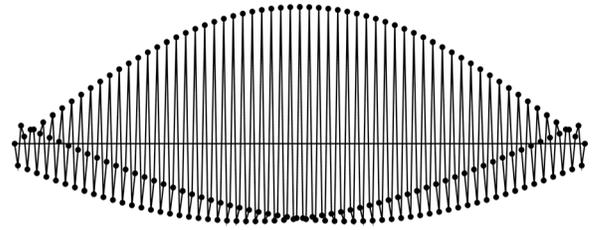


FIG. 4. Modulation at $\omega = (\sqrt{3} - 0.01) \omega_0$ calculated on a system constituted by 160 Helmholtz resonators. The amplitude A of the mode $(2/\sqrt{6}, -1/\sqrt{6}, -1/\sqrt{6})$ is imposed null at both extremities. Note the coexistence of the mode $(0, 1/\sqrt{2}, -1/\sqrt{2})$ with maximum amplitude at the extremities and zero amplitude in the center when $\text{grad}(A) = 0$.

III. LARGE MODULATION OF HIGH FREQUENCY WAVES IN PERIODIC POROUS MEDIA

Let us now consider the case of a periodic 3D porous medium. The period Ω of characteristic size l contains a pore domain Ω_f of boundary $\partial\Omega_f = \Gamma \cup \Gamma_f$, where Γ is the fluid/solid boundary of normal \underline{n} and Γ_f the fluid boundary of Ω . The porosity ϕ equals $|\Omega_f|/|\Omega|$ (Fig. 5). We focus on small acoustic perturbations in the high frequency range where the local dynamics induce the loss of scale separation for the pressure. In a first approach, we neglect the thermal and viscous dissipation effects as in the previous section. In order of magnitude, this is physically justified by the fact that for most of sound absorbing materials, the first eigenfrequency of the irreducible period occurs with weak visco-thermal dissipation compared with the compresso-inertial effects (cf. Ref. 14). Nevertheless the dissipation effects (confined in viscous and thermal “skins” in the range of the first eigenfrequency) certainly influence the modulation phenomena, but a detailed analysis of these effects is beyond the scope of this paper.

Then we consider that the local description within the pores is that of a perfect gas in adiabatic dynamic linear regime governed by the following constitutive and balance equations (by linearity, the harmonic time dependence $\exp(i\omega t)$ is skipped):

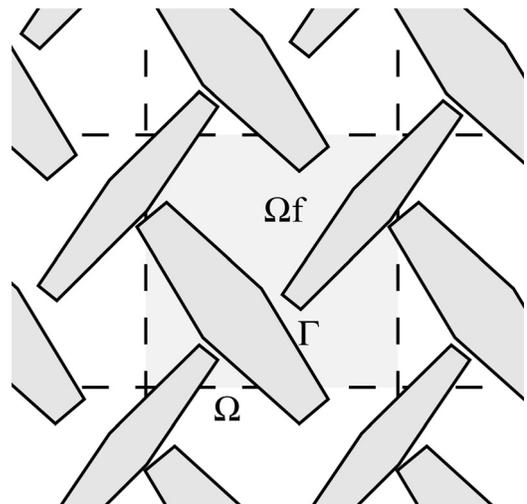


FIG. 5. Description of the irreducible period Ω_0 and multiple cells period Ω of a periodic porous media. Notations.

$$P = -\gamma P^e \operatorname{div}(\underline{u}), \quad -\underline{\operatorname{grad}}(P) + \rho^e \omega^2 \underline{u} = \underline{0},$$

where \underline{u} is the gas motion and P the pressure perturbation. Introducing the differential operator $\underline{\mathcal{F}}$, the impenetrability conditions on Γ and the periodic condition on Γ_f , the local problem reads:

$$\begin{cases} \mathcal{C}^2 \underline{\mathcal{F}}(\underline{u}) + \omega^2 \underline{u} = \underline{0}, & \text{on } \Omega_f; \quad \underline{\mathcal{F}}(\underline{u}) = \underline{\operatorname{grad}}(\operatorname{div}(\underline{u})) \\ \underline{u} \cdot \underline{n} = 0, & \text{on } \Gamma; \quad \underline{u} \Omega - \text{periodic}. \end{cases} \quad (24)$$

Following Courant and Hilbert,¹⁵ the problem (24) has a discrete and positive spectrum (conversely to spring-mass chains, there is no high frequency limitation because the mass is here continuously distributed)

$$0 \leq \omega_1^2 \leq \omega_2^2 \leq \omega_3^2 \leq \dots$$

Each eigenvalue ω_J^2 is associated with eigenmode Φ^J (both denoted with capital subscripts). The $\omega_J = 2\pi f_J$, J integer, corresponds to the resonance frequencies of the gas in the periodic porous matrix (hence the periodic boundary conditions). In other words, the series of solutions $\{\omega_J^2; \Phi^J\}$ satisfy

$$\begin{cases} \mathcal{C}^2 \underline{\mathcal{F}}(\Phi^J) + \omega_J^2 \Phi^J = \underline{0}, & \text{on } \Omega_f \\ \Phi^J \cdot \underline{n} = 0, & \text{on } \Gamma; \quad \Phi^J \Omega - \text{periodic}. \end{cases}$$

Recall that the series of normalized eigenmodes $\{\Phi^J\}$ (of zero mean value by periodicity) constitute an orthonormal basis for the Ω -periodic motions \underline{u} defined in Ω_f , with $\underline{u} \cdot \underline{n} = 0$ on Γ . Thus:

$$\underline{u} = \sum_{J=1}^{\infty} u^J \Phi^J, \quad \text{with } \langle \Phi^J, \Phi^J \rangle = \delta_{JJ} \quad \text{and} \quad \langle \Phi^J \rangle = 0$$

where δ is the Kronecker symbol, and the mean values and local norm are denoted by

$$\langle a \rangle = \frac{1}{\Omega_f} \int_{\Omega_f} a \, dv; \quad |\Phi|^2 = \Phi \cdot \Phi.$$

Remark: A given eigenvalue ω_J generally corresponds to a single mode Φ^J . However, it happens, especially in periodic media, several orthonormal modes constituting the sub-vectorial space denoted $\{\Phi\}^J$ are associated with the same eigenvalue ω_J . To lighten the equations, we do not distinguish in the notations the situations of single from those of multiple modes, writing Φ^J (instead of the orthonormal mode of $\{\Phi\}^J$). In the sequel, simple multiple modes will be specified when necessary.

Before going further, it is necessary to clarify the notion of period used in the sequel.

A. Irreducible period and selected period

A periodic media is generally defined by its irreducible period Ω_0 . However, any integer combination (p, q, r) in the three directions of space of Ω_0 define another period

$\Omega = \cup_{p,q,r} \Omega_0$ of the media with $|\Omega| = p \times q \times r \times |\Omega_0|$. In the usual homogenization approach, the periodic problems (and solutions) at the local scale are shown to be independent of the definition of the period. This results from the quasi-static regime at the local scale.

Now when considering dynamics at the local scale, the modes necessarily depend on the definition of the volume (the period Ω) to which the periodic boundary conditions apply (on $\partial\Omega$). Indeed, the “family” of mode increases as the number of irreducible periods included in the period Ω grows. For instance, the series of modes of double period, $\Omega = \cup_{2,1,1} \Omega_0$, includes the Ω_0 modes (“duplicated” along the first direction) and the new modes specific to the double period. These latter modes respect the periodicity condition on $\partial\Omega$ but not on the internal boundary defined by $\partial\Omega_0$.

Therefore we have to specify the period Ω on which the periodic mode is defined: Contrary to low frequency classic homogenization, the selected period Ω becomes a parameter of the description. This is implicitly included in sequel, where the analysis is performed for any type of $\Omega = \cup_{p,q,r} \Omega_0$.

This procedure guarantees that the whole family of large modulation phenomena, associated to modes existing in the collection of periods, will be reached.

Notice that a subgroup of Ω -modes can be determined by using specific boundary condition on the irreducible period Ω_0 as proposed by Craster *et al.*⁴ For instance, a subgroup of modes of double cell period, $\Omega = \cup_{2,1,1} \Omega_0$ can be obtained from an antiperiodic condition on Ω_0 (in the first direction), and periodicity on Ω_0 (in the two other directions). Similarly, for $\Omega = \cup_{p,1,2} \Omega_0$, a subgroup of modes can be derived applying on Ω_0 , a “ $e^{\pm i2\pi J/(pl_0)}$ -phase shifted” condition in the first direction, the periodic condition in the second direction, the antiperiodic condition in the third direction, etc. These conditions will be also obtained in the classical Floquet–Bloch framework, for rational Bloch parameter $k_B = \pm 2\pi J/(pl_0)$. Remark also that if the modes of the multicells period Ω contain the modes of another multicells period Ω' , the first eigenfrequency of Ω is lower than the one of Ω' (and therefore than the one of the irreducible cell Ω_0).

B. Modulation and asymptotic method

Choose a period Ω and select for example the K th mode, i.e., the couple (ω_K, Φ^K) . By construction, mode Φ^K repeated Ω -periodically, gives rise to a high frequency wave of constant modal amplitude or equivalently to a high frequency wave of infinite modulation length. Contrary to infinite length, a large modulation length implies (i) a non-constant amplitude (of the mode Φ^K) varying at a scale L much larger than the period l and (ii) a frequency ω distinct but nevertheless close to the eigenfrequency ω_K because $\varepsilon = l/L \rightarrow 0$, when $\omega \rightarrow \omega_K$.

To describe the situation of modulated K th mode when the modulation to period scale ratio is small, i.e., $\varepsilon \ll 1$, we apply the multi-scale asymptotic method well established in the framework of homogenization¹ of periodic media. The physics introduces two dimensionless space variables, $x/l = y^*$ and $x/L = x^*$ associated, respectively, to the variations at the period and modulation scales, where x stands for

the usual space variable. Equivalently, two space variables, $y = ly^* = x$ and $x = lx^* = \varepsilon y = \varepsilon x$ will be used, the usual derivative being therefore changed into $\partial/\partial y + \varepsilon\partial/\partial x$. Consequently, the differential operator $\underline{\mathcal{F}}$ becomes

$$\begin{cases} \underline{\mathcal{F}} \rightarrow \underline{\mathcal{F}}_{y^2} + \varepsilon \underline{\mathcal{F}}_{yx} + \varepsilon^2 \underline{\mathcal{F}}_{x^2} \\ \underline{\mathcal{F}}_{y^2}(\underline{u}) = \underline{\text{grad}}_y(\text{div}_y(\underline{u})) \\ \underline{\mathcal{F}}_{yx}(\underline{u}) = \underline{\text{grad}}_y(\text{div}_x(\underline{u})) + \underline{\text{grad}}_x(\text{div}_y(\underline{u})) \\ \underline{\mathcal{F}}_{x^2}(\underline{u}) = \underline{\text{grad}}_x(\text{div}_x(\underline{u})). \end{cases}$$

The motion \underline{u} is expanded in power of ε , each term (specified by exponents in brackets) being Ω periodic, and the frequency is also expanded in ε -power around ω_K :

$$\underline{u}(x, y) = \sum_0^\infty \varepsilon^i \underline{u}^{(i)}(x, y) \quad \omega = \omega_K + \sum_1^\infty \varepsilon^i \omega^{(i)}.$$

The process, similar to Craster *et al.*⁴ is achieved as in usual homogenization:^{1,3} The expansions are introduced in Eq. (24) rewritten with the two-scale derivatives, then the terms of the same power in ε are identified, and the problems obtained in series are solved until the equation governing the large scale modulation at the leading order is obtained.

C. Modulation of a simple mode

We focus here on the case where the considered mode Φ^K is simple (double and multiple modes are addressed in Secs. III D and III E, respectively).

1. Leading order

The problem encountered at the leading order reads

$$\begin{cases} \mathcal{C}^2 \underline{\mathcal{F}}_{y^2}(\underline{u}^{(0)}) + \omega_K^2 \underline{u}^{(0)} = 0, & \text{on } \Omega_f \\ \underline{u}^{(0)} \cdot \underline{n} = 0, & \text{on } \Gamma; \underline{u}^{(0)} \Omega - \text{periodic.} \end{cases} \quad (25)$$

Mode Φ_K being simple, the solution is in the form

$$\underline{u}^{(0)}(x, y) = A(x) \Phi^K(y),$$

where $A(x)$ is the slow varying amplitude of mode Φ^K .

2. First order

The following problem, governing $\underline{u}^{(1)}$, takes the form of the previous eigenvalue problem with an additional source term $\mathcal{S}^{(1)}$ induced by the field $\underline{u}^{(0)}$:

$$\begin{cases} \mathcal{C}^2 \underline{\mathcal{F}}_{y^2}(\underline{u}^{(1)}) + \omega_K^2 \underline{u}^{(1)} + \mathcal{S}^{(1)}(\underline{u}^{(0)}) = 0, & \text{on } \Omega_f \\ \mathcal{S}^{(1)}(\underline{u}^{(0)}) = \mathcal{C}^2 \underline{\mathcal{F}}_{yx}(\underline{u}^{(0)}) + \omega^{(1)}(2\omega_K \underline{u}^{(0)}) \\ \underline{u}^{(1)} \cdot \underline{n} = 0, & \text{on } \Gamma; \underline{u}^{(1)} \Omega - \text{periodic.} \end{cases} \quad (26)$$

To handle this problem, let us first establish that, following the Fredholm alternative, we have

$$\langle \mathcal{S}^{(1)}(\underline{u}^{(0)}) \cdot \underline{u}^{(0)} \rangle = 0. \quad (27)$$

In this aim, notice that from the divergence theorem, the impenetrability condition (on Γ) and the periodic condition (on Γ_f):

$$\begin{aligned} \langle \underline{\mathcal{F}}_{y^2}(\underline{u}^{(1)}) \cdot \underline{u}^{(0)} \rangle &= - \langle \text{div}_y(\underline{u}^{(1)}) \cdot \text{div}_y(\underline{u}^{(0)}) \rangle \\ &= \langle \underline{\mathcal{F}}_{y^2}(\underline{u}^{(0)}) \cdot \underline{u}^{(1)} \rangle. \end{aligned}$$

Then take the scalar product of Eq. (26) by $\underline{u}^{(0)}$ and of Eq. (25) by $\underline{u}^{(1)}$, integrating on Ω_f and subtracting provides equality (27), which can be rewritten as

$$\mathcal{C}^2 \langle \underline{\mathcal{F}}_{yx}(\underline{u}^{(0)}) \cdot \underline{u}^{(0)} \rangle + \omega^{(1)}(2\omega_K \langle |\underline{u}^{(0)}|^2 \rangle) = 0.$$

The first term in brackets can further be simplified. Indeed

$$\begin{aligned} &\langle \underline{\mathcal{F}}_{yx}(\underline{u}^{(0)}) \cdot \underline{u}^{(0)} \rangle \\ &= A(x) \left\langle \left[\underline{\text{grad}}_y(\text{div}_x(\underline{u}^{(0)})) + \underline{\text{grad}}_x(\text{div}_y(\underline{u}^{(0)})) \right] \cdot \Phi^K \right\rangle \\ &= A(x) \left\langle \Phi_{j,y_i}^K \Phi_i^K + \Phi_{i,y_j}^K \Phi_j^K \right\rangle A_{,xj}, \end{aligned}$$

and from the divergence theorem with the impenetrability on Γ and periodic conditions on Γ_f (usual subscripts stand for 3D-space directions), we have

$$\begin{aligned} &|\Omega_f| \langle \underline{\mathcal{F}}_{yx}(\underline{u}^{(0)}) \cdot \underline{u}^{(0)} \rangle \\ &= A(x) \int_{\partial\Omega_f} (\Phi_j^K \Phi_i^K + \Phi_i^K \Phi_j^K) A_{,xj} n_i ds \\ &= 2A(x) \underline{\text{grad}}_x(A) \cdot \left[\int_{\partial\Omega_f} (\Phi^K \otimes \Phi^K) \cdot \underline{n} ds \right] = 0. \end{aligned}$$

Consequently, Eq. (27) simply reduces to

$$\omega^{(1)} 2\omega_K (A(x))^2 = 0 \quad \text{thus} \quad \omega^{(1)} = 0$$

which means that the the corrector of frequency is necessarily of the second order. Because, with this result,

$$\mathcal{S}^{(1)}(\underline{u}^{(0)}) = \mathcal{C}^2 \underline{\mathcal{F}}_{yx}(A(x) \Phi^K),$$

the set (26) is a dynamic problem with a forcing term linearly dependent on the components of $\underline{\text{grad}}_x(A)$. By linearity, considering the specific solutions $\underline{\xi}^\alpha(\underline{y})$, $\alpha = 1, 2, 3$ corresponding to unit $\underline{\text{grad}}_x(A)$ in the three spatial directions (i.e., $\underline{\text{grad}}_x(A) = \delta_{ix} \underline{e}_i$), we deduce that

$$\underline{u}^{(1)} = B(x) \Phi^K + \underline{\xi}^\alpha(\underline{y}) A_{,x\alpha}.$$

The solutions $\underline{\xi}^\alpha(\underline{y})$ can be expressed using the fact that $\{\Phi^j\}$ forms an orthonormal basis. In this aim, rewrite $\underline{u}^{(1)}$ as

a linear combination (depending on the macrovariable \underline{x}) of the Φ^J :

$$\underline{u}^{(1)}(\underline{x}, \underline{y}) = B(\underline{x})\Phi^K(\underline{y}) + \sum_{J \neq K} a^J(\underline{x}) \Phi^J(\underline{y}).$$

Using a basic result of modal analysis (see the Appendix), $B(\underline{x})$ is undetermined, while the coefficients $a^J(\underline{x})$ are given by

$$\begin{aligned} a^J(\underline{x}) &= -\left\langle \mathcal{S}^{(1)}(\underline{u}^{(0)}) \cdot \Phi^J \right\rangle \frac{1}{\omega_K^2 - \omega_J^2} \\ &= -\left\langle \mathcal{F}_{yx}(A(\underline{x})\Phi^K) \cdot \Phi^J \right\rangle \frac{\mathcal{C}^2}{\omega_K^2 - \omega_J^2}. \end{aligned}$$

The term in brackets is given as

$$\begin{aligned} \left\langle \mathcal{F}_{yx}(A(\underline{x})\Phi^K) \cdot \Phi^J \right\rangle &= \left\langle \Phi_{j,y_i}^K \Phi_i^J + \text{div}_y(\Phi^K)\Phi_j^K \right\rangle A_{,x_j} \\ &= \left\langle -\Phi_j^K \text{div}_y(\Phi^J) + \text{div}_y(\Phi^K)\Phi_j^J \right\rangle A_{,x_j} \\ &\quad + \left\langle (\Phi_j^K \Phi_i^J)_{,y_i} \right\rangle A_{,x_j}. \end{aligned}$$

Moreover, from the divergence theorem, the impenetrability on Γ and the periodic condition on Γ_f , we have

$$\left\langle (\Phi_j^K \Phi_i^J)_{,y_i} \right\rangle = \int_{\partial\Omega_f} \Phi_j^K(\Phi_i^J \cdot \underline{n}) ds = 0.$$

Introducing the following notation for the combined vector built on two vectors Φ and Ψ :

$$\widehat{\Phi\Psi} = \langle \Phi \text{div}_y(\Psi) - \Psi \text{div}_y(\Phi) \rangle = -\widehat{\Psi\Phi},$$

we derive

$$\left\langle \mathcal{F}_{yx}(A(\underline{x})\Phi^K) \cdot \Phi^J \right\rangle = \underline{\text{grad}}_x(A) \cdot \widehat{\Phi^J \Phi^K}, \quad (28)$$

which gives

$$\underline{u}^{(1)} = B(\underline{x})\Phi^K - \sum_{J \neq K} \frac{\mathcal{C}^2}{\omega_K^2 - \omega_J^2} \left(\underline{\text{grad}}_x(A) \cdot \widehat{\Phi^J \Phi^K} \right) \Phi^J,$$

and by identification,

$$\underline{\xi}^{\alpha} = -\sum_{J \neq K} \frac{\mathcal{C}^2}{\omega_K^2 - \omega_J^2} \left(\widehat{\Phi^J \Phi^K} \right)_\alpha \Phi^J. \quad (29)$$

The presence of the modes $J \neq K$ in $\underline{u}^{(1)}$, means that the perturbation generated by the modulation of the amplitude of the mode Φ^K , expressed by $\underline{\text{grad}}_x(A)$, is balanced by the other modes of the same cell Ω . Expanding $\underline{\xi}^{\alpha}$ in eigenmodes provides a clear physical insight of the nature of energy transfer related to the modulation of amplitude. Indeed from their definition, vectors $\widehat{\Phi\Psi}$ express (up to the compressibility modulus P^c) the energy transferred from a mode to another. Further, it may help the numerical computation of $\underline{\xi}^{\alpha}$ through relevant truncation in the series (around the eigenfrequency ω_K).

3. Second order

Owing to the fact that $\omega^{(1)} = 0$, the problem at the second order reads

$$\begin{cases} \mathcal{C}^2 \mathcal{F}_{y^2}(\underline{u}^{(2)}) + \omega_K^2 \underline{u}^{(2)} + \mathcal{S}^{(1)}(\underline{u}^{(1)}) + \mathcal{S}^{(2)}(\underline{u}^{(0)}) = \underline{0}, & \text{on } \Omega_f \\ \mathcal{S}^{(1)}(\underline{u}^{(1)}) = \mathcal{C}^2 \mathcal{F}_{yx}(\underline{u}^{(1)}) \\ \mathcal{S}^{(2)}(\underline{u}^{(0)}) = \mathcal{C}^2 \mathcal{F}_{xx}(\underline{u}^{(0)}) + \omega^{(2)} 2\omega_K \underline{u}^{(0)} \\ \underline{u}^{(2)} \cdot \underline{n} = 0, & \text{on } \Gamma; \quad \underline{u}^{(2)} \Omega - \text{periodic}. \end{cases}$$

Following the same reasoning as for the first order, it is straightforward to establish that

$$\left\langle \left[\mathcal{S}^{(2)}(\underline{u}^{(0)}) + \mathcal{S}^{(1)}(\underline{u}^{(1)}) \right] \cdot \underline{u}^{(0)} \right\rangle = 0,$$

which becomes after introducing the expressions of $\underline{u}^{(0)}$ and $\underline{u}^{(1)}$:

$$\begin{aligned} &\left\langle \left[\mathcal{S}^{(2)}(A(\underline{x})\Phi^K) \cdot \Phi^K \right] + \sum_{J \neq K} \left\langle \mathcal{S}^{(1)}(a^J(\underline{x})\Phi^J) \cdot \Phi^K \right\rangle \right. \\ &\quad \left. + \left\langle \mathcal{S}^{(1)}(B(\underline{x})\Phi^K) \cdot \Phi^K \right\rangle \right\rangle A(\underline{x}) = 0. \end{aligned}$$

The amplitude $A(\underline{x})$ can be simplified, and each term in brackets can be calculated. As in Eq. (27), we have $\langle \mathcal{S}^{(1)}(B(\underline{x})\Phi^K) \cdot \Phi^K \rangle = 0$. Further, each $\langle \mathcal{S}^{(1)}(a^J(\underline{x})\Phi^J) \cdot \Phi^K \rangle$ is determined similarly as the term $\langle \mathcal{S}^{(1)}(A(\underline{x})\Phi^K) \cdot \Phi^K \rangle$, see Eq. (28):

$$\begin{aligned} &\left\langle \mathcal{S}^{(1)}(a^J(\underline{x})\Phi^J) \cdot \Phi^K \right\rangle \\ &= \mathcal{C}^2 \underline{\text{grad}}_x(a^J(\underline{x})) \cdot \widehat{\Phi^J \Phi^K} \\ &= A(\underline{x})_{,x_p x_q} \frac{\mathcal{C}^4}{\omega_K^2 - \omega_J^2} \left(\widehat{\Phi^J \Phi^K} \right)_p \left(\widehat{\Phi^J \Phi^K} \right)_q. \end{aligned}$$

Finally,

$$\begin{aligned} &\left\langle \mathcal{S}^{(2)}(A(\underline{x})\Phi^K) \cdot \Phi^K \right\rangle \\ &= \mathcal{C}^2 \left\langle \underline{\text{grad}}_x(\text{div}_x(A(\underline{x})\Phi^K)) \cdot \Phi^K \right\rangle + \omega^{(2)} 2\omega_K A(\underline{x}) \\ &= \mathcal{C}^2 A(\underline{x})_{,x_p x_q} \left\langle \Phi_p^K \Phi_q^K \right\rangle + \omega^{(2)} 2\omega_K A(\underline{x}). \end{aligned}$$

Thus we obtain the following macroscopic equation driving $A(\underline{x})$:

$$\begin{aligned} &\mathcal{C}^2 \left(\left\langle \Phi_p^K \Phi_q^K \right\rangle + \sum_{J \neq K} \frac{\mathcal{C}^2}{\omega_K^2 - \omega_J^2} \left(\widehat{\Phi^J \Phi^K} \right)_p \left(\widehat{\Phi^J \Phi^K} \right)_q \right) A_{,x_p x_q} \\ &\quad + \omega^{(2)} 2\omega_K A = 0. \end{aligned}$$

Which, from expression (29) of $\underline{\xi}^{\alpha}$ can be rewritten as

$$\mathcal{C}^2 \left(\left\langle \Phi_p^K \Phi_q^K \right\rangle - \left(\widehat{\underline{\xi}^{\alpha} \Phi^K} \right)_q \right) A_{,x_p x_q} + \omega^{(2)} 2\omega_K A = 0.$$

4. Features of the modulation of a simple mode

The preceding results can be summarized as follows. Coming back to the usual space variable $\underline{x} = \underline{x}/\varepsilon$, the leading order governing equation for the modulation at the frequency $\omega = \omega_K + \omega^{(2)}\varepsilon^2$ of the simple mode (ω_K, Φ^K) is in the form:

$$\mathcal{C}^2 \operatorname{div}_{\underline{x}} \left(\underline{\underline{T}}^K \cdot \underline{\underline{\operatorname{grad}}}_{\underline{x}} (A(\underline{x})) \right) + (\omega^2 - \omega_K^2) A(\underline{x}) = o(\varepsilon^2),$$

where we use the $o(\varepsilon^2)$ -approximation: $\omega_2 - \omega_K^2 = \omega^{(2)}(\varepsilon^2 2\omega_K)$ and where the tensor $\underline{\underline{T}}$ associated to mode Φ^K is given by

$$\underline{\underline{T}}^K = \langle \Phi^K \otimes \Phi^K \rangle + \mathcal{C}^2 \sum_{j \neq K} \frac{\widehat{\Phi^j \Phi^K} \otimes \widehat{\Phi^j \Phi^K}}{\omega_K^2 - \omega_j^2}.$$

Note that the modulation is driven by a strictly macroscopic equation. The parameters of this equation are fully determined by the dynamic properties of the period (eigenmodes) and the eigenfrequencies, independently of the macroscopic boundary conditions. These results are consistent with the scale separation assumption relating the long scale modulation and the carrying local mode. They constitute a 3D generalization of the result established in Sec. II on the 1D network of resonators for the modulation of the simple mode.

This description is of the same nature as that established for elastic composites in Ref. 4, but it is formulated in different manner:

- (1) Considering multicells Ω (with periodic conditions) instead of the irreducible cell Ω_0 solely (with nonperiodic conditions) enables a simple and unified synthetic formalism for all possible situations.
- (2) The decomposition of tensor $\underline{\underline{T}}^K$ on the intrinsic physical basis made of the periodic modes of the considered cell Ω reveals its fundamental nature.
- (3) Additionally, in this form the symmetry of $\underline{\underline{T}}^K$ is trivially proved, and the parallel with the multiple modes case (see next sections) is straightforward.
- (4) Because of the specificity of porous media—i.e., homogeneous scalar elastic property P^e and uniform density ρ^e —the expression of $\underline{\underline{T}}^K$ is significantly simplified compared to the general case of elastic composites.

From its expression, the tensor $\underline{\underline{T}}^K$ is $O(1)$ (by construction, $|\widehat{\Phi^j \Phi^K}| = O(2\pi/\lambda_K) = O(\omega_K/\mathcal{C})$, where λ_K is the wavelength in air at ω_K) and symmetric, therefore diagonalizable. Thus any 3D modulation of the considered mode can be decomposed into specific modulations along the three principal directions of $\underline{\underline{T}}^K$. The principal values, denoted T_α^K , $\alpha = 1, 2, 3$, may be positive or negative. They are most likely to be different, which results in an anisotropy of the modulation phenomena. This was already mentioned and numerically evidenced for elastic composites in Ref. 4; however, the origin of the anisotropy could have been attributed to the vector nature of the displacement field. The poroacoustic case, proves that the anisotropy is intrinsically

related to the “geometry of modes” even if they concern a scalar variable (the pressure).

At the leading order, the amplitude of the modulation in the α -principal direction is governed by ($'$ stands for spatial derivative in the α -direction):

$$\mathcal{C}^2 T_\alpha^K A'' + (\omega^2 - \omega_K^2) A = 0.$$

This equation is of the same nature as Eq. (19) established in the discrete case for the amplitude modulation of the single mode of Ω_2 . The amplitude variation takes the classic form (x stands here for the macro variable in principal direction α):

$$A(x) = A_+ \exp(+i\kappa_\alpha x) + A_- \exp(-i\kappa_\alpha x),$$

where the “modulation-number” κ_α is given up to $o(\varepsilon)$ by

$$\kappa_\alpha(\omega) = \frac{1}{\mathcal{C}} \sqrt{\frac{\omega^2 - \omega_K^2}{T_\alpha^K}} = \frac{\omega_K}{\mathcal{C}} \sqrt{\frac{2}{T_\alpha^K} \left(\frac{\omega}{\omega_K} - 1 \right)}.$$

Thus provided that $(\omega - \omega_K)T_\alpha^K > 0$, $\kappa_\alpha(\omega)$ is real, the modulation oscillates and propagates with a “modulation-length” $\Lambda_\alpha \gg \lambda_K$ highly dependent on the frequency:

$$\Lambda_\alpha(\omega) = \lambda_K \sqrt{\frac{T_\alpha^K}{2} \frac{\omega_K}{\omega - \omega_K}}.$$

Conversely, when $(\omega - \omega_K)T_\alpha^K < 0$, $\kappa_\alpha(\omega)$ is imaginary, then modulation behaves as “diffuse modulation.” It presents oscillations with exponential decay, and the penetration depth, of the order of $|\Lambda_\alpha|$, is highly frequency dependent.

Note the drastic change of the modulation behavior when frequencies move from slightly lower to slightly higher than the modal frequency ω_K . At this frequency, the group velocity $(d\kappa_\alpha/d\omega)^{-1}$ tends to infinity. This situation corresponds to a border of a frequency band gap that may be situated either before or after ω_K according to the sign of T^K . Further, the asymmetric behavior can be inverted for different principal directions (anisotropy may result in two principal values of opposite signs). Clearly, propagative modulations enable long correlation lengths of high frequency waves.

D. Modulation of double modes

We now address the case of double modes at the frequency ω_K , say Φ^K and Ψ^K with $\langle \Phi^K, \Psi^K \rangle = 0$ and $\langle |\Phi^K|^2 \rangle = \langle |\Psi^K|^2 \rangle = 1$.

1. Derivation of the modulation equation

In presence of double modes, the solution of the leading order problem (25) becomes

$$\underline{u}^{(0)}(x, y) = A(x) \Phi^K(y) + B(x) \Psi^K(y).$$

The next problem is identical to Eq. (26) and introduces the same term $\underline{S}^{(1)}(\underline{u}^{(0)})$. However, two orthogonality conditions

are derived from the Fredholm alternative. Taking the scalar product of Eq. (26) by Φ^K —respectively, by Ψ^K —and of Eq. (25) (with $\underline{u}^{(0)} = \Phi^K$, respectively, $\underline{u}^{(0)} = \Psi^K$) by $\underline{u}^{(1)}$, integrating on Ω_f and subtracting, provides

$$\langle \underline{\mathcal{S}}^{(1)}(\underline{u}^{(0)}) \cdot \Phi^K \rangle = 0; \quad \langle \underline{\mathcal{S}}^{(1)}(\underline{u}^{(0)}) \cdot \Psi^K \rangle = 0.$$

As seen in Sec. III C 2:

$$\langle \mathcal{F}_{yx}(A(\underline{x})\Phi^K) \cdot \Phi^K \rangle = \langle \mathcal{F}_{yx}(B(\underline{x})\Psi^K) \cdot \Psi^K \rangle = 0,$$

thus, the orthogonality conditions simplify into

$$\begin{cases} \langle \mathcal{F}_{yx}(B(\underline{x})\Psi^K) \cdot \Phi^K \rangle + \omega^{(1)}2\omega_K A(\underline{x}) = 0 \\ \langle \mathcal{F}_{yx}(A(\underline{x})\Phi^K) \cdot \Psi^K \rangle + \omega^{(1)}2\omega_K B(\underline{x}) = 0 \end{cases}$$

which gives the two following coupled equations:

$$\begin{cases} \mathcal{C}^2 \underline{\text{grad}}_{\underline{x}}(B(\underline{x})) \cdot \overbrace{\Phi^K \Psi^K} = -\omega^{(1)}2\omega_K A(\underline{x}) \\ \mathcal{C}^2 \underline{\text{grad}}_{\underline{x}}(A(\underline{x})) \cdot \overbrace{\Psi^K \Phi^K} = -\omega^{(1)}2\omega_K B(\underline{x}). \end{cases}$$

Eliminating B (for example) in the preceding set, yields

$$\mathcal{C}^2 \text{div}_{\underline{x}}(\underline{R} \underline{\text{grad}}_{\underline{x}}(A(\underline{x}))) + 4(\omega^{(1)})^2 A(\underline{x}) = 0,$$

where the tensor \underline{R} is defined by (note the parallel—and the differences—with the tensor \underline{T}^K of simple modes)

$$\underline{R} = \frac{\mathcal{C}^2}{\omega_K^2} \overbrace{\Phi^K \Psi^K} \otimes \overbrace{\Psi^K \Phi^K}. \quad (30)$$

2. Features of double mode modulation

Using the $o(\varepsilon)$ -approximation: $\varepsilon\omega^{(1)} = \omega - \omega_K$, and coming back to the usual space variable $\underline{x} = \underline{x}/\varepsilon$, we derive the following leading order set of equations for the modulation at frequency $\omega = \omega_K + \omega^{(1)}\varepsilon$ of the double mode $(\omega_K, \Phi^K, \Psi^K)$:

$$\begin{cases} \mathcal{C}^2 \text{div}_{\underline{x}}(\underline{R} \underline{\text{grad}}_{\underline{x}}(A(\underline{x}))) + 4(\omega - \omega_K)^2 A(\underline{x}) = 0 \\ B(\underline{x}) = \underline{\text{grad}}_{\underline{x}}(A(\underline{x})) \cdot \overbrace{\Phi^K \Psi^K} \frac{\mathcal{C}^2}{2\omega_K(\omega - \omega_K)}. \end{cases}$$

The relation between $B(\underline{x})$ and $\underline{\text{grad}}_{\underline{x}}(A(\underline{x}))$ indicates that the modulation of a double mode necessarily involves the modulation of both modes. This coupling means that the perturbation induced by the modulation of the amplitude of one mode is balanced by the second mode (and reciprocally). As both modes interfere at the same eigenfrequency, the characteristics of the modulation of double and simple modes differ significantly.

Expression (30) of the symmetric tensor \underline{R} implies that there is a *single non-zero* principal value R^K associated with the principal vector $\overbrace{\Phi^K \Psi^K}$. This latter defines the unique

principal direction of modulation. In this direction, the amplitude of the modulation is driven by

$$\mathcal{C}^2 R^K A'' + 4(\omega - \omega_K)^2 A = 0,$$

where

$$R^K = \frac{\mathcal{C}^2}{\omega_K^2} |\overbrace{\Phi^K \Psi^K}|^2 > 0; \quad O(R^K) = O(1)$$

This equation is of the same nature as Eq. (23) established in the discrete case for the modulation of the double mode in Ω_3 . The variations of amplitude of A and B take the form (\underline{x} stands here for the macro variable in the principal direction $\overbrace{\Phi^K \Psi^K}$):

$$\begin{aligned} A(\underline{x}) &= A_+ \exp(+i\kappa \underline{x}) + A_- \exp(-i\kappa \underline{x}); \\ B(\underline{x}) &= i\kappa(A_+ \exp(+i\kappa \underline{x}) - A_- \exp(-i\kappa \underline{x})) \frac{\mathcal{C}^2 |\overbrace{\Phi^K \Psi^K}|}{2\omega_K(\omega - \omega_K)} \end{aligned}$$

with the real valued “modulation-number”:

$$\kappa(\omega) = \frac{1}{\mathcal{C}} \sqrt{\frac{4(\omega - \omega_K)^2}{R^K}} = \frac{2|\omega - \omega_K|}{\mathcal{C} \sqrt{R^K}}. \quad (31)$$

Thus the modulation is always a propagating phenomena. The modulation-length Λ_K is linked to the frequency by the relation

$$\Lambda_K(\omega) = \lambda_K \sqrt{\frac{R^K}{4}} \frac{\omega_K}{|\omega - \omega_K|} \gg \lambda_K.$$

Conversely to simple modes, the modulation behavior is symmetric on both sides of the modal frequency ω_K , and the group velocity $|(d\kappa/d\omega)^{-1}| = \mathcal{C}\sqrt{R^K}/2$ remains finite. The modulation length is inversely proportional to the frequency shift $|\omega - \omega_K|$ for double mode and to the square root of the frequency shift $\sqrt{|\omega - \omega_K|}$ for simple mode. Consequently, the frequency band of occurrence of double modes modulation is wider than for simple mode and longer correlation lengths can be expected.

E. Modulation of triple and multiple modes

We carry on the study considering now multiple modes at the frequency ω_K , beginning first with the case of triple modes (say Φ^K, Ψ^K, Θ^K with the normality and orthogonality properties).

1. Modulation of triple modes

As for the leading order problem Eq. (25), the solution reads

$$\underline{u}^{(0)}(\underline{x}, \underline{y}) = A(\underline{x})\Phi^K(\underline{y}) + B(\underline{x})\Psi^K(\underline{y}) + C(\underline{x})\Theta^K(\underline{y}).$$

Again, the next problem is identical to Eq. (26) and introduces the same term $\underline{\mathcal{S}}^{(1)}(\underline{u}^{(0)})$. However, three orthogonality conditions are now derived from the Fredholm alternative. Following the same reasoning as in the preceding text, we derive the three coupled equations (expressed with the usual variable \underline{x} and using the $o(\varepsilon)$ -approximation: $\varepsilon\omega^{(1)}2\omega_K = (\omega - \omega_K)^2$):

$$\begin{cases} C^2(\underline{\text{grad}}_{\underline{x}}(B) \cdot \overbrace{\Phi^K \Psi^K} + \underline{\text{grad}}_{\underline{x}}(C) \cdot \overbrace{\Phi^K \Theta^K}) = -2(\omega - \omega_K)\omega_K A \\ C^2(\underline{\text{grad}}_{\underline{x}}(A) \cdot \overbrace{\Psi^K \Phi^K} + \underline{\text{grad}}_{\underline{x}}(C) \cdot \overbrace{\Psi^K \Theta^K}) = -2(\omega - \omega_K)\omega_K B \\ C^2(\underline{\text{grad}}_{\underline{x}}(A) \cdot \overbrace{\Theta^K \Phi^K} + \underline{\text{grad}}_{\underline{x}}(B) \cdot \overbrace{\Theta^K \Psi^K}) = -2(\omega - \omega_K)\omega_K C. \end{cases}$$

Hence the modulation of the three components A, B, C is governed by an antisymmetric differential set. It is convenient to consider this problem in a vectorial form. For this purpose, we introduce the vectorial space \mathcal{V} of the three components “vectors” \mathbf{v} constituted by the amplitude of each mode and denote by $\mathbf{e}_i, i = 1, 2, 3$ the basis of this vectorial space, i.e., $\mathcal{V} = \{\mathbf{v}(\underline{x}); \mathbf{v}(\underline{x}) = A(\underline{x})\mathbf{e}_1 + B(\underline{x})\mathbf{e}_2 + C(\underline{x})\mathbf{e}_3\}$. The features of the modulation are investigated in the spatial Fourier domain by setting

$$\mathbf{v}(\underline{x}) = \mathbf{v}_0 \exp(i\kappa \underline{x} \cdot \underline{\alpha}); \quad \mathbf{v}_0 = (A_0 \mathbf{e}_1 + B_0 \mathbf{e}_2 + C_0 \mathbf{e}_3);$$

$$|\underline{\alpha}| = 1,$$

which corresponds to a “plane” modulation “traveling” in the direction $\underline{\alpha}$ with a “modulation”-number $\kappa(\omega, \underline{\alpha})$. Reporting this expression in the differential set with the following simplified notations:

$$a = \overbrace{\Psi^K \Theta^K}; \quad b = \overbrace{\Theta^K \Phi^K}; \quad c = \overbrace{\Phi^K \Psi^K}$$

leads to the following antisymmetric system of linear equations expressed in (3×3) matrix form:

$$iC^2 \kappa [M_3] \cdot \mathbf{v}_0 = -2(\omega - \omega_K)\omega_K \mathbf{v}_0;$$

$$[M_3] = \begin{pmatrix} 0 & c \cdot \underline{\alpha} & -b \cdot \underline{\alpha} \\ -c \cdot \underline{\alpha} & 0 & a \cdot \underline{\alpha} \\ b \cdot \underline{\alpha} & -a \cdot \underline{\alpha} & 0 \end{pmatrix}.$$

Therefore applying $[M_3]$ to the system gives

$$\begin{cases} -(C^2 \kappa)^2 [M_3]^2 \cdot \mathbf{v}_0 = 4(\omega - \omega_K)^2 \omega_K^2 \mathbf{v}_0 \\ [M_3]^2 = -|\mathbf{p}_3|^2 I + \mathbf{p}_3 \otimes \mathbf{p}_3 \\ \mathbf{p}_3(\underline{\alpha}) = (a \cdot \underline{\alpha} \mathbf{e}_1 + b \cdot \underline{\alpha} \mathbf{e}_2 + c \cdot \underline{\alpha} \mathbf{e}_3). \end{cases}$$

$[M_3]^2$ is a symmetric matrix, the principal directions of which are (i) $\mathbf{p}_3(\underline{\alpha})$ with zero principal value and (ii) any vector \mathbf{p}_\perp orthogonal to $\mathbf{p}_3(\underline{\alpha})$ with $-|\mathbf{p}_3(\underline{\alpha})|^2$ as principal value. Consequently, the modulation “traveling” in the given direction $\underline{\alpha}$ presents the following features:

(1) The amplitudes of the three modes are related by

$$A(\underline{x}) \left(\underline{\alpha} \cdot \overbrace{\Psi^K \Theta^K} \right) + B(\underline{x}) \left(\underline{\alpha} \cdot \overbrace{\Theta^K \Phi^K} \right) + C(\underline{x}) \left(\underline{\alpha} \cdot \overbrace{\Phi^K \Psi^K} \right) = 0.$$

(2) The common modulation number $\kappa(\omega, \underline{\alpha})$ and modulation-length Λ_K of the three modes are linked to the frequency and to the direction by

$$\kappa(\omega, \underline{\alpha}) = \frac{2\pi}{\Lambda_K(\omega, \underline{\alpha})} = \frac{2|\omega - \omega_K|}{C}$$

$$\times \sqrt{\frac{\omega_K^2 / C^2}{\left(\underline{\alpha} \cdot \overbrace{\Psi^K \Theta^K} \right)^2 + \left(\underline{\alpha} \cdot \overbrace{\Theta^K \Phi^K} \right)^2 + \left(\underline{\alpha} \cdot \overbrace{\Phi^K \Psi^K} \right)^2}}.$$

Recalling that $|\overbrace{\Phi^K \Psi^K}| = O(\omega_K / C)$ (and the same for other modes), the frequency dependence of modulation of triple modes is of the same nature as for double mode [see Eq. (31)], in particular the phenomenon is always propagating. However, modulation can travel in any direction with anisotropic properties.

2. Modulation of multiple modes

Multiple modes (e.g., n modes) and triple modes can be addressed in a similar manner. The equivalent to the matrix $[M_3]$ is a $n \times n$ antisymmetric matrix $[M_n]$. The diagonalizable matrix $[M_n]^2$ satisfies the relation $-(C^2 \kappa)^2 [M_n]^2 \cdot \mathbf{v}_0 = 4(\omega - \omega_K)^2 \omega_K^2 \mathbf{v}_0$, where $\mathbf{v}_0 = \sum_{m=1}^M A_0 \mathbf{e}_m$. Hence the characteristics of multiple mode modulation are defined by n eigenvectors and their associated eigenvalues.

IV. CONCLUSION

The governing equations describing the modulation at large scale of high frequency acoustic waves in periodic porous media has been established through the assumption of scale separation. Indeed, although high frequency implies that the physical variables vary locally according to periodic modes, long modulations of the mode amplitude consist of a particular type of scale separation enabling the application of asymptotic multi-scale methods.

Due to the specific type of scale separation, the study of modulation phenomena departs from the classic framework of homogenization of periodic media. In fact, the scale separation is commonly understood in the sense that a physical variable, relevant for the phenomena (pressure in poroacoustics), varies at the macroscale. It follows as essential consequences that in usual homogenization approaches:

- (1) The local regime in the period is quasi-static at the leading order (or at least in subdomains of the period), which implies a “low frequency” range of validity.
- (2) The relevant physical variable(s) is preserved identical though the upscaling.

- (3) A unique “equivalent media” description applies to the whole low frequency range.
- (4) The homogenized description is independent of the selected period (irreducible or not).
- (5) The properties of symmetry and coercivity of the physical tensors at the microscale are preserved at the macroscale for the effective tensors.

The derived description of modulation is also a macro-scale formulation, but it differs significantly in nature with classic macro-representations:

- (1) Through the upscaling process, the type of variable is changed (from pressure to mode amplitude) in addition to the change of differential operator.
- (2) Contrary to a single description, a *family* of descriptions is derived for modulation phenomena.
- (3) Each description is attached to a specific eigenmode carrying the high frequency modulated wave.
- (4) Consequently, a given description depends on the selected period (irreducible or not) and is valid only in the vicinity of the considered eigenfrequency.
- (5) The property of coercivity of physical tensors, which exist at the microscale, may be lost for the effective tensors; however, the symmetry properties are preserved at the macroscale.
- (6) Consistently with the macroscopic nature of the description, the parameters of the modulation equation are fully determined through the knowledge of the considered period (namely its eigenmodes).

Despite the differences in the results, the usual homogenization and the modulation approach are nevertheless closely related when observing that the zero frequency is an eigenvalue of the local problem with uniform pressure as associated eigenmode. Hence usual homogenization can be seen as a specific case of modulation (of the uniform pressure at low frequency) or the modulation appears as the natural extension of the homogenization (considering eigenmodes at higher frequency). Let us also underscore that the mathematical process of upscaling is identical for low or high frequencies. Further, we have seen that the Fredholm alternative plays a determining role in the modulation approach. This also the case in usual homogenization either for establishing the leading order balance equation¹ or for investigating Rayleigh scattering through higher order expansions, e.g., Ref. 16.

These comparative comments concern the asymptotic multi-scale method itself, hence they should be of general application, independently of the type of physical phenomena. Actually, elastic waves in composites⁴ and acoustic waves in porous media provide two examples of these methodological aspects.

Those principles are implemented by studying two types of periodic porous media. The academic case of networks of identical Helmholtz resonators presents the advantage to be reducible to a 1D-spring-mass system enabling analytical description of long range modulations. It is shown that these latter can be recovered independently through a multi-scale method devoted to periodic discrete media. Then following the multi-scale asymptotic method in the context of contin-

uum mechanics, we establish the modulation equation for 3D-porous media.

These several cases enable us to identify the general common features of modulations phenomena and their specificity attached to the 1D or 3D aspect of the media as well as its discrete or continuous nature. The derived macroscopic formulations demonstrate the existence of two types of modulation according to the simple or multiple nature of the considered mode with significant difference in terms of modulation length and width of the frequency band of occurrence. In particular, the frequencies of simple modes (of irreducible or multicells period) coincide with the beginning or the end of frequency band gaps that determine the nature of the modulation.

In the 3D case, the anisotropic feature of modulation is evidenced: Simple modes present three principal directions of modulation, double modes modulations are characterized by a single (non trivial) principal direction, and multiple modes travel in any direction with anisotropic characteristics.

Periodicity is a key assumption of the present approach. As for usual homogenization, from a practical point of view, the periodicity may actually exist or may be simply a convenient mathematical formulation of the statistically invariant representative volume element.³ Conversely, the periodicity is here required to define the periodic eigenmodes. Thus this assumption seems difficult to relax. However, the analysis should apply to imperfect periodic media (periodic with noise) for a limited number of eigenmodes (provided that the geometric imperfections remain negligible compared to the wave length).

The present study disregards the viscous and thermal dissipation effects. Nevertheless, the integration of dissipation would be of interest for practical concern. This can be handled through boundary layer approximations and complementary developments are in progress on this topic.

Modulation phenomena involve “full” dynamics at the local scale. This differs from situations of partial inner dynamics observed in double porosity media¹⁷ (eigenmodes within the weak permeability domain only), in meta-materials made of reticulated media¹⁸ (eigenmodes within the most deformable beams only), or made with highly contrasted composites,^{19–21} (eigenmodes within the softer constituent only). Beside, when focusing on the first mode of large multicells period Ω , the inertia becomes weak in each of the irreducible period Ω_0 . In this case, the modulation approach and the Rayleigh scattering description should tend to identical effects.

“Full” dynamics situation in periodic media can be described through multi-scattering approaches, e.g., Ref. 22 or by means of the Floquet–Bloch theory.²³ The specificity of the present approach is to extract, from the comprehensive Floquet–Bloch modal space, the particular frequency bands enabling large modulations, therefore large correlation, of high frequency acoustic perturbations. It also provides the governing modulation equation. This latter is expressed in a quite simple form that should provide simplifications in the formulation of high frequency propagation in periodic systems with reduced cost of calculations. The extension of this work to other domains of dynamics in heterogeneous media is underway.

APPENDIX

Recall some basic properties of the eigenvalue problem:

$$\begin{cases} \gamma P^e \underline{\text{grad}}_y (\text{div}_y(\underline{u})) + \rho^e \omega^2 \underline{u} = \underline{0}, & \text{on } \Omega_f \\ \underline{u} \cdot \underline{n} = 0, & \text{on } \Gamma; \quad \underline{u} \text{ } \Omega - \text{periodic} \end{cases}$$

whose the series of eigenfrequency and eigenmodes $\{\omega_j^2; \Phi^J\}$ of which satisfy

$$\begin{cases} \gamma P^e \underline{\text{grad}}_y (\text{div}_y(\Phi^J)) + \rho^e \omega_j^2 \Phi^J = \underline{0}, & \text{on } \Omega_f \\ \Phi^J \cdot \underline{n} = 0, & \text{on } \Gamma; \quad \Phi^J \text{ } \Omega - \text{periodic.} \end{cases} \quad (\text{A1})$$

The normalized eigenmodes $\{\Phi^J\}$ constitute an orthonormal basis for the Ω -periodic motions \underline{u} defined in Ω_f , with $\underline{u} \cdot \underline{n} = 0$ on Γ . For any field \underline{u} , \underline{w} satisfying the periodic and impenetrability condition on Γ_f and Γ , respectively, note the equality deduced from the divergence theorem:

$$\begin{aligned} \int_{\Omega_f} \underline{\text{grad}}_y \text{div}_y(\underline{v}) \cdot \underline{w} d\Omega &= - \int_{\Omega_f} \text{div}_y(\underline{v}) \text{div}_y(\underline{w}) d\Omega \\ &= \int_{\Omega_f} \underline{\text{grad}}_y (\text{div}_y(\underline{w})) \cdot \underline{v} d\Omega. \end{aligned} \quad (\text{A2})$$

Consider now the problem in presence of a source term \underline{S} (that may depend on y):

$$\begin{cases} \gamma P^e \underline{\text{grad}}_y (\text{div}_y(\underline{u})) + \rho^e \omega_K^2 \underline{u} + \underline{S} = \underline{0}, & \text{on } \Omega_f \\ \underline{u} \cdot \underline{n} = 0, & \text{on } \Gamma; \quad \underline{u} \text{ } \Omega - \text{periodic.} \end{cases} \quad (\text{A3})$$

Then taking the scalar product of Eq. (A3) by Φ^J ($J \neq K$) and of Eq. (A1) by \underline{u} , integrating on Ω_f and subtracting provides, using equality (A2):

$$\langle \underline{S}, \Phi^J \rangle + \rho^e (\omega_K^2 - \omega_j^2) \langle \underline{u}, \Phi^J \rangle = 0.$$

Because $\{\Phi^J\}$ constitutes an orthonormal basis on which \underline{u} can be decomposed, we derive the expression of the solution:

$$\underline{u} = \langle \underline{u}, \Phi^J \rangle \Phi^J + B \Phi^K = - \sum_{J \neq K} \frac{\langle \underline{S}, \Phi^J \rangle}{\rho^e (\omega_K^2 - \omega_j^2)} \Phi^J + B \Phi^K,$$

where B is an arbitrary constant.

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