

# Similarities and differences between the control theory of discrete and continuous time chaotic systems

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# Outline

- 1 Continuous-time systems analysis
- 2 Discrete-time system analysis
- 3 Graph-based approach for discrete-time systems

Let us consider the following continuous-time system

$$\dot{x} = f(x) + g(x)u, \quad x \in \mathbb{R}^n, u \in \mathbb{R}^{n_{\text{in}}}, \quad (1)$$

$$\xi = h(x), \quad \xi \in \mathbb{R}^{n_{\text{out}}}, \quad (2)$$

where  $f(x)$ ,  $g(x)$  and  $h(x)$  are sufficiently smooth vector functions.

Linear system i.e.  $f(x) = Ax$ ,  $g(x) = B$  and  $h(x) = Cx$ , there exists an observability condition:

The linear system is observable if and only if

$$\text{Rank} \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-2} \\ CA^{n-1} \end{pmatrix} = n \quad (3)$$

For nonlinear systems, the observability depends on the input, is local and there are no stopping criteria. Besides, there exist several definitions of observability as local [5], generic [3] etc. In a very simplified way, we can refer to the implicit functions theorem and find out on which order the output function  $\xi$  must be derived with respect to time. This leads to the simple test:

$$\text{Rank} \begin{pmatrix} d\xi \\ d\xi^{(1)} \\ \vdots \\ d\xi^{(n)} \\ \vdots \end{pmatrix} = n \quad (4)$$

where  $\xi^{(j)}$  denotes the  $i^{\text{th}}$  derivative of  $\xi$  with respect to time and  $d\bullet := \left( \frac{\partial \bullet}{\partial x_1}, \dots, \frac{\partial \bullet}{\partial x_n} \right)$ .

Let us consider the well-known Rossler dynamics [11]:

$$\begin{aligned}\dot{x}_1 &= -x_2 - x_3 \\ \dot{x}_2 &= x_1 + ax_2 \\ \dot{x}_3 &= b + x_3(x_1 - c)\end{aligned}\tag{5}$$

If we consider  $\xi = x_1$  as output, then (4) becomes

$$\text{Rank} \begin{pmatrix} d\xi \\ d\xi^{(1)} \\ d\xi^{(2)} \end{pmatrix} = \text{Rank} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & -1 \\ -(1+x_3) & -a & -(x_1-c) \end{pmatrix}$$

which has a state singularity in  $x_1 = a + c$  due to the nonlinearity.

Now, if we consider  $\xi = x_2$  as output, then (4) becomes

$$\text{Rank} \begin{pmatrix} d\xi \\ d\xi^{(1)} \\ d\xi^{(2)} \end{pmatrix} = \text{Rank} \begin{pmatrix} 0 & 1 & 0 \\ 1 & a & 0 \\ a & 1-a^2 & -1 \end{pmatrix}.$$

Since the seminal work of Fliess and co-authors [4], the flatness concept is used in many applications.

**Definition 1:**

The dynamic (1) with  $x \in \mathbb{R}^n$  and  $u \in \mathbb{R}^{n_{in}}$ , is *zero-flat* if there locally exist  $n_{out} = n_{in}$  smooth functions  $h_i = h_i(x)$ , where  $1 \leq i \leq n_{out}$  having the following property:

there exist an integer  $q$  and smooth functions  $\gamma_i$ ,  $1 \leq i \leq n$ , and  $\delta_j$ ,  $1 \leq j \leq n_{in}$ , such that locally

$$x_i = \gamma_i(h, \dot{h}, \dots, h^{(q-1)}) \text{ and } u_j = \delta_j(h, \dot{h}, \dots, h^{(q)}), \quad (6)$$

The function  $h = (h_1, \dots, h_{n_{out}})^T$  is called a *flat output*.

Example, let us consider again the Rossler dynamic (7) with  $\xi = x_2$  and add an input that appears linearly in the equation describing the time derivative of  $x_3$ . The resulting controlled system reads:

$$\begin{aligned}\dot{x}_1 &= -x_2 - x_3 \\ \dot{x}_2 &= x_1 + ax_2 \\ \dot{x}_3 &= b + x_3(x_1 - c) + u \\ \xi &= x_2\end{aligned}\tag{7}$$

It holds that  $\frac{\partial \xi}{\partial u} = 0$ ,  $\frac{\partial \xi^{(1)}}{\partial u} = 0$ ,  $\frac{\partial \xi^{(2)}}{\partial u} = 0$  and  $\frac{\partial \xi^{(3)}}{\partial u} = -1$ . All the properties like observability, controllability, input-output link hold without state singularity as it would hold for a linear system. In [10, 9]... a symbolic algorithm is proposed to check whether a system is flat or not along with an assessment of the quality of flatness.

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Let us now consider the invariant discrete-time system

$$x^+ = G(x, u), \quad x \in \mathbb{R}^n, u \in \mathbb{R}^{n_{\text{in}}} \quad (8)$$

$$\xi = h(x), \quad \xi \in \mathbb{R}^{n_{\text{out}}}. \quad (9)$$

where  $x^+$  stands for  $x(k+1)$  and  $x$  stands for  $x(k)$ . Moreover, we consider discrete-time dynamic with nonlinearity in  $u$ , because the composition of functions generally kills the input linearity e.g.  $x_1^+ = x_2^2$  and  $x_2^+ = u$  give  $x_1^{++} = u^2$ .

**Definition 2:** The invariant discrete system (8)-(9) is globally universally causally observable, if  $\forall x \in \mathbb{R}^n$  and  $\forall u \in \mathbb{R}^{n_{\text{in}}}$  there exists  $l \in \mathbb{R}^*$  and a function  $F$  such that

$$x^{-l} = F(\xi, \xi^-, \dots, \xi^{-l}, u, u^-, \dots, u^{-(l-1)}) \quad (10)$$

where  $\bullet^-$  stands for  $\bullet(k-1)$  and  $\bullet^{-l}$  stands for  $\bullet(k-l)$ .

**Definition 3:**

The invariant discrete system (8)-(9) is locally universally causally observable at  $x_0 := x(0)$ , if  $\forall u \in \mathbb{R}^{n_{\text{in}}}$ , there exist  $V_{x_0}$  a neighborhood of  $x_0$ ,  $l \in \mathbb{R}^*$  and a function  $F_{x_0}$  such that  $\forall x \in V_{x_0}$

$$x^{-l} = F_{x_0}(\xi, \xi^-, \dots, \xi^{-l}, u, u^-, \dots, u^{-(l-1)}) \quad (11)$$

**Proposition:**

The pair  $[G, \xi]$  is locally universally observable at  $x_0$  if and only if there exists  $l > 0$  such that  $\forall u \in \mathbb{R}^{n_{\text{in}}}$  the observability matrix

$$\mathcal{O}_{\xi^l}(x_0) = \begin{pmatrix} d(h(x_0)) \\ d(h \circ G(x_0, u)) \\ \vdots \\ d(h \circ G^{ol}(x_0, u)) \end{pmatrix} \quad (12)$$

is of rank  $n$  where  $G^{\circ 2}(x_0, u)$  is equal to  $G(G(x_0, u), u^+)$ ,  $G^{\circ 3}(x_0, u) = G(G(G(x_0, u), u^+), u^{++})$ , and so on.

As a discrete-time counterpart of flatness for continuous-time systems (let us recall [4]), the definition of a zero-flat nonlinear discrete-time system was given in [7]:

**Definition 4:**

The nonlinear discrete-time input-output system (8)-(9) is zero-flat if

- 1 There exists an integer  $k$  and a function  $F_{\text{state}}$ , such that, the state  $x$  can be rewritten as a function of the output, that is  $x = F_{\text{state}}(\xi, \xi^+, \dots, \xi^{+k})$
- 2 There exists an integer  $k$  and a function  $F_{\text{in}}$ , such that, the control input can be rewritten as a function of the output that is  $u = F_{\text{in}}(\xi, \xi^+, \dots, \xi^{+(k+1)})$ .

The discrete-time nonlinear canonical form given in [8] makes a connection between controllability, observability and flatness [7, 2]:

$$\begin{aligned}
 x_1^+ &= x_2 + a_1(x_1) \\
 x_2^+ &= x_3 + a_2(x_1, x_2) \\
 &\vdots \\
 &\vdots \\
 x_{n-1}^+ &= x_n + a_{n-1}(x_1, x_2, \dots, x_{n-1}) \\
 x_n^+ &= a_n(x_1, x_2, \dots, x_n, u)
 \end{aligned} \tag{13}$$

Systems (13) with  $a_n(x_1, x_2, \dots, x_n, u) = a(x_1, x_2, \dots, x_n) + bu$  and  $x_1$  as output  $\xi$  are zero-flat. In fact  $x_1 = \xi$ ,  $x_2 = \xi^+ - a_1(\xi)$ ,  $x_3 = \xi^{++} - a_1(\xi^+) - a_2(\xi, \xi^+ - a_1(\xi))$  and so on recursively. Consequently, from the fact that every  $x_i$  is function of  $y, y^+, \dots, y^{i+}$  and that  $x_n^+ = a_n(x_1, \dots, x_n) + bu$ , there exists a function  $F_{\text{in}}(\xi, \xi^+, \dots, \xi^{(n+1)+})$  such that  $u = F_{\text{in}}(\xi, \xi^+, \dots, \xi^{(n+1)+})$ .

Let us consider the nonlinear discrete-time system:

$$\begin{aligned}x_1^+ &= x_1 + x_2^2 \\x_2^+ &= u \\ \xi &= x_1\end{aligned}\tag{14}$$

The change of coordinates  $\zeta_1 = \xi$  and  $\zeta_2 = \xi + x_2^2$  gives

$$\begin{aligned}\zeta_1^+ &= \zeta_2 \\\zeta_2^+ &= \zeta_2 + u^2 \\ \xi &= \zeta_1\end{aligned}\tag{15}$$

This coordinate change is not a global diffeomorphism, the singularity is at  $x_2 = 0$ . Thus, even if the system (15) is observable the system (14) is not observable. Moreover, there is also a commandability in both system representations (i.e. (14) and (15) where  $x_2$  respectively  $\zeta_2$  can only increase.

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The aim is to propose a graph-based methodology [6, 1] in order to verify that for a given discrete map (1) with  $u = 0$  (autonomous system), the controlled system with input  $u \in \mathbb{R}^{n_{in}}$  and measurement  $\xi \in \mathbb{R}^{n_{in}}$  admits  $\xi$  as a flat output. This approach is restricted to discrete-time systems that can be written in the controlled form:

$$\Sigma : \begin{cases} x^+ & = Ax + Bu, \\ \xi & = Cx \end{cases} \quad (16)$$

where the entries of matrices  $A \in \mathbb{R}^{n \times n}$  and  $B \in \mathbb{R}^{n \times n_{in}}$  may be non constant but may depend on the output  $\xi \in \{x_1, x_2, \dots, x_n\}$ .

### Proposition

Consider the structured linear discrete-time system  $\Sigma$  described by (16). The output denoted by  $\xi \in \{x_1, x_2, \dots, x_n\}$ , associated to set of vertices  $\Xi$ , is generically a flat output if and only if, in the associated digraph  $\mathcal{G}(\Sigma)$ , the following three conditions hold:

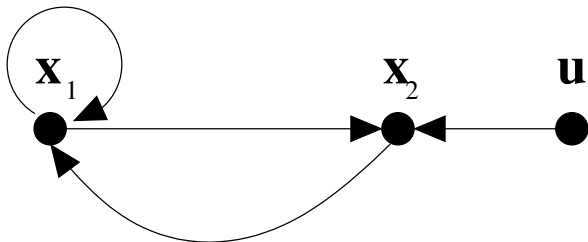
- ①  $\eta(\mathbf{U}, \Xi) = n_{in}$ .
- ② All the maximum  $\mathbf{U}$ - $\Xi$  linkings have the same length.
- ③ Every cycle in the digraph  $\mathcal{G}(\Sigma)$  covers at least an element of  $V_{ess}(\mathbf{U}, \Xi)$ .

Let us consider the Henon map described by

$$\begin{aligned}x_1^+ &= 1 - ax_1^2 + x_2 \\x_2^+ &= bx_1 \\ \xi &= x_1\end{aligned}\tag{17}$$

where  $a$  and  $b$  are real numbers. When applying a control input  $u$  on the second component  $x_2$ , the resulting dynamics can be rewritten like  $x^+ = Ax + Bu + f(\xi)$  with

$$A = \begin{bmatrix} -a\xi & 1 \\ b & 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$





$$\begin{aligned}x_1 &= \xi = F_{state,1}(\xi) \\x_2 &= \xi^+ - 1 + a\xi^2 = F_{state,2}(\xi, \xi^+) \\u &= \xi^{++} - 1 + a\xi^{++} - b\xi = F_{in}(\xi, \xi^+, \xi^{++})\end{aligned}$$



T. Boukhobza, J. van der Woude, C. Commault, and G. Millérioux.

A graph-oriented approach to address generically flat outputs in structured LTI discrete-time systems.

*Automatica*, 142:110344, 2022.



J. Diwold, B. Kolar, and M. Schöberl.

A trajectory-based approach to discrete-time flatness.

*IEE Control Systems Letters*, 6:289–294, 2022.



M. Fliess and S. T. Glad.

*An Algebraic Approach to Linear and Nonlinear Control*, pages 223–267.

Birkhäuser Boston, Boston, MA, 1993.



M. Fliess, J. L. Lévine, P. Martin, and P. Rouchon.

Flatness and defect of non-linear systems: introductory theory and examples.

*International Journal of Control*, 61(6):1327–1361, 1995.



Madalin Frunzete, Jean-Pierre Barbot, and Christophe Letellier.

Influence of the singular manifold of nonobservable states in reconstructing chaotic attractors.

*Physical Review E*, 86(2):026205, 2012.



Philippe Guillot and Gilles Millerioux.

Flatness and submersivity of discrete-time dynamical systems.

*IEEE Control Systems Letters*, 4(2):337–342, 2020.



A. Kaldmäe and U. Kotta.

On flatness of discrete-time nonlinear systems.

*9th IFAC Symposium on Nonlinear Control Systems, Toulouse, France*, pages 588–593, 2013.



U. Kotta.

Controller and controllability canonical forms for discrete-time non-linear systems.

*5th IFAC Symposium on Nonlinear Control Systems, St Petersburg, Russia*, pages 391–395, 2001.



C. Letellier and J.-P. Barbot.

Optimal flatness placement of sensors and actuators for controlling chaotic systems.

*Chaos*, 31(10):103–114, 2021.



Christophe Letellier, I. Leyva, and I. Sendiña Nadal.

Dynamical complexity measure to distinguish organized from disorganized dynamics.

*Physical Review E*, 101(2):022204, 2020.



O. E. RöSSLer.

An equation for continuous chaos.

*Physics Letters A*, 57(5):397–398, 1976.