



**HAL**  
open science

# Similarities and differences between the control theory of discrete and continuous time chaotic systems

Jean-Pierre Barbot, Christophe Letellier, Gilles Millerioux

## ► To cite this version:

Jean-Pierre Barbot, Christophe Letellier, Gilles Millerioux. Similarities and differences between the control theory of discrete and continuous time chaotic systems. *OttoChaos: From the nonlinear dynamical systems theory to observational chaos*, Oct 2023, Toulouse, France. hal-04302051

**HAL Id: hal-04302051**

**<https://hal.science/hal-04302051v1>**

Submitted on 23 Nov 2023

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Similarities and differences between the control theory of discrete and continuous time chaotic systems

JP Barbot<sup>1,2</sup>, C. Letellier<sup>3</sup> and G. Millerioux<sup>4</sup>

<sup>1</sup> Ecole Centrale de Nantes-LS2N, UMR 6004 CNRS, France

<sup>2</sup> ENSEA Quartz Laboratory EA 7393, France

<sup>3</sup> Rouen Normandy University, Av. de l'Université, F-76801 Saint-Etienne du Rouvray, France

<sup>4</sup> Université de Lorraine, CNRS, CRAN, F-54000 Nancy, France

# Outline

- 1 Continuous-time systems analysis
- 2 Discrete-time system analysis
- 3 Graph-based approach for discrete-time systems

Let us consider the following continuous-time system

$$\dot{x} = f(x) + g(x)u, \quad x \in \mathbb{R}^n, u \in \mathbb{R}^{n_{\text{in}}}, \quad (1)$$

$$\xi = h(x), \quad \xi \in \mathbb{R}^{n_{\text{out}}}, \quad (2)$$

where  $f(x)$ ,  $g(x)$  and  $h(x)$  are sufficiently smooth vector functions.

Linear system i.e.  $f(x) = Ax$ ,  $g(x) = B$  and  $h(x) = Cx$ , there exists an observability condition:

The linear system is observable if and only if

$$\text{Rank} \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-2} \\ CA^{n-1} \end{pmatrix} = n \quad (3)$$

For nonlinear systems, the observability depends on the input, is local and there are no stopping criteria. Besides, there exist several definitions of observability as local [5], generic [3] etc. In a very simplified way, we can refer to the implicit functions theorem and find out on which order the output function  $\xi$  must be derived with respect to time. This leads to the simple test:

$$\text{Rank} \begin{pmatrix} d\xi \\ d\xi^{(1)} \\ \vdots \\ d\xi^{(n)} \\ \vdots \end{pmatrix} = n \quad (4)$$

where  $\xi^{(j)}$  denotes the  $i^{\text{th}}$  derivative of  $\xi$  with respect to time and  $d\bullet := \left( \frac{\partial \bullet}{\partial x_1}, \dots, \frac{\partial \bullet}{\partial x_n} \right)$ .

Let us consider the well-known Rossler dynamics [11]:

$$\begin{aligned}\dot{x}_1 &= -x_2 - x_3 \\ \dot{x}_2 &= x_1 + ax_2 \\ \dot{x}_3 &= b + x_3(x_1 - c)\end{aligned}\tag{5}$$

If we consider  $\xi = x_1$  as output, then (4) becomes

$$\text{Rank} \begin{pmatrix} d\xi \\ d\xi^{(1)} \\ d\xi^{(2)} \end{pmatrix} = \text{Rank} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & -1 \\ -(1+x_3) & -a & -(x_1-c) \end{pmatrix}$$

which has a state singularity in  $x_1 = a + c$  due to the nonlinearity.

Now, if we consider  $\xi = x_2$  as output, then (4) becomes

$$\text{Rank} \begin{pmatrix} d\xi \\ d\xi^{(1)} \\ d\xi^{(2)} \end{pmatrix} = \text{Rank} \begin{pmatrix} 0 & 1 & 0 \\ 1 & a & 0 \\ a & 1-a^2 & -1 \end{pmatrix}.$$

Since the seminal work of Fliess and co-authors [4], the flatness concept is used in many applications.

**Definition 1:**

The dynamic (1) with  $x \in \mathbb{R}^n$  and  $u \in \mathbb{R}^{n_{in}}$ , is *zero-flat* if there locally exist  $n_{out} = n_{in}$  smooth functions  $h_i = h_i(x)$ , where  $1 \leq i \leq n_{out}$  having the following property:

there exist an integer  $q$  and smooth functions  $\gamma_i$ ,  $1 \leq i \leq n$ , and  $\delta_j$ ,  $1 \leq j \leq n_{in}$ , such that locally

$$x_i = \gamma_i(h, \dot{h}, \dots, h^{(q-1)}) \text{ and } u_j = \delta_j(h, \dot{h}, \dots, h^{(q)}), \quad (6)$$

The function  $h = (h_1, \dots, h_{n_{out}})^T$  is called a *flat output*.

Example, let us consider again the Rossler dynamic (7) with  $\xi = x_2$  and add an input that appears linearly in the equation describing the time derivative of  $x_3$ . The resulting controlled system reads:

$$\begin{aligned}\dot{x}_1 &= -x_2 - x_3 \\ \dot{x}_2 &= x_1 + ax_2 \\ \dot{x}_3 &= b + x_3(x_1 - c) + u \\ \xi &= x_2\end{aligned}\tag{7}$$

It holds that  $\frac{\partial \xi}{\partial u} = 0$ ,  $\frac{\partial \xi^{(1)}}{\partial u} = 0$ ,  $\frac{\partial \xi^{(2)}}{\partial u} = 0$  and  $\frac{\partial \xi^{(3)}}{\partial u} = -1$ . All the properties like observability, controllability, input-output link hold without state singularity as it would hold for a linear system. In [10, 9]... a symbolic algorithm is proposed to check whether a system is flat or not along with an assessment of the quality of flatness.



# Outline

- 1 Continuous-time systems analysis
- 2 Discrete-time system analysis**
- 3 Graph-based approach for discrete-time systems

Let us now consider the invariant discrete-time system

$$x^+ = G(x, u), \quad x \in \mathbb{R}^n, u \in \mathbb{R}^{n_{\text{in}}} \quad (8)$$

$$\xi = h(x), \quad \xi \in \mathbb{R}^{n_{\text{out}}}. \quad (9)$$

where  $x^+$  stands for  $x(k+1)$  and  $x$  stands for  $x(k)$ . Moreover, we consider discrete-time dynamic with nonlinearity in  $u$ , because the composition of functions generally kills the input linearity e.g.  $x_1^+ = x_2^2$  and  $x_2^+ = u$  give  $x_1^{++} = u^2$ .

**Definition 2:** The invariant discrete system (8)-(9) is globally universally causally observable, if  $\forall x \in \mathbb{R}^n$  and  $\forall u \in \mathbb{R}^{n_{\text{in}}}$  there exists  $l \in \mathbb{R}^*$  and a function  $F$  such that

$$x^{-l} = F(\xi, \xi^-, \dots, \xi^{-l}, u, u^-, \dots, u^{-(l-1)}) \quad (10)$$

where  $\bullet^-$  stands for  $\bullet(k-1)$  and  $\bullet^{-l}$  stands for  $\bullet(k-l)$ .

**Definition 3:**

The invariant discrete system (8)-(9) is locally universally causally observable at  $x_0 := x(0)$ , if  $\forall u \in \mathbb{R}^{n_{\text{in}}}$ , there exist  $V_{x_0}$  a neighborhood of  $x_0$ ,  $l \in \mathbb{R}^*$  and a function  $F_{x_0}$  such that  $\forall x \in V_{x_0}$

$$x^{-l} = F_{x_0}(\xi, \xi^-, \dots, \xi^{-l}, u, u^-, \dots, u^{-(l-1)}) \quad (11)$$

**Proposition:**

The pair  $[G, \xi]$  is locally universally observable at  $x_0$  if and only if there exists  $l > 0$  such that  $\forall u \in \mathbb{R}^{n_{\text{in}}}$  the observability matrix

$$\mathcal{O}_{\xi^l}(x_0) = \begin{pmatrix} d(h(x_0)) \\ d(h \circ G(x_0, u)) \\ \vdots \\ d(h \circ G^{ol}(x_0, u)) \end{pmatrix} \quad (12)$$

is of rank  $n$  where  $G^{\circ 2}(x_0, u)$  is equal to  $G(G(x_0, u), u^+)$ ,  $G^{\circ 3}(x_0, u) = G(G(G(x_0, u), u^+), u^{++})$ , and so on.

As a discrete-time counterpart of flatness for continuous-time systems (let us recall [4]), the definition of a zero-flat nonlinear discrete-time system was given in [7]:

**Definition 4:**

The nonlinear discrete-time input-output system (8)-(9) is zero-flat if

- 1 There exists an integer  $k$  and a function  $F_{\text{state}}$ , such that, the state  $x$  can be rewritten as a function of the output, that is  $x = F_{\text{state}}(\xi, \xi^+, \dots, \xi^{+k})$
- 2 There exists an integer  $k$  and a function  $F_{\text{in}}$ , such that, the control input can be rewritten as a function of the output that is  $u = F_{\text{in}}(\xi, \xi^+, \dots, \xi^{+(k+1)})$ .

The discrete-time nonlinear canonical form given in [8] makes a connection between controllability, observability and flatness [7, 2]:

$$\begin{aligned}
 x_1^+ &= x_2 + a_1(x_1) \\
 x_2^+ &= x_3 + a_2(x_1, x_2) \\
 &\vdots \\
 &\vdots \\
 x_{n-1}^+ &= x_n + a_{n-1}(x_1, x_2, \dots, x_{n-1}) \\
 x_n^+ &= a_n(x_1, x_2, \dots, x_n, u)
 \end{aligned} \tag{13}$$

Systems (13) with  $a_n(x_1, x_2, \dots, x_n, u) = a(x_1, x_2, \dots, x_n) + bu$  and  $x_1$  as output  $\xi$  are zero-flat. In fact  $x_1 = \xi$ ,  $x_2 = \xi^+ - a_1(\xi)$ ,  $x_3 = \xi^{++} - a_1(\xi^+) - a_2(\xi, \xi^+ - a_1(\xi))$  and so on recursively. Consequently, from the fact that every  $x_i$  is function of  $y, y^+, \dots, y^{i+}$  and that  $x_n^+ = a_n(x_1, \dots, x_n) + bu$ , there exists a function  $F_{\text{in}}(\xi, \xi^+, \dots, \xi^{(n+1)+})$  such that  $u = F_{\text{in}}(\xi, \xi^+, \dots, \xi^{(n+1)+})$ .

Let us consider the nonlinear discrete-time system:

$$\begin{aligned}x_1^+ &= x_1 + x_2^2 \\x_2^+ &= u \\ \xi &= x_1\end{aligned}\tag{14}$$

The change of coordinates  $\zeta_1 = \xi$  and  $\zeta_2 = \xi + x_2^2$  gives

$$\begin{aligned}\zeta_1^+ &= \zeta_2 \\\zeta_2^+ &= \zeta_2 + u^2 \\ \xi &= \zeta_1\end{aligned}\tag{15}$$

This coordinate change is not a global diffeomorphism, the singularity is at  $x_2 = 0$ . Thus, even if the system (15) is observable the system (14) is not observable. Moreover, there is also a commandability in both system representations (i.e. (14) and (15) where  $x_2$  respectively  $\zeta_2$  can only increase.

# Outline

- 1 Continuous-time systems analysis
- 2 Discrete-time system analysis
- 3 Graph-based approach for discrete-time systems**

The aim is to propose a graph-based methodology [6, 1] in order to verify that for a given discrete map (1) with  $u = 0$  (autonomous system), the controlled system with input  $u \in \mathbb{R}^{n_{in}}$  and measurement  $\xi \in \mathbb{R}^{n_{in}}$  admits  $\xi$  as a flat output. This approach is restricted to discrete-time systems that can be written in the controlled form:

$$\Sigma : \begin{cases} x^+ & = Ax + Bu, \\ \xi & = Cx \end{cases} \quad (16)$$

where the entries of matrices  $A \in \mathbb{R}^{n \times n}$  and  $B \in \mathbb{R}^{n \times n_{in}}$  may be non constant but may depend on the output  $\xi \in \{x_1, x_2, \dots, x_n\}$ .

### Proposition

Consider the structured linear discrete-time system  $\Sigma$  described by (16). The output denoted by  $\xi \in \{x_1, x_2, \dots, x_n\}$ , associated to set of vertices  $\Xi$ , is generically a flat output if and only if, in the associated digraph  $\mathcal{G}(\Sigma)$ , the following three conditions hold:

- ①  $\eta(\mathbf{U}, \Xi) = n_{in}$ .
- ② All the maximum  $\mathbf{U}$ - $\Xi$  linkings have the same length.
- ③ Every cycle in the digraph  $\mathcal{G}(\Sigma)$  covers at least an element of  $V_{ess}(\mathbf{U}, \Xi)$ .

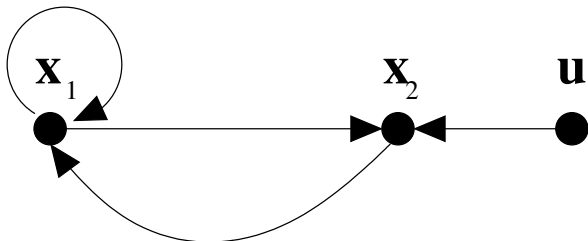


Let us consider the Henon map described by

$$\begin{aligned}x_1^+ &= 1 - ax_1^2 + x_2 \\x_2^+ &= bx_1 \\ \xi &= x_1\end{aligned}\tag{17}$$

where  $a$  and  $b$  are real numbers. When applying a control input  $u$  on the second component  $x_2$ , the resulting dynamics can be rewritten like  $x^+ = Ax + Bu + f(\xi)$  with

$$A = \begin{bmatrix} -a\xi & 1 \\ b & 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$



$$\begin{aligned}x_1 &= \xi = F_{state,1}(\xi) \\x_2 &= \xi^+ - 1 + a\xi^2 = F_{state,2}(\xi, \xi^+) \\u &= \xi^{++} - 1 + a\xi^{++} - b\xi = F_{in}(\xi, \xi^+, \xi^{++})\end{aligned}$$



T. Boukhobza, J. van der Woude, C. Commault, and G. Millérioux.

A graph-oriented approach to address generically flat outputs in structured LTI discrete-time systems.

*Automatica*, 142:110344, 2022.



J. Diwold, B. Kolar, and M. Schöberl.

A trajectory-based approach to discrete-time flatness.

*IEE Control Systems Letters*, 6:289–294, 2022.



M. Fliess and S. T. Glad.

*An Algebraic Approach to Linear and Nonlinear Control*, pages 223–267.

Birkhäuser Boston, Boston, MA, 1993.



M. Fliess, J. L. Lévine, P. Martin, and P. Rouchon.

Flatness and defect of non-linear systems: introductory theory and examples.

*International Journal of Control*, 61(6):1327–1361, 1995.



Madalin Frunzete, Jean-Pierre Barbot, and Christophe Letellier.

Influence of the singular manifold of nonobservable states in reconstructing chaotic attractors.

*Physical Review E*, 86(2):026205, 2012.



Philippe Guillot and Gilles Millerioux.

Flatness and submersivity of discrete-time dynamical systems.

*IEEE Control Systems Letters*, 4(2):337–342, 2020.



A. Kaldmäe and U. Kotta.

On flatness of discrete-time nonlinear systems.

*9th IFAC Symposium on Nonlinear Control Systems, Toulouse, France*, pages 588–593, 2013.



U. Kotta.

Controller and controllability canonical forms for discrete-time non-linear systems.

*5th IFAC Symposium on Nonlinear Control Systems, St Petersburg, Russia*, pages 391–395, 2001.



C. Letellier and J.-P. Barbot.

Optimal flatness placement of sensors and actuators for controlling chaotic systems.

*Chaos*, 31(10):103–114, 2021.



Christophe Letellier, I. Leyva, and I. Sendiña Nadal.

Dynamical complexity measure to distinguish organized from disorganized dynamics.

*Physical Review E*, 101(2):022204, 2020.



O. E. RöSSLer.

An equation for continuous chaos.

*Physics Letters A*, 57(5):397–398, 1976.